DEFORMATIONS OF MODIFIED r-MATRICES AND COHOMOLOGIES OF RELATED ALGEBRAIC STRUCTURES

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Abstract. Modified r-matrices are solutions of the modified classical Yang-Baxter equation, introduced by Semenov-Tian-Shansky, and play important roles in mathematical physics. In this paper, first we introduce a cohomology theory for modified r-matrices. Then we study three kinds of deformations of modified r-matrices using the established cohomology theory, including algebraic deformations, geometric deformations and linear deformations. We give the differential graded Lie algebra that governs algebraic deformations of modified r-matrices. For geometric deformations, we prove the rigidity theorem and study when is a neighborhood of a modified r-matrix smooth in the space of all modified r-matrix structures. In the study of trivial linear deformations, we introduce the notion of a Nijenhuis element for a modified r-matrix. Finally, applications are given to study deformations of complement of the diagonal Lie algebra and compatible Poisson structures.

Contents

1. Introduction

In the seminal work [25], Semenov-Tian-Shansky showed that solutions of the modified classical Yang-Baxter equation, which we call modified r-matrices in this paper, play an important role in studying solutions of Lax equations [23, 25, 26]. Furthermore, modified r-matrices are intimately related to particular factorization problems in the corresponding Lie algebras and Lie groups. This factorization problem was considered by Reshetikhin and Semenov-Tian-Shansky in the framework of the enveloping algebra of a Lie algebra with a modified r-matrix to study quantum integrable systems [24]. Any modified r-matrix induces a post-Lie algebra [1], and a factorization theorem for group-like elements of the completion of the Lie enveloping algebra of a post-Lie algebra was established by Ebrahimi-Fard, Mencattini and Munthe-Kaas in [9, 10]. Recently, the global factorization theorem for a Rota-Baxter Lie group was given in [15]. Moreover, modified r-matrices are also useful for the construction of flat metrics and Frobenius manifolds

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[27], and compatible Poisson structures [18]. Note that in the associative algebra context, such objects are called modified Rota-Baxter algebras by Zhang, Gao and Guo [30, 31].

A classical approach to study a mathematical structure is to associate to it invariants. Among these, cohomology theories occupy a central position as they enable for example to control deformations or extension problems. Note that the cohomology theory for a skew-symmetric classical r-matrix was studied in [29] under the general framework of relative Rota-Baxter operators (also called $\mathcal{O}$-operators [17]). The first purpose of this paper is to study the cohomology theory for a modified r-matrix. In [25], Semenov-Tian-Shansky showed that a modified r-matrix $R:\mathfrak{g}\to\mathfrak{g}$ on a Lie algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$ induces a new Lie algebra $\mathfrak{g}_R$ in which the Lie bracket $[\cdot,\cdot]_R$ is given by

$$[x,y]_R=[R(x),y]_{\mathfrak{g}}+[x,R(y)]_{\mathfrak{g}},\quad \forall x,y\in\mathfrak{g}.$$

In [2], Bordemann showed that the induced Lie algebra $\mathfrak{g}_R$ represents on $\mathfrak{g}$. We use the corresponding Chevalley-Eilenberg cohomology [4] of the Lie algebra $\mathfrak{g}_R$ with coefficients in $\mathfrak{g}$ to define the cohomology of the modified r-matrix R. It is well known that there is a one-to-one correspondence between modified r-matrix R and Rota-Baxter operator B of weight 1 via the relation $R=\mathrm{Id}+2B$. The cohomology theory of the latter was given in [16] and the Van Est type theorem was established. We also show that the cohomology of the modified r-matrix $R=\mathrm{Id}+2B$ and the cohomology of the Rota-Baxter operator B are isomorphic.

The concept of a formal deformation of an algebraic structure began with the seminal work of Gerstenhaber [13, 14] for associative algebras. Nijenhuis and Richardson extended this study to Lie algebras [20, 21]. There is a well known slogan, often attributed to Deligne, Drinfeld and Kontsevich: every reasonable deformation theory is controlled by a differential graded Lie algebra, determined up to quasi-isomorphism. This slogan has been made into a rigorous theorem by Lurie and Pridham [19, 22]. It is also meaningful to deform *maps* compatible with given algebraic structures. Recently, the deformation theory of morphisms was developed in [3, 11, 12], the deformation theories of $\mathcal{O}$-operators on Lie algebras and associative algebras were developed in [29, 6]. The second purpose of the paper is to study deformation theories of modified r-matrices. We study three kinds of deformations of a modified r-matrix R:

- (algebraic deformations) first we consider algebraic deformation $R+R'$ for certain linear map R', and show that this kind of deformations are governed by a differential graded Lie algebra. This fulfill the general slogan for the deformation theory proposed by Deligne, Drinfeld and Kontsevich;
- (geometric deformations) then we consider smooth geometric deformation R_t such that $R_0=R$ using the approach developed by Crainic, Schatz and Struchiner in [5]. We show that the tangent space $T_R\mathrm{Orb}_R$ of the orbit Orb_R is the space of 2-coboundaries $B^2(R)$. Consequently, the condition $H^2(R)=0$ will imply certain rigidity theorem, and the condition $H^3(R)=0$ will imply the space of modified r-matrices on the Lie algebra $\mathfrak{g}$ is a manifold in a neighborhood of R. We also give the necessary and sufficient condition on a 2-cocycle giving a geometric deformation using the Kuranishi map;
- (linear deformation) next we study linear deformation $R+t\hat{R}$. In particular, trivial linear deformations leads to the concept of Nijenhuis elements for a modified r-matrix. If $x\in\mathfrak{g}$ is a Nijenhuis element, then ad_x is a Nijenhuis operator on the Lie algebra $\mathfrak{g}_R$.

Note that certain particular deformation of classical r-matrices are considered in [28] in the study of integrable infinite-dimensional systems.

The papers is organized as follows. In Section 2, we define the cohomology of a modified r-matrix R using the Chevalley-Eilenberg cohomology of the Lie algebra $\mathfrak{g}_R$ with coefficients in $\mathfrak{g}$. In Section 3, we construct a differential graded Lie algebra that governs algebraic deformations of a modified r-matrix. In Section 4, we study geometric deformations of a modified r-matrix. In Section 5, we study linear deformations of a modified r-matrix. In Section 6, we study deformations of complement of the diagonal Lie algebra and compatible Poisson structures as applications.

2. Cohomologies of modified r-matrices

In this section, we establish the cohomology theory of a modified r-matrix R using the Chevalley-Eilenberg cohomology of the Lie algebra $\mathfrak{g}_R$ with coefficients in $\mathfrak{g}$.

Definition 2.1. ([25]) Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a Lie algebra. A linear map $R : \mathfrak{g} \to \mathfrak{g}$ is called a **modified r-matrix** if it is a solution of the following **modified classical Yang-Baxter equation**:

$$[R(x), R(y)]_{\mathfrak{g}} = R([R(x), y]_{\mathfrak{g}} + [x, R(y)]_{\mathfrak{g}}) - [x, y]_{\mathfrak{g}}, \quad \forall x, y \in \mathfrak{g}. \tag{1}$$

Definition 2.2. *Let R and R' be modified r-matrices on a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$. A* **homomorphism** *from R to R' is a Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{g}$ such that*

$$\varphi \circ R = R' \circ \varphi.$$

Remark 2.3. *The notion of a modified Rota-Baxter operator of weight -1 on an associative algebra was introduced in [8]. More precisely, it is a linear map $P : A \to A$ on an associative algebra $(A, \cdot_A)$ satisfying*

$$P(u) \cdot_A P(v) = P(P(u) \cdot_A v + u \cdot_A P(v)) - u \cdot_A v, \quad \forall u, v \in A.$$

It is straightforward to see that if a linear map $P : A \to A$ is a modified Rota-Baxter operator of weight -1 on an associative algebra $(A, \cdot_A)$, then P is a modified r-matrix on the Lie algebra $(A, [\cdot,\cdot]_A)$, where $[\cdot,\cdot]_A$ is the commutator Lie bracket.

Remark 2.4. *Let $R : \mathfrak{g} \to \mathfrak{g}$ be a linear map on a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$. Under the condition $R^2 = \mathrm{Id}$, the following structures are equivalent:*

- *R is a modified r-matrix;*
- *R is a Nijenhuis operator;*
- *R is a product structure;*
- *There is a vector space direct sum decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ of $\mathfrak{g}$ into subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ such that R is given by*

$$R(x, u) = (x, -u), \quad \forall x \in \mathfrak{g}_1, u \in \mathfrak{g}_2.$$

Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a Lie algebra and R be a modified r-matrix. Semenov-Tian-Shansky showed that $(\mathfrak{g}, [\cdot,\cdot]_R)$ is a Lie algebra which plays important roles in the study of integrable systems [25], where

$$[x, y]_R = [R(x), y]_{\mathfrak{g}} + [x, R(y)]_{\mathfrak{g}}, \quad \forall x, y \in \mathfrak{g}. \tag{2}$$

Recall that a matched pair of Lie algebras consists of Lie algebras $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$, $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}})$, a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h})$ of $\mathfrak{g}$ on $\mathfrak{h}$ and a representation $\varrho : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g})$ of $\mathfrak{h}$ on $\mathfrak{g}$, such that some compatibility conditions are satisfied. Bordemann further showed that the induced Lie algebra $\mathfrak{g}_R$ represents on $\mathfrak{g}$ which leads to a matched pair of Lie algebras $((\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}), (\mathfrak{g}, [\cdot,\cdot]_R))$ [2]. Here we give a direct proof to be self-contained.

Proposition 2.5. *Let R be a modified r-matrix on a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$. Define a linear map $\rho : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ by*

$$\rho(x)y = [R(x), y]_{\mathfrak{g}} - R([x, y]_{\mathfrak{g}}), \quad \forall x, y \in \mathfrak{g}. \tag{3}$$

Then ρ is a representation of the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_R)$ on the vector space $\mathfrak{g}$.

