

TROPICAL INVARIANTS FOR BINARY QUINTICS AND REDUCTION TYPES OF PICARD CURVES

PAUL ALEXANDER HELMINCK, YASSINE EL MAAZOUZ, AND ENIS KAYA

ABSTRACT. In this paper, we express the reduction types of Picard curves in terms of tropical invariants associated to binary quintics. We also give a general framework for tropical invariants associated to group actions on arbitrary varieties. The problem of finding tropical invariants for binary forms fits in this general framework by mapping the space of binary forms to symmetrized versions of the Deligne–Mumford compactification $\overline{M}_{0,n}$.

1. INTRODUCTION

Invariant theory studies quantities in geometry that are invariant under group actions. This theory sparked many developments in commutative algebra, leading to the Hilbert basis theorem and many other results. In this paper, we study invariants of binary forms $f(x, z) = a_0x^n + a_1x^{n-1}z + \cdots + a_nz^n$ defined over an algebraically closed field K of characteristic 0. The group in question is $\mathrm{GL}_2(K)$ and it acts on these binary forms through *Möbius transformations*; that is

$$f^\sigma(x, z) = f(ax + bz, cx + dz), \quad \text{for } \sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(K).$$

By mapping the coefficients a_i of f to those of f^σ , this then also gives an action of $\mathrm{GL}_2(K)$ on the ring $A := K[a_0, \dots, a_n]$. The *invariants* for this action are homogeneous polynomials $H \in A$ such that for any $\sigma \in \mathrm{GL}_2(K)$ one has $H^\sigma = \det(\sigma)^k H$ for some $k \in \mathbb{Z}$. Similarly, we say that a homogeneous polynomial $H \in A$ is $\mathrm{SL}_2(K)$ -invariant if $H^\sigma = H$ for all $\sigma \in \mathrm{SL}_2(K)$. Note that these two types of invariants are the same. We will from now on restrict our attention to the $G := \mathrm{SL}_2(K)$ action. By well-known results from commutative algebra, the invariant polynomials form a finitely generated subring A^G of A , called the ring of invariants. There are algorithms that can explicitly calculate the generators of this ring and a full list of generators is known for all binary forms of degree $n \leq 10$. These generators of the ring of invariants satisfy the pleasant property that two separable binary forms lie in the same $\mathrm{GL}_2(K)$ -orbit if and only if their invariants define the same point in the weighted projective space associated to the generators, where the weights are the degrees of the generators. This equivalence gives a strong connection between geometry on the one hand and algebra on the other.

We now turn to the non-archimedean side of this story and consider a complete non-archimedean algebraically closed field K of characteristic zero with non-trivial valuation $v: K^* \rightarrow \mathbb{R}$. Let $f(x, z)$ be a separable binary form of degree n over K . The zeroes of this form give a canonical metric tree on n leaves by connecting the corresponding points in the Berkovich analytification of \mathbb{P}^1 . There are only finitely many phylogenetic types for any given degree n (see Figure 1 for the case $n = 5$), and the possible types give a partition of the space of all non-archimedean binary forms of degree n . Since

the tree is invariant under projective isomorphisms, this also partitions the space of all invariants. A natural question now arises: what are the equations for these partitions?

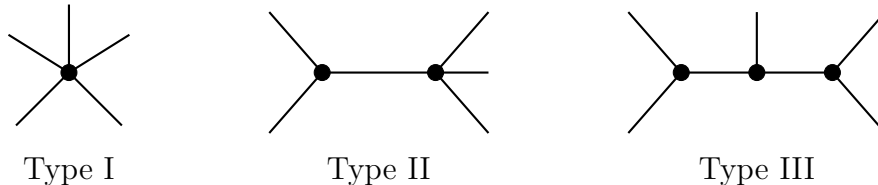


FIGURE 1. The three types of unmarked phylogenetic trees with 5 leaves

There are two instances where we know the equations: $n = 4$ and $n = 6$. For $n = 4$ there are two tree types: a trivial one with four leaves connected to a single vertex and a non-trivial one. We can distinguish between these two using the valuation of the j -invariant of the quartic. Namely, a quartic has trivial tree type if and only if $v(j) \geq 0$, see [20, Chapter VII]. For $n = 6$, there are seven tree types and one can distinguish between them using the valuations of Igusa invariants, see [14] and [10]. Our main goal in this paper is to fill up this gap and find the invariants for *binary quintics*; that is, for $n = 5$.

