ON DENSENESS OF CERTAIN DIRECTION AND GENERALIZED DIRECTION SETS

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ABSTRACT. Direction sets, recently introduced by Leonetti and Sanna, are generalization of ratio sets of subsets of positive integers. In this article, we generalize the notion of direction sets and define k-generalized direction sets and distinct k-generalized direction sets for subsets of positive integers. We prove a necessary condition for a subset of $S^{k-1} := \{\underline{x} \in [0,1]^k : ||\underline{x}|| = 1\}$ to be realized as the set of accumulation points of a distinct k-generalized direction set. We provide sufficient conditions for some particular subsets of positive integers so that the corresponding k-generalized direction sets are dense in S^{k-1} . We also consider the denseness properties of certain direction sets and give a partial answer to a question posed by Leonetti and Sanna. Finally we consider a similar question in the framework of an algebraic number field.

1. INTRODUCTION AND STATEMENTS OF RESULTS

For a non-empty set $A \subseteq \mathbb{N}$, the *ratio set* of A is defined by $R(A) := \{\frac{a}{b} \in \mathbb{Q} : a, b \in A\}$. One of the most fundamental results in real analysis, viz. \mathbb{Q} is dense in \mathbb{R} , when rephrased in terms of ratio sets, reads as the ratio set of \mathbb{N} is dense in $\mathbb{R}_{>0}$. This reformulation of the denseness of \mathbb{Q} in \mathbb{R} has spurred a lot of research in recent times. In particular, the classification of subsets of \mathbb{N} having dense ratio sets in $\mathbb{R}_{>0}$ has been a central question of investigation. In what follows, we say that A is *fractionally dense* in $\mathbb{R}_{>0}$ if R(A) is dense in $\mathbb{R}_{>0}$.

One of the most natural choices for A is the set \mathbb{P} of prime numbers and it is known to be fractionally dense (cf. [16], [19]). Several generalizations of this result have been proven over the years and several interesting subsets of natural numbers have been shown to be fractionally dense (cf. [3] - [7], [11], [14] - [16], [19] - [21], [24] - [27]). In [8], [11] and [23], analogous questions have been dealt with in the set up of algebraic number fields. Very recently, the denseness of ratio sets in the *p*-adic completion \mathbb{Q}_p have also been considered (cf. [1], [2], [12], [13], [18], [22]).

Very recently, Leonetti and Sanna [17] introduced the notion of *direction sets*, which generalizes the notion of ratio sets as follows. For an integer $k \ge 2$ and $\emptyset \ne A \subseteq \mathbb{N}$, they considered the following sets:

$$\mathcal{S}^{k-1} := \{ \underline{x} \in [0,1]^k : ||\underline{x}|| = 1 \}, \ \mathcal{D}^k(A) := \{ \rho(\underline{a}) : \underline{a} \in A^k \} \text{ and } \mathcal{D}^{\underline{k}}(A) := \{ \rho(\underline{a}) : \underline{a} \in A^{\underline{k}} \},$$

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where $\rho : \mathbb{R}_{\geq 0}^k \to \mathcal{S}^{k-1}$ is the map defined by $\rho(\underline{x}) = \frac{x}{||\underline{x}||}$ and $A^{\underline{k}} = \{\underline{a} \in A^k : a_i \neq a_j \text{ for all } i \neq j\}$. The sets $\mathcal{D}^k(A)$ and $\mathcal{D}^{\underline{k}}(A)$ are called the k-direction sets of A. We note that, for k = 2, we can identify \mathcal{S}^1 with $[0, +\infty]$ via a bijective map and thus the question of denseness in $\mathbb{R}_{>0}$ can be translated into that in \mathcal{S}^1 . Therefore, direction sets are indeed generalizations of ratio sets. Leonetti and Sanna [17, Theorem 1.2] proved a necessary and sufficient criterion that determines whether a set $X \subseteq \mathcal{S}^{k-1}$ can be realized as the set of accumulation points of $\mathcal{D}^{\underline{k}}(A)$ for some $A \subseteq \mathbb{N}$. Moreover, they proved a sufficient condition (cf. [17, Theorem 1.5]) that asserts whether $\mathcal{D}^k(A)$ is dense in \mathcal{S}^{k-1} .

