

NO UNIVERSAL GRAPHS AT UNCOUNTABLE REGULAR CARDINALS

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ABSTRACT. Assuming the existence of a strong cardinal, we find a model of ZFC in which for each uncountable regular cardinal λ , there is no universal graph of size λ .

1. INTRODUCTION

The existence of universal objects in an infinite cardinal is of great interest in both set theory and model theory. In this paper we concentrate on universal graphs. It is well-known that there is a countable universal graph and that if λ is an uncountable cardinal with $2^{<\lambda} = \lambda$, then there exists a universal graph of size λ . On the other hand, by a result of Shelah (see [7]), if λ is a regular cardinal with $2^\lambda = \lambda^+$, then after forcing with $\text{Add}(\lambda, \lambda^{++})$ for adding λ^{++} -many Cohen subsets to λ , there are no universal graphs of size λ^+ . Friedman and Thompson [3] obtained further results in this direction, in particular, they showed it is consistent, modulo the existence of strong cardinals, that there are no universal graphs at the successor of a singular strong limit cardinal of countable cofinality. In this paper we continue this line of research further and prove the following global result:

Theorem 1.1. *Assume GCH holds and κ is a $(\kappa + 3)$ -strong cardinal. Then there exists a generic extension W of the universe in which κ remains inaccessible and for each uncountable regular cardinal $\lambda < \kappa$, there is no universal graph of size λ , in particular the rank initial segment of W at κ is a model of ZFC in which for each uncountable regular cardinal λ , there is no universal graph of size λ .*

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This is the preliminary version of the paper which only sketches the proof. The details will appear in the next version of the paper.

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The theorem in particular answers Question 6.1 from [3]. The rest of the paper is devoted to the proof of the above theorem. In Section 2 we present some preliminaries. In Section 3 we construct the model W , and then in Section 4 we show that W is as required.

2. SOME PRELIMINARIES

We will start by stating some general non-existing results for universal graphs.

Lemma 2.1. (*Shelah, see [7]*) *Suppose λ is a regular cardinal and $2^\lambda = \lambda^+$. Then in the generic extension by $\text{Add}(\lambda, \mu)$, for $\text{cf}(\mu) \geq \lambda^{++}$, there are no universal graphs of size λ^+ .*

Friedman and Thompson [3] isolated the main properties used in the proof of the above lemma and they have given an abstract lemma extending the above lemma, we will need the following special case.

Lemma 2.2. (*see [3]*) *Suppose λ is a regular cardinal and $2^\lambda = \lambda^+$. Then in the generic extension by $\text{Sacks}(\lambda, \mu)$, for $\text{cf}(\mu) \geq \lambda^{++}$, there are no universal graphs of size λ^+ .*

3. BUILDING THE MODEL W

In this section we construct the generic extension W which will be the desired model as requested in Theorem 1.1. We construct the model W in essentially two steps. The first step is a reverse Easton iteration of suitable forcing notions of length $\kappa + 1$. The second step is a variant of Radin forcing using the existence of suitable guiding generics.

Assume GCH holds, κ is a $(\kappa+3)$ -strong cardinal and let $j : V \rightarrow M$ witness κ is $(\kappa+3)$ -strong. Let also $i : V \rightarrow N$ be the ultrapower embedding by $U = \{X \subseteq \kappa : \kappa \in j(X)\}$. Also factor j through i to get $k : N \rightarrow M$ defined by $k([f]_U) = j(f)(\kappa)$. Note that $\text{crit}(k) = \kappa_N^{++} < \kappa_M^{++} = \kappa^{++}$.