For quintics, there are three tree types; see Figure 1. We wish to distinguish between these tree types using the valuations of suitable invariants. To that end, we start with a set of generators I_4, I_8, I_{12}, I_{18} for the ring of invariants together with the discriminant Δ . The valuations of these invariants are not sufficient to determine the tree type of the quintic, so we introduce a new invariant H , giving the set $S = \{I_4, I_8, I_{12}, I_{18}, \Delta, H\}$. In our first theorem, we show that the valuations of these invariants determine the tree type of a quintic. We call the valuations of these invariants the *tropical invariants* of the quintic. For technical reasons, we assume for the remainder of this section that the residue characteristic p of K is not equal to 2, 3, 11. Note however that Remark 33 explains how to deal with the case where $p = 11$.

Theorem 1 (Tree types of binary quintics). *Let f be a separable binary quintic over K . Then, the tree type of f is determined by the tropical invariants as follows:*

- (I) *The tree is of Type I if and only if $8v(I) - \deg(I)v(\Delta) \geq 0$ for all $I \in S$.*
- (II) *The tree is of Type II if and only if $v(\Delta) - 2v(I_4) > 0$ or $9v(\Delta) - 4v(I_{18}) > 0$, and $12v(I) - \deg(I)v(H) \geq 0$ for all $I \in S$.*
- (III) *The tree is of Type III if and only if $v(\Delta) - 2v(I_4) > 0$ and $v(H) - 3v(I_4) > 0$.*

Finding the equations for $n = 4$ and $n = 6$ is partially motivated by applications to reduction types of hyperelliptic curves. For $n = 5$, the motivation comes from Picard curves X , which are smooth plane quartics of the form

$$y^3 \ell(x, z) = q(x, z),$$

where $\deg(\ell(x, z)) = 1$ and $\deg(q(x, z)) = 4$. We are interested in obtaining the *minimal skeleton* of the Berkovich analytification of such a curve, which codifies the different possible semistable models for X , see [4, Section 4.16]. The results in [11] show that this skeleton can be recovered from the *marked tree type* of the quintic $f(x, z) = \ell(x, z) \cdot q(x, z)$. More precisely, this quintic has five distinct roots, giving a metric tree with

five leaves, and the root of $\ell(x, z)$ gives the marking. We can assume by a projective transformation that this marked point is ∞ . There are exactly five marked tree types, see Figure 2, giving rise to five reduction types of Picard curves. In terms of invariant theory, the natural object to consider here is the binary $(4, 1)$ -form $(q(x, z), \ell(x, z))$. These binary $(4, 1)$ -forms similarly have a finitely generated ring of invariants and in our second theorem we show that we can find a set of invariants for $(4, 1)$ -forms that distinguish between the five marked tree types. This set consists of the set S from Theorem 1 together with a new set $S' = \{j_2, j_3, j_5, j_6, j_9\}$ of $(4, 1)$ -invariants. As above, we call the valuations of these invariants the *tropical invariants* of the $(4, 1)$ -form.

