HYPERCRITICAL DEFORMED HERMITIAN-YANG-MILLS EQUATION REVISITED

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Abstract. In this paper, we study the hypercritical deformed Hermitian-Yang-Mills equation on compact Kähler manifolds and resolve two conjectures of Collins-Yau [6].

1. Introduction

Let (X^n, ω) be a compact Kähler manifold and α be a closed real $(1,1)$ form on X so that $\int_X (\alpha + \sqrt{-1}\omega)^n \neq 0$ and therefore we might write

$$\int_X (\alpha + \sqrt{-1}\omega)^n = \mathbb{R}_{>0} \cdot e^{\sqrt{-1}\theta_0} \tag{1.1}$$

for some $e^{\sqrt{-1}\theta_0} \in \mathbb{S}^1$. In particular, the angle θ_0 is well-defined modulo 2π. The deformed Hermitian-Yang-Mills (dHYM) equation seeks for $\varphi \in C^\infty(X)$ such that $\alpha_\varphi = \alpha + \sqrt{-1}\partial\bar{\partial}\varphi$ satisfies

$$\mathrm{Im}\left(e^{-\sqrt{-1}\theta_0}(\alpha_\varphi + \sqrt{-1}\omega)^n\right) = 0. \tag{1.2}$$

The dHYM equation first appeared in [10] from the mathematical side drawing from the physics literature [11] which is corresponding to the special Lagrangian equation under the setting of the Strominger-Yau-Zaslow mirror symmetry [14].

One of the main topic in the study of dHYM equation is to characterize the solvability in terms of certain algebraic conditions on the class $[\alpha]$. In [3, Conjecture 1.4], Collins-Jacob-Yau predicted that the existence of solution to the supercritical dHYM equation is equivalent to a stability condition in terms of holomorphic intersection numbers for any irreducible subvarieties $V \subset X$, modeled on the Nakai-Moishezon criterion, and confirmed it for complex surfaces. In [2], the authors and Takahashi confirmed the conjecture in the projective case building on the works of Chen [1] and Song [12], see also [7, 9].

On the other hand, motivated by the GIT (Geometric Invariant Theory) approach for special Lagrangian [15, 13], Collins-Yau [6] proposed to study

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the dHYM equation using the space $\mathcal{H}_\omega$ of almost calibrated $(1,1)$ forms in the class $[\alpha]$:

$$\mathcal{H}_\omega = \left\{\varphi \in C^\infty(X) : \mathrm{Re}\left(e^{-\sqrt{-1}\theta_0}(\alpha_\varphi + \sqrt{-1}\omega)^n\right) > 0\right\}. \tag{1.3}$$

The space $\mathcal{H}_\omega$ is a (possibly empty) open subset of the space of smooth, real valued functions on X. By studying the geodesic and functional on $\mathcal{H}$, Collins-Yau [6] discovered a number of algebraic obstruction to the dHYM solution in the hypercritical phase. We refer interested readers to the survey article [4] for a comprehensive discussion.

When $\mathcal{H}_\omega \neq \emptyset$, a maximum principle shows that

$$\mathcal{H}_\omega = \left\{\varphi \in C^\infty(X) : |Q_\omega(\alpha_\varphi) - \beta| < \frac{\pi}{2}\right\} \tag{1.4}$$

where $Q_\omega(\alpha_\varphi)$ is the special Lagrangian operator defined by (2.2) and β is some lift of θ_0 from $\mathbb{R}/2\pi\mathbb{Z}$ to $(0, n\pi)$. The lift β is usually referred to the analytic lifted angle. To determine the non-emptiness of $\mathcal{H}_\omega$ using algebraic information of $[\alpha]$, Collins-Yau [6, Section 8] introduced an algebraic approach in determining the lifted angle, see Definition 2.1. Particularly, using a Chern number inequality in [5], it was shown that the algebraic lifted angle coincides with the analytic lifted angle in three dimensional whenever a supercritical dHYM solution exists. Moreover, the following was shown.

Proposition 1.1 (Proposition 8.4 in [6])**.** *Suppose (X^3, ω) is a compact three-dimensional Kähler manifold and $[\alpha] \in H^{1,1}(X, \mathbb{R})$. If the dHYM equation admits a solution with $\theta \in (0, \frac{\pi}{2})$*[1] *then the followings hold:*

(i) *The Chern number satisfies*

$$\left(\int_X \alpha^3\right)\left(\int_X \omega^3\right) < 9\left(\int_X \alpha\wedge\omega^2\right)\left(\int_X \alpha^2\wedge\omega\right),$$

in particular the algebraic lifted angle $\hat{\theta}_X([\alpha])$ is well-defined;

(ii) $\mathrm{Im}(Z_X([\alpha]) > 0$ *and* $\varphi_X([\alpha]) \in (\frac{\pi}{2}, \pi)$*;*

(iii) *For any irreducible subvariety $V \subsetneq X$,*

$$\mathrm{Im}(Z_V([\alpha])) > 0, \quad \varphi_V([\alpha]) > \varphi_X([\alpha]).$$

It is conjectured that the converse should also hold, see [6, Conjecture 8.5]. In this work, we give an affirmative answer to this question.

