ERGODIC BEHAVIORS OF COMPOSITION OPERATORS ACTING ON SPACE OF BOUNDED HOLOMORPHIC FUNCTIONS

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Abstract. We completely characterize the mean ergodic composition operators on $H^\infty(\mathbb{B}_n)$. In particular, we show that a composition operator acting on this space is mean ergodic if and only if it is uniformly mean ergodic.
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1. Introduction and main results

The purpose of this paper is to prove the following theorem:

Theorem 1.1. *Let φ be a holomorphic self-map of $\mathbb{B}_n$. Then, the following statements are equivalent.*

(i) C_φ *is mean ergodic on* $H^\infty(\mathbb{B}_n)$.
(ii) C_φ *is uniformly mean ergodic on* $H^\infty(\mathbb{B}_n)$.
(iii) φ *has a fixed point in* $\mathbb{B}_n$ *and there is a* $k \in \mathbb{N}$ *such that* $\|\varphi_{kj} - \rho_\varphi\|_\infty \to 0$ *as* $j \to \infty$.

Where ρ_φ is the holomorphic retraction associated with φ and is defined below. We prove this theorem in two parts: Theorems 1.3 and 1.4. Moreover, Theorem 1.2 plays a key role in our method. However, we believe that Theorem 1.2 has an independent interest.

Throughout the paper, n is a fixed positive integer. Here is some notations:

- $\mathbb{C}$: the complex plane.
- $\mathbb{B}_n = \{z \in \mathbb{C}^n : \ |z| < 1\}$: the unit ball of $\mathbb{C}^n$.
- $\mathbb{D} = \mathbb{B}_1$: the unit disk in $\mathbb{C}$.
- $H(\mathbb{B}_n)$: the space of all holomorphic functions from $\mathbb{B}_n$ into $\mathbb{C}$
- $H^\infty(\mathbb{B}_n)$: the subspace of all bounded functions in $H(\mathbb{B}_n)$.
- $Hol(\mathbb{B}_n, \mathbb{B}_n)$: the set of all holomorphic self-maps of $\mathbb{B}_n$

Consider $\varphi \in Hol(\mathbb{B}_n, \mathbb{B}_n)$. The iterates of φ are the functions $\varphi_k := \varphi \circ \overset{(k)}{\cdots} \circ \varphi$. We denote by φ^i, $1 \le i \le n$ the components of φ, that is, $\varphi = (\varphi^1, ...\varphi^n)$ where $\varphi^i : \mathbb{B}_n \to \mathbb{C}$ are holomorphic functions. Moreover, the composition operator C_φ on $H(\mathbb{B}_n)$ is defined as $C_\varphi f = f \circ \varphi$.

When we say that $\rho \in Hol(\mathbb{B}_n, \mathbb{B}_n)$ is holomorphic retraction, it means that it is an idempotent, that is, $\rho_2 = \rho$. Clearly, if $\varphi : \mathbb{B}_n \to \mathbb{B}_n$ be holomorphic such that the sequence of its iterates converges to a holomorphic function $h : \mathbb{B}_n \to \mathbb{B}_n$. Then, $h_2 = h$, that is, h is a holomorphic retraction of $\mathbb{B}_n$. For more details about the holomorphic self-maps of the unit ball and their iterates see [1, Chapter 2].

Let $\varphi : \mathbb{B}_n \to \mathbb{B}_n$ be holomorphic and have an interior fixed point. Then, from [1, Theorem 2.1.29 and Proposition 2.2.30], there exist a unique submanifold M_φ of $\mathbb{B}_n$ and a unique holomorphic retraction $\rho_\varphi : \mathbb{B}_n \to M_\varphi$ such that every limit point $h \in Hol(\mathbb{B}_n, \mathbb{B}_n)$ of $\{\varphi_j\}$ is of the form $h = \gamma \circ \rho_\varphi$, where γ is an automorphism of M_φ. Moreover, even ρ_φ is a limit point of the sequence $\{\varphi_j\}$. This implies that $\rho_\varphi \circ \varphi = \varphi \circ \rho_\varphi$.

Let $\{e_1, ..., e_n\}$ be the standard basis of $\mathbb{C}^n$.

Theorem 1.2. *Let φ be a holomorphic self-map of the unit ball with converging iterates and $\varphi(0) = 0$. Then, there is an invertible matrix V so that:*

$$V^{-1}\varphi_j V = \Big((V^{-1}\varphi_j V)^1, ..., (V^{-1}\varphi_j V)^s\Big) \oplus P_{n-s},$$

where $\dim M_\varphi = n - s$, the functions $(V^{-1}\varphi_j V)^1$,...,$(V^{-1}\varphi_j V)^s$ are the components of $V^{-1}\varphi_j V$, and P_{n-s} is the orthogonal projection from $\mathbb{C}^n$ onto $U = e_{s+1} \oplus ... \oplus e_n$. Moreover, $V^{-1}\varphi_j V$ coverges to P_{n-s} uniformly on the compact subsets of $V^{-1}\mathbb{B}_n$.

