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Smooth subvarieties of Jacobians

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À Claire, pour son soixantième anniversaire

Abstract. We give new examples of algebraic integral cohomology classes on smooth projective complex varieties that are not integral linear combinations of classes of smooth subvarieties. Some of our examples have dimension 6, the lowest possible. The classes that we consider are minimal cohomology classes on Jacobians of very general curves. Our main tool is complex cobordism.

Keywords. Algebraic cycles, Jacobians, complex cobordism

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Contents

1. Introduction

Let c, d be nonnegative integers and let X be a smooth projective complex variety of dimension $n := c + d$. An important geometric invariant of X is the subgroup $H^{2c}(X,\mathbb{Z})_{\mathrm{alg}} \subset H^{2c}(X,\mathbb{Z})$ of algebraic cohomology classes, which is generated by the cycle classes of codimension c algebraic subvarieties $Y \subset X$. Since the subvarieties Y may be singular, the following question going back to Borel and Haefliger [BH61, Section 5.17] naturally arises.

Question 1.1. Is the group $H^{2c}(X,\mathbb{Z})_{\mathrm{alg}}$ generated by classes of smooth subvarieties of X?

The answer to Question 1.1 is obviously positive if $d = 0$ or if $c \leq 1$. Further positive answers were obtained by Hironaka [Hir68, Theorem, Section 5, p. 50] when $d \leq \min(3, c-1)$ and by Kleiman [Kle69, Theorem 5.8] when $c = 2$ and $d \in \{2, 3\}$; the answer is therefore positive for all $n \leq 5$.

A first counterexample was then constructed by Hartshorne, Rees, and Thomas [HRT74, Theorem 1] when $c = 2$ and $d \geq 7$ (and their method should yield counterexamples for all $c \geq 2$ and $d \gg c$). Other counterexamples were given by Debarre [Deb95, Théorème 6] when $c = 2$ and $d \geq 5$, and by Benoist [Ben20, Theorem 0.3] when $d \geq c$ and $\alpha(c+1) \geq 3$ (where $\alpha(m)$ denotes the number of ones in the binary expansion of m).

In this article, we build on the counterexample given in [Deb95]. There, Debarre considers the Jacobian X of a smooth projective complex curve C of genus n, polarized by its theta divisor class $\theta \in H^2(X,\mathbb{Z})$. The minimal cohomology class $\frac{\theta^c}{c!} \in H^{2c}(X,\mathbb{Z})$ is the cycle class of the image $W_{n-c}(C) \subset X$ of the symmetric power $C^{(n-c)}$ by the Abel–Jacobi map, hence is algebraic. However, when $n \geq 2c+2$, the variety $W_{n-c}(C)$ is singular and there is no general reason why its class $\frac{\theta^c}{c!}$ should be a $\mathbb{Z}$-linear combination of classes of smooth subvarieties of X. Debarre shows that this is indeed not the case if $n \geq 7$, $c = 2$, and X is very general.

Our main theorem extends this result to many other values of (c, n). Recall that $\alpha(m)$ is the number of ones in the binary expansion of m.

Theorem 1.2 (Theorem 3.7). *Let (X,θ) be a very general complex Jacobian of dimension n. Let c be a nonnegative integer such that $\alpha(c+\alpha(c)) > \alpha(c)$ and $n \geq 4c-2$. Then the integral classes $\lambda\frac{\theta^c}{c!}$ with λ odd are algebraic but are not $\mathbb{Z}$-linear combinations of cycle classes of smooth subvarieties of X.*

The weird condition $\alpha(c+\alpha(c)) > \alpha(c)$ springs naturally from our proof. It holds for integers c in the set $\{2, 4, 5, 8, 9, 12, 16, 17, \dots\}$. We have nothing to say when $c = 3$: we do not know if there exist Jacobians (X,θ) such that $\frac{\theta^3}{3!}$ is not a $\mathbb{Z}$-linear combination of cycle classes of smooth subvarieties of X. Applied with $c = 2$ and $n = 6$, Theorem 1.2 implies the following result.

Corollary 1.3. *There exists a smooth projective complex variety X of dimension 6 such that $H^4(X,\mathbb{Z})_{\mathrm{alg}}$ is not generated by classes of smooth subvarieties of X.*

As noted above, the groups of algebraic cohomology classes of smooth projective varieties of dimension at most 5 are generated by classes of smooth subvarieties. The dimension of the variety X in Corollary 1.3 is thus the lowest possible. We do not know, however, if Question 1.1 has a positive answer if $c = d = 3$.

The proof of Theorem 1.2 when $c = 2$ and $n \geq 7$ given in [Deb95] relies on a Barth–Lefschetz-type theorem for abelian varieties, on the Hirzebruch–Riemann–Roch theorem, and on the Serre construction. We can still rely on the same Barth–Lefschetz-type theorem under our more general hypotheses. However, computations based on the Hirzebruch–Riemann–Roch theorem become untractable in high codimension (and do not yield a proof when $c = 2$ and $n = 6$), and we cannot use the Serre construction in general because it is specific to codimension 2 subvarieties.

Both difficulties are overcome by resorting to complex cobordism, whose use in the theory of algebraic cycles was pioneered by Totaro [Tot97]. We replace the integrality results derived from the Hirzebruch–Riemann–Roch theorem with divisibility properties of Chern numbers (Proposition 2.5). Those are obtained in Section 2 as a consequence of a detailed understanding of the structure of the complex cobordism ring. In [Deb95], the Serre construction was used to infer restrictions on the cohomology class of a smooth subvariety $Y \subset X$. Instead, we consider the class of Y in the complex cobordism of X and use the description of the complex cobordism of an abelian variety (in Sections 3.1–3.2). These tools are combined in Section 3.3 to prove Theorem 1.2.