In this article, we further generalize the notion of direction sets and introduce generalized k-direction sets as follows.

Definition 1. Let $k \geq 2$ be an integer and let U_1, \ldots, U_k be non-empty subsets of N. We define the k-generalized direction set for the k-tuple (U_1, \ldots, U_k) to be $\mathcal{D}^k(U_1, \ldots, U_k) := \{\rho(u_1, \ldots, u_k) : u_j \in U_j \text{ for } j = 1, \ldots, k\}$. Also, we define the distinct k-generalized direction set to be $\mathcal{D}^{\underline{k}}(U_1, \ldots, U_k) := \{\rho(u_1, \ldots, u_k) : u_j \in U_j \text{ for } j = 1, \ldots, k\}$. Also, we define the distinct k-generalized direction set to be $\mathcal{D}^{\underline{k}}(U_1, \ldots, U_k) := \{\rho(u_1, \ldots, u_k) : u_j \in U_j \text{ for } j = 1, \ldots, k \text{ and } u_i \neq u_j \text{ for all } i \neq j\}$.

Our first theorem is an analogue of Theorem 1.2 of [17] for distinct k-generalized direction sets. For any set $X \subseteq S^{k-1}$, we denote by X' the set of accumulation points of X. Also, we denote by S_k the symmetric group on k elements $\{1, \ldots, k\}$. For a permutation $\pi \in S_k$, we define $\pi(x_1, \ldots, x_k) := (x_{\pi(1)}, \ldots, x_{\pi(k)})$ for all $\underline{x} = (x_1, \ldots, x_k)$ in S^{k-1} . Also, for any subset I of $\{1, \ldots, k\}$, we define $\rho_I(\underline{x}) := \rho(\underline{y})$ where $\underline{y} = (y_1, \ldots, y_k)$ is defined as $y_i := x_i$ if $i \in I$ and the other coordinates as 0. We say that I meets \underline{x} if $x_i \neq 0$ for some $i \in I$. We state our first theorem as follows.

Theorem 1. Let $k \geq 2$ be an integer. For subsets U_1, \ldots, U_k of \mathbb{N} , let $X = \mathcal{D}^{\underline{k}}(U_1, \ldots, U_k)'$. Then, we have:

- (i) X is a closed subset of \mathcal{S}^{k-1} .
- (ii) If $U_{i_1} = \cdots = U_{i_m}$ for some $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, k\}$, then for $\pi \in S_k$ with $\pi(j) = j$ for all $j \notin \{i_1, \ldots, i_m\}$, we have $\pi(\underline{x}) \in X$ for every $\underline{x} \in X$.
- (iii) If $|U_i| \ge k$ for each $i \in \{1, \ldots, k\}$, then for every $I \subseteq \{1, \ldots, k\}$ that meets \underline{x} , we have $\rho_I(\underline{x}) \in X$.

We recall that for a non-empty set $A \subseteq \mathbb{N}$, the natural density of A is defined as $d(A) := \lim_{X \to \infty} \frac{\#\{n \in A : n \leq X\}}{X}$, provided the limit exists. The next theorem provides a sufficient condition for $\mathcal{D}^k(U_1, \ldots, U_k)$ to be dense in \mathcal{S}^{k-1} .

Theorem 2. Let $k \ge 2$ be an integer and let $U_1, \ldots, U_k \subseteq \mathbb{N}$ be such that $d(U_i)$ exists and equals $\delta_i > 0$ for all $i = 1, \ldots, k$. Assume that $\bigcap_{i=1}^k U_i$ is an infinite set. Then $\mathcal{D}^k(U_1, \ldots, U_k)$ is dense in \mathcal{S}^{k-1} .