Proof. For all $x, y, z \in \mathfrak{g}$, by (1) and (3), we have

$$\begin{aligned}
&[\rho(x), \rho(y)]z\\
= &\ \rho(x)\rho(y)z - \rho(y)\rho(x)z\\
= &\ \rho(x)([R(y), z]_{\mathfrak{g}} - R([y, z]_{\mathfrak{g}})) - \rho(y)([R(x), z]_{\mathfrak{g}} - R([x, z]_{\mathfrak{g}}))\\
= &\ [R(x), [R(y), z]_{\mathfrak{g}}]_{\mathfrak{g}} - [R(x), R([y, z]_{\mathfrak{g}})]_{\mathfrak{g}} - R([x, [R(y), z]_{\mathfrak{g}}]_{\mathfrak{g}}) + R([x, R([y, z]_{\mathfrak{g}})]_{\mathfrak{g}})\\
&-[R(y), [R(x), z]_{\mathfrak{g}}]_{\mathfrak{g}} + [R(y), R([x, z]_{\mathfrak{g}})]_{\mathfrak{g}} + R([y, [R(x), z]_{\mathfrak{g}}]_{\mathfrak{g}}) - R([y, R([x, z]_{\mathfrak{g}})]_{\mathfrak{g}})\\
= &\ [[R(x), R(y)]_{\mathfrak{g}}, z]_{\mathfrak{g}} - R([R(x), [y, z]_{\mathfrak{g}}]_{\mathfrak{g}}) - R([x, [R(y), z]_{\mathfrak{g}}]_{\mathfrak{g}}) + [x, [y, z]_{\mathfrak{g}}]_{\mathfrak{g}}\\
&+R([R(y), [x, z]_{\mathfrak{g}}]_{\mathfrak{g}}) - [y, [x, z]_{\mathfrak{g}}]_{\mathfrak{g}} + R([y, [R(x), z]_{\mathfrak{g}}]_{\mathfrak{g}})\\
= &\ [[R(x), R(y)]_{\mathfrak{g}}, z]_{\mathfrak{g}} + [[x, y]_{\mathfrak{g}}, z]_{\mathfrak{g}} - R([[R(x), y]_{\mathfrak{g}}, z]_{\mathfrak{g}}) - R([[x, R(y)]_{\mathfrak{g}}, z]_{\mathfrak{g}}),
\end{aligned}$$

and

$$\begin{aligned}
&\rho([x, y]_R)z\\
= &\ \rho([R(x), y]_{\mathfrak{g}} + [x, R(y)]_{\mathfrak{g}})z\\
= &\ [R([R(x), y]_{\mathfrak{g}} + [x, R(y)]_{\mathfrak{g}}), z]_{\mathfrak{g}} - R([[R(x), y]_{\mathfrak{g}} + [x, R(y)]_{\mathfrak{g}}, z]_{\mathfrak{g}})\\
= &\ [[R(x), R(y)]_{\mathfrak{g}}, z]_{\mathfrak{g}} + [[x, y]_{\mathfrak{g}}, z]_{\mathfrak{g}} - R([[R(x), y]_{\mathfrak{g}}, z]_{\mathfrak{g}}) - R([[x, R(y)]_{\mathfrak{g}}, z]_{\mathfrak{g}}).
\end{aligned}$$

Thus we have $\rho([x, y]_R) = [\rho(x), \rho(y)]$, which means that ρ is a representation of $(\mathfrak{g}, [\cdot,\cdot]_R)$ on the vector space $\mathfrak{g}$. □

Let $\mathrm{d}^R_{\mathsf{CE}} : \mathrm{Hom}(\wedge^k\mathfrak{g}, \mathfrak{g}) \to \mathrm{Hom}(\wedge^{k+1}\mathfrak{g}, \mathfrak{g})$ be the corresponding Chevalley-Eilenberg coboundary operator of the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_R)$ with coefficients in the representation $(\mathfrak{g}, \rho)$. More precisely, for all $f \in \mathrm{Hom}(\wedge^k\mathfrak{g}, \mathfrak{g})$ and $x_1, \cdots, x_{k+1} \in \mathfrak{g}$, we have

$$\begin{aligned}
&\mathrm{d}^R_{\mathsf{CE}} f(x_1, \cdots, x_{k+1}) \qquad (4)\\
= &\ \sum_{i=1}^{k+1}(-1)^{i+1}\rho(x_i)f(x_1, \cdots, \hat{x}_i, \cdots, x_{k+1})\\
&+ \sum_{i<j}(-1)^{i+j} f([x_i, x_j]_R, x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_{k+1})\\
= &\ \sum_{i=1}^{k+1}(-1)^{i+1}[R(x_i), f(x_1, \cdots, \hat{x}_i, \cdots, x_{k+1})]_{\mathfrak{g}}\\
&- \sum_{i=1}^{k+1}(-1)^{i+1}R([x_i, f(x_1, \cdots, \hat{x}_i, \cdots, x_{k+1})]_{\mathfrak{g}}) \qquad (5)\\
&+ \sum_{i<j}(-1)^{i+j} f([R(x_i), x_j]_{\mathfrak{g}} + [x_i, R(x_j)]_{\mathfrak{g}}, x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_{k+1}).
\end{aligned}$$

Now, we define the cohomology of a modified r-matrix $R : \mathfrak{g} \to \mathfrak{g}$. Define the space of 0-cochains $C^0(R)$ to be 0 and define the space of 1-cochains $C^1(R)$ to be $\mathfrak{g}$. For $n \geq 2$, define the space of n-cochains $C^n(R)$ by $C^n(R) = \mathrm{Hom}(\wedge^{n-1}\mathfrak{g}, \mathfrak{g})$.

Definition 2.6. *Let* $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$ *be a Lie algebra and* R *be a modified* r*-matrix. The cohomology of the cochain complex* $(\oplus_{i=0}^{+\infty}C^i(R),\mathrm{d}_{\mathsf{CE}}^R)$ *is defined to be the* **cohomology for the modified r-matrix** R.

Denote the set of n-cocycles by $Z^n(R)$, the set of n-coboundaries by $B^n(R)$ and the n-th cohomology group by

$$H^n(R)=Z^n(R)/B^n(R),\quad n\geq 0.$$

It is obvious that $x\in\mathfrak{g}$ is closed if and only if

$$\mathrm{ad}_x\circ R=R\circ\mathrm{ad}_x,$$

and $f\in\mathrm{Hom}(\mathfrak{g},\mathfrak{g})$ is closed if and only if

$$[R(x),f(y)]_{\mathfrak{g}}-R([x,f(y)]_{\mathfrak{g}})-[R(y),f(x)]_{\mathfrak{g}}+R([y,f(x)]_{\mathfrak{g}})=f([R(x),y]_{\mathfrak{g}}+[x,R(y)]_{\mathfrak{g}}),\tag{6}$$

for all $x,y\in\mathfrak{g}$.

At the end of this section, we recall the cohomology theory of Rota-Baxter operators given in [16], and establish its relation with the cohomology theory of modified r-matrices.

Definition 2.7. *Let* $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$ *be a Lie algebra. A linear map* $B:\mathfrak{g}\longrightarrow\mathfrak{g}$ *is called a* **Rota-Baxter operator of weight** λ *if*

$$[B(x),B(y)]_{\mathfrak{g}}=B([B(x),y]_{\mathfrak{g}}+[x,B(y)]_{\mathfrak{g}}+\lambda[x,y]_{\mathfrak{g}}),\quad\forall x,y\in\mathfrak{g}.$$

The following result is well known.

Proposition 2.8. *Let* $\mathfrak{g}$ *be a Lie algebra and* $B\in\mathfrak{gl}(\mathfrak{g})$. *The linear map* $\mathrm{Id}+2B$ *is a modified* r*-matrix on* $\mathfrak{g}$ *if and only if* B *is a Rota-Baxter operator of weight* 1 *on* $\mathfrak{g}$.

Let B be a Rota-Baxter operator of weight 1 on a Lie algebra $\mathfrak{g}$. Consider the cochain complex $(\oplus_{k=1}^{+\infty}C^k(B),\mathrm{d}_{\mathsf{CE}}^B)$, where $C^1(B)=\mathfrak{g}$ and $C^k(B)=\mathrm{Hom}(\wedge^{k-1}\mathfrak{g},\mathfrak{g})$ for $k\geq 2$, and $\mathrm{d}_{\mathsf{CE}}^B$ is defined by

$$\begin{aligned}
&\mathrm{d}_{\mathsf{CE}}^B f(u_1,\cdots,u_{k+1})\\
=\;&\sum_{i=1}^{k+1}(-1)^{i+1}B([f(u_1,\cdots,\hat{u}_i,\cdots,u_{k+1}),u_i]_{\mathfrak{g}})\\
&+\sum_{i=1}^{k+1}(-1)^{i+1}[B(u_i),f(u_1,\cdots,\hat{u}_i,\cdots,u_{k+1})]_{\mathfrak{g}}\\
&+\sum_{i<j}(-1)^{i+j}f([B(u_i),u_j]_{\mathfrak{g}}-[B(u_j),u_i]_{\mathfrak{g}}+[u_i,u_j]_{\mathfrak{g}},u_1,\cdots,\hat{u}_i,\cdots,\hat{u}_j,\cdots,u_{k+1}),
\end{aligned}$$

where $f\in C^{k+1}(B)$ and $u_i\in\mathfrak{g}$, $1\leq i\leq k+1$.

It was proved in [16] that $(\mathrm{d}_{\mathsf{CE}}^B)^2=0$. The cohomology of the cochain complex $(\oplus_{k=1}^{+\infty}C^k(B),\mathrm{d}_{\mathsf{CE}}^B)$ is defined to be the cohomology of the Rota-Baxter operator B.

Theorem 2.9. *With the above notations, we have*

$$\mathrm{d}_{\mathsf{CE}}^R=2\mathrm{d}_{\mathsf{CE}}^B.$$

Consequently, for $k\geq 1$, *the* k*-th cohomology group* $H^k(B)$ *of a Rota-Baxter operator* B *is isomorphic with the* k*-th cohomology group* $H^k(R)$ *of the modified* r*-matrix* $R=\mathrm{Id}+2B$.