Force over V with the reverse Easton iteration

$$\langle \langle \mathbb{P}_\lambda : \lambda \leq \kappa + 1 \rangle, \langle \mathbb{Q}_\lambda : \lambda \leq \kappa \rangle \rangle$$

where we force with the trivial forcing notion except $\lambda \leq \kappa$ is an inaccessible cardinal in which case

$$\mathbb{H}_{\mathbb{P}_\lambda} \text{ “} \mathbb{Q}_\lambda = \text{Sacks}(\lambda, \lambda^{++}) \times \text{Add}(\lambda^+, \lambda^{+3}) \times \text{Add}(\lambda^{++}, \lambda^{+4}). \text{”}$$

Let $\langle\langle G_\lambda : \lambda \leq \kappa + 1 \rangle\rangle, \langle\langle H_\lambda : \lambda \leq \kappa \rangle\rangle$ be generic for the forcing, so each G_λ is \mathbb{P}_λ -generic over V and H_λ is $\mathbb{Q}_\lambda = \mathbb{Q}_\lambda[G_\lambda]$ -generic over $V[G_\lambda]$. Let us also write, for each inaccessible $\lambda \leq \kappa$, $H_\lambda = H_\lambda^0 \times H_\lambda^1 \times H_\lambda^2$, which corresponds to $\mathbb{Q}_\lambda = \text{Sacks}(\lambda, \lambda^{++}) \times \text{Add}(\lambda^+, \lambda^{+3}) \times \text{Add}(\lambda^{++}, \lambda^{+4})$.

By arguments similar to [2] and [1], the following hold in $V[G_{\kappa+1}]$:

- (1) κ is a $(\kappa + 2)$ -strong cardinal,
- (2) $j : V \rightarrow M$ extends to some $j^* : V[G_{\kappa+1}] \rightarrow M[G_{\kappa+1} * H]$ with ${}^\kappa M[G_{\kappa+1} * H] \subseteq M[G_{\kappa+1} * H], V_{\kappa+3}[G_{\kappa+1}] \subseteq M[G_{\kappa+1} * H]$, and j^* is generated by a (κ, κ^{+3}) -extender,
- (3) $i : V \rightarrow N$ extends to $i^* : V[G_{\kappa+1}] \rightarrow N[i^*(G_{\kappa+1})]$ and i^* is the ultrapower of $V[G_{\kappa+1}]$ by $U^* = \{X \subseteq \kappa : \kappa \in j^*(X)\}$
- (4) For each inaccessible cardinal $\lambda \leq \kappa$, we have

$$2^\lambda = \lambda^{++} + 2^{\lambda^+} = \lambda^{+3} + 2^{\lambda^{++}} = \lambda^{+4},$$

$$(5) M[G_{\kappa+1} * H] \models "2^\kappa = \kappa^{++} + 2^{\kappa^+} = \kappa^{+3} + 2^{\kappa^{++}} = \kappa^{+4}",$$

- (6) There is an $F \in V[G_{\kappa+1}]$ which is a generic filter over $N[i^*(G_{\kappa+1})]$, by the forcing notion

$$(\text{Col}(\kappa^{+4}, < i(\kappa)) \times \text{Add}(\kappa^{+3}, i(\kappa)) \times \text{Add}(\kappa^{+4}, i(\kappa)^+))_{N[i^*(G_{\kappa+1})]}$$

Note that $F \in M[G_{\kappa+1} * H]$, as it can be coded by an element of $V_{\kappa+2}$, also, if $k^* : N[i^*(G_{\kappa+1})] \rightarrow M[G_{\kappa+1} * H]$ is the induced elementary embedding, defined by $k^*([f]_{U^*}) = j^*(f)(\kappa)$, then $\text{crit}(k^*) = \kappa_N^{++} < \kappa_M^{++} = \kappa^{++}$ and F can be transferred along k^* , in the sense that $\langle k^{*''}[F] \rangle$, the filter generated by $k^{*''}[F]$, is $(\text{Col}(\kappa^{+4}, < i(\kappa)) \times \text{Add}(\kappa^{+3}, i(\kappa)) \times \text{Add}(\kappa^{+4}, i(\kappa)^+))_{M[G_{\kappa+1} * H]}$ -generic over $M[G_{\kappa+1} * H]$.

For notational simplicity let us denote the models $V[G_{\kappa+1}]$, $M[G_{\kappa+1} * H]$ and $N[i^*(G_{\kappa+1})]$ by V^* , M^* and N^* respectively.