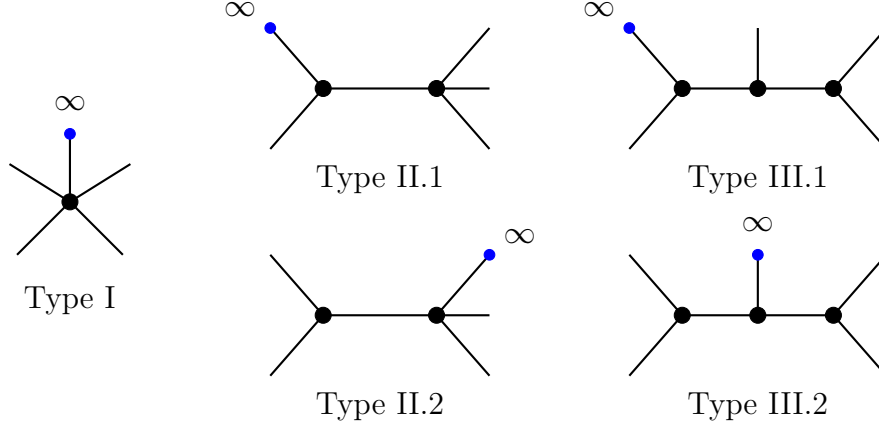


FIGURE 2. Tree types of binary $(4, 1)$ -forms

Theorem 2 (Tree types of $(4, 1)$ -forms). *Let (q, ℓ) be a $(4, 1)$ -form over K such that the associated binary quintic $f = q \cdot \ell$ is separable. The tree type of (q, ℓ) is determined by the tropical invariants as follows:*

- (I) *If f has tree Type I, then (q, ℓ) also has Type I.*
- (II) *If f has tree Type II, then (q, ℓ) has Type II.1 (resp. Type II.2) if and only if the quantity $5v(j_2) - 2v(j_5)$ is strictly positive (resp. zero).*
- (III) *If f has tree Type III, then (q, ℓ) has Type III.1 (resp. Type III.2) if and only if the quantity $5v(j_2) - 2v(j_5)$ is strictly positive (resp. zero).*

To determine the skeleton of a Picard curve, we also need to know the associated weights and edge lengths. The weights of the skeleton are completely determined by Theorem 2, but the lengths are not. In our third theorem, we give formulas for the edge lengths of a $(4, 1)$ -form in terms of its tropical invariants. For trees of Type II and Type III.2, we are able to give these in terms of invariants of quintics. For trees of Type III.1, we express the marked edge lengths in terms of $(4, 1)$ -invariants. This difference is quite natural, since there is a natural symmetry on trees of Type III.2.

Theorem 3 (Edge lengths). *The edge lengths of the trees in Theorem 2 are given by the tropical invariants as follows:*

- (I) *If f has Type I, then there are no non-trivial edges.*

(II) If f has Type II, then there is only one edge e_1 . Its length, both in the cases of unmarked and marked trees, is given by

$$L(e_1) = \max \left(\frac{1}{2}(v(\Delta) - 2v(I_4)), \frac{1}{3}(2v(\Delta) - v(I_{18})) \right).$$

(III) If f has Type III, then there are two edges e_1 and e_2 . Assume the length of e_1 is less than or equal to that of e_2 . The unmarked edge lengths are given by

$$(1) \quad \begin{aligned} L(e_1) &= \min \left(\frac{1}{2} \left(v(I_{18}) - \frac{9}{2}v(I_4) \right), \frac{1}{4} (v(\Delta) - 2v(I_4)) \right), \\ L(e_2) &= v(\Delta) - 2v(I_4) - 2L(e_1). \end{aligned}$$

If (q, ℓ) has Type III.1, then we write e_1 for the edge adjacent to the marked point and e_2 for the other edge. The edge lengths are then given by

$$\begin{aligned} L(e_1) &= \frac{1}{10}(5v(j_2) - 2v(j_5)), \\ L(e_2) &= \frac{1}{2}(v(\Delta) - 2v(I_4)) - L(e_1). \end{aligned}$$

For trees of Type III.2, they are as in (1).

Combining these theorems, we then immediately obtain a description of the reduction types of Picard curves in terms of quintic and $(4, 1)$ -invariants.