Proof. For $k \geq 1$, define linear maps $\Phi_k : C^k(B) \to C^k(R)$ by $\Phi_k = 2^{k-2}\mathrm{Id}$. Then the following diagram is commutative:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathfrak{g} & \xrightarrow{\mathrm{d}^B_{\mathsf{CE}}} & \mathrm{Hom}(\mathfrak{g},\mathfrak{g}) & \xrightarrow{\mathrm{d}^B_{\mathsf{CE}}} & \cdots & \xrightarrow{\mathrm{d}^B_{\mathsf{CE}}} & \mathrm{Hom}(\wedge^k\mathfrak{g},\mathfrak{g}) & \xrightarrow{\mathrm{d}^B_{\mathsf{CE}}} & \cdots \\ & & \downarrow{\scriptstyle \frac{1}{2}\mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id}} & & & & \downarrow{\scriptstyle 2^{k-1}\mathrm{Id}} & & \\ 0 & \longrightarrow & \mathfrak{g} & \xrightarrow{\mathrm{d}^R_{\mathsf{CE}}} & \mathrm{Hom}(\mathfrak{g},\mathfrak{g}) & \xrightarrow{\mathrm{d}^R_{\mathsf{CE}}} & \cdots & \xrightarrow{\mathrm{d}^R_{\mathsf{CE}}} & \mathrm{Hom}(\wedge^k\mathfrak{g},\mathfrak{g}) & \xrightarrow{\mathrm{d}^R_{\mathsf{CE}}} & \cdots . \end{array}$$

In fact, for any $f \in \mathrm{Hom}(\wedge^k\mathfrak{g},\mathfrak{g})$, $x_i \in \mathfrak{g}$, $1 \leq i \leq k+1$, we have

$$\begin{aligned}
& \mathrm{d}^R_{\mathsf{CE}}(\Phi_k f)(x_1,\cdots,x_{k+1}) \\
= \ & 2^{k-1}\Big(\sum_{i=1}^{k+1}(-1)^{i+1}([x_i, f(x_1,\cdots,\hat{x}_i,\cdots,x_{k+1})]_{\mathfrak{g}} \\
& +2[B(x_i), f(x_1,\cdots,\hat{x}_i,\cdots,x_{k+1})]_{\mathfrak{g}}) \\
& -\sum_{i=1}^{k+1}(-1)^{i+1}[x_i, f(x_1,\cdots,\hat{x}_i,\cdots,x_{k+1})]_{\mathfrak{g}} \\
& -\sum_{i=1}^{k+1}(-1)^{i+1}2B([x_i, f(x_1,\cdots,\hat{x}_i,\cdots,x_{k+1})]_{\mathfrak{g}}) \\
& +\sum_{i<j}(-1)^{i+j}2f([x_i,x_j]_{\mathfrak{g}}, x_1,\cdots,\hat{x}_i,\cdots,\hat{x}_j,\cdots,x_{k+1}) \\
& +\sum_{i<j}(-1)^{i+j}2f([B(x_i),x_j]_{\mathfrak{g}}+[x_i,B(x_j)]_{\mathfrak{g}}, x_1,\cdots,\hat{x}_i,\cdots,\hat{x}_j,\cdots,x_{k+1})\Big) \\
= \ & 2^{k}\Big(\sum_{i=1}^{k+1}(-1)^{i+1}([B(x_i), f(x_1,\cdots,\hat{x}_i,\cdots,x_{k+1})]_{\mathfrak{g}} - B([x_i, f(x_1,\cdots,\hat{x}_i,\cdots,x_{k+1})]_{\mathfrak{g}})) \\
& +\sum_{i<j}(-1)^{i+j}f([B(x_i),x_j]_{\mathfrak{g}}+[x_i,B(x_j)]_{\mathfrak{g}}+[x_i,x_j]_{\mathfrak{g}}, x_1,\cdots,\hat{x}_i,\cdots,\hat{x}_j,\cdots,x_{k+1})\Big) \\
= \ & \Phi_{k+1}(\mathrm{d}^B_{\mathsf{CE}}f)(x_1,\cdots,x_{k+1}),
\end{aligned}$$

which implies that $\mathrm{d}^R_{\mathsf{CE}} = 2\mathrm{d}^B_{\mathsf{CE}}$ and $H^k(B) \cong H^k(R), k \geq 1$. □

Example 2.10. Consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$. It is well known that the Cartan subalgebra of $\mathfrak{sl}(n,\mathbb{R})$ is $H = \mathrm{span}\{E_{ii} - E_{i+1i+1} | 1 \leq i \leq n-1\}$. Denote the Borel subalgebra of $\mathfrak{sl}(n,\mathbb{R})$ by $B(\mathfrak{sl}(n,\mathbb{R}))$. It is well known that $B(\mathfrak{sl}(n,\mathbb{R})) = H \oplus \mathrm{span}\{E_{ij} | i < j\}$. Thus $\mathfrak{sl}(n,\mathbb{R}) = B(\mathfrak{sl}(n,\mathbb{R})) \oplus A$ as vector spaces, where $A = \mathrm{span}\{E_{ij} | i > j\}$. Define a linear map $R : \mathfrak{sl}(n,\mathbb{R}) \to \mathfrak{sl}(n,\mathbb{R})$ by

$$R(x+u) = x - u, \quad \forall x \in B(\mathfrak{sl}(n,\mathbb{R})), u \in A.$$

By Remark 2.4, we obtain that R is a modified r-matrix on the Lie algebra $\mathfrak{sl}(n,\mathbb{R})$. Assume that $a = x + u \in \mathfrak{sl}(n,\mathbb{R})$ where $x \in B(\mathfrak{sl}(n,\mathbb{R}))$ and $u \in A$, such that $\mathrm{d}^R_{\mathsf{CE}}a = 0$, that is

$$\mathrm{d}^R_{\mathsf{CE}}a(y) = [R(y), a] - R([y,a]) = 0, \quad \forall y \in \mathfrak{sl}(n,\mathbb{R}).$$

- For any $y \in A$, $[R(y), a] - R([y,a]) = 0$ implies that $x \in H$.
- For any $y \in B(\mathfrak{sl}(n,\mathbb{R}))$, $[R(y), a] - R([y,a]) = 0$ implies that $u = 0$.

Thus $\mathrm{d}^R_{\mathsf{CE}}a = 0$ if and only if $a \in H$. Therefore, $H^1(R) \cong \mathbb{R}^{n-1}$.

Example 2.11. Consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$, where the Lie bracket is given by $[e, f] = h, [h, e] = 2e$ and $[h, f] = -2f$ with respect to the basis $\{e, f, h\}$. Then $R : \mathfrak{sl}(2,\mathbb{R}) \rightarrow \mathfrak{sl}(2,\mathbb{R})$ defined by

$$R(e, f, h) = (e, f, h)\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

is a modified r-matrix. Let $T = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} : \mathfrak{sl}(2,\mathbb{R}) \rightarrow \mathfrak{sl}(2,\mathbb{R})$ satisfy $\mathrm{d}^R_{\mathsf{CE}}T = 0$. Then we obtain

$$\begin{aligned} 0 &= [e, T(f)] + [f, T(e)] - R([e, T(f)]) + R([f, T(e)]), \\ 0 &= [e, T(h)] - [h, T(e)] - R([e, T(h)]) + R([h, T(e)]) + 4T(e) \end{aligned}$$

and

$$0 = -[f, T(h)] - [h, T(f)] - R([f, T(h)]) + R([h, T(f)]).$$

Thus we have $t_{11} = t_{21} = t_{31} = 0$ and $t_{22} = t_{13} = 0$. By Example 2.10, we have $B^2(R) = \mathrm{Im}\mathrm{d}^R_{\mathsf{CE}} \cong \frac{\mathfrak{g}}{\ker \mathrm{d}^R_{\mathsf{CE}}} = \frac{\mathfrak{g}}{H^1(R)} \cong \mathbb{R}^2$. Thus $H^2(R) \simeq \mathbb{R}^2$.

3. Algebraic deformations of modified r-matrices

In this section, we construct a differential graded Lie algebra that governs algebraic deformations of a modified r-matrix.

Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a Lie algebra. We consider the graded vector space $C^*(\mathfrak{g}) = \oplus_{k=1}^{+\infty}\mathrm{Hom}(\wedge^k\mathfrak{g}, \mathfrak{g})$. Define a skew-symmetric bracket operation

$$[\![\cdot,\cdot]\!] : \mathrm{Hom}(\wedge^p\mathfrak{g}, \mathfrak{g}) \times \mathrm{Hom}(\wedge^q\mathfrak{g}, \mathfrak{g}) \rightarrow \mathrm{Hom}(\wedge^{p+q}\mathfrak{g}, \mathfrak{g})$$

by

$$\begin{aligned} &[\![f, g]\!](x_1, x_2, \cdots, x_{p+q}) \\ = &\sum_{\sigma\in S(q,1,p-1)} (-1)^\sigma f([g(x_{\sigma(1)}, \cdots, x_{\sigma(q)}), x_{\sigma(q+1)}]_{\mathfrak{g}}, x_{\sigma(q+2)}, \cdots, x_{\sigma(p+q)}) \\ &-(-1)^{pq} \sum_{\sigma\in S(p,1,q-1)} (-1)^\sigma g([f(x_{\sigma(1)}, \cdots, x_{\sigma(p)}), x_{\sigma(p+1)}]_{\mathfrak{g}}, x_{\sigma(p+2)}, \cdots, x_{\sigma(p+q)}) \\ &+(-1)^{pq} \sum_{\sigma\in S(p,q)} (-1)^\sigma [f(x_{\sigma(1)}, \cdots, x_{\sigma(p)}), g(x_{\sigma(p+1)}, \cdots, x_{\sigma(p+q)})]_{\mathfrak{g}}, \end{aligned} \tag{7}$$

for all $f \in \mathrm{Hom}(\wedge^p\mathfrak{g}, \mathfrak{g}), g \in \mathrm{Hom}(\wedge^q\mathfrak{g}, \mathfrak{g})$.

Then we have the following theorem characterizing modified r-matrices.

Theorem 3.1. *Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a Lie algebra. Then $(C^*(\mathfrak{g}), [\![\cdot,\cdot]\!])$ is a graded Lie algebra and its Maurer-Cartan elements are precisely Rota-Baxter operators of weight* 0.

Moreover, a linear map $R \in \mathfrak{gl}(\mathfrak{g})$ is a modified r-matrix on the Lie algebra $\mathfrak{g}$ if and only if R satisfies the equation

$$[\![R, R]\!] = 2\pi. \tag{8}$$

where denote $[\cdot,\cdot]_{\mathfrak{g}}$ by π.

Proof. By [29, Corollary 6.1], $(C^*(\mathfrak{g}), [\![\cdot,\cdot]\!])$ is a graded Lie algebra.