Work in V^* . Let

$$\mathcal{U} = \langle \mathcal{U}(\alpha, \beta) : \alpha \leq \kappa, \beta < o^\mathcal{U}(\alpha) \rangle$$

be a coherent sequence of measures of length $\ell^\mathcal{U} = \kappa + 1$ and $o^\mathcal{U}(\kappa) = \kappa^+$. For each such α, β let

$$i_{\alpha, \beta}^* : V^* \rightarrow M_{\alpha, \beta}^* \simeq \text{Ult}(V^*, \mathcal{U}(\alpha, \beta))$$

be the induced embedding. Let also

$$\mathcal{K} = \langle K_{\alpha,\beta} : \alpha \leq \kappa, \beta < o^{\mathcal{U}}(\alpha) \rangle$$

be a sequence of filters such that:

$$(7) \ K_{\alpha,\beta} \text{ is } \left(\text{Col}(\kappa^{+4}, < i_{\alpha,\beta}(\kappa)) \times \text{Add}(\kappa^{+3}, i_{\alpha,\beta}(\kappa)) \times \text{Add}(\kappa^{+4}, i_{\alpha,\beta}(\kappa)^+) \right)_{M_{\alpha,\beta}^*} \text{-generic}$$

over $M_{\alpha,\beta}^*$,

(8) the sequence is coherent in the sense that

$$\langle K_{\alpha,\tau} : \tau < \beta \rangle = [\bar{\alpha} \mapsto \langle K_{\bar{\alpha},\tau} : \tau < \beta \rangle]_{\mathcal{U}(\alpha,\beta)}.$$

For each $\alpha \leq \kappa$ set $\mathcal{F}(\alpha) = \bigcap_{\beta < o^{\mathcal{U}}(\alpha)} \mathcal{U}(\alpha, \beta)$ if $o^{\mathcal{U}}(\alpha) > 0$, and set $\mathcal{F}(\alpha) = \{\emptyset\}$ otherwise.

We now use \mathcal{U} and \mathcal{K} to define a variant of Radin forcing.

Definition 3.1. Let $\mathbb{R} = \mathbb{R}_{\mathcal{U},\mathcal{K}}$ be the forcing notion consisting of conditions of the form

$$p = \langle \alpha_{-1}^p, (\alpha_0^p, f_0^p, A_n^p, F_0^p), \dots, (\alpha_{n^p}^p, f_{n^p}^p, A_{n^p}^p, F_{n^p}^p) \rangle$$

where

- (1) $n^p \geq 0$,
- (2) $\alpha_{-1}^p < \alpha_0^p < \dots < \alpha_{n^p}^p = \kappa$,
- (3) $f_i^p \in \text{Col}((\alpha_{i-1}^p)^{+4}, < \alpha_i^p) \times \text{Add}((\alpha_{i-1}^p)^{+3}, \alpha_i^p) \times \text{Add}((\alpha_{i-1}^p)^{+4}, (\alpha_i^p)^+)$,
- (4) $A_i^p \in \mathcal{F}(\alpha_i^p)$,
- (5) F_i^p is a function with domain A_i^p such that:
 - (A) if $o^{\mathcal{U}}(\alpha_i^p) > 0$, then:
 - (i) for each $\theta \in A_i^p, \theta > \alpha_{i-1}^p$,
 - (ii) $f_i^p \in \text{Col}((\alpha_{i-1}^p)^{+4}, < \theta) \times \text{Add}((\alpha_{i-1}^p)^{+3}, \theta) \times \text{Add}((\alpha_{i-1}^p)^{+4}, \theta^+)$, for each $\theta \in A_i^p$,
 - (iii) for all $\theta \in A_i^p, F_i^p(\theta) \in \text{Col}(\theta^{+4}, < \alpha_i^p) \times \text{Add}(\theta^{+3}, \alpha_i^p) \times \text{Add}(\theta^{+4}, (\alpha_i^p)^+)$,
 - (iv) $[F_i^p]_{\mathcal{U}(\alpha_i^p, \beta)} \in K_{\alpha_i^p, \beta}$, for all $\beta < \mathcal{U}(\alpha)$.
 - (B) if $o^{\mathcal{U}}(\alpha_i^p) = 0$, then $F_i^p(\emptyset) = f_i^p$.

