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A new generalization of a system of two-sided coupled Sylvester-like quaternion tensor equations[1]

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Abstract: This study establishes consistency conditions and a general solution for a coupled system that consists of five two-sided Sylvester-like tensor equations in ten quaternion variables throughout the Einstein tensor product. Certain specific cases are thus established. In a direct application, we investigate certain necessary and sufficient conditions for the existence of an η-Hermitian solution to five coupled two-sided Sylvester-like quaternion tensor equations. Finally, we present an algorithm and a numerical example to validate the main result.

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1. Introduction

We introduce certain notations and definitions for convenience. Consider I_1, ..., I_M to be positive integers for the positive integer M. An M order tensor D with entry $D_{i_1..i_M}$ $(1 \leq i_j \leq I_j,\ i = 1, ..., M)$ is a multidimensional array with the subscripts $i_1, i_2, ..i_M$ [1, 9–12, 33–36, 41]. We utilize the notation that $I(M)$ represents $I_1 \times I_2 \times ... \times I_M$. The quaternion concept was investigated by Hamilton in [19], and quaternion algebra can be considered a non-commutative skew field. Let $\mathbb{R}$ and $\mathbb{C}$ be the fields of real numbers and complex numbers, respectively, and let $\mathbb{H}$ be the quaternion algebra

$$\mathbb{H} = \{d_0 + d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k} \mid \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,\ d_0, d_1, d_2, d_3 \in \mathbb{R}\}.$$

Let $\mathbb{H}^{I(M)}$ be the set of the order M dimension $I(M)$ tensors over the quaternion algebra $\mathbb{H}$. Tensors are the natural expansions of vectors and matrices. Tensor equations and computations have applications in machine learning, signal processing, mechanics, physics, Markov processes, control theory, numerical analysis, partial differential equations, and engineering problems [5,37]. Tensor decompositions, tensor eigenvalue, and non-negative tensors [33], [1], [13] have implementations in signal processing, color image processing [28], quantum mechanics [7], quaternion tensor computing [15], Iterative algorithms for solving some tensor equations [16–18, 30–32, 53, 56]. Let $\mathcal{A} \in \mathbb{H}^{I(N)\times J(N)}$ and $\mathcal{B} \in \mathbb{H}^{J(N)\times K(M)}$, then the Einstein tensor product [42] of tensors $\mathcal{A}$

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and $\mathcal{B}$ is denoted by $\mathcal{A} *_N \mathcal{B} \in \mathbb{H}^{I(N)\times K(M)}$, where

$$(\mathcal{A} *_N \mathcal{B})_{i_1..i_N k_1..k_M} = \sum_{j_1\ldots j_N} a_{i_1\ldots i_N j_1\ldots j_N} b_{j_1\ldots j_N k_1\ldots k_M}.$$

The operation $*_N$ is associative over the set of all quaternion tensors with qualified order.

Let ψ be a nonstandard involution of the quaternion algebra $\mathbb{H}$ (Definition 3.4.5 [38]). If $D \in \mathbb{H}^{m\times n}$, then $(D)_\psi$ is an $n \times m$ matrix over $\mathbb{H}$ obtained by applying ψ entrywise to the transpose of D. let D be an $n\times n$ matrix over $\mathbb{H}$. D is called a ψ-Hermitian matrix if $(D)_\psi = D$ (Definition 3.6.1 [38]). Took et al. [43] introduced an example of a ψ-Hermitian matrix called an η-Hermitian matrix. For fixed $\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, A square matrix A is called an η-Hermitian matrix if $D^{\eta^*} = D$, where $D^{\eta^*} = -\eta D^* \eta$. An η-Hermitian matrix has applications in linear modeling and statistical signal processing [43–46]. He [25] gave a generalization of an η-Hermitian matrix. A square quaternion tensor $\mathcal{D}$ is called an η-Hermitian tensor if $\mathcal{D} = \mathcal{D}^{\eta^*}$, where $\mathcal{D}^{\eta^*} = -\eta\mathcal{D}^*\eta$. We [25] investigated the consistency conditions and the exact general solution formula for the following two-sided quaternion tensor equations:

$$\begin{gathered}\mathcal{A}_1 *_N \mathcal{X}_1 *_M \mathcal{B}_1 + \mathcal{A}_2 *_N \mathcal{X}_2 *_M \mathcal{B}_2 \\ +\mathcal{A}_2 *_N (\mathcal{C}_3 *_N \mathcal{X}_3 *_M \mathcal{D}_3 + \mathcal{C}_4 *_N \mathcal{X}_4 *_M \mathcal{D}_4) *_M \mathcal{B}_1 = \mathcal{E}_1.\end{gathered} \tag{1.1}$$

where $\mathcal{A}_i,\ \mathcal{B}_i,\ \mathcal{C}_j,\ \mathcal{D}_j\ (i = 1, 2,\ j = 3, 4)$, and $\mathcal{E}_1$ are given quaternion tensors. A tensor equation (1.1) has applications in the discretization of higher-dimension linear partial differential equations, even including its generalizations [29]. Recently, Wang et al. [51] gave a proper extension to the quaternion tensor equation (1.1). They established consistency conditions and a general solution to the following coupled two-sided Sylvester-type quaternion system of tensor equations in terms of the Moore–Penrose inverses for certain given tensors:

$$\begin{cases}\mathcal{A}_1 *_N \mathcal{X}_1 *_M \mathcal{B}_1 + \mathcal{A}_2 *_N \mathcal{W} *_M \mathcal{B}_2 = \mathcal{E}_1 \\ \mathcal{A}_3 *_N \mathcal{Y}_1 *_M \mathcal{B}_3 + \mathcal{A}_4 *_N \mathcal{W} *_M \mathcal{B}_4 = \mathcal{E}_2.\end{cases} \tag{1.2}$$

This is based on the various uses of quaternions, rank characterizations of some matrix expressions, matrix decompositions, the coupled Sylvester-like quaternion systems of matrix equations [2–4,6,8,14,20–24,26,27,39,40,47–50,52,54,55,57], and the theoretical studies surrounding Sylvester-like quaternion tensor equations. This paper investigates the consistency of and general solution to the following coupled two-sided Sylvester-like quaternion system of tensor equations:

$$\begin{cases}\mathcal{F}_4 *_N \mathcal{Z}_1 *_M \mathcal{G}_4 = \mathcal{E}_4, \\ \mathcal{A}_i *_N \mathcal{X}_i *_M \mathcal{B}_i + \mathcal{C}_i *_N \mathcal{Y}_i *_M \mathcal{D}_i \\ +\mathcal{C}_i *_N (\mathcal{F}_i *_N \mathcal{Z}_i *_M \mathcal{G}_i + \mathcal{H}_i *_N \mathcal{Z}_{i+1} *_M \mathcal{J}_i) *_M \mathcal{B}_i = \mathcal{E}_i, \\ \mathcal{H}_4 *_N \mathcal{Z}_4 *_M \mathcal{J}_4 = \mathcal{E}_5,\end{cases} \tag{1.3}$$

$(i = \overline{1, 3})$, which gives us a proper generalization of both systems, (1.1) and (1.2). As a direct conclusion, we derive certain necessary and sufficient conditions for the consistency of:

$$\begin{cases}\mathcal{F}_4 *_N \mathcal{Z}_1 *_M \mathcal{G}_4 = \mathcal{E}_4, \\ \mathcal{F}_1 *_N \mathcal{Z}_1 *_M \mathcal{G}_1 + \mathcal{H}_1 *_N \mathcal{Z}_2 *_M \mathcal{J}_1 = \mathcal{E}_1, \\ \mathcal{F}_2 *_N \mathcal{Z}_2 *_M \mathcal{G}_2 + \mathcal{H}_2 *_N \mathcal{Z}_3 *_M \mathcal{J}_2 = \mathcal{E}_2, \\ \mathcal{F}_3 *_N \mathcal{Z}_3 *_M \mathcal{G}_3 + \mathcal{H}_3 *_N \mathcal{Z}_4 *_M \mathcal{J}_3 = \mathcal{E}_3, \\ \mathcal{H}_4 *_N \mathcal{Z}_4 *_M \mathcal{J}_4 = \mathcal{E}_5.\end{cases} \tag{1.4}$$

As an implementation of (1.4), we obtain the consistency conditions for the existence of an η-Hermitian solution to the following two-sided quaternion system of tensor equations:

$$\begin{cases} \mathcal{F}_4 *_N \mathcal{Z}_1 *_N \mathcal{F}_4^{\eta^*} = \mathcal{E}_4, \\ \mathcal{F}_1 *_N \mathcal{Z}_1 *_N \mathcal{F}_1^{\eta^*} + \mathcal{H}_1 *_N \mathcal{Z}_2 *_N \mathcal{H}_1^{\eta^*} = \mathcal{E}_1, \\ \mathcal{F}_2 *_N \mathcal{Z}_2 *_N \mathcal{F}_2^{\eta^*} + \mathcal{H}_2 *_N \mathcal{Z}_3 *_N \mathcal{H}_2^{\eta^*} = \mathcal{E}_2, \\ \mathcal{F}_3 *_N \mathcal{Z}_3 *_N \mathcal{F}_3^{\eta^*} + \mathcal{H}_3 *_N \mathcal{Z}_4 *_N \mathcal{H}_3^{\eta^*} = \mathcal{E}_3, \\ \mathcal{H}_4 *_N \mathcal{Z}_4 *_N \mathcal{H}_4^{\eta^*} = \mathcal{E}_5. \end{cases} \tag{1.5}$$

If we set $\mathcal{C}_i = \mathcal{B}_i = \mathcal{I}$ in (1.3) where $i = \overline{1,3}$, we obtain the following Sylvester-like quaternion system of tensor equations:

$$\begin{cases} \mathcal{A}_i *_N \mathcal{X}_i + \mathcal{Y}_i *_M \mathcal{D}_i + \mathcal{F}_i *_N \mathcal{Z}_i *_M \mathcal{G}_i + \mathcal{H}_i *_N \mathcal{Z}_{i+1} *_M \mathcal{J}_i = \mathcal{E}_i, \\ \mathcal{F}_4 *_N \mathcal{Z}_1 *_M \mathcal{G}_4 = \mathcal{E}_4, \ \mathcal{H}_4 *_N \mathcal{Z}_4 *_M \mathcal{J}_4 = \mathcal{E}_5 \ . \end{cases} \tag{1.6}$$

The remainder of this manuscript is described as follows. The concept of an η-Hermitian quaternion tensor and the Moore—Penrose inverse for a general tensor are reminiscent of Section 2. Section 3 expresses the general solution to the two-sided Sylvester-type quaternion system of tensor equations (1.3) when the solvability conditions are applicable. In Section 4, we provide the necessary and sufficient conditions for the existence of a η-Hermitian solution to a system (1.5) as a system (1.4) application. We briefly summarize the key results in Section 5.

2. **Preliminaries**

Throughout this paper tensors are considered quaternion tensors. A tensor $\mathcal{C} \in \mathbb{H}^{I(N)\times J(N)}$ is called an even-order tensor. An even-order tensor $\mathcal{C} \in \mathbb{H}^{I(N)\times I(N)}$ is called an even-order square tensor. Let $c \in \mathbb{H}$, then $\overline{c}$ stands for the conjugate of c. A quaternion tensor $\mathcal{C}^*= (\overline{c}_{j_1..j_M i_1..i_N}) \in \mathbb{H}^{J(M)\times I(N)}$ calls the conjugate transpose of the tensor $\mathcal{C}= (c_{i_1..i_N j_1..j_M}) \in \mathbb{H}^{I(N)\times J(M)}$. If $\mathcal{C} = \mathcal{C}^*$, then $\mathcal{C}$ is called Hermitian tensor.

Definition 2.1. *[42] An even order square tensor $\mathcal{C} = (c_{i_1...i_M i_1...i_M}) \in \mathbb{H}^{I(M)\times I(M)}$ is called a diagonal tensor if $c_{i_1...i_M i_1...i_M} \neq 0$ and all its entries are zero. A diagonal tensor is said to be a unit tensor if $c_{i_1...i_M i_1...i_M} = 1$, which denotes by $\mathcal{I}$.*

Definition 2.2. *[42] Let $\mathcal{C} = (c_{i_1...i_N j_1...j_M}) \in \mathbb{H}^{I(N)\times J(M)}$, $\mathcal{D} = (d_{i_1...i_N k_1...k_M}) \in \mathbb{H}^{I(N)\times K(M)}$. The "row block tensor" of $\mathcal{C}$ and $\mathcal{D}$ is denoted by*

$$\begin{pmatrix}\mathcal{C} & \mathcal{D}\end{pmatrix} \in \mathbb{H}^{I(N)\times L(M)}, \tag{2.1}$$

where $L_s = J_s + K_s$, $s = 1, ..., M$ define as

$$\begin{pmatrix}\mathcal{C} & \mathcal{D}\end{pmatrix}_{i_1...i_N l_1...l_M} = \begin{cases} c_{i_1...i_N l_1...l_M}, \ if \ i_1...i_N \in [I_1] \times ... \times [I_N], \ l_1...l_M \in [J_1] \times ... \times [J_M], \\ d_{i_1...i_N l_1...l_M}, \ if \ i_1...i_N \in [I_1] \times ... \times [I_N], \ l_1...l_M \in \Gamma_1 \times ... \times \Gamma_M, \\ 0, \quad otherwise, \end{cases}$$

where $\Gamma_s = \{J_s + 1, ..., J_s + K_s\}$, $s = 1, ..., M$. For a given tensors $\mathcal{A} = (a_{j_1...j_M i_1...i_N}) \in \mathbb{H}^{J(M)\times I(N)}$, $\mathcal{B} = (b_{k_1...k_M i_1...i_N}) \in \mathbb{H}^{K(M)\times I(N)}$. The "column block tensor" of $\mathcal{A}$ and $\mathcal{B}$ is *denoted by*

$$\begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix} \in \mathbb{H}^{L(M)\times I(N)}, \tag{2.2}$$

where $L_s = J_s + K_s,\ s = 1, ..., M$ *define as*

$$\begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix}_{l_1...l_M i_1...i_N} = \begin{cases} a_{l_1...l_M i_1...i_N},\ if\ l_1...l_M \in [J_1] \times ... \times [J_M],\ i_1...i_N \in [I_1] \times ... \times [I_N], \\ b_{l_1...l_M i_1...i_N},\ if\ l_1...l_M \in \Gamma_1 \times ... \times \Gamma_M,\ i_1...i_N \in [I_1] \times ... \times [I_N], \\ 0, \quad otherwise, \end{cases}$$

where $\Gamma_s = \{J_s + 1, ..., J_s + K_s\},\ s = 1, ..., M.$

Proposition 2.1. *[42] Let* $\mathcal{A} \in \mathbb{H}^{I(P)\times K(N)}$ *and* $\mathcal{B} \in \mathbb{H}^{K(N)\times J(M)}$*. Then*

(1) $(\mathcal{A} *_N \mathcal{B})^* = \mathcal{B}^* *_N \mathcal{A}^*$*;*
(2) $\mathcal{I_N} *_N \mathcal{B} = \mathcal{B}$, $\mathcal{B} *_M \mathcal{I}_M = \mathcal{B}$*, where* $\mathcal{I_N} \in \mathbb{H}^{K(N)\times K(N)}$ *and* $\mathcal{I_M} \in \mathbb{H}^{J(M)\times J(M)}$ *are units.*

Proposition 2.2. *[42] Consider the tensors* $\begin{pmatrix} \mathcal{A} & \mathcal{B} \end{pmatrix}$ *and* $\begin{pmatrix} \mathcal{C} \\ \mathcal{D} \end{pmatrix}$ *given in* (2.1) *and* (2.2)*. For a given quaternion tensor* $\mathcal{G} \in \mathbb{H}^{I(N)\times I(N)}$*, we have that*

(1) $\mathcal{G} *_N \begin{pmatrix} \mathcal{A} & \mathcal{B} \end{pmatrix} = \begin{pmatrix} \mathcal{G} *_N \mathcal{A} & \mathcal{G} *_N \mathcal{B} \end{pmatrix} \in \mathbb{H}^{I(N)\times L(M)}$,
(2) $\begin{pmatrix} \mathcal{C} \\ \mathcal{D} \end{pmatrix} *_N \mathcal{G} = \begin{pmatrix} \mathcal{C} *_N \mathcal{G} \\ \mathcal{D} *_N \mathcal{G} \end{pmatrix} \in \mathbb{H}^{L(M)\times I(N)}$,
(3) $\begin{pmatrix} \mathcal{A} & \mathcal{B} \end{pmatrix} *_M \begin{pmatrix} \mathcal{C} \\ \mathcal{D} \end{pmatrix} = \mathcal{A} *_M \mathcal{C} + \mathcal{B} *_M \mathcal{D} \in \mathbb{H}^{I(N)\times I(N)}$.

Definition 2.3. *[25] For a given quaternion tensor* $\mathcal{D} \in \mathbb{H}^{I(N)\times J(N)}$*. The Moore-Penrose inverse of* $\mathcal{D}$ *is the unique quaternion tensor* $\mathcal{X} \in \mathbb{H}^{J(N)\times I(N)}$ *satisfies the following axioms:*

(1) $\mathcal{D} *_N \mathcal{X} *_N \mathcal{D} = \mathcal{D}$,
(2) $\mathcal{X} *_N \mathcal{D} *_N \mathcal{X} = \mathcal{X}$,
(3) $(\mathcal{D} *_N \mathcal{X})^* = \mathcal{D} *_N \mathcal{X}$,
(4) $(\mathcal{X} *_N \mathcal{D})^* = \mathcal{X} *_N \mathcal{D}$.

which denotes by $\mathcal{D}^\dagger$*. Furthermore,* $\mathcal{R_D}$ *and* $\mathcal{L_D}$ *denote the projections along* $\mathcal{D}$*.*

Proposition 2.3. *[25] Let* $\mathcal{D} \in \mathbb{H}^{I(N)\times I(N)}$*. Then*

(1) $\mathcal{L_D} *_N \mathcal{D}^\dagger = \mathcal{D} *_N \mathcal{L_D} = 0,\ \mathcal{R_D} *_N \mathcal{D} = \mathcal{D}^\dagger *_N \mathcal{R_D} = 0$,
(2) $(\mathcal{D}^*)^\dagger = (\mathcal{D}^\dagger)^*,\ (\mathcal{D}^{\eta^*})^\dagger = (\mathcal{D}^\dagger)^{\eta^*}$,
(3) $(\mathcal{L_D})^{\eta^*} = \mathcal{R}_{\mathcal{D}^{\eta^*}},\ (\mathcal{R_D})^{\eta^*} = \mathcal{L}_{\mathcal{D}^{\eta^*}}$,
(4) $(\mathcal{D}^* *_N \mathcal{D})^\dagger = \mathcal{D}^\dagger *_N (\mathcal{D}^*)^\dagger,\ (\mathcal{D} *_N \mathcal{D}^*)^\dagger = (\mathcal{D}^*)^\dagger *_N \mathcal{D}^\dagger$.

Lemma 2.4. *[25] Let* $\mathcal{A}_1 \in \mathbb{H}^{I(N)\times J(N)}$, $\mathcal{A}_2 \in \mathbb{H}^{I(N)\times G(N)}$, $\mathcal{B}_1 \in \mathbb{H}^{K(M)\times L(M)}$, $\mathcal{B}_2 \in \mathbb{H}^{H(M)\times L(M)}$, $\mathcal{C}_3 \in \mathbb{H}^{G(N)\times Q(N)}$, $\mathcal{C}_4 \in \mathbb{H}^{G(N)\times T(N)}$, $\mathcal{D}_3 \in \mathbb{H}^{S(M)\times K(M)}$, $\mathcal{D}_4 \in \mathbb{H}^{P(M)\times K(M)}$ *and* $\mathcal{E}_1 \in \mathbb{H}^{I(N)\times L(M)}$

be given. Set

$$\mathcal{M}_1=\mathcal{R}_{\mathcal{A}_1}*_N\mathcal{A}_2,\ \mathcal{N}_1=\mathcal{B}_2*_M\mathcal{L}_{\mathcal{B}_1},\ \mathcal{S}_1=\mathcal{A}_2*_N\mathcal{L}_{\mathcal{M}_1},\ \widehat{\mathcal{A}}_1=\mathcal{M}_1*_N\mathcal{C}_3,$$
$$\widehat{\mathcal{A}}_2=\mathcal{M}_1*_N\mathcal{C}_4,\ \widehat{\mathcal{B}}_1=\mathcal{D}_3*_M\mathcal{B}_1*_M\mathcal{L}_{\mathcal{B}_2},\ \widehat{\mathcal{B}}_2=\mathcal{D}_4*_M\mathcal{B}_1*_M\mathcal{L}_{\mathcal{B}_2},$$
$$\widehat{\mathcal{M}}_1=\mathcal{R}_{\widehat{\mathcal{A}}_1}*_N\widehat{\mathcal{A}}_2,\ \widehat{\mathcal{N}}_1=\widehat{\mathcal{B}}_2*_M\mathcal{L}_{\widehat{\mathcal{B}}_1},\ \widehat{\mathcal{S}}_1=\widehat{\mathcal{A}}_2*_N\mathcal{L}_{\widehat{\mathcal{M}}_1},\ \widehat{\mathcal{E}}_1=\mathcal{R}_{\mathcal{A}_1}*_N\mathcal{E}_1*_M\mathcal{L}_{\mathcal{B}_2},$$
$$\grave{\mathcal{E}}_1=\mathcal{E}_1-\mathcal{A}_2*_N(\mathcal{C}_3*_N\mathcal{X}_3*_M\mathcal{D}_3+\mathcal{C}_4*_N\mathcal{W}*_M\mathcal{D}_4)*_M\mathcal{B}_1.$$

Then the following statements are equivalent:
(1) (1.1) *is solvable.*
(2)

$$\mathcal{R}_{\mathcal{M}_1}*_N\mathcal{R}_{\mathcal{A}_1}*_N\mathcal{E}_1=0,\ \mathcal{E}_1*_M\mathcal{L}_{\mathcal{B}_1}*_M\mathcal{L}_{\mathcal{N}_1}=0,\ \mathcal{R}_{\mathcal{A}_2}*_N\mathcal{E}_1*_M\mathcal{L}_{\mathcal{B}_1}=0,$$
$$\mathcal{R}_{\widehat{\mathcal{M}}_1}*_N\mathcal{R}_{\widehat{\mathcal{A}}_1}*_N\widehat{\mathcal{E}}_1=0,\ \widehat{\mathcal{E}}_1*_M\mathcal{L}_{\widehat{\mathcal{B}}_1}*_M\mathcal{L}_{\widehat{\mathcal{N}}_1}=0,$$
$$\mathcal{R}_{\widehat{\mathcal{A}}_1}*_N\widehat{\mathcal{E}}_1*_M\mathcal{L}_{\widehat{\mathcal{B}}_2}=0,\ \mathcal{R}_{\widehat{\mathcal{A}}_2}*_N\widehat{\mathcal{E}}_1*_M\mathcal{L}_{\widehat{\mathcal{B}}_1}=0.$$

In that case, the general solution to (1.1) *can be expressed as follows:*

$$\begin{aligned}\mathcal{X}_1=&\mathcal{A}_1^\dagger*_N\grave{\mathcal{E}}_1*_M\mathcal{B}_1^\dagger-\mathcal{A}_1^\dagger*_N\mathcal{A}_2*_N\mathcal{M}_1^\dagger*_N\grave{\mathcal{E}}_1*_M\mathcal{B}_1^\dagger-\mathcal{A}_1^\dagger*_N\mathcal{S}_1*_N\mathcal{A}_2^\dagger\\&*_N\grave{\mathcal{E}}_1*_M\mathcal{N}_1^\dagger*_M\mathcal{B}_2*_M\mathcal{B}_1^\dagger-\mathcal{A}_1^\dagger*_N\mathcal{S}_1*_N\mathcal{U}_2*_M\mathcal{R}_{\mathcal{N}_1}*_M\mathcal{B}_2*_M\mathcal{B}_1^\dagger\\&+\mathcal{L}_{\mathcal{A}_1}*_N\mathcal{U}_4+\mathcal{U}_5*_M\mathcal{R}_{\mathcal{B}_1},\\ \mathcal{X}_2=&\mathcal{M}_1^\dagger*_N\grave{\mathcal{E}}_1*_M\mathcal{B}_2^\dagger+\mathcal{S}_1^\dagger*_N\mathcal{S}_1*_N\mathcal{A}_2^\dagger*_N\grave{\mathcal{E}}*_M\mathcal{N}_1^\dagger+\mathcal{L}_{\mathcal{M}_1}*_N\mathcal{L}_{\mathcal{S}_1}\\&*_N\mathcal{U}_1+\mathcal{L}_{\mathcal{M}_1}*_N\mathcal{U}_2*_M\mathcal{R}_{\mathcal{N}_1}+\mathcal{U}_3*_M\mathcal{R}_{\mathcal{B}_2},\\ \mathcal{X}_3=&\widehat{\mathcal{A}}_1^\dagger*_N\widehat{\mathcal{E}}_1*_M\widehat{\mathcal{B}}_1^\dagger-\widehat{\mathcal{A}}_1^\dagger*_N\widehat{\mathcal{A}}_2*_N\widehat{\mathcal{M}}_1^\dagger*_N\widehat{\mathcal{E}}_1*_M\widehat{\mathcal{B}}_1^\dagger-\widehat{\mathcal{A}}_1^\dagger*_N\widehat{\mathcal{S}}_1*_N\widehat{\mathcal{A}}_2^\dagger\\&{}_N\widehat{\mathcal{E}}_1*_M\widehat{\mathcal{N}}_1^\dagger*_M\widehat{\mathcal{B}}_2*_M\widehat{\mathcal{B}}_1^\dagger-\widehat{\mathcal{A}}_1^\dagger*_N\widehat{\mathcal{S}}_1*_N\widehat{\mathcal{U}}_2*_M\mathcal{R}_{\widehat{\mathcal{N}}_1}*_M\widehat{\mathcal{B}}_2*_M\widehat{\mathcal{B}}_1^\dagger\\&+\mathcal{L}_{\widehat{\mathcal{A}}_1}*_N\widehat{\mathcal{U}}_4+\widehat{\mathcal{U}}_5*_M\mathcal{R}_{\widehat{\mathcal{B}}_1},\\ \mathcal{X}_4=&\widehat{\mathcal{M}}_1^\dagger*_N\widehat{\mathcal{E}}_1*_M\widehat{\mathcal{B}}_2^\dagger+\widehat{\mathcal{S}}_1^\dagger*_N\widehat{\mathcal{S}}_1*_N\widehat{\mathcal{A}}_2^\dagger*_N\widehat{\mathcal{E}}*_M\widehat{\mathcal{N}}_1^\dagger+\mathcal{L}_{\widehat{\mathcal{M}}_1}*_N\mathcal{L}_{\widehat{\mathcal{S}}_1}\\&*_N\widehat{\mathcal{U}}_1+\mathcal{L}_{\widehat{\mathcal{M}}_1}*_N\widehat{\mathcal{U}}_2*_M\mathcal{R}_{\widehat{\mathcal{N}}_1}+\widehat{\mathcal{U}}_3*_M\mathcal{R}_{\widehat{\mathcal{B}}_2},\end{aligned}$$

where $\mathcal{U}_i,\widehat{\mathcal{U}}_i\ (i=\overline{1,5})$ *are arbitrary tensors with suitable orders.*

In case of $\mathcal{A}_2=\mathcal{B}_1=\mathcal{I}$ *and* $\mathcal{A}_1=\mathcal{B}_2=0$*, we have the following special case of* (1.1)

$$\mathcal{C}_3*_N\mathcal{X}_3*_M\mathcal{D}_3+\mathcal{C}_4*_N\mathcal{X}_4*_M\mathcal{D}_4=\mathcal{E}_1,$$

which is solvable if and only if

$$\mathcal{R}_{\widehat{\mathcal{M}}_1}*_N\mathcal{R}_{\mathcal{C}_3}*_N\mathcal{E}_1=0,\ \mathcal{E}_1*_M\mathcal{L}_{\mathcal{D}_3}*_M\mathcal{L}_{\widehat{\mathcal{N}}_1}=0,$$
$$\mathcal{R}_{\mathcal{C}_3}*_N\mathcal{E}_1*_M\mathcal{L}_{D_4}=0,\ \mathcal{R}_{\mathcal{C}_4}*_N\mathcal{E}_1*_M\mathcal{L}_{\mathcal{D}_3}=0.$$

In that case, the general solution can be expressed as follows:

$$\begin{aligned}\mathcal{X}_3=&\mathcal{C}_3^\dagger*_N\mathcal{E}_1*_M\mathcal{D}_3^\dagger-\mathcal{C}_3^\dagger*_N\mathcal{C}_4*_N\widehat{\mathcal{M}}_1^\dagger*_N\mathcal{E}_1*_M\mathcal{D}_3^\dagger-\mathcal{C}_3^\dagger*_N\widehat{\mathcal{S}}_1*_N\mathcal{C}_4^\dagger\\&{}_N\mathcal{E}_1*_M\widehat{\mathcal{N}}_1^\dagger*_M\mathcal{D}_4*_M\mathcal{D}_3^\dagger-\mathcal{C}_3^\dagger*_N\widehat{\mathcal{S}}_1*_N\widehat{\mathcal{U}}_2*_M\mathcal{R}_{\widehat{\mathcal{N}}_1}*_M\mathcal{D}_4*_M\mathcal{D}_3^\dagger\\&+\mathcal{L}_{\mathcal{C}_3}*_N\widehat{\mathcal{U}}_4+\widehat{\mathcal{U}}_5*_M\mathcal{R}_{\mathcal{D}_3},\\ \mathcal{X}_4=&\widehat{\mathcal{M}}_1^\dagger*_N\mathcal{E}_1*_M\mathcal{D}_4^\dagger+\widehat{\mathcal{S}}_1^\dagger*_N\widehat{\mathcal{S}}_1*_N\mathcal{C}_4^\dagger*_N\mathcal{E}_1*_M\widehat{\mathcal{N}}_1^\dagger+\mathcal{L}_{\widehat{\mathcal{M}}_1}*_N\mathcal{L}_{\widehat{\mathcal{S}}_1}\\&*_N\widehat{\mathcal{U}}_1+\mathcal{L}_{\widehat{\mathcal{M}}_1}*_N\widehat{\mathcal{U}}_2*_M\mathcal{R}_{\widehat{\mathcal{N}}_1}+\widehat{\mathcal{U}}_3*_M\mathcal{R}_{\mathcal{D}_4},\end{aligned}$$

3. The consistency conditions and the general Solution to (1.4)

In the following Theorem, we provide consistency conditions and general solution of a coupled Two-sided Sylvester-like quaternion system of tensor equations (1.3).