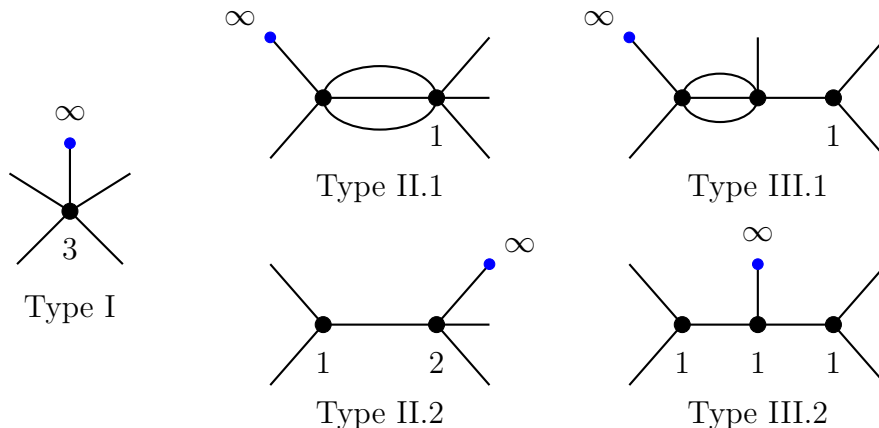


FIGURE 3. Reduction types of Picard curves

Corollary 4 (Reduction types of Picard curves). *The reduction type of the Picard curve $y^3\ell(x, z) = q(x, z)$ is completely determined by the tropical invariants of the $(4, 1)$ -form (q, ℓ) . The tree types given in Figure 2 correspond to the reduction types in Figure 3.*

We thus obtain a description of the moduli space of tropical Picard curves in terms of invariants of quintics and $(4, 1)$ -forms. The work in [6] shows that these invariants in fact give rise to Picard modular forms. This is completely analogous to the cases $n = 4$ and $n = 6$ mentioned above. For instance, for $n = 4$ we have the invariants c_4 and c_6 corresponding to Eisenstein modular forms for $\mathrm{SL}_2(\mathbb{Z})$. The j -invariant is a rational function in terms of these modular forms and the classical criterion an elliptic curve E

has (potential) good reduction if and only if its j -invariant is non-negative expresses the tropical moduli space in terms of these modular forms. For $n = 6$ the Igusa invariants similarly give rise to modular forms (see [23, Section 6]) and the criteria given in [14] and [10] again express the tropical moduli space in terms of modular forms. We view our results as extensions of those for Picard curves.

1.1. Comparison with existing literature. The results obtained in this paper can be seen as a natural continuation of [14] and [10]. In the first, criteria for the seven reduction types of curves of genus two were given in terms of the Igusa invariants. In [10], this result was extended to arbitrary complete non-archimedean fields and an easier proof was given. This paper in turn was based on [11], where skeleta of general superelliptic curves are studied. In the latter, it was shown that one can recover the skeleton from tropicalizations of certain functions in the coefficients of $f(x)$. The key difference between this paper and the latter is that the functions we give here are projective invariants of binary forms. This can be used to interpret the criteria in terms of Picard modular forms, giving a stronger connection to various moduli spaces in the literature.

We also define a general notion of a set of tropical invariants using tools from non-archimedean geometry. A notion that seems distantly related to this one appears in [13], where spherical varieties and invariant valuations are studied. We do not restrict ourselves to spherical varieties, as we can phrase everything in terms of G -invariant subsets of the analytification of an algebraic variety for some group G .

From a geometric point of view, our paper fits into the literature as follows. Suppose that we have a group action on a variety that admits a geometric quotient. This quotient admits many possible compactifications and for every compactification we obtain a natural definition of a tropical invariant. For instance, for separable binary forms we can compactify the space using either stable binary forms (see [17]) or the symmetrized Deligne–Mumford compactification $\overline{M}_{0,n}/S_n$. We are mostly interested in the latter, since this has direct applications to reduction types of Picard curves. That is, if we write $\overline{\mathcal{N}}$ for the space of admissible $\mathbb{Z}/3\mathbb{Z}$ -coverings with ramification signature $(4, 1)$, then there is a natural map $\overline{\mathcal{N}} \rightarrow \overline{M}_{0,5}/(S_4 \times S_1)$ sending a covering to its branch locus and this map respects the boundary loci. Recent work by Cléry and van der Geer [6] shows that this map can be used to connect $(4, 1)$ -invariants to Picard modular forms, mirroring the classical case of elliptic curves. It would be interesting to see how tropical invariants of binary forms are connected to other moduli spaces.