Theorem 1.1. *The converse of Proposition 1.1 is true.*

The resolution of the conjecture is based on a Nakai-Moishezon type criterion proved by the authors and Takahashi [2]. The most crucial observation is to show that the assumptions (i)-(iii) indeed give rise to the Kählerity of $[\alpha]$ and a stability in terms of intersection number of subvariety in X.

[1]The convention taken here is slightly different from that in [6]. The range of $\theta \in (0, \frac{\pi}{2})$ is equivalent to $\hat{\theta} \in (\pi, \frac{3\pi}{2})$ there.

In [6], it is also conjectured that the non-emptiness of $\mathcal{H}_\omega$ is equivalent to certain Nakai-Moishezon type criterion.

Conjecture 1.1 (Conjecture 8.7 in [6]). *The followings are equivalent:*

(A) *The space $\mathcal{H}_\omega$ is non-empty and $[\alpha]$ has hypercritical phase;*

(B) *For any irreducible subvariety $V \subset X$, $\mathrm{Im}(Z_{V,[\alpha]}) > 0$.*

The implication $(A) \implies (B)$ has been established in [6, Corollary 8.6]. Though an example in blow-up of $\mathbb{CP}^2$ at one point, we find that the converse is not necessarily true.

Proposition 1.2. *On $X = \mathrm{Bl}_p(\mathbb{CP}^2)$, there exist Kähler class $[\omega]$ and $[\alpha] \in H^{1,1}(X, \mathbb{R})$ such that (B) in Conjecture 1.1 holds but $\mathcal{H}_\omega = \emptyset$.*

In contrast, we can provide an alternative criteria of $\mathcal{H}_\omega \neq \emptyset$ in terms of stability condition on holomorphic intersection numbers for any irreducible subvariety $V \subset X$ based on the work in [2], see Theorem 5.1 and Remark 5.1.

The paper is organized as follows: In Section 2, we will collect some preliminaries and notations that will be used throughout this work. In Section 3, we will give the proof of Theorem 1.1. In Section 4, we will prove Proposition 1.2 which gives a counter-example of Conjecture 1.1. In Section 5, we will discuss a criteria of $\mathcal{H}_\omega \neq \emptyset$.

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2. Preliminaries and notations

In this section, we will introduce the necessary notations in this work. The ultimate goal is to understand the existence of solution to the dHYM equation (1.2). Locally, if we choose a local holomorphic coordinate around $p \in X$ so that $\alpha_\varphi(p)$ is diagonal with respect to $\omega(p)$ with eigenvalues λ_i, then

$$\frac{(\alpha_\varphi + \sqrt{-1}\omega)^n}{\omega^n} = \sqrt{\prod_{i=1}^n (1+\lambda_i^2)} \cdot e^{\sqrt{-1}\sum_{i=1}^n \operatorname{arccot}(\lambda_i)}. \tag{2.1}$$

In this way, we define the Lagrangian phase operator[2] as

$$Q_\omega(\alpha_\varphi) = \sum_{i=1}^n \operatorname{arccot}(\lambda_i). \tag{2.2}$$

In other words, the dHYM equation seeks for $\varphi \in C^\infty(X)$ so that

$$Q_\omega(\alpha_\varphi) = \theta_0 \mod 2\pi. \tag{2.3}$$

[2]In the literature, it is sometime convenient to consider the integral $\int_X (\omega + \sqrt{-1}\alpha)^n$ instead and the corresponding Lagrangian phase operator will be defined as $\hat{Q}_\omega(\alpha_\varphi) = \sum_{i=1}^n \arctan(\lambda_i)$ instead.

where $e^{\sqrt{-1}\theta_0}$ is a cohomological constant determined by the class $[\omega]$ and $[\alpha]$.

The space of almost calibrated $(1,1)$ forms in the class $[\alpha]$ is given by

$$\mathcal{H}_\omega = \left\{\varphi \in C^\infty(X) : \mathrm{Re}\left(e^{-\sqrt{-1}\theta_0}(\alpha_\varphi + \sqrt{-1}\omega)^n\right) > 0\right\}. \tag{2.4}$$

In general, the space $\mathcal{H}_\omega$ depends also on the representative ω of $[\omega]$.