Theorem 3.1. *Consider the quaternion system of tensor equations* (1.3)*, where*

$$\begin{aligned}
&\mathcal{F}_4 \in \mathbb{H}^{I(N)\times J(N)},\ \mathcal{G}_4 \in \mathbb{H}^{L(M)\times K(M)},\ \mathcal{H}_4 \in \mathbb{H}^{I(N)\times Q(N)},\ \mathcal{J}_4 \in \mathbb{H}^{S(M)\times K(M)},\\
&\mathcal{E}_4 \in \mathbb{H}^{I(N)\times K(M)},\ \mathcal{E}_5 \in \mathbb{H}^{I(N)\times K(M)},\ \mathcal{A}_1 \in \mathbb{H}^{I(N)\times J(N)},\ \mathcal{A}_2 \in \mathbb{H}^{I(N)\times Q(N)},\\
&\mathcal{A}_3 \in \mathbb{H}^{I(N)\times P(N)},\ \mathcal{B}_1 \in \mathbb{H}^{F(M)\times K(M)},\ \mathcal{B}_2 \in \mathbb{H}^{G(M)\times K(M)},\ \mathcal{B}_3 \in \mathbb{H}^{H(M)\times K(M)},\\
&\mathcal{C}_1 \in \mathbb{H}^{I(N)\times A(N)},\ \mathcal{C}_2 \in \mathbb{H}^{I(N)\times B(N)},\ \mathcal{C}_3 \in \mathbb{H}^{I(N)\times C(N)},\ \mathcal{D}_1 \in \mathbb{H}^{L(M)\times K(M)},\\
&\mathcal{D}_2 \in \mathbb{H}^{L(M)\times K(M)},\ \mathcal{D}_3 \in \mathbb{H}^{L(M)\times K(M)},\ \mathcal{F}_1 \in \mathbb{H}^{A(N)\times J(N)},\ \mathcal{F}_2 \in \mathbb{H}^{B(N)\times P(N)},\\
&\mathcal{F}_3 \in \mathbb{H}^{C(N)\times J(N)},\ \mathcal{G}_1 \in \mathbb{H}^{L(M)\times F(M)},\ \mathcal{G}_2 \in \mathbb{H}^{Q(M)\times G(M)},\ \mathcal{G}_3 \in \mathbb{H}^{L(M)\times H(M)},\\
&\mathcal{H}_1 \in \mathbb{H}^{A(N)\times P(N)},\ \mathcal{H}_2 \in \mathbb{H}^{B(N)\times J(N)},\ \mathcal{H}_3 \in \mathbb{H}^{C(N)\times Q(N)},\ \mathcal{J}_1 \in \mathbb{H}^{Q(M)\times F(M)},\\
&\mathcal{J}_2 \in \mathbb{H}^{L(M)\times J(M)},\ \mathcal{J}_3 \in \mathbb{H}^{S(M)\times H(M)},\ \mathcal{E}_i \in \mathbb{H}^{I(N)\times K(M)},\ (i=\overline{1,3}).
\end{aligned}$$

are given tensors over $\mathbb{H}$*. Set*

$$\grave{\mathcal{E}}_i = \mathcal{E}_i - \mathcal{C}_i *_N (\mathcal{F}_i *_N \mathcal{Z}_i *_M \mathcal{G}_i - \mathcal{H}_i *_N \mathcal{Z}_{i+1} *_M \mathcal{J}_i) *_M \mathcal{B}_i, \tag{3.1a}$$

$$\mathcal{M}_i = \mathcal{R}_{\mathcal{A}_i} *_N \mathcal{C}_i,\ \mathcal{N}_i = \mathcal{D}_i *_M \mathcal{L}_{\mathcal{B}_i},\ \mathcal{S}_i = \mathcal{C}_i *_N \mathcal{L}_{\mathcal{M}_i},\ \widehat{\mathcal{A}}_i = \mathcal{M}_i *_N \mathcal{F}_i, \tag{3.1b}$$

$$\widehat{\mathcal{C}}_i = \mathcal{M}_i *_N \mathcal{H}_i,\ \widehat{\mathcal{B}}_i = \mathcal{G}_i *_M \mathcal{B}_i *_M \mathcal{L}_{\mathcal{D}_i},\ \widehat{\mathcal{D}}_i = \mathcal{J}_i *_M \mathcal{B}_i *_M \mathcal{L}_{\mathcal{D}_i},\ \widehat{\mathcal{M}}_i = \mathcal{R}_{\widehat{\mathcal{A}}_i} *_N \widehat{\mathcal{C}}_i, \tag{3.1c}$$

$$\widehat{\mathcal{N}}_i = \widehat{\mathcal{D}}_i *_M \mathcal{L}_{\widehat{\mathcal{B}}_i},\ \widehat{\mathcal{S}}_i = \widehat{\mathcal{C}}_i *_N \mathcal{L}_{\widehat{\mathcal{M}}_i},\ \widehat{\mathcal{E}}_i = \mathcal{R}_{\mathcal{A}_i} *_N \mathcal{E}_i *_M \mathcal{L}_{\mathcal{D}_i},\ (i=\overline{1,3}), \tag{3.1d}$$

$$\mathcal{A}_{11} = \begin{bmatrix}\mathcal{L}_{\mathcal{F}_4} & -\mathcal{L}_{\widehat{\mathcal{A}}_1}\end{bmatrix},\ \mathcal{D}_{11} = \begin{bmatrix}\mathcal{R}_{\mathcal{G}_4}\\ -\mathcal{R}_{\widehat{\mathcal{B}}_1}\end{bmatrix},\ \widehat{\mathcal{A}}_{11} = \widehat{\mathcal{A}}_1^{\dagger} *_N \widehat{\mathcal{S}}_1,\ \widehat{\mathcal{B}}_{11} = R_{\widehat{\mathcal{N}}_1} *_M \widehat{\mathcal{D}}_1 *_M \widehat{\mathcal{B}}_1^{\dagger}, \tag{3.1e}$$

$$\begin{aligned}\mathcal{E}_{11} = {} & \widehat{\mathcal{A}}_1^{\dagger} *_N \widehat{\mathcal{E}}_1 *_M \widehat{\mathcal{B}}_1^{\dagger} - \widehat{\mathcal{A}}_1^{\dagger} *_N \widehat{\mathcal{C}}_1 *_N \widehat{\mathcal{M}}_1^{\dagger} *_N \widehat{\mathcal{E}}_1 *_M \widehat{\mathcal{B}}_1^{\dagger} - \widehat{\mathcal{A}}_1^{\dagger} *_N \widehat{\mathcal{S}}_1 *_N \widehat{\mathcal{C}}_1^{\dagger} *_N \widehat{\mathcal{E}}_1\\ & *_M \widehat{\mathcal{N}}_1^{\dagger} *_M \widehat{\mathcal{D}}_1 *_M \widehat{\mathcal{B}}_1^{\dagger} - \mathcal{F}_4^{\dagger} *_N \mathcal{E}_4 *_M \mathcal{G}_4^{\dagger},\end{aligned} \tag{3.1f}$$

$$\mathcal{A}_{22} = \begin{bmatrix}\mathcal{L}_{\mathcal{H}_4} & -\mathcal{L}_{\widehat{\mathcal{M}}_3} *_N \mathcal{L}_{\widehat{\mathcal{S}}_3}\end{bmatrix},\ \mathcal{D}_{22} = \begin{bmatrix}\mathcal{R}_{\mathcal{J}_4}\\ -\mathcal{R}_{\widehat{\mathcal{D}}_3}\end{bmatrix},\ \widehat{\mathcal{A}}_{22} = \mathcal{L}_{\widehat{\mathcal{M}}_3},\ \widehat{\mathcal{B}}_{22} = R_{\widehat{\mathcal{N}}_3}, \tag{3.1g}$$

$$\mathcal{E}_{22} = \widehat{\mathcal{M}}_3^{\dagger} *_N \widehat{\mathcal{E}}_3 *_M \widehat{\mathcal{D}}_3^{\dagger} + \widehat{\mathcal{S}}_3^{\dagger} *_N \widehat{\mathcal{S}}_3 *_N \widehat{\mathcal{C}}_3^{\dagger} *_N \widehat{\mathcal{E}}_3 *_M \widehat{\mathcal{N}}_3^{\dagger} - \mathcal{H}_4^{\dagger} *_N \mathcal{E}_5 *_M \mathcal{J}_4^{\dagger}, \tag{3.1h}$$

$$\widehat{\widehat{\mathcal{A}}}_{ii} = \mathcal{R}_{\mathcal{A}_{ii}} *_N \widehat{\mathcal{A}}_{ii},\ \widehat{\widehat{\mathcal{B}}}_{ii} = \widehat{\mathcal{B}}_{ii} *_M \mathcal{L}_{\mathcal{D}_{ii}},\ \widehat{\widehat{\mathcal{E}}}_{ii} = \mathcal{R}_{\mathcal{A}_{ii}} *_N \mathcal{E}_{ii} *_M \mathcal{L}_{\mathcal{D}_{ii}},\ (i=1,2), \tag{3.1i}$$

$$\overline{\mathcal{A}}_1 = \begin{bmatrix}-\mathcal{L}_{\widehat{\mathcal{M}}_1} *_N \mathcal{L}_{\widehat{\mathcal{S}}_1} & \mathcal{L}_{\widehat{\mathcal{A}}_2}\end{bmatrix},\ \overline{\mathcal{A}}_2 = \begin{bmatrix}-\mathcal{L}_{\widehat{\mathcal{M}}_2} *_N \mathcal{L}_{\widehat{\mathcal{S}}_2} & \mathcal{L}_{\widehat{\mathcal{A}}_3}\end{bmatrix},\ \overline{\mathcal{F}}_1 = \widehat{\mathcal{A}}_2^{\dagger} *_N \widehat{\mathcal{S}}_2, \tag{3.1j}$$

$$\overline{\mathcal{B}}_1 = \begin{bmatrix}-\mathcal{R}_{\widehat{\mathcal{D}}_1}\\ \mathcal{R}_{\widehat{\mathcal{B}}_2}\end{bmatrix},\ \overline{\mathcal{B}}_2 = \begin{bmatrix}-\mathcal{R}_{\widehat{\mathcal{D}}_2}\\ \mathcal{R}_{\widehat{\mathcal{B}}_3}\end{bmatrix}, \overline{\mathcal{F}}_2 = \widehat{\mathcal{A}}_3^{\dagger} *_N \widehat{\mathcal{S}}_3,\ \overline{\mathcal{G}}_1 = \widehat{\mathcal{D}}_2 *_N \widehat{\mathcal{B}}_2^{\dagger},\ \overline{\mathcal{G}}_2 = \widehat{\mathcal{D}}_3 *_N \widehat{\mathcal{B}}_3^{\dagger}, \tag{3.1k}$$

$$\overline{\mathcal{H}}_1 = \mathcal{L}_{\widehat{\mathcal{M}}_1},\ \overline{\mathcal{J}}_1 = \mathcal{R}_{\widehat{\mathcal{N}}_1},\ \overline{\mathcal{H}}_2 = \mathcal{L}_{\widehat{\mathcal{M}}_2},\ \overline{\mathcal{J}}_2 = \mathcal{R}_{\widehat{\mathcal{N}}_2}, \tag{3.1l}$$

$$\begin{aligned}\overline{\mathcal{E}}_1 = {} & -\widehat{\mathcal{M}}_1^{\dagger} *_N \widehat{\mathcal{E}}_1 *_M \widehat{\mathcal{D}}_1^{\dagger} - \widehat{\mathcal{S}}_1^{\dagger} *_N \widehat{\mathcal{S}}_1 *_N \widehat{\mathcal{C}}_1^{\dagger} *_N \widehat{\mathcal{E}} *_M \widehat{\mathcal{N}}_1^{\dagger} + \widehat{\mathcal{A}}_2^{\dagger} *_N \widehat{\mathcal{E}}_2 *_M \widehat{\mathcal{B}}_2^{\dagger} - \widehat{\mathcal{A}}_2^{\dagger} *_N \widehat{\mathcal{C}}_2\\ & *_N \widehat{\mathcal{M}}_2^{\dagger} *_N \widehat{\mathcal{E}}_2 *_M \widehat{\mathcal{B}}_2^{\dagger} - \widehat{\mathcal{A}}_2^{\dagger} *_N \widehat{\mathcal{S}}_2 *_N \widehat{\mathcal{C}}_2^{\dagger} *_N \widehat{\mathcal{E}}_2 *_M \widehat{\mathcal{N}}_2^{\dagger} *_M \widehat{\mathcal{D}}_2 *_M \widehat{\mathcal{B}}_2^{\dagger},\end{aligned} \tag{3.1m}$$

$$\begin{aligned}\overline{\mathcal{E}}_2 = &-\widehat{\mathcal{M}}_2^\dagger *_N \widehat{\mathcal{E}}_2 *_M \widehat{\mathcal{D}}_2^\dagger - \widehat{\mathcal{S}}_2^\dagger *_N \widehat{\mathcal{S}}_2 *_N \widehat{\mathcal{C}}_2^\dagger *_N \widehat{\mathcal{E}} *_M \widehat{\mathcal{N}}_2^\dagger + \widehat{\mathcal{A}}_3^\dagger *_N \widehat{\mathcal{E}}_3 *_M \widehat{\mathcal{B}}_3^\dagger - \widehat{\mathcal{A}}_3^\dagger *_N \widehat{\mathcal{C}}_3 \\ &*_N \widehat{\mathcal{M}}_3^\dagger *_N \widehat{\mathcal{E}}_3 *_M \widehat{\mathcal{B}}_3^\dagger - \widehat{\mathcal{A}}_3^\dagger *_N \widehat{\mathcal{S}}_3 *_N \widehat{\mathcal{C}}_3^\dagger *_N \widehat{\mathcal{E}}_3 *_M \widehat{\mathcal{N}}_3^\dagger *_M \widehat{\mathcal{D}}_3 *_M \widehat{\mathcal{B}}_3^\dagger,\end{aligned} \tag{3.2a}$$

$$\overline{\mathcal{F}}_{ii} = \mathcal{R}_{\overline{\mathcal{A}}_i} *_N \overline{\mathcal{F}}_i,\ \overline{\mathcal{G}}_{ii} = \overline{\mathcal{G}}_i *_M \mathcal{L}_{\overline{\mathcal{B}}_i},\ \overline{\mathcal{H}}_{ii} = \mathcal{R}_{\overline{\mathcal{A}}_i} *_N \overline{\mathcal{H}}_i,\ \overline{\mathcal{J}}_{ii} = \overline{\mathcal{J}}_i *_M \mathcal{L}_{\overline{\mathcal{B}}_i}, \tag{3.2b}$$

$$\overline{\mathcal{E}}_{ii} = \mathcal{R}_{\overline{\mathcal{A}}_i} *_N \overline{\mathcal{E}}_i *_M \mathcal{L}_{\overline{\mathcal{B}}_i},\ \overline{\mathcal{M}}_{ii} = \mathcal{R}_{\overline{\mathcal{F}}_{ii}} *_N \overline{\mathcal{H}}_{ii},\ \overline{\mathcal{N}}_{ii} = \overline{\mathcal{J}}_{ii} *_M \mathcal{L}_{\overline{\mathcal{G}}_{ii}},\ \overline{\mathcal{S}}_{ii} = \overline{\mathcal{H}}_{ii} *_N \mathcal{L}_{\overline{\mathcal{M}}_{ii}}, \tag{3.2c}$$

$$\overline{\overline{\mathcal{A}}}_1 = \begin{bmatrix}\mathcal{L}_{\overline{\mathcal{F}}_{11}} & -\mathcal{L}_{\overline{\mathcal{M}}_{22}} *_N \mathcal{L}_{\overline{\mathcal{S}}_{22}}\end{bmatrix},\ \overline{\overline{\mathcal{B}}}_1 = \begin{bmatrix}\mathcal{R}_{\overline{\mathcal{G}}_{11}} \\ -\mathcal{R}_{\overline{\mathcal{J}}_{11}}\end{bmatrix},\ \overline{\overline{\mathcal{F}}}_1 = \overline{\mathcal{F}}_{11}^\dagger *_N \overline{\mathcal{S}}_{11}, \tag{3.2d}$$

$$\overline{\overline{\mathcal{G}}}_1 = R_{\overline{\mathcal{N}}_{11}} *_M \overline{\mathcal{J}}_{11} *_M \overline{\mathcal{G}}_{11}^\dagger,\ \overline{\overline{\mathcal{H}}}_1 = \mathcal{L}_{\overline{\mathcal{M}}_{22}},\ \overline{\overline{\mathcal{J}}}_1 = \mathcal{R}_{\overline{\mathcal{N}}_{22}}, \tag{3.2e}$$

$$\begin{aligned}\overline{\overline{\mathcal{E}}}_1 = &\overline{\mathcal{F}}_{11}^\dagger *_N \overline{\mathcal{E}}_{11} *_M \overline{\mathcal{G}}_{11}^\dagger - \overline{\mathcal{F}}_{11}^\dagger *_N \overline{\mathcal{H}}_{11} *_N \overline{\mathcal{M}}_{11}^\dagger *_N \overline{\mathcal{E}}_{11} *_M \overline{\mathcal{G}}_{11}^\dagger - \overline{\mathcal{F}}_{11}^\dagger *_N \overline{\mathcal{S}}_{11} *_N \overline{\mathcal{H}}_{11}^\dagger \\ &*_N \overline{\mathcal{E}}_{11} *_M \overline{\mathcal{N}}_{11}^\dagger *_M \overline{\mathcal{J}}_{11} *_M \overline{\mathcal{G}}_{11}^\dagger - \overline{\mathcal{M}}_{22}^\dagger *_N \overline{\mathcal{E}}_{22} *_M \overline{\mathcal{J}}_{22}^\dagger - \overline{\mathcal{S}}_{22}^\dagger *_N \overline{\mathcal{S}}_{22} *_N \overline{\mathcal{H}}_{22}^\dagger \\ &*_N \overline{\mathcal{E}}_{22} *_M \overline{\mathcal{N}}_{22}^\dagger\end{aligned} \tag{3.2f}$$

$$\overline{\overline{\mathcal{F}}}_{11} = \mathcal{R}_{\overline{\overline{\mathcal{A}}}_1} *_N \overline{\overline{\mathcal{F}}}_1,\ \overline{\overline{\mathcal{G}}}_{11} = \overline{\overline{\mathcal{G}}}_1 *_M \mathcal{L}_{\overline{\overline{\mathcal{B}}}_1},\ \overline{\overline{\mathcal{H}}}_{11} = \mathcal{R}_{\overline{\overline{\mathcal{A}}}_1} *_N \overline{\overline{\mathcal{H}}}_1,\ \overline{\overline{\mathcal{J}}}_{11} = \overline{\overline{\mathcal{J}}}_1 *_M \mathcal{L}_{\overline{\overline{\mathcal{B}}}_1}, \tag{3.2g}$$

$$\overline{\overline{\mathcal{E}}}_{11} = \mathcal{R}_{\overline{\overline{\mathcal{A}}}_1} *_N \overline{\overline{\mathcal{E}}}_1 *_M \mathcal{L}_{\overline{\overline{\mathcal{B}}}_1},\ \overline{\overline{\mathcal{M}}}_{11} = \mathcal{R}_{\overline{\overline{\mathcal{F}}}_{11}} *_N \overline{\overline{\mathcal{H}}}_{11},\ \overline{\overline{\mathcal{N}}}_{11} = \overline{\overline{\mathcal{J}}}_{11} *_M \mathcal{L}_{\overline{\overline{\mathcal{G}}}_{11}}, \tag{3.2h}$$

$$\overline{\overline{\mathcal{S}}}_{11} = \overline{\overline{\mathcal{H}}}_{11} *_N \mathcal{L}_{\overline{\overline{\mathcal{M}}}_{11}},\ \widetilde{\mathcal{A}}_1 = \begin{bmatrix}\mathcal{L}_{\overline{\mathcal{M}}_{11}} *_N \mathcal{L}_{\overline{\mathcal{S}}_{11}} & -\mathcal{L}_{\widehat{\widehat{\mathcal{A}}}_{11}}\end{bmatrix},\ \widetilde{\mathcal{A}}_2 = \begin{bmatrix}\mathcal{L}_{\widehat{\widehat{\mathcal{A}}}_{22}} & -\mathcal{L}_{\overline{\mathcal{F}}_{22}}\end{bmatrix}, \tag{3.2i}$$

$$\widetilde{\mathcal{B}}_1 = \begin{bmatrix}\mathcal{R}_{\overline{\mathcal{J}}_{11}} \\ -\mathcal{R}_{\widehat{\widehat{\mathcal{B}}}_{11}}\end{bmatrix},\ \widetilde{\mathcal{B}}_2 = \begin{bmatrix}\mathcal{R}_{\widehat{\widehat{\mathcal{B}}}_{22}} \\ -\mathcal{R}_{\overline{\mathcal{G}}_{22}}\end{bmatrix}, \widetilde{\mathcal{C}}_1 = \mathcal{L}_{\overline{\mathcal{M}}_{11}} \widetilde{\mathcal{D}}_1 = \mathcal{R}_{\overline{\mathcal{N}}_{11}}, \tag{3.2j}$$

$$\widetilde{\mathcal{C}}_2 = \overline{\mathcal{F}}_{22}^\dagger *_N \overline{\mathcal{S}}_{22},\ \widetilde{\mathcal{D}}_2 = \mathcal{R}_{\overline{\mathcal{N}}_{22}} *_M \overline{\mathcal{J}}_{22} *_M \overline{\mathcal{G}}_{22}^\dagger, \tag{3.2k}$$

$$\widetilde{\mathcal{E}}_1 = \widehat{\widehat{\mathcal{A}}}_{11}^\dagger *_N \widehat{\widehat{\mathcal{E}}}_{11} *_M \widehat{\widehat{\mathcal{B}}}_{11}^\dagger - \overline{\mathcal{M}}_{11}^\dagger *_N \overline{\mathcal{E}}_{11} *_M \overline{\mathcal{J}}_{11}^\dagger - \overline{\mathcal{S}}_{11}^\dagger *_N \overline{\mathcal{S}}_{11} *_N \overline{\mathcal{H}}_{11}^\dagger *_N \overline{\mathcal{E}}_{11} *_M \overline{\mathcal{N}}_{11}^\dagger, \tag{3.2l}$$

$$\begin{aligned}\widetilde{\mathcal{E}}_2 = &\overline{\mathcal{F}}_{22}^\dagger *_N \overline{\mathcal{E}}_{22} *_M \overline{\mathcal{G}}_{22}^\dagger - \overline{\mathcal{F}}_{22}^\dagger *_N \overline{\mathcal{H}}_{22} *_N \overline{\mathcal{M}}_{22}^\dagger *_N \overline{\mathcal{E}}_{22} *_M \overline{\mathcal{G}}_{22}^\dagger - \overline{\mathcal{F}}_{22}^\dagger *_N \overline{\mathcal{S}}_{22} *_N \overline{\mathcal{H}}_{22}^\dagger \\ &*_N \overline{\mathcal{E}}_{22} *_M \overline{\mathcal{N}}_{22}^\dagger *_M \overline{\mathcal{J}}_{22} *_M \overline{\mathcal{G}}_{22}^\dagger - \widehat{\widehat{\mathcal{A}}}_{22}^\dagger *_N \widehat{\widehat{\mathcal{E}}}_{22} *_M \widehat{\widehat{\mathcal{B}}}_{22}^\dagger,\end{aligned} \tag{3.2m}$$

$$\widetilde{\mathcal{F}}_1 = \begin{bmatrix}\mathcal{L}_{\widetilde{\mathcal{C}}_{11}} & -\mathcal{L}_{\overline{\overline{\mathcal{F}}}_{11}}\end{bmatrix},\ \widetilde{\mathcal{F}}_2 = \begin{bmatrix}\mathcal{L}_{\overline{\overline{\mathcal{M}}}_{11}} *_N \mathcal{L}_{\overline{\overline{\mathcal{S}}}_{11}} & -\mathcal{L}_{\widetilde{\mathcal{C}}_{22}}\end{bmatrix},\ \widetilde{\mathcal{H}}_1 = \overline{\overline{\mathcal{F}}}_{11}^\dagger *_N \overline{\overline{\mathcal{S}}}_{11}, \tag{3.2n}$$

$$\widetilde{\mathcal{J}}_1 = \mathcal{R}_{\overline{\overline{\mathcal{N}}}_{11}} *_M \overline{\overline{\mathcal{J}}}_{11} *_M \overline{\overline{\mathcal{G}}}_{11}^\dagger\ \widetilde{\mathcal{G}}_1 = \begin{bmatrix}\mathcal{R}_{\widetilde{\mathcal{D}}_{11}} \\ -\mathcal{R}_{\overline{\overline{\mathcal{G}}}_{11}}\end{bmatrix},\ \widetilde{\mathcal{G}}_2 = \begin{bmatrix}\mathcal{R}_{\overline{\overline{\mathcal{J}}}_{11}} \\ -\mathcal{R}_{\widetilde{\mathcal{D}}_{22}}\end{bmatrix}, \widetilde{\mathcal{H}}_2 = \mathcal{L}_{\overline{\overline{\mathcal{M}}}_{11}},\ \widetilde{\mathcal{J}}_2 = \mathcal{R}_{\overline{\overline{\mathcal{N}}}_{11}}, \tag{3.2o}$$

$$\begin{aligned}\widetilde{\widetilde{\mathcal{E}}}_1 = &\overline{\overline{\mathcal{F}}}_{11}^\dagger *_N \overline{\overline{\mathcal{E}}}_{11} *_M \overline{\overline{\mathcal{G}}}_{11}^\dagger - \overline{\overline{\mathcal{F}}}_{11}^\dagger *_N \overline{\overline{\mathcal{H}}}_{11} *_N \overline{\overline{\mathcal{M}}}_{11}^\dagger *_N \overline{\overline{\mathcal{E}}}_{11} *_M \overline{\overline{\mathcal{G}}}_{11}^\dagger - \overline{\overline{\mathcal{F}}}_{11}^\dagger *_N \overline{\overline{\mathcal{S}}}_{11} \\ &*_N \overline{\overline{\mathcal{H}}}_{11}^\dagger *_N \overline{\overline{\mathcal{E}}}_{11} *_M \overline{\overline{\mathcal{N}}}_{11}^\dagger *_M \overline{\overline{\mathcal{J}}}_{11} *_M \overline{\overline{\mathcal{G}}}_{11}^\dagger - \widetilde{\mathcal{C}}_{11}^\dagger *_N \widetilde{\mathcal{E}}_{11} *_M \widetilde{\mathcal{D}}_{11}^\dagger,\end{aligned} \tag{3.2p}$$

$$\widetilde{\widetilde{\mathcal{E}}}_2 = \widetilde{\mathcal{C}}_{22}^\dagger *_N \widetilde{\mathcal{E}}_{22} *_M \widetilde{\mathcal{D}}_{22}^\dagger - \overline{\overline{\mathcal{M}}}_{11}^\dagger *_N \overline{\overline{\mathcal{E}}}_{11} *_M \overline{\overline{\mathcal{J}}}_{11}^\dagger - \overline{\overline{\mathcal{S}}}_{11}^\dagger *_N \overline{\overline{\mathcal{S}}}_{11} *_N \overline{\overline{\mathcal{H}}}_{11}^\dagger *_N \overline{\overline{\mathcal{E}}}_{11} *_M \overline{\overline{\mathcal{N}}}_{11}^\dagger, \tag{3.2q}$$

$$\widetilde{\mathcal{H}}_{11} = \mathcal{R}_{\widetilde{\mathcal{F}}_1} *_N \widetilde{\mathcal{H}}_1,\ \widetilde{\mathcal{H}}_{22} = \mathcal{R}_{\widetilde{\mathcal{F}}_2} *_N \widetilde{\mathcal{H}}_2,\ \widetilde{\mathcal{J}}_{11} = \widetilde{\mathcal{J}}_1 *_M \mathcal{L}_{\widetilde{\mathcal{G}}_1},\ \widetilde{\mathcal{J}}_{22} = \widetilde{\mathcal{J}}_2 *_M \mathcal{L}_{\widetilde{\mathcal{G}}_2}, \tag{3.2r}$$

$$\begin{aligned}&\widetilde{\widetilde{\mathcal{E}}}_{11} = \mathcal{R}_{\widetilde{\mathcal{F}}_1} *_N \widetilde{\widetilde{\mathcal{E}}}_1 *_M \mathcal{L}_{\widetilde{\mathcal{G}}_1},\ \widetilde{\widetilde{\mathcal{E}}}_{22} = \mathcal{R}_{\widetilde{\mathcal{F}}_2} *_N \widetilde{\widetilde{\mathcal{E}}}_2 *_M \mathcal{L}_{\widetilde{\mathcal{G}}_2},\ \widetilde{\mathcal{A}} = \begin{bmatrix}\mathcal{L}_{\widetilde{\mathcal{H}}_{11}} & -\mathcal{L}_{\widetilde{\mathcal{H}}_{22}}\end{bmatrix}, \\ &\widetilde{\mathcal{B}} = \begin{bmatrix}\mathcal{R}_{\widetilde{\mathcal{J}}_{11}} & -\mathcal{R}_{\widetilde{\mathcal{J}}_{22}}\end{bmatrix},\ \widetilde{\mathcal{E}} = \widetilde{\mathcal{H}}_{22}^\dagger *_N \widetilde{\widetilde{\mathcal{E}}}_{22} *_M \widetilde{\mathcal{J}}_{22}^\dagger - \widetilde{\mathcal{H}}_{11}^\dagger *_N \widetilde{\widetilde{\mathcal{E}}}_{11} *_M \widetilde{\mathcal{J}}_{11}^\dagger.\end{aligned} \tag{3.2s}$$

Then the system (1.3) *is consistent if and only if*

$$\mathcal{R}_{\mathcal{M}_i} *_N \mathcal{R}_{\mathcal{A}_i} *_N \mathcal{E}_i = 0,\ \mathcal{E}_i *_M \mathcal{L}_{\mathcal{B}_i} *_M \mathcal{L}_{\mathcal{N}_i} = 0,\ \mathcal{R}_{\mathcal{C}_i} *_N \mathcal{E}_i *_M \mathcal{L}_{\mathcal{B}_i} = 0, \tag{3.3}$$

$$\mathcal{R}_{\widehat{\mathcal{M}}_i} *_N \mathcal{R}_{\widehat{\mathcal{A}}_i} *_N \widehat{\mathcal{E}}_i = 0,\ \widehat{\mathcal{E}}_i *_M \mathcal{L}_{\widehat{\mathcal{B}}_i} *_M \mathcal{L}_{\widehat{\mathcal{N}}_i} = 0, \tag{3.4}$$

$$\mathcal{R}_{\widehat{\mathcal{A}}_i} *_N \widehat{\mathcal{E}}_i *_M \mathcal{L}_{\widehat{\mathcal{D}}_i} = 0,\ \mathcal{R}_{\widehat{\mathcal{C}}_i} *_N \widehat{\mathcal{E}}_i *_M \mathcal{L}_{\widehat{\mathcal{B}}_i} = 0,\ (i = \overline{1,3}), \tag{3.5}$$

$$\mathcal{R}_{\mathcal{F}_4} *_N \mathcal{E}_4 = 0,\ \mathcal{E}_4 *_M \mathcal{L}_{\mathcal{G}_4} = 0,\ \mathcal{R}_{\mathcal{H}_4} *_N \mathcal{E}_5 = 0,\ \mathcal{E}_5 *_M \mathcal{L}_{\mathcal{J}_4} = 0, \tag{3.6}$$

$$\mathcal{R}_{\widehat{\widehat{\mathcal{A}}}_{kk}} *_N \widehat{\widehat{\mathcal{E}}}_{kk} = 0,\ \widehat{\widehat{\mathcal{E}}}_{kk} *_M \mathcal{L}_{\widehat{\widehat{\mathcal{B}}}_{kk}} = 0, \tag{3.7}$$

$$\mathcal{R}_{\overline{\mathcal{M}}_{kk}} *_N \mathcal{R}_{\overline{\mathcal{F}}_{kk}} *_N \overline{\mathcal{E}}_{kk} = 0,\ \overline{\mathcal{E}}_{kk} *_M \mathcal{L}_{\overline{\mathcal{G}}_{kk}} *_M \mathcal{L}_{\overline{\mathcal{N}}_{kk}} = 0, \tag{3.8}$$

$$\mathcal{R}_{\overline{\mathcal{F}}_{kk}} *_N \overline{\mathcal{E}}_{kk} *_M \mathcal{L}_{\overline{\mathcal{J}}_{kk}} = 0,\ \mathcal{R}_{\overline{\mathcal{H}}_{kk}} *_N \overline{\mathcal{E}}_{kk} *_M \mathcal{L}_{\overline{\mathcal{G}}_{kk}} = 0,\ (k = 1, 2), \tag{3.9}$$

$$\mathcal{R}_{\overline{\overline{\mathcal{M}}}_{11}} *_N \mathcal{R}_{\overline{\overline{\mathcal{F}}}_{11}} *_N \overline{\overline{\mathcal{E}}}_{11} = 0,\ \overline{\overline{\mathcal{E}}}_{11} *_M \mathcal{L}_{\overline{\overline{\mathcal{G}}}_{11}} *_M \mathcal{L}_{\overline{\overline{\mathcal{N}}}_{11}} = 0, \tag{3.10}$$

$$\mathcal{R}_{\overline{\overline{\mathcal{F}}}_{11}} *_N \overline{\overline{\mathcal{E}}}_{11} *_M \mathcal{L}_{\overline{\overline{\mathcal{J}}}_{11}} = 0,\ \mathcal{R}_{\overline{\overline{\mathcal{H}}}_{11}} *_N \overline{\overline{\mathcal{E}}}_{11} *_M \mathcal{L}_{\overline{\overline{\mathcal{G}}}_{11}} = 0, \tag{3.11}$$

$$\mathcal{R}_{\widetilde{\mathcal{C}}_{jj}} *_N \widetilde{\mathcal{E}}_{jj} = 0,\ \widetilde{\mathcal{E}}_{jj} *_M \mathcal{L}_{\widetilde{\mathcal{D}}_{jj}} = 0\ (j = 1, 2), \tag{3.12}$$

$$\mathcal{R}_{\widetilde{\mathcal{H}}_{ll}} *_N \widetilde{\widetilde{\mathcal{E}}}_{ll} = 0,\ \widetilde{\widetilde{\mathcal{E}}}_{ll} *_M \mathcal{L}_{\widetilde{\mathcal{G}}_{ll}} = 0,\ \mathcal{R}_{\widetilde{\mathcal{A}}} *_N \widetilde{\mathcal{E}} *_M \mathcal{L}_{\widetilde{\mathcal{B}}},\ (l = 1, 2), \tag{3.13}$$

$$\mathcal{R}_{\widetilde{\mathcal{A}}} *_N \widetilde{\mathcal{E}} *_M \mathcal{L}_{\widetilde{\mathcal{B}}} = 0. \tag{3.14}$$