In classical invariant theory, many important concepts such as Hilbert series and generators of invariant rings can be computed explicitly using Gröbner bases. In tropical geometry, Gröbner bases have similarly led to many explicit algorithms as well. This analogy suggests the existence of an algorithm to compute tropical invariants using Gröbner-theoretic methods. We leave an in-depth study of this topic for future work.

1.2. Outline. This paper is organized as follows. In Section 2, we review some background on invariant theory for binary forms and Picard curves. Section 3 introduces and discusses the notion of tropical invariants. Finally, we prove our main results, and in particular discuss the edge lengths (or thickness of singular points), in Section 4. Finally we note that several results in this article were found or proved by symbolic

computations. The codes and computations are made available at

(2)

<https://mathrepo.mis.mpg.de/TropicalInvariantsPicardCurves/index.html>.

Our code is implemented in SageMath [22].

1.3. Notation. Throughout this paper, unless explicitly stated otherwise, we will use the following notation:

- K – Complete, non-archimedean and algebraically closed valued field of characteristic 0.
- v – Non-trivial valuation on K .
- R – The valuation ring of K .
- \mathfrak{m} – The unique maximal ideal of R .
- k – The residue field R/\mathfrak{m} .
- p – The residue characteristic of K ; that is $p = \text{char}(k)$.
- \mathbb{A}^n – Affine space over K of dimension n .
- \mathbb{P}^n – Projective space over K of dimension n .
- $\overline{\mathbb{R}}$ – $\mathbb{R} \cup \{\infty\}$.

In Section 4, we will require that $p \neq 2, 3, 11$.

Acknowledgements

This work started during the first and second named authors' visit to the Max Planck Institute for Mathematics in the Sciences. The authors thank the aforementioned institute for the generous hospitality and welcoming environment. The authors are grateful to Bernd Sturmfels for illuminating discussions. They also would like to thank the anonymous referee for several useful suggestions which improved the readability of this paper. The first author was supported by UK Research and Innovation under the Future Leaders Fellowship programme MR/S034463/2. The third author is supported by grant GYN-D9843-G0B1721N of the Fund for Scientific Research–Flanders.

UKRI Data Access Statement: the code associated to this article is publicly available at the link (2). Apart from this, there are no further associated research materials.

2. BACKGROUND

In this section, we review the necessary background and preliminaries on invariant theory for binary forms and $(4, 1)$ -forms. We also recall some notions on Picard curves.

2.1. Invariant theory of binary forms. In recall some facts and results from invariant theory of binary forms. For detailed treatments of invariant theory, we refer the reader to [21, 7, 19]. Fix an algebraically closed field K such that $\text{char}(K) = 0$.

2.1.1. A group action on binary forms. Let $n \geq 0$ be a fixed integer. Let $A = K[a_0, \dots, a_n]$ be the polynomial ring in $n + 1$ variables. We view A as a graded ring with the standard grading $\deg(a_i) = 1$. Let V_n be the A -submodule of $A[x, z]$ consisting of homogeneous polynomials in x and z of total degree n . The reductive group

$G := \mathrm{SL}_2(K)$ acts (as a right action) on the A -module V_n as follows:

$$(3) \quad g^\sigma(x, z) := g(ax + bz, cx + dz), \quad \text{for } g \in V_n \text{ and } \sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G.$$

Definition 5. [Universal binary form] We define the *universal* binary form f of degree n over K as the binary form given by

$$f(x, z) = a_0x^n + a_1x^{n-1}z + \cdots + a_nz^n \in V_n.$$

A binary form over K of degree n is obtained by specializing the coefficients to K .