In low codimension, a different method based on complex topological K-theory and on the Grothendieck–Riemann–Roch theorem, closer to the one used in [Deb95], gives small improvements on Theorem 1.2. We use this method in Section 4 to prove the following.

Theorem 1.4 (Theorem 4.3). *Let (X,θ) be a very general complex Jacobian of dimension n. Then $\lambda\frac{\theta^4}{4!}$ is algebraic but not a $\mathbb{Z}$-linear combination of classes of smooth subvarieties of X*

(a) *if $n \geq 12$ and λ is odd;*
(b) *if $n \geq 14$ and λ is not divisible by 4.*

Conventions

A complex variety is a separated scheme of finite type over $\mathbb{C}$. If M is a compact oriented manifold of dimension d, we let $[M] \in H_d(M,\mathbb{Z})$ be the fundamental class of M and denote by $\deg_M : H^d(M,\mathbb{Z}) \to \mathbb{Z}$ the morphism $\omega \mapsto \deg(\omega \frown [M])$. We let $\alpha(m)$ denote the number of ones in the binary expansion of m.

2. Congruences of Chern numbers

2.1. The Hurewicz morphism of MU

In this paragraph, we recall the structure of the Hurewicz morphism

$$H : \pi_*(\mathbf{MU}) \longrightarrow H_*(\mathbf{MU},\mathbb{Z}) \tag{2.1}$$

of the spectrum **MU** representing complex cobordism.

The computation of the cohomology of complex Grassmannians and the Thom isomorphism combine to show that $H_*(\mathbf{MU},\mathbb{Z})$ is a polynomial ring with integral coefficients with one generator in degree $2i$ for each $i \geq 1$ (see [Ada74, Section I.3]). A deep theorem of Milnor shows that $\pi_*(\mathbf{MU})$ is also a polynomial ring with integral coefficients with one generator in degree $2i$ for each $i \geq 1$ and that H is injective (see [Ada74, Theorem II.8.1 and Corollary II.8.11]). Quillen's theorem identifying $\pi_*(\mathbf{MU})$ with the coefficient ring of the universal formal group law [Ada74, Theorem II.8.2] and Hazewinkel's explicit construction of a universal formal group law [Haz78] allow us to be more precise.

Proposition 2.1. *For each $i \geq 1$, there exist $u_i \in \pi_{2i}(\mathbf{MU})$ and $v_i \in H_{2i}(\mathbf{MU}, \mathbb{Z})$ such that*

(a) $\pi_*(\mathbf{MU}) = \mathbb{Z}[u_i]_{i\geq 1}$;
(b) $H_*(\mathbf{MU}, \mathbb{Z}) = \mathbb{Z}[v_i]_{i\geq 1}$;
(c) $H(u_i) = \lambda_i v_i$, *where $\lambda_i = p$ if $i = p^t - 1$ for some $t \geq 1$ and some prime number p, and $\lambda_i = 1$ otherwise.*

Proof. Let $u_i \in \pi_{2i}(\mathbf{MU})$ be the polynomial generators of $\pi_*(\mathbf{MU})$ specified in [Haz78, Section 34.4.1]. Induction on i using the formula [Haz78, (34.4.3)] shows that there exist $v_i \in H_{2i}(\mathbf{MU}, \mathbb{Z})$ such that $H(u_i) = \lambda_i v_i$. It then follows from [Ada74, Lemmas II.7.9(iii) and II.8.10] that v_i generates $H_{2i}(\mathbf{MU}, \mathbb{Z})$ modulo its decomposable elements. Consequently, $H_*(\mathbf{MU}, \mathbb{Z}) = \mathbb{Z}[v_i]_{i\geq 1}$. □

For each $e \geq 1$, define the ideal $I_e \subset \pi_*(\mathbf{MU})$ to be the kernel of the composition

$$\pi_*(\mathbf{MU}) \xrightarrow{H} H_*(\mathbf{MU}, \mathbb{Z}) \longrightarrow H_*(\mathbf{MU}, \mathbb{Z})/2^e$$

of H and the reduction modulo 2^e. Working in the monomial bases associated with the u_i and the v_i given by Proposition 2.1 shows at once the following result.

Lemma 2.2. *One has $I_1 = \langle 2, u_{2^t-1}\rangle_{t\geq 1}$ and $I_e = (I_1)^e$.*

2.2. Chern numbers

Let MU_* be the complex cobordism ring, whose degree d elements are complex cobordism classes of d-dimensional compact stably almost complex manifolds. The Thom–Pontrjagin construction gives an identification

$$\xi\colon \mathrm{MU}_* \xrightarrow{\sim} \pi_*(\mathbf{MU}) \tag{2.2}$$

(apply [Koc96, Theorem 1.5.10] with $\mathfrak{B} = \mathrm{BU}$).