Under these conditions, the general solution to system (1.3) *can be expressed as follows:*

$$\begin{aligned}\mathcal{X}_i = {} & \mathcal{A}_i^{\dagger} *_N \grave{\mathcal{E}}_i *_M \mathcal{B}_i^{\dagger} - \mathcal{A}_i^{\dagger} *_N \mathcal{C}_i *_N \mathcal{M}_i^{\dagger} *_N \grave{\mathcal{E}}_i *_M \mathcal{B}_i^{\dagger} - \mathcal{A}_i^{\dagger} *_N \mathcal{S}_i *_N \mathcal{C}_i^{\dagger} *_N \grave{\mathcal{E}}_i \\ & *_M \mathcal{N}_i^{\dagger} *_M \mathcal{D}_i *_M \mathcal{B}_i^{\dagger} - \mathcal{A}_i^{\dagger} *_N \mathcal{S}_i *_N \mathcal{U}_{2i} *_M \mathcal{R}_{\mathcal{N}_i} *_M \mathcal{D}_i *_M \mathcal{B}_i^{\dagger} + \mathcal{L}_{\mathcal{A}_i} *_N \mathcal{U}_{4i} \\ & + \mathcal{U}_{5i} *_M \mathcal{R}_{\mathcal{B}_i},\end{aligned} \tag{3.15}$$

$$\begin{aligned}\mathcal{Y}_i = {} & \mathcal{M}_i^{\dagger} *_N \grave{\mathcal{E}}_i *_M \mathcal{D}_i^{\dagger} + \mathcal{S}_i^{\dagger} *_N \mathcal{S}_i *_N \mathcal{C}_i^{\dagger} *_N \grave{\mathcal{E}}_i *_M \mathcal{N}_i^{\dagger} + \mathcal{L}_{\mathcal{M}_i} *_N \mathcal{L}_{\mathcal{S}_i} *_N \mathcal{U}_{1i} \\ & + \mathcal{L}_{\mathcal{M}_i} *_N \mathcal{U}_{2i} *_M \mathcal{R}_{\mathcal{N}_i} + \mathcal{U}_{3i} *_M \mathcal{R}_{\mathcal{D}_1},\end{aligned} \tag{3.16}$$

$$\mathcal{Z}_1 = \mathcal{F}_4^{\dagger} *_N \mathcal{E}_4 *_M \mathcal{G}_4^{\dagger} + \mathcal{L}_{\mathcal{F}_4} *_N \mathcal{W}_1 + \mathcal{W}_2 *_M \mathcal{R}_{\mathcal{G}_4}, \tag{3.17}$$

$$\mathcal{Z}_4 = \mathcal{H}_4^{\dagger} *_N \mathcal{E}_5 *_M \mathcal{J}_4^{\dagger} + \mathcal{L}_{\mathcal{H}_4} *_N \acute{\mathcal{W}}_1 + \mathcal{W}_3 *_M \mathcal{R}_{\mathcal{J}_4}, \tag{3.18}$$

$$\begin{aligned}\mathcal{Z}_2 = {} & \widehat{\mathcal{M}}_1^{\dagger} *_N \widehat{\mathcal{E}}_1 *_M \widehat{\mathcal{D}}_1^{\dagger} + \widehat{\mathcal{S}}_1^{\dagger} *_N \widehat{\mathcal{S}}_1 *_N \widehat{\mathcal{C}}_1^{\dagger} *_N \widehat{\mathcal{E}}_1 *_M \widehat{\mathcal{N}}_1^{\dagger} + \mathcal{L}_{\widehat{\mathcal{M}}_1} *_N \mathcal{L}_{\widehat{\mathcal{S}}_1} *_N \widehat{\mathcal{U}}_1 \\ & + \mathcal{L}_{\widehat{\mathcal{M}}_1} *_N \widehat{\mathcal{U}}_2 *_M \mathcal{R}_{\widehat{\mathcal{N}}_1} + \widehat{\mathcal{U}}_3 *_M \mathcal{R}_{\widehat{\mathcal{D}}_1},\end{aligned} \tag{3.19}$$

$$\begin{aligned} \textit{or } \mathcal{Z}_2 = {} & \widehat{\mathcal{A}}_2^{\dagger} *_N \widehat{\mathcal{E}}_2 *_M \widehat{\mathcal{B}}_2^{\dagger} - \widehat{\mathcal{A}}_2^{\dagger} *_N \widehat{\mathcal{C}}_2 *_N \widehat{\mathcal{M}}_2^{\dagger} *_N \widehat{\mathcal{E}}_2 *_M \widehat{\mathcal{B}}_2^{\dagger} - \widehat{\mathcal{A}}_2^{\dagger} *_N \widehat{\mathcal{S}}_2 *_N \widehat{\mathcal{C}}_2^{\dagger} *_N \widehat{\mathcal{E}}_2 \\ & *_M \widehat{\mathcal{N}}_2^{\dagger} *_M \widehat{\mathcal{D}}_2 *_M \widehat{\mathcal{B}}_2^{\dagger} - \widehat{\mathcal{A}}_2^{\dagger} *_N \widehat{\mathcal{S}}_2 *_N \widehat{\mathcal{V}}_2 *_M \mathcal{R}_{\widehat{\mathcal{N}}_2} *_M \widehat{\mathcal{D}}_2 *_M \widehat{\mathcal{B}}_2^{\dagger} + \mathcal{L}_{\widehat{\mathcal{A}}_2} *_N \widehat{\mathcal{V}}_4 \\ & + \widehat{\mathcal{V}}_5 *_M \mathcal{R}_{\widehat{\mathcal{B}}_2},\end{aligned} \tag{3.20}$$

$$\begin{aligned}\mathcal{Z}_3 = {} & \widehat{\mathcal{M}}_2^{\dagger} *_N \widehat{\mathcal{E}}_2 *_M \widehat{\mathcal{D}}_2^{\dagger} + \widehat{\mathcal{S}}_2^{\dagger} *_N \widehat{\mathcal{S}}_2 *_N \widehat{\mathcal{C}}_2^{\dagger} *_N \widehat{\mathcal{E}}_2 *_M \widehat{\mathcal{N}}_2^{\dagger} + \mathcal{L}_{\widehat{\mathcal{M}}_2} *_N \mathcal{L}_{\widehat{\mathcal{S}}_2} *_N \widehat{\mathcal{V}}_1 \\ & + \mathcal{L}_{\widehat{\mathcal{M}}_2} *_N \widehat{\mathcal{V}}_2 *_M \mathcal{R}_{\widehat{\mathcal{N}}_2} + \widehat{\mathcal{V}}_3 *_M \mathcal{R}_{\widehat{\mathcal{D}}_2},\end{aligned} \tag{3.21}$$

$$\begin{aligned} \textit{or } \mathcal{Z}_3 = {} & \widehat{\mathcal{A}}_3^{\dagger} *_N \widehat{\mathcal{E}}_3 *_M \widehat{\mathcal{B}}_3^{\dagger} - \widehat{\mathcal{A}}_3^{\dagger} *_N \widehat{\mathcal{C}}_3 *_N \widehat{\mathcal{M}}_3^{\dagger} *_N \widehat{\mathcal{E}}_3 *_M \widehat{\mathcal{B}}_3^{\dagger} - \widehat{\mathcal{A}}_3^{\dagger} *_N \widehat{\mathcal{S}}_3 *_N \widehat{\mathcal{C}}_3^{\dagger} *_N \widehat{\mathcal{E}}_3 \\ & *_M \widehat{\mathcal{N}}_3^{\dagger} *_M \widehat{\mathcal{D}}_3 *_M \widehat{\mathcal{B}}_3^{\dagger} - \widehat{\mathcal{A}}_3^{\dagger} *_N \widehat{\mathcal{S}}_3 *_N \widehat{\mathcal{K}}_2 *_M \mathcal{R}_{\widehat{\mathcal{N}}_3} *_M \widehat{\mathcal{D}}_3 *_M \widehat{\mathcal{B}}_3^{\dagger} + \mathcal{L}_{\widehat{\mathcal{A}}_3} *_N \widehat{\mathcal{K}}_4 \\ & + \widehat{\mathcal{K}}_5 *_M \mathcal{R}_{\widehat{\mathcal{B}}_3}, \ (i = \overline{1,3}), \quad \textit{where}\end{aligned} \tag{3.22}$$

$$\mathcal{W}_1 = \begin{bmatrix} \mathcal{I} & 0 \end{bmatrix} *_N [\mathcal{A}_{11}^{\dagger} *_N (\mathcal{E}_{11} - \widehat{\mathcal{A}}_{11} *_N \widehat{\mathcal{U}}_2 *_M \widehat{\mathcal{B}}_{11}) - \mathcal{V}_{11} *_M \mathcal{D}_{11} + \mathcal{L}_{\mathcal{A}_{11}} *_N \mathcal{V}_{22}], \tag{3.23}$$

$$\widehat{\mathcal{U}}_4 = \begin{bmatrix} 0 & \mathcal{I} \end{bmatrix} *_N [\mathcal{A}_{11}^{\dagger} *_N (\mathcal{E}_{11} - \widehat{\mathcal{A}}_{11} *_N \widehat{\mathcal{U}}_2 *_M \widehat{\mathcal{B}}_{11}) - \mathcal{V}_{11} *_M \mathcal{D}_{11} + \mathcal{L}_{\mathcal{A}_{11}} *_N \mathcal{V}_{22}], \tag{3.24}$$

$$\begin{aligned}\mathcal{W}_2 = {} & [\mathcal{R}_{\mathcal{A}_{11}} *_N (\mathcal{E}_{11} - \widehat{\mathcal{A}}_{11} *_N \widehat{\mathcal{U}}_2 *_M \widehat{\mathcal{B}}_{11}) *_M \mathcal{D}_{11}^{\dagger} + \mathcal{A}_{11} *_N \mathcal{V}_{11} \\ & + \mathcal{V}_{33} *_M \mathcal{R}_{\mathcal{D}_{11}}] *_M \begin{bmatrix} \mathcal{I} \\ 0 \end{bmatrix},\end{aligned} \tag{3.25}$$

$$\widehat{\mathcal{U}}_5 = [\mathcal{R}_{\mathcal{A}_{11}} *_N (\mathcal{E}_{11} - \widehat{\mathcal{A}}_{11} *_N \widehat{\mathcal{U}}_2 *_M \widehat{\mathcal{B}}_{11}) *_M \mathcal{D}_{11}^{\dagger} + \mathcal{A}_{11} *_N \mathcal{V}_{11} \\ + \mathcal{V}_{33} *_M \mathcal{R}_{\mathcal{D}_{11}}] *_M \begin{bmatrix} 0 \\ \mathcal{I} \end{bmatrix}, \tag{3.26a}$$

$$\widehat{\mathcal{W}}_1 = \begin{bmatrix} \mathcal{I} & 0 \end{bmatrix} *_N [\mathcal{A}_{22}^{\dagger} *_N (\mathcal{E}_{22} - \widehat{\mathcal{A}}_{22} *_N \widehat{\mathcal{K}}_2 *_M \widehat{\mathcal{B}}_{22}) - \mathcal{V}_{44} *_M \mathcal{D}_{22} + \mathcal{L}_{\mathcal{A}_{22}} *_N \mathcal{V}_{55}], \tag{3.26b}$$

$$\widehat{\mathcal{K}}_1 = \begin{bmatrix} 0 & \mathcal{I} \end{bmatrix} *_N [\mathcal{A}_{22}^{\dagger} *_N (\mathcal{E}_{22} - \widehat{\mathcal{A}}_{22} *_N \widehat{\mathcal{K}}_2 *_M \widehat{\mathcal{B}}_{22}) - \mathcal{V}_{44} *_M \mathcal{D}_{22} + \mathcal{L}_{\mathcal{A}_{22}} *_N \mathcal{V}_{55}], \tag{3.26c}$$

$$\mathcal{W}_3 = [\mathcal{R}_{\mathcal{A}_{22}} *_N (\mathcal{E}_{22} - \widehat{\mathcal{A}}_{22} *_N \widehat{\mathcal{K}}_2 *_M \widehat{\mathcal{B}}_{22}) *_M \mathcal{D}_{22}^{\dagger} + \mathcal{A}_{22} *_N \mathcal{V}_{44} \\ + \mathcal{V}_{66} *_M \mathcal{R}_{\mathcal{D}_{22}}] *_M \begin{bmatrix} \mathcal{I} \\ 0 \end{bmatrix}, \tag{3.26d}$$

$$\widehat{\mathcal{K}}_3 = [\mathcal{R}_{\mathcal{A}_{22}} *_N (\mathcal{E}_{22} - \widehat{\mathcal{A}}_{22} *_N \widehat{\mathcal{K}}_2 *_M \widehat{\mathcal{B}}_{22}) *_M \mathcal{D}_{22}^{\dagger} + \mathcal{A}_{22} *_N \mathcal{V}_{44} \\ + \mathcal{V}_{66} *_M \mathcal{R}_{\mathcal{D}_{22}}] *_M \begin{bmatrix} 0 \\ \mathcal{I} \end{bmatrix}, \tag{3.26e}$$

$$\widehat{\mathcal{U}}_2 = \widehat{\widehat{\mathcal{A}}}_{11}^{\dagger} *_N \widehat{\widehat{\mathcal{E}}}_{11} *_M \widehat{\widehat{\mathcal{B}}}_{11}^{\dagger} + \mathcal{L}_{\widehat{\widehat{\mathcal{A}}}_{11}} *_N \mathcal{V}_{77} + \mathcal{V}_{88} *_M \mathcal{R}_{\widehat{\widehat{\mathcal{B}}}_{11}}, \tag{3.26f}$$

$$\widehat{\mathcal{K}}_2 = \widehat{\widehat{\mathcal{A}}}_{22}^{\dagger} *_N \widehat{\widehat{\mathcal{E}}}_{22} *_M \widehat{\widehat{\mathcal{B}}}_{22}^{\dagger} + \mathcal{L}_{\widehat{\widehat{\mathcal{A}}}_{22}} *_N \mathcal{V}_{99} + \mathcal{W}_{11} *_M \mathcal{R}_{\widehat{\widehat{\mathcal{B}}}_{22}}, \tag{3.26g}$$

$$\widehat{\mathcal{U}}_1 = \begin{bmatrix} \mathcal{I} & 0 \end{bmatrix} *_N [\overline{\mathcal{A}}_1 *_N (\overline{\mathcal{F}}_1 *_N \widehat{\mathcal{V}}_2 *_M \overline{\mathcal{G}}_1 + \overline{\mathcal{H}}_1 *_N \widehat{\mathcal{U}}_2 *_M \overline{\mathcal{J}}_1 - \overline{\mathcal{E}}_1) \\ + \mathcal{P}_{11} *_M \overline{\mathcal{B}}_1 + \mathcal{L}_{\overline{\mathcal{A}}_1} *_N \mathcal{Q}_{11}], \tag{3.26h}$$

$$\widehat{\mathcal{V}}_4 = \begin{bmatrix} 0 & \mathcal{I} \end{bmatrix} *_N [\overline{\mathcal{A}}_1 *_N (\overline{\mathcal{F}}_1 *_N \widehat{\mathcal{V}}_2 *_M \overline{\mathcal{G}}_1 + \overline{\mathcal{H}}_1 *_N \widehat{\mathcal{U}}_2 *_M \overline{\mathcal{J}}_1 - \overline{\mathcal{E}}_1) \\ + \mathcal{P}_{11} *_M \overline{\mathcal{B}}_1 + \mathcal{L}_{\overline{\mathcal{A}}_1} *_N \mathcal{Q}_{11}], \tag{3.26i}$$

$$\widehat{\mathcal{U}}_3 = [\mathcal{R}_{\overline{\mathcal{A}}_1} *_N (\overline{\mathcal{F}}_1 *_N \widehat{\mathcal{V}}_2 *_M \overline{\mathcal{G}}_1 + \overline{\mathcal{H}}_1 *_N \widehat{\mathcal{U}}_2 *_M \overline{\mathcal{J}}_1 - \overline{\mathcal{E}}_1) \\ *_M \overline{\mathcal{B}}_1^{\dagger} + \overline{\mathcal{A}}_1 *_N \mathcal{P}_{11} + \mathcal{P}_{33} *_M \mathcal{R}_{\overline{\mathcal{B}}_1}] *_M \begin{bmatrix} \mathcal{I} \\ 0 \end{bmatrix}, \tag{3.26j}$$

$$\widehat{\mathcal{V}}_5 = [\mathcal{R}_{\overline{\mathcal{A}}_1} *_N (\overline{\mathcal{F}}_1 *_N \widehat{\mathcal{V}}_2 *_M \overline{\mathcal{G}}_1 + \overline{\mathcal{H}}_1 *_N \widehat{\mathcal{U}}_2 *_M \overline{\mathcal{J}}_1 - \overline{\mathcal{E}}_1) \\ *_M \overline{\mathcal{B}}_1^{\dagger} + \overline{\mathcal{A}}_1 *_N \mathcal{P}_{11} + \mathcal{P}_{33} *_M \mathcal{R}_{\overline{\mathcal{B}}_1}] *_M \begin{bmatrix} 0 \\ \mathcal{I} \end{bmatrix}, \tag{3.26k}$$

$$\widehat{\mathcal{V}}_1 = \begin{bmatrix} \mathcal{I} & 0 \end{bmatrix} *_N [\overline{\mathcal{A}}_2 *_N (\overline{\mathcal{F}}_2 *_N \widehat{\mathcal{K}}_2 *_M \overline{\mathcal{G}}_2 + \overline{\mathcal{H}}_2 *_N \widehat{\mathcal{V}}_2 *_M \overline{\mathcal{J}}_2 - \overline{\mathcal{E}}_2) \\ + \mathcal{P}_{22} *_M \overline{\mathcal{B}}_2 + \mathcal{L}_{\overline{\mathcal{A}}_2} *_N \mathcal{Q}_{22}], \tag{3.26l}$$

$$\widehat{\mathcal{K}}_4 = \begin{bmatrix} 0 & \mathcal{I} \end{bmatrix} *_N [\overline{\mathcal{A}}_2 *_N (\overline{\mathcal{F}}_2 *_N \widehat{\mathcal{K}}_2 *_M \overline{\mathcal{G}}_2 + \overline{\mathcal{H}}_2 *_N \widehat{\mathcal{V}}_2 *_M \overline{\mathcal{J}}_2 - \overline{\mathcal{E}}_2) \\ + \mathcal{P}_{22} *_M \overline{\mathcal{B}}_2 + \mathcal{L}_{\overline{\mathcal{A}}_2} *_N \mathcal{Q}_{22}], \tag{3.26m}$$

$$\widehat{\mathcal{V}}_3 = [\mathcal{R}_{\overline{\mathcal{A}}_2} *_N (\overline{\mathcal{F}}_2 *_N \widehat{\mathcal{K}}_2 *_M \overline{\mathcal{G}}_2 + \overline{\mathcal{H}}_2 *_N \widehat{\mathcal{V}}_2 *_M \overline{\mathcal{J}}_2 - \overline{\mathcal{E}}_2) \\ *_M \overline{\mathcal{B}}_2^{\dagger} + \overline{\mathcal{A}}_2 *_N \mathcal{P}_{22} + \mathcal{Q}_{33} *_M \mathcal{R}_{\overline{\mathcal{B}}_2}] *_M \begin{bmatrix} \mathcal{I} \\ 0 \end{bmatrix}, \tag{3.26n}$$

$$\widehat{\mathcal{K}}_5 = [\mathcal{R}_{\overline{\mathcal{A}}_2} *_N (\overline{\mathcal{E}}_2 + \overline{\mathcal{F}}_2 *_N \widehat{\mathcal{K}}_2 *_M \overline{\mathcal{G}}_2 + \overline{\mathcal{H}}_2 *_N \widehat{\mathcal{V}}_2 *_M \overline{\mathcal{J}}_2) \\ *_M \overline{\mathcal{B}}_2^{\dagger} + \overline{\mathcal{A}}_2 *_N \mathcal{P}_{22} + \mathcal{Q}_{33} *_M \mathcal{R}_{\overline{\mathcal{B}}_2}] *_M \begin{bmatrix} 0 \\ \mathcal{I} \end{bmatrix}, \tag{3.26o}$$

$$\widehat{\mathcal{V}}_2 = \overline{\mathcal{F}}_{11}^{\dagger} *_N \overline{\mathcal{E}}_{11} *_M \overline{\mathcal{G}}_{11}^{\dagger} - \overline{\mathcal{F}}_{11}^{\dagger} *_N \overline{\mathcal{H}}_{11} *_N \overline{\mathcal{M}}_{11}^{\dagger} *_N \overline{\mathcal{E}}_{11} *_M \overline{\mathcal{G}}_{11}^{\dagger} - \overline{\mathcal{F}}_{11}^{\dagger} *_N \overline{\mathcal{S}}_{11} *_N \overline{\mathcal{H}}_{11}^{\dagger} \\ *_N \overline{\mathcal{E}}_{11} *_M \overline{\mathcal{N}}_{11}^{\dagger} *_M \overline{\mathcal{J}}_{11} *_M \overline{\mathcal{G}}_{11}^{\dagger} - \overline{\mathcal{F}}_{11}^{\dagger} *_N \overline{\mathcal{S}}_{11} *_N \mathcal{P}_{44} *_M \mathcal{R}_{\overline{\mathcal{N}}_{11}} *_M \overline{\mathcal{J}}_{11} *_M \overline{\mathcal{G}}_{11}^{\dagger} \\ + \mathcal{L}_{\overline{\mathcal{F}}_{11}} *_N \mathcal{P}_{55} + \mathcal{P}_{66} *_M \mathcal{R}_{\overline{\mathcal{G}}_{11}}, \tag{3.26p}$$

$$\begin{aligned} or\ \widehat{\mathcal{V}}_2 = \overline{\mathcal{M}}_{22}^{\dagger} *_N \overline{\mathcal{E}}_{22} *_M \overline{\mathcal{J}}_{22}^{\dagger} + \overline{\mathcal{S}}_{22}^{\dagger} *_N \overline{\mathcal{S}}_{22} *_N \overline{\mathcal{H}}_{22}^{\dagger} *_N \overline{\mathcal{E}}_{22} *_M \overline{\mathcal{N}}_{22}^{\dagger} + \mathcal{L}_{\overline{\mathcal{M}}_{22}} \\ *_N \mathcal{L}_{\overline{\mathcal{S}}_{22}} *_N \mathcal{Q}_{77} + \mathcal{L}_{\overline{\mathcal{M}}_{22}} *_N \mathcal{Q}_{55} *_M \mathcal{R}_{\overline{\mathcal{N}}_{22}} + \mathcal{Q}_{88} *_M \mathcal{R}_{\overline{\mathcal{J}}_{22}}, \end{aligned} \tag{3.26q}$$

$$\begin{aligned} \mathcal{P}_{55} = \begin{bmatrix} 0 & \mathcal{I} \end{bmatrix} *_N [\overline{\mathcal{A}}_1 *_N (-\overline{\overline{\overline{\mathcal{E}}}}_1 + \overline{\overline{\overline{\mathcal{F}}}}_1 *_N \mathcal{P}_{44} *_M \overline{\overline{\overline{\mathcal{G}}}}_1 + \overline{\overline{\overline{\mathcal{H}}}}_1 *_N \mathcal{Q}_{55} *_M \overline{\overline{\overline{\mathcal{J}}}}_1) \\ + \mathcal{K}_{11} *_M \overline{\overline{\overline{\mathcal{B}}}}_1 + \mathcal{L}_{\overline{\overline{\mathcal{A}}}_1} *_N \mathcal{K}_{22}], \end{aligned} \tag{3.26r}$$

$$\begin{aligned} \mathcal{Q}_{77} = \begin{bmatrix} \mathcal{I} & 0 \end{bmatrix} *_N [\overline{\mathcal{A}}_1 *_N (-\overline{\overline{\overline{\mathcal{E}}}}_1 + \overline{\overline{\overline{\mathcal{F}}}}_1 *_N \mathcal{P}_{44} *_M \overline{\overline{\overline{\mathcal{G}}}}_1 + \overline{\overline{\overline{\mathcal{H}}}}_1 *_N \mathcal{Q}_{55} *_M \overline{\overline{\overline{\mathcal{J}}}}_1) \\ + \mathcal{K}_{11} *_M \overline{\overline{\overline{\mathcal{B}}}}_1 + \mathcal{L}_{\overline{\overline{\mathcal{A}}}_1} *_N \mathcal{K}_{22}], \end{aligned} \tag{3.26s}$$

$$\begin{aligned} \mathcal{P}_{66} = [\mathcal{R}_{\overline{\overline{\mathcal{A}}}_1} *_N (-\overline{\overline{\overline{\mathcal{E}}}}_1 + \overline{\overline{\overline{\mathcal{F}}}}_1 *_N \mathcal{P}_{44} *_M \overline{\overline{\overline{\mathcal{G}}}}_1 + \overline{\overline{\overline{\mathcal{H}}}}_1 *_N \mathcal{Q}_{55} *_M \overline{\overline{\overline{\mathcal{J}}}}_1) \\ *_M \overline{\overline{\mathcal{B}}}_1^{\dagger} + \overline{\overline{\mathcal{A}}}_1 *_N \mathcal{K}_{11} + \mathcal{K}_{33} *_M \mathcal{R}_{\overline{\overline{\mathcal{B}}}_1}] *_M \begin{bmatrix} \mathcal{I} \\ 0 \end{bmatrix}, \end{aligned} \tag{3.26t}$$

$$\begin{aligned} \mathcal{Q}_{88} = [\mathcal{R}_{\overline{\overline{\mathcal{A}}}_1} *_N (-\overline{\overline{\overline{\mathcal{E}}}}_1 + \overline{\overline{\overline{\mathcal{F}}}}_1 *_N \mathcal{P}_{44} *_M \overline{\overline{\overline{\mathcal{G}}}}_1 + \overline{\overline{\overline{\mathcal{H}}}}_1 *_N \mathcal{Q}_{55} *_M \overline{\overline{\overline{\mathcal{J}}}}_1) \\ *_M \overline{\overline{\mathcal{B}}}_1^{\dagger} + \overline{\overline{\mathcal{A}}}_1 *_N \mathcal{K}_{11} + \mathcal{K}_{33} *_M \mathcal{R}_{\overline{\overline{\mathcal{B}}}_1}] *_M \begin{bmatrix} 0 \\ \mathcal{I} \end{bmatrix}, \end{aligned} \tag{3.26u}$$

$$\mathcal{Q}_{44} = \begin{bmatrix} \mathcal{I} & 0 \end{bmatrix} *_N [\widetilde{\mathcal{A}}_1 *_N (\widetilde{\mathcal{E}}_1 - \widetilde{\mathcal{C}}_1 *_N \mathcal{P}_{44} *_M \widetilde{\mathcal{D}}_1) + \mathcal{W}_{22} *_M \widetilde{\mathcal{B}}_1 + \mathcal{L}_{\widetilde{\mathcal{A}}_1} *_N \mathcal{W}_{33}], \tag{3.26v}$$

$$\mathcal{V}_{77} = \begin{bmatrix} 0 & \mathcal{I} \end{bmatrix} *_N [\widetilde{\mathcal{A}}_1 *_N (\widetilde{\mathcal{E}}_1 - \widetilde{\mathcal{C}}_1 *_N \mathcal{P}_{44} *_M \widetilde{\mathcal{D}}_1) + \mathcal{W}_{22} *_M \widetilde{\mathcal{B}}_1 + \mathcal{L}_{\widetilde{\mathcal{A}}_1} *_N \mathcal{W}_{33}], \tag{3.26w}$$

$$\mathcal{Q}_{66} = [\mathcal{R}_{\widetilde{\mathcal{A}}_1} *_N (\widetilde{\mathcal{E}}_1 - \widetilde{\mathcal{C}}_1 *_N \mathcal{P}_{44} *_M \widetilde{\mathcal{D}}_1) *_M \widetilde{\mathcal{B}}_1^{\dagger} + \widetilde{\mathcal{A}}_1 *_N \mathcal{W}_{22} + \mathcal{W}_{44} *_M \mathcal{R}_{\widetilde{\mathcal{B}}_1}] *_M \begin{bmatrix} \mathcal{I} \\ 0 \end{bmatrix}, \tag{3.26x}$$

$$\mathcal{V}_{88} = [\mathcal{R}_{\widetilde{\mathcal{A}}_1} *_N (\widetilde{\mathcal{E}}_1 - \widetilde{\mathcal{C}}_1 *_N \mathcal{P}_{44} *_M \widetilde{\mathcal{D}}_1) *_M \widetilde{\mathcal{B}}_1^{\dagger} + \widetilde{\mathcal{A}}_1 *_N \mathcal{W}_{22} + \mathcal{W}_{44} *_M \mathcal{R}_{\widetilde{\mathcal{B}}_1}] *_M \begin{bmatrix} 0 \\ \mathcal{I} \end{bmatrix}, \tag{3.26y}$$

$$\mathcal{V}_{99} = \begin{bmatrix} \mathcal{I} & 0 \end{bmatrix} *_N [\widetilde{\mathcal{A}}_2 *_N (\widetilde{\mathcal{E}}_2 - \widetilde{\mathcal{C}}_2 *_N \mathcal{P}_{44} *_M \widetilde{\mathcal{D}}_2) + \mathcal{W}_{55} *_M \widetilde{\mathcal{B}}_2 + \mathcal{L}_{\widetilde{\mathcal{A}}_2} *_N \mathcal{W}_{66}], \tag{3.26z}$$

$$\mathcal{P}_{77} = \begin{bmatrix} 0 & \mathcal{I} \end{bmatrix} *_N [\widetilde{\mathcal{A}}_2 *_N (\widetilde{\mathcal{E}}_2 - \widetilde{\mathcal{C}}_2 *_N \mathcal{P}_{44} *_M \widetilde{\mathcal{D}}_2) + \mathcal{W}_{55} *_M \widetilde{\mathcal{B}}_2 + \mathcal{L}_{\widetilde{\mathcal{A}}_2} *_N \mathcal{W}_{66}], \tag{3.27a}$$

$$\mathcal{W}_{11} = [\mathcal{R}_{\widetilde{\mathcal{A}}_2} *_N (\widetilde{\mathcal{E}}_2 - \widetilde{\mathcal{C}}_2 *_N \mathcal{P}_{44} *_M \widetilde{\mathcal{D}}_2) *_M \widetilde{\mathcal{B}}_2^{\dagger} + \widetilde{\mathcal{A}}_2 *_N \mathcal{W}_{55} + \mathcal{W}_{77} *_M \mathcal{R}_{\widetilde{\mathcal{B}}_2}] *_M \begin{bmatrix} \mathcal{I} \\ 0 \end{bmatrix}, \tag{3.27b}$$

$$\mathcal{P}_{88} = [\mathcal{R}_{\widetilde{\mathcal{A}}_2} *_N (\widetilde{\mathcal{E}}_2 - \widetilde{\mathcal{C}}_2 *_N \mathcal{P}_{44} *_M \widetilde{\mathcal{D}}_2) *_M \widetilde{\mathcal{B}}_2^{\dagger} + \widetilde{\mathcal{A}}_2 *_N \mathcal{W}_{55} + \mathcal{W}_{77} *_M \mathcal{R}_{\widetilde{\mathcal{B}}_2}] *_M \begin{bmatrix} 0 \\ \mathcal{I} \end{bmatrix}, \tag{3.27c}$$

$$\mathcal{P}_{44} = \widetilde{\mathcal{C}}_{11}^{\dagger} *_N \widetilde{\mathcal{E}}_{11} *_M \widetilde{\mathcal{D}}_{11}^{\dagger} + \mathcal{L}_{\widetilde{\mathcal{C}}_{11}} *_N \mathcal{W}_{88} + \mathcal{W}_{99} *_M \mathcal{R}_{\widetilde{\mathcal{D}}_{11}}, \tag{3.27d}$$

$$\mathcal{Q}_{55} = \widetilde{\mathcal{C}}_{22}^{\dagger} *_N \widetilde{\mathcal{E}}_{22} *_M \widetilde{\mathcal{D}}_{22}^{\dagger} + \mathcal{L}_{\widetilde{\mathcal{C}}_{22}} *_N \mathcal{T}_{11} + \mathcal{T}_{22} *_M \mathcal{R}_{\widetilde{\mathcal{D}}_{22}}, \tag{3.27e}$$

$$\mathcal{W}_{88} = \begin{bmatrix} \mathcal{I} & 0 \end{bmatrix} *_N [\widetilde{\mathcal{F}}_1 *_N (\widetilde{\widetilde{\mathcal{E}}}_1 - \widetilde{\mathcal{H}}_1 *_N \mathcal{K}_{44} *_M \widetilde{\mathcal{J}}_1) + \mathcal{T}_{33} *_M \widetilde{\mathcal{G}}_1 + \mathcal{L}_{\widetilde{\mathcal{F}}_1} *_N \mathcal{T}_{44}], \tag{3.27f}$$

$$\mathcal{K}_{55} = \begin{bmatrix} 0 & \mathcal{I} \end{bmatrix} *_N [\widetilde{\mathcal{F}}_1 *_N (\widetilde{\widetilde{\mathcal{E}}}_1 - \widetilde{\mathcal{H}}_1 *_N \mathcal{K}_{44} *_M \widetilde{\mathcal{J}}_1) + \mathcal{T}_{33} *_M \widetilde{\mathcal{G}}_1 + \mathcal{L}_{\widetilde{\mathcal{F}}_1} *_N \mathcal{T}_{44}], \tag{3.27g}$$

$$\mathcal{W}_{99} = [\mathcal{R}_{\widetilde{\mathcal{F}}_1} *_N (\widetilde{\widetilde{\mathcal{E}}}_1 - \widetilde{\mathcal{H}}_1 *_N \mathcal{K}_{44} *_M \widetilde{\mathcal{J}}_1) *_M \widetilde{\mathcal{G}}_1^{\dagger} + \widetilde{\mathcal{F}}_1 *_N \mathcal{T}_{33} + \mathcal{T}_{55} *_M \mathcal{R}_{\widetilde{\mathcal{G}}_1}] *_M \begin{bmatrix} \mathcal{I} \\ 0 \end{bmatrix}, \tag{3.27h}$$

$$\mathcal{K}_{66} = [\mathcal{R}_{\widetilde{\mathcal{F}}_1} *_N (\widetilde{\widetilde{\mathcal{E}}}_1 - \widetilde{\mathcal{H}}_1 *_N \mathcal{K}_{44} *_M \widetilde{\mathcal{J}}_1) *_M \widetilde{\mathcal{G}}_1^{\dagger} + \widetilde{\mathcal{F}}_1 *_N \mathcal{T}_{33} + \mathcal{T}_{55} *_M \mathcal{R}_{\widetilde{\mathcal{G}}_1}] *_M \begin{bmatrix} 0 \\ \mathcal{I} \end{bmatrix}, \tag{3.27i}$$