The next theorem extends Theorem 1.5 of [17], which asserts that if for a set $A \subseteq \mathbb{N}$, there exists an increasing sequence $\{a_n\}_{n=1}^{\infty} \subseteq A$ with $\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = 1$, then $\mathcal{D}^k(A)$ is dense in \mathcal{S}^{k-1} . We generalize this for $\mathcal{D}^k(U_1, \ldots, U_k)$ as follows.

Theorem 3. Let $k \ge 2$ be an integer and let U_1, U_2, \ldots, U_k be non-empty subsets of \mathbb{N} . If there exist increasing sequences $u_i^{(n)} \subseteq U_i$ for all $i \in \{1, \ldots, k\}$ such that $\lim_{n \to \infty} \frac{u_i^{(n-1)}}{u_i^{(n)}} = 1$, then $\mathcal{D}^k(U_1, \ldots, U_k)$ is dense in \mathcal{S}^{k-1} .

Remark 1. For an integer $k \geq 2$ and for each $i \in \{1, \ldots, k\}$, let a_i and m_i be integers with $gcd(a_i, m_i) = 1$. Let $\mathbb{P}_{m_i} := \{p \in \mathbb{P} : p \equiv a_i \pmod{m_i}\}$. For $U_i = \mathbb{P}_{m_i}$, using Dirichlet's theorem for primes in arithmetic progressions, we see that the hypotheses of Theorem 3 are satisfied. Therefore, $\mathcal{D}^k(\mathbb{P}_{m_1}, \ldots, \mathbb{P}_{m_k})$ is dense in \mathcal{S}^{k-1} .

Theorem 4. Let $k \geq 2$ be an integer and for each $i \in \{1, \ldots, k\}$, let $f_i(X_1, \ldots, X_m) \in \mathbb{Z}[X_1, \ldots, X_m]$ be polynomials of total degree d_i such that the sum of the coefficients of degree d_i terms is positive. Let $U_i := \{f_i(n_1, \ldots, n_m) | (n_1, \ldots, n_m) \in \mathbb{N}^m\} \cap \mathbb{N}$. Then $\mathcal{D}^k(U_1, \ldots, U_k)$ is dense in \mathcal{S}^{k-1} .

In [5], it is proven that there is a 3-partition of $\mathbb{N} = A \cup B \cup C$, such that none of R(A), R(B)and R(C) is dense in $\mathbb{R}_{>0}$. That is, none of $\mathcal{D}^2(A), \mathcal{D}^2(B)$ and $\mathcal{D}^2(C)$ is dense in \mathcal{S}^1 . In [17], Leonetti and Sanna asked for a possible generalization of this result for $k \geq 3$ [17, Question 1.9]. We give a partial answer to their question in the next theorem.

Theorem 5. Let $k \geq 3$ be an integer. Then there exists a 3-partition $\mathbb{N} = A \cup B \cup C$ of \mathbb{N} such that none of $\mathcal{D}^k(A), \mathcal{D}^k(B)$ or $\mathcal{D}^k(C)$ is dense in \mathcal{S}^{k-1} .

Remark 2. In view of Theorem 5, it remains to be seen whether for a 2-partition $\mathbb{N} = A \cup B$, either $\mathcal{D}^k(A)$ or $\mathcal{D}^k(B)$ is dense in \mathcal{S}^{k-1} or not. We note that Theorem 3 cannot be directly applied to address this issue. This can be seen by considering $A = \bigcup_{k=0}^{\infty} [3^k, 2 \cdot 3^k) \cap \mathbb{N}$ and

 $B = \bigcup_{k=0}^{\infty} [2 \cdot 3^k, 3^{k+1}) \cap \mathbb{N}.$ For, if $\{a_n\}_{n=1}^{\infty} \subseteq A$ is an infinite sequence, then there are infinitely

many indices *i* for which $a_i \in [3^k, 2 \cdot 3^k)$ and $a_{i+1} \in [3^\ell, 2 \cdot 3^\ell)$ for $k < \ell$. Then it follows that $\frac{a_i}{a_{i+1}} < \frac{2 \cdot 3^k}{3^\ell} \le \frac{2}{3}$. Therefore, the elements of the sequence $\{\frac{a_n}{a_{n+1}}\}_{n=1}^{\infty}$ cannot get arbitrarily close to 1. Similar argument works for *B* as well. Thus there exist a 2-partition of N, none of which contains a sequence with the ratio of consecutive terms converging to 1.