Since θ_0 is a-priori only defined in $\mathbb{R}/2\pi\mathbb{Z}$, $\mathcal{H}_\omega$ will be a disjoint union of branches. It is an application of maximum principle [5] that if $\mathcal{H}_\omega \neq \emptyset$, then we have

$$\mathcal{H}_\omega = \left\{\varphi \in C^\infty(X) : |Q_\omega(\alpha_\varphi) - \beta| < \frac{\pi}{2}\right\} \tag{2.5}$$

for an unique $\beta \in (0, n\pi)$ so that $\beta = \theta_0 \pmod{2\pi}$. The lift β is usually referred to the analytic lifted angle. For notational convenience, if $\mathcal{H}_\omega \neq \emptyset$, we will use θ_0 to denote this uniquely defined lifted phase β. And thus, the dHYM equation can be rewritten as

$$Q_\omega(\alpha_\varphi) = \theta_0 \in \mathbb{R}. \tag{2.6}$$

When the lifted phase $\theta_0 \in (0, \frac{\pi}{2})$, we say that $[\alpha]$ has the hypercritical phase, while if $\theta_0 \in (0,\pi)$, $[\alpha]$ is said to have supercritical phase. When the lifted phase lies inside the region of supercritical phase, the dHYM equation is known to be well-behaved in the analytic point of view. It is therefore important to determine the lifted angle. In [6], Collins-Yau proposed a purely algebraic approach to determine the lift. They introduced the following.

Definition 2.1. *Let (X,ω) be a compact n-dimensional Kähler manifold. For $[\alpha] \in H^{1,1}(X,\mathbb{R})$ and p-dimensional irreducible subvariety $V \subset X$, define*

$$\begin{cases} Z_{V,[\alpha]}(t) = -\displaystyle\int_V e^{-\sqrt{-1}(t\omega + \sqrt{-1}\alpha)} = -\frac{(-\sqrt{-1})^p}{p!}\int_V (t\omega + \sqrt{-1}\alpha)^p; \\ Z_V([\alpha]) = Z_{V,[\alpha]}(1) \end{cases}$$

for $t \in [1,+\infty]$. Suppose that $Z_{V,[\alpha]}(t) \in \mathbb{C}^*$ for all $t \in [1,+\infty]$.

(i) *The algebraic lifted angle $\hat{\theta}_V([\alpha])$ is defined as the winding angle of the curve $Z_{V,[\alpha]}(t)$ as t runs from $+\infty$ to 1.*

(ii) *The slicing angle $\varphi_V([\alpha])$ is defined as*

$$\varphi_V([\alpha]) = \hat{\theta}_V([\alpha]) - (p-2)\cdot\frac{\pi}{2}.$$

3. Proof of Theorem 3.1

In this section, we will establish the characterization of existence of hypercritical dHYM solution in three dimension, namely Theorem 3.1. We start with some preparation lemmas.

Lemma 3.1. *Under the assumption (i), (ii) and (iii) in Theorem 3.1, the following holds. For any proper p-dimensional irreducible subvariety $V \subsetneq X$, we have*

$$\frac{\pi}{2} < \varphi_X([\alpha]) < \varphi_V([\alpha]) < \pi. \tag{3.1}$$

Moreover, $Z_V([\alpha]) \in \mathbb{R}_{>0} \cdot e^{\sqrt{-1}\varphi_V([\alpha])}$.

Proof. The first two inequalities follows from assumption (ii) and (iii). It suffices to show $\varphi_V([\alpha]) < \pi$. Indeed, this follows from the following simple observation. By definition, the algebraic lifted angle $\hat{\theta}_V([\alpha])$ is given by

$$\lim_{t \to +\infty} \frac{Z_{V,[\alpha]}(1)}{Z_{V,[\alpha]}(t)} \in \mathbb{R}_{>0} \cdot e^{\sqrt{-1}\hat{\theta}_V([\alpha])}. \tag{3.2}$$

Together with the fact that as $t \to +\infty$,

$$Z_{V,[\alpha]}(t) \approx e^{-\sqrt{-1}(p-2)\frac{\pi}{2}} \cdot \frac{t^p}{p!} \int_V \omega^p, \tag{3.3}$$

this shows that

$$Z_V([\alpha]) = Z_{V,[\alpha]}(1) \in \mathbb{R}_{>0} \cdot e^{\sqrt{-1}\varphi_V([\alpha])}. \tag{3.4}$$

When $p = 1$,

$$Z_{V,[\alpha]}(t) = -\int_V \alpha + \sqrt{-1}t \int_V \omega. \tag{3.5}$$

For $t \in [1, +\infty]$, it is clear that

$$\mathrm{Im}(Z_{V,[\alpha]}(t)) > 0. \tag{3.6}$$

This implies $\hat{\theta}_V([\alpha]) \in (0, \pi)$ and

$$\varphi_V([\alpha]) = \hat{\theta}_V([\alpha]) + \frac{\pi}{2} \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right). \tag{3.7}$$

Combining this with (3.4) and $\mathrm{Im}(Z_{V,[\alpha]}(1)) = \int_V \omega > 0$, we see that $\varphi_V([\alpha]) < \pi$.