Let X be a Banach space and $T : X \to X$ be an operator. Then, We say that T is mean ergodic if

$$M_j(T) = \frac{1}{j}\sum_{i=1}^{j} T^i.$$

converges to a bounded operator defined on X for the strong operator topology. Uniformly mean ergodicity will define in a same way with convergence in the operator norm.

Lotz [13] proved that: If X is a Grothendieck Banach space with Dunford-Pettis property (GDP space), and $T \in L(X)$ satisfies $\|T^n/n\| \to 0$, then T is mean ergodic if and only if it is uniformly mean ergodic. For the definition of GDP spaces see [13, Pages 208-209].

For some work on the mean ergodicity of composition operators see [2, 3, 4, 6, 10, 11, 12]. The (uniformly) mean ergodicity of composition operators on $H^\infty(\mathbb{D})$ have been characterized in [4]. It is well-known that $H^\infty(\mathbb{D})$ is a GDP space. Thus, a composition operator, acting on $H^\infty(\mathbb{D})$, is mean ergodic if and only if it is uniformly mean ergodic. However, we do not know whether $H^\infty(\mathbb{B}_n)$ is a GDP space or not. In [12], the first author has proved that if φ is a holomorphic self-map of the unit ball with converging iterates and an interior fixed point, then the mean ergodicity and the uniformly mean ergodicity of C_φ are equivalent. In the following theorem, we give this equivalence for all $\varphi \in Hol(\mathbb{B}_n, \mathbb{B}_n)$ with an interior fixed point.

Theorem 1.3. *Let φ be a holomorphic self-map of the unit ball with a fixed point in $\mathbb{B}_n$. Then, the following statements are equivalent.*

(i) *C_φ is mean ergodic on $H^\infty(\mathbb{B}_n)$.*
(ii) *C_φ is uniformly mean ergodic on $H^\infty(\mathbb{B}_n)$.*
(iii) *There is a $k \in \mathbb{N}$ such that $\|\varphi_{kj} - \rho_\varphi\|_\infty \to 0$, as $j \to \infty$.*

As the final result, we prove that every holomorphic self-map of $\mathbb{B}_n$ which has no interior fixed point induces a composition that is not mean ergodic on $H^\infty(\mathbb{B}_n)$. This theorem gives the answer to [12, Question 3.16].

Theorem 1.4. *Let the holomorphic function $\varphi : \mathbb{B}_n \to \mathbb{B}_n$ has no interior fixed point. Then, C_φ is not mean ergodic on $H^\infty(\mathbb{B}_n)$.*

2. Basic results

Every automorphism φ of $\mathbb{B}_n$ is of the form $\varphi = U\varphi_a = \varphi_b V$, where U and V are unitary matrices of $\mathbb{C}^n$ and

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle}, \qquad z \in \mathbb{B}_n, \tag{2.1}$$

where $a \neq 0$, $s_a = \sqrt{1-|a|^2}$, P_a is the projection from $\mathbb{C}^n$ onto the subspace $\langle a\rangle$ spanned by a, and Q_a is the projection from $\mathbb{C}^n$ onto $\mathbb{C}^n \ominus \langle a\rangle$. Clearly, $\varphi_a(0) = a$, $\varphi_a(a) = 0$, and $\varphi_a \circ \varphi_a(z) = z$. It is well-known that an automorphism φ of $\mathbb{B}_n$ is a unitary matrix of $\mathbb{C}^n$ if and only if $\varphi(0) = 0$.

Let Ω be a strongly pseudoconvex bounded domain. The infinitesimal Kobayashi metric $F_K : \Omega \times \mathbb{C}^n \to [0, \infty)$ is defined as:

$$F_K(z, w) = \inf\Big\{C > 0 :\ \exists f \in H(\mathbb{D}, \Omega)\ with\ f(0) = z,\ f'(0) = \frac{w}{C}\Big\},$$

where $H(\mathbb{D}, \Omega)$ is the space of analytic functions from $\mathbb{D}$ to Ω. Let $\gamma : [0, 1] \to \Omega$ be a C^1-curve. The Kobayashi length of γ is defined as:

$$L_K(\gamma) = \int_0^1 F_K(\gamma(t), \gamma'(t))dt.$$

For $z, w \in \Omega$, the Kobayashi metric function is defined as:

$$k_\Omega(z, w) = \inf\Big\{L_K(\gamma);\ \gamma\ is\ C^1 - curve\ with\ \gamma(0) = z\ and\ \gamma(1) = w\Big\}.$$

If Ω and Λ are two strongly pseudoconvex bounded domains and $\varphi : \Omega \to \Lambda$ is a holomorphic function, then from [1, Proposition 2.3.1], we have:

$$k_\Lambda(\varphi(z), \varphi(w)) \le k_\Omega(z, w), \qquad \forall z, w \in \Omega. \tag{2.2}$$

Thus, k_Ω is invariant under automorphisms, that is,

$$k_\Omega(\varphi(z), \varphi(w)) = k_\Omega(z, w),$$

for all $z, w \in \mathbb{B}_n$ and $\varphi : \Omega \to \Omega$ is an automorphism.