The order relations \leq and \leq^* (the Prikry order) on \mathbb{R} are defined in the natural way, see for example [1] or [5].

Let $\mathbb{R} = \mathbb{R}_{\mathcal{U},\mathcal{K}}$ and let K be \mathbb{R} -generic over V^* . Let C be the Radin club added by K and let $\langle \lambda_i : i < \kappa \rangle$ be an increasing enumeration of C . For each $i < \kappa$ let

$K_i = K_i^C \times K_i^3 \times K_i^4$ be $\text{Col}(\lambda_i^{+4}, < \lambda_{i+1}) \times \text{Add}(\lambda_i^{+3}, \lambda_{i+1}) \times \text{Add}(\lambda_i^{+4}, \lambda_{i+1}^+)$ -generic filter over V^* added by K . In the next lemma we collect the main properties of the above forcing notion.

Lemma 3.2. (1) *(the chain condition)* \mathbb{R} satisfies the κ^+ -Knaster,

(2) *(Prikrý property)* The forcing notion $(\mathbb{R}, \leq, \leq^*)$ satisfies the Prikrý property.

(3) *(the factorization lemma):* Suppose $p \in \mathbb{R}$ is of length $n > 0$ and let $0 \leq m < n$.

Then

(a) there exists an isomorphism

$$\mathbb{R}/p \simeq \mathbb{R}/p^{\leq m} \times \mathbb{R}/p^{> m},$$

where $p^{\leq m} = p \upharpoonright m + 1 = \langle \alpha_{-1}^p, (\alpha_0^p, f_0^p, A_n^p, F_0^p), \dots, (\alpha_m^p, f_m^p, A_m^p, F_m^p) \rangle$ and $p^{> m} = \langle \alpha_{m+1}^p, (\alpha_{m+1}^p, f_{m+1}^p, A_{m+1}^p, F_{m+1}^p), \dots, (\alpha_{n^p}^p, f_{n^p}^p, A_{n^p}^p, F_{n^p}^p) \rangle$,

(b) $(\mathbb{R}/p^{\leq m}, \leq)$ satisfies the $(\alpha_m^p)^+$ -c.c. and $(\mathbb{R}/p^{> m}, \leq^*)$ is $(\alpha_m^p)^{+3}$ -closed.

(4) κ remains an inaccessible cardinal in $V^*[K]$.

(5) Let C and $\langle \lambda_i : i < \kappa \rangle$ be as above. Then

(a) for each $i < \kappa$, we have

$$2^{\lambda_i} = \lambda_i^{++} + 2^{\lambda_i^+} = \lambda_i^{+3} + 2^{\lambda_i^{++}} = \lambda_i^{+4} + 2^{\lambda_i^{+3}} = \lambda_i^{+5} = \lambda_{i+1} + 2^{\lambda_i^{+4}} = \lambda_{i+1}^+.$$

(b) If we force with $\text{Col}(\aleph_0, < \lambda_0) \times \text{Add}(\aleph_0, \lambda_0^+) \times \mathbb{R}$, then in the generic extension, κ is the least inaccessible cardinal and for all $\lambda < \kappa$, $2^\lambda = \lambda^{++}$.

(6) $V^*[K] = V^*[\langle K_i : i < \kappa \rangle]$.

Now suppose that $L_0 \times L_1 \times K$ is $\text{Col}(\aleph_0, < \lambda_0) \times \text{Add}(\aleph_0, \lambda_0^+) \times \mathbb{R}$ -generic over V^* and set

$$W = V^*[L_0 \times L_1 \times K] = V[G_{\kappa+1}][L_0 \times L_1 \times K].$$

This completes our construction of the model W . Let us note that by Lemma 3.2,

$$\text{Card}^W \cap [\aleph_0, \kappa) = \{\aleph_0\} \cup \{\lambda_i, \lambda_i^+, \lambda_i^{++}, \lambda_i^{+3}, \lambda_i^{+4} : i < \kappa\}.$$

In the next section we show that the model W satisfies the conclusion of Theorem 1.1.