One of the interesting questions in the literature of fractionally dense sets is to look for sets $A \subseteq \mathbb{N}$ such that the ratio set R(A) is dense in $\mathbb{R}_{>0}$ but A contains no 3-term arithmetic progressions. One such set is $A = \{2^m : m \ge 2\} \cup \{3^n : n \ge 2\}$, which is known to be fractionally dense in $\mathbb{R}_{>0}$ but A contains no 3-term arithmetic progressions (cf. [3, Proposition 6]). In view of this, we may ask the following question. Question 1. For an integer $k \ge 2$, does there exist a set $A \subseteq \mathbb{N}$ such that A contains no 3-term arithmetic progressions and $\mathcal{D}^k(A)$ is dense in \mathcal{S}^{k-1} ?

We answer Question 1 assertively in the following theorem.

Theorem 6. There exists a set $A \subseteq \mathbb{N}$ such that A contains no 3-term arithmetic progressions and $\mathcal{D}^k(A)$ is dense in \mathcal{S}^{k-1} .

Remark 3. We shall see in the proof of Theorem 6 that we can obtain infinitely many sets $A \subseteq \mathbb{N}$ having no arithmetic progression of length 3 such that $\mathcal{D}^k(A)$ is dense in \mathcal{S}^{k-1} .

Next, we discuss the denseness of some particular type of sets whose properties have been recently considered in [10]. For an arithmetic function $f : \mathbb{N} \to \mathbb{N}$ and a positive real number X, let $f_X := \#\{n \leq X : n = kf(k) \text{ for some } k \in \mathbb{N}\}$. Keeping this notation, we state the results of [10] as follows.

Theorem 7. [10] (i) Let
$$\omega(n) = \sum_{\substack{p|n\\p \in \mathbb{P}}} 1$$
 be the prime divisor function. Then

$$\omega_X = \frac{X}{\log \log X} + o\left(\frac{X}{\log \log X}\right).$$

(ii) Let $\phi(n) = \#\{1 \le k \le n : \gcd(k, n) = 1\}$ be the Euler's totient function. Then

$$\phi_X = cX^{\frac{1}{2}} + o(X^{\frac{1}{2}}),$$

where $c = \prod_p \left(1 + \frac{1}{p(p-1+\sqrt{p^2-p})}\right) \sim 1.365....$

Now, we state our result as follows.

Theorem 8. Let $A = \{n\omega(n) : n \in \mathbb{N}\}$ and $B = \{n\phi(n) : n \in \mathbb{N}\}$. Then for any integer $k \ge 2$, we have that both $\mathcal{D}^k(A)$ and $\mathcal{D}^k(B)$ are dense in \mathcal{S}^{k-1} .

2. Proof of Theorems

In this section, we prove our theorems. We first prove Theorem 1.

Proof of Theorem 1. Since X is the set of accumulation points of a subset of \mathcal{S}^{k-1} , we immediately conclude that X is closed and (i) is satisfied.