Let β from $\mathbb{B}_n \times \mathbb{B}_n$ to $[0, \infty)$ be the Bergman metric. From [1, Corollary 2.3.6], the Kobayashi metric and the Bergman metric coincide on $\mathbb{B}_n$. We have:

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}, \qquad z, w \in \mathbb{B}_n. \tag{2.3}$$

We shall denote by $B(a, r)$ the Bergman ball centered at $a \in \mathbb{B}_n$ with radius $r > 0$, that is,

$$B(a, r) = \{z \in \mathbb{B}_n :\ \beta(a, z) < r\}.$$

It is well-known (see [1, page 134]) that $B(a, r)$ is the ellipsoid

$$\frac{|P_a(\zeta) - a_r|^2}{R^2 s^2} + \frac{|Q_a(\zeta)|^2}{R^2 s} < 1, \tag{2.4}$$

where $R = \tanh r$, $a_r = \frac{1-R^2}{1-R^2|a|^2} a$ and $s = \frac{1-|a|^2}{1-R^2|a|^2}$.

Let P_k be the space homogeneous polynomial $P : \mathbb{B}_n \to \mathbb{C}$ of degree k. The Taylor series expansions of functions in $H^\infty(\mathbb{B}_n)$ yield a direct sum decomposition of

$$H^\infty(\mathbb{B}_n) = P_0 \oplus P_1 \oplus ... \oplus P_m \oplus R_m;$$

where the remaining space R_m consists of the functions $h \in H^\infty(\mathbb{B}_n)$ such that $|h(z)|/\|z\|^m$ is bounded for z near 0. Similarly, $f : \mathbb{B}_n \to \mathbb{C}^n$ admits a homogeneous expansion:

$$f(z) = \sum_{k=0}^{\infty} F_k(z) = f(0) + f'(0)z + ...,$$

where all n component functions of each F_k are homogeneous polynomial of degree k.

It should be noted that $d_z\varphi = \varphi'(z)$. Note that $d_z\varphi$ is a matrix:

$$d_z\varphi := \begin{bmatrix} \frac{\partial \varphi^1}{\partial z_1} & \cdots & \frac{\partial \varphi^1}{\partial z_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial \varphi^n}{\partial z_1} & \cdots & \frac{\partial \varphi^n}{\partial z_n} \end{bmatrix} (z).$$

3. Proof of Theorem 1.2

Let $n - s$ be the dimension of M_φ.

If $s = 0$, then from [1, Proposition 2.2.14] and [12, Proposition 3.8], φ is a unitary matrix. Since the iterates of φ are convergent, φ is the identity matrix. If $s = n$, then from [1, Theorem 2.2.32], $M_\varphi = \{0\}$ and $\rho_\varphi \equiv 0$. Therefore, for $s = 0$ or n, the result is obtained by considering V as the identity matrix.

Thus, let $1 \le s \le n - 1$. We give the proof in three steps:

Step 1. There is an invertible matrix V so that $V^{-1} d_0\rho V = P_{n-s}$.

Proof. Recall that P_{n-s} is the orthogonal projection from $\mathbb{C}^n$ onto $e_{s+1} \oplus ... \oplus e_n$.

Let V be an invertible matrix so that $V^{-1}d_0\rho V$ be the Jordan canonical form of $d_0\rho$. Since, $\rho^2 = \rho$ and $\rho(0) = 0$, the matrix $d_0\rho$ is also an idempotent. Thus, the eigenvalues of $V^{-1}d_0\rho V$ are in $\{0, 1\}$. Note that since $\rho(\mathbb{B}_n) = M$ and ρ is identity on M, it is easy to show that 0 and 1 will be repeated s and $n - s$ times as the eigenvalues of $d_0\rho$, respectively.

We have

$$V^{-1}d_0\rho V = J_1(0) \oplus ... \oplus J_k(0) \oplus I_1(1) \oplus ... \oplus I_l(1),$$

where $J_i(0)$ and $I_i(1)$ are the blocks associated with the eigenvalues 0 and 1, respectively. Now since $d_0\rho$ is an idempotent, the blocks $J_i(0)$ and $I_i(1)$ must be 1×1. That is,

$$V^{-1}d_0\rho V = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-s} \end{bmatrix},$$

where I_{n-s} is the $(n - s) \times (n - s)$ identity matrix. Hence, $V^{-1}d_0\rho V = P_{n-s}$. □

From [1, Theorem 2.1.21], we know that if $f : \mathbb{B}_n \to \mathbb{B}_n$ is holomorphic, $f(0) = 0$ and d_0f is identity, then so is f. In the next step, we want to show that if $d_0f = 0_s \oplus I_{n-s}$, then $f = 0_s \oplus I_{n-s}$.

Step 2. For the matrix V, obtained in step 1, we have $V^{-1}\rho V = P_{n-s}$.

Proof. Let $V^{-1}\rho V \neq P_{n-s}$. Consider the function $\psi = V^{-1}\rho V - P_{n-s} : \mathbb{B}_n \to \mathbb{C}^n$. Since $d_0\psi = V^{-1}d_0\rho V - d_0P_{n-s} = 0$, $\psi(0) = 0$, but $\psi \neq 0$, we can write:

$$V^{-1}\rho V(z) = P_{n-s}(z) + F_k(z) + \sum_{j=k+1}^{\infty} F_j(z),$$

where F_k is a homogeneous polynomial of degree $k \geq 2$. In summation, F_j is zero or a homogeneous polynomial of degree j.