4. NO UNCOUNTABLE UNIVERSAL GRAPHS IN W

In this section we complete the proof of Theorem 1.1, by showing that in W , there is no universal graph of size λ , where λ is an uncountable regular cardinal less than κ .

The proof is divided into several cases:

Case 1. $\lambda = \aleph_1$:

We have $W = V^*[K, L_0][L_1]$, where $V^*[K, L_0]$ satisfies CH and W is an extension of $V^*[K, L_0]$ by L_1 which is $\text{Add}(\aleph_0, \lambda_0^+)$ -generic over $V^*[K, L_0]$. Thus by Lemma 2.1, there are no universal graphs of size \aleph_1 .

Case 2. $\lambda = \lambda_{i+1}^{+n}$ for some $i < \kappa$ and some $n < 3$:

Assume on the contrary that T is a universal graph of size λ in W . Now we can write W as

$$W = V^*[\langle K_j : j \leq i \rangle][\langle K_j : i < j < \kappa \rangle][L_0 \times L_1]$$

and that by Lemma 3.2(2), $\mathcal{P}^W(\lambda) = \mathcal{P}^{V^*[\langle K_j : j \leq i \rangle][L_0 \times L_1]}(\lambda)$. It immediately follow that $T \in V^*[\langle K_j : j \leq i \rangle][L_0 \times L_1]$.

Now let i_* be a limit ordinal, with possibly $i_* = 0$, such that $i = i_* + k$, for some $k < \omega$. So we have

$$V^*[\langle K_j : j \leq i \rangle][L_0 \times L_1] = V^*[\langle K_j : j < i_* \rangle][L_0 \times L_1][\prod_{\ell \leq k} K_{i_* + \ell}].$$

Now $V^* = V[G_{\lambda_{i+1}}][X]$, where X is generic over $V[G_{\lambda_{i+1}}]$ for a forcing notion which is λ_{i+1}^* -closed and λ_{i+1}^* is the least inaccessible above λ_{i+1} . It immediately follow that $\mathcal{U} \upharpoonright \lambda_{i_*} + 1 \in V[G_{\lambda_{i+1}}]$ and hence by Lemma 3.2(2), we may assume that T belongs to

$$V[G_{\lambda_{i+1}}][\langle K_j : j \leq i \rangle][H_{\lambda_{i+1}}][L_0 \times L_1] = V[G_{\lambda_{i+1}}][\langle K_j : j \leq i \rangle][L_0 \times L_1][\prod_{\ell < 3, \ell \neq n} H_{\lambda_{i+1}}^\ell][H_{\lambda_{i+1}}^n].$$

Now we have $V[G_{\lambda_{i+1}}][\langle K_j : j \leq i \rangle][L_0 \times L_1][\prod_{\ell < 3, \ell \neq n} H_{\lambda_{i+1}}^\ell]$ satisfies $2^\lambda = \lambda^+$ and $V[G_{\lambda_{i+1}}][\langle K_j : j \leq i \rangle][H_{\lambda_{i+1}}][L_0 \times L_1]$ is a generic extension of it, by either Sacks(λ, λ^{++}) if $\lambda = \lambda_{i+1}$ or $\text{Add}(\lambda, \lambda^{++})$ if $\lambda \in \{\lambda_{i+1}^+, \lambda_{i+1}^{++}\}$. This contradicts either Lemma 2.1 or Lemma 2.2.

Case 3. $\lambda = \lambda_i^{+n}$ for some $i < \kappa$ and some $2 < n < 5$:

The argument is essentially the same as above. Suppose towards a contradiction that T is a universal graph of size λ in W . As in Case 2, we will get that $T \in V[G_{\lambda_{i+1}}][\langle K_j : j < i \rangle][K_i][L_0 \times L_1]$ and this model is an extension of $V[G_{\lambda_{i+1}}][\langle K_j : j < i \rangle][K_i^C \times K_i^\ell][L_0 \times L_1]$, where $\ell \in \{3, 4\}, \ell \neq n$. This later model satisfies GCH at λ , and as in Case 2 we get a contradiction to Lemma 2.1.