For $R \in \mathfrak{gl}(\mathfrak{g})$, we have

$$[\![R,R]\!](x,y) = 2(R([R(x),y]_{\mathfrak{g}}) - R([R(y),x]_{\mathfrak{g}}) - [R(x),R(y)]_{\mathfrak{g}}), \quad \forall x,y \in \mathfrak{g}.$$

By this equality, we can deduce that on the one hand R is a Rota-Baxter operator of weight 0 if and only if $[\![R,R]\!] = 0$, i.e. R is a Maurer-Cartan element. On the other hand, R is a modified r-matrix on the Lie algebra $\mathfrak{g}$ if and only if R satisfies (8). □

Proposition 3.2. *Let R be a modified r-matrix on a Lie algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$. Then $[\![R,R]\!]$ is in the center of the graded Lie algebra $(C^*(\mathfrak{g}), [\![\cdot,\cdot]\!])$.*

Proof. Denote the Lie bracket $[\cdot,\cdot]_{\mathfrak{g}}$ by π. Since R is a modified r-matrix on the Lie algebra $\mathfrak{g}$, we have $[\![R,R]\!] = 2\pi$ via Theorem 3.1. For all $f \in \mathrm{Hom}(\wedge^k\mathfrak{g},\mathfrak{g})$, by (7), we have

$$\begin{aligned}
& [\![2\pi, f]\!](x_1,\cdots,x_k,x_{k+1},x_{k+2}) \\
= & \ 2\Big(\sum_{\sigma\in S(k,1,1)} (-1)^{|\sigma|}\pi(\pi(f(x_{\sigma(1)},\cdots,x_{\sigma(k)}),x_{\sigma(k+1)}),x_{\sigma(k+2)}) \\
& - \sum_{\sigma\in S(2,1,k-1)} (-1)^{|\sigma|} f(\pi(\pi(x_{\sigma(1)},x_{\sigma(2)}),x_{\sigma(3)}),x_{\sigma(4)},\cdots,x_{\sigma(k+2)}) \\
& + \sum_{\sigma\in S(2,k)} (-1)^{|\sigma|}\pi(\pi(x_{\sigma(1)},x_{\sigma(2)}),f(x_{\sigma(3)},\cdots,x_{\sigma(k+2)}))\Big) \\
= & \ 0,
\end{aligned}$$

which implies that $[\![R,R]\!]$ is in the center of $C^*(\mathfrak{g})$. □

We denote $[\![R,\cdot]\!]$ by d_R. Now we obtain the differential graded Lie algebra that governs algebraic deformations of a modified r-matrix.

Theorem 3.3. *With the above notations, $(C^*(\mathfrak{g}), [\![\cdot,\cdot]\!], \mathrm{d}_R)$ is a differential graded Lie algebra.*

Furthermore, $R + R'$ is still a modified r-matrix on the Lie algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$ if and only if R' is a Maurer-Cartan element of the differential graded Lie algebra $(C^*(\mathfrak{g}), [\![\cdot,\cdot]\!], \mathrm{d}_R)$.

Proof. It follows from the graded Jacobi identity that d_R is a graded derivation on the graded Lie algebra $(C^*(\mathfrak{g}), [\![\cdot,\cdot]\!])$. By Proposition 3.2, we have

$$\mathrm{d}_R^2 f = [\![R,[\![R,f]\!]]\!] = [\![[\![R,R]\!],f]\!] - [\![R,[\![R,f]\!]]\!],$$

which implies that

$$\mathrm{d}_R^2 f = [\![R,[\![R,f]\!]]\!] = \frac{1}{2}[\![[\![R,R]\!],f]\!] = 0.$$

Therefore, $(C^*(\mathfrak{g}), [\![\cdot,\cdot]\!], \mathrm{d}_R)$ is a differential graded Lie algebra.

Let R' be a linear map from $\mathfrak{g}$ to $\mathfrak{g}$. Then $R + R'$ is a modified r-matrix if and only if

$$[\![R+R', R+R']\!] = 2\pi,$$

that is

$$0 = [\![R,R']\!] + \frac{1}{2}[\![R',R']\!] = \mathrm{d}_R R' + \frac{1}{2}[\![R',R']\!].$$

Thus $R + R'$ is still a modified r-matrix on the Lie algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$ if and only if R' is a Maurer-Cartan element of the differential graded Lie algebra $(C^*(\mathfrak{g}), [\![\cdot,\cdot]\!], \mathrm{d}_R)$. □

At the end of this section, we establish the relationship between the coboundary operator $\mathrm{d}_{\mathsf{CE}}^R$ and the differential d_R.

Proposition 3.4. *Let R be a modified r-matrix on a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$. Then we have*

$$\mathrm{d}^R_{\mathsf{CE}}(f) = (-1)^{n-1}\,[\![R, f]\!], \quad \forall f \in \mathrm{Hom}(\wedge^{n-1}\mathfrak{g}, \mathfrak{g}).$$

Proof. For any $f \in \mathrm{Hom}(\wedge^{n-1}\mathfrak{g}, \mathfrak{g})$ and x_i, $1 \le i \le n$, by (7), we have

$$\begin{aligned}
& (-1)^{n-1}\,[\![R, f]\!]\,(x_1, \cdots, x_n) \\
= & (-1)^{n-1}\Big(\sum_{\sigma\in S(n-1,1)} (-1)^{\sigma} R([f(x_{\sigma(1)}, \cdots, x_{\sigma(n-1)}), x_{\sigma(n)}]_{\mathfrak{g}}) \\
& -(-1)^{n-1} \sum_{\sigma\in S(1,1,n-2)} (-1)^{\sigma} f([R(x_{\sigma(1)}), x_{\sigma(2)}]_{\mathfrak{g}}, x_{\sigma(3)}, \cdots, x_{\sigma(n)}) \\
& +(-1)^{n-1} \sum_{\sigma\in S(1,n-1)} (-1)^{\sigma} [R(x_{\sigma(1)}), f(x_{\sigma(2)}, \cdots, x_{\sigma(n)})]_{\mathfrak{g}}\Big) \\
= & \sum_{i=1}^{n} (-1)^{i+1} R([f(x_1, \cdots, \hat{x}_i, \cdots, x_n), x_i]_{\mathfrak{g}}) \\
& + \sum_{i<j} (-1)^{i+j}\Big(f([R(x_i), x_j]_{\mathfrak{g}}, x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_n) \\
& - f([R(x_j), x_i]_{\mathfrak{g}}, x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_n)\Big) \\
& + \sum_{i=1}^{n} (-1)^{i+1} [R(x_i), f(x_1, \cdots, \hat{x}_i, \cdots, x_n)]_{\mathfrak{g}} \\
= & \mathrm{d}^R_{\mathsf{CE}}(f)(x_1, \cdots, x_n).
\end{aligned}$$

We finish the proof. □

4. Geometric deformations of modified r-matrices

In this section, we study geometric deformations of modified r-matrices following the approach developed by Crainic, Schatz and Struchiner. We show that the condition $H^2(R) = 0$ will imply certain rigidity theorem, and the condition $H^3(R) = 0$ will imply the space of modified r-matrices on the Lie algebra $\mathfrak{g}$ is a manifold in a neighborhood of R. We also give the necessary and sufficient condition on a 2-cocycle giving a geometric deformation using the Kuranishi map.

Definition 4.1. *Let R be a modified r-matrix on a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$. A* **geometric deformation** *of R is a smooth one parameter family of modified r-matrices R_t on the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ such that $R_0 = R$.*

Definition 4.2. *Two geometric deformations R_t and R'_t of R are called* **equivalent** *if there exists a smooth family of modified r-matrices isomorphism $\varphi_t : R_t \to R'_t$ such that $\varphi_0 = \mathrm{Id}$, where φ_t are inner automorphisms of the Lie algebra $\mathfrak{g}$.*

Let R_t be a geometric deformation of R. Denote $\frac{d}{dt}|_{t=0} R_t$ by $\dot{R}_0$. Then there is the following proposition.

Proposition 4.3. *With the above notations, $\dot{R}_0$ is a 2-cocycle in $C^2(R)$. Moreover if R_t and R'_t are equivalent geometric deformations of R, then $[\dot{R}_0] = [\dot{R}'_0]$ in $H^2(R)$.*

Proof. Since R_t is a geometric deformation of R, for any $x, y \in \mathfrak{g}$, we have

(9) $$[R(x), \dot{R}_0(y)]_{\mathfrak{g}} + [\dot{R}_0(x), R(y)]_{\mathfrak{g}}$$

$$
\begin{aligned}
&= \frac{d}{dt}|_{t=0}[R_t(x), R_t(y)]_{\mathfrak{g}} \\
&= \frac{d}{dt}|_{t=0}(R_t([R_t(x), y]_{\mathfrak{g}} + [x, R_t(y)]_{\mathfrak{g}}) - [x, y]_{\mathfrak{g}}) \\
&= \dot{R}_0([R(x), y]_{\mathfrak{g}} + [x, R(y)]_{\mathfrak{g}}) + R([\dot{R}_0(x), y]_{\mathfrak{g}} + [x, \dot{R}_0(y)]_{\mathfrak{g}}).
\end{aligned}
$$

Thus by (6) and (9), we have $\mathrm{d}^R_{\mathsf{CE}}(\dot{R}_0) = 0$.

Assume that φ_t is an isomorphism from R_t to R'_t, that is

$$\varphi_t(R_t(x)) = R'_t(\varphi_t(x)), \quad \forall x \in \mathfrak{g}.$$

Denote $\frac{d}{dt}|_{t=0}\varphi_t$ by $\dot{\varphi}_0$. Then we have $\dot{\varphi}_0(R(x)) + \dot{R}_0(x) = R(\dot{\varphi}_0(x)) + \dot{R}'_0(x)$. Since φ_t are inner automorphisms of the Lie algebra $\mathfrak{g}$, it follows that $\dot{\varphi}_0$ is an inner derivation of the Lie algebra $\mathfrak{g}$. Thus there exists $y \in \mathfrak{g}$ such that $\dot{\varphi}_0 = \mathrm{ad}_y$. Therefore, we have

$$[y, R(x)]_{\mathfrak{g}} + \dot{R}_0(x) = R([y, x]_{\mathfrak{g}}) + \dot{R}'_0(x),$$

which implies $\dot{R}_0 - \dot{R}'_0 = \mathrm{d}^R_{\mathsf{CE}}(y)$. Thus $[\dot{R}_0] = [\dot{R}'_0]$ in $H^2(R)$. □

Next, we consider under which conditions does a cocycle $f \in Z^2(R)$ determine a geometric deformation R_t. Define the Kuranishi map $K : Z^2(R) \longrightarrow H^3(R)$ by

$$K(f) = [[\![f, f]\!]], \quad \forall f \in Z^2(R).$$

Now we give a necessary condition of the above question. The sufficient condition need some preparations and will be given at the end of this section.