When $p = 2$,

$$Z_{V,[\alpha]}(t) = \frac{1}{2} \int_V (t^2\omega^2 - \alpha^2) + \sqrt{-1}t \int_V \alpha \wedge \omega. \tag{3.8}$$

By assumption (iii), we obtain $\int_V \alpha \wedge \omega > 0$, and then for $t \in [1, +\infty]$,

$$\mathrm{Im}(Z_{V,[\alpha]}(t)) > 0. \tag{3.9}$$

This implies $\hat{\theta}_V([\alpha]) \in (0, \pi)$ and $\varphi_V([\alpha]) = \hat{\theta}_V([\alpha]) < \pi$. □

Next, we wish to show that $[\alpha] \in H^{1,1}(X, \mathbb{R})$ is in fact a Kähler class. This is analogous to the Kählerity of $[\alpha]$ if it is a sub-solution in the hypercritical phase.

Lemma 3.2. *Under the assumption (i), (ii) and (iii) in Theorem 3.1, $[\alpha] \in H^{1,1}(X, \mathbb{R})$ is a Kähler class.*

Proof. By [8, Theorem 4.2], it suffices to show that for any p-dimensional irreducible subvariety $V \subset X$ and $k = 1, 2, \ldots, p$, we have

$$\int_V \alpha^k \wedge \omega^{p-k} > 0. \tag{3.10}$$

When $p = 1$, Lemma 3.1 implies that $\varphi_V([\alpha]) \in (\frac{\pi}{2}, \pi)$ and hence $\mathrm{Re}\,(Z_V([\alpha])) < 0$. Since

$$Z_V([\alpha]) = \sqrt{-1} \cdot \left(\int_V \omega + \sqrt{-1}\alpha \right), \tag{3.11}$$

this gives $\displaystyle\int_V \alpha > 0$.

When $p = 2$,

$$\begin{aligned} 2 \cdot Z_V([\alpha]) &= \int_V (\omega + \sqrt{-1}\alpha)^2 \\ &= \left(\int_V \omega^2 - \alpha^2 + \sqrt{-1} \int_V 2\alpha \wedge \omega \right). \end{aligned} \tag{3.12}$$

Hence, Lemma 3.1 implies

$$\int_V \alpha \wedge \omega > 0 \quad \text{and} \quad \int_V \alpha^2 > \int_V \omega^2 > 0. \tag{3.13}$$

When $p = 3$, $V = X$ and hence

$$\begin{aligned} 6 \cdot Z_X([\alpha]) &= -\sqrt{-1} \int_X (\omega + \sqrt{-1}\alpha)^3 \\ &= -\sqrt{-1} \left(\int_X \omega^3 - 3\alpha^2 \wedge \omega + \sqrt{-1} \int_X 3\alpha \wedge \omega^2 - \alpha^3 \right) \\ &= \left(\int_X 3\alpha \wedge \omega^2 - \alpha^3 \right) + \sqrt{-1} \left(\int_X 3\alpha^2 \wedge \omega - \omega^3 \right). \end{aligned} \tag{3.14}$$

Since $\varphi_X([\alpha]) \in (\frac{\pi}{2}, \pi)$, we have

$$\int_X 3\alpha \wedge \omega^2 < \int_X \alpha^3 \quad \text{and} \quad \int_X 3\alpha^2 \wedge \omega > \int_X \omega^3 > 0. \tag{3.15}$$

Therefore, it remains to show that $\int_X \omega^2 \wedge \alpha > 0$. Using the assumption (i) on the Chern number,

$$\begin{aligned} 3 \left(\int_X \alpha \wedge \omega^2 \right) \left(\int_X \omega^3 \right) &< \left(\int_X \alpha^3 \right) \left(\int_X \omega^3 \right) \\ &< 9 \left(\int_X \alpha \wedge \omega^2 \right) \left(\int_X \alpha^2 \wedge \omega \right). \end{aligned} \tag{3.16}$$

The integral $\int_X \alpha \wedge \omega^2$ is clearly non-zero. If it is negative, we will have

$$\int_X \omega^3 > \int_X 3\alpha^2 \wedge \omega > \int_X \omega^3, \tag{3.17}$$

which is impossible. In conclusion, we have

$$\int_X \alpha \wedge \omega^2, \quad \int_X \alpha^2 \wedge \omega, \quad \int_X \alpha^3 > 0. \tag{3.18}$$

This completes the proof. □

Next, we show that the class $[\alpha]$ will satisfy a kind of intersection number. This is in the same spirit as the numerical criterion of the Kähler class proved by Demailly-Păun [8].