Consider the polynomial ring $\mathbb{Z}[c_j]_{j\geq 1}$ in the Chern classes c_j, graded so that c_j has degree j. A degree i element $P \in \mathbb{Z}[c_j]_{j\geq 1}$ may be evaluated on a $2i$-dimensional compact stably almost complex manifold M by setting $P(M) := \deg_M(P(c_j(M)))$. This integer only depends on the complex cobordism class of M (see [Koc96, Section 4.3, p. 135], with $\mathfrak{B} = \mathrm{BU}$ and E the ordinary integral cohomology spectrum), so we get a morphism $P\colon \mathrm{MU}_{2i} \to \mathbb{Z}$, called the Chern number associated with P. The following lemma is an immediate consequence of [Koc96, Proposition 4.3.8].

Lemma 2.3. *Let $i \geq 0$. A morphism $\mathrm{MU}_{2i} \to \mathbb{Z}$ is a Chern number if and only if it may be written as $\psi \circ H \circ \xi$ for some group morphism $\psi\colon H_{2i}(\mathbf{MU}, \mathbb{Z}) \to \mathbb{Z}$, where H and ξ are as in (2.1) and (2.2).*

We define the Segre classes $s_i \in \mathbb{Z}[c_j]_{j\geq 1}$ to be the unique elements such that s_i has degree i and

$$\Big(\sum\nolimits_j c_j\Big)\Big(\sum\nolimits_i s_i\Big) = 1. \tag{2.3}$$

The Chern numbers associated with s_i have the following properties.

Lemma 2.4.

(a) *If $x \in \mathrm{MU}_{2i}$ and $x' \in \mathrm{MU}_{2i'}$, then $s_{i+i'}(xx') = s_i(x)s_{i'}(x')$.*
(b) *For $i \geq 0$ and $h \geq 1$, the function $s_i\colon \mathrm{MU}_{2i} \to \mathbb{Z}$ is divisible by 2^h if and only if $\alpha(i + h - 1) > 2h - 2$.*
(c) *For $i \geq 1$, the function $s_i\colon \mathrm{MU}_{2i} \to \mathbb{Z}$ only takes even values and only takes values divisible by 4 on decomposable elements.*
(d) *For $i = 2^t - 1 \geq 1$ and $u_i \in \mathrm{MU}_{2i}$ as in Proposition 2.1, $s_i(u_i) \equiv 2 \pmod 4$.*

Proof. For assertion (a), see [Ben20, Lemma 3.3]. Assertions (b) and (c) follow from a theorem of Rees and Thomas ([RT77, Theorem 3]; see [Ben20, Theorem 3.4 and Corollary 3.5] for these exact statements). The same result of Rees and Thomas ([RT77, Theorem 3] applied with $r = 0$ and $n = 2^t - 1$) shows that not all the values of $s_i\colon \mathrm{MU}_{2i} \to \mathbb{Z}$ are divisible by 4 if $i = 2^t - 1$. This fact combined with (c) implies (d). □

2.3. A congruence result for the top Segre class

The next proposition is the main goal of this section. It simultaneously generalizes the theorem of Rees and Thomas recalled in Lemma 2.4(b) (when $e = 0$) and [Ben20, Theorem 3.6] (when $e = 1$).

Proposition 2.5. *Fix $e \geq 0$, $h \geq 1$, and $i \geq 1$. The following assertions are equivalent:*

(i) *There exists a degree i element $Q \in \mathbb{Z}[c_j]_{j\geq 1}$ such that the Chern number*

$$s_i + 2^h Q \colon \mathrm{MU}_{2i} \longrightarrow \mathbb{Z}$$

only takes values divisible by 2^{e+h}.

(ii) *One has $\alpha(i+e+h-1) > e+2h-2$.*

Proof. Assertion (i) implies that $s_i \colon \mathrm{MU}_{2i} \to \mathbb{Z}$ is divisible by 2^h. So does assertion (ii) by Lemma 2.4(b). We may thus assume that the function $s_i \colon \mathrm{MU}_{2i} \to \mathbb{Z}$ is indeed divisible by 2^h.

Identify MU_{2i} and $\pi_{2i}(\mathbf{MU})$ using (2.2). Consider the following statements:

(a) The function $\frac{s_i}{2^h} \colon \mathrm{MU}_{2i} \to \mathbb{Z}$ coincides modulo 2^e with a Chern number.
(b) The function $\frac{s_i}{2^h} \colon \mathrm{MU}_{2i} \to \mathbb{Z}/2^e$ factors through $H_{2i}(\mathbf{MU}, \mathbb{Z})$ via H.
(c) The function $\frac{s_i}{2^h} \colon \mathrm{MU}_{2i} \to \mathbb{Z}/2^e$ factors through $H_{2i}(\mathbf{MU}, \mathbb{Z})/2^e$ via H.
(d) The function $\frac{s_i}{2^h} \colon \mathrm{MU}_{2i} \to \mathbb{Z}/2^e$ vanishes on $(I_e)_{2i}$.
(e) The function $s_i \colon \mathrm{MU}_{2i} \to \mathbb{Z}$ only takes values divisible by 2^{e+h} on $(I_e)_{2i}$.
(f) If $i = (\sum_{k=1}^{e'} 2^{t_k} - 1) + j$ for some $e' \leq e$, then $s_j \colon \mathrm{MU}_{2j} \to \mathbb{Z}$ is divisible by 2^h.