Now, let $\underline{x} = (x_1, x_2, \dots, x_k) \in X = \mathcal{D}^{\underline{k}}(U_1, \dots, U_k)'$. Then there exists a sequence $\rho(\underline{a}^{(n)}) \in \mathcal{D}^{\underline{k}}(U_1, \dots, U_k)$ converging to \underline{x} such that $\rho(\underline{a}^{(n)}) \neq \underline{x}$ for infinitely many n, where $\underline{a}^{(n)} \in \prod_{i=1}^k U_i$. For $\pi \in S_k$ with $\pi(j) = j$ for all $j \notin \{i_1, \dots, i_m\}$, we consider $\underline{b}^{(n)} := \pi(\underline{a}^{(n)}) \in \mathcal{D}^{\underline{k}}(U_1, \dots, U_k)$. Then $\rho(\underline{b}^{(n)})$ converges to $\pi(\underline{x})$. Consequently, we have $\pi(\underline{x}) \in X$ for every $\underline{x} \in X$ and thus (ii) is satisfied.

Now, assume that I is a non-empty subset of $\{1, \ldots, k\}$ that meets \underline{x} . We can consider a subsequence of $\underline{a}^{(n)}$ such that each $a_i^{(n)}$ is non-decreasing for each $i \in \{1, \ldots, k\}$. If $j \in \{1, \ldots, k\} \setminus I$, then we can choose distinct $c_j \in U_j$ such that for sufficiently large positive integer n_0 , a sequence $\underline{d}^{(n)} \in U_1 \times \cdots \times U_k$ with distinct coordinates can be defined for all $n \ge n_0$ with $d_i^{(n)} := a_i^{(n)}$ for $i \in I$ and $d_i^{(n)} := c_i$ for $i \notin I$. This choice is possible because of the assumption $|U_i| \ge k$ for each i. It then follows that $\rho(\underline{d}^{(n)})$ converges to $\rho_I(\underline{x})$. Thus (iii) holds. This completes the proof of Theorem 1.

Proof of Theorem 2. Let $\underline{x} \in (x_1, \ldots, x_k) \in \mathcal{S}^{k-1}$ and let $I_i = (a_i, b_i)$ be open intervals such that $x_i \in I_i$ for each $i \in \{1, \ldots, k\}$. Then $\prod_{i=1}^k (a_i, b_i) \cap \mathcal{S}^{k-1}$ is a basic open set in \mathcal{S}^{k-1} containing \underline{x} . For a real number X > 1, let $U_i(X) := \#\{u_i \in U_i | u_i \leq X\}$. By the hypothesis, we have that $\lim_{X \to \infty} \frac{U_i(X)}{X} = \delta_i > 0$. This implies that $U_i(X) = \delta_i X + o(X)$. Therefore,

$$\lim_{X \to \infty} \frac{U_i(a_i X)}{U_i(b_i X)} = \lim_{X \to \infty} \frac{\delta_i a_i X + o(a_i X)}{\delta_i b_i X + o(b_i X)} = \frac{a_i}{b_i} < 1.$$

Thus for all sufficiently large real number X, there exists $u_i \in U_i$ such that $a_i X < u_i \leq b_i X$. That is, $a_i < \frac{u_i}{X} \leq b_i$. Since $\bigcap_{i=1}^k U_i$ is an infinite set, we can choose a large enough element $u \in \bigcap_{i=1}^k U_i$ such that $a_i u < u_i \leq b_i u$ for all i = 1, ..., k. This, in turn, implies that $\frac{u_i}{u} \in (a_i, b_i)$. Using the fact that $\rho(\underline{\alpha}) = \frac{\alpha}{\|\underline{\alpha}\|}$ is continuous function, we see that $\rho(u_1, ..., u_k) \in \prod_{i=1}^k I_i \cap S^{k-1}$.

In other words, $\mathcal{D}^k(U_1,\ldots,U_k)$ is dense in \mathcal{S}^{k-1} .

We next prove Theorem 3 which extends Theorem 1.5 of [17].