Lemma 3.3. *Under the assumption (i), (ii) and (iii) in Theorem 3.1, the following holds. There is $\theta_0 \in (0, \frac{\pi}{2})$ such that*

$$\int_X \mathrm{Re}\left(\alpha + \sqrt{-1}\omega\right)^n - \cot\theta_0 \cdot \mathrm{Im}\left(\alpha + \sqrt{-1}\omega\right)^n = 0. \tag{3.19}$$

And for all p-dimensional irreducible subvariety $V \subsetneq X$, we have

$$\int_V \mathrm{Re}\left(\alpha + \sqrt{-1}\omega\right)^p - \cot\theta_0 \cdot \mathrm{Im}\left(\alpha + \sqrt{-1}\omega\right)^p > 0. \tag{3.20}$$

Proof. Clearly, θ_0 is determined by the class of ω and α. We first show that θ_0 is in the desired range. Direct computation and the computation in the proof of Lemma 3.2 shows that

$$\begin{cases} \int_X \mathrm{Re}(\alpha + \sqrt{-1}\omega)^3 = \int_X \alpha^3 - 3\alpha \wedge \omega^2 > 0; \\ \int_X \mathrm{Im}(\alpha + \sqrt{-1}\omega)^3 = \int_X 3\alpha^2 \wedge \omega - \omega^3 > 0. \end{cases} \tag{3.21}$$

If $\theta_0 \in (0, 2\pi)$ is chosen so that

$$\int_X \mathrm{Re}(\alpha + \sqrt{-1}\omega)^3 = \cot\theta_0 \cdot \int_X \mathrm{Im}(\alpha + \sqrt{-1}\omega)^3, \tag{3.22}$$

then assumption (ii) forces $\theta_0 \in (0, \frac{\pi}{2})$. This proves the first assertion.

It remains to consider the integral on the irreducible subvariety $V \subsetneq X$. We first relate $Z_V([\alpha])$ with $\mathrm{Arg}\left(\int_V (\alpha + \sqrt{-1}\omega)^p\right)$. For any p-dimensional irreducible subvariety $V \subset X$, by using $\varphi_X([\alpha]) < \varphi_V([\alpha])$,

$$\begin{aligned} p! \cdot Z_V([\alpha]) &= -(-\sqrt{-1})^p \cdot \int_V (\omega + \sqrt{-1}\alpha)^p \\ &= -\int_V (\alpha - \sqrt{-1}\omega)^p \\ &= e^{\sqrt{-1}\pi} \cdot \overline{\int_V (\alpha + \sqrt{-1}\omega)^p}. \end{aligned} \tag{3.23}$$

Since $\varphi_V([\alpha]) \in (\frac{\pi}{2}, \pi)$ by Lemma 3.1,

$$\operatorname{Arg}\left(\int_V (\alpha + \sqrt{-1}\omega)^p\right) = \pi - \varphi_V([\alpha]). \tag{3.24}$$

In particular, $\theta_0 = \pi - \varphi_X([\alpha])$ and therefore for any irreducible subvariety $V \subsetneq X$,

$$0 < \operatorname{Arg}\left(\int_V (\alpha + \sqrt{-1}\omega)^p\right) < \theta_0 < \frac{\pi}{2}. \tag{3.25}$$

We complete the proof. □

We remark here that if [3, Conjecture 1.4] holds, then the main result will follow from Lemma 3.3. Now we are ready to prove the main theorem.

Theorem 3.1. *Suppose (X^3, ω) is a compact three-dimensional Kähler manifold and $[\alpha] \in H^{1,1}(X, \mathbb{R})$. Then the dHYM equation admits a solution with $\theta \in (0, \frac{\pi}{2})$ if and only if the followings hold:*

(i) *The Chern number satisfies*

$$\left(\int_X \alpha^3\right)\left(\int_X \omega^3\right) < 9\left(\int_X \alpha \wedge \omega^2\right)\left(\int_X \alpha^2 \wedge \omega\right),$$

in particular the algebraic lifted angle $\hat{\theta}_X([\alpha])$ is well-defined;

(ii) $\operatorname{Im}(Z_X([\alpha]) > 0$ *and* $\varphi_X([\alpha]) \in (\frac{\pi}{2}, \pi)$*;*

(iii) *For any irreducible subvariety $V \subsetneq X$,*

$$\operatorname{Im}(Z_V([\alpha])) > 0, \quad \varphi_V([\alpha]) > \varphi_X([\alpha]).$$

Proof. We begin by noting that if there exists a dHYM solution with lifted angle $\theta_0 \in (0, \frac{\pi}{2})$, then

$$\sum_{i=1}^{3} \arctan \lambda_i = \hat{\theta}_0 \in \left(\pi, \frac{3\pi}{2}\right). \tag{3.26}$$

Then (i)-(iii) follows from the same argument as in [6, Proposition 8.4]. In [6], $[\alpha]$ is assumed to be $c_1(L)$ for some line bundle L. It is clear from the proof that $[\alpha] \in H^{1,1}(X, \mathbb{R})$ suffices, see also [5].

It remains to prove the existence of dHYM solution under assumption (i)-(iii). We fix $\theta_0 \in (0, \frac{\pi}{2})$ from Lemma 3.3.