The equivalence (i)⇔(a) is clear, and (a)⇔(b) follows from Lemma 2.3. The equivalence (b)⇔(c) holds because MU_{2i} is $\mathbb{Z}$-free. The implication (c)⇒(d) is a consequence of the definition of I_e, and the converse holds because $\mathbb{Z}/2^e$ is an injective $\mathbb{Z}/2^e$-module [Wei94, Exercise 2.3.1]. The equivalence (d)⇔(e) is elementary. As for (e)⇔(f), it is a consequence of the description of I_e given in Lemma 2.2 and of the properties of the Segre classes given in Lemma 2.4(a) and (d). Finally, applying Lemma 2.4(b) shows the equivalence (f)⇔(ii). □

3. Smooth cycles on abelian varieties

3.1. Complex cobordism

We denote by $\mathrm{MU}_*(X)$ the complex cobordism homology theory represented by the spectrum **MU**, evaluated on a topological space X. The group $\mathrm{MU}_d(X)$ has a geometric interpretation as the group of complex cobordism classes of continuous maps $f \colon M \to X$, where M is a d-dimensional compact stably almost complex manifold (see for instance [Swi75, Proposition 12.35]). We denote by $[f] \in \mathrm{MU}_d(X)$ the class of f. When X is a point, one recovers the ring $\mathrm{MU}_* = \pi_*(\mathbf{MU})$ which we studied in Sections 2.1 and 2.2. In general, the group $\mathrm{MU}_*(X)$ is naturally an MU_*-module.

If $f \colon M \to X$ is as above and if $P \in \mathbb{Z}[c_j]_{j\geq 1}$ of degree p and $\omega \in H^{d-2p}(X, \mathbb{Z})$ are given, the integer $\deg_M\left(P(c_j(M)) \cdot f^*\omega\right)$ only depends on $[f] \in \mathrm{MU}_d(X)$ (the argument found in [Con79, Section 17, p. 54] for unoriented cobordism also works for complex cobordism). It is called the *characteristic number* of f associated with P and ω.

Fix a point $o \in \mathbb{S}^1$. The compatibility of the homology theory MU_* with suspension shows that $\mathrm{MU}_*(\mathbb{S}^1)$ is free of rank 2 over MU_*, generated by the classes of the inclusion $\{o\} \to \mathbb{S}^1$ and of the identity $\mathbb{S}^1 \to \mathbb{S}^1$ (where $\mathbb{S}^1$ is endowed with the stably almost complex structure induced by a trivialization of its tangent bundle). For $N \geq 0$ and $E \subset \{1, \ldots, N\}$, we consider the inclusion $f_E \colon T_E \hookrightarrow (\mathbb{S}^1)^N$ of the subtorus $T_E := \prod_{e\in E} \mathbb{S}^1 \times \prod_{e\in\{1,\ldots,N\}\setminus E} \{o\}$.

Lemma 3.1. *Fix $N \geq 1$. When E describes the set of all subsets of $\{1, \ldots, N\}$,*

(a) *the* $f_{E,*}[T_E]\in H_{|E|}((\mathbb{S}^1)^N,\mathbb{Z})$ *form an additive basis of* $H_*((\mathbb{S}^1)^N,\mathbb{Z})$;
(b) *the* $[f_E]\in \mathrm{MU}_{|E|}((\mathbb{S}^1)^N)$ *form an* MU_**-basis of* $\mathrm{MU}_*((\mathbb{S}^1)^N)$.

Proof. The first assertion follows from the Künneth formula in ordinary homology and the second assertion from the Künneth formula in complex cobordism (apply [Swi75, Theorem 13.75 i)] with $E=\mathbf{MU}$). ☐

Lemma 3.2. *Let* $x\in \mathrm{MU}_{2i}$, *and let* $E\subset\{1,\dots,N\}$ *be such that* $|E|=d-2i$. *Fix* $P\in\mathbb{Z}[c_j]_{j\geq 1}$ *of degree* l *and* $\omega\in H^{d-2l}((\mathbb{S}^1)^N,\mathbb{Z})$. *The characteristic number of* $x\cdot[f_E]\in\mathrm{MU}_d((\mathbb{S}^1)^N)$ *associated with* P *and* ω *is equal to* 0 *if* $l\neq i$ *and to* $P(x)\deg_{(\mathbb{S}^1)^N}\left(f_{E,*}(1)\cdot\omega\right)$ *if* $l=i$.

Proof. Let M be a $2i$-dimensional compact stably almost complex manifold representing $x\in\mathrm{MU}_{2i}$. Let $g_E\colon M\times T_E\to X$ be the composition of the second projection and $f_E\colon T_E\to X$. We will also let $h_E\colon M\times T_E\to M$ denote the first projection. Since the stably almost complex structure on the tangent bundle on the torus T_E is stably trivial, one has $c_j(T_E)=0$ for $j>0$. It follows from the Whitney sum formula that $P(c_j(M\times T_E))=h_E^*P(c_j(M))\in H^{2l}(M\times T_E,\mathbb{Z})$. As a consequence, $P(c_j(M\times T_E))\cdot g_E^*\omega\in H^d(M\times T_E,\mathbb{Z})$ vanishes unless $l=i$. When $l=i$, we use the projection formula to compute

$$\begin{aligned}\deg_{M\times T_E}\left(P(c_j(M\times T_E))\cdot g_E^*\omega\right)&=\deg_{M\times T_E}\left(h_E^*P(c_j(M))\cdot g_E^*\omega\right)\\&=\deg_M\left(P(c_j(M))\right)\deg_{T_E}(f_E^*\omega)\\&=P(x)\deg_{(\mathbb{S}^1)^N}(f_{E,*}(1)\cdot\omega).\end{aligned}$$ ☐