Proof of Theorem 3. Let $\underline{x} = (x_1, \ldots, x_k) \in \mathcal{S}^{k-1}$ with $x_i > 0 \ \forall \ i \in \{1, \ldots, k\}$. We pick an integer m such that $m > \frac{u_i^{(1)}}{\min\{x_1, \ldots, x_k\}} \ \forall \ i \in \{1, \ldots, k\}$. Then there exist integers m_i for each $i \in \{1, \ldots, k\}$ such that $u_i^{(m_i-1)} \leq mx_i < u_i^{(m_i)}$. That is, $x_i < \frac{u_i^{(m_i)}}{m} \leq \frac{u_i^{(m_i)}}{u_i^{(m_i-1)}} x_i$. Since $m_i \to \infty$ as $m \to \infty$, it follows that $\lim_{m \to \infty} \frac{u_i^{(m_i)}}{m} = x_i$. Consequently, $\underline{u} = (u_1^{(m_1)}, \ldots, u_k^{(m_k)})$ converges to \underline{x} . Since ρ is a continuous map, $\rho(\underline{u})$ converges to \underline{x} . Consequently, $\mathcal{D}^k(U_1, \ldots, U_k)$ is dense in \mathcal{S}^{k-1} .

Proof of Theorem 4. For a fixed integer $i \in \{1, ..., k\}$, we consider the polynomial $g_i(X)$ obtained by replacing all variables of g_i by the variable X. We get, $g_i(X) = a_{d_i}X^{d_i} + a_{d_i-1}X^{d_i-1} + a_{d_i-1}X^{d_i-1}$

 $\dots + a_0 \in \mathbb{Z}[X]$. Since $a_{d_i} > 0$, we conclude that for a sufficiently large positive real number X, we have $g_i(X) > 0$. Let $B_i := \{g_i(n) | n \in \mathbb{N}\} \cap \mathbb{N}$. We have $\frac{g_i(X-1)}{g_i(X)} = \frac{a_{d_i}(X-1)^{d_i} + \dots + a_0}{a_{d_i}X^{d_i} + \dots + a_0}$ which tends to 1 as X tends to ∞ . Also, since $g_i(X)$ is a polynomial in one variable, the sequence $\{g_i(n)\}_{n=1}^{\infty}$ is eventually increasing. Therefore, by using Theorem 3, we obtain that $\mathcal{D}^k(B_i)$ is dense in \mathcal{S}^{k-1} . Since $B_i \subseteq U_i$, we conclude that $\mathcal{D}^k(U_1, \dots, U_k)$ is dense in \mathcal{S}^{k-1} .

We now prove Theorem 5 which gives a partial answer to [17, Question 1.9].

Proof of Theorem 5. We consider the following three sets as in [5] (see also [3]).

$$A := \bigcup_{k=0}^{\infty} [5^k, 2 \cdot 5^k) \cap \mathbb{N},$$
$$B := \bigcup_{k=0}^{\infty} [2 \cdot 5^k, 3 \cdot 5^k) \cap \mathbb{N},$$
$$C := \bigcup_{k=0}^{\infty} [3 \cdot 5^k, 5 \cdot 5^k) \cap \mathbb{N}.$$

If $\mathcal{D}^{k}(A)$, $\mathcal{D}^{k}(B)$ or $\mathcal{D}^{k}(C)$ is dense in \mathcal{S}^{k-1} , then by Theorem 1.4 of [17], which states that if $\mathcal{D}^{k}(A)$ is dense in \mathcal{S}^{k-1} for some $A \subseteq \mathbb{N}$, then $\mathcal{D}^{k-1}(A)$ is dense in \mathcal{S}^{k-2} , we see inductively that $\mathcal{D}^{2}(A)$ (or $\mathcal{D}^{2}(B)$ or $\mathcal{D}^{2}(C)$) is dense in \mathcal{S}^{1} , which is false (cf. [3, Proposition 3]). Therefore, we get a 3-partition of \mathbb{N} such that none of $\mathcal{D}^{k}(A)$, $\mathcal{D}^{k}(B)$ or $\mathcal{D}^{k}(C)$ is dense in \mathcal{S}^{k-1} . This completes the proof of Theorem 5.