We obtain an action of G on A by sending a_i to the coefficient $c_i(f^\sigma)$ of $x^{n-i}z^i$ in f^σ for $0 \leq i \leq n$. This means that G acts on A as follows:

$$(4) \quad F^\sigma(a_0, \dots, a_n) := F(c_0(f^\sigma), \dots, c_n(f^\sigma)), \quad \text{for } F \in A \text{ and } \sigma \in G.$$

Example 6. When $n = 2$, from the definition

$$f^\sigma(x, z) = f(ax + bz, cx + dz), \quad \text{for } \sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$$

we get

$$\begin{bmatrix} c_0(f^\sigma) \\ c_1(f^\sigma) \\ c_2(f^\sigma) \end{bmatrix} = \begin{bmatrix} a^2 & ac & c^2 \\ 2ab & ad + bc & 2cd \\ b^2 & bd & d^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}.$$

We can then see how σ acts on the generators a_0, a_1, a_2 of $A = K[a_0, a_1, a_2]$ and we have $a_0^\sigma = a^2a_0 + aca_1 + c^2a_2$, $a_1^\sigma = 2aba_0 + (ad + bc)a_1 + 2cda_2$ and $a_2^\sigma = b^2a_0 + bda_1 + d^2a_2$.

Definition 7. [Separable binary forms] A binary form is said to be *separable* if its discriminant does not vanish.

Definition 8. [Invariants of binary forms] A homogeneous polynomial $F \in A$ is called *G -invariant* if

$$F^\sigma = F \quad \text{for all } \sigma \in G = \mathrm{SL}_2(K).$$

We denote the graded ring generated by all homogeneous G -invariant polynomials by A^G .

Remark 9. Notice that there is also an action of $\mathrm{GL}_2(K)$ on A as in (4). Since K is algebraically closed, a homogeneous polynomial $F \in A$ is SL_2 -invariant if and only if for any $\sigma \in \mathrm{GL}_2$ we have

$$F^\sigma = \det(\sigma)^{\deg(F)} F.$$

We say that a homogeneous polynomial $F \in A$ is *GL_2 -invariant* if it satisfies the identity above. This immediately implies that the ring of invariants for GL_2 and SL_2 are the same. We will use these interchangeably.

Since $G = \mathrm{SL}_2(K)$ is reductive, it is a well known fact from invariant theory that the ring A^G is finitely generated over K ; see, for example, [7, Corollary 2.2.11]. To find generators of A^G we can use the notion of transvectants, or *Überschiebung*, which we

explain briefly here. Let $g \in V_m$ and $h \in V_n$ with $m \geq n$ and let r be an integer with $0 \leq r \leq n$. We define the bilinear map

$$\langle \cdot, \cdot \rangle_r : V_m \times V_n \rightarrow V_{m+n-2r}, \quad (g, h) \mapsto \sum_{i=0}^r (-1)^i \binom{r}{i} \frac{\partial^r g}{\partial^{r-i} x \partial^i z} \frac{\partial^r h}{\partial^i x \partial^{r-i} z}.$$

It turns out that this map is G -invariant, i.e.,

$$\langle g^\sigma, h^\sigma \rangle_r = \langle g, h \rangle_r, \quad \text{for } (g, h) \in V_m \times V_n \text{ and } \sigma \in G.$$

The quantity $\langle g, h \rangle_r$ is called the r -th *transvectant* of g and h . Notice that when $n = m = r$, the transvectant $\langle g, h \rangle_r$ has degree 0 in x, z so $\langle g, h \rangle_r \in A^G$. A theorem by Gordan [8] shows that one can obtain all invariants by taking (iterated if necessary) transvectants of f , so this makes the set generators of A^G explicit, at least in theory. In practice, the generators have only been found for binary forms of degrees up to 10; see [19, Chapter 1] or [7, Example 2.1.2].

Example 10. (1) For binary quadrics the invariant ring A^G is generated by $\langle f, f \rangle_2$, which is equal to the discriminant of f .