3.2. Complex cobordism of abelian varieties

Now let X be a complex abelian variety of dimension n with a principal polarization $\theta\in H^2(X,\mathbb{Z})$. We identify X and $(\mathbb{S}^1)^N$ for $N=2n$ by means of a Lie group isomorphism $X\simeq(\mathbb{S}^1)^N$. By Lemma 3.1(a), there exists for each $k\geq 0$ a unique $\mathbb{Z}$-linear combination

$$\tau_k=\sum_{\substack{E\subset\{1,\dots,N\}\\|E|=2k}}\mu_E[f_E]\in\mathrm{MU}_{2k}(X)\tag{3.1}$$

such that $\sum_{|E|=2k}\mu_E f_{E,*}[T_E]\in H_{2k}(X,\mathbb{Z})$ is Poincaré-dual to the integral class $\frac{\theta^{n-k}}{(n-k)!}$ or, in other words, such that

$$\sum_{|E|=2k}\mu_E f_{E,*}(1)=\frac{\theta^{n-k}}{(n-k)!}.\tag{3.2}$$

Proposition 3.3. *Let* (X,θ) *be a principally polarized complex abelian variety of dimension* n. *Assume that the group* $\mathrm{Hdg}^{2k}(X,\mathbb{Z})$ *of Hodge classes is generated by* $\frac{\theta^k}{k!}$ *for each* $k\geq 0$, *and let* $\tau_k\in\mathrm{MU}_{2k}(X)$ *be as in* (3.1). *Let* $f\colon Y\to X$ *be a morphism of smooth projective varieties with* Y *of pure dimension* d. *Then there exists, for each* $i\in\{0,\dots,d\}$, *an element* $x_i\in\mathrm{MU}_{2i}$ *such that*

$$[f]=\sum_{i=0}^{d}x_i\cdot\tau_{d-i}\in\mathrm{MU}_{2d}(X).\tag{3.3}$$

Proof. Let R_i be the rank of the free $\mathbb{Z}$-module MU_{2i}, and let $(y_{i,r})_{1\leq r\leq R_i}$ be a $\mathbb{Z}$-basis of it. Since the MU_*-module $\mathrm{MU}_*(X)$ is free with basis $([f_E])_{E\subset\{1,\dots,N\}}$ by Lemma 3.1(b), there exist unique integers $\nu_{i,r,E}$ such that

$$[f]=\sum_{i=0}^{d}\sum_{r=1}^{R_i}\left(y_{i,r}\cdot\sum_{|E|=2d-2i}\nu_{i,r,E}[f_E]\right)\in\mathrm{MU}_{2d}(X).\tag{3.4}$$

Fix $0 \le i \le d$ and $1 \le r \le R_i$. As $\mathrm{MU}_{2i} \xrightarrow{\sim} \pi_{2i}(\mathbf{MU}) \xrightarrow{H} H_{2i}(\mathbf{MU},\mathbb{Z})$ is an inclusion of free $\mathbb{Z}$-modules of the same rank R_i by Proposition 2.1, it follows from Lemma 2.3 that there exists a degree i element $P \in \mathbb{Z}[c_j]_{j\ge 1}$ such that $P(y_{i,s})$ is nonzero if and only if $s = r$. In view of Lemma 3.2 and the projection formula, the characteristic number of (3.4) associated with P and $\omega \in H^{2d-2i}(X,\mathbb{Z})$ reads

$$\deg_X\Big(f_*(P(c_j(Y)))\cdot\omega\Big) = P(y_{i,r})\deg_X\left(\sum_{|E|=2d-2i} \nu_{i,r,E}\cdot f_{E,*}(1)\cdot\omega\right).$$

As ω is arbitrary, Poincaré duality on X implies

$$f_*(P(c_j(Y))) = P(y_{i,r})\sum_{|E|=2d-2i}\nu_{i,r,E} f_{E,*}(1) \in H^{2n-2d+2i}(X,\mathbb{Z}). \tag{3.5}$$

The left side of (3.5) is algebraic, hence is a Hodge class. As a consequence, so is the right side. Since $P(y_{i,r}) \ne 0$, we deduce from our hypothesis that the class $\sum_{|E|=2d-2i}\nu_{i,r,E} f_{E,*}(1)$ is an integral multiple of $\frac{\theta^{n-d+i}}{(n-d+i)!}$. Equation (3.2) now implies $\sum_{|E|=2d-2i}\nu_{i,r,E}[f_E] = \xi_{i,r}\tau_{d-i}$ for some $\xi_{i,r}\in\mathbb{Z}$.

It finally follows from (3.4) that (3.3) holds with $x_i = \sum_{r=1}^{R_i}\xi_{i,r}y_{i,r}$. □

Proposition 3.4. *Keep the hypotheses and notation of Proposition 3.3. Let $P \in \mathbb{Z}[c_j]_{j\ge1}$ be of degree l for some $0 \le l \le d$. Then,*

$$\deg_Y\left(P(c_j(Y))\cdot f^*\left(\frac{\theta^{d-l}}{(d-l)!}\right)\right) = \binom{n}{d-l}P(x_l).$$

Proof. Let us compute the characteristic number of f associated with P and $\frac{\theta^{d-l}}{(d-l)!}$. Combining (3.3), (3.1), and Lemma 3.2 shows that it is

$$\begin{aligned}\deg_Y\left(P(c_j(Y))\cdot f^*\left(\frac{\theta^{d-l}}{(d-l)!}\right)\right) &= P(x_l)\sum_{|E|=2d-2l}\mu_E\deg_X\left(f_{E,*}(1)\cdot\frac{\theta^{d-l}}{(d-l)!}\right)\\ &= P(x_l)\deg_X\left(\frac{\theta^{n-d+l}}{(n-d+l)!}\cdot\frac{\theta^{d-l}}{(d-l)!}\right).\end{aligned}$$

Since $\deg_X(\theta^n) = n!$, the proposition is proven. □

3.3. Smooth subvarieties of abelian varieties

The next proposition is an application of a Barth–Lefschetz-type theorem proved by Sommese [Som82].