Proof of Theorem 6. In [9], it has been proven that the equation $x^n + y^n = 2z^n$ has no non-trivial solution in \mathbb{Z} if $n \geq 3$. In other words, the set $A := \{m^r : r, m \in \mathbb{Z}, r \geq 3\}$ does not contain any 3-term arithmetic progressions. Since for a fixed value of $r \geq 3$, we have $\frac{m^r}{(m+1)^r} \to 1$ as $m \to \infty$, by Theorem 1.5 of [17], we conclude that $\mathcal{D}^k(A)$ is dense in \mathcal{S}^{k-1} .

Proof of Theorem 8. Let $\underline{x} = (x_1, \ldots, x_k) \in \mathcal{S}^{k-1}$ and let $\prod_{i=1}^{\kappa} (a_i, b_i)$ be a basic neighborhood of \underline{x} . Then by Theorem 7, we see that

$$\lim_{X \to \infty} \frac{\omega_{a_i X}}{\omega_{b_i X}} = \lim_{X \to \infty} \frac{a_i X}{\log \log a_i X} \cdot \frac{\log \log b_i X}{b_i X} = \frac{a_i}{b_i} < 1 \text{ for all } i \text{ with } 1 \le i \le k.$$

Therefore, for sufficiently large X, there exists $\alpha_i \in A$ such that $a_i X < \alpha_i < b_i X$ for all *i*. That is, $\left(\frac{\alpha_1}{X}, \ldots, \frac{\alpha_k}{X}\right) \in \prod_{i=1}^k (a_i, b_i)$. Hence $\rho(\alpha_1, \ldots, \alpha_k) = \rho\left(\frac{\alpha_1}{X}, \ldots, \frac{\alpha_k}{X}\right) \in \prod_{i=1}^k (a_i, b_i)$. Consequently, $\mathcal{D}^k(A)$ is dense in \mathcal{S}^{k-1} .

Similarly, for $\mathcal{D}^k(B)$, we note that

$$\lim_{X \to \infty} \frac{\phi_{a_i X}}{\phi_{b_i X}} = \frac{\sqrt{a_i}}{\sqrt{b_i}} < 1 \text{ for all } i \text{ with } 1 \le i \le k$$

and thereafter it follows a similar line of argument.

3. Concluding Remarks : Case of Algebraic Number fields

The ratio sets have been studied in the context of algebraic number fields in [8], [11] and [23]. It is interesting to extend the notion of direction sets in the set up of number fields and formulate analogous questions for the same.

Let $K \subsetneq \mathbb{R}$ be a number field of degree $d \ge 2$ and let \mathcal{O}_K be its ring of integers. Let $\mathcal{O}_K^0 := \{\alpha \in \mathcal{O}_K : \operatorname{Tr}_{K/\mathbb{Q}}(\alpha) = 0\}$ be the set of elements in \mathcal{O}_K with trace 0. Since \mathcal{O}_K is a free \mathbb{Z} -module of rank d and Tr is an additive group homomorphism from \mathcal{O}_K to \mathbb{Z} , we see that $\mathcal{O}_K \cong \mathcal{O}_K^0 \oplus \mathbb{Z}$. In particular, \mathcal{O}_K^0 is a free \mathbb{Z} -module of rank d-1. Therefore, \mathcal{O}_K^0 itself is dense in \mathbb{R} whenever $d \ge 3$. Also, for d = 2, we see that the ratio set of \mathcal{O}_K^0 is \mathbb{Q} . Consequently, the direction set of \mathcal{O}_K^0 is dense in \mathcal{S}^{k-1} for any integer $k \ge 2$.

We note that $\mathcal{O}_{K}^{0} \cap \mathbb{N} = \emptyset$. In view of this, we ask the following question.

Question 2. Let $d \geq 2$ and $k \geq 2$ be integers and let K be a number field of degree d. Characterize the sets $\mathcal{A} \subseteq \mathcal{O}_K$ such that $\mathcal{A} \cap \mathbb{N}$ is finite and $\mathcal{D}^{k-1}(\mathcal{A})$ is dense in \mathcal{S}^{k-1} .

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