Proposition 4.4. *Assume that there exists a geometric deformation R_t of R on a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ such that $\dot{R}_0 = f \in Z^2(R)$, then $K(f) = 0$.*

Proof. Consider the Taylor expansion of R_t around $t = 0$, then we have

$$R_t(x) = R(x) + tf(x) + \frac{t^2}{2}g(x) + o(t^3).$$

Since $[R_t(x), R_t(y)]_{\mathfrak{g}} = R_t([R_t(x), y]_{\mathfrak{g}} + [x, R_t(y)]_{\mathfrak{g}}) - [x, y]_{\mathfrak{g}}$ and $f \in Z^2(R)$, we have

$$\frac{t^2}{2}\Big(\mathrm{d}^R_{\mathsf{CE}}(g)(x, y) + 2[f(x), f(y)]_{\mathfrak{g}} - 2f([f(x), y]_{\mathfrak{g}} + [x, f(y)]_{\mathfrak{g}})\Big) + o(t^3) = 0. \tag{10}$$

Thus by (7) and (10), we obtain $[\![f, f]\!] = \mathrm{d}^R_{\mathsf{CE}}g$, which implies that $K(f) = 0$. □

Let $E \stackrel{\pi}{\longrightarrow} M$ be a vector bundle. Assume that there is a smooth action $\cdot : G \times E \longrightarrow E$ of a Lie group G on E preserving the zero-section $Z : M \longrightarrow E$. It follows that M inherits a G-action. We also denote the action of G on M by $\cdot : G \times M \longrightarrow M$. For all $x \in M$, define a smooth map $\mu_x : G \longrightarrow M$ by $\mu_x(g) = g \cdot x$. Denote the tangent map from $\mathfrak{g}$ to T_xM by $D(\mu_x)_{e_G}$, where e_G is the unit of G.

Definition 4.5. ([5]) A section $s : M \longrightarrow E$ is called **equivariant** if s satisfies

$$s(g \cdot x) = g \cdot s(x), \quad \forall g \in G, x \in M.$$

Denote the zero set of a section $s : M \longrightarrow E$ by $z(s) = \{x \in M | s(x) = 0\}$. A zero $x \in M$ of s is called **non-degenerate** if the sequence

$$\mathfrak{g} \stackrel{D(\mu_x)_{e_G}}{\longrightarrow} T_xM \stackrel{D^v(s)_x}{\longrightarrow} E_x$$

is exact, where $D^v(s)_x$ is the vertical derivative of s at x.

Proposition 4.6. ([5]) Let s be an equivariant section of the vector bundle $E \overset{\pi}{\to} M$ and x be a non-degenerate zero of s. Then there is an open neighborhood U of x and a smooth map $p : U \to G$ such that for all $m \in U$ with $s(m) = 0$, one has $p(m) \cdot x = m$. In particular, the orbit of x under the action of G and the zero set of s coincide in an open neighborhood of x.

Proposition 4.7. ([5]) Let E and F be vector bundles over a smooth manifold M. Let $s \in \Gamma(E)$ be a section and $\phi \in \Gamma(\mathrm{Hom}(E, F))$ be a vector bundle map such that $\phi \circ s = 0$. Suppose that $x \in M$ is $s(x) = 0$ such that

$$T_xM \overset{D^v(s)_x}{\longrightarrow} E_x \overset{\phi_x}{\longrightarrow} F_x$$

is exact. Then $s^{-1}(0)$ is locally a manifold around x of dimension $\dim\ker(D^v(s)_x)$.

Denote the group whose elements are inner automorphisms of a Lie algebra $\mathfrak{g}$ by $\mathrm{InnAut}(\mathfrak{g})$. Then its Lie algebra is the Lie algebra of inner derivations of $\mathfrak{g}$ and denote it by $\mathrm{InnDer}(\mathfrak{g})$. Define an action of $\mathrm{InnAut}(\mathfrak{g})$ on $\mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$ by

$$\cdot : \mathrm{InnAut}(\mathfrak{g}) \times \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}) \to \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}), \quad A \cdot f = AfA^{-1},$$

for all $A \in \mathrm{InnAut}(\mathfrak{g}), f \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$. Assume that R is a modified r-matrix on a Lie algebra $\mathfrak{g}$, then the orbit $\mathrm{Orb}_R = \{A \cdot R | A \in \mathrm{InnAut}(\mathfrak{g})\}$ of R is a manifold. Define a map $\mu_R : \mathrm{InnAut}(\mathfrak{g}) \to \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$ by $\mu_R(A) = A \cdot R$. Then $T_R\mathrm{Orb}_R$ is $D(\mu_R)_{e_G}(\mathrm{InnDer}(\mathfrak{g}))$, where $D(\mu_R)_{e_G}$ is the tangent map of μ_R at e_G.

Proposition 4.8. *With the above notations, $T_R\mathrm{Orb}_R$ is $B^2(R)$.*

Proof. Since $T_R\mathrm{Orb}_R = D(\mu_R)_{e_G}(\mathrm{InnDer}(\mathfrak{g}))$, for any $v \in T_R\mathrm{Orb}_R$, there exists $x \in \mathfrak{g}$ such that

$$\begin{aligned} v &= \frac{d}{dt}|_{t=0} \exp(t\mathrm{ad}_x) \cdot R \\ &= \frac{d}{dt}|_{t=0}(\exp(t\mathrm{ad}_x)R\exp(-t\mathrm{ad}_x)) \\ &= \mathrm{ad}_xR - R\mathrm{ad}_x \\ &= -\mathrm{d}^R_{\mathsf{CE}}x. \end{aligned}$$

Thus we have $T_R\mathrm{Orb}_R = B^2(R)$. □

Theorem 4.9. *Let R be a modified r-matrix on a Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g})$. If $H^2(R) = 0$, then there exists an open neighborhood $U \subset \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$ of R and a smooth map $p : U \to \mathrm{InnAut}(\mathfrak{g})$ such that $p(R') \cdot R = R'$ for every modified r-matrix $R' \in U$.*

Proof. Denote by $M = \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$ and $E = \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}) \times \mathrm{Hom}(\wedge^2\mathfrak{g}, \mathfrak{g})$. Then E is a trivial vector bundle over M with fiber $\mathrm{Hom}(\wedge^2\mathfrak{g}, \mathfrak{g})$. Define an action of $\mathrm{InnAut}(\mathfrak{g})$ on the manifold E by

$$\cdot : \mathrm{InnAut}(\mathfrak{g}) \times E \to E, \quad A \cdot (f, \alpha) = (AfA^{-1}, A\alpha \circ A^{-1}),$$

for $(f, \alpha) \in E$, $A \in \mathrm{InnAut}(\mathfrak{g})$, where $A\alpha \circ A^{-1}(x, y) = A\alpha(A^{-1}x, A^{-1}y)$ for any $x, y \in \mathfrak{g}$. Define a section $s : M \to E$ by

$$s(f) = (f, S(f)), \quad \forall f \in M,$$

where $S : \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}) \to \mathrm{Hom}(\wedge^2\mathfrak{g}, \mathfrak{g})$ is given by

$$S(f)(x, y) = [f(x), f(y)]_\mathfrak{g} - f([f(x), y]_\mathfrak{g} + [x, f(y)]_\mathfrak{g}) + [x, y]_\mathfrak{g},$$

for all $f \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$, $x, y \in \mathfrak{g}$. Then for any $A \in \mathrm{InnAut}(\mathfrak{g})$, $f \in M$ and $x, y \in \mathfrak{g}$, we have

$$AS(f) \circ A^{-1}(x, y)$$

$$
\begin{aligned}
&= A([f(A^{-1}x), f(A^{-1}y)]_{\mathfrak{g}} - f([f(A^{-1}x), A^{-1}y]_{\mathfrak{g}} + [A^{-1}x, f(A^{-1}y)]_{\mathfrak{g}}) + [A^{-1}x, A^{-1}y]_{\mathfrak{g}})\\
&= [Af(A^{-1}x), Af(A^{-1}y)]_{\mathfrak{g}} - AfA^{-1}([Af(A^{-1}x), y]_{\mathfrak{g}} + [x, Af(A^{-1}y)]_{\mathfrak{g}}) + [x, y]_{\mathfrak{g}}\\
&= S(AfA^{-1})(x, y).
\end{aligned}
$$

Thus we have

$$A \cdot s(f) = (AfA^{-1}, AS(f) \circ A^{-1}) = s(A \cdot f),$$

which implies that s is an equivariant section.

Since R is a modified r-matrix on the Lie algebra $\mathfrak{g}$, it follows that $R \in z(s)$. Moreover, since E is a trivial vector bundle, we have $D^v(s)_R = D(S)_R : T_R M \longrightarrow E_R$. For any $g \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$, $x, y \in \mathfrak{g}$,

$$
\begin{aligned}
(11)\qquad & D(S)_R(g)(x, y)\\
&= \frac{d}{dt}|_{t=0} S(R + tg)(x, y)\\
&= \frac{d}{dt}|_{t=0}\Big([R(x) + tg(x), R(y) + tg(y)]_{\mathfrak{g}}\\
&\quad -(R + tg)([R(x) + tg(x), y]_{\mathfrak{g}} + [x, R(y) + tg(y)]_{\mathfrak{g}}) + [x, y]_{\mathfrak{g}}\Big)\\
&= [g(x), R(y)]_{\mathfrak{g}} + [R(x), g(y)]_{\mathfrak{g}} - R([g(x), y]_{\mathfrak{g}} + [x, g(y)]_{\mathfrak{g}}) - g([R(x), y]_{\mathfrak{g}} + [x, R(y)]_{\mathfrak{g}})\\
&= \mathrm{d}^R_{\mathsf{CE}}(g)(x, y).
\end{aligned}
$$

By Proposition 4.8 and $H^2(R) = 0$, we have that R is a non-degenerate zero of s. By Proposition 4.6, there exists an open neighborhood $U \subset \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$ of R and a smooth map $p : U \longrightarrow \mathrm{InnAut}(\mathfrak{g})$ such that $p(R) \cdot R = R'$ for every modified r-matrix $R' \in U$. □

Theorem 4.10. *Let R be a modified r-matrix on a Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$. If $H^3(R) = 0$, then the space of modified matrices on the Lie algebra $\mathfrak{g}$ is a manifold in a neighborhood of R, whose dimension is* $\dim Z^2(R)$.