Case 4. $\lambda = \lambda_i^{+n}$ for some limit ordinal $i < \kappa$ and some $0 < n < 3$:

Arguing as above, if $T \in W$ is universal graph of size λ , then $T \in V[G_{\lambda_i}][\langle K_j : j < i \rangle][H_{\lambda_i}^0 \times H_{\lambda_i}^1 \times H_{\lambda_i}^2][L_0 \times L_1]$ and this model extends $V[G_{\lambda_i}][\langle K_j : j < i \rangle][H_{\lambda_i}^0 \times H_{\lambda_i}^\ell][L_0 \times L_1]$, where $\ell \in \{1, 2\}, \ell \neq n$, by $\text{Add}(\lambda, \lambda^{++})$, so again we get a contradiction using Lemma 2.1.

Case 5. $\lambda = \lambda_i^+$ for some limit ordinal $i < \kappa$:

This is the hardest part of the proof. We follow very closely the argument given in [3]. Suppose towards a contradiction that T is a universal graph on λ . As before, we can conclude that $T \in V[G_{\lambda_i}][\langle K_j : j < i \rangle][L_0 \times L_1][H_{\lambda_i}^0 \times H_{\lambda_i}^1]$. Let us identify $H_{\lambda_i}^0$ with the κ^{++} -sequences $\langle A_\sigma : \sigma < \kappa^{++} \rangle$ of mutually Sacks-generic subsets of λ_i .

For notational simplicity, set

$$\tilde{V} = V[G_{\lambda_i}][\langle K_j : j < i \rangle][L_0 \times L_1][H_{\lambda_i}^1],$$

and for each $\sigma \leq \lambda_i^{++}$ set

$$\tilde{V}_\sigma = V[G_{\lambda_i}][\langle K_j : j < i \rangle][L_0 \times L_1][H_{\lambda_i}^1][\langle A_\alpha : \alpha < \sigma \rangle].$$

Thus $\tilde{V} = \tilde{V}_0$ and $\tilde{V}_{\lambda_i^{++}} = V[G_{\lambda_i}][\langle K_j : j < i \rangle][L_0 \times L_1][H_{\lambda_i}^0 \times H_{\lambda_i}^1]$. Also for each $\sigma < \lambda_i^{++}$, the model \tilde{V}_σ satisfies $2^{\lambda_i} = \lambda_i^+$. The proof of the next claim is essentially the same as in [6].

Claim 4.1. Let \mathcal{S} be the set of ordinals $\sigma < \lambda_i^{++}$ such that:

- (1) The restriction of \mathcal{U} to \tilde{V}_σ , i.e.,

$$\mathcal{U} \cap \tilde{V}_\sigma = \langle \mathcal{U}(\alpha, \beta) \cap \tilde{V}_\sigma : \alpha \leq \lambda_i, \beta < o^{\mathcal{U}}(\alpha) \rangle$$

is a coherent sequence of measures in \tilde{V}_σ ,

- (2) For each σ as above, $\alpha \leq \lambda_i$ and $\beta < o^{\mathcal{U}}(\alpha)$ set $i_{\alpha, \beta}^\sigma : V^* \rightarrow M_{\alpha, \beta}^{*, \sigma} \simeq \text{Ult}(V^*, \mathcal{U}(\alpha, \beta) \cap \tilde{V}_\sigma)$. Then the sequence

$$\mathcal{K} \cap \tilde{V}_\sigma = \langle K_{\alpha, \beta} \cap \tilde{V}_\sigma : \alpha \leq \lambda_i, \beta < o^{\mathcal{U}}(\alpha) \rangle$$

is a sequence of filters such that:

- (a) $K_{\alpha, \beta} \cap \tilde{V}_\sigma$ is $\left(\text{Col}(\kappa^{+4}, < i_{\alpha, \beta}(\kappa)) \times \text{Add}(\kappa^{+3}, i_{\alpha, \beta}(\kappa)) \times \text{Add}(\kappa^{+4}, i_{\alpha, \beta}(\kappa)^+) \right) \underset{M_{\alpha, \beta}^{*, \sigma}}{-}$
generic over $M_{\alpha, \beta}^{*, \sigma}$,