Proposition 3.5. *Let (X,θ) be a principally polarized complex abelian variety of dimension n such that $\mathrm{Hdg}^{2k}(X,\mathbb{Z})$ is generated by $\frac{\theta^k}{k!}$ for all $k\ge0$. Let $f: Y\to X$ be the inclusion of a smooth projective subvariety of pure codimension c. Assume that $n \ge 4c-2l$ for some $l\ge1$. Then there exist $a_0,\dots,a_{c-l},a_c\in\mathbb{Z}$ such that*

(a) *$s_i(Y) = a_i f^*(\frac{\theta^i}{i!})$ for $i\in\{0,\dots,c-l,c\}$;*
(b) *$f_*[Y]$ is Poincaré-dual to $a_c\frac{\theta^c}{c!}\in H^{2c}(X,\mathbb{Z})$.*

Proof. Since the subvariety $Y\subset X$ is algebraic, the homology class $f_*[Y]$ is Poincaré-dual to a Hodge class, which is necessarily of the form $a_c\frac{\theta^c}{c!}$ by hypothesis. This proves (b). By the self-intersection formula [Ful98, Corollary 6.3], the top Chern class of the normal bundle $N_{Y/X}$ is $a_c f^*(\frac{\theta^c}{c!})$. Since the tangent bundle T_X is trivial, one has $c(N_{Y/X}) = c(Y)^{-1} = s(Y)$. This shows (a) for $i=c$.

If the abelian variety X were not simple, pulling back an ample divisor from a nontrivial quotient would produce a nonample divisor on X, contradicting the fact that $\mathrm{Hdg}^2(X,\mathbb{Z})$ is generated by θ. We deduce that X is simple. One may thus apply [Som82, Corollary 3.5 and (3.6.1)] with $B=Y$ to obtain $\pi_j(X,Y,y)=0$ for $j\le n-2c+1$ and all $y\in Y$. It follows from the version [Hat02, Theorem 4.37] of the Hurewicz theorem, from the universal coefficient theorem, and from the long exact sequence of relative cohomology of the pair (X,Y) that the restriction map $H^j(X,\mathbb{Z})\to H^j(Y,\mathbb{Z})$ is an isomorphism for $j\le n-2c$. For $0\le i\le c-l$, one may apply this fact with $j=2i$ because $n\ge 4c-2l$. This shows that the class $s_i(Y)\in H^{2i}(Y,\mathbb{Z})$, which is Hodge because it is algebraic, is the restriction to Y of a class in $\mathrm{Hdg}^{2i}(X,\mathbb{Z})$. The latter is necessarily of the form $a_i\frac{\theta^i}{i!}$ by hypothesis. The proof is now complete. □

Proposition 3.6. *Keep the hypotheses and notation of Proposition 3.5, assume that $l=1$, and suppose in addition that $\alpha(c+\alpha(c))>\alpha(c)$. Then a_c is even.*

Proof. Let $Q\in\mathbb{Z}[c_j]_{j\ge 1}$ be the degree c homogeneous polynomial obtained by applying Proposition 2.5 with $i=c$, $e=\alpha(c)$, and $h=1$. Applying Proposition 3.4 with $l=c$ and $P=s_c+2Q$ yields the identity

$$\deg_Y\left(P(c_j(Y))\cdot f^*\left(\frac{\theta^{n-2c}}{(n-2c)!}\right)\right)=\binom{n}{n-2c}P(x_c). \tag{3.6}$$

Using that Chern classes may be expressed as polynomials with integral coefficients in Segre classes by (2.3), it follows from Proposition 3.5(a) that $Q(c_j(Y))\in H^{2c}(Y,\mathbb{Z})$ is an integral multiple of $f^*(\frac{\theta^c}{c!})$, say $Q(c_j(Y))=bf^*(\frac{\theta^c}{c!})$ for some $b\in\mathbb{Z}$. Applying Proposition 3.5(a) again, we get $P(c_j(Y))=(a_c+2b)f^*(\frac{\theta^c}{c!})$. Rewriting the left side of (3.6) using the projection formula and Proposition 3.5(b), we obtain

$$\deg_X\left(a_c\frac{\theta^c}{c!}\cdot(a_c+2b)\frac{\theta^c}{c!}\cdot\frac{\theta^{n-2c}}{(n-2c)!}\right)=\binom{n}{n-2c}P(x_c).$$

Using $\deg_X(\theta^n)=n!$, we finally get

$$P(x_c)=\binom{2c}{c}a_c(a_c+2b). \tag{3.7}$$

Our choice of Q implies that the left side of (3.7) is divisible by $2^{\alpha(c)+1}$. The formula for the 2-adic valuation of the factorial given in [Rob00, Lemma, Section 5.3.1, p. 241] implies that the 2-adic valuation of $\binom{2c}{c}$ is equal to $\alpha(c)$. We deduce that $a_c(a_c+2b)$ is even, hence that a_c is even. □

Theorem 3.7. *Let (X,θ) be a very general complex Jacobian of dimension n. Let $c\ge 0$ be such that $\alpha(c+\alpha(c))>\alpha(c)$ and $n\ge 4c-2$. Then the classes $\lambda\frac{\theta^c}{c!}$ with λ odd are algebraic but are not $\mathbb{Z}$-linear combinations of cycle classes of smooth subvarieties of X.*

Proof. The integral class $\frac{\theta^c}{c!}$ is algebraic by [BL04, Poincaré's formula 11.2.1].