$$\mathcal{K}_{77} = \begin{bmatrix} \mathcal{I} & 0 \end{bmatrix} *_N [\widetilde{\mathcal{F}}_2 *_N (\widetilde{\widetilde{\mathcal{E}}}_2 - \widetilde{\mathcal{H}}_2 *_N \mathcal{K}_{44} *_M \widetilde{\mathcal{J}}_2) + \mathcal{T}_{66} *_M \widetilde{\mathcal{B}}_2 + \mathcal{L}_{\widetilde{\mathcal{F}}_2} *_N \mathcal{T}_{77}], \tag{3.27j}$$

$$\mathcal{T}_{11} = \begin{bmatrix} 0 & \mathcal{I} \end{bmatrix} *_N [\widetilde{\mathcal{F}}_2 *_N (\widetilde{\widetilde{\mathcal{E}}}_2 - \widetilde{\mathcal{H}}_2 *_N \mathcal{K}_{44} *_M \widetilde{\mathcal{J}}_2) + \mathcal{T}_{66} *_M \widetilde{\mathcal{B}}_2 + \mathcal{L}_{\widetilde{\mathcal{F}}_2} *_N \mathcal{T}_{77}], \tag{3.27k}$$

$$\mathcal{K}_{88} = [\mathcal{R}_{\widetilde{\mathcal{F}}_2} *_N (\widetilde{\widetilde{\mathcal{E}}}_2 - \widetilde{\mathcal{H}}_2 *_N \mathcal{K}_{44} *_M \widetilde{\mathcal{J}}_2) *_M \widetilde{\mathcal{G}}_2^{\dagger} + \widetilde{\mathcal{F}}_2 *_N \mathcal{T}_{66} + \mathcal{T}_{88} *_M \mathcal{R}_{\widetilde{\mathcal{G}}_2}] *_M \begin{bmatrix} \mathcal{I} \\ 0 \end{bmatrix}, \tag{3.27l}$$

$$\mathcal{T}_{22} = [\mathcal{R}_{\widetilde{\mathcal{F}}_2} *_N (\widetilde{\widetilde{\mathcal{E}}}_2 - \widetilde{\mathcal{H}}_2 *_N \mathcal{K}_{44} *_M \widetilde{\mathcal{J}}_2) *_M \widetilde{\mathcal{G}}_2^{\dagger} + \widetilde{\mathcal{F}}_2 *_N \mathcal{T}_{66} + \mathcal{T}_{88} *_M \mathcal{R}_{\widetilde{\mathcal{G}}_2}] *_M \begin{bmatrix} 0 \\ \mathcal{I} \end{bmatrix}, \tag{3.27m}$$

$$\mathcal{K}_{44} = \widetilde{\mathcal{H}}_{11}^{\dagger} *_N \widetilde{\widetilde{\mathcal{E}}}_{11} *_M \widetilde{\mathcal{J}}_{11}^{\dagger} + \mathcal{L}_{\widetilde{\mathcal{H}}_{11}} *_N \acute{\mathcal{W}}_2 + \acute{\mathcal{W}}_3 *_M \mathcal{R}_{\widetilde{\mathcal{J}}_{11}}, \tag{3.27n}$$

$$\textit{or } \mathcal{K}_{44} = \widetilde{\mathcal{H}}_{22}^{\dagger} *_N \widetilde{\widetilde{\mathcal{E}}}_{22} *_M \widetilde{\mathcal{J}}_{22}^{\dagger} + \mathcal{L}_{\widetilde{\mathcal{H}}_{22}} *_N \acute{\mathcal{W}}_4 + \acute{\mathcal{W}}_5 *_M \mathcal{R}_{\widetilde{\mathcal{J}}_{22}}, \tag{3.27o}$$

$$\mathcal{T}_{33} = \begin{bmatrix} \mathcal{I} & 0 \end{bmatrix} *_N [\widetilde{\mathcal{A}} *_N \widetilde{\mathcal{E}} + \acute{\mathcal{W}}_6 *_M \widetilde{\mathcal{B}} + \mathcal{L}_{\widetilde{\mathcal{A}}} *_N \acute{\mathcal{W}}_7], \tag{3.27p}$$

$$\mathcal{T}_{55} = \begin{bmatrix} 0 & \mathcal{I} \end{bmatrix} *_N [\widetilde{\mathcal{A}} *_N \widetilde{\mathcal{E}} + \acute{\mathcal{W}}_6 *_M \widetilde{\mathcal{B}} + \mathcal{L}_{\widetilde{\mathcal{A}}} *_N \acute{\mathcal{W}}_7], \tag{3.27q}$$

$$\mathcal{T}_{44} = [\mathcal{R}_{\widetilde{\mathcal{A}}} *_N \widetilde{\mathcal{E}} *_M \widetilde{\mathcal{B}}^{\dagger} + \widetilde{\mathcal{A}} *_N \acute{\mathcal{W}}_6 + \acute{\mathcal{W}}_8 *_M \mathcal{R}_{\widetilde{\mathcal{B}}}] *_M \begin{bmatrix} \mathcal{I} \\ 0 \end{bmatrix}, \tag{3.27r}$$

$$\mathcal{T}_{66} = [\mathcal{R}_{\widetilde{\mathcal{A}}} *_N \widetilde{\mathcal{E}} *_M \widetilde{\mathcal{B}}^{\dagger} + \widetilde{\mathcal{A}} *_N \acute{\mathcal{W}}_6 + \acute{\mathcal{W}}_8 *_M \mathcal{R}_{\widetilde{\mathcal{B}}}] *_M \begin{bmatrix} 0 \\ \mathcal{I} \end{bmatrix}. \tag{3.27s}$$

Where $\mathcal{U}_{ji}$, $\mathcal{W}_{kk}$, $\mathcal{V}_{kk}$, $\acute{\mathcal{W}}_k$, $\mathcal{P}_{ii}$, $\mathcal{Q}_{ii}$, $\mathcal{K}_{ii}$ *and* $\mathcal{T}_{ss}$, $(i = \overline{1,3},\ j = \overline{1,5},\ k = \overline{1,6},\ s = \overline{3,8})$ *are arbitrary quaternion tensors with appropriate sizes.*

Proof. The system (1.3) can divide to the following two-sided Sylvester-like tensor equations:

$$\mathcal{A}_1 *_N \mathcal{X}_1 *_M \mathcal{B}_1 + \mathcal{C}_1 *_N \mathcal{Y}_1 *_M \mathcal{D}_1 + \mathcal{C}_1 *_N (\mathcal{F}_1 *_N \mathcal{Z}_1 *_M \mathcal{G}_1 + \mathcal{H}_1 *_N \mathcal{Z}_2 *_M \mathcal{J}_1) *_M \mathcal{B}_1 = \mathcal{E}_1, \tag{3.28}$$

$$\mathcal{A}_2 *_N \mathcal{X}_2 *_M \mathcal{B}_2 + \mathcal{C}_2 *_N \mathcal{Y}_2 *_M \mathcal{D}_2 + \mathcal{C}_2 *_N (\mathcal{F}_2 *_N \mathcal{Z}_2 *_M \mathcal{G}_2 + \mathcal{H}_2 *_N \mathcal{Z}_3 *_M \mathcal{J}_2) *_M \mathcal{B}_2 = \mathcal{E}_2, \tag{3.29}$$

$$\mathcal{A}_3 *_N \mathcal{X}_3 *_M \mathcal{B}_3 + \mathcal{C}_3 *_N \mathcal{Y}_3 *_M \mathcal{D}_3 + \mathcal{C}_3 *_N (\mathcal{F}_3 *_N \mathcal{Z}_3 *_M \mathcal{G}_3 + \mathcal{H}_3 *_N \mathcal{Z}_4 *_M \mathcal{J}_3) *_M \mathcal{B}_3 = \mathcal{E}_3, \tag{3.30}$$

$$\mathcal{F}_4 *_N \mathcal{Z}_1 *_M \mathcal{G}_4 = \mathcal{E}_4, \tag{3.31}$$

$$\mathcal{H}_4 *_N \mathcal{Z}_4 *_M \mathcal{J}_4 = \mathcal{E}_5. \tag{3.32}$$

The main idea is to implement the conditions of consistency to enable this group to have a solution, and hence, we establish an expression of this solution. Applying *Lemma* 2.4, we have that the Sylvester-like tensor equation (3.28) is consistent if and only if

$$\begin{aligned} \mathcal{R}_{\mathcal{M}_1} *_N \mathcal{R}_{\mathcal{A}_1} *_N \mathcal{E}_1 = 0,\ \mathcal{E}_1 *_M \mathcal{L}_{\mathcal{B}_1} *_M \mathcal{L}_{\mathcal{N}_1} = 0,\ \mathcal{R}_{\mathcal{C}_1} *_N \mathcal{E}_1 *_M \mathcal{L}_{\mathcal{B}_1} = 0 \\ \mathcal{R}_{\widehat{\mathcal{M}}_1} *_N \mathcal{R}_{\widehat{\mathcal{A}}_1} *_N \widehat{\mathcal{E}}_1 = 0,\ \widehat{\mathcal{E}}_1 *_M \mathcal{L}_{\widehat{\mathcal{B}}_1} *_M \mathcal{L}_{\widehat{\mathcal{N}}_1} = 0, \\ \mathcal{R}_{\widehat{\mathcal{A}}_1} *_N \widehat{\mathcal{E}}_1 *_M \mathcal{L}_{\widehat{\mathcal{D}}_1} = 0,\ \mathcal{R}_{\widehat{\mathcal{C}}_1} *_N \widehat{\mathcal{E}}_1 *_M \mathcal{L}_{\widehat{\mathcal{B}}_1} = 0. \end{aligned} \tag{3.33}$$

In that case, the general solution can be expressed as

$$\begin{aligned}\mathcal{X}_1 = {} & \mathcal{A}_1^{\dagger} *_N \grave{\mathcal{E}}_1 *_M \mathcal{B}_1^{\dagger} - \mathcal{A}_1^{\dagger} *_N \mathcal{C}_1 *_N \mathcal{M}_1^{\dagger} *_N \grave{\mathcal{E}}_1 *_M \mathcal{B}_1^{\dagger} - \mathcal{A}_1^{\dagger} *_N \mathcal{S}_1 *_N \mathcal{C}_1^{\dagger} *_N \grave{\mathcal{E}}_1 \\ & *_M \mathcal{N}_1^{\dagger} *_M \mathcal{D}_1 *_M \mathcal{B}_1^{\dagger} - \mathcal{A}_1^{\dagger} *_N \mathcal{S}_1 *_N \mathcal{U}_{21} *_M \mathcal{R}_{\mathcal{N}_1} *_M \mathcal{D}_1 *_M \mathcal{B}_1^{\dagger} + \mathcal{L}_{\mathcal{A}_1} *_N \mathcal{U}_{41} \\ & + \mathcal{U}_{51} *_M \mathcal{R}_{\mathcal{B}_1},\end{aligned} \tag{3.34a}$$

$$\begin{aligned}\mathcal{Y}_1 = {} & \mathcal{M}_1^{\dagger} *_N \grave{\mathcal{E}}_1 *_M \mathcal{D}_1^{\dagger} + \mathcal{S}_1^{\dagger} *_N \mathcal{S}_1 *_N \mathcal{C}_1^{\dagger} *_N \grave{\mathcal{E}}_1 *_M \mathcal{N}_1^{\dagger} + \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{L}_{\mathcal{S}_1} *_N \mathcal{U}_{11} \\ & + \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{U}_{21} *_M \mathcal{R}_{\mathcal{N}_1} + \mathcal{U}_{31} *_M \mathcal{R}_{\mathcal{D}_1}.\end{aligned} \tag{3.34b}$$

$$\begin{aligned}\mathcal{Z}_1 = {} & \widehat{\mathcal{A}}_1^{\dagger} *_N \widehat{\mathcal{E}}_1 *_M \widehat{\mathcal{B}}_1^{\dagger} - \widehat{\mathcal{A}}_1^{\dagger} *_N \widehat{\mathcal{C}}_1 *_N \widehat{\mathcal{M}}_1^{\dagger} *_N \widehat{\mathcal{E}}_1 *_M \widehat{\mathcal{B}}_1^{\dagger} - \widehat{\mathcal{A}}_1^{\dagger} *_N \widehat{\mathcal{S}}_1 *_N \widehat{\mathcal{C}}_1^{\dagger} *_N \widehat{\mathcal{E}}_1 \\ & *_M \widehat{\mathcal{N}}_1^{\dagger} *_M \widehat{\mathcal{D}}_1 *_M \widehat{\mathcal{B}}_1^{\dagger} - \widehat{\mathcal{A}}_1^{\dagger} *_N \widehat{\mathcal{S}}_1 *_N \widehat{\mathcal{U}}_2 *_M \mathcal{R}_{\widehat{\mathcal{N}}_1} *_M \widehat{\mathcal{D}}_1 *_M \widehat{\mathcal{B}}_1^{\dagger} + \mathcal{L}_{\widehat{\mathcal{A}}_1} *_N \widehat{\mathcal{U}}_4 \\ & + \widehat{\mathcal{U}}_5 *_M \mathcal{R}_{\widehat{\mathcal{B}}_1},\end{aligned} \tag{3.34c}$$

$$\begin{aligned}\mathcal{Z}_2 = {} & \widehat{\mathcal{M}}_1^{\dagger} *_N \widehat{\mathcal{E}}_1 *_M \widehat{\mathcal{D}}_1^{\dagger} + \widehat{\mathcal{S}}_1^{\dagger} *_N \widehat{\mathcal{S}}_1 *_N \widehat{\mathcal{C}}_1^{\dagger} *_N \widehat{\mathcal{E}} *_M \widehat{\mathcal{N}}_1^{\dagger} + \mathcal{L}_{\widehat{\mathcal{M}}_1} *_N \mathcal{L}_{\widehat{\mathcal{S}}_1} *_N \widehat{\mathcal{U}}_1 \\ & + \mathcal{L}_{\widehat{\mathcal{M}}_1} *_N \widehat{\mathcal{U}}_2 *_M \mathcal{R}_{\widehat{\mathcal{N}}_1} + \widehat{\mathcal{U}}_3 *_M \mathcal{R}_{\widehat{\mathcal{D}}_1},\end{aligned} \tag{3.34d}$$

where $\widehat{\mathcal{A}}_1$, $\widehat{\mathcal{B}}_1$, $\widehat{\mathcal{C}}_1$, $\widehat{\mathcal{D}}_1$, $\widehat{\mathcal{E}}_1$, $\widehat{\mathcal{M}}_1$, $\widehat{\mathcal{N}}_1$ and $\widehat{\mathcal{S}}_1$ given by (3.1b)-(3.1d) whenever $i = 1$. It can follow the same technique to determine the consistency conditions and the general solution to the Sylvester-like quaternion tensor equation (3.29). So, we have that Eq.(3.29) is solvable if and only if the conditions (3.3)-(3.5) satisfy whenever $i = 2$. In this case, the quaternion tensors $\mathcal{X}_2$ and $\mathcal{Y}_2$ can be given by (3.15)-(3.16) whenever $i = 2$ and

$$\begin{aligned}\mathcal{Z}_2 = {} & \widehat{\mathcal{A}}_2^{\dagger} *_N \widehat{\mathcal{E}}_2 *_M \widehat{\mathcal{B}}_2^{\dagger} - \widehat{\mathcal{A}}_2^{\dagger} *_N \widehat{\mathcal{C}}_2 *_N \widehat{\mathcal{M}}_2^{\dagger} *_N \widehat{\mathcal{E}}_2 *_M \widehat{\mathcal{B}}_2^{\dagger} - \widehat{\mathcal{A}}_2^{\dagger} *_N \widehat{\mathcal{S}}_2 *_N \widehat{\mathcal{C}}_2^{\dagger} *_N \widehat{\mathcal{E}}_2 \\ & *_M \widehat{\mathcal{N}}_2^{\dagger} *_M \widehat{\mathcal{D}}_2 *_M \widehat{\mathcal{B}}_2^{\dagger} - \widehat{\mathcal{A}}_2^{\dagger} *_N \widehat{\mathcal{S}}_2 *_N \widehat{\mathcal{V}}_2 *_M \mathcal{R}_{\widehat{\mathcal{N}}_2} *_M \widehat{\mathcal{D}}_2 *_M \widehat{\mathcal{B}}_2^{\dagger} + \mathcal{L}_{\widehat{\mathcal{A}}_2} *_N \widehat{\mathcal{V}}_4 \\ & + \widehat{\mathcal{V}}_5 *_M \mathcal{R}_{\widehat{\mathcal{B}}_2},\end{aligned} \tag{3.35a}$$

$$\begin{aligned}\mathcal{Z}_3 = {} & \widehat{\mathcal{M}}_2^{\dagger} *_N \widehat{\mathcal{E}}_2 *_M \widehat{\mathcal{D}}_2^{\dagger} + \widehat{\mathcal{S}}_2^{\dagger} *_N \widehat{\mathcal{S}}_2 *_N \widehat{\mathcal{C}}_2^{\dagger} *_N \widehat{\mathcal{E}}_2 *_M \widehat{\mathcal{N}}_2^{\dagger} + \mathcal{L}_{\widehat{\mathcal{M}}_2} *_N \mathcal{L}_{\widehat{\mathcal{S}}_2} *_N \widehat{\mathcal{V}}_1 \\ & + \mathcal{L}_{\widehat{\mathcal{M}}_2} *_N \widehat{\mathcal{V}}_2 *_M \mathcal{R}_{\widehat{\mathcal{N}}_2} + \widehat{\mathcal{V}}_3 *_M \mathcal{R}_{\widehat{\mathcal{D}}_2},\end{aligned} \tag{3.35b}$$

where $\widehat{\mathcal{A}}_2$, $\widehat{\mathcal{B}}_2$, $\widehat{\mathcal{C}}_2$, $\widehat{\mathcal{D}}_2$, $\widehat{\mathcal{E}}_2$, $\widehat{\mathcal{M}}_2$, $\widehat{\mathcal{N}}_2$ and $\widehat{\mathcal{S}}_2$ given by (3.1b)-(3.1d) whenever $i = 2$.

Similarly, we can provide that E.q. (3.30) is solvable if and only if the conditions (3.3)-(3.5) are satisfying whenever $i = 3$. In this case, the quaternion tensors $\mathcal{X}_3$ and $\mathcal{Y}_3$ can be given by (3.15)-(3.16), whenever $i = 3$ and

$$\begin{aligned}\mathcal{Z}_3 = {} & \widehat{\mathcal{A}}_3^{\dagger} *_N \widehat{\mathcal{E}}_3 *_M \widehat{\mathcal{B}}_3^{\dagger} - \widehat{\mathcal{A}}_3^{\dagger} *_N \widehat{\mathcal{C}}_3 *_N \widehat{\mathcal{M}}_3^{\dagger} *_N \widehat{\mathcal{E}}_3 *_M \widehat{\mathcal{B}}_3^{\dagger} - \widehat{\mathcal{A}}_3^{\dagger} *_N \widehat{\mathcal{S}}_3 *_N \widehat{\mathcal{C}}_3^{\dagger} *_N \widehat{\mathcal{E}}_3 \\ & *_M \widehat{\mathcal{N}}_3^{\dagger} *_M \widehat{\mathcal{D}}_3 *_M \widehat{\mathcal{B}}_3^{\dagger} - \widehat{\mathcal{A}}_3^{\dagger} *_N \widehat{\mathcal{S}}_3 *_N \widehat{\mathcal{K}}_2 *_M \mathcal{R}_{\widehat{\mathcal{N}}_3} *_M \widehat{\mathcal{D}}_3 *_M \widehat{\mathcal{B}}_3^{\dagger} + \mathcal{L}_{\widehat{\mathcal{A}}_3} *_N \widehat{\mathcal{K}}_4 \\ & + \widehat{\mathcal{K}}_5 *_M \mathcal{R}_{\widehat{\mathcal{B}}_3},\end{aligned} \tag{3.36a}$$

$$\begin{aligned}\mathcal{Z}_4 = {} & \widehat{\mathcal{M}}_3^{\dagger} *_N \widehat{\mathcal{E}}_3 *_M \widehat{\mathcal{D}}_3^{\dagger} + \widehat{\mathcal{S}}_3^{\dagger} *_N \widehat{\mathcal{S}}_3 *_N \widehat{\mathcal{C}}_3^{\dagger} *_N \widehat{\mathcal{E}}_3 *_M \widehat{\mathcal{N}}_3^{\dagger} + \mathcal{L}_{\widehat{\mathcal{M}}_3} *_N \mathcal{L}_{\widehat{\mathcal{S}}_3} *_N \widehat{\mathcal{K}}_1 \\ & + \mathcal{L}_{\widehat{\mathcal{M}}_3} *_N \widehat{\mathcal{K}}_2 *_M \mathcal{R}_{\widehat{\mathcal{N}}_3} + \widehat{\mathcal{K}}_3 *_M \mathcal{R}_{\widehat{\mathcal{D}}_3},\end{aligned} \tag{3.36b}$$

where $\widehat{\mathcal{A}}_3$, $\widehat{\mathcal{B}}_3$, $\widehat{\mathcal{C}}_3$, $\widehat{\mathcal{D}}_3$, $\widehat{\mathcal{E}}_3$, $\widehat{\mathcal{M}}_3$, $\widehat{\mathcal{N}}_3$ and $\widehat{\mathcal{S}}_3$ given by (3.1b)-(3.1d), whenever $i = 3$.

It follows from *Lemma* 2.4 that the necessary and sufficient conditions for the Sylvester-like quaternion tensor equation (3.31) and (3.32) to be consistent are given by (3.6), respectively. Consequently, the solutions to these two equations are expressed as

$$\mathcal{Z}_1 = \mathcal{F}_4^{\dagger} *_N \mathcal{E}_4 *_M \mathcal{G}_4^{\dagger} + \mathcal{L}_{\mathcal{F}_4} *_N \mathcal{W}_1 + \mathcal{W}_2 *_M \mathcal{R}_{\mathcal{G}_4}, \tag{3.37a}$$

$$\mathcal{Z}_4 = \mathcal{H}_4^{\dagger} *_N \mathcal{E}_5 *_M \mathcal{J}_4^{\dagger} + \mathcal{L}_{\mathcal{H}_4} *_N \acute{\mathcal{W}}_1 + \acute{\mathcal{W}}_2 *_M \mathcal{R}_{\mathcal{J}_4}, \tag{3.37b}$$

Let Z_1 in (3.34c) be equal to Z_1 in (3.37a), and Z_4 in (3.36b) be equal to Z_4 in (3.37b). Then we have the following equations:

$$\mathcal{A}_{11} *_N \begin{bmatrix} \mathcal{W}_1 \\ \widehat{\mathcal{U}}_4 \end{bmatrix} + \begin{bmatrix} \mathcal{W}_2 & \widehat{\mathcal{U}}_5 \end{bmatrix} *_M \mathcal{D}_{11} = \mathcal{E}_{11} - \widehat{\mathcal{A}}_{11} *_N \widehat{\mathcal{U}}_2 *_M \widehat{\mathcal{B}}_{11}, \tag{3.38}$$

$$\mathcal{A}_{22} *_N \begin{bmatrix} \grave{\mathcal{W}}_1 \\ \widehat{\mathcal{K}}_1 \end{bmatrix} + \begin{bmatrix} \mathcal{W}_3 & \widehat{\mathcal{K}}_3 \end{bmatrix} *_M \mathcal{D}_{22} = \mathcal{E}_{22} - \widehat{\mathcal{A}}_{22} *_N \widehat{\mathcal{K}}_2 *_M \widehat{\mathcal{B}}_{22}, \tag{3.39}$$

Apply *Lemma* 2.4, to Eq.(3.38), we have that it is solvable if and only if there exists a quaternion tensor $\widehat{\mathcal{U}}_2$ satisfies

$$\widehat{\widehat{\mathcal{A}}}_{11} *_N \widehat{\mathcal{U}}_2 *_M \widehat{\widehat{\mathcal{B}}}_{11} = \widehat{\widehat{\mathcal{E}}}_{11}. \tag{3.40}$$

In that case, the general solution can be express as

$$\begin{bmatrix} \mathcal{W}_1 \\ \widehat{\mathcal{U}}_4 \end{bmatrix} = \mathcal{A}_{11}^{\dagger} *_N (\mathcal{E}_{11} - \widehat{\mathcal{A}}_{11} *_N \widehat{\mathcal{U}}_2 *_M \widehat{\mathcal{B}}_{11}) - \mathcal{V}_{11} *_M \mathcal{D}_{11} + \mathcal{L}_{\mathcal{A}_{11}} *_N \mathcal{V}_{22}, \tag{3.41}$$

$$\begin{bmatrix} \mathcal{W}_2 & \widehat{\mathcal{U}}_5 \end{bmatrix} = \mathcal{R}_{\mathcal{A}_{11}} *_N (\mathcal{E}_{11} - \widehat{\mathcal{A}}_{11} *_N \widehat{\mathcal{U}}_2 *_M \widehat{\mathcal{B}}_{11}) *_M \mathcal{D}_{11}^{\dagger} + \mathcal{A}_{11} *_N \mathcal{V}_{11} + \mathcal{V}_{33} *_M \mathcal{R}_{\mathcal{D}_{11}}. \tag{3.42}$$

By applying *Proposition* 2.2 to equations (3.41)-(3.42), we can find expressions for quaternion tensors $\mathcal{W}_1, \widehat{\mathcal{U}}_4, \mathcal{W}_2$, and $\widehat{\mathcal{U}}_5$ in (3.23)-(3.26a). In the same way, we have that Eq.(3.39) is solvable if and only if there exists a quaternion tensor $\widehat{\mathcal{K}}_2$ satisfies

$$\widehat{\widehat{\mathcal{A}}}_{22} *_N \widehat{\mathcal{K}}_2 *_M \widehat{\widehat{\mathcal{B}}}_{22} = \widehat{\widehat{\mathcal{E}}}_{22}. \tag{3.43}$$

In that case, the general solution can be express as

$$\begin{bmatrix} \grave{\mathcal{W}}_1 \\ \widehat{\mathcal{K}}_1 \end{bmatrix} = \mathcal{A}_{22}^{\dagger} *_N (\mathcal{E}_{22} - \widehat{\mathcal{A}}_{22} *_N \widehat{\mathcal{K}}_2 *_M \widehat{\mathcal{B}}_{22}) - \mathcal{V}_{44} *_M \mathcal{D}_{22} + \mathcal{L}_{\mathcal{A}_{22}} *_N \mathcal{V}_{55}, \tag{3.44}$$

$$\begin{bmatrix} \mathcal{W}_3 & \widehat{\mathcal{K}}_3 \end{bmatrix} = \mathcal{R}_{\mathcal{A}_{22}} *_N (\mathcal{E}_{22} - \widehat{\mathcal{A}}_{22} *_N \widehat{\mathcal{K}}_2 *_M \widehat{\mathcal{B}}_{22}) *_M \mathcal{D}_{22}^{\dagger} + \mathcal{A}_{22} *_N \mathcal{V}_{44} + \mathcal{V}_{66} *_M \mathcal{R}_{\mathcal{D}_{22}}. \tag{3.45}$$

It can be utilized *Proposition* 2.2 to equations (3.44)-(3.45), we can get quaternion tensors $\acute{\mathcal{W}}_1$, $\widehat{\mathcal{K}}_1$, $\acute{\mathcal{W}}_2$ and $\widehat{\mathcal{K}}_3$ in (3.26b)-(3.26e). Meanwhile, the quaternion tensor equations (3.40) and (3.43) are solvable if and only if the conditions (3.7) are satisfying, respectively, for $k = 1, 2$. In that case, the general solution can be written as

$$\widehat{\mathcal{U}}_2 = \widehat{\widehat{\mathcal{A}}}_{11}^{\dagger} *_N \widehat{\widehat{\mathcal{E}}}_{11} *_M \widehat{\widehat{\mathcal{B}}}_{11}^{\dagger} + \mathcal{L}_{\widehat{\widehat{\mathcal{A}}}_{11}} *_N \mathcal{V}_{77} + \mathcal{V}_{88} *_M \mathcal{R}_{\widehat{\widehat{\mathcal{B}}}_{11}}, \tag{3.46a}$$

$$\widehat{\mathcal{K}}_2 = \widehat{\widehat{\mathcal{A}}}_{22}^{\dagger} *_N \widehat{\widehat{\mathcal{E}}}_{22} *_M \widehat{\widehat{\mathcal{B}}}_{22}^{\dagger} + \mathcal{L}_{\widehat{\widehat{\mathcal{A}}}_{22}} *_N \mathcal{V}_{99} + \mathcal{W}_{11} *_M \mathcal{R}_{\widehat{\widehat{\mathcal{B}}}_{22}}. \tag{3.46b}$$

Now, Z_2 in (3.34d) should be equal to Z_2 in (3.35a) and Z_3 in (3.35b) should be equal to Z_3 in (3.36a), yields:

$$\overline{\mathcal{A}}_1 *_N \begin{bmatrix} \widehat{\mathcal{U}}_1 \\ \widehat{\mathcal{V}}_4 \end{bmatrix} + \begin{bmatrix} \widehat{\mathcal{U}}_3 & \widehat{\mathcal{V}}_5 \end{bmatrix} *_M \overline{\mathcal{B}}_1 = -\overline{\mathcal{E}}_1 + \overline{\mathcal{F}}_1 *_N \widehat{\mathcal{V}}_2 *_M \overline{\mathcal{G}}_1 + \overline{\mathcal{H}}_1 *_N \widehat{\mathcal{U}}_2 *_M \overline{\mathcal{J}}_1, \tag{3.47}$$

$$\overline{\mathcal{A}}_2 *_N \begin{bmatrix} \widehat{\mathcal{V}}_1 \\ \widehat{\mathcal{K}}_4 \end{bmatrix} + \begin{bmatrix} \widehat{\mathcal{V}}_3 & \widehat{\mathcal{K}}_5 \end{bmatrix} *_M \overline{\mathcal{B}}_2 = -\overline{\mathcal{E}}_2 + \overline{\mathcal{F}}_2 *_N \widehat{\mathcal{K}}_2 *_M \overline{\mathcal{G}}_2 + \overline{\mathcal{H}}_2 *_N \widehat{\mathcal{V}}_2 *_M \overline{\mathcal{J}}_2. \tag{3.48}$$

The system of tensor equations which consists of the two equations (3.47) and (3.48) is consistent if and only if there exists quaternion tensors $\widehat{\mathcal{V}}_2$, $\widehat{\mathcal{U}}_2$, and $\widehat{\mathcal{K}}_2$ satisfy the following system:

$$\overline{\mathcal{F}}_{11} *_N \widehat{\mathcal{V}}_2 *_M \overline{\mathcal{G}}_{11} + \overline{\mathcal{H}}_{11} *_N \widehat{\mathcal{U}}_2 *_M \overline{\mathcal{J}}_{11} = \overline{\mathcal{E}}_{11}, \tag{3.49}$$

$$\overline{\mathcal{F}}_{22} *_N \widehat{\mathcal{K}}_2 *_M \overline{\mathcal{G}}_{22} + \overline{\mathcal{H}}_{22} *_N \widehat{\mathcal{V}}_2 *_M \overline{\mathcal{J}}_{22} = \overline{\mathcal{E}}_{22}, \tag{3.50}$$