Proof. Denote by $M = \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$, $E = \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}) \times \mathrm{Hom}(\wedge^2\mathfrak{g}, \mathfrak{g})$ and $F = \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}) \times \mathrm{Hom}(\wedge^3\mathfrak{g}, \mathfrak{g})$. Then E and F are trivial vector bundles over M with fiber $\mathrm{Hom}(\wedge^2\mathfrak{g}, \mathfrak{g})$ and $\mathrm{Hom}(\wedge^3\mathfrak{g}, \mathfrak{g})$ respectively. Define a smooth map $\phi : E \longrightarrow F$ by

$$\phi(f, \alpha) = (f, [\![f, \alpha]\!]), \quad \forall f \in M, \alpha \in \mathrm{Hom}(\wedge^2\mathfrak{g}, \mathfrak{g}).$$

Thus ϕ is a vector bundle map.

Moreover, denote the Lie bracket $[\cdot, \cdot]_{\mathfrak{g}}$ by π, define $s(f) = \pi - \frac{1}{2}[\![f, f]\!]$. By Proposition 3.2, we know that π lies in the center, we have $\phi \circ s(f) = (f, [\![f, \pi]\!] - \frac{1}{2}[\![f, [\![f, f]\!]]\!]) = (f, 0)$, which implies $\phi \circ s = 0$. Moreover, denote $\phi_R : E_R \longrightarrow F_R$ by $\phi(R, \cdot)$, then $\phi_R = \mathrm{d}^R_{\mathsf{CE}}$. By (11) and $H^3(R) = 0$, we have that

$$T_R M \xrightarrow{D^v(s)_R} E_R \xrightarrow{\phi_R} F_R$$

is exact. By Proposition 4.7, we obtain that the space of modified r-matrices on the Lie algebra $\mathfrak{g}$ is a manifold in a neighborhood of R, whose dimension is $\dim Z^2(R)$. □

At the end of this section, we give the sufficient condition on a 2-cocycle to give a geometric deformation. Recall that the necessary condition is given in Proposition 4.4 using the Kuranishi map.

Corollary 4.11. *With the above notations, if $H^3(R) = 0$, then any $f \in Z^2(R)$ gives rise to a geometric deformation of R.*

Proof. Since R is a modified r-matrix and $H^3(R) = 0$, we have that the space W of modified r-matrices on the Lie algebra $\mathfrak{g}$ is a manifold in a neighborhood of R, whose dimension is $\dim Z^2(R)$. Assume $\gamma(t) \in W$, by Proposition 4.3, we have $\dot{\gamma}(0) \in Z^2(R)$. Moreover, $\dim W = \dim Z^2(R)$, then $T_R W = Z^2(R)$. Thus any $f \in Z^2(R)$ gives rise to a geometric deformation of R. □

5. Linear deformations of modified r-matrices

In this section, we study linear deformations of a modified r-matrix using the established cohomology theory. In particular, a trivial linear deformation leads to a Nijenhuis element for a modified r-matrix R.

Definition 5.1. *Let R be a modified r-matrix on the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ and $\hat{R} : \mathfrak{g} \to \mathfrak{g}$ be a linear map. If there exists a positive number $\epsilon \in \mathbb{R}$ such that $R_t = R + t\hat{R}$ is still a modified r-matrix on the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ for all $t \in (-\epsilon, \epsilon)$, we say that $\hat{R}$ generates a* **linear deformation** *of the modified r-matrix R.*

Definition 5.2. *Let $R : \mathfrak{g} \to \mathfrak{g}$ be a modified r-matrix on $\mathfrak{g}$. Two linear deformations $R^1_t = R + t\hat{R}_1$ and $R^2_t = R + t\hat{R}_2$ are said to be* **equivalent** *if there exists an $x \in \mathfrak{g}$ such that*

$$\varphi_t = \mathrm{Id}_\mathfrak{g} + t\mathrm{ad}_x,$$

satisfies the following conditions:

(i) $\varphi_t([y, z]_\mathfrak{g}) = [\varphi_t(y), \varphi_t(z)]_\mathfrak{g}, \quad \forall y, z \in \mathfrak{g}$,
(ii) $R^2_t \circ \varphi_t = \varphi_t \circ R^1_t$.

Theorem 5.3. *Let $\hat{R} : \mathfrak{g} \to \mathfrak{g}$ generate a linear deformation of the modified r-matrix R. Then $\hat{R}$ is a 2-cocycle.*

Let R^1_t and R^2_t be equivalent linear deformations of R generated by $\hat{R}_1$ and $\hat{R}_2$ respectively. Then $[\hat{R}_1] = [\hat{R}_2]$ in $H^2(R)$.

Proof. Since $R_t = R + t\hat{R}$ is a modified r-matrix on the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$, we have

$$[R_t(x), R_t(y)]_\mathfrak{g} = R_t([R_t(x), y]_\mathfrak{g} + [x, R_t(y)]_\mathfrak{g}) - [x, y]_\mathfrak{g}, \quad \forall x, y \in \mathfrak{g}.$$

Consider the coefficients of t and t^2 respectively, we have

$$\begin{aligned} & [\hat{R}(x), R(y)]_\mathfrak{g} + [R(x), \hat{R}(y)]_\mathfrak{g} \\ = \ & R([\hat{R}(x), y]_\mathfrak{g} + [x, \hat{R}(y)]_\mathfrak{g}) + \hat{R}([R(x), y]_\mathfrak{g} + [x, R(y)]_\mathfrak{g}), \quad \forall x, y \in \mathfrak{g}, \end{aligned} \tag{12}$$

and

$$[\hat{R}(x), \hat{R}(y)]_\mathfrak{g} = \hat{R}([\hat{R}(x), y]_\mathfrak{g} + [x, \hat{R}(y)]_\mathfrak{g}). \tag{13}$$

By (12), we deduce that $\hat{R}$ is a 2-cocycle of the modified r-matrix R.

If R^1_t and R^2_t are equivalent linear deformations of R, then there exists $x \in \mathfrak{g}$ such that

$$(\mathrm{Id}_\mathfrak{g} + t\mathrm{ad}_x)(R + t\hat{R}_1)(u) = (R + t\hat{R}_2)(\mathrm{Id}_\mathfrak{g} + t\mathrm{ad}_x)(u), \quad \forall u \in \mathfrak{g},$$

which implies

$$\hat{R}_1(u) - \hat{R}_2(u) = [R(u), x]_\mathfrak{g} - R([u, x]_\mathfrak{g}), \quad \forall u \in \mathfrak{g}. \tag{14}$$

By (14), we have

$$\hat{R}_1 - \hat{R}_2 = \mathrm{d}^R_{\mathsf{CE}} x,$$

where $\mathrm{d}^R_{\mathsf{CE}}$ is given by (4). Thus $[\hat{R}_1] = [\hat{R}_2]$ in $H^2(R)$. □

Definition 5.4. *A linear deformation of a modified r-matrix R generated by $\hat{R}$ is* **trivial** *if there exists an $x \in \mathfrak{g}$ such that $\mathrm{Id} + t\mathrm{ad}_x$ is an isomorphism from $R_t = R + t\hat{R}$ to R.*

Definition 5.5. *Let R be a modified r-matrix on a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$. An element $x \in \mathfrak{g}$ is called a* **Nijenhuis element** *associated to R if x satisfies*

$$[[x,y]_{\mathfrak{g}}, [x,z]_{\mathfrak{g}}]_{\mathfrak{g}} = 0, \tag{15}$$

$$[x,[x,R(y)]_{\mathfrak{g}}]_{\mathfrak{g}} = [x, R([x,y]_{\mathfrak{g}})]_{\mathfrak{g}}, \tag{16}$$

for all $y, z \in \mathfrak{g}$.

Let $\hat{R}$ generate a trivial linear deformation of a modified r-matrix R on a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$. Then there exists $x \in \mathfrak{g}$ such that

$$\begin{aligned}(\mathrm{Id} + t\mathrm{ad}_x)[y,z]_{\mathfrak{g}} &= [y + t[x,y]_{\mathfrak{g}}, z + t[x,z]_{\mathfrak{g}}]_{\mathfrak{g}},\\ R(y + t[x,y]_{\mathfrak{g}}) &= (\mathrm{Id} + t\mathrm{ad}_x)(R(y) + t\hat{R}(y)),\end{aligned}$$

for all $y, z \in \mathfrak{g}$. Therefore, we have

$$[[x,y]_{\mathfrak{g}}, [x,z]_{\mathfrak{g}}]_{\mathfrak{g}} = 0, \quad [x, \hat{R}(y)]_{\mathfrak{g}} = 0, \quad R([x,y]_{\mathfrak{g}}) = [x, R(y)]_{\mathfrak{g}} + \hat{R}(y).$$

Thus a trivial linear deformation gives rise to a Nijenhuis element.