Note that every component of a homogeneous polynomial of degree j is a summation of polynomials

$$z^m = z_1^{m_1}...z_n^{m_n},$$

where $z = (z_1, ..., z_n)$, $m = (m_1, ..., m_n) \in \mathbb{N}^n$, and $m_1 + ... + m_n = j$. Thus, for $j \geq k$ if $F_j = (F_j^1, ..., F_j^n)$ is non-zero, then each component of $F_j(V^{-1}\rho V(z))$ is a summation of polynomials

$$\begin{aligned}\Big(P_{n-s}(z) + F_k(z) + \sum_{j=k+1}^{\infty} F_j(z)\Big)^m \\ = \Big(F_k^1(z) + \sum_{j=k+1}^{\infty} F_j^1(z)\Big)^{m_1} ... \Big(F_k^s(z) + \sum_{j=k+1}^{\infty} F_j^s(z)\Big)^{m_s} \\ \times \Big(z_{s+1} + F_k^{s+1}(z) + \sum_{j=k+1}^{\infty} F_j^{s+1}(z)\Big)^{m_{s+1}} ... \Big(z_n + F_k^n(z) + \sum_{j=k+1}^{\infty} F_j^n(z)\Big)^{m_n}\end{aligned}$$

Thus, from the above statement and the assumption $1 \leq s \leq n-1$, if F_j is non-zero for $j \geq k$, then each component of $F_j(V^{-1}\rho V(z))$ is a polynomial with a degree greater than or equal to:

$$km_1 + ... + km_s + m_{s+1} + ... + m_n.$$

On the other hand, since $k \geq 2$, we have

$$km_1 + ... + km_s + m_{s+1} + ... + m_n > \sum_{i=1}^{n} m_i = j.$$

Thus,

$$V^{-1}\rho^2 V(z) = P_{n-s}(z) + P_{n-s}F_k(z) + \sum_{j=k+1}^{\infty} G_j(z),$$

where each G_j is zero or a homogeneous polynomial of degree j. Since $\rho^2 = \rho$, we must have $F_k = P_{n-s}F_k$ which contradicts the assumption that $s \neq 0, n$. □

Indeed, we proved the following result in steps 1 and 2 as well as the paragraph before them:

Corollary 3.1. *Every holomorphic retraction ρ on $\mathbb{B}_n$ which fixes the origin is a matrix.*

Step 3. $(V^{-1}\varphi_j V)^i(z^1, ..., z^n) = z^i$, for $i = s+1, ..., n$ and $j \in \mathbb{N}$.

Proof. From step 2,

$$V^{-1}(\rho \circ \varphi)V = V^{-1} \circ \rho \circ V(V^{-1} \circ \varphi \circ V) = 0_s \oplus \begin{bmatrix} (V^{-1}\varphi V)^{s+1}(z) \\ \vdots \\ (V^{-1}\varphi V)^n(z) \end{bmatrix}. \tag{3.1}$$

Moreover, since ρ and $\varphi \circ \rho$ are the limit points of the convergent sequence $\{\varphi_j\}$, we have:

$$V^{-1}\rho \circ \varphi V = V^{-1}\varphi \circ \rho V = V^{-1}\rho V. \tag{3.2}$$

Thus, 3.1, 3.2, and step 2 imply that

$$\begin{bmatrix} (V^{-1}\varphi V)^{s+1}(z) \\ \vdots \\ (V^{-1}\varphi V)^{n}(z) \end{bmatrix} = \begin{bmatrix} z^{s+1} \\ \vdots \\ z^{n} \end{bmatrix}.$$

Again, by a similar argument, we can see that $\rho \circ \varphi_j = \varphi_j \circ \rho = \rho$. Thus,

$$\begin{bmatrix} (V^{-1}\varphi_j V)^{s+1}(z) \\ \vdots \\ (V^{-1}\varphi_j V)^{n}(z) \end{bmatrix} = \begin{bmatrix} z^{s+1} \\ \vdots \\ z^{n} \end{bmatrix}.$$

The proof is complete. □

4. Proof of Theorem 1.3

If φ has an interior fixed point $a \in \mathbb{B}$, then $\psi := \varphi_a \circ \varphi \circ \varphi_a$ is a holomorphic self-map of $\mathbb{B}_n$ that $\psi(0) = 0$. Hence, without loss of generality, we assume that $\varphi(0) = 0$. (ii)$\Rightarrow$ (i) is obvious.