In utilizing *Lemma* 2.4, we have that the general solution to quaternion tensor equations (3.47)-(3.48) can be given by (3.26h)-(3.26o). The quaternion system of tensor equations (3.49)-(3.50) is solvable if and only if Eq. (3.49) is solvable, Eq. (3.50) is solvable, and the quaternion tensor $\widehat{\mathcal{V}}_2$ in (3.49) coincide with $\widehat{\mathcal{V}}_2$ in (3.50). So, Eq. (3.49) and Eq. (3.50) are solvable if and only if the conditions (3.8)-(3.9) are satisfying, respectively, for $k = 1, 2$. In this case, the general solution can be express as

$$\begin{aligned}\widehat{\mathcal{V}}_2 = {} & \overline{\mathcal{F}}_{11}^{\dagger} *_N \overline{\mathcal{E}}_{11} *_M \overline{\mathcal{G}}_{11}^{\dagger} - \overline{\mathcal{F}}_{11}^{\dagger} *_N \overline{\mathcal{H}}_{11} *_N \overline{\mathcal{M}}_{11}^{\dagger} *_N \overline{\mathcal{E}}_{11} *_M \overline{\mathcal{G}}_{11}^{\dagger} - \overline{\mathcal{F}}_{11}^{\dagger} *_N \overline{\mathcal{S}}_{11} *_N \overline{\mathcal{H}}_{11}^{\dagger} \\ & *_N \overline{\mathcal{E}}_{11} *_M \overline{\mathcal{N}}_{11}^{\dagger} *_M \overline{\mathcal{J}}_{11} *_M \overline{\mathcal{G}}_{11}^{\dagger} - \overline{\mathcal{F}}_{11}^{\dagger} *_N \overline{\mathcal{S}}_{11} *_N \mathcal{P}_{44} *_M \mathcal{R}_{\overline{\mathcal{N}}_{11}} *_M \overline{\mathcal{J}}_{11} *_M \overline{\mathcal{G}}_{11}^{\dagger} \\ & + \mathcal{L}_{\overline{\mathcal{F}}_{11}} *_N \mathcal{P}_{55} + \mathcal{P}_{66} *_M \mathcal{R}_{\overline{\mathcal{G}}_{11}},\end{aligned} \tag{3.51a}$$

$$\begin{aligned}\widehat{\mathcal{U}}_2 = {} & \overline{\mathcal{M}}_{11}^{\dagger} *_N \overline{\mathcal{E}}_{11} *_M \overline{\mathcal{J}}_{11}^{\dagger} + \overline{\mathcal{S}}_{11}^{\dagger} *_N \overline{\mathcal{S}}_{11} *_N \overline{\mathcal{H}}_{11}^{\dagger} *_N \overline{\mathcal{E}}_{11} *_M \overline{\mathcal{N}}_{11}^{\dagger} + \mathcal{L}_{\overline{\mathcal{M}}_{11}} \\ & *_N \mathcal{L}_{\overline{\mathcal{S}}_{11}} *_N \mathcal{Q}_{44} + \mathcal{L}_{\overline{\mathcal{M}}_{11}} *_N \mathcal{P}_{44} *_M \mathcal{R}_{\overline{\mathcal{N}}_{11}} + \mathcal{Q}_{66} *_M \mathcal{R}_{\overline{\mathcal{J}}_{11}},\end{aligned} \tag{3.51b}$$

$$\begin{aligned}\widehat{\mathcal{K}}_2 = {} & \overline{\mathcal{F}}_{22}^{\dagger} *_N \overline{\mathcal{E}}_{22} *_M \overline{\mathcal{G}}_{22}^{\dagger} - \overline{\mathcal{F}}_{22}^{\dagger} *_N \overline{\mathcal{H}}_{22} *_N \overline{\mathcal{M}}_{22}^{\dagger} *_N \overline{\mathcal{E}}_{22} *_M \overline{\mathcal{G}}_{22}^{\dagger} - \overline{\mathcal{F}}_{22}^{\dagger} *_N \overline{\mathcal{S}}_{22} *_N \overline{\mathcal{H}}_{22}^{\dagger} \\ & *_N \overline{\mathcal{E}}_{22} *_M \overline{\mathcal{N}}_{22}^{\dagger} *_M \overline{\mathcal{J}}_{22} *_M \overline{\mathcal{G}}_{22}^{\dagger} - \overline{\mathcal{F}}_{22}^{\dagger} *_N \overline{\mathcal{S}}_{22} *_N \mathcal{Q}_{55} *_M \mathcal{R}_{\overline{\mathcal{N}}_{22}} *_M \overline{\mathcal{J}}_{22} *_M \overline{\mathcal{G}}_{22}^{\dagger} \\ & + \mathcal{L}_{\overline{\mathcal{F}}_{22}} *_N \mathcal{P}_{77} + \mathcal{P}_{88} *_M \mathcal{R}_{\overline{\mathcal{G}}_{22}},\end{aligned} \tag{3.51c}$$

$$\begin{aligned}\widehat{\mathcal{V}}_2 = {} & \overline{\mathcal{M}}_{22}^{\dagger} *_N \overline{\mathcal{E}}_{22} *_M \overline{\mathcal{J}}_{22}^{\dagger} + \overline{\mathcal{S}}_{22}^{\dagger} *_N \overline{\mathcal{S}}_{22} *_N \overline{\mathcal{H}}_{22}^{\dagger} *_N \overline{\mathcal{E}}_{22} *_M \overline{\mathcal{N}}_{22}^{\dagger} + \mathcal{L}_{\overline{\mathcal{M}}_{22}} \\ & *_N \mathcal{L}_{\overline{\mathcal{S}}_{22}} *_N \mathcal{Q}_{77} + \mathcal{L}_{\overline{\mathcal{M}}_{22}} *_N \mathcal{Q}_{55} *_M \mathcal{R}_{\overline{\mathcal{N}}_{22}} + \mathcal{Q}_{88} *_M \mathcal{R}_{\overline{\mathcal{J}}_{22}},\end{aligned} \tag{3.51d}$$

By equating $\widehat{\mathcal{V}}_2$ in (3.51a) with $\widehat{\mathcal{V}}_2$ in (3.51d), we have the following equation:

$$\overline{\overline{\mathcal{A}}}_1 *_N \begin{bmatrix} \mathcal{P}_{55} \\ \mathcal{Q}_{77} \end{bmatrix} + \begin{bmatrix} \mathcal{P}_{66} & \mathcal{Q}_{88} \end{bmatrix} *_M \overline{\overline{\mathcal{B}}}_1 = -\overline{\overline{\mathcal{E}}}_1 + \overline{\overline{\mathcal{F}}}_1 *_N \mathcal{P}_{44} *_M \overline{\overline{\mathcal{G}}}_1 + \overline{\overline{\mathcal{H}}}_1 *_N \mathcal{Q}_{55} *_M \overline{\overline{\mathcal{J}}}_1, \tag{3.52}$$

It follows from *Lemma* 2.4 that Eq.(3.52) is solvable if and only if there exist quaternion tensors $\mathcal{P}_{44}$ and $\mathcal{Q}_{55}$ satisfy

$$\overline{\overline{\mathcal{F}}}_{11} *_N \mathcal{P}_{44} *_M \overline{\overline{\mathcal{G}}}_{11} + \overline{\overline{\mathcal{H}}}_{11} *_N \mathcal{Q}_{55} *_M \overline{\overline{\mathcal{J}}}_{11} = \overline{\overline{\mathcal{E}}}_{11}, \tag{3.53}$$

In utilizing *Lemma* 2.4, we have that the general solution to quaternion tensor equation (3.52) can be given by (3.26r)-(3.26u). On applying *Lemma* 2.4, we have that Eq.(3.53) is solvable if and only if conditions (3.10)-(3.11) satisfy. In that case, the general solution can be given by

$$\begin{aligned}\mathcal{P}_{44} = {} & \overline{\overline{\mathcal{F}}}_{11}^{\dagger} *_N \overline{\overline{\mathcal{E}}}_{11} *_M \overline{\overline{\mathcal{G}}}_{11}^{\dagger} - \overline{\overline{\mathcal{F}}}_{11}^{\dagger} *_N \overline{\overline{\mathcal{H}}}_{11} *_N \overline{\overline{\mathcal{M}}}_{11}^{\dagger} *_N \overline{\overline{\mathcal{E}}}_{11} *_M \overline{\overline{\mathcal{G}}}_{11}^{\dagger} - \overline{\overline{\mathcal{F}}}_{11}^{\dagger} *_N \overline{\overline{\mathcal{S}}}_{11} \\ & *_N \overline{\overline{\mathcal{H}}}_{11}^{\dagger} *_N \overline{\overline{\mathcal{E}}}_{11} *_M \overline{\overline{\mathcal{N}}}_{11}^{\dagger} *_M \overline{\overline{\mathcal{J}}}_{11} *_M \overline{\overline{\mathcal{G}}}_{11}^{\dagger} - \overline{\overline{\mathcal{F}}}_{11}^{\dagger} *_N \overline{\overline{\mathcal{S}}}_{11} *_N \mathcal{K}_{44} *_M \mathcal{R}_{\overline{\overline{\mathcal{N}}}_{11}} \\ & *_M \overline{\overline{\mathcal{J}}}_{11} *_M \overline{\overline{\mathcal{G}}}_{11}^{\dagger} + \mathcal{L}_{\overline{\overline{\mathcal{F}}}_{11}} *_N \mathcal{K}_{55} + \mathcal{K}_{66} *_M \mathcal{R}_{\overline{\overline{\mathcal{G}}}_{11}},\end{aligned} \tag{3.54}$$

$$\begin{aligned}\mathcal{Q}_{55} = {} & \overline{\overline{\mathcal{M}}}_{11}^{\dagger} *_N \overline{\overline{\mathcal{E}}}_{11} *_M \overline{\overline{\mathcal{J}}}_{11}^{\dagger} + \overline{\overline{\mathcal{S}}}_{11}^{\dagger} *_N \overline{\overline{\mathcal{S}}}_{11} *_N \overline{\overline{\mathcal{H}}}_{11}^{\dagger} *_N \overline{\overline{\mathcal{E}}}_{11} *_M \overline{\overline{\mathcal{N}}}_{11}^{\dagger} + \mathcal{L}_{\overline{\overline{\mathcal{M}}}_{11}} \\ & *_N \mathcal{L}_{\overline{\overline{\mathcal{S}}}_{11}} *_N \mathcal{K}_{77} + \mathcal{L}_{\overline{\overline{\mathcal{M}}}_{11}} *_N \mathcal{K}_{44} *_M \mathcal{R}_{\overline{\overline{\mathcal{N}}}_{11}} + \mathcal{K}_{88} *_M \mathcal{R}_{\overline{\overline{\mathcal{J}}}_{11}}.\end{aligned} \tag{3.55}$$

Quaternion tensors $\widehat{\mathcal{U}}_2$ in (3.46a) and $\widehat{\mathcal{K}}_2$ in (3.46b) should coincide with $\widehat{\mathcal{U}}_2$ in (3.51b) and $\widehat{\mathcal{K}}_2$ in (3.51c), respectively. In that case, we have the following equations:

$$\widetilde{\mathcal{A}}_1 *_N \begin{bmatrix} \mathcal{Q}_{44} \\ \mathcal{V}_{77} \end{bmatrix} + \begin{bmatrix} \mathcal{Q}_{66} & \mathcal{V}_{88} \end{bmatrix} *_M \widetilde{\mathcal{B}}_1 = \widetilde{\mathcal{E}}_1 - \widetilde{\mathcal{C}}_1 *_N \mathcal{P}_{44} *_M \widetilde{\mathcal{D}}_1, \tag{3.56}$$

$$\widetilde{\mathcal{A}}_2 *_N \begin{bmatrix} \mathcal{V}_{99} \\ \mathcal{P}_{77} \end{bmatrix} + \begin{bmatrix} \mathcal{W}_{11} & \mathcal{P}_{88} \end{bmatrix} *_M \widetilde{\mathcal{B}}_2 = \widetilde{\mathcal{E}}_2 - \widetilde{\mathcal{C}}_2 *_N \mathcal{Q}_{55} *_M \widetilde{\mathcal{D}}_2, \tag{3.57}$$

Apply *Lemma* 2.4 to Eq. (3.56) and Eq.(3.57). we have that (3.56) and Eq.(3.57) are solvable if and only if there are quaternion tensors $\mathcal{P}_{44}$ and $\mathcal{Q}_{55}$ satisfy

$$\widetilde{\mathcal{C}}_{11} *_N \mathcal{P}_{44} *_M \widetilde{\mathcal{D}}_{11} = \widetilde{\mathcal{E}}_{11}, \tag{3.58}$$

$$\widetilde{\mathcal{C}}_{22} *_N \mathcal{Q}_{55} *_M \widetilde{\mathcal{D}}_{22} = \widetilde{\mathcal{E}}_{22}, \tag{3.59}$$

In that case, the general solution to (3.56)-(3.57) can be given by (3.26w)-(3.27c). Meanwhile, the quaternion tensor equations (3.58) and (3.59) are solvable if and only if the conditions (3.12) are satisfying, respectively, for $j = 1, 2$. In that case, the general solution can be given by

$$\mathcal{P}_{44} = \widetilde{\mathcal{C}}_{11}^{\dagger} *_N \widetilde{\mathcal{E}}_{11} *_M \widetilde{\mathcal{D}}_{11}^{\dagger} + \mathcal{L}_{\widetilde{\mathcal{C}}_{11}} *_N \mathcal{W}_{88} + \mathcal{W}_{99} *_M \mathcal{R}_{\widetilde{\mathcal{D}}_{11}}, \tag{3.60}$$

$$\mathcal{Q}_{55} = \widetilde{\mathcal{C}}_{22}^{\dagger} *_N \widetilde{\mathcal{E}}_{22} *_M \widetilde{\mathcal{D}}_{22}^{\dagger} + \mathcal{L}_{\widetilde{\mathcal{C}}_{22}} *_N \mathcal{T}_{11} + \mathcal{T}_{22} *_M \mathcal{R}_{\widetilde{\mathcal{D}}_{22}}. \tag{3.61}$$

Quaternion tensors $\mathcal{P}_{44}$ in (3.54) and $\mathcal{Q}_{55}$ in (3.55) should be equal to quaternion tensors $\mathcal{P}_{44}$ in (3.60) and $\mathcal{Q}_{55}$ in (3.61), respectively. Then we have the following system of tensor equations:

$$\widetilde{\mathcal{F}}_1 *_N \begin{bmatrix} \mathcal{W}_{88} \\ \mathcal{K}_{55} \end{bmatrix} + \begin{bmatrix} \mathcal{W}_{99} & \mathcal{K}_{66} \end{bmatrix} *_M \widetilde{\mathcal{G}}_1 = \widetilde{\widetilde{\mathcal{E}}}_1 - \widetilde{\mathcal{H}}_1 *_N \mathcal{K}_{44} *_M \widetilde{\mathcal{J}}_1, \tag{3.62}$$

$$\widetilde{\mathcal{F}}_2 *_N \begin{bmatrix} \mathcal{K}_{77} \\ \mathcal{T}_{11} \end{bmatrix} + \begin{bmatrix} \mathcal{K}_{88} & \mathcal{T}_{22} \end{bmatrix} *_M \widetilde{\mathcal{G}}_2 = \widetilde{\widetilde{\mathcal{E}}}_2 - \widetilde{\mathcal{H}}_2 *_N \mathcal{K}_{44} *_M \widetilde{\mathcal{J}}_2, \tag{3.63}$$

Apply *Lemma* 2.4, to Eq.(3.62) and Eq.(3.63). Consequently, we have that

$$\widetilde{\mathcal{H}}_{11} *_N \mathcal{K}_{44} *_M \widetilde{\mathcal{J}}_{11} = \widetilde{\widetilde{\mathcal{E}}}_{11}, \tag{3.64}$$

$$\widetilde{\mathcal{H}}_{22} *_N \mathcal{K}_{44} *_M \widetilde{\mathcal{J}}_{22} = \widetilde{\widetilde{\mathcal{E}}}_{22}, \tag{3.65}$$

In that case, the general solution to Equations (3.62) and (3.63) can be given by (3.27f)-(3.27m). Meanwhile, the quaternion system of tensor equations (3.64)-(3.65) is solvable if and only if Eq. (3.64), Eq. (3.65) are solvable, and $\mathcal{K}_{44}$ in (3.64) coincide with $\mathcal{K}_{44}$ in (3.65). Eq. (3.64) and Eq. (3.65) are solvable if and only if the conditions (3.13) are satisfying, respectively, for $l = 1, 2$. In that case, the general solution to (3.64) and (3.65) can be given by

$$\mathcal{K}_{44} = \widetilde{\mathcal{H}}_{11}^{\dagger} *_N \widetilde{\widetilde{\mathcal{E}}}_{11} *_M \widetilde{\mathcal{J}}_{11}^{\dagger} + \mathcal{L}_{\widetilde{\mathcal{H}}_{11}} *_N \acute{\mathcal{W}}_2 + \acute{\mathcal{W}}_3 *_M \mathcal{R}_{\widetilde{\mathcal{J}}_{11}}, \tag{3.66}$$

$$\mathcal{K}_{44} = \widetilde{\mathcal{H}}_{22}^{\dagger} *_N \widetilde{\widetilde{\mathcal{E}}}_{22} *_M \widetilde{\mathcal{J}}_{22}^{\dagger} + \mathcal{L}_{\widetilde{\mathcal{H}}_{22}} *_N \acute{\mathcal{W}}_4 + \acute{\mathcal{W}}_5 *_M \mathcal{R}_{\widetilde{\mathcal{J}}_{22}}. \tag{3.67}$$

Ultimately, equating $\mathcal{K}_{44}$ in (3.66) by $\mathcal{K}_{44}$ in (3.67), yield:

$$\widetilde{\mathcal{A}} *_N \begin{bmatrix} \mathcal{T}_{33} \\ \mathcal{T}_{55} \end{bmatrix} + \begin{bmatrix} \mathcal{T}_{44} & \mathcal{T}_{66} \end{bmatrix} *_M \widetilde{\mathcal{B}} = \widetilde{\mathcal{E}}. \tag{3.68}$$

Apply *Lemma* 2.4 to Eq.(3.68), we have that Eq.(3.68) is solvable if and only if condition (3.13) satisfies. In that case the general solution can be given by (3.27p)-(3.27s). □

Algorithm 3.2. *The general solution to the system of two-sided four coupled Sylvester-like quaternion tensor equations* (1.3) *gives by the following:*

(1) ***Input*** *the system of two-sided four coupled Sylvester-like quaternion tensor equations* (1.3) *with viable orders over* $\mathbb{H}$.
(2) *Compute all quaternion tensors, which appeared in* (3.1a)-(3.2s).
(3) *Check whether the Moore-Penrose inverses conditions in Theorem* 3.1 *are satisfied or not. If not, return "The system* (1.3) *is inconsistent".*
(4) *Else compute the quaternion unknowns* $\mathcal{X}_i, \mathcal{Y}_i, \mathcal{Z}_j$, *where* $(i = \overline{1,3})$ *and* $(j = \overline{1,4})$ *by* (3.15)-(3.22).
(5) ***Output*** *the general solution of the system* (1.3) *is* $\mathcal{X}_i, \mathcal{Y}_i, \mathcal{Z}_j$.

We give an example to illustrate *Theorem* 3.1.

Example 3.3. *Consider the two-sided four coupled Sylvester-like quaternion system of tensor equations* (1.3), *where*

$$\mathcal{F}_4(:,:,1,1) = \begin{bmatrix} \mathbf{0} & \mathbf{i} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},\ \mathcal{F}_4(:,:,1,2) = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{j} \end{bmatrix},\ \mathcal{F}_4(:,:,2,1) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{k} \end{bmatrix},\ \mathcal{F}_4(:,:,2,2) = \begin{bmatrix} \mathbf{1} & \mathbf{i} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\mathcal{G}_4(:,:,1,1) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{2} & \mathbf{i} \end{bmatrix},\ \mathcal{G}_4(:,:,1,2) = \begin{bmatrix} \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{2i} \end{bmatrix},\ \mathcal{G}_4(:,:,2,1) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{k} \end{bmatrix},\ \mathcal{G}_4(:,:,2,2) = \begin{bmatrix} \mathbf{0} & \mathbf{3-j} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\mathcal{E}_4(:,:,1,1) = \begin{bmatrix} -\mathbf{2j} & \mathbf{2i} \\ \mathbf{0} & -\mathbf{2j} \end{bmatrix},\ \mathcal{E}_4(:,:,1,2) = \begin{bmatrix} \mathbf{0} & -\mathbf{4i} \\ \mathbf{0} & \mathbf{2i-6j} \end{bmatrix},\ \mathcal{E}_4(:,:,2,1) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{2+k} \end{bmatrix},$$

$$\mathcal{E}_4(:,:,2,2) = \begin{bmatrix} \mathbf{6-2j} & -\mathbf{1-3j} \\ \mathbf{0} & \mathbf{2-3i+6j+k} \end{bmatrix},\ \mathcal{H}_4(:,:,1,1) = \begin{bmatrix} \mathbf{i} & \mathbf{j} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},\ \mathcal{H}_4(:,:,1,2) = \begin{bmatrix} \mathbf{j} & \mathbf{k} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\mathcal{H}_4(:,:,2,1) = \begin{bmatrix} \mathbf{i} & \mathbf{k} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},\ \mathcal{H}_4(:,:,2,2) = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{2-i} & \mathbf{0} \end{bmatrix},\ \mathcal{J}_4(:,:,1,1) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{k} & \mathbf{i} \end{bmatrix},\ \mathcal{J}_4(:,:,1,2) = \begin{bmatrix} \mathbf{0} & \mathbf{2} \\ \mathbf{0} & -\mathbf{j} \end{bmatrix},$$

$$\mathcal{J}_4(:,:,2,1) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{2-i} & \mathbf{2+j} \end{bmatrix},\ \mathcal{J}_4(:,:,2,2) = \begin{bmatrix} \mathbf{0} & \mathbf{3-k} \\ \mathbf{0} & \mathbf{3+k} \end{bmatrix},\ \mathcal{E}_5(:,:,1,1) = \begin{bmatrix} \mathbf{i+2j-k} & \mathbf{1+i} \\ \mathbf{1+2i+2j-k} & \mathbf{0} \end{bmatrix},$$

$$\mathcal{E}_5(:,:,1,2) = \begin{bmatrix} \mathbf{10i+2j} & -\mathbf{2i+6j+k} \\ \mathbf{3+6i} & \mathbf{0} \end{bmatrix},\ \mathcal{E}_5(:,:,2,1) = \begin{bmatrix} -\mathbf{2+i+4j+5k} & \mathbf{1-2i+2j+k} \\ \mathbf{9j+3k} & \mathbf{0} \end{bmatrix},$$

$$\mathcal{E}_5(:,:,2,2) = \begin{bmatrix} -\mathbf{2+11i+7j+6k} & -\mathbf{9i+9j} \\ \mathbf{1+7i+5j+5k} & \mathbf{0} \end{bmatrix},\ \mathcal{A}_1(:,:,1,1) = \begin{bmatrix} \mathbf{i} & -\mathbf{i} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},\ \mathcal{A}_1(:,:,1,2) = \begin{bmatrix} \mathbf{j} & -\mathbf{j} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\mathcal{A}_1(:,:,2,1) = \begin{bmatrix} \mathbf{k} & -\mathbf{k} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},\ \mathcal{A}_1(:,:,2,2) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{i} & -\mathbf{i} \end{bmatrix},\ \mathcal{B}_1(:,:,1,1) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{j} & -\mathbf{j} \end{bmatrix},\ \mathcal{B}_1(:,:,1,2) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{k} & -\mathbf{k} \end{bmatrix},$$

$$\mathcal{B}_1(:,:,2,1) = \begin{bmatrix} \mathbf{i} & \mathbf{i+j} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},\ \mathcal{B}_1(:,:,2,2) = \begin{bmatrix} \mathbf{j} & \mathbf{j+k} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},\ \mathcal{C}_1(:,:,1,1) = \begin{bmatrix} \mathbf{k} & \mathbf{i+k} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \mathcal{C}_1(:,:,2,1) = \begin{bmatrix} \mathbf{5i} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\mathcal{C}_1(:,:,1,2) = \begin{bmatrix} \mathbf{2-i} & \mathbf{0} \\ \mathbf{0} & \mathbf{2k} \end{bmatrix},\ \mathcal{C}_1(:,:,2,2) = \begin{bmatrix} \mathbf{0} & \mathbf{i} \\ \mathbf{k} & \mathbf{0} \end{bmatrix},\ \mathcal{D}_1(:,:,1,1) = \begin{bmatrix} \mathbf{k} & \mathbf{2-k} \\ \mathbf{j} & \mathbf{0} \end{bmatrix}, \mathcal{D}_1(:,:,1,2) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{3i} \end{bmatrix},$$

$$\mathcal{D}_1(:,:,2,1)=\begin{bmatrix}\mathbf{i} & \mathbf{j}\\ \mathbf{k} & \mathbf{2}\end{bmatrix},\ \mathcal{D}_1(:,:,2,2)=\begin{bmatrix}\mathbf{2} & \mathbf{i}\\ \mathbf{0} & \mathbf{j}\end{bmatrix},\ \mathcal{F}_1(:,:,1,1)=\begin{bmatrix}\mathbf{0} & \mathbf{k}\\ \mathbf{0} & \mathbf{2i}\end{bmatrix},\mathcal{F}_1(:,:,1,2)=\begin{bmatrix}\mathbf{-1} & \mathbf{-1+i}\\ \mathbf{0} & \mathbf{0}\end{bmatrix},$$

$$\mathcal{F}_1(:,:,2,1)=\begin{bmatrix}\mathbf{0} & \mathbf{1}\\ \mathbf{0} & \mathbf{1-i}\end{bmatrix},\ \mathcal{F}_1(:,:,2,2)=\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{2} & \mathbf{2-i}\end{bmatrix},\ \mathcal{G}_1(:,:,1,2)=\begin{bmatrix}\mathbf{i} & \mathbf{2-i}\\ \mathbf{j} & \mathbf{0}\end{bmatrix},\ \mathcal{H}_1(:,:,1,2)=\begin{bmatrix}\mathbf{0} & \mathbf{i}\\ \mathbf{j} & \mathbf{0}\end{bmatrix},$$

$$\mathcal{G}_1(:,:,1,1)=\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{i+j+k}\end{bmatrix},\ \mathcal{G}_1(:,:,2,1)=\begin{bmatrix}\mathbf{j} & \mathbf{2-j}\\ \mathbf{k} & \mathbf{0}\end{bmatrix},\ \mathcal{H}_1(:,:,2,2)=\begin{bmatrix}\mathbf{0} & \mathbf{k}\\ \mathbf{i} & \mathbf{0}\end{bmatrix},\ \mathcal{J}_1(:,:,1,1)=\begin{bmatrix}\mathbf{0} & \mathbf{j}\\ \mathbf{k} & \mathbf{0}\end{bmatrix},$$

$$\mathcal{G}_1(:,:,2,2)=\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{i+k}\end{bmatrix},\ \mathcal{H}_1(:,:,1,1)=\begin{bmatrix}\mathbf{i-k} & \mathbf{0}\\ \mathbf{0} & \mathbf{-2k}\end{bmatrix},\ \mathcal{J}_1(:,:,1,2)=\begin{bmatrix}\mathbf{j} & \mathbf{0}\\ \mathbf{0} & \mathbf{k}\end{bmatrix},\ \mathcal{J}_1(:,:,2,1)=\begin{bmatrix}\mathbf{i} & \mathbf{0}\\ \mathbf{0} & \mathbf{j}\end{bmatrix},$$

$$\mathcal{H}_1(:,:,2,1)=\begin{bmatrix}\mathbf{i} & \mathbf{j-k}\\ \mathbf{0} & \mathbf{0}\end{bmatrix},\ \mathcal{E}_1(:,:,1,1)=\begin{bmatrix}\mathbf{49+19i+5j+23k} & \mathbf{3-10i-3j-14k}\\ \mathbf{1+3i+2j} & \mathbf{-12+6i+3j-7k}\end{bmatrix},$$

$$\mathcal{J}_1(:,:,2,2)=\begin{bmatrix}\mathbf{1-i} & \mathbf{0}\\ \mathbf{0} & \mathbf{j}\end{bmatrix},\ \mathcal{E}_1(:,:,1,2)=\begin{bmatrix}\mathbf{14-44i-19j-2k} & \mathbf{-6-i+16j}\\ \mathbf{5-5i-3j+3k} & \mathbf{12+10i+7j+3k}\end{bmatrix},$$

$$\mathcal{E}_1(:,:,2,1)=\begin{bmatrix}\mathbf{-2-4i-39j+82k} & \mathbf{-3-i+7j+27k}\\ \mathbf{3-15i5j-k} & \mathbf{-14-2i-14j-4k}\end{bmatrix},\ \mathcal{A}_2(:,:,1,1)=\begin{bmatrix}\mathbf{2} & \mathbf{i}\\ \mathbf{0} & \mathbf{0}\end{bmatrix},$$

$$\mathcal{E}_1(:,:,2,2)=\begin{bmatrix}\mathbf{58-18i-45j+34k} & \mathbf{14-21i+11k}\\ \mathbf{-4-10i-8j-12k} & \mathbf{-2+22i-14j-4k}\end{bmatrix},\ \mathcal{A}_2(:,:,1,2)=\begin{bmatrix}\mathbf{0} & \mathbf{-5k}\\ \mathbf{0} & \mathbf{k}\end{bmatrix},$$

$$\mathcal{A}_2(:,:,2,1)=\begin{bmatrix}\mathbf{3} & \mathbf{0}\\ \mathbf{-i} & \mathbf{0}\end{bmatrix},\ \mathcal{A}_2(:,:,2,2)=\begin{bmatrix}\mathbf{0} & \mathbf{-6}\\ \mathbf{0} & \mathbf{-i}\end{bmatrix},\ \mathcal{B}_2(:,:,1,1)=\begin{bmatrix}\mathbf{4} & \mathbf{j}\\ \mathbf{0} & \mathbf{0}\end{bmatrix},\ \mathcal{B}_2(:,:,1,2)=\begin{bmatrix}\mathbf{-i} & \mathbf{-7}\\ \mathbf{0} & \mathbf{0}\end{bmatrix},$$

$$\mathcal{B}_2(:,:,2,1)=\begin{bmatrix}\mathbf{5} & \mathbf{0}\\ \mathbf{-j} & \mathbf{0}\end{bmatrix},\ \mathcal{B}_2(:,:,2,2)=\begin{bmatrix}\mathbf{j} & \mathbf{-8}\\ \mathbf{0} & \mathbf{0}\end{bmatrix},\ \mathcal{C}_2(:,:,1,1)=\begin{bmatrix}\mathbf{6} & \mathbf{k}\\ \mathbf{0} & \mathbf{-k}\end{bmatrix},\ \mathcal{C}_2(:,:,1,2)=\begin{bmatrix}\mathbf{k} & \mathbf{0}\\ \mathbf{9} & \mathbf{0}\end{bmatrix},$$

$$\mathcal{C}_2(:,:,2,1)=\begin{bmatrix}\mathbf{7} & \mathbf{0}\\ \mathbf{0} & \mathbf{-3j}\end{bmatrix},\ \mathcal{C}_2(:,:,2,2)=\begin{bmatrix}\mathbf{2k} & \mathbf{0}\\ \mathbf{8} & \mathbf{0}\end{bmatrix},\ \mathcal{D}_2(:,:,1,1)=\begin{bmatrix}\mathbf{8} & \mathbf{0}\\ \mathbf{0} & \mathbf{3j}\end{bmatrix},\ \mathcal{D}_2(:,:,1,2)=\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{7} & \mathbf{0}\end{bmatrix},$$

$$\mathcal{D}_2(:,:,2,1)=\begin{bmatrix}\mathbf{9} & \mathbf{i-k}\\ \mathbf{0} & \mathbf{0}\end{bmatrix},\ \mathcal{D}_2(:,:,2,2)=\begin{bmatrix}\mathbf{i} & \mathbf{0}\\ \mathbf{6} & \mathbf{-k}\end{bmatrix},\ \mathcal{F}_2(:,:,1,1)=\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{i} & \mathbf{i-k}\end{bmatrix},$$

$$\mathcal{F}_2(:,:,2,1)=\begin{bmatrix}\mathbf{0} & \mathbf{2i}\\ \mathbf{k} & \mathbf{0}\end{bmatrix},\ \mathcal{F}_2(:,:,2,2)=\begin{bmatrix}\mathbf{0} & \mathbf{k-j}\\ \mathbf{0} & \mathbf{j}\end{bmatrix},\ \mathcal{G}_2(:,:,1,1)=\begin{bmatrix}\mathbf{0} & \mathbf{-k}\\ \mathbf{2k} & \mathbf{0}\end{bmatrix},$$

$$\mathcal{G}_2(:,:,2,1)=\begin{bmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{2i} & \mathbf{0}\end{bmatrix},\ \mathcal{G}_2(:,:,2,2)=\begin{bmatrix}\mathbf{0} & \mathbf{5}\\ \mathbf{0} & \mathbf{3j}\end{bmatrix},\ \mathcal{H}_2(:,:,1,1)=\begin{bmatrix}\mathbf{2i} & \mathbf{0}\\ \mathbf{2j} & \mathbf{0}\end{bmatrix},\ \mathcal{H}_2(:,:,1,2)=\begin{bmatrix}\mathbf{0} & \mathbf{i-k}\\ \mathbf{0} & \mathbf{i+k}\end{bmatrix},$$

$$\mathcal{H}_2(:,:,2,1)=\begin{bmatrix}\mathbf{-i+j} & \mathbf{0}\\ \mathbf{0} & \mathbf{i+k}\end{bmatrix},\ \mathcal{H}_2(:,:,2,2)=\begin{bmatrix}\mathbf{0} & \mathbf{3}\\ \mathbf{3-i} & \mathbf{j}\end{bmatrix},\ \mathcal{J}_2(:,:,1,1)=\begin{bmatrix}\mathbf{i} & \mathbf{0}\\ \mathbf{0} & \mathbf{2j}\end{bmatrix},$$

$$\mathcal{J}_2(:,:,2,1)=\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{2i}\end{bmatrix},\ \mathcal{E}_2(:,:,1,1)=\begin{bmatrix}\mathbf{-89+649i-668j+5k} & \mathbf{-2+88i+25j-20k}\\ \mathbf{-271-523i-241j+6k} & \mathbf{-33-94i+46j+252k}\end{bmatrix},$$

$$\mathcal{J}_2(:,:,2,2)=\begin{bmatrix}\mathbf{3i} & \mathbf{0}\\ \mathbf{-j} & \mathbf{0}\end{bmatrix},\ \mathcal{E}_2(:,:,1,2)=\begin{bmatrix}\mathbf{146+329i-635j-147k} & \mathbf{42+33j+35k}\\ \mathbf{108-95i-860j+77k} & \mathbf{-318-74i-179j+15k}\end{bmatrix},$$

$$\mathcal{E}_2(:,:,2,1)=\begin{bmatrix}\mathbf{-131+821i-879j+151k} & \mathbf{-20+105i+27j-25k}\\ \mathbf{19-682i-160j-3k} & \mathbf{-24-113i+62j+309k}\end{bmatrix},\ \mathcal{A}_3(:,:,1,1)=\begin{bmatrix}\mathbf{0} & \mathbf{i-k}\\ \mathbf{0} & \mathbf{j}\end{bmatrix},$$