Theorem 5.6. *Let R be a modified r-matrix on a Lie algebra $\mathfrak{g}$. Then for any Nijenhuis element $x \in \mathfrak{g}$, $R_t = R + t\mathrm{d}^R_{\mathsf{CE}}x$ is a trivial linear deformation of the modified r-matrix R.*

Proof. Denote by $\hat{R} = \mathrm{d}^R_{\mathsf{CE}}x$. To show that R_t is a linear deformation of R, it suffices to show that (12) and (13) hold. Note that (12) means that $\hat{R}$ is closed, which holds naturally since now $\hat{R} = \mathrm{d}^R_{\mathsf{CE}}x$ is exact. Thus, we need to verify that Equation (13) holds. For any $y, z \in \mathfrak{g}$, by (4), we obtain $\hat{R}(y) = [R(y), x]_{\mathfrak{g}} - R([y,x]_{\mathfrak{g}})$. Moreover, by (1), (15) and (16), it follows that

$$\begin{aligned}
& [R([y,x]_{\mathfrak{g}}), R([z,x]_{\mathfrak{g}})]_{\mathfrak{g}}\\
\overset{(1),(15)}{=}\ & R\Big([R([y,x]_{\mathfrak{g}}), [z,x]_{\mathfrak{g}}]_{\mathfrak{g}} + [[y,x]_{\mathfrak{g}}, R([z,x]_{\mathfrak{g}})]_{\mathfrak{g}}\Big)\\
=\ & R\Big([R([y,x]_{\mathfrak{g}}), [z,x]_{\mathfrak{g}}]\Big) + R\Big([[y,x]_{\mathfrak{g}}, R([z,x]_{\mathfrak{g}})]_{\mathfrak{g}}\Big)\\
=\ & R\Big([[R([y,x]_{\mathfrak{g}}), z]_{\mathfrak{g}}, x]_{\mathfrak{g}}\Big) + R\Big([z, [R([y,x]_{\mathfrak{g}}), x]_{\mathfrak{g}}]\Big)\\
& + R\Big([[y, R([z,x]_{\mathfrak{g}})]_{\mathfrak{g}}, x]_{\mathfrak{g}}\Big) + R\Big([y, [x, R([z,x]_{\mathfrak{g}})]_{\mathfrak{g}}]_{\mathfrak{g}}\Big)\\
\overset{(15),(16)}{=}\ & R\Big([[R([y,x]_{\mathfrak{g}}), z]_{\mathfrak{g}}, x]_{\mathfrak{g}}\Big) + R\Big([x, [z, [x, R(y)]_{\mathfrak{g}}]_{\mathfrak{g}}]\Big)\\
& + R\Big([[y, R([z,x]_{\mathfrak{g}})]_{\mathfrak{g}}, x]_{\mathfrak{g}}\Big) - R\Big([x, [y, [x, R(z)]_{\mathfrak{g}}]_{\mathfrak{g}}]_{\mathfrak{g}}\Big),
\end{aligned}$$

$$\begin{aligned}
& -[[R(y), x]_{\mathfrak{g}}, R([z,x]_{\mathfrak{g}})]_{\mathfrak{g}}\\
=\ & -[[R(y), R([z,x]_{\mathfrak{g}})]_{\mathfrak{g}}, x]_{\mathfrak{g}} - [R(y), [x, R([z,x]_{\mathfrak{g}})]_{\mathfrak{g}}]_{\mathfrak{g}}\\
\overset{(1)}{=}\ & -[R([R(y), [z,x]_{\mathfrak{g}}]_{\mathfrak{g}}), x]_{\mathfrak{g}} - [R([y, R([z,x]_{\mathfrak{g}})]_{\mathfrak{g}}), x]_{\mathfrak{g}}\\
& + [[y, [z,x]_{\mathfrak{g}}]_{\mathfrak{g}}, x]_{\mathfrak{g}} - [R(y), [x, R([z,x]_{\mathfrak{g}})]_{\mathfrak{g}}]_{\mathfrak{g}}\\
\overset{(15),(16)}{=}\ & -[x, [x, R([R(y), z]_{\mathfrak{g}})]_{\mathfrak{g}}]_{\mathfrak{g}} - [R([z, [R(y), x]_{\mathfrak{g}}]_{\mathfrak{g}}), x]_{\mathfrak{g}} - [R([y, R([z,x]_{\mathfrak{g}})]_{\mathfrak{g}}), x]_{\mathfrak{g}}\\
& + [[y, [z,x]_{\mathfrak{g}}]_{\mathfrak{g}}, x]_{\mathfrak{g}} + [x, [x, [R(y), R(z)]_{\mathfrak{g}}]_{\mathfrak{g}}]_{\mathfrak{g}} + [[x, [R(y), x]_{\mathfrak{g}}]_{\mathfrak{g}}, R(z)]_{\mathfrak{g}}\\
\overset{(1),(16)}{=}\ & [x, [x, R([y, R(z)]_{\mathfrak{g}})]_{\mathfrak{g}}]_{\mathfrak{g}} - [x, [x, [y,z]_{\mathfrak{g}}]_{\mathfrak{g}}]_{\mathfrak{g}} + [[y, [z,x]_{\mathfrak{g}}]_{\mathfrak{g}}, x]_{\mathfrak{g}}
\end{aligned}$$

$$-[R([z,[R(y),x]_{\mathfrak{g}}]_{\mathfrak{g}}),x]_{\mathfrak{g}}-[R([y,R([z,x]_{\mathfrak{g}})]_{\mathfrak{g}}),x]_{\mathfrak{g}}-[[x,R([x,y]_{\mathfrak{g}}]_{\mathfrak{g}}),R(z)]_{\mathfrak{g}}$$

and

$$\begin{aligned}&-[R([y,x]_{\mathfrak{g}}),[R(z),x]_{\mathfrak{g}}]_{\mathfrak{g}}\\ \overset{(1)}{=}\ &[R([[R(z),y]_{\mathfrak{g}},x]_{\mathfrak{g}}),x]_{\mathfrak{g}}+[R([y,[R(z),x]_{\mathfrak{g}}]_{\mathfrak{g}}),x]_{\mathfrak{g}}+[R([z,R([y,x]_{\mathfrak{g}})]_{\mathfrak{g}}),x]_{\mathfrak{g}}\\ &-[[z,[y,x]_{\mathfrak{g}}]_{\mathfrak{g}},x]_{\mathfrak{g}}+[R(z),[x,R([y,x]_{\mathfrak{g}})]_{\mathfrak{g}}]_{\mathfrak{g}}.\end{aligned}$$

By (15) and above equations, we have

$$\begin{aligned}&[\hat{R}(y),\hat{R}(z)]_{\mathfrak{g}}-\hat{R}([\hat{R}(y),z]_{\mathfrak{g}}+[y,\hat{R}(z)]_{\mathfrak{g}})\\ =\ &[[R(y),x]_{\mathfrak{g}}-R([y,x]_{\mathfrak{g}}),[R(z),x]_{\mathfrak{g}}-R([z,x]_{\mathfrak{g}})]_{\mathfrak{g}}\\ &-[R([[R(y),x]_{\mathfrak{g}}-R([y,x]_{\mathfrak{g}}),z]),x]_{\mathfrak{g}}+R([[[R(y),x]_{\mathfrak{g}}-R([y,x]_{\mathfrak{g}}),z]_{\mathfrak{g}},x]_{\mathfrak{g}})\\ &-[R([y,[R(z),x]_{\mathfrak{g}}-R([z,x]_{\mathfrak{g}})]_{\mathfrak{g}}),x]_{\mathfrak{g}}+R([[y,[R(z),x]_{\mathfrak{g}}-R([z,x]_{\mathfrak{g}})]_{\mathfrak{g}},x]_{\mathfrak{g}})\\ =\ &-[[R(y),x]_{\mathfrak{g}},R([z,x]_{\mathfrak{g}})]_{\mathfrak{g}}+[R([y,x]_{\mathfrak{g}}),R([z,x]_{\mathfrak{g}})]_{\mathfrak{g}}-[R([y,x]_{\mathfrak{g}}),[R(z),x]_{\mathfrak{g}}]_{\mathfrak{g}}\\ &-[R([[R(y),x]_{\mathfrak{g}}-R([y,x]_{\mathfrak{g}}),z]),x]_{\mathfrak{g}}+R([[[R(y),x]_{\mathfrak{g}}-R([y,x]_{\mathfrak{g}}),z]_{\mathfrak{g}},x]_{\mathfrak{g}})\\ &-[R([y,[R(z),x]_{\mathfrak{g}}-R([z,x]_{\mathfrak{g}})]_{\mathfrak{g}}),x]_{\mathfrak{g}}+R([[y,[R(z),x]_{\mathfrak{g}}-R([z,x]_{\mathfrak{g}})]_{\mathfrak{g}},x]_{\mathfrak{g}})\\ =\ &0.\end{aligned}$$

Thus $R_t = R + t\mathrm{d}^R_{\mathsf{CE}}x$ is a linear deformation of the modified r-matrix R. Since $x\in\mathfrak{g}$ is a Nijenhuis element, we have $(\mathrm{Id}+t\mathrm{ad}_x)[y,z]_{\mathfrak{g}} = [y+t[x,y]_{\mathfrak{g}}, z+t[x,z]_{\mathfrak{g}}]_{\mathfrak{g}}$ and $R\circ(\mathrm{Id}+t\mathrm{ad}_x) = (\mathrm{Id}+t\mathrm{ad}_x)\circ(R+t\mathrm{d}^R_{\mathsf{CE}}x)$. Thus for any Nijenhuis element $x\in\mathfrak{g}$, $R_t = R+t\mathrm{d}^R_{\mathsf{CE}}x$ is a trivial linear deformation of the modified r-matrix R. □

At the end of this section, we consider the relation between linear deformations of modified r-matrices and linear deformations of the induced Lie algebras. Recall that a skew-symmetric bilinear map $\omega:\wedge^2\mathfrak{g}\to\mathfrak{g}$ generates a linear deformation of a Lie algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$ if $[\cdot,\cdot]_t = [\cdot,\cdot]_{\mathfrak{g}}+t\omega$ defines a Lie algebra structure on $\mathfrak{g}$ for all $t\in(-\epsilon,\epsilon)$.

Proposition 5.7. *Let $\hat{R}$ generate a linear deformation of a modified r-matrix R on a Lie algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$. Then ω defined by*

$$\omega(x,y) = [\hat{R}(x),y]_{\mathfrak{g}}+[x,\hat{R}(y)]_{\mathfrak{g}},\quad \forall x,y\in\mathfrak{g},$$

generates a linear deformation of the Lie algebra $(\mathfrak{g},[\cdot,\cdot]_R)$ given by the modified r-matrix R, which is exactly the one associated to the linear deformation of the modified r-matrix R.

Proof. It is obvious that

$$[x,y]_{R_t} = [R(x),y]_{\mathfrak{g}}+[x,R(y)]_{\mathfrak{g}}+t([\hat{R}(x),y]_{\mathfrak{g}}+[x,\hat{R}(y)]_{\mathfrak{g}}) = [x,y]_R+t\omega(x,y).$$

Since $[\cdot,\cdot]_{R_t}$ are Lie algebra structures, we have that ω generates a linear deformation of the Lie algebra $(\mathfrak{g},[\cdot,\cdot]_R)$ given by the modified r-matrix R. □

The notion of a Nijenhuis operator on a Lie algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$ was given in [7], which gives rise to a trivial linear deformation of the Lie algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$.