4.1. **(iii)$\Rightarrow$ (ii).** Since $\varphi_{kj} \to \rho$, from Theorem (1.2), there is an invertible matrix V so that

$$V^{-1}\varphi_{kj}V = \Big((V^{-1}\varphi_{kj}V)^1, ..., (V^{-1}\varphi_{kj}V)^s\Big) \oplus P_{n-s},$$

and

$$V^{-1}\rho V = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-s} \end{bmatrix} = P_{n-s}.$$

From the continuity of V^{-1} and (iii), there is a $C > 0$ so that

$$\begin{aligned} \lim_{j\to\infty} \sup_{z\in V^{-1}\mathbb{B}_n} &\Big|((V^{-1}\varphi_{kj}V)^1, ..., (V^{-1}\varphi_{kj}V)^s)(z)\Big| \\ &= \lim_{j\to\infty} \|V^{-1}(\varphi_{kj} - \rho)\|_\infty \\ &\leq C \lim_{j\to\infty} \|\varphi_{kj} - \rho\|_\infty = 0. \end{aligned} \tag{4.1}$$

It is easy to see that $V^{-1}\mathbb{B}_n$ is a taut manifold. Thus, from 2.2 we have:

$$\begin{aligned} \sup_{z\in\mathbb{B}_n} \beta(\varphi_{kj}(z), \rho(z)) &\leq \sup_{z\in\mathbb{B}_n} k_{V^{-1}\mathbb{B}_n}(V^{-1}\varphi_{kj}(z), V^{-1}\rho(z)) \\ &= \sup_{z\in V^{-1}\mathbb{B}_n} k_{V^{-1}\mathbb{B}_n}(V^{-1}\varphi_{kj}V(z), V^{-1}\rho V(z)). \end{aligned}$$

Hence, from [12, Lemma 4.1] and Equation 4.1, we obtain:

$$\begin{aligned} \sup_{z\in\mathbb{B}_n} \beta(\varphi_{kj}(z), \rho(z)) &\leq \sup_{z\in V^{-1}\mathbb{B}_n} \omega(\Big|((V^{-1}\varphi_{kj}V)^1, ..., (V^{-1}\varphi_{kj}V)^s)(z)\Big|, 0) \\ &= \frac{1}{2} \sup_{z\in V^{-1}\mathbb{B}_n} \tanh^{-1}(\Big|((V^{-1}\varphi_{kj}V)^1, ..., (V^{-1}\varphi_{kj}V)^s)(z)\Big|) \to 0, \end{aligned}$$

as $j \to \infty$. Therefore, (ii) follows from [12, Theorem 3.6].

4.2. **(i)⇒ (iii).** Before presenting the proof, we state some auxiliary results.

For $k > 0$ and $\zeta \in \partial\mathbb{B}_n$, we define the ellipsoid

$$E(k, \zeta) = \{z \in \mathbb{B}_n : \ |1 - \langle z, \zeta\rangle|^2 \leq k(1 - |z|^2)\}.$$

Let ρ be a holomorphic self-map of the unit ball and $\eta > 0$. Set

$$L(\rho, \eta) = \{z \in \mathbb{B}_n, \quad \beta(z, \rho(z)) \geq \eta\}.$$

The following lemma is an extension of [12, Lemma 3.9]. Since the proof is the same, we omit it.

Lemma 4.1. *Let φ be a holomorphic self-map of the unit ball, $\varphi(0) = 0$, and ρ be the holomorphic retraction associated with φ. If $\eta > 0$ be such that $L(\rho, \eta) \neq \emptyset$, then there is some $A > 1$ such that*

$$\frac{1 - |\varphi(z)|}{1 - |z|} > A, \qquad \forall z \in L(\rho, \eta).$$

Proposition 4.2. $\beta(z, w) \geq \dfrac{1}{2}|z - w|$, *for all $z, w \in \mathbb{B}_n$.*

Proof. The case $z = w$ is clear. Let $z \neq w$. Then $\beta(z, w) = r > 0$. Note from (2.4) that $B(w, r)$ is the ellipsoid

$$\frac{|P_w(\zeta) - w_R|^2}{R^2 s^2} + \frac{|Q_w(\zeta)|^2}{R^2 s} < 1,$$

where

$$R = \tanh r = \frac{e^r - e^{-r}}{e^r + e^{-r}} < 1,$$

$w_R = \frac{1-R^2}{1-R^2|w|^2} w$ and $s = \frac{1-|w|^2}{1-R^2|w|^2} < 1$. Thus,

$$\frac{|P_w(z) - w_R|^2}{R^2 s^2} + \frac{|Q_w(z)|^2}{R^2 s} = 1,$$

Since $s < 1$ and $Q_w(z)$ is orthogonal to $P_w(z)$ and $P_w(z) - w_R$, we obtain

$$\begin{aligned}
|z - w_R|^2 &= |P_w(z) - w_R|^2 + |Q_w(z)|^2 \\
&= R^2 s\Big(\frac{|P_w(z) - w_R|^2}{R^2 s} + \frac{|Q_w(z)|^2}{R^2 s}\Big) \\
&< R^2 s\Big(\frac{|P_w(z) - w_R|^2}{R^2 s^2} + \frac{|Q_w(z)|^2}{R^2 s}\Big) = R^2 s.
\end{aligned}$$

From the mean value theorem, there is a $0 \leq t \leq r$ so that:

$$R = \tanh r = r sech^2 t \leq r.$$

Note that the last inequality comes from $sech\ t = \frac{2}{e^t + e^{-t}} \leq 1$.