$$\mathcal{E}_2(:,:,2,2)=\begin{bmatrix}\mathbf{166+368i-705j+192k} & \mathbf{17+3i+32j+53k}\\ \mathbf{402-114i-975j-15k} & \mathbf{-337-148i-146j-8k}\end{bmatrix},\ \mathcal{A}_3(:,:,1,2)=\begin{bmatrix}\mathbf{i+j} & \mathbf{0}\\ \mathbf{i-2j} & \mathbf{0}\end{bmatrix},$$

$$\mathcal{A}_3(:,:,2,1)=\begin{bmatrix}\mathbf{0}&\mathbf{0}\\\mathbf{j}+\mathbf{k}&\mathbf{i}\end{bmatrix},\ \mathcal{A}_3(:,:,2,2)=\begin{bmatrix}\mathbf{j}&-\mathbf{k}\\\mathbf{0}&\mathbf{0}\end{bmatrix},\ \mathcal{B}_3(:,:,1,1)=\begin{bmatrix}\mathbf{i}&\mathbf{0}\\\mathbf{j}&\mathbf{0}\end{bmatrix},\ \mathcal{B}_3(:,:,1,2)=\begin{bmatrix}\mathbf{0}&\mathbf{j}\\\mathbf{k}&\mathbf{0}\end{bmatrix},$$

$$\mathcal{B}_3(:,:,2,1)=\begin{bmatrix}\mathbf{k}&\mathbf{i}\\\mathbf{0}&\mathbf{0}\end{bmatrix},\ \mathcal{B}_3(:,:,2,2)=\begin{bmatrix}\mathbf{i}&\mathbf{0}\\\mathbf{0}&\mathbf{i}\end{bmatrix},\ \mathcal{C}_3(:,:,1,1)=\begin{bmatrix}\mathbf{0}&\mathbf{j}\\\mathbf{j}&\mathbf{0}\end{bmatrix},\ \mathcal{C}_3(:,:,1,2)=\begin{bmatrix}\mathbf{0}&\mathbf{k}\\\mathbf{0}&\mathbf{k}\end{bmatrix},$$

$$\mathcal{C}_3(:,:,2,1)=\begin{bmatrix}\mathbf{i}+\mathbf{j}&\mathbf{0}\\\mathbf{0}&\mathbf{k}\end{bmatrix},\ \mathcal{C}_3(:,:,2,2)=\begin{bmatrix}\mathbf{0}&\mathbf{0}\\\mathbf{j}+\mathbf{k}&\mathbf{i}\end{bmatrix},\ \mathcal{D}_3(:,:,1,1)=\begin{bmatrix}\mathbf{0}&\mathbf{i}+\mathbf{k}\\\mathbf{0}&\mathbf{j}\end{bmatrix},$$

$$\mathcal{D}_3(:,:,2,1)=\begin{bmatrix}\mathbf{0}&\mathbf{0}\\\mathbf{j}-\mathbf{k}&\mathbf{i}\end{bmatrix},\ \mathcal{D}_3(:,:,2,2)=\begin{bmatrix}\mathbf{k}-\mathbf{i}&\mathbf{j}\\\mathbf{0}&\mathbf{0}\end{bmatrix},\ \mathcal{F}_3(:,:,1,1)=\begin{bmatrix}\mathbf{2j}&\mathbf{0}\\\mathbf{3k}&\mathbf{0}\end{bmatrix},$$

$$\mathcal{F}_3(:,:,2,1)=\begin{bmatrix}\mathbf{0}&\mathbf{0}\\\mathbf{i}+\mathbf{k}&-\mathbf{k}\end{bmatrix},\ \mathcal{F}_3(:,:,2,2)=\begin{bmatrix}\mathbf{k}&\mathbf{0}\\\mathbf{2k}&\mathbf{0}\end{bmatrix},\ \mathcal{G}_3(:,:,1,1)=\begin{bmatrix}\mathbf{0}&\mathbf{j}\\\mathbf{2j}&\mathbf{0}\end{bmatrix},\ \mathcal{G}_3(:,:,1,2)=\begin{bmatrix}\mathbf{0}&\mathbf{i}\\\mathbf{3i}&\mathbf{0}\end{bmatrix},$$

$$\mathcal{G}_3(:,:,2,1)=\begin{bmatrix}\mathbf{i}-\mathbf{j}&\mathbf{0}\\\mathbf{0}&-\mathbf{j}\end{bmatrix},\ \mathcal{G}_3(:,:,2,2)=\begin{bmatrix}\mathbf{j}+\mathbf{k}&\mathbf{0}\\\mathbf{0}&-\mathbf{k}\end{bmatrix},\ \mathcal{H}_3(:,:,1,1)=\begin{bmatrix}\mathbf{0}&\mathbf{i}+\mathbf{j}\\\mathbf{0}&\mathbf{k}\end{bmatrix},$$

$$\mathcal{H}_3(:,:,2,1)=\begin{bmatrix}\mathbf{i}+\mathbf{j}&\mathbf{i}\\\mathbf{0}&\mathbf{0}\end{bmatrix},\ \mathcal{H}_3(:,:,2,2)=\begin{bmatrix}\mathbf{j}+\mathbf{k}&\mathbf{0}\\\mathbf{j}&\mathbf{k}\end{bmatrix},\ \mathcal{J}_3(:,:,1,1)=\begin{bmatrix}\mathbf{i}+\mathbf{j}&\mathbf{0}\\\mathbf{k}&\mathbf{0}\end{bmatrix},$$

$$\mathcal{J}_3(:,:,2,1)=\begin{bmatrix}\mathbf{0}&\mathbf{0}\\\mathbf{j}-\mathbf{k}&-\mathbf{j}\end{bmatrix},\ \mathcal{E}_3(:,:,1,1)=\begin{bmatrix}\mathbf{4}-\mathbf{7i}-\mathbf{2j}+\mathbf{10k}&\mathbf{9}+\mathbf{2i}+\mathbf{2j}-\mathbf{9k}\\\mathbf{21}-\mathbf{4i}-\mathbf{5j}-\mathbf{13k}&-\mathbf{1}-\mathbf{5i}+\mathbf{4j}-\mathbf{4k}\end{bmatrix},$$

$$\mathcal{J}_3(:,:,2,2)=\begin{bmatrix}\mathbf{i}+\mathbf{j}&\mathbf{i}\\\mathbf{0}&\mathbf{0}\end{bmatrix},\ \mathcal{E}_3(:,:,1,2)=\begin{bmatrix}-\mathbf{31}+\mathbf{20i}-\mathbf{18j}+\mathbf{12k}&-\mathbf{17}-\mathbf{9i}+\mathbf{4j}+\mathbf{10k}\\-\mathbf{14}-\mathbf{13i}+\mathbf{11j}&-\mathbf{22}-\mathbf{22i}+\mathbf{4j}+\mathbf{7k}\end{bmatrix},$$

$$\mathcal{E}_3(:,:,2,1)=\begin{bmatrix}-\mathbf{10}+\mathbf{4i}+\mathbf{18j}-\mathbf{28k}&-\mathbf{5}+\mathbf{5i}+\mathbf{14j}-\mathbf{10k}\\\mathbf{1}+\mathbf{4i}+\mathbf{23j}-\mathbf{11k}&\mathbf{3i}+\mathbf{20j}+\mathbf{10k}\end{bmatrix},\ \mathcal{F}_2(:,:,1,2)=\begin{bmatrix}\mathbf{0}&\mathbf{j}\\\mathbf{0}&\mathbf{j}-\mathbf{k}\end{bmatrix},$$

$$\mathcal{E}_3(:,:,2,1)=\begin{bmatrix}\mathbf{24}+\mathbf{15i}+\mathbf{8j}-\mathbf{6k}&\mathbf{11}+\mathbf{11i}-\mathbf{7j}-\mathbf{k}\\\mathbf{23}+\mathbf{2j}+\mathbf{6k}&\mathbf{27i}-\mathbf{7j}+\mathbf{6k}\end{bmatrix},\ \mathcal{J}_2(:,:,1,2)=\begin{bmatrix}\mathbf{k}&-\mathbf{2k}\\\mathbf{0}&\mathbf{0}\end{bmatrix},$$

$$\mathcal{D}_3(:,:,1,2)=\begin{bmatrix}\mathbf{i}-\mathbf{j}&\mathbf{0}\\\mathbf{k}&\mathbf{0}\end{bmatrix},\ \mathcal{F}_3(:,:,1,2)=\begin{bmatrix}\mathbf{i}-\mathbf{k}&-\mathbf{k}\\\mathbf{0}&\mathbf{0}\end{bmatrix},\ \mathcal{H}_3(:,:,1,2)=\begin{bmatrix}\mathbf{0}&\mathbf{j}+\mathbf{k}\\\mathbf{j}&\mathbf{k}\end{bmatrix},$$

$$\mathcal{J}_3(:,:,1,2)=\begin{bmatrix}\mathbf{0}&\mathbf{i}+\mathbf{k}\\\mathbf{i}&\mathbf{k}\end{bmatrix},\ \mathcal{G}_2(:,:,1,2)=\begin{bmatrix}-\mathbf{2}&\mathbf{0}\\\mathbf{i}-\mathbf{k}&\mathbf{0}\end{bmatrix}.$$

We now look at the system (1.3). *Rendering of direct calculations*

$$\mathcal{R}_{\mathcal{M}_i}*_2\mathcal{R}_{\mathcal{A}_i}*_2\mathcal{E}_i=0,\ \mathcal{E}_i*_2\mathcal{L}_{\mathcal{B}_i}*_2\mathcal{L}_{\mathcal{N}_i}=0,\ \mathcal{R}_{\mathcal{C}_i}*_2\mathcal{E}_i*_2\mathcal{L}_{\mathcal{B}_i}=0,$$

$$\mathcal{R}_{\widehat{\mathcal{M}}_i}*_2\mathcal{R}_{\widehat{\mathcal{A}}_i}*_2\widehat{\mathcal{E}}_i=0,\ \widehat{\mathcal{E}}_i*_2\mathcal{L}_{\widehat{\mathcal{B}}_i}*_2\mathcal{L}_{\widehat{\mathcal{N}}_i}=0,$$

$$\mathcal{R}_{\widehat{\mathcal{A}}_i}*_2\widehat{\mathcal{E}}_i*_2\mathcal{L}_{\widehat{\mathcal{D}}_i}=0,\ \mathcal{R}_{\widehat{\mathcal{C}}_i}*_2\widehat{\mathcal{E}}_i*_2\mathcal{L}_{\widehat{\mathcal{B}}_i}=0,\ (i=\overline{1,3}),$$

$$\mathcal{R}_{\mathcal{F}_4}*_2\mathcal{E}_4=0,\ \mathcal{E}_4*_2\mathcal{L}_{\mathcal{G}_4}=0,\ \mathcal{R}_{\mathcal{H}_4}*_2\mathcal{E}_5=0,\ \mathcal{E}_5*_2\mathcal{L}_{\mathcal{J}_4}=0,$$

$$\mathcal{R}_{\widehat{\widehat{\mathcal{A}}}_{kk}}*_2\widehat{\widehat{\mathcal{E}}}_{kk}=0,\ \widehat{\widehat{\mathcal{E}}}_{kk}*_2\mathcal{L}_{\widehat{\widehat{\mathcal{B}}}_{kk}}=0,$$

$$\mathcal{R}_{\overline{\mathcal{M}}_{kk}}*_2\mathcal{R}_{\overline{\mathcal{F}}_{kk}}*_2\overline{\mathcal{E}}_{kk}=0,\ \overline{\mathcal{E}}_{kk}*_2\mathcal{L}_{\overline{\mathcal{G}}_{kk}}*_2\mathcal{L}_{\overline{\mathcal{N}}_{kk}}=0,$$

$$\mathcal{R}_{\overline{\mathcal{F}}_{kk}}*_2\overline{\mathcal{E}}_{kk}*_2\mathcal{L}_{\overline{\mathcal{J}}_{kk}}=0,\ \mathcal{R}_{\overline{\mathcal{H}}_{kk}}*_2\overline{\mathcal{E}}_{kk}*_2\mathcal{L}_{\overline{\mathcal{G}}_{kk}}=0,\ (k=1,2),$$

$$\mathcal{R}_{\overline{\overline{\mathcal{M}}}_{11}}*_2\mathcal{R}_{\overline{\overline{\mathcal{F}}}_{11}}*_2\overline{\overline{\mathcal{E}}}_{11}=0,\ \overline{\overline{\mathcal{E}}}_{11}*_2\mathcal{L}_{\overline{\overline{\mathcal{G}}}_{11}}*_2\mathcal{L}_{\overline{\overline{\mathcal{N}}}_{11}}=0,$$

$$\mathcal{R}_{\overline{\overline{\mathcal{F}}}_{11}} *_2 \overline{\overline{\mathcal{E}}}_{11} *_2 \mathcal{L}_{\overline{\overline{\mathcal{J}}}_{11}} = 0,\ \mathcal{R}_{\overline{\overline{\mathcal{H}}}_{11}} *_2 \overline{\overline{\mathcal{E}}}_{11} *_2 \mathcal{L}_{\overline{\overline{\mathcal{G}}}_{11}} = 0,$$
$$\mathcal{R}_{\widetilde{\mathcal{C}}_{jj}} *_2 \widetilde{\mathcal{E}}_{jj} = 0,\ \widetilde{\mathcal{E}}_{jj} *_2 \mathcal{L}_{\widetilde{\mathcal{D}}_{jj}} = 0\ (j = 1,2),$$
$$\mathcal{R}_{\widetilde{\mathcal{H}}_{ll}} *_2 \widetilde{\widetilde{\mathcal{E}}}_{ll} = 0,\ \widetilde{\widetilde{\mathcal{E}}}_{ll} *_2 \mathcal{L}_{\widetilde{\mathcal{G}}_{ll}} = 0,\ \mathcal{R}_{\widetilde{\mathcal{A}}} *_2 \widetilde{\mathcal{E}} *_2 \mathcal{L}_{\widetilde{\mathcal{B}}},\ (l = 1,2),$$
$$\mathcal{R}_{\widetilde{\mathcal{A}}} *_2 \widetilde{\mathcal{E}} *_2 \mathcal{L}_{\widetilde{\mathcal{B}}} = 0.$$

Consequently, in Theorem 3.1, *all Moore-Penrose inverse conditions hold, and the system* (1.3) *is thus consistent. Moreover it is simple to show that the following structures satisfy the system:*

$$\mathcal{Z}_1(:,:,1,1) = \begin{bmatrix} -\mathbf{2} & \mathbf{0} \\ \mathbf{i} & \mathbf{0} \end{bmatrix},\ \mathcal{Z}_1(:,:,1,2) = \begin{bmatrix} \mathbf{0} & \mathbf{2-k} \\ \mathbf{0} & \mathbf{k} \end{bmatrix},\ \mathcal{Z}_1(:,:,2,1) = \begin{bmatrix} \mathbf{1+j} & \mathbf{0} \\ \mathbf{0} & -\mathbf{j} \end{bmatrix},$$
$$\mathcal{Z}_4(:,:,1,1) = \begin{bmatrix} \mathbf{2} & \mathbf{i} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},\ \mathcal{Z}_4(:,:,1,2) = \begin{bmatrix} \mathbf{3-k} & \mathbf{0} \\ \mathbf{0} & \mathbf{i} \end{bmatrix},\ \mathcal{Z}_4(:,:,2,1) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{i} & \mathbf{j} \end{bmatrix},\ \mathcal{Z}_4(:,:,2,2) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{j} & \mathbf{k} \end{bmatrix},$$
$$\mathcal{X}_1(:,:,1,1) = \begin{bmatrix} \mathbf{i-k} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},\ \mathcal{X}_1(:,:,1,2) = \begin{bmatrix} \mathbf{j-k} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},\ \mathcal{X}_1(:,:,2,1) = \begin{bmatrix} \mathbf{1+i} & \mathbf{0} \\ \mathbf{0} & \mathbf{1-i} \end{bmatrix},$$
$$\mathcal{Y}_1(:,:,1,1) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{i-K} & \mathbf{0} \end{bmatrix},\ \mathcal{Y}_1(:,:,1,2) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{j} & \mathbf{j-k} \end{bmatrix},\ \mathcal{Y}_1(:,:,2,1) = \begin{bmatrix} \mathbf{0} & \mathbf{i} \\ \mathbf{0} & \mathbf{2j} \end{bmatrix},\ \mathcal{Y}_1(:,:,2,2) = \begin{bmatrix} \mathbf{0} & \mathbf{j} \\ \mathbf{0} & \mathbf{2k} \end{bmatrix},$$
$$\mathcal{Z}_2(:,:,1,1) = \begin{bmatrix} \mathbf{k} & \mathbf{0} \\ \mathbf{0} & \mathbf{i} \end{bmatrix},\ \mathcal{Z}_2(:,:,1,2) = \begin{bmatrix} \mathbf{j-1} & \mathbf{j} \\ \mathbf{0} & \mathbf{1} \end{bmatrix},\ \mathcal{Z}_2(:,:,2,1) = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{i} \end{bmatrix},\ \mathcal{Z}_2(:,:,2,2) = \begin{bmatrix} \mathbf{1} & \mathbf{3} \\ \mathbf{0} & \mathbf{j} \end{bmatrix},$$
$$\mathcal{X}_2(:,:,1,1) = \begin{bmatrix} -\mathbf{1} & -\mathbf{j+k} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},\ \mathcal{X}_2(:,:,1,2) = \begin{bmatrix} -\mathbf{3i} & \mathbf{0} \\ \mathbf{5} & \mathbf{0} \end{bmatrix},\ \mathcal{X}_2(:,:,2,1) = \begin{bmatrix} \mathbf{2i} & -\mathbf{2} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$
$$\mathcal{Y}_2(:,:,1,1) = \begin{bmatrix} \mathbf{0} & -\mathbf{3} \\ \mathbf{0} & \mathbf{i} \end{bmatrix},\ \mathcal{Y}_2(:,:,1,2) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{3} & \mathbf{0} \end{bmatrix},\ \mathcal{Y}_2(:,:,2,1) = \begin{bmatrix} \mathbf{0} & -\mathbf{4} \\ \mathbf{0} & \mathbf{j} \end{bmatrix},\ \mathcal{Y}_2(:,:,2,2) = \begin{bmatrix} \mathbf{2i} & \mathbf{0} \\ \mathbf{2} & \mathbf{j} \end{bmatrix},$$
$$\mathcal{Z}_3(:,:,1,1) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{2i} & \mathbf{0} \end{bmatrix},\ \mathcal{Z}_3(:,:,1,2) = \begin{bmatrix} \mathbf{i-k} & \mathbf{0} \\ \mathbf{0} & \mathbf{2i} \end{bmatrix},\ \mathcal{Z}_3(:,:,2,1) = \begin{bmatrix} \mathbf{0} & \mathbf{3i} \\ \mathbf{0} & \mathbf{3j} \end{bmatrix},\ \mathcal{Z}_3(:,:,2,2) = \begin{bmatrix} \mathbf{5i} & \mathbf{4j} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$
$$\mathcal{X}_3(:,:,1,1) = \begin{bmatrix} \mathbf{0} & \mathbf{i} \\ \mathbf{j} & \mathbf{0} \end{bmatrix},\ \mathcal{X}_3(:,:,1,2) = \begin{bmatrix} \mathbf{0} & \mathbf{2j} \\ \mathbf{k} & \mathbf{0} \end{bmatrix},\ \mathcal{X}_3(:,:,2,1) = \begin{bmatrix} -\mathbf{i+k} & \mathbf{0} \\ \mathbf{0} & \mathbf{j} \end{bmatrix},\ \mathcal{X}_3(:,:,2,2) = \begin{bmatrix} \mathbf{j-k} & \mathbf{0} \\ \mathbf{0} & \mathbf{i} \end{bmatrix},$$
$$\mathcal{Y}_3(:,:,1,1) = \begin{bmatrix} \mathbf{0} & \mathbf{i+j+k} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},\ \mathcal{Y}_3(:,:,1,2) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{i+j} & \mathbf{j+k} \end{bmatrix},\ \mathcal{Y}_3(:,:,2,1) = \begin{bmatrix} \mathbf{i+k} & \mathbf{j+k} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$
$$\mathcal{Z}_1(:,:,2,2) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{2+k} & \mathbf{0} \end{bmatrix},\ \mathcal{X}_1(:,:,2,2) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{2j} \end{bmatrix},\ \mathcal{X}_2(:,:,2,2) = \begin{bmatrix} \mathbf{i} & \mathbf{0} \\ \mathbf{4} & -\mathbf{k} \end{bmatrix},\ \mathcal{Y}_3(:,:,2,2) = \begin{bmatrix} \mathbf{0} & \mathbf{2i} \\ \mathbf{0} & \mathbf{3j} \end{bmatrix}.$$

Remark 3.4. *If we set* $\mathcal{C}_i = \mathcal{B}_i = \mathcal{I}$ *in* (1.3) *where* $i = \overline{1,3}$, *we obtain the Sylvester-like quaternion system of tensor equations* (1.6).

Remark 3.5. *If we set* $\mathcal{A}_i = \mathcal{D}_i = 0$ *in* (1.6) *where* $i = \overline{1,3}$, *we derive the Sylvester-like quaternion system of tensor equations* (1.4).

Remark 3.6. *If we set* $\mathcal{G}_i = \mathcal{F}_i^{\eta^*} = 0$, $\mathcal{J}_i = \mathcal{H}_i^{\eta^*} = 0$ *and* $\mathcal{E}_i = \mathcal{E}_i^{\eta^*} = 0$ *in* (1.4) *where* $i = \overline{1,3}$, *we investigate* η*-Hermitian solution for* (1.5).

In the following *Section*, we establish the consistency conditions and the general solution to (1.4). In a direct implementation, we investigate some necessary and sufficient conditions for the existence of a common η-Hermitian solution of (1.5).

4. Some implementations of the central system (1.4)

Theorem 4.1. *Consider the quaternion system of tensor equations* (1.4)*, where*

$$\mathcal{F}_4 \in \mathbb{H}^{I(N)\times J(N)},\ \mathcal{G}_4 \in \mathbb{H}^{L(M)\times K(M)},\ \mathcal{H}_4 \in \mathbb{H}^{I(N)\times Q(N)},\ \mathcal{J}_4 \in \mathbb{H}^{S(M)\times K(M)},$$
$$\mathcal{E}_4 \in \mathbb{H}^{I(N)\times K(M)},\ \mathcal{E}_5 \in \mathbb{H}^{I(N)\times K(M)},\ \mathcal{F}_i \in \mathbb{H}^{A(N)\times J(N)},\ \mathcal{G}_i \in \mathbb{H}^{L(M)\times F(M)},$$
$$\mathcal{H}_i \in \mathbb{H}^{A(N)\times P(N)},\ \mathcal{J}_i \in \mathbb{H}^{S(M)\times F(M)},\ \mathcal{E}_i \in \mathbb{H}^{A(N)\times F(M)}\ (i=\overline{1,3})$$

are given tensors over $\mathbb{H}$. *Set*

$$\widehat{\mathcal{M}_i} = \mathcal{R}_{\mathcal{F}_i} *_N \mathcal{H}_i,\ \widehat{\mathcal{N}_i} = \mathcal{J}_i *_M \mathcal{L}_{\mathcal{G}_i},\ \widehat{\mathcal{S}}_i = \mathcal{H}_i *_N \mathcal{L}_{\widehat{\mathcal{M}_i}},\ (i=\overline{1,3}), \tag{4.1a}$$

$$\mathcal{A}_{11} = \begin{bmatrix}\mathcal{L}_{\mathcal{F}_4} & -\mathcal{L}_{\mathcal{F}_1}\end{bmatrix},\ \mathcal{D}_{11} = \begin{bmatrix}\mathcal{R}_{\mathcal{G}_4}\\ -\mathcal{R}_{\mathcal{G}_1}\end{bmatrix},\ \widehat{\mathcal{A}}_{11} = \mathcal{F}_1^\dagger *_N \widehat{\mathcal{S}}_1,\ \widehat{\mathcal{B}}_{11} = R_{\widehat{\mathcal{N}}_1} *_M \mathcal{J}_1 *_M \mathcal{G}_1^\dagger, \tag{4.1b}$$

$$\begin{aligned}\mathcal{E}_{11} &= \mathcal{F}_1^\dagger *_N \mathcal{E}_1 *_M \mathcal{G}_1^\dagger - \mathcal{F}_1^\dagger *_N \mathcal{H}_1 *_N \widehat{\mathcal{M}}_1^\dagger *_N \mathcal{E}_1 *_M \mathcal{G}_1^\dagger - \mathcal{F}_1^\dagger *_N \widehat{\mathcal{S}}_1 *_N \mathcal{H}_1^\dagger *_N \mathcal{E}_1\\ &\quad *_M \widehat{\mathcal{N}}_1^\dagger *_M \mathcal{J}_1 *_M \mathcal{G}_1^\dagger - \mathcal{F}_4^\dagger *_N \mathcal{E}_4 *_M \mathcal{G}_4^\dagger,\end{aligned} \tag{4.1c}$$

$$\mathcal{A}_{22} = \begin{bmatrix}\mathcal{L}_{\mathcal{H}_4} & -\mathcal{L}_{\widehat{\mathcal{M}}_3} *_N \mathcal{L}_{\widehat{\mathcal{S}}_3}\end{bmatrix},\ \mathcal{D}_{22} = \begin{bmatrix}\mathcal{R}_{\mathcal{J}_4}\\ -\mathcal{R}_{\mathcal{J}_3}\end{bmatrix},\ \widehat{\mathcal{A}}_{22} = \mathcal{L}_{\widehat{\mathcal{M}}_3},\ \widehat{\mathcal{B}}_{22} = R_{\widehat{\mathcal{N}}_3}, \tag{4.1d}$$

$$\mathcal{E}_{22} = \widehat{\mathcal{M}}_3^\dagger *_N \mathcal{E}_3 *_M \mathcal{J}_3^\dagger + \widehat{\mathcal{S}}_3^\dagger *_N \widehat{\mathcal{S}}_3 *_N \mathcal{H}_3^\dagger *_N \mathcal{E}_3 *_M \widehat{\mathcal{N}}_3^\dagger - \mathcal{H}_4^\dagger *_N \mathcal{E}_5 *_M \mathcal{J}_4^\dagger, \tag{4.1e}$$

$$\widehat{\widehat{\mathcal{A}}}_{ii} = \mathcal{R}_{\mathcal{A}_{ii}} *_N \widehat{\mathcal{A}}_{ii},\ \widehat{\widehat{\mathcal{B}}}_{ii} = \widehat{\mathcal{B}}_{ii} *_M \mathcal{L}_{\mathcal{D}_{ii}},\ \widehat{\widehat{\mathcal{E}}}_{ii} = \mathcal{R}_{\mathcal{A}_{ii}} *_N \mathcal{E}_{ii} *_M \mathcal{L}_{\mathcal{D}_{ii}},\ (i=1,2), \tag{4.1f}$$

$$\overline{\mathcal{A}}_1 = \begin{bmatrix}-\mathcal{L}_{\widehat{\mathcal{M}}_1} *_N \mathcal{L}_{\widehat{\mathcal{S}}_1} & \mathcal{L}_{\mathcal{F}_2}\end{bmatrix},\ \overline{\mathcal{A}}_2 = \begin{bmatrix}-\mathcal{L}_{\widehat{\mathcal{M}}_2} *_N \mathcal{L}_{\widehat{\mathcal{S}}_2} & \mathcal{L}_{\mathcal{F}_3}\end{bmatrix},\ \overline{\mathcal{F}}_1 = \mathcal{F}_2^\dagger *_N \widehat{\mathcal{S}}_2, \tag{4.1g}$$

$$\overline{\mathcal{B}}_1 = \begin{bmatrix}-\mathcal{R}_{\mathcal{J}_1}\\ \mathcal{R}_{\mathcal{G}_2}\end{bmatrix},\ \overline{\mathcal{B}}_2 = \begin{bmatrix}-\mathcal{R}_{\mathcal{J}_2}\\ \mathcal{R}_{\mathcal{G}_3}\end{bmatrix}, \overline{\mathcal{F}}_2 = \mathcal{F}_3^\dagger *_N \widehat{\mathcal{S}}_3,\ \overline{\mathcal{G}}_1 = \mathcal{J}_2 *_N \mathcal{G}_2^\dagger,\ \overline{\mathcal{G}}_2 = \mathcal{J}_3 *_N \mathcal{G}_3^\dagger, \tag{4.1h}$$

$$\overline{\mathcal{H}}_1 = \mathcal{L}_{\widehat{\mathcal{M}}_1},\ \overline{\mathcal{J}}_1 = \mathcal{R}_{\widehat{\mathcal{N}}_1},\ \overline{\mathcal{H}}_2 = \mathcal{L}_{\widehat{\mathcal{M}}_2},\ \overline{\mathcal{J}}_2 = \mathcal{R}_{\widehat{\mathcal{N}}_2}, \tag{4.1i}$$

$$\begin{aligned}\overline{\mathcal{E}}_1 &= -\widehat{\mathcal{M}}_1^\dagger *_N \mathcal{E}_1 *_M \mathcal{J}_1^\dagger - \widehat{\mathcal{S}}_1^\dagger *_N \widehat{\mathcal{S}}_1 *_N \mathcal{H}_1^\dagger *_N \mathcal{E}_1 *_M \widehat{\mathcal{N}}_1^\dagger + \mathcal{F}_2^\dagger *_N \mathcal{E}_2 *_M \mathcal{G}_2^\dagger - \mathcal{F}_2^\dagger\\ &\quad *_N \mathcal{H}_2 *_N \widehat{\mathcal{M}}_2^\dagger *_N \mathcal{E}_2 *_M \mathcal{G}_2^\dagger - \mathcal{F}_2^\dagger *_N \widehat{\mathcal{S}}_2 *_N \mathcal{H}_2^\dagger *_N \mathcal{E}_2 *_M \widehat{\mathcal{N}}_2^\dagger *_M \mathcal{J}_2 *_M \mathcal{G}_2^\dagger,\end{aligned} \tag{4.1j}$$

$$\begin{aligned}\overline{\mathcal{E}}_2 &= -\widehat{\mathcal{M}}_2^\dagger *_N \mathcal{E}_2 *_M \mathcal{J}_2^\dagger - \widehat{\mathcal{S}}_2^\dagger *_N \widehat{\mathcal{S}}_2 *_N \mathcal{H}_2^\dagger *_N \mathcal{E}_2 *_M \widehat{\mathcal{N}}_2^\dagger + \mathcal{F}_2^\dagger *_N \mathcal{E}_2 *_M \mathcal{G}_2^\dagger - \mathcal{F}_3^\dagger\\ &\quad *_N \mathcal{H}_3 *_N \widehat{\mathcal{M}}_3^\dagger *_N \mathcal{E}_3 *_M \mathcal{G}_3^\dagger - \mathcal{F}_3^\dagger *_N \widehat{\mathcal{S}}_3 *_N \mathcal{H}_3^\dagger *_N \mathcal{E}_3 *_M \widehat{\mathcal{N}}_3^\dagger *_M \mathcal{J}_3 *_M \mathcal{G}_3^\dagger,\end{aligned} \tag{4.1k}$$

$$\overline{\mathcal{F}}_{ii} = \mathcal{R}_{\overline{\mathcal{A}}_i} *_N \overline{\mathcal{F}}_i,\ \overline{\mathcal{G}}_{ii} = \overline{\mathcal{G}}_i *_M \mathcal{L}_{\overline{\mathcal{B}}_i},\ \overline{\mathcal{H}}_{ii} = \mathcal{R}_{\overline{\mathcal{A}}_i} *_N \overline{\mathcal{H}}_i,\ \overline{\mathcal{J}}_{ii} = \overline{\mathcal{J}}_i *_M \mathcal{L}_{\overline{\mathcal{B}}_i}, \tag{4.1l}$$

$$\overline{\mathcal{E}}_{ii} = \mathcal{R}_{\overline{\mathcal{A}}_i} *_N \overline{\mathcal{E}}_i *_M \mathcal{L}_{\overline{\mathcal{B}}_i},\ \overline{\mathcal{M}}_{ii} = \mathcal{R}_{\overline{\mathcal{F}}_{ii}} *_N \overline{\mathcal{H}}_{ii},\ \overline{\mathcal{N}}_{ii} = \overline{\mathcal{J}}_{ii} *_M \mathcal{L}_{\overline{\mathcal{G}}_{ii}},\ \overline{\mathcal{S}}_{ii} = \overline{\mathcal{H}}_{ii} *_N \mathcal{L}_{\overline{\mathcal{M}}_{ii}}, \tag{4.1m}$$

$$\overline{\overline{\mathcal{A}}}_1 = \begin{bmatrix}\mathcal{L}_{\overline{\mathcal{F}}_{11}} & -\mathcal{L}_{\overline{\mathcal{M}}_{22}} *_N \mathcal{L}_{\overline{\mathcal{S}}_{22}}\end{bmatrix},\ \overline{\overline{\mathcal{B}}}_1 = \begin{bmatrix}\mathcal{R}_{\overline{\mathcal{G}}_{11}}\\ -\mathcal{R}_{\overline{\mathcal{J}}_{11}}\end{bmatrix},\ \overline{\overline{\mathcal{F}}}_1 = \overline{\mathcal{F}}_{11}^\dagger *_N \overline{\mathcal{S}}_{11}, \tag{4.1n}$$