(2) For binary cubics, the invariant ring is again generated by the discriminant

$$\Delta = \langle \langle f, f \rangle_2, \langle f, f \rangle_2 \rangle_2.$$

(3) For binary quartics $n = 4$, the invariant ring A^G is generated by two algebraically independent invariants I_2 and I_3 of degrees 2 and 3 respectively. In terms of transvectants, they are given by

$$I_2 = \langle f, f \rangle_4 \quad \text{and} \quad I_3 = \langle f, \langle f, f \rangle_2 \rangle_4.$$

(4) For binary quintics, the invariant ring A^G is generated by four generators I_4, I_8, I_{12} and I_{18} of degrees 4, 8, 12 and 18, respectively. These are not algebraically independent, as I_{18}^2 can be expressed in terms of the other three. We refer the reader to Section 2.1.2 for a more detailed exposition in this case.

2.1.2. *Invariants of binary quintics.* As discussed in Example 10, the ring of invariants for quintic binary forms is generated by four elements I_4, I_8, I_{12} and I_{18} of degrees 4, 8, 12 and 18, respectively. Let f be the universal quintic

$$f(x, z) = a_0 x^5 + a_1 x^4 z + a_2 x^3 z^2 + a_3 x^2 z^3 + a_4 x z^4 + a_5 z^5.$$

The four invariants I_4, I_8, I_{12} and I_{18} can be obtained by taking transvectants of f in the following way:

$$(5) \quad \begin{aligned} I_4 &= \langle f^2, f^2 \rangle_{10}, & I_8 &= \langle f^4, f^4 \rangle_{20}, \\ I_{12} &= \langle f^6, f^6 \rangle_{30}, & I_{18} &= \langle \langle f^5, f^6 \rangle_{10}, f^7 \rangle_{35}. \end{aligned}$$

Note that these transvectants are different from the ones considered in [19, Section 4.4]¹. They still yield generators however, as one easily checks using the Poincaré series. The discriminant Δ of f can be expressed in terms of the previous invariants as

$$(6) \quad \Delta = c_0 I_4^2 + c_1 I_8$$

¹The universal binary form used in the literature is of the shape $f = \sum_{j=0}^n \binom{n}{j} a_j x^{n-j} z^j$. In this paper we use $f = \sum_{j=0}^n a_j x^{n-j} z^j$, since in our field K some binomial coefficients $\binom{n}{j}$ might have a positive valuation. Since $\text{char}(K) = 0$, the invariant theory remains unchanged.

where c_0, c_1 are the constants

$$c_0 = -1/2746158938062848000000, \quad c_1 = 1/46987474647852089270599680000000.$$

Note that $\deg(\Delta) = 8$.

In Section 3, we will assume that K is a complete non-archimedean field and associate a metric tree to a binary form over K . This tree is invariant under the action of GL_2 . For quintics, there are three different types, see Figure 1. The invariants I_4, I_8, I_{12}, I_{18} and the discriminant Δ are however not enough to distinguish between the different types, as we will see in Example 32. In order to distinguish tree types, we need to introduce another invariant, which we call the *H-invariant*.

Definition 11. [*H-invariant*] The *H-invariant* of the quintic $f = a_0x^5 + a_1x^4z + \cdots + a_5z^5$ is defined as follows:

$$(7) \quad H = \beta I_{12} - 396\alpha^3 I_4^3,$$

where α and β are given by

$$\begin{aligned} \alpha &= 2^{-17} \cdot 3^{-7} \cdot 5^{-3} \cdot 7^{-1}, \\ \beta &= 2^{-50} \cdot 3^{-27} \cdot 5^{-14} \cdot 7^{-7} \cdot 11^{-4} \cdot 13^{-3} \cdot 17^{-1} \cdot 19^{-1} \cdot 23^{-1} \cdot 29^{-1}. \end{aligned}$$

This invariant is of degree 12 and constructed to satisfy the following property:

$$H(0, 0, 1, 0, a_4, a_5) \quad \text{is equal to the discriminant of the cubic} \quad x^3 + a_4x + a_5.$$

The motivation behind creating this invariant is as follows. For binary quintics, there are three unmarked tree types, see Figure 1. For the two non-trivial ones, we choose a non-trivial vertex of valency two. By applying a projective linear transformation, we can ensure that the roots are grouped as $\{\infty, \lambda_1\}$ and $\{0, 1, \lambda_2\}$, where $v(\lambda_1) < v(\lambda_2) = 0$. After scaling the binary form so that the smallest valuation of its coefficients is zero, the coefficients of x^5 and x^4z have positive valuation. This binary form hence reduces to a cubic and the discriminant of this cubic distinguishes between the two remaining types.