Combining the above estimates, we deduce that:

$$\begin{aligned}
|z - w| &\leq |z - w_R| + |w_R - w| \\
&< R\sqrt{s} + R^2\Big(\frac{1 - |w|^2}{1 - R^2|w|^2}\Big) \\
&< 2R \leq 2r = 2\beta(z, w).
\end{aligned}$$

The proof is complete. ∎

Now we proceed to the proof of (i)⇒ (iii). From [12, Lemma 3.3], there is a positive integer k so that $\varphi_{kj} \to \rho$ uniformly on the compact subsets of $\mathbb{B}_n$ and

$$\lim_{j\to\infty} M_j(C_\varphi) = \frac{1}{k}\sum_{i=0}^{k-1} C_{\rho\circ\varphi_i}. \tag{4.2}$$

for the strong operator topology. Let (iii) not hold.

Claim 4.3. *There is an $\varepsilon > 0$ so that $\|\varphi_{kj} - \rho\|_\infty \geq \varepsilon$ for all j.*

Proof. Since (iii) does not hold, there is a sequence m_j in $\mathbb{N}$ such that $\|\varphi_{km_j} - \rho\|_\infty \geq \varepsilon$ for all j. Consider an arbitrary positive integer j. Then, there is a j_0 so that $m_{j_0} \geq j$. Thus, from the fact that $\rho\circ\varphi_{kl} = \rho$ for all $l \in \mathbb{N}$, we have:

$$\begin{aligned}\varepsilon &\leq \|\varphi_{km_{j_0}} - \rho\|_\infty \\ &= \|\varphi_{kj}\circ\varphi_{k(m_{j_0}-j)} - \rho\circ\varphi_{k(m_{j_0}-j)}\|_\infty \\ &= \sup_{z\in\mathbb{B}_n} |(\varphi_{kj} - \rho)(\varphi_{k(m_{j_0}-j)}(z))| \\ &\leq \|\varphi_{kj} - \rho\|_\infty.\end{aligned}$$

The proof is complete. □

Claim 4.4. *For every $0 < r < 1$, we can find $a \in \mathbb{B}_n$ and $m \in \mathbb{N}$ such that:*

$$|\varphi_{2km}(a) - \rho(a)| \geq \varepsilon, \quad \textit{and} \quad |\varphi_{km}(a)| > r.$$

Proof. If the claim do not hold, then there is an $0 < r < 1$ so that

$$\sup\{|\varphi_{2kj}(z) - \rho(z)|;\ z\in\mathbb{B}_n,\ |\varphi_{kj}(z)| > r\} \leq \varepsilon. \tag{4.3}$$

for all $j \in \mathbb{N}$. On the other hand, there is a j_0 so that

$$\sup\{|\varphi_{kj_0}(z) - \rho(z)|;\ z\in\mathbb{B}_n,\ |z| \leq r\} \leq \varepsilon. \tag{4.4}$$

We have:

$$\begin{aligned}\|\varphi_{2kj_0} - \rho\|_\infty = \max\Big\{ &\sup\{|\varphi_{2kj}(z) - \rho(z)|;\ z\in\mathbb{B}_n,\ |\varphi_{kj}(z)| > r\}, \\ &\sup\{|\varphi_{2kj}(z) - \rho(z)|;\ z\in\mathbb{B}_n,\ |\varphi_{kj}(z)| \leq r\}\Big\}.\end{aligned}$$

From (4.3), the first supremum is less than or equal to ε. For the second one, from the fact $\rho = \rho\circ\varphi_{kj_0}$ and (4.4), we have

$$\begin{aligned}&\sup\{|\varphi_{2kj_0}(z) - \rho(z)|;\ z\in\mathbb{B}_n,\ |\varphi_{kj_0}(z)| \leq r\} \\ &\qquad = \sup\{|\varphi_{kj_0}\circ\varphi_{kj_0}(z) - \rho\circ\varphi_{kj_0}(z)|;\ z\in\mathbb{B}_n,\ |\varphi_{kj}(z)| \leq r\} \\ &\qquad \leq \sup\{|\varphi_{kj_0}(z) - \rho(z)|;\ z\in\mathbb{B}_n,\ |z| \leq r\} \leq \varepsilon\end{aligned}$$

Therefore, $\|\varphi_{2kj_0} - \rho\|_\infty \leq \varepsilon$, which contradicts Claim (4.3). □

Claim 4.5. *There are two sequences $\{m_j\} \subseteq \mathbb{N}$ and $\{a_j\} \subset \mathbb{B}_n$ and some f in $H^\infty(\mathbb{B}_n)$ such that $|\varphi_{2km_j}(a_j) - \rho(a_j)| \geq \varepsilon$ for all j, and*

$$f\circ\rho \equiv 0, \quad f(\varphi_l(a_j)) = |\varphi_{2km_j}(a_j) - \rho(a_j)|^2, \qquad 1 \leq l \leq km_j,\ \forall j \in \mathbb{N}.$$