$$\overline{\overline{\mathcal{G}}}_1 = R_{\overline{\mathcal{N}}_{11}} *_M \overline{\mathcal{J}}_{11} *_M \overline{\mathcal{G}}_{11}^\dagger,\ \overline{\overline{\mathcal{H}}}_1 = \mathcal{L}_{\overline{\mathcal{M}}_{22}},\ \overline{\overline{\mathcal{J}}}_1 = \mathcal{R}_{\overline{\mathcal{N}}_{22}}, \tag{4.1o}$$

$$\begin{aligned}\overline{\overline{\mathcal{E}}}_1 &= \overline{\mathcal{F}}_{11}^\dagger *_N \overline{\mathcal{E}}_{11} *_M \overline{\mathcal{G}}_{11}^\dagger - \overline{\mathcal{F}}_{11}^\dagger *_N \overline{\mathcal{H}}_{11} *_N \overline{\mathcal{M}}_{11}^\dagger *_N \overline{\mathcal{E}}_{11} *_M \overline{\mathcal{G}}_{11}^\dagger - \overline{\mathcal{F}}_{11}^\dagger *_N \overline{\mathcal{S}}_{11} *_N \overline{\mathcal{H}}_{11}^\dagger\\ &\quad *_N \overline{\mathcal{E}}_{11} *_M \overline{\mathcal{N}}_{11}^\dagger *_M \overline{\mathcal{J}}_{11} *_M \overline{\mathcal{G}}_{11}^\dagger - \overline{\mathcal{M}}_{22}^\dagger *_N \overline{\mathcal{E}}_{22} *_M \overline{\mathcal{J}}_{22}^\dagger - \overline{\mathcal{S}}_{22}^\dagger *_N \overline{\mathcal{S}}_{22} *_N \overline{\mathcal{H}}_{22}^\dagger\\ &\quad *_N \overline{\mathcal{E}}_{22} *_M \overline{\mathcal{N}}_{22}^\dagger\end{aligned} \tag{4.1p}$$

$$\overline{\overline{\mathcal{F}}}_{11} = \mathcal{R}_{\overline{\overline{\mathcal{A}}}_1} *_N \overline{\overline{\mathcal{F}}}_1,\ \overline{\overline{\mathcal{G}}}_{11} = \overline{\overline{\mathcal{G}}}_1 *_M \mathcal{L}_{\overline{\overline{\mathcal{B}}}_1},\ \overline{\overline{\mathcal{H}}}_{11} = \mathcal{R}_{\overline{\overline{\mathcal{A}}}_1} *_N \overline{\overline{\mathcal{H}}}_1,\ \overline{\overline{\mathcal{J}}}_{11} = \overline{\overline{\mathcal{J}}}_1 *_M \mathcal{L}_{\overline{\overline{\mathcal{B}}}_1}, \tag{4.1q}$$

$$\overline{\overline{\mathcal{E}}}_{11}=\mathcal{R}_{\overline{\overline{\mathcal{A}}}_1}*_N\overline{\overline{\mathcal{E}}}_1*_M\mathcal{L}_{\overline{\overline{\mathcal{B}}}_1},\ \overline{\overline{\mathcal{M}}}_{11}=\mathcal{R}_{\overline{\overline{\mathcal{F}}}_{11}}*_N\overline{\overline{\mathcal{H}}}_{11},\ \overline{\overline{\mathcal{N}}}_{11}=\overline{\overline{\mathcal{J}}}_{11}*_M\mathcal{L}_{\overline{\overline{\mathcal{G}}}_{11}},\tag{4.1r}$$

$$\overline{\overline{\mathcal{S}}}_{11}=\overline{\overline{\mathcal{H}}}_{11}*_N\mathcal{L}_{\overline{\overline{\mathcal{M}}}_{11}},\ \widetilde{\mathcal{A}}_1=\begin{bmatrix}\mathcal{L}_{\overline{\mathcal{M}}_{11}}*_N\mathcal{L}_{\overline{\mathcal{S}}_{11}} & -\mathcal{L}_{\widehat{\widehat{\mathcal{A}}}_{11}}\end{bmatrix},\ \widetilde{\mathcal{A}}_2=\begin{bmatrix}\mathcal{L}_{\widehat{\widehat{\mathcal{A}}}_{22}} & -\mathcal{L}_{\overline{\mathcal{F}}_{22}}\end{bmatrix},\tag{4.1s}$$

$$\widetilde{\mathcal{B}}_1=\begin{bmatrix}\mathcal{R}_{\overline{\mathcal{J}}_{11}}\\-\mathcal{R}_{\widehat{\widehat{\mathcal{B}}}_{11}}\end{bmatrix},\ \widetilde{\mathcal{B}}_2=\begin{bmatrix}\mathcal{R}_{\widehat{\widehat{\mathcal{B}}}_{22}}\\-\mathcal{R}_{\overline{\mathcal{G}}_{22}}\end{bmatrix},\widetilde{\mathcal{C}}_1=\mathcal{L}_{\overline{\mathcal{M}}_{11}}\widetilde{\mathcal{D}}_1=\mathcal{R}_{\overline{\mathcal{N}}_{11}},\tag{4.1t}$$

$$\widetilde{\mathcal{C}}_2=\overline{\mathcal{F}}_{22}^{\dagger}*_N\overline{\mathcal{S}}_{22},\ \widetilde{\mathcal{D}}_2=\mathcal{R}_{\overline{\mathcal{N}}_{22}}*_M\overline{\mathcal{J}}_{22}*_M\overline{\mathcal{G}}_{22}^{\dagger},\tag{4.1u}$$

$$\widetilde{\mathcal{E}}_1=\widehat{\widehat{\mathcal{A}}}_{11}^{\dagger}*_N\widehat{\widehat{\mathcal{E}}}_{11}*_M\widehat{\widehat{\mathcal{B}}}_{11}^{\dagger}-\overline{\mathcal{M}}_{11}^{\dagger}*_N\overline{\mathcal{E}}_{11}*_M\overline{\mathcal{J}}_{11}^{\dagger}-\overline{\mathcal{S}}_{11}^{\dagger}*_N\overline{\mathcal{S}}_{11}*_N\overline{\mathcal{H}}_{11}^{\dagger}*_N\overline{\mathcal{E}}_{11}*_M\overline{\mathcal{N}}_{11}^{\dagger},\tag{4.1v}$$

$$\begin{aligned}\widetilde{\mathcal{E}}_2=&\overline{\mathcal{F}}_{22}^{\dagger}*_N\overline{\mathcal{E}}_{22}*_M\overline{\mathcal{G}}_{22}^{\dagger}-\overline{\mathcal{F}}_{22}^{\dagger}*_N\overline{\mathcal{H}}_{22}*_N\overline{\mathcal{M}}_{22}^{\dagger}*_N\overline{\mathcal{E}}_{22}*_M\overline{\mathcal{G}}_{22}^{\dagger}-\overline{\mathcal{F}}_{22}^{\dagger}*_N\overline{\mathcal{S}}_{22}*_N\overline{\mathcal{H}}_{22}^{\dagger}\\&*_N\overline{\mathcal{E}}_{22}*_M\overline{\mathcal{N}}_{22}^{\dagger}*_M\overline{\mathcal{J}}_{22}*_M\overline{\mathcal{G}}_{22}^{\dagger}-\widehat{\widehat{\mathcal{A}}}_{22}^{\dagger}*_N\widehat{\widehat{\mathcal{E}}}_{22}*_M\widehat{\widehat{\mathcal{B}}}_{22}^{\dagger},\end{aligned}\tag{4.1w}$$

$$\widetilde{\mathcal{F}}_1=\begin{bmatrix}\mathcal{L}_{\widetilde{\mathcal{C}}_{11}} & -\mathcal{L}_{\overline{\overline{\mathcal{F}}}_{11}}\end{bmatrix},\ \widetilde{\mathcal{F}}_2=\begin{bmatrix}\mathcal{L}_{\overline{\overline{\mathcal{M}}}_{11}}*_N\mathcal{L}_{\overline{\overline{\mathcal{S}}}_{11}} & -\mathcal{L}_{\widetilde{\mathcal{C}}_{22}}\end{bmatrix},\ \widetilde{\mathcal{H}}_1=\overline{\overline{\mathcal{F}}}_{11}^{\dagger}*_N\overline{\overline{\mathcal{S}}}_{11},\tag{4.1x}$$

$$\widetilde{\mathcal{J}}_1=\mathcal{R}_{\overline{\overline{\mathcal{N}}}_{11}}*_M\overline{\overline{\mathcal{J}}}_{11}*_M\overline{\overline{\mathcal{G}}}_{11}^{\dagger}\ \widetilde{\mathcal{G}}_1=\begin{bmatrix}\mathcal{R}_{\widetilde{\mathcal{D}}_{11}}\\-\mathcal{R}_{\overline{\overline{\mathcal{G}}}_{11}}\end{bmatrix},\ \widetilde{\mathcal{G}}_2=\begin{bmatrix}\mathcal{R}_{\overline{\overline{\mathcal{J}}}_{11}}\\-\mathcal{R}_{\widetilde{\mathcal{D}}_{22}}\end{bmatrix},\widetilde{\mathcal{H}}_2=\mathcal{L}_{\overline{\overline{\mathcal{M}}}_{11}},\ \widetilde{\mathcal{J}}_2=\mathcal{R}_{\overline{\overline{\mathcal{N}}}_{11}},\tag{4.1y}$$

$$\begin{aligned}\widetilde{\widetilde{\mathcal{E}}}_1=&\overline{\overline{\mathcal{F}}}_{11}^{\dagger}*_N\overline{\overline{\mathcal{E}}}_{11}*_M\overline{\overline{\mathcal{G}}}_{11}^{\dagger}-\overline{\overline{\mathcal{F}}}_{11}^{\dagger}*_N\overline{\overline{\mathcal{H}}}_{11}*_N\overline{\overline{\mathcal{M}}}_{11}^{\dagger}*_N\overline{\overline{\mathcal{E}}}_{11}*_M\overline{\overline{\mathcal{G}}}_{11}^{\dagger}-\overline{\overline{\mathcal{F}}}_{11}^{\dagger}*_N\overline{\overline{\mathcal{S}}}_{11}\\&*_N\overline{\overline{\mathcal{H}}}_{11}^{\dagger}*_N\overline{\overline{\mathcal{E}}}_{11}*_M\overline{\overline{\mathcal{N}}}_{11}^{\dagger}*_M\overline{\overline{\mathcal{J}}}_{11}*_M\overline{\overline{\mathcal{G}}}_{11}^{\dagger}-\widetilde{\mathcal{C}}_{11}^{\dagger}*_N\widetilde{\mathcal{E}}_{11}*_M\widetilde{\mathcal{D}}_{11}^{\dagger},\end{aligned}\tag{4.1z}$$

$$\widetilde{\widetilde{\mathcal{E}}}_2=\widetilde{\mathcal{C}}_{22}^{\dagger}*_N\widetilde{\mathcal{E}}_{22}*_M\widetilde{\mathcal{D}}_{22}^{\dagger}-\overline{\overline{\mathcal{M}}}_{11}^{\dagger}*_N\overline{\overline{\mathcal{E}}}_{11}*_M\overline{\overline{\mathcal{J}}}_{11}^{\dagger}-\overline{\overline{\mathcal{S}}}_{11}^{\dagger}*_N\overline{\overline{\mathcal{S}}}_{11}*_N\overline{\overline{\mathcal{H}}}_{11}^{\dagger}*_N\overline{\overline{\mathcal{E}}}_{11}*_M\overline{\overline{\mathcal{N}}}_{11}^{\dagger},\tag{4.2a}$$

$$\widetilde{\mathcal{H}}_{11}=\mathcal{R}_{\widetilde{\mathcal{F}}_1}*_N\widetilde{\mathcal{H}}_1,\ \widetilde{\mathcal{H}}_{22}=\mathcal{R}_{\widetilde{\mathcal{F}}_2}*_N\widetilde{\mathcal{H}}_2,\ \widetilde{\mathcal{J}}_{11}=\widetilde{\mathcal{J}}_1*_M\mathcal{L}_{\widetilde{\mathcal{G}}_1},\ \widetilde{\mathcal{J}}_{22}=\widetilde{\mathcal{J}}_2*_M\mathcal{L}_{\widetilde{\mathcal{G}}_2},\tag{4.2b}$$

$$\widetilde{\widetilde{\mathcal{E}}}_{11}=\mathcal{R}_{\widetilde{\mathcal{F}}_1}*_N\widetilde{\widetilde{\mathcal{E}}}_1*_M\mathcal{L}_{\widetilde{\mathcal{G}}_1},\ \widetilde{\widetilde{\mathcal{E}}}_{22}=\mathcal{R}_{\widetilde{\mathcal{F}}_2}*_N\widetilde{\widetilde{\mathcal{E}}}_2*_M\mathcal{L}_{\widetilde{\mathcal{G}}_2},\ \widetilde{\mathcal{A}}=\begin{bmatrix}\mathcal{L}_{\widetilde{\mathcal{H}}_{11}} & -\mathcal{L}_{\widetilde{\mathcal{H}}_{22}}\end{bmatrix},\tag{4.2c}$$

$$\widetilde{\mathcal{B}}=\begin{bmatrix}\mathcal{R}_{\widetilde{\mathcal{J}}_{11}} & -\mathcal{R}_{\widetilde{\mathcal{J}}_{22}}\end{bmatrix},\ \widetilde{\mathcal{E}}=\widetilde{\mathcal{H}}_{22}^{\dagger}*_N\widetilde{\widetilde{\mathcal{E}}}_{22}*_M\widetilde{\mathcal{J}}_{22}^{\dagger}-\widetilde{\mathcal{H}}_{11}^{\dagger}*_N\widetilde{\widetilde{\mathcal{E}}}_{11}*_M\widetilde{\mathcal{J}}_{11}^{\dagger}.\tag{4.2d}$$

Then the system (1.3) is consistent if and only if

$$\mathcal{R}_{\widehat{\mathcal{M}}_i}*_N\mathcal{R}_{\mathcal{F}_i}*_N\mathcal{E}_i=0,\ \mathcal{E}_i*_M\mathcal{L}_{\mathcal{G}_i}*_M\mathcal{L}_{\widehat{\mathcal{N}}_i}=0,\tag{4.3}$$

$$\mathcal{R}_{\mathcal{F}_i}*_N\mathcal{E}_i*_M\mathcal{L}_{\mathcal{J}_i}=0,\ \mathcal{R}_{\mathcal{H}_i}*_N\mathcal{E}_i*_M\mathcal{L}_{\mathcal{G}_i}=0,\ (i=\overline{1,3}),\tag{4.4}$$

$$\mathcal{R}_{\mathcal{F}_4}*_N\mathcal{E}_4=0,\ \mathcal{E}_4*_M\mathcal{L}_{\mathcal{G}_4}=0,\ \mathcal{R}_{\mathcal{H}_4}*_N\mathcal{E}_5=0,\ \mathcal{E}_5*_M\mathcal{L}_{\mathcal{J}_4}=0,\tag{4.5}$$

$$\mathcal{R}_{\widehat{\widehat{\mathcal{A}}}_{kk}}*_N\widehat{\widehat{\mathcal{E}}}_{kk}=0,\ \widehat{\widehat{\mathcal{E}}}_{kk}*_M\mathcal{L}_{\widehat{\widehat{\mathcal{B}}}_{kk}}=0,\tag{4.6}$$

$$\mathcal{R}_{\overline{\mathcal{M}}_{kk}}*_N\mathcal{R}_{\overline{\mathcal{F}}_{kk}}*_N\overline{\mathcal{E}}_{kk}=0,\ \overline{\mathcal{E}}_{kk}*_M\mathcal{L}_{\overline{\mathcal{G}}_{kk}}*_M\mathcal{L}_{\overline{\mathcal{N}}_{kk}}=0,\tag{4.7}$$

$$\mathcal{R}_{\overline{\mathcal{F}}_{kk}}*_N\overline{\mathcal{E}}_{kk}*_M\mathcal{L}_{\overline{\mathcal{J}}_{kk}}=0,\ \mathcal{R}_{\overline{\mathcal{H}}_{kk}}*_N\overline{\mathcal{E}}_{kk}*_M\mathcal{L}_{\overline{\mathcal{G}}_{kk}}=0,\ (k=1,2),\tag{4.8}$$

$$\mathcal{R}_{\overline{\overline{\mathcal{M}}}_{11}}*_N\mathcal{R}_{\overline{\overline{\mathcal{F}}}_{11}}*_N\overline{\overline{\mathcal{E}}}_{11}=0,\ \overline{\overline{\mathcal{E}}}_{11}*_M\mathcal{L}_{\overline{\overline{\mathcal{G}}}_{11}}*_M\mathcal{L}_{\overline{\overline{\mathcal{N}}}_{11}}=0,\tag{4.9}$$

$$\mathcal{R}_{\overline{\overline{\mathcal{F}}}_{11}}*_N\overline{\overline{\mathcal{E}}}_{11}*_M\mathcal{L}_{\overline{\overline{\mathcal{J}}}_{11}}=0,\ \mathcal{R}_{\overline{\overline{\mathcal{H}}}_{11}}*_N\overline{\overline{\mathcal{E}}}_{11}*_M\mathcal{L}_{\overline{\overline{\mathcal{G}}}_{11}}=0,\tag{4.10}$$

$$\mathcal{R}_{\widetilde{\mathcal{C}}_{jj}}*_N\widetilde{\mathcal{E}}_{jj}=0,\ \widetilde{\mathcal{E}}_{jj}*_M\mathcal{L}_{\widetilde{\mathcal{D}}_{jj}}=0\ (j=1,2),\tag{4.11}$$

$$\mathcal{R}_{\widetilde{\mathcal{H}}_{ll}}*_N\widetilde{\widetilde{\mathcal{E}}}_{ll}=0,\ \widetilde{\widetilde{\mathcal{E}}}_{ll}*_M\mathcal{L}_{\widetilde{\mathcal{G}}_{ll}}=0,\ \mathcal{R}_{\widetilde{\mathcal{A}}}*_N\widetilde{\mathcal{E}}*_M\mathcal{L}_{\widetilde{\mathcal{B}}},\ (l=1,2),\tag{4.12}$$

$$\mathcal{R}_{\widetilde{\mathcal{A}}}*_N\widetilde{\mathcal{E}}*_M\mathcal{L}_{\widetilde{\mathcal{B}}}=0.\tag{4.13}$$

Under these conditions, the general solution to system (1.3) *can be expressed as follows:*

$$\mathcal{Z}_1 = \mathcal{F}_4^{\dagger} *_N \mathcal{E}_4 *_M \mathcal{G}_4^{\dagger} + \mathcal{L}_{\mathcal{F}_4} *_N \mathcal{W}_1 + \mathcal{W}_2 *_M \mathcal{R}_{\mathcal{G}_4}, \tag{4.14}$$

$$\mathcal{Z}_4 = \mathcal{H}_4^{\dagger} *_N \mathcal{E}_5 *_M \mathcal{J}_4^{\dagger} + \mathcal{L}_{\mathcal{H}_4} *_N \acute{\mathcal{W}}_1 + \mathcal{W}_3 *_M \mathcal{R}_{\mathcal{J}_4}, \tag{4.15}$$

$$\begin{aligned}\mathcal{Z}_2 = {}& \widehat{\mathcal{M}}_1^{\dagger} *_N \widehat{\mathcal{E}}_1 *_M \mathcal{J}_1^{\dagger} + \widehat{\mathcal{S}}_1^{\dagger} *_N \widehat{\mathcal{S}}_1 *_N \mathcal{H}_1^{\dagger} *_N \mathcal{E}_1 *_M \widehat{\mathcal{N}}_1^{\dagger} + \mathcal{L}_{\widehat{\mathcal{M}}_1} *_N \mathcal{L}_{\widehat{\mathcal{S}}_1} *_N \widehat{\mathcal{U}}_1 \\ & + \mathcal{L}_{\widehat{\mathcal{M}}_1} *_N \widehat{\mathcal{U}}_2 *_M \mathcal{R}_{\widehat{\mathcal{N}}_1} + \widehat{\mathcal{U}}_3 *_M \mathcal{R}_{\mathcal{J}_1},\end{aligned} \tag{4.16}$$

$$\begin{aligned}\text{or } \mathcal{Z}_2 = {}& \mathcal{F}_2^{\dagger} *_N \mathcal{E}_2 *_M \mathcal{G}_2^{\dagger} - \mathcal{F}_2^{\dagger} *_N \mathcal{H}_2 *_N \widehat{\mathcal{M}}_2^{\dagger} *_N \mathcal{E}_2 *_M \mathcal{G}_2^{\dagger} - \mathcal{F}_2^{\dagger} *_N \widehat{\mathcal{S}}_2 *_N \mathcal{H}_2^{\dagger} *_N \mathcal{E}_2 \\ & *_M \widehat{\mathcal{N}}_2^{\dagger} *_M \mathcal{J}_2 *_M \mathcal{G}_2^{\dagger} - \mathcal{F}_2^{\dagger} *_N \widehat{\mathcal{S}}_2 *_N \widehat{\mathcal{V}}_2 *_M \mathcal{R}_{\widehat{\mathcal{N}}_2} *_M \mathcal{J}_2 *_M \mathcal{G}_2^{\dagger} + \mathcal{L}_{\mathcal{F}_2} *_N \widehat{\mathcal{V}}_4 \\ & + \widehat{\mathcal{V}}_5 *_M \mathcal{R}_{\mathcal{F}_2},\end{aligned} \tag{4.17}$$

$$\begin{aligned}\mathcal{Z}_3 = {}& \widehat{\mathcal{M}}_2^{\dagger} *_N \mathcal{E}_2 *_M \mathcal{J}_2^{\dagger} + \widehat{\mathcal{S}}_2^{\dagger} *_N \widehat{\mathcal{S}}_2 *_N \mathcal{H}_2^{\dagger} *_N \mathcal{E}_2 *_M \widehat{\mathcal{N}}_2^{\dagger} + \mathcal{L}_{\widehat{\mathcal{M}}_2} *_N \mathcal{L}_{\widehat{\mathcal{S}}_2} *_N \widehat{\mathcal{V}}_1 \\ & + \mathcal{L}_{\widehat{\mathcal{M}}_2} *_N \widehat{\mathcal{V}}_2 *_M \mathcal{R}_{\widehat{\mathcal{N}}_2} + \widehat{\mathcal{V}}_3 *_M \mathcal{R}_{\mathcal{J}_2},\end{aligned} \tag{4.18}$$

$$\begin{aligned}\text{or } \mathcal{Z}_3 = {}& \mathcal{F}_3^{\dagger} *_N \mathcal{E}_3 *_M \mathcal{G}_3^{\dagger} - \mathcal{F}_3^{\dagger} *_N \mathcal{H}_3 *_N \widehat{\mathcal{M}}_3^{\dagger} *_N \mathcal{E}_3 *_M \mathcal{G}_3^{\dagger} - \mathcal{F}_3^{\dagger} *_N \widehat{\mathcal{S}}_3 *_N \mathcal{H}_3^{\dagger} *_N \mathcal{E}_3 \\ & *_M \widehat{\mathcal{N}}_3^{\dagger} *_M \mathcal{J}_3 *_M \mathcal{G}_3^{\dagger} - \mathcal{F}_3^{\dagger} *_N \widehat{\mathcal{S}}_3 *_N \widehat{\mathcal{K}}_2 *_M \mathcal{R}_{\widehat{\mathcal{N}}_3} *_M \mathcal{J}_3 *_M \mathcal{G}_3^{\dagger} + \mathcal{L}_{\mathcal{F}_3} *_N \widehat{\mathcal{K}}_4 \\ & + \widehat{\mathcal{K}}_5 *_M \mathcal{R}_{\mathcal{G}_3}, \ (i = \overline{1,3}).\end{aligned} \tag{4.19}$$

Where the arbitrary tensors $\mathcal{W}_j,\ \widehat{\mathcal{V}}_i,\ \widehat{\mathcal{U}}_j,\ \widehat{\mathcal{K}}_k$ *and* $\acute{\mathcal{W}}_1$ $(j = \overline{1,3},\ i = \overline{1,5},\ k \in \{2,4,5\})$ *can be reduced by* (3.23)-(3.26q).

Proof. See *Remark* (3.4)-*Remark* (3.5). □

Corollary 4.2. *Consider the quaternion system of tensor equations* (1.4)*, where*

$$\mathcal{F}_4 \in \mathbb{H}^{I(N)\times J(N)},\ \mathcal{H}_4 \in \mathbb{H}^{I(N)\times Q(N)},\ \mathcal{E}_4 \in \mathbb{H}^{I(N)\times I(N)},\ \mathcal{E}_5 \in \mathbb{H}^{I(N)\times I(N)},$$
$$\mathcal{F}_i \in \mathbb{H}^{A(N)\times J(N)},\ \mathcal{H}_i \in \mathbb{H}^{A(N)\times I(N)},\ \mathcal{E}_i \in \mathbb{H}^{A(N)\times A(N)}\ (i = \overline{1,3})$$

are given tensors over $\mathbb{H}$*. Set*

$$\widehat{\mathcal{M}}_i = \mathcal{R}_{\mathcal{F}_i} *_N \mathcal{H}_i,\ \widehat{\mathcal{N}}_i = (\widehat{\mathcal{M}}_i)^{\eta^*},\ \widehat{\mathcal{S}}_i = \mathcal{H}_i *_N \mathcal{L}_{\widehat{\mathcal{M}}_i},\ (i = \overline{1,3})\ \mathcal{A}_{11} = \begin{bmatrix}\mathcal{L}_{\mathcal{F}_4} & -\mathcal{L}_{\mathcal{F}_1}\end{bmatrix}, \tag{4.20a}$$

$$\mathcal{D}_{11} = \begin{bmatrix}\mathcal{R}_{\mathcal{F}_4^{\eta^*}} \\ -\mathcal{R}_{\mathcal{F}_1^{\eta^*}}\end{bmatrix},\ \widehat{\mathcal{A}}_{11} = \mathcal{F}_1^{\dagger} *_N \widehat{\mathcal{S}}_1,\ \widehat{\mathcal{B}}_{11} = R_{\widehat{\mathcal{N}}_1} *_N \mathcal{H}_1^{\eta^*} *_N (\mathcal{F}_1^{\eta^*})^{\dagger}, \tag{4.20b}$$

$$\begin{aligned}\mathcal{E}_{11} = {}& \mathcal{F}_1^{\dagger} *_N \mathcal{E}_1 *_N (\mathcal{F}_1^{\eta^*})^{\dagger} - \mathcal{F}_1^{\dagger} *_N \mathcal{H}_1 *_N \widehat{\mathcal{M}}_1^{\dagger} *_N \mathcal{E}_1 *_N (\mathcal{F}_1^{\eta^*})^{\dagger} - \mathcal{F}_1^{\dagger} *_N \widehat{\mathcal{S}}_1 \\ & *_N \mathcal{H}_1^{\dagger} *_N \mathcal{E}_1 *_N \widehat{\mathcal{N}}_1^{\dagger} *_M \mathcal{H}_1^{\eta^*} *_N (\mathcal{F}_1^{\eta^*})^{\dagger} - \mathcal{F}_4^{\dagger} *_N \mathcal{E}_4 *_N (\mathcal{F}_4^{\eta^*})^{\dagger},\end{aligned} \tag{4.20c}$$

$$\mathcal{A}_{22} = \begin{bmatrix}\mathcal{L}_{\mathcal{H}_4} & -\mathcal{L}_{\widehat{\mathcal{M}}_3} *_N \mathcal{L}_{\widehat{\mathcal{S}}_3}\end{bmatrix},\ \mathcal{D}_{22} = \begin{bmatrix}\mathcal{R}_{\mathcal{H}_4^{\eta^*}} \\ -\mathcal{R}_{\mathcal{H}_3^{\eta^*}}\end{bmatrix},\ \widehat{\mathcal{A}}_{22} = \mathcal{L}_{\widehat{\mathcal{M}}_3},\ \widehat{\mathcal{B}}_{22} = R_{\widehat{\mathcal{N}}_3}, \tag{4.20d}$$

$$\mathcal{E}_{22} = \widehat{\mathcal{M}}_3^{\dagger} *_N \mathcal{E}_3 *_N (\mathcal{H}_3^{\eta^*})^{\dagger} + \widehat{\mathcal{S}}_3^{\dagger} *_N \widehat{\mathcal{S}}_3 *_N \mathcal{H}_3^{\dagger} *_N \mathcal{E}_3 *_N \widehat{\mathcal{N}}_3^{\dagger} - \mathcal{H}_4^{\dagger} *_N \mathcal{E}_5 *_N (\mathcal{J}_4^{\eta^*})^{\dagger}, \tag{4.20e}$$

$$\widehat{\widehat{\mathcal{A}}}_{ii} = \mathcal{R}_{\mathcal{A}_{ii}} *_N \widehat{\mathcal{A}}_{ii},\ \widehat{\widehat{\mathcal{B}}}_{ii} = \widehat{\mathcal{B}}_{ii} *_N \mathcal{L}_{\mathcal{D}_{ii}},\ \widehat{\widehat{\mathcal{E}}}_{ii} = \mathcal{R}_{\mathcal{A}_{ii}} *_N \mathcal{E}_{ii} *_N \mathcal{L}_{\mathcal{D}_{ii}},\ (i = 1,2), \tag{4.20f}$$

$$\overline{\mathcal{A}}_1 = \begin{bmatrix}-\mathcal{L}_{\widehat{\mathcal{M}}_1} *_N \mathcal{L}_{\widehat{\mathcal{S}}_1} & \mathcal{L}_{\mathcal{F}_2}\end{bmatrix},\ \overline{\mathcal{A}}_2 = \begin{bmatrix}-\mathcal{L}_{\widehat{\mathcal{M}}_2} *_N \mathcal{L}_{\widehat{\mathcal{S}}_2} & \mathcal{L}_{\mathcal{F}_3}\end{bmatrix},\ \overline{\mathcal{F}}_1 = \mathcal{F}_2^{\dagger} *_N \widehat{\mathcal{S}}_2, \tag{4.20g}$$

$$\overline{\mathcal{B}}_1 = \begin{bmatrix}-\mathcal{R}_{\mathcal{H}_1^{\eta^*}} \\ \mathcal{R}_{\mathcal{F}_2^{\eta^*}}\end{bmatrix},\ \overline{\mathcal{B}}_2 = \begin{bmatrix}-\mathcal{R}_{\mathcal{H}_2^{\eta^*}} \\ \mathcal{R}_{\mathcal{F}_3^{\eta^*}}\end{bmatrix}, \overline{\mathcal{F}}_2 = \mathcal{F}_3^{\dagger} *_N \widehat{\mathcal{S}}_3,\ \overline{\mathcal{G}}_1 = \mathcal{J}_2 *_N (\mathcal{G}_2^{\eta^*})^{\dagger}, \tag{4.20h}$$

$$\overline{\mathcal{G}}_2 = \mathcal{J}_3 *_N (\mathcal{G}_3^{\eta^*})^{\dagger},\ \overline{\mathcal{H}}_1 = \mathcal{L}_{\widehat{\mathcal{M}}_1},\ \overline{\mathcal{J}}_1 = \mathcal{R}_{\widehat{\mathcal{N}}_1},\ \overline{\mathcal{H}}_2 = \mathcal{L}_{\widehat{\mathcal{M}}_2},\ \overline{\mathcal{J}}_2 = \mathcal{R}_{\widehat{\mathcal{N}}_2}, \tag{4.20i}$$

$$\begin{aligned}\overline{\mathcal{E}}_1 = &-\widehat{\mathcal{M}}_1^{\dagger} *_N \mathcal{E}_1 *_M (\mathcal{H}_1^{\eta^*})^{\dagger} - \widehat{\mathcal{S}}_1^{\dagger} *_N \widehat{\mathcal{S}}_1 *_N \mathcal{H}_1^{\dagger} *_N \mathcal{E}_1 *_N \widehat{\mathcal{N}}_1^{\dagger} + \mathcal{F}_2^{\dagger} *_N \mathcal{E}_2 *_N \\ &(\mathcal{F}_2^{\eta^*})^{\dagger} - \mathcal{F}_2^{\dagger} *_N \mathcal{H}_2 *_N \widehat{\mathcal{M}}_2^{\dagger} *_N \mathcal{E}_2 *_N (\mathcal{F}_2^{\eta^*})^{\dagger} - \mathcal{F}_2^{\dagger} *_N \widehat{\mathcal{S}}_2 *_N \mathcal{H}_2^{\dagger} *_N \mathcal{E}_2 \\ &*_N \widehat{\mathcal{N}}_2^{\dagger} *_N \mathcal{J}_2^{\eta^*} *_N (\mathcal{F}_2^{\eta^*})^{\dagger},\end{aligned} \tag{4.20j}$$

$$\begin{aligned}\overline{\mathcal{E}}_2 = &-\widehat{\mathcal{M}}_2^{\dagger} *_N \mathcal{E}_2 *_N (\mathcal{H}_2^{\eta^*})^{\dagger} - \widehat{\mathcal{S}}_2^{\dagger} *_N \widehat{\mathcal{S}}_2 *_N \mathcal{H}_2^{\dagger} *_N \mathcal{E}_2 *_N \widehat{\mathcal{N}}_2^{\dagger} + \mathcal{F}_2^{\dagger} *_N \mathcal{E}_2 *_N \\ &(\mathcal{F}_2^{\eta^*})^{\dagger} - \mathcal{F}_3^{\dagger} *_N \mathcal{H}_3 *_N \widehat{\mathcal{M}}_3^{\dagger} *_N \mathcal{E}_3 *_N (\mathcal{F}_3^{\eta^*})^{\dagger} - \mathcal{F}_3^{\dagger} *_N \widehat{\mathcal{S}}_3 *_N \mathcal{H}_3^{\dagger} *_N \mathcal{E}_3 \\ &*_N \widehat{\mathcal{N}}_3^{\dagger} *_M \mathcal{H}_3^{\eta^*} *_N (\mathcal{F}_3^{\eta^*})^{\dagger},\end{aligned} \tag{4.20k}$$