(b) the sequence is coherent in the sense that

$$\langle K_{\alpha,\tau} \cap \tilde{V}_\sigma : \tau < \beta \rangle = [\bar{\alpha} \mapsto \langle K_{\bar{\alpha},\tau} \cap \tilde{V}_\sigma : \tau < \beta \rangle]_{\mathcal{U}(\alpha,\beta) \cap \tilde{V}_\sigma}.$$

Then \mathcal{S} is a stationary subset of λ_i^{++} .

Now take $\sigma \in \mathcal{S}$ such that $T \in \tilde{V}_\sigma$. Let us define the graph $T^* \in \tilde{V}_{\lambda_i^{++}}$ as follows:

- T^* has the universe $\lambda_i \cup Y$, where $Y \subseteq \lambda_i^{++}$ has size $\lambda = \lambda_i^+$, $\min(Y) > \sigma$ and $Y \cap \mathcal{S}$ is cofinal in $\sup(Y)$.
- the pair (γ, β) is an edge in T^* iff $\gamma \in Y$ and $\beta \in A_\gamma$ or symmetrically $\beta \in Y$ and $\gamma \in A_\beta$.

Since T is universal, we can find an embedding $f : T^* \rightarrow T$ with $f \in \tilde{V}_{\lambda_i^{++}}$. Take $X \subseteq \lambda_i^{++}$ of size $\leq \lambda_i$ such that $f \upharpoonright \lambda_i \in \tilde{V}_\sigma[\langle A_\alpha : \alpha \in X \rangle]$. The same arguments as in [3] show that there exists $\gamma \in Y \cap \mathcal{S} \setminus X$ such that $A_\gamma \notin \tilde{V}_\sigma[\langle A_\alpha : \alpha \in X \rangle]$. By [3, Lemma 2.6] we can recover A_γ using $f \upharpoonright \lambda_i$ and T , which implies $A_\gamma \in \tilde{V}_\sigma[\langle A_\alpha : \alpha \in X \rangle]$, which is impossible.

Remark 4.2. In the model W , for every singular cardinal λ , we have $2^{<\lambda} = \lambda$ and hence by classical results in model theory, there is a universal graph of size λ in W .

Remark 4.3. In the model W , for each uncountable regular cardinal λ , the universality number for graphs on λ is λ^{++} .

REFERENCES

- [1] Cummings, James A model in which GCH holds at successors but fails at limits. *Trans. Amer. Math. Soc.* 329 (1992), no. 1, 1-39.
- [2] Friedman, Sy-David; Honzik, Radek Easton's theorem and large cardinals. *Ann. Pure Appl. Logic* 154 (2008), no. 3, 191-208.
- [3] Friedman, S.-D.; Thompson, K. Negative universality results for graphs. *Fund. Math.* 210 (2010), no. 3, 269-283.
- [4] Gitik, Moti Prikry-type forcings. *Handbook of set theory*. Vols. 1, 2, 3, 1351-1447, Springer, Dordrecht, 2010.
- [5] Golshani, Mohammad (Weak) diamond can fail at the least inaccessible cardinal. *Fund. Math.* 256 (2022), no. 2, 113-129.
- [6] Golshani, Mohammad; Mohammadpour, Rahman; The tree property at double successors of singular cardinals of uncountable cofinality, *Ann. Pure Appl. Logic*, 169 (2018), no. 2, 164-175.

- [7] Kojman, Menachem; Shelah, Saharon Nonexistence of universal orders in many cardinals. *J. Symbolic Logic* 57 (1992), no. 3, 875–891.
- [8] Shelah, Saharon On universal graphs without instances of CH. *Ann. Pure Appl. Logic* 26 (1984), no. 1, 75–87.
- [9] Shelah, Saharon Universal graphs without instances of CH: revisited. *Israel J. Math.* 70 (1990), no. 1, 69–81.

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