The hypothesis that $\mathrm{Hdg}^{2k}(X,\mathbb{Z})$ is generated by $\frac{\theta^k}{k!}$ for all $k\ge 0$ is satisfied by [BL04, Theorem 17.5.1] and because the integral class $\frac{\theta^k}{k!}$ is primitive. One may thus combine Propositions 3.5(b) and 3.6 to show that the cycle classes of smooth codimension c subvarieties $Y\subset X$ are even multiples of $\frac{\theta^c}{c!}$. This concludes the proof. □

Remarks 3.8.

(i) In Theorem 3.7, the hypothesis that (X,θ) is very general is only used to ensure that $\mathrm{Hdg}^{2k}(X,\mathbb{Z})$ is generated by $\frac{\theta^k}{k!}$ for $k\ge 0$. As there exist Jacobians over $\overline{\mathbb{Q}}$ whose Mumford–Tate group is the full symplectic group (see [And96, Théorème 5.2 3) and Remarque (vii) below it] which apply as all Hodge classes on abelian varieties are absolute Hodge by [Del82, Main Theorem 2.11]), one may find such an (X,θ) that is defined over $\overline{\mathbb{Q}}$.

(ii) The proof of Theorem 3.7 actually shows that the class of any smooth subvariety of codimension c of X is divisible by 2 in the group $H^{2c}(X,\mathbb{Z})_{\mathrm{alg}}$.

4. Codimension 4 cycles

Theorem 3.7 is not optimal in several respects. When c is fixed, it is sometimes possible to give stronger restrictions on the cycle classes of smooth subvarieties of codimension c of X or results for lower values of the dimension n of X.

To obtain such improvements, one may work with complex topological K-theory instead of complex cobordism and replace the divisibility result for Chern numbers given in Proposition 2.5 with an application of the Grothendieck–Riemann–Roch theorem and an integrality property of the Chern character (*cf.* Lemma 4.1). This works well when c is low but becomes intractable for high values of c. We illustrate the method when $c = 4$.

We start with a lemma. Let X be a topological space, and let $K^*(X) = K^0(X) \oplus K^1(X)$ be its $\mathbb{Z}/2$-graded complex topological K-theory defined in [AH61, Section 1.9]. We consider the Chern character $\mathrm{ch} : K^*(X) \to H^*(X, \mathbb{Q})$ as in [AH61, Section 1.10].

Lemma 4.1. *For $N \geq 1$, the Chern character $\mathrm{ch} : K^*((\mathbb{S}^1)^N) \to H^*((\mathbb{S}^1)^N, \mathbb{Q})$ is an isomorphism onto $H^*((\mathbb{S}^1)^N, \mathbb{Z})$.*

Proof. First, suppose that $N = 1$. The morphism $\mathrm{ch} : K^0(\mathbb{S}^1) \to H^0(\mathbb{S}^1, \mathbb{Q}) = \mathbb{Q}$ has image $\mathbb{Z}$ because it associates with each vector bundle its rank, and it is injective because $K^0(\mathbb{S}^1) \simeq \mathbb{Z}$. That $\mathrm{ch} : K^1(\mathbb{S}^1) \to H^1(\mathbb{S}^1, \mathbb{Q})$ is an isomorphism onto $H^1(\mathbb{S}^1, \mathbb{Z})$ follows from the definition of this morphism using suspension and Bott periodicity (see [AH61, Section 1.10]) and from the fact the Chern character sends the Bott element which is a generator of $\widetilde{K}^0(\mathbb{S}^2)$ to a generator of $H^2(\mathbb{S}^2, \mathbb{Z})$ (see [AH61, Section 1.10, p. 206]).

It now follows from the Künneth formula in cohomology and in complex topological K-theory (for which see [Ati62, Lemma 1]) and from the multiplicativity of the Chern character that $\mathrm{ch} : K^*((\mathbb{S}^1)^N) \to H^*((\mathbb{S}^1)^N, \mathbb{Q})$ is an isomorphism onto $H^*((\mathbb{S}^1)^N, \mathbb{Z})$. □

Proposition 4.2. *Let (X, θ) be a principally polarized complex abelian variety of dimension n such that $\mathrm{Hdg}^{2k}(X, \mathbb{Z})$ is generated by $\frac{\theta^k}{k!}$ for all $k \geq 0$. Let $Y \subset X$ be a smooth projective subvariety of pure codimension 4.*

(a) *If $n \geq 12$, the cohomology class of Y is an integral multiple of $2\frac{\theta^4}{4!}$.*

(b) *If $n \geq 14$, the cohomology class of Y is an integral multiple of $4\frac{\theta^4}{4!}$.*

Proof. Let $f : Y \to X$ be the inclusion map. Proposition 3.5 shows the existence of integers a_i such that $s_i(Y) = a_i f^*(\frac{\theta^i}{i!})$ for $i \in \{0, 1, 2, 4\}$ (if $n \geq 12$), or $i \in \{0, 1, 2, 3, 4\}$ (if $n \geq 14$), and such that, in addition, the cohomology class of Y in X is equal to $a_4 \frac{\theta^4}{4!}$. As the class $f_* s_3(Y)$ is Hodge, we may also write $f_* s_3(Y) = b\frac{\theta^7}{7!}$ for some $b \in \mathbb{Z}$.