Claim 3.1. *For any $k = 1, 2, 3$, we have*

$$\int_X \left(\operatorname{Re}(\alpha + \sqrt{-1}\omega)^k - \cot\theta_0 \cdot \operatorname{Im}(\alpha + \sqrt{-1}\omega)^k\right) \wedge \alpha^{3-k} \geq 0; \tag{3.27}$$

and for any p-dimensional irreducible subvariety $V \subsetneq X$ and $k = 1, 2, \ldots, p$,

$$\int_V \left(\operatorname{Re}(\alpha + \sqrt{-1}\omega)^k - \cot\theta_0 \cdot \operatorname{Im}(\alpha + \sqrt{-1}\omega)^k\right) \wedge \alpha^{p-k} > 0. \tag{3.28}$$

Proof of Claim. By Lemma 3.3, it remains to consider the following cases: $(p,k)=(2,1)$, $(3,2)$ and $(3,1)$.

When $(p,k)=(2,1)$, we have from (3.25) and $[\alpha]>0$ that

$$\begin{aligned}\int_V \alpha^2-\omega^2 &= \cot\left(\operatorname{Arg}\Big(\int_V(\alpha+\sqrt{-1}\omega)^2\Big)\right)\cdot 2\int_V \alpha\wedge\omega \\ &> \cot\theta_0\cdot 2\int_V \alpha\wedge\omega.\end{aligned} \tag{3.29}$$

Therefore,

$$\begin{aligned}&\int_V \big[\mathrm{Re}(\alpha+\sqrt{-1}\omega)-\cot\theta_0\cdot\mathrm{Im}(\alpha+\sqrt{-1}\omega)\big]\wedge\alpha \\ =&\int_V \alpha^2-\cot\theta_0\cdot\alpha\wedge\omega>\int_V\omega^2+\cot\theta_0\cdot\alpha\wedge\omega>0.\end{aligned} \tag{3.30}$$

We proceed to consider $p=3$. For notational convenience, we denote

$$a_i=\int_X \alpha^i\wedge\omega^{3-i},\quad \text{for } i=0,1,2,3. \tag{3.31}$$

Then the assumption (i), $\theta_0\in(0,\frac{\pi}{2})$ and Kählerity of $[\alpha]$ can be reduced to

$$\begin{cases} a_0a_3<9a_1a_2; \\ 0<3a_1<a_3; \\ 0<3a_2>a_0; \\ \cot\theta_0=\dfrac{a_3-3a_1}{3a_2-a_0}\in\mathbb{R}_{>0}. \end{cases} \tag{3.32}$$

If $k=1$,

$$\begin{aligned}&\int_X \big(\mathrm{Re}(\alpha+\sqrt{-1}\omega)-\cot\theta_0\cdot\mathrm{Im}(\alpha+\sqrt{-1}\omega)\big)\wedge\alpha^2 \\ =&\, a_3-\frac{a_3-3a_1}{3a_2-a_0}\cdot a_2 \\ =&\,\frac{2a_2a_3-a_3a_0+3a_1a_2}{3a_2-a_0} \\ >&\,\frac{2a_2(a_3-3a_1)}{3a_2-a_0}>0.\end{aligned} \tag{3.33}$$

If $k = 2$,

$$\begin{aligned}
&\int_X \left(\mathrm{Re}(\alpha+\sqrt{-1}\omega)^2 - \cot\theta_0 \cdot \mathrm{Im}(\alpha+\sqrt{-1}\omega)^2\right) \wedge \alpha \\
={}& \int_X \left[(\alpha^2-\omega^2) - \cot\theta_0 \cdot (2\alpha\wedge\omega)\right] \wedge \alpha \\
={}& (a_3 - a_1) - \left(\frac{a_3 - 3a_1}{3a_2 - a_0}\right) \cdot 2a_2 \\
={}& \frac{3a_1a_2 + a_2a_3 + a_1a_0 - a_0a_3}{3a_2 - a_0} \\
>{}& \frac{1}{3a_2 - a_0}\left(-\frac{2}{3}a_0a_3 + a_1a_0 + \frac{a_0a_3^2}{9a_1}\right) \\
={}& \frac{a_0a_1}{3a_2 - a_0}\left(\frac{a_3}{3a_1} - 1\right)^2 \geq 0.
\end{aligned} \tag{3.34}$$

□

Since $[\alpha]$ is a Kähler class by Lemma 3.2, the existence of dHYM solution with hypercritical phase follows from the Claim and [2, Corollary 1.4]. This completes the proof. □