Definition 5.8. ([7]) Let $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$ be a Lie algebra. A linear map $N:\mathfrak{g}\to\mathfrak{g}$ is called **Nijenhuis operator** if

$$[N(x),N(y)]_{\mathfrak{g}} = N([N(x),y]_{\mathfrak{g}}+[x,N(y)]_{\mathfrak{g}})-N^2([x,y]_{\mathfrak{g}}),\quad \forall x,y\in\mathfrak{g}.$$

Theorem 5.9. *Let $x\in\mathfrak{g}$ be a Nijenhuis element associated to a modified matrix R. Then ad_x is a Nijenhuis operator on the Lie algebra $(\mathfrak{g},[\cdot,\cdot]_R)$.*

Proof. For any $x, y, z \in \mathfrak{g}$, by (15) and (16), we have

$$
\begin{aligned}
& [\mathrm{ad}_x y, \mathrm{ad}_x z]_R \\
= & [R([x,y]_\mathfrak{g}), [x,z]_\mathfrak{g}]_\mathfrak{g} + [[x,y]_\mathfrak{g}, R([x,z]_\mathfrak{g})]_\mathfrak{g} \\
= & [[R([x,y]_\mathfrak{g}), x]_\mathfrak{g}, z]_\mathfrak{g} + [x, [R([x,y]_\mathfrak{g}), z]_\mathfrak{g}]_\mathfrak{g} + [[x, R([x,z]_\mathfrak{g})]_\mathfrak{g}, y]_\mathfrak{g} + [x, [y, R([x,z]_\mathfrak{g})]_\mathfrak{g}]_\mathfrak{g} \\
= & -[[x,[x,R(y)]_\mathfrak{g}]_\mathfrak{g}, z]_\mathfrak{g} + [x, [R([x,y]_\mathfrak{g}), z]_\mathfrak{g}]_\mathfrak{g} + [[x,[x,R(z)]_\mathfrak{g}]_\mathfrak{g}, y]_\mathfrak{g} + [x, [y, R([x,z]_\mathfrak{g})]_\mathfrak{g}]_\mathfrak{g} \\
= & -[x, [[x,R(y)]_\mathfrak{g}, z]_\mathfrak{g}]_\mathfrak{g} + [x, [R([x,y]_\mathfrak{g}), z]_\mathfrak{g}]_\mathfrak{g} + [x, [[x,R(z)]_\mathfrak{g}, y]_\mathfrak{g}]_\mathfrak{g} + [x, [y, R([x,z]_\mathfrak{g})]_\mathfrak{g}]_\mathfrak{g}
\end{aligned}
$$

and

$$
\begin{aligned}
& \mathrm{ad}_x([\mathrm{ad}_x y, z]_R + [y, \mathrm{ad}_x z]_R) - \mathrm{ad}_x^2([y,z]_R) \\
= & [x, [R([x,y]_\mathfrak{g}), z]_\mathfrak{g}]_\mathfrak{g} + [[x,y]_\mathfrak{g}, R(z)]_\mathfrak{g} + [x, [R(y), [x,z]_\mathfrak{g}]_\mathfrak{g}]_\mathfrak{g} \\
& [x, [y, R([x,z]_\mathfrak{g})]_\mathfrak{g}]_\mathfrak{g} - [x, [x, [R(y), z]_\mathfrak{g}]_\mathfrak{g}]_\mathfrak{g} - [x, [x, [y, R(z)]_\mathfrak{g}]_\mathfrak{g}]_\mathfrak{g}.
\end{aligned}
$$

Thus $[\mathrm{ad}_x y, \mathrm{ad}_x z]_R = \mathrm{ad}_x([\mathrm{ad}_x y, z]_R + [y, \mathrm{ad}_x z]_R) - \mathrm{ad}_x^2([y,z]_R)$, which implies that ad_x is a Nijenhuis operator on the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_R)$. □

6. Applications

In this section, we give some applications of the above deformation theories, including deformations of complement of the diagonal Lie algebra $\mathfrak{g}_\Delta$ and compatible Poisson structures.

6.1. **Deformations of complements.** Let $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ be a Lie algebra, then we have a direct-product Lie algebra structure $[\cdot,\cdot]_\oplus$ on $\mathfrak{g} \oplus \mathfrak{g}$, where

$$
[(x_1, y_1), (x_2, y_2)]_\oplus = ([x_1, x_2]_\mathfrak{g}, [y_1, y_2]_\mathfrak{g}), \quad \forall x_i, y_i \in \mathfrak{g}, i = 1, 2.
$$

Define the subspace $\mathfrak{g}_\Delta$ by $\mathfrak{g}_\Delta = \{(x,x) | \forall x \in \mathfrak{g}\}$ and the subspace $\mathfrak{g}_{-\Delta} = \{(x,-x) | \forall x \in \mathfrak{g}\}$. It is obvious that $\mathfrak{g}_\Delta$ is a Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$, while $\mathfrak{g}_{-\Delta}$ is not a Lie subalgebra. To find a complement of $\mathfrak{g}_\Delta$ which is also a Lie subalgebra, it is natural to consider the graph of certain linear map from $\mathfrak{g}_{-\Delta}$ to $\mathfrak{g}_\Delta$. It is known that a complement of $\mathfrak{g}_\Delta$ is isomorphic to a graph of a linear map from $\mathfrak{g}_{-\Delta}$ to $\mathfrak{g}_\Delta$. Let $R \in \mathfrak{gl}(\mathfrak{g})$ be a linear map. Define a linear map $\hat{R} : \mathfrak{g}_{-\Delta} \to \mathfrak{g}_\Delta$ by

$$
\hat{R}(x, -x) = (-R(x), -R(x)), \quad \forall x \in \mathfrak{g}.
$$

Proposition 6.1. *With the above notations, the graph $\mathcal{G}(\hat{R}) := \{\hat{R}u + u | u \in \mathfrak{g}_{-\Delta}\}$ is a Lie subalgebra of $(\mathfrak{g} \oplus \mathfrak{g}, [\cdot,\cdot]_\oplus)$ if and only if R is a modified r-matrix.*

Proof. For all $x, y \in \mathfrak{g}$, we have

$$
\begin{aligned}
& [(-R(x), -R(x)) + (x, -x), (-R(y), -R(y)) + (y, -y)]_\oplus \\
= & ([x,y]_\mathfrak{g}, [x,y]_\mathfrak{g}) + ([R(x), R(y)]_\mathfrak{g}, [R(x), R(y)]_\mathfrak{g}) \\
& + (-[R(x), y]_\mathfrak{g}, [R(x), y]_\mathfrak{g}) + (-[x, R(y)]_\mathfrak{g}, [x, R(y)]_\mathfrak{g}) \\
= & ([x,y]_\mathfrak{g} + [R(x), R(y)]_\mathfrak{g}, [x,y]_\mathfrak{g} + [R(x), R(y)]_\mathfrak{g}) \\
& + (-[R(x), y]_\mathfrak{g} - [x, R(y)]_\mathfrak{g}, [R(x), y]_\mathfrak{g} + [x, R(y)]_\mathfrak{g}).
\end{aligned}
$$

Thus $\mathcal{G}(\hat{R})$ is a Lie subalgebra if and only if

$$
R([R(x), y]_\mathfrak{g} + [x, R(y)]_\mathfrak{g}) = [x,y]_\mathfrak{g} + [R(x), R(y)]_\mathfrak{g},
$$

i.e. R is a modified r-matrix. □

Proposition 6.2. *Let R be a modified r-matrix. Then $(\mathfrak{g}_\Delta, \mathcal{G}(\hat{R}))$ is a matched pair of Lie algebras.*

Proof. It is obvious that $\mathfrak{g}\oplus\mathfrak{g}=\mathfrak{g}_\Delta\oplus\mathcal{G}(\hat{R})$ since $\mathfrak{g}_\Delta\bigcap\mathcal{G}(\hat{R})=0$. Then the conclusion follows from the fact that both $\mathfrak{g}_\Delta$ and $\mathcal{G}(\hat{R})$ are Lie subalgebras. □

Summarizing the above studies, we have the following conclusion.

Theorem 6.3. *Let R_t be a geometric deformation of a modified r-matrix R. Then $\mathcal{G}(\hat{R}_t)$ is a deformation of the complement $\mathcal{G}(\hat{R})$. Moreover, $(\mathfrak{g}_\Delta,\mathcal{G}(\hat{R}_t))$ are matched pairs of Lie algebras.*

6.2. **Compatible Poisson structures.** A **compatible Poisson structure** consists of two Poisson structures π,π' on a manifold M such that $\pi+\pi'$ is also a Poisson structure on the manifold M.

Let R be a modified r-matrix on a Lie algebra $\mathfrak{g}$. Then $(\mathfrak{g},[\cdot,\cdot]_R)$ is a Lie algebra and we denote by $(\mathfrak{g}^*,\{\cdot,\cdot\}_R)$ the corresponding linear Poisson manifold.

Proposition 6.4. *Let R be a modified r-matrix on a Lie algebra $\mathfrak{g}$ and $R_t=R+t\hat{R}$ be a linear deformation of R. For any $t_1,t_2\in\mathbb{R}$, $\{\cdot,\cdot\}_{R_{t_1}}$ and $\{\cdot,\cdot\}_{R_{t_2}}$ are compatible Poisson structures on $\mathfrak{g}^*$.*

Proof. By the fact that $R+\frac{t_1+t_2}{2}\hat{R}$ is also a modified r-matrix on the Lie algebra $\mathfrak{g}$, we have

$$[x,y]_{R_{t_1}}+[x,y]_{R_{t_2}}=2([R(x)+\frac{t_1+t_2}{2}\hat{R}(x),y]_\mathfrak{g}+[x,R(y)+\frac{t_1+t_2}{2}\hat{R}(y)]),$$

which implies that $[\cdot,\cdot]_{R_{t_1}}+[\cdot,\cdot]_{R_{t_2}}$ is also a Lie bracket on the Lie algebra $\mathfrak{g}$ by (2). Thus, for any $t_1,t_2\in\mathbb{R}$, $\{\cdot,\cdot\}_{R_{t_1}}$ and $\{\cdot,\cdot\}_{R_{t_2}}$ are compatible Poisson structures on $\mathfrak{g}^*$. □

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