$$\overline{\mathcal{F}}_{ii} = \mathcal{R}_{\overline{\mathcal{A}}_i} *_N \overline{\mathcal{F}}_i,\ \overline{\mathcal{G}}_{ii} = \overline{\mathcal{G}}_i *_N \mathcal{L}_{\overline{\mathcal{B}}_i},\ \overline{\mathcal{H}}_{ii} = \mathcal{R}_{\overline{\mathcal{A}}_i} *_N \overline{\mathcal{H}}_i,\ \overline{\mathcal{J}}_{ii} = \overline{\mathcal{J}}_i *_N \mathcal{L}_{\overline{\mathcal{B}}_i}, \tag{4.20l}$$

$$\overline{\mathcal{E}}_{ii} = \mathcal{R}_{\overline{\mathcal{A}}_i} *_N \overline{\mathcal{E}}_i *_N \mathcal{L}_{\overline{\mathcal{B}}_i},\ \overline{\mathcal{M}}_{ii} = \mathcal{R}_{\overline{\mathcal{F}}_{ii}} *_N \overline{\mathcal{H}}_{ii},\ \overline{\mathcal{N}}_{ii} = \overline{\mathcal{J}}_{ii} *_N \mathcal{L}_{\overline{\mathcal{G}}_{ii}},\ \overline{\mathcal{S}}_{ii} = \overline{\mathcal{H}}_{ii} *_N \mathcal{L}_{\overline{\mathcal{M}}_{ii}}, \tag{4.20m}$$

$$\overline{\overline{\mathcal{A}}}_1 = \begin{bmatrix}\mathcal{L}_{\overline{\mathcal{F}}_{11}} & -\mathcal{L}_{\overline{\mathcal{M}}_{22}} *_N \mathcal{L}_{\overline{\mathcal{S}}_{22}}\end{bmatrix},\ \overline{\overline{\mathcal{B}}}_1 = \begin{bmatrix}\mathcal{R}_{\overline{\mathcal{G}}_{11}} \\ -\mathcal{R}_{\overline{\mathcal{J}}_{11}}\end{bmatrix},\ \overline{\overline{\mathcal{F}}}_1 = \overline{\mathcal{F}}_{11}^{\dagger} *_N \overline{\mathcal{S}}_{11}, \tag{4.20n}$$

$$\overline{\overline{\mathcal{G}}}_1 = R_{\overline{\mathcal{N}}_{11}} *_N \overline{\mathcal{J}}_{11} *_M \overline{\mathcal{G}}_{11}^{\dagger},\ \overline{\overline{\mathcal{H}}}_1 = \mathcal{L}_{\overline{\mathcal{M}}_{22}},\ \overline{\overline{\mathcal{J}}}_1 = \mathcal{R}_{\overline{\mathcal{N}}_{22}}, \tag{4.20o}$$

$$\begin{aligned}\overline{\overline{\mathcal{E}}}_1 = &\overline{\mathcal{F}}_{11}^{\dagger} *_N \overline{\mathcal{E}}_{11} *_N \overline{\mathcal{G}}_{11}^{\dagger} - \overline{\mathcal{F}}_{11}^{\dagger} *_N \overline{\mathcal{H}}_{11} *_N \overline{\mathcal{M}}_{11}^{\dagger} *_N \overline{\mathcal{E}}_{11} *_M \overline{\mathcal{G}}_{11}^{\dagger} - \overline{\mathcal{F}}_{11}^{\dagger} *_N \overline{\mathcal{S}}_{11} *_N \overline{\mathcal{H}}_{11}^{\dagger} \\ &*_N \overline{\mathcal{E}}_{11} *_N \overline{\mathcal{N}}_{11}^{\dagger} *_N \overline{\mathcal{J}}_{11} *_N \overline{\mathcal{G}}_{11}^{\dagger} - \overline{\mathcal{M}}_{22}^{\dagger} *_N \overline{\mathcal{E}}_{22} *_M \overline{\mathcal{J}}_{22}^{\dagger} - \overline{\mathcal{S}}_{22}^{\dagger} *_N \overline{\mathcal{S}}_{22} *_N \overline{\mathcal{H}}_{22}^{\dagger} \\ &*_N \overline{\mathcal{E}}_{22} *_N \overline{\mathcal{N}}_{22}^{\dagger}\end{aligned} \tag{4.20p}$$

$$\overline{\overline{\mathcal{F}}}_{11} = \mathcal{R}_{\overline{\overline{\mathcal{A}}}_1} *_N \overline{\overline{\mathcal{F}}}_1,\ \overline{\overline{\mathcal{G}}}_{11} = \overline{\overline{\mathcal{G}}}_1 *_N \mathcal{L}_{\overline{\overline{\mathcal{B}}}_1},\ \overline{\overline{\mathcal{H}}}_{11} = \mathcal{R}_{\overline{\overline{\mathcal{A}}}_1} *_N \overline{\overline{\mathcal{H}}}_1,\ \overline{\overline{\mathcal{J}}}_{11} = \overline{\overline{\mathcal{J}}}_1 *_N \mathcal{L}_{\overline{\overline{\mathcal{B}}}_1}, \tag{4.20q}$$

$$\overline{\overline{\mathcal{E}}}_{11} = \mathcal{R}_{\overline{\overline{\mathcal{A}}}_1} *_N \overline{\overline{\mathcal{E}}}_1 *_N \mathcal{L}_{\overline{\overline{\mathcal{B}}}_1},\ \overline{\overline{\mathcal{M}}}_{11} = \mathcal{R}_{\overline{\overline{\mathcal{F}}}_{11}} *_N \overline{\overline{\mathcal{H}}}_{11},\ \overline{\overline{\mathcal{N}}}_{11} = \overline{\overline{\mathcal{J}}}_{11} *_N \mathcal{L}_{\overline{\overline{\mathcal{G}}}_{11}}, \tag{4.20r}$$

$$\overline{\overline{\mathcal{S}}}_{11} = \overline{\overline{\mathcal{H}}}_{11} *_N \mathcal{L}_{\overline{\overline{\mathcal{M}}}_{11}},\ \widetilde{\mathcal{A}}_1 = \begin{bmatrix}\mathcal{L}_{\overline{\mathcal{M}}_{11}} *_N \mathcal{L}_{\overline{\mathcal{S}}_{11}} & -\mathcal{L}_{\widehat{\widehat{\mathcal{A}}}_{11}}\end{bmatrix},\ \widetilde{\mathcal{A}}_2 = \begin{bmatrix}\mathcal{L}_{\widehat{\widehat{\mathcal{A}}}_{22}} & -\mathcal{L}_{\overline{\mathcal{F}}_{22}}\end{bmatrix}, \tag{4.20s}$$

$$\widetilde{\mathcal{B}}_1 = \begin{bmatrix}\mathcal{R}_{\overline{\mathcal{J}}_{11}} \\ -\mathcal{R}_{\widehat{\widehat{\mathcal{B}}}_{11}}\end{bmatrix},\ \widetilde{\mathcal{B}}_2 = \begin{bmatrix}\mathcal{R}_{\widehat{\widehat{\mathcal{B}}}_{22}} \\ -\mathcal{R}_{\overline{\mathcal{G}}_{22}}\end{bmatrix}, \widetilde{\mathcal{C}}_1 = \mathcal{L}_{\overline{\mathcal{M}}_{11}} \widetilde{\mathcal{D}}_1 = \mathcal{R}_{\overline{\mathcal{N}}_{11}}, \tag{4.20t}$$

$$\widetilde{\mathcal{C}}_2 = \overline{\mathcal{F}}_{22}^{\dagger} *_N \overline{\mathcal{S}}_{22},\ \widetilde{\mathcal{D}}_2 = \mathcal{R}_{\overline{\mathcal{N}}_{22}} *_N \overline{\mathcal{J}}_{22} *_N \overline{\mathcal{G}}_{22}^{\dagger}, \tag{4.20u}$$

$$\widetilde{\mathcal{E}}_1 = \widehat{\widehat{\mathcal{A}}}_{11}^{\dagger} *_N \widehat{\widehat{\mathcal{E}}}_{11} *_M \widehat{\widehat{\mathcal{B}}}_{11}^{\dagger} - \overline{\mathcal{M}}_{11}^{\dagger} *_N \overline{\mathcal{E}}_{11} *_N \overline{\mathcal{J}}_{11}^{\dagger} - \overline{\mathcal{S}}_{11}^{\dagger} *_N \overline{\mathcal{S}}_{11} *_N \overline{\mathcal{H}}_{11}^{\dagger} *_N \overline{\mathcal{E}}_{11} *_M \overline{\mathcal{N}}_{11}^{\dagger}, \tag{4.20v}$$

$$\begin{aligned}\widetilde{\mathcal{E}}_2 = &\overline{\mathcal{F}}_{22}^{\dagger} *_N \overline{\mathcal{E}}_{22} *_N \overline{\mathcal{G}}_{22}^{\dagger} - \overline{\mathcal{F}}_{22}^{\dagger} *_N \overline{\mathcal{H}}_{22} *_N \overline{\mathcal{M}}_{22}^{\dagger} *_N \overline{\mathcal{E}}_{22} *_M \overline{\mathcal{G}}_{22}^{\dagger} - \overline{\mathcal{F}}_{22}^{\dagger} *_N \\ &\overline{\mathcal{S}}_{22} *_N \overline{\mathcal{H}}_{22}^{\dagger} *_N \overline{\mathcal{E}}_{22} *_N \overline{\mathcal{N}}_{22}^{\dagger} *_N \overline{\mathcal{J}}_{22} *_M \overline{\mathcal{G}}_{22}^{\dagger} - \widehat{\widehat{\mathcal{A}}}_{22}^{\dagger} *_N \widehat{\widehat{\mathcal{E}}}_{22} *_N \widehat{\widehat{\mathcal{B}}}_{22}^{\dagger},\end{aligned} \tag{4.20w}$$

$$\widetilde{\mathcal{F}}_1 = \begin{bmatrix}\mathcal{L}_{\widetilde{\mathcal{C}}_{11}} & -\mathcal{L}_{\overline{\overline{\mathcal{F}}}_{11}}\end{bmatrix},\ \widetilde{\mathcal{F}}_2 = \begin{bmatrix}\mathcal{L}_{\overline{\overline{\mathcal{M}}}_{11}} *_N \mathcal{L}_{\overline{\overline{\mathcal{S}}}_{11}} & -\mathcal{L}_{\widetilde{\mathcal{C}}_{22}}\end{bmatrix},\ \widetilde{\mathcal{H}}_1 = \overline{\overline{\mathcal{F}}}_{11}^{\dagger} *_N \overline{\overline{\mathcal{S}}}_{11}, \tag{4.20x}$$

$$\widetilde{\mathcal{J}}_1 = \mathcal{R}_{\overline{\overline{\mathcal{N}}}_{11}} *_N \overline{\overline{\mathcal{J}}}_{11} *_N \overline{\overline{\mathcal{G}}}_{11}^{\dagger}\ \widetilde{\mathcal{G}}_1 = \begin{bmatrix}\mathcal{R}_{\widetilde{\mathcal{D}}_{11}} \\ -\mathcal{R}_{\overline{\overline{\mathcal{G}}}_{11}}\end{bmatrix},\ \widetilde{\mathcal{G}}_2 = \begin{bmatrix}\mathcal{R}_{\overline{\overline{\mathcal{J}}}_{11}} \\ -\mathcal{R}_{\widetilde{\mathcal{D}}_{22}}\end{bmatrix}, \widetilde{\mathcal{H}}_2 = \mathcal{L}_{\overline{\overline{\mathcal{M}}}_{11}},\ \widetilde{\mathcal{J}}_2 = \mathcal{R}_{\overline{\overline{\mathcal{N}}}_{11}}, \tag{4.20y}$$

$$\begin{aligned}\widetilde{\widetilde{\mathcal{E}}}_1 = &\overline{\overline{\mathcal{F}}}_{11}^{\dagger} *_N \overline{\overline{\mathcal{E}}}_{11} *_N \overline{\overline{\mathcal{G}}}_{11}^{\dagger} - \overline{\overline{\mathcal{F}}}_{11}^{\dagger} *_N \overline{\overline{\mathcal{H}}}_{11} *_N \overline{\overline{\mathcal{M}}}_{11}^{\dagger} *_N \overline{\overline{\mathcal{E}}}_{11} *_N \overline{\overline{\mathcal{G}}}_{11}^{\dagger} - \overline{\overline{\mathcal{F}}}_{11}^{\dagger} *_N \overline{\overline{\mathcal{S}}}_{11} \\ &*_N \overline{\overline{\mathcal{H}}}_{11}^{\dagger} *_N \overline{\overline{\mathcal{E}}}_{11} *_N \overline{\overline{\mathcal{N}}}_{11}^{\dagger} *_N \overline{\overline{\mathcal{J}}}_{11} *_M \overline{\overline{\mathcal{G}}}_{11}^{\dagger} - \widetilde{\mathcal{C}}_{11}^{\dagger} *_N \widetilde{\mathcal{E}}_{11} *_M \widetilde{\mathcal{D}}_{11}^{\dagger},\end{aligned} \tag{4.20z}$$

$$\widetilde{\widetilde{\mathcal{E}}}_2 = \widetilde{\mathcal{C}}_{22}^{\dagger} *_N \widetilde{\mathcal{E}}_{22} *_M \widetilde{\mathcal{D}}_{22}^{\dagger} - \overline{\overline{\mathcal{M}}}_{11}^{\dagger} *_N \overline{\overline{\mathcal{E}}}_{11} *_N \overline{\overline{\mathcal{J}}}_{11}^{\dagger} - \overline{\overline{\mathcal{S}}}_{11}^{\dagger} *_N \overline{\overline{\mathcal{S}}}_{11} *_N \overline{\overline{\mathcal{H}}}_{11}^{\dagger} *_N \overline{\overline{\mathcal{E}}}_{11} *_M \overline{\overline{\mathcal{N}}}_{11}^{\dagger}, \tag{4.21a}$$

$$\widetilde{\mathcal{H}}_{11} = \mathcal{R}_{\widetilde{\mathcal{F}}_1} *_N \widetilde{\mathcal{H}}_1,\ \widetilde{\mathcal{H}}_{22} = \mathcal{R}_{\widetilde{\mathcal{F}}_2} *_N \widetilde{\mathcal{H}}_2,\ \widetilde{\mathcal{J}}_{11} = \widetilde{\mathcal{J}}_1 *_N \mathcal{L}_{\widetilde{\mathcal{G}}_1},\ \widetilde{\mathcal{J}}_{22} = \widetilde{\mathcal{J}}_2 *_N \mathcal{L}_{\widetilde{\mathcal{G}}_2}, \tag{4.21b}$$

$$\widetilde{\widetilde{\mathcal{E}}}_{11} = \mathcal{R}_{\widetilde{\mathcal{F}}_1} *_N \widetilde{\widetilde{\mathcal{E}}}_1 *_N \mathcal{L}_{\widetilde{\mathcal{G}}_1},\ \widetilde{\widetilde{\mathcal{E}}}_{22} = \mathcal{R}_{\widetilde{\mathcal{F}}_2} *_N \widetilde{\widetilde{\mathcal{E}}}_2 *_N \mathcal{L}_{\widetilde{\mathcal{G}}_2},\ \widetilde{\mathcal{A}} = \begin{bmatrix}\mathcal{L}_{\widetilde{\mathcal{H}}_{11}} & -\mathcal{L}_{\widetilde{\mathcal{H}}_{22}}\end{bmatrix}, \tag{4.21c}$$

$$\widetilde{\mathcal{B}} = \begin{bmatrix}\mathcal{R}_{\widetilde{\mathcal{J}}_{11}} & -\mathcal{R}_{\widetilde{\mathcal{J}}_{22}}\end{bmatrix},\ \widetilde{\mathcal{E}} = \widetilde{\mathcal{H}}_{22}^{\dagger} *_N \widetilde{\widetilde{\mathcal{E}}}_{22} *_M \widetilde{\mathcal{J}}_{22}^{\dagger} - \widetilde{\mathcal{H}}_{11}^{\dagger} *_N \widetilde{\widetilde{\mathcal{E}}}_{11} *_N \widetilde{\mathcal{J}}_{11}^{\dagger}. \tag{4.21d}$$

Then the system (1.3) *is consistent if and only if*

$$\mathcal{R}_{\widehat{\mathcal{M}}_i} *_N \mathcal{R}_{\mathcal{F}_i} *_N \mathcal{E}_i = 0,\ \mathcal{R}_{\mathcal{F}_i} *_N \mathcal{E}_i *_M \mathcal{L}_{\mathcal{H}_i^{\eta^*}} = 0,\ (i = \overline{1,3}), \tag{4.22}$$

$$\mathcal{R}_{\mathcal{F}_4} *_N \mathcal{E}_4 = 0,\ \mathcal{R}_{\mathcal{H}_4} *_N \mathcal{E}_5 = 0,\ \mathcal{R}_{\widehat{\widehat{\mathcal{A}}}_{kk}} *_N \widehat{\widehat{\mathcal{E}}}_{kk} = 0,\ \widehat{\widehat{\mathcal{E}}}_{kk} *_M \mathcal{L}_{\widehat{\widehat{\mathcal{B}}}_{kk}} = 0, \tag{4.23}$$

$$\mathcal{R}_{\overline{\mathcal{M}}_{kk}} *_N \mathcal{R}_{\overline{\mathcal{F}}_{kk}} *_N \overline{\mathcal{E}}_{kk} = 0,\ \overline{\mathcal{E}}_{kk} *_M \mathcal{L}_{\overline{\mathcal{G}}_{kk}} *_M \mathcal{L}_{\overline{\mathcal{N}}_{kk}} = 0, \tag{4.24}$$

$$\mathcal{R}_{\overline{\mathcal{F}}_{kk}} *_N \overline{\mathcal{E}}_{kk} *_M \mathcal{L}_{\overline{\mathcal{J}}_{kk}} = 0,\ \mathcal{R}_{\overline{\mathcal{H}}_{kk}} *_N \overline{\mathcal{E}}_{kk} *_M \mathcal{L}_{\overline{\mathcal{G}}_{kk}} = 0,\ (k = 1,2), \tag{4.25}$$

$$\mathcal{R}_{\overline{\overline{\mathcal{M}}}_{11}} *_N \mathcal{R}_{\overline{\overline{\mathcal{F}}}_{11}} *_N \overline{\overline{\mathcal{E}}}_{11} = 0,\ \overline{\overline{\mathcal{E}}}_{11} *_M \mathcal{L}_{\overline{\overline{\mathcal{G}}}_{11}} *_M \mathcal{L}_{\overline{\overline{\mathcal{N}}}_{11}} = 0, \tag{4.26}$$

$$\mathcal{R}_{\overline{\overline{\mathcal{F}}}_{11}} *_N \overline{\overline{\mathcal{E}}}_{11} *_M \mathcal{L}_{\overline{\overline{\mathcal{J}}}_{11}} = 0,\ \mathcal{R}_{\overline{\overline{\mathcal{H}}}_{11}} *_N \overline{\overline{\mathcal{E}}}_{11} *_M \mathcal{L}_{\overline{\overline{\mathcal{G}}}_{11}} = 0, \tag{4.27}$$

$$\mathcal{R}_{\widetilde{\mathcal{C}}_{jj}} *_N \widetilde{\mathcal{E}}_{jj} = 0,\ \widetilde{\mathcal{E}}_{jj} *_M \mathcal{L}_{\widetilde{\mathcal{D}}_{jj}} = 0\ (j = 1,2),\ \mathcal{R}_{\widetilde{\mathcal{H}}_{ll}} *_N \widetilde{\widetilde{\mathcal{E}}}_{ll} = 0, \tag{4.28}$$

$$\widetilde{\widetilde{\mathcal{E}}}_{ll} *_M \mathcal{L}_{\widetilde{\mathcal{G}}_{ll}} = 0,\ \mathcal{R}_{\widetilde{\mathcal{A}}} *_N \widetilde{\mathcal{E}} *_M \mathcal{L}_{\widetilde{\mathcal{B}}},\ (l = 1,2),\ \mathcal{R}_{\widetilde{\mathcal{A}}} *_N \widetilde{\mathcal{E}} *_M \mathcal{L}_{\widetilde{\mathcal{B}}} = 0. \tag{4.29}$$

Under these conditions, the general solution to system (1.3) *can be expressed as follows:*

$$\mathcal{Z}_k = \frac{\acute{\mathcal{Z}}_k + \acute{\mathcal{Z}}_k^{\eta^*}}{2}, \quad (k = \overline{1,4}), \tag{4.30}$$

where

$$\acute{\mathcal{Z}}_1 = \mathcal{F}_4^{\dagger} *_N \mathcal{E}_4 *_M (\mathcal{F}_4^{\eta^*})^{\dagger} + \mathcal{L}_{\mathcal{F}_4} *_N \mathcal{W}_1 + \mathcal{W}_2 *_M \mathcal{R}_{\mathcal{F}_4^{\eta^*}}, \tag{4.31}$$

$$\acute{\mathcal{Z}}_4 = \mathcal{H}_4^{\dagger} *_N \mathcal{E}_5 *_M (\mathcal{H}_4^{\eta^*})^{\dagger} + \mathcal{L}_{\mathcal{H}_4} *_N \acute{\mathcal{W}}_1 + \mathcal{W}_3 *_M \mathcal{R}_{\mathcal{H}_4^{\eta^*}}, \tag{4.32}$$

$$\begin{aligned}\acute{\mathcal{Z}}_2 = {} & \widehat{\mathcal{M}}_1^{\dagger} *_N \widehat{\mathcal{E}}_1 *_M (\mathcal{H}_1^{\eta^*})^{\dagger} + \widehat{\mathcal{S}}_1^{\dagger} *_N \widehat{\mathcal{S}}_1 *_N \mathcal{H}_1^{\dagger} *_N \mathcal{E}_1 *_N \widehat{\mathcal{N}}_1^{\dagger} + \mathcal{L}_{\widehat{\mathcal{M}}_1} *_N \mathcal{L}_{\widehat{\mathcal{S}}_1} *_N \widehat{\mathcal{U}}_1 \\ & + \mathcal{L}_{\widehat{\mathcal{M}}_1} *_N \widehat{\mathcal{U}}_2 *_M \mathcal{R}_{\widehat{\mathcal{N}}_1} + \widehat{\mathcal{U}}_3 *_M \mathcal{R}_{\mathcal{H}_1^{\eta^*}},\end{aligned} \tag{4.33}$$

$$\begin{aligned}\text{or } \acute{\mathcal{Z}}_2 = {} & \mathcal{F}_2^{\dagger} *_N \mathcal{E}_2 *_M (\mathcal{F}_2^{\eta^*})^{\dagger} - \mathcal{F}_2^{\dagger} *_N \mathcal{H}_2 *_N \widehat{\mathcal{M}}_2^{\dagger} *_N \mathcal{E}_2 *_N (\mathcal{F}_2^{\eta^*})^{\dagger} - \mathcal{F}_2^{\dagger} *_N \widehat{\mathcal{S}}_2 *_N \mathcal{H}_2^{\dagger} \\ & *_N \mathcal{E}_2 *_N \widehat{\mathcal{N}}_2^{\dagger} *_M \mathcal{H}_2^{\eta^*} *_N (\mathcal{F}_2^{\eta^*})^{\dagger} - \mathcal{F}_2^{\dagger} *_N \widehat{\mathcal{S}}_2 *_N \widehat{\mathcal{V}}_2 *_M \mathcal{R}_{\widehat{\mathcal{N}}_2} *_M \mathcal{H}_2^{\eta^*} *_M (\mathcal{G}_2^{\eta^*})^{\dagger} \\ & + \mathcal{L}_{\mathcal{F}_2} *_N \widehat{\mathcal{V}}_4 + \widehat{\mathcal{V}}_5 *_M \mathcal{R}_{\mathcal{F}_2},\end{aligned} \tag{4.34}$$

$$\begin{aligned}\acute{\mathcal{Z}}_3 = {} & \widehat{\mathcal{M}}_2^{\dagger} *_N \mathcal{E}_2 *_N (\mathcal{H}_2^{\eta^*})^{\dagger} + \widehat{\mathcal{S}}_2^{\dagger} *_N \widehat{\mathcal{S}}_2 *_N \mathcal{H}_2^{\dagger} *_N \mathcal{E}_2 *_M \widehat{\mathcal{N}}_2^{\dagger} + \mathcal{L}_{\widehat{\mathcal{M}}_2} *_N \mathcal{L}_{\widehat{\mathcal{S}}_2} *_N \widehat{\mathcal{V}}_1 \\ & + \mathcal{L}_{\widehat{\mathcal{M}}_2} *_N \widehat{\mathcal{V}}_2 *_M \mathcal{R}_{\widehat{\mathcal{N}}_2} + \widehat{\mathcal{V}}_3 *_M \mathcal{R}_{\mathcal{H}_2^{\eta^*}},\end{aligned} \tag{4.35}$$

$$\begin{aligned}\text{or } \acute{\mathcal{Z}}_3 = {} & \mathcal{F}_3^{\dagger} *_N \mathcal{E}_3 *_M (\mathcal{F}_3^{\eta^*})^{\dagger} - \mathcal{F}_3^{\dagger} *_N \mathcal{H}_3 *_N \widehat{\mathcal{M}}_3^{\dagger} *_N \mathcal{E}_3 *_N (\mathcal{F}_3^{\eta^*})^{\dagger} - \mathcal{F}_3^{\dagger} *_N \widehat{\mathcal{S}}_3 *_N \mathcal{H}_3^{\dagger} \\ & *_N \mathcal{E}_3 *_N \widehat{\mathcal{N}}_3^{\dagger} *_N (\mathcal{H}_3^{\eta^*}) *_N (\mathcal{F}_3^{\eta^*})^{\dagger} - \mathcal{F}_3^{\dagger} *_N \widehat{\mathcal{S}}_3 *_N \widehat{\mathcal{K}}_2 *_M \mathcal{R}_{\widehat{\mathcal{N}}_3} *_M \mathcal{H}_3^{\eta^*} *_N (\mathcal{F}_3^{\eta^*})^{\dagger} \\ & + \mathcal{L}_{\mathcal{F}_3} *_N \widehat{\mathcal{K}}_4 + \widehat{\mathcal{K}}_5 *_M \mathcal{R}_{\mathcal{F}_3^{\eta^*}},\ (i = \overline{1,3}).\end{aligned} \tag{4.36}$$

Where the arbitrary tensors $\mathcal{W}_j$, $\widehat{\mathcal{V}}_i$, $\widehat{\mathcal{U}}_j$, $\widehat{\mathcal{K}}_k$ *and* $\acute{\mathcal{W}}_1$ $(j = \overline{1,3},\ i = \overline{1,5},\ k \in \{2,4,5\})$ *can be reduced by* (3.23)-(3.26q) *under the definitions* (4.20a)-(4.21d).

Proof. Consider the following quaternion system of tensor equations:

$$\begin{cases} \mathcal{F}_4 *_N \acute{\mathcal{Z}}_1 *_N \mathcal{F}_4^{\eta^*} = \mathcal{E}_4, \\ \mathcal{F}_i *_N \acute{\mathcal{Z}}_i *_N \mathcal{F}_i^{\eta^*} + \mathcal{H}_i *_N \acute{\mathcal{Z}}_{i+1} *_N \mathcal{H}_i^{\eta^*} = \mathcal{E}_i, \\ \mathcal{H}_4 *_N \acute{\mathcal{Z}}_4 *_N \mathcal{H}_4^{\eta^*} = \mathcal{E}_5, \end{cases} \tag{4.37}$$

where $(i = \overline{1,3})$. Suppose that the system (1.5) is consistent. Claim that $(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4)$ is a solution to the quaternion system of tensor equations (1.5), then it is evident that $(\acute{\mathcal{Z}}_1, \acute{\mathcal{Z}}_2, \acute{\mathcal{Z}}_3, \acute{\mathcal{Z}}_4)$

$= (\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4)$ is a solution to the system (4.37). Conversely, if the system (4.37) has a solution $(\acute{\mathcal{Z}}_1, \acute{\mathcal{Z}}_2, \acute{\mathcal{Z}}_3, \acute{\mathcal{Z}}_4)$. It is sufficient to show that

$$(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4) = \left(\frac{\acute{\mathcal{Z}}_1 + \acute{\mathcal{Z}}_1^{\eta^*}}{2}, \frac{\acute{\mathcal{Z}}_2 + \acute{\mathcal{Z}}_2^{\eta^*}}{2}, \frac{\acute{\mathcal{Z}}_3 + \acute{\mathcal{Z}}_3^{\eta^*}}{2}, \frac{\acute{\mathcal{Z}}_4 + \acute{\mathcal{Z}}_4^{\eta^*}}{2} \right), \tag{4.38}$$

is a solution to system (1.5). Clearly, the quaternion tensors $\mathcal{Z}_i$, $(i = \overline{1,4})$ are η-Hermitian tensors. By Applying (4.38) on the system (1.5) yields:

$$\begin{aligned} &\mathcal{F}_4 *_N \mathcal{Z}_1 *_N \mathcal{F}_4^{\eta^*} = \mathcal{F}_4 *_N \left(\frac{\acute{\mathcal{Z}}_1 + \acute{\mathcal{Z}}_1^{\eta^*}}{2} \right) *_N F_4^{\eta^*} \\ &= \frac{1}{2} \mathcal{F}_4 *_N \acute{\mathcal{Z}}_1 *_N \mathcal{F}_4^{\eta^*} + \frac{1}{2} \left(\mathcal{F}_4 *_N \acute{\mathcal{Z}}_1 *_N \mathcal{F}_4^{\eta^*} \right)^{\eta^*} = \mathcal{E}_4. \end{aligned}$$

Similarly, it can be shown that

$$\mathcal{H}_4 *_N \mathcal{Z}_4 *_N \mathcal{H}_4^{\eta^*} = \mathcal{E}_5.$$

Moreover,

$$\begin{aligned} &\mathcal{F}_i *_N \mathcal{Z}_i *_N \mathcal{F}_i^{\eta^*} + \mathcal{H}_i *_N \mathcal{Z}_{i+1} *_N \mathcal{H}_i^{\eta^*} \\ &= \mathcal{F}_i *_N \left(\frac{\acute{\mathcal{Z}}_i + \acute{\mathcal{Z}}_i^{\eta^*}}{2} \right) *_N \mathcal{F}_i^{\eta^*} + \mathcal{H}_i *_N \left(\frac{\acute{\mathcal{Z}}_{i+1} + \acute{\mathcal{Z}}_{i+1}^{\eta^*}}{2} \right) *_N \mathcal{H}_i^{\eta^*} \\ &= \frac{1}{2} \left[\mathcal{F}_i *_N \acute{\mathcal{Z}}_i *_N \mathcal{F}_i^{\eta^*} + \mathcal{H}_i *_N \acute{\mathcal{Z}}_{i+1} *_N \mathcal{H}_i^{\eta^*} \right] \\ &+ \frac{1}{2} \left[\mathcal{F}_i *_N \acute{\mathcal{Z}}_i *_N \mathcal{F}_i^{\eta^*} + \mathcal{H}_i *_N \acute{\mathcal{Z}}_{i+1} *_N \mathcal{H}_i^{\eta^*} \right]^{\eta^*} = \mathcal{E}_i, \ (i = \overline{1,3}). \end{aligned}$$

Therefore, (4.38) is a solution to the system (1.5). Consequently, apply *Theorem* 4.1, on the system (4.37), we can establish the solvability conditions and the general solution to the quaternion system (1.5). □

5. **Conclusion**

Having first established the necessary and sufficient conditions for the presence of a solution to (1.3), we, therefore, manifest an expression of its general solution. If $\mathcal{A}_i = \mathcal{D}_i = 0$ in (1.6), where $(i = \overline{1,3})$, we obtain the Sylvester-like quaternion system of tensor equations (1.4). As an application of system (1.4), we investigate an η-Hermitian solution to system (1.5). We also construct a numerical example to validate the system (1.3). It is notable that the primary conclusions of this study are particularly beneficial for the corresponding systems over the real and complex number fields. These conclusions can also obtain the corresponding matrix equation systems to (1.3)-(1.5).

All results are valid over an arbitrary division ring. As a direct consequence, the corresponding systems of quaternion matrix equations to the systems (1.3), (1.4), (1.5), and (1.6) can be described by rank equalities and Moore-Penrose inverses of matrices whenever $N = M = 1$.

In further future work, we infer the solvability constraints to the n-system of the matrix equations

$$\begin{cases} A_1X_1B_1 + C_1Y_1D_1 + C_1(G_1Z_1F_1 + H_1Z_2J_1)B_1 = E_1 \\ A_2X_2B_2 + C_2Y_2D_2 + C_2(G_2Z_2F_2 + H_2Z_3J_2)B_2 = E_2 \\ \vdots \\ A_nX_nB_n + C_nY_nD_n + C_n(G_nZ_nF_n + H_nZ_{n+1}J_n)B_n = E_n \end{cases}$$

can be characterized by rank equalities and Moore-Penrose inverses of some known matrices and hence, we can derive a formula of its general solution. Moreover, we intend to study that system over an arbitrary regular ring.

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