4. Counter-example on Blow-up of $\mathbb{CP}^2$

In this section, we will prove Proposition 1.2. Let X be the blow-up of $\mathbb{CP}^2$ at one point, H be the pull-back of the hyperplane divisor, and E be the exceptional divisor. It is well-known that

$$H^2 = 1, \quad E^2 = -1, \quad H \cdot E = 0, \tag{4.1}$$

and $a[H] - [E]$ is Kähler when $a > 1$. Now we choose

$$[\omega] = 2[H] - [E], \quad [\alpha] = 6[H] + [E]. \tag{4.2}$$

Proof of Proposition 1.2. By direct calculation,

$$\int_X (\alpha + \sqrt{-1}\omega)^2 = \int_X (\alpha^2 - \omega^2) + 2\sqrt{-1}\int_X \alpha\wedge\omega = 32 + 26\sqrt{-1}. \tag{4.3}$$

Then the complex number $\int_X (\alpha+\sqrt{-1}\omega)^2$ lies in the first quadrant of $\mathbb{C}$. For any 1-dimensional irreducible subvariety $V \subset X$,

$$Z_V([\alpha]) = -\int_V \alpha + \sqrt{-1}\int_V \omega, \tag{4.4}$$

and hence $\mathrm{Im}(Z_V([\alpha])) > 0$. When $V = X$,

$$Z_X([\alpha]) = -\frac{1}{2}\int_V (\alpha^2 - \omega^2) + \sqrt{-1}\int_X \alpha\wedge\omega = -16 + 13\sqrt{-1}. \tag{4.5}$$

Now we show that $\mathcal{H}_\omega$ is empty. If $\mathcal{H}_\omega \neq \emptyset$, then by $\dim X = 2$, there exists $\theta_0 \in (0, n\pi) = (0, 2\pi)$ such that

$$\int_X (\alpha + \sqrt{-1}\omega)^2 \in \mathbb{R}_{>0} \cdot e^{\sqrt{-1}\theta_0} \tag{4.6}$$

and

$$\mathcal{H}_\omega = \left\{ \varphi \in C^\infty(X) \;\middle|\; |Q(\alpha_\varphi) - \theta_0| < \frac{\pi}{2} \right\}. \tag{4.7}$$

By (4.3), we see that $\theta_0 \in (0, \frac{\pi}{2})$ and $\tan\theta_0 = \frac{13}{16}$. For $\varphi \in \mathcal{H}_\omega$, let λ_1 and λ_2 be the eigenvalues of α_φ with respect to ω. It then follows that for $i = 1, 2$,

$$0 < \operatorname{arccot}(\lambda_i) < \operatorname{arccot}(\lambda_1) + \operatorname{arccot}(\lambda_2) = Q_\omega(\alpha_\varphi) < \theta_0 + \frac{\pi}{2} < \pi \tag{4.8}$$

and so $\lambda_i > -\tan\theta_0 \cdot \omega$. This implies $\alpha_\varphi + \tan\theta_0 \cdot \omega > 0$. In particular,

$$0 < \int_E \alpha_\varphi + \tan\theta_0 \cdot \omega = \int_X (\alpha + \tan\theta_0 \cdot \omega) \wedge [E] = -1 + \tan\theta_0 = -\frac{3}{16}, \tag{4.9}$$

which is impossible. $\square$

5. Non-emptiness of $\mathcal{H}_\omega$ under test family condition

In [2], it is proved that the dHYM equation admits a supercritical phase solution if and only if the triple (X, ω, α) is stable along some test family. In this section, we find that a similar type of stability also give rise to non-emptiness of the space $\mathcal{H}_\omega$ of almost calibrated $(1,1)$ forms. We start by recalling the concept of test family defined by Chen [1].

Definition 5.1. *A family of $(1,1)$ forms α_t, $t \in [0, +\infty)$ is said to be a θ-test family (emanating from a real $(1,1)$ form α) if*

(a) *$\alpha_0 = \alpha$;*
(b) *$\alpha_t > \alpha_s$ if $t > s$;*
(c) *there exists $T \geq 0$ such that $\alpha_T > \cot\left(\frac{\theta}{n}\right) \cdot \omega$ for all $t > T$.*

Now we are ready to state the criteria in terms of test family.

Theorem 5.1. *Suppose (1.1) holds and there exists a $(\theta_0 + \frac{\pi}{2})$-test family α_t for some $\theta_0 \in (0, \frac{\pi}{2})$ such that for any p-dimensional subvariety $V \subset X$,*

$$\int_V \operatorname{Re}(\alpha_t + \sqrt{-1}\omega)^p - \cot\left(\theta_0 + \frac{\pi}{2}\right) \cdot \operatorname{Im}(\alpha_t + \sqrt{-1}\omega)^p > 0. \tag{5.1}$$

then $\mathcal{H}_\omega \neq \emptyset$. Conversely, if $\mathcal{H}_\omega \neq \emptyset$ and $[\alpha]$ has hypercritical phase $\theta_0 \in (0, \frac{\pi}{2})$, then (5.1) holds for any p-dimensional irreducible subvariety $V \subset X$.