As noted during the proof of Proposition 3.5, one has $c(-N_{Y/X}) = s(Y)^{-1}$, hence

$$\begin{aligned} c(-N_{Y/X}) = 1 &- a_1 f^*\theta + (a_1^2 - a_2/2) f^*\theta^2 + ((a_1 a_2 - a_1^3) f^*\theta^3 - s_3(Y)) \\ &+ ((a_1^4 - 3a_1^2 a_2/2 + a_2^2/4 - a_4/24) f^*\theta^4 + 2a_1 s_3(Y) f^*\theta) + \cdots. \end{aligned}$$

One can then compute the Todd class (see [Ful98, Example 3.2.4]):

$$\begin{aligned} \mathrm{td}(-N_{Y/X}) = 1 &- a_1 f^*\theta/2 + (4a_1^2 - a_2) f^*\theta^2/24 + (a_1 a_2 - 2a_1^3) f^*\theta^3/48 \\ &+ (144a_1^4 - 108a_1^2 a_2 + 12a_2^2 + a_4) f^*\theta^4/17\,280 - a_1 s_3(Y) f^*\theta/720 + \cdots. \end{aligned} \tag{4.1}$$

Let $\chi \in K^0(X)$ be the image in complex topological K-theory of the algebraic K-theory class $f_*[\mathcal{O}_Y]$. The Grothendieck–Riemann–Roch theorem [Ful98, Theorem 15.2] applied to the inclusion f and to the algebraic K-theory class $[\mathcal{O}_Y]$ shows that $\mathrm{ch}(\chi) = f_* \mathrm{td}(-N_{Y/X})$. Since $f_* f^*$ is the cup-product with the cohomology class of Y which equals $a_4\frac{\theta^4}{4!}$, this identity may be combined with (4.1) to give

$$\begin{aligned}\mathrm{ch}(\chi) = a_4\theta^4/24 - a_1a_4\theta^5/48 + (4a_1^2 - a_2)a_4\theta^6/576 + (a_1a_2 - 2a_1^3)a_4\theta^7/1\,152 \\ + (144a_1^4 - 108a_1^2a_2 + 12a_2^2 + a_4)a_4\theta^8/414\,720 - ba_1\theta^8/3\,628\,800 + \cdots.\end{aligned} \tag{4.2}$$

By Lemma 4.1, which applies because X is diffeomorphic to $(\mathbb{S}^1)^{2n}$, the left side $\mathrm{ch}(\chi)$ of (4.2) is an integral cohomology class, hence so is its right side.

Assume by way of contradiction that a_4 is odd. The integrality of the class $\mathrm{ch}_5(\chi) = a_1a_4\theta^5/48$ implies that a_1 is even, and the integrality of the class $\mathrm{ch}_6(\chi) = (4a_1^2 - a_2)a_4\theta^6/576$ shows that a_2 is divisible by 4. All terms appearing in the expression for $\mathrm{ch}_8(\chi)$ given in (4.2) are multiples of $\frac{\theta^8}{8!}$ with coefficients a rational number of nonnegative 2-adic valuation, with the exception of $a_4^2\theta^8/414\,720 = \frac{7a_4^2}{72}\frac{\theta^8}{8!}$ since a_4 is assumed to be odd. It follows that $\mathrm{ch}_8(\chi)$ is not an integral cohomology class, which gives a contradiction. This proves (a).

Now suppose $n \geq 14$. Then $b\theta^7/7! = f_*s_3(Y) = f_*f^*(a_3\theta^3/3!) = a_3a_4\theta^7/144$, so that $b = 35a_3a_4$. Assume by way of contradiction that a_4 is not divisible by 4. By the integrality of $\mathrm{ch}_5(\chi) = a_1a_4\theta^5/48$, we see that a_1a_4 is even. The integrality of $\mathrm{ch}_6(\chi) = (4a_1^2 - a_2)a_4\theta^6/576$ shows that a_2 is even. These pieces of information imply that in the right side of the equality

$$\mathrm{ch}_8(\chi) = (144a_1^4 - 108a_1^2a_2 + 12a_2^2 + a_4)a_4\theta^8/414\,720 - 35a_1a_3a_4\theta^8/3\,628\,800,$$

all terms are multiples of $\frac{\theta^8}{8!}$ with coefficients a rational number of nonnegative 2-adic valuation, with the exception of $a_4^2\theta^8/414\,720 = \frac{7a_4^2}{72}\frac{\theta^8}{8!}$ since we assumed that a_4 is not a multiple of 4. This contradicts the integrality of $\mathrm{ch}_8(\chi)$ and proves (b). □

Combining Proposition 4.2 with [BL04, 11.2.1 and Theorem 17.5.1], we get the following.

Theorem 4.3. *Let (X, θ) be a very general complex Jacobian of dimension n. For $\lambda \in \mathbb{Z}$, the class $\lambda\frac{\theta^4}{4!}$ is algebraic but is not a $\mathbb{Z}$-linear combination of classes of smooth subvarieties of X*

(a) *if $n \geq 12$ and λ is odd;*
(b) *if $n \geq 14$ and λ is not divisible by 4.*

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