Proof. From Lemma 4.1, there is a constant $0 < a < 1$ such that if $\beta(z, \rho(z)) \geq \varepsilon/2$, then

$$\frac{1-|z|}{1-|\varphi(z)|} < a. \tag{4.5}$$

Let a_1 in $\mathbb{B}_n$ be such that $|\varphi_{2k}(a_1) - \rho(a_1)| \geq \varepsilon$. Then, from Proposition (4.2), the fact that $\rho \circ \varphi_{kl} = \rho$ and $\rho \circ \varphi_l = \varphi_l \circ \rho$ for al $l \in \mathbb{N}$, and inequality (2.2), we obtain:

$$\frac{\varepsilon}{2} \leq \beta(\varphi_{2k}(a_1), \rho(a_1)) = \beta(\varphi_{2k}(a_1), \rho \circ \varphi_{2k}(a_1)) \leq \beta(\varphi_i(a_1), \rho \circ \varphi_i(a_1)).$$

for all $1 \leq i \leq 2k$. Thus, from 4.5, we have

$$\frac{1-|\varphi_i(a_1)|}{1-|\varphi_{i+1}(a_1)|} < a, \quad 1 \leq i \leq k-1.$$

Put $m_1 = 1$. Using Claim 4.4, we can find $a_2 \in \mathbb{B}_n$ and $m_2 \in \mathbb{N}$ such that $|\varphi_{km_2}(a_2)|$ is large enough so that

$$|\varphi_{2km_2}(a_2) - \rho(a_2)| \geq \varepsilon,$$

and

$$\frac{1-|\varphi_{km_2}(a_2)|}{1-|\varphi(a_1)|} < a.$$

Again,

$$\frac{\varepsilon}{2} \leq \beta(\varphi_{2km_2}(a_2), \rho(a_2)) \leq \beta(\varphi_i(a_2), \rho \circ \varphi_i(a_2))$$

for all $0 \leq i \leq 2km_2$. Thus, from 4.5, we obtain:

$$\frac{1-|\varphi_i(a_2)|}{1-|\varphi_{i+1}(a_2)|} < a, \quad 1 \leq i \leq km_2 - 1.$$

By repeating this process we will construct the sequence

$$\begin{array}{cccc} x_1 = \varphi_k(a_1), & x_2 = \varphi_{k-1}(a_1), & \ldots, & x_{km_1} = \varphi(a_1) \\ x_{km_1+1} = \varphi_{km_2}(a_2), & x_{km_1+2} = \varphi_{km_2-1}(a_2), & \ldots, & x_{k(m_2+m_1)} = \varphi(a_2) \\ x_{k(m_2+m_1)+1} = \varphi_{km_3}(a_3), & x_{k(m_2+m_1)+2} = \varphi_{km_3-1}(a_3), & \ldots, & x_{k(m_3+m_2+m_1)} = \varphi(a_3), \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

which satisfies condition (i) of [12, Lemma 3.11]. Thus, there are some $M > 0$ and a sequence $\{f_{l,j}\}_{j,l=1}^{\infty, km_j} \subset H^\infty(\mathbb{B}_n)$ such that

(a) $f_{l,j}(\varphi_l(a_j)) = 1$, and $f_{l,j}(\varphi_r(a_s)) = 0$ whenever $l \neq r$ or $j \neq s$.

(b) $\sum_{j=1}^\infty \sum_{l=1}^{km_j} |f_{l,j}(z)| \leq M$, for all $z \in \mathbb{B}_n$.

Define

$$f(z) = \sum_{j=1}^{\infty} \sum_{l=1}^{km_j} \langle \varphi_{2km_j - l}(z) - \rho \circ \varphi_{2km_j - l}(z), \varphi_{2km_j}(a_j) - \rho(a_j) \rangle f_{l,j}(z).$$

Hence, from the Lebesgue dominated convergence theorem, (a), (b), and the fact that $\rho \circ \varphi = \varphi \circ \rho$, we deduce that $f \in H^\infty(\mathbb{B}_n)$, $f(\rho) = 0$, and

$$f(\varphi_l(a_j)) = |\varphi_{2km_j}(a_j) - \rho(a_j)|^2, \quad 1 \leq j < \infty, \ 1 \leq l \leq km_j.$$

The proof is complete. ☐

Using Claim 4.5, we have:

$$\begin{aligned}\Big\|\frac{1}{m_j}\sum_{l=1}^{m_j}C_{\varphi_l}-\frac{1}{k}\sum_{i=0}^{k-1}C_{\rho\circ\varphi_i}\Big\| &\geq \frac{1}{\|f\|_\infty}\Big\|\frac{1}{m_j}\sum_{l=1}^{m_j}C_{\varphi_l}f-\frac{1}{k}\sum_{i=0}^{k-1}C_{\rho\circ\varphi_i}f\Big\|_\infty \\ &\geq \frac{1}{\|f\|_\infty}\Big|\frac{1}{m_j}\sum_{l=1}^{m_j}f(\varphi_l(a_j))-\frac{1}{k}\sum_{i=0}^{k-1}f(\rho\circ\varphi_i(a_j)\Big| \\ &= \frac{1}{\|f\|_\infty}.\frac{1}{m_j}\sum_{l=1}^{m_j}|\varphi_{2km_j}(a_j)-\rho(a_j)|^2\geq\frac{\varepsilon^2}{\|f\|_\infty}.\end{aligned}$$

From the above estimate, we deduce that $\{M_j(C_\varphi)\}_{j=1}^\infty$ does not converge to $\frac{1}{k}\sum_{i=0}^{k-1}C_{\rho\circ\varphi_i}$ for the strong operator topology, which contradicts 4.2. Thus, (iii) holds.