Proof. Suppose (5.1) holds for some Θ_0-test family α_t where $\Theta_0 = \frac{\pi}{2} + \theta_0$. The non-emptiness of $\mathcal{H}_\omega$ follows from the argument of [2, Theorem 1.3] on the existence of dHYM solution under stability assumption, see also [1, Section 5].

Since the proof is almost identical, we only point out the modifications. As in [2, (7.2)], we consider the twisted dHYM equation for $\alpha_{t,\varphi} = \alpha_t + \sqrt{-1}\partial\bar{\partial}\varphi(t)$:

$$\mathrm{Re}(\alpha_{t,\varphi} + \sqrt{-1}\omega)^n - \cot\Theta_0 \cdot \mathrm{Im}(\alpha_{t,\varphi} + \sqrt{-1}\omega)^n = c_t\omega^n \tag{5.2}$$

where c_t is the normalization constant so that their integral over X coincides. Define also the continuity path:

$$\mathcal{T} = \{t \in [0, +\infty) : (5.2) \text{ admits a solution } \alpha_{t,\varphi} \in \Gamma_{\omega,\alpha_t,\Theta_0,\tilde{\Theta}_0}\} \tag{5.3}$$

where $\tilde{\Theta}_0 \in (\Theta_0, \pi)$ is some constant as in the proof of [2, Theorem 1.3]. By assumption (ii), $c_t > 0$ for all $t \in [0, +\infty)$. The openness and closeness of $\mathcal{T}$ follows from the same argument. Since c_0 is strictly positive in this case (which is the only distinction from [2]), we obtain a $\varphi_0 \in C^\infty(X)$ so that

$$\mathrm{Re}(\alpha_{\varphi_0} + \sqrt{-1}\omega)^n - \cot\Theta_0 \cdot \mathrm{Im}(\alpha_{\varphi_0} + \sqrt{-1}\omega)^n = c_0\omega^n > 0. \tag{5.4}$$

In particular, $Q_\omega(\alpha_{\varphi_0}) \in (0, \theta_0 + \frac{\pi}{2})$ and hence $\varphi_0 \in \mathcal{H}_\omega$.

Conversely, if $\mathcal{H}_\omega \neq \emptyset$ and $[\alpha]$ has hypercritical phase $\theta_0 \in (0, \frac{\pi}{2})$. Then there is $\varphi \in C^\infty(X)$ such that $Q_\omega(\alpha_\varphi) \in (0, \Theta_0)$ where $\Theta_0 = \frac{\pi}{2} + \theta_0 < \pi$. By the same argument of [2, Lemma 2.3] (see also [3, Lemma 8.2]), for any $p = 1, 2, \ldots, n$, we see that

$$\mathrm{Im}\left(e^{-\sqrt{-1}\Theta_0}(\alpha_\varphi + \sqrt{-1}\omega)^p\right) < 0. \tag{5.5}$$

We define the test family $\alpha_t = \alpha + t\omega$. Since $[\alpha_\varphi] = [\alpha] = [\alpha_0]$, for any p-dimensional subvariety $V \subset X$,

$$\int_V \mathrm{Re}(\alpha_0 + \sqrt{-1}\omega)^p - \cot\Theta_0 \cdot \mathrm{Im}(\alpha_0 + \sqrt{-1}\omega)^p > 0. \tag{5.6}$$

Since

$$\begin{aligned} &\frac{d}{dt}\int_V \mathrm{Re}(\alpha_t + \sqrt{-1}\omega)^p - \cot\Theta_0 \cdot \mathrm{Im}(\alpha_t + \sqrt{-1}\omega)^p \\ &= p\int_V \left(\mathrm{Re}(\alpha_t + \sqrt{-1}\omega)^{p-1} - \cot\Theta_0 \cdot \mathrm{Im}(\alpha_t + \sqrt{-1}\omega)^{p-1}\right) \wedge \omega > 0. \end{aligned} \tag{5.7}$$

The assertion follows. This completes the proof. □

Remark 5.1. As in [2, Corollary 1.4, Corollary 1.5], the stability condition (5.1) in terms of test family can also be ensured by requiring: for some Kähler class χ in X such that for any p-dimensional subvariety $V \subset X$ and $0 \leq m \leq p$,

$$\int_V \left\{\mathrm{Re}(\alpha + \sqrt{-1}\omega)^{p-m} - \cot\left(\theta_0 + \frac{\pi}{2}\right) \cdot \mathrm{Im}(\alpha + \sqrt{-1}\omega)^{p-m}\right\} \wedge \chi^m > 0.$$

In particular, if X is projective, then the above condition can be weaken as

$$\int_V \mathrm{Re}(\alpha + \sqrt{-1}\omega)^p - \cot\left(\theta_0 + \frac{\pi}{2}\right) \cdot \mathrm{Im}(\alpha + \sqrt{-1}\omega)^p > 0. \tag{5.8}$$

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