5. Proof of Theorem 1.4

First, we define the sequence of operators $T_j: H^\infty(\mathbb{D})\to H^\infty(\mathbb{D})$ as follows

$$T_jf(z):=f\circ\varphi_j^1(z,0,...,0),$$

where φ_j^1 is the first component of φ_j. Note that if we consider $f\in H^\infty(\mathbb{D})$ as a function in $H^\infty(\mathbb{B}_n)$, then $T_jf=C_{\varphi_j}f$. Thus, if C_φ is mean ergodic on $H^\infty(\mathbb{B}_n)$, then

$$N_j(\varphi):=\frac{1}{j}\sum_{i=1}^{j}T_i: H^\infty(\mathbb{D})\to H^\infty(\mathbb{D}),$$

converges for the strong operator topology.

We give the proof in two steps. In the first step, we show that if $N_j(\varphi)$ is SOT-convergent, then it must converge in the norm operator. Then, in the second step, we prove that $N_j(\varphi)$ does not converge in the norm operator. Therefore, the proof will be complete.

Step 1. From the ergodic theorem, $M_j(\varphi)$ converges to a projection P so that $PC_\varphi=C_\varphi P=P$. Since $N_j(\varphi)$ converges for the strong operator topology to $P\mid_{H^\infty(\mathbb{D})}$ and $H^\infty(\mathbb{D})$ is a GDP space, from [13, Theorem 2] the spectral radius of $N_j(\varphi)-P$ converges to 0 as $j\to\infty$. That is, $I-N_j(\varphi)+P$ is invertible for a large enough j.

Now, we show that $I-T_1+P$ is bounded below. If not, then there is a sequence of unit vectors $\{f_l\}$ in $H^\infty(\mathbb{B}_n)$ so that:

$$\|(I-T_1+P)f_l\|_\infty\to 0 \qquad as\ j\to\infty.$$

Since $P=PT_1=P^2$, we obtain

$$\|Pf_l\|_\infty=\|P(I-T_1+P)f_l\|_\infty\to 0 \qquad as\ j\to\infty.$$

Thus,

$$\|(I-T_1)f_l\|_\infty\to 0 \qquad as\ j\to\infty.$$

Therefore,

$$\begin{aligned}(I - N_j(\varphi) + P)f_l &= (I - M_j(\varphi) + P)f_l \\ &= \frac{1}{n}\sum_{i=1}^{j}(I - C_{\varphi_i})f_l + Pf_l \\ &= \frac{1}{n}\sum_{i=1}^{j}(I + C_\varphi + ... + C_{\varphi_{i-1}})(I - C_\varphi)f_l + Pf_l \\ &= \frac{1}{n}\sum_{i=1}^{j}(I + T_1 + ... + T_{i-1})(I - T_1)f_l + Pf_l \to 0,\end{aligned}$$

as $l \to \infty$. This contradicts the invertibility of $I - N_j(\varphi) + P$.

Now, since $I - T_1 + P$ is bounded below, there is a bounded operator S on $H^\infty(\mathbb{D})$ so that $S(I - T_1 + P) = I$. Therefore,

$$\begin{aligned}(N_j(\varphi) - P) &= S(I - T_1 + P)(N_j(\varphi) - P) \\ &= S(I - C_\varphi + P)(M_j(\varphi) - P) \\ &= \frac{1}{j}S(C_\varphi - C_{\varphi_{j+1}}) \to 0,\end{aligned}$$

as $j \to \infty$.

Step 2. The proof of this step is similar to that of [4, Theorem 3.6] and also [12, Theorem 3.14].

From [1, Theorem 2.2.31], there is a $z_0 \in \partial\mathbb{B}_n$ such that $\varphi_j \to z_0$ uniformly on the compact subsets of $\mathbb{B}_n$. By a unitary equivalent, we can let $z_0 = e_1$. Thus, if $\varphi_j = (\varphi_j^1, ..., \varphi_j^n)$, then $\varphi_j^1 \to 1$ and $\varphi_j^i \to 0$ for $2 \leq i \leq n$ uniformly on the compact subsets of $\mathbb{B}_n$ as $j \to \infty$.

Thus, if $N_j(\varphi)$ converges in operator norm, then $N_j(\varphi) \to K_1$ on

$$A(\mathbb{D}) = H(\mathbb{D}) \cap \{f : \overline{\mathbb{D}} \to \mathbb{C},\ \textit{continuous}\},$$

where $K_1(f) = f(1)$ on $A(\mathbb{D})$. The remaining of the proof is similar to that of [4, Theorem 3.6], by considering $g(z) = \frac{1+z}{2} \in A(\mathbb{B}_n)$.

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