2.1.3. Invariants of (4, 1)-forms. Let $B = K[b_0, \dots, b_4, c_0, c_1]$ be a polynomial ring. For an integer $n \geq 0$, we denote by W_n the space of homogeneous polynomials in $B[x, z]$ of degree n . Here, we are interested in (4, 1)-binary forms, which are the elements (q, ℓ) of $W_4 \oplus W_1$. There is a natural group action of $G = \mathrm{SL}_2(K)$ on (4, 1)-forms as follows

$$(q(x, z), \ell(x, z))^\sigma = (q^\sigma(x, z), \ell^\sigma(x, z)) = (q(\sigma(x, z)), \ell(\sigma(x, z))).$$

Definition 12. [**Universal (4, 1)-form**] We define the *universal* (4, 1)-form (q, ℓ) as

$$q(x, z) = b_0x^4 + b_1x^3z + \cdots + b_4z^4 \quad \text{and} \quad \ell(x, z) = c_0x + c_1z.$$

The action of G on (q, ℓ) defines an action of G on B where $\sigma \in G$ acts on B by sending $b_0, \dots, b_4, c_0, c_1$ to the coefficients of $(q, \ell)^\sigma$. We denote by B^G the ring of G -invariant polynomials in B . This ring is again finitely generated since G is reductive. It has 5 generators j_2, j_3, j_5, j_6 and j_9 of respective degrees 2, 3, 5, 6 and 9, and they can be obtained using transvectants as follows:

$$(8) \quad \begin{aligned} j_2 &= \langle q, q \rangle_4, & j_3 &= \langle \langle q, q \rangle_2, q \rangle_4, \\ j_5 &= \langle q, \ell^4 \rangle_4, & j_6 &= \langle \langle q, q \rangle_2, \ell^4 \rangle_4, \\ j_9 &= \langle \langle q, \langle q, q \rangle_2 \rangle_1, \ell^6 \rangle_6. \end{aligned}$$

We refer the reader to [19, Section 5.4.1] for more details.

The invariants $I_4, I_8, I_{12}, I_{18}, \Delta, H$ for binary quintics and the invariants j_2, j_3, j_5, j_6, j_9 for $(4, 1)$ -forms are polynomials with rational coefficients. Scaling these invariants, we may assume that their coefficients are integer and are coprime; recall that $\text{char}(K) = 0$. The reason behind this is that we are working with a valued field K in which integer scalars might have a positive valuation. This justifies the following assumption:

Assumption 13. We scale all the invariants so that the greatest common divisor of their coefficients is 1, and we do not change the notation. These scaled invariants are computed in (2), and are the ones used in our results.

Write $B_{4,1} = B$ for the coefficient ring corresponding to $(4, 1)$ -forms and B_5 for the coefficient ring corresponding to quintics. The universal $(4, 1)$ -form (q, ℓ) gives a canonical quintic $f = q \cdot \ell$. From this, we obtain an injective ring homomorphism

$$B_5 \rightarrow B_{4,1}.$$

This ring homomorphism is $G = \text{SL}_2(K)$ -equivariant, so we obtain an induced injective ring homomorphism

$$B_5^G \rightarrow B_{4,1}^G$$

of invariants. We identify B_5^G with its image, so that we have an inclusion $B_5^G \subset B_{4,1}^G$. The generators from Section 2.1.2 can then be expressed in terms of the j_i explicitly as follows:

$$(9) \quad \begin{aligned} I_4 &= \frac{2}{3}j_2j_6 - \frac{1}{2}j_3j_5, \\ I_8 &= \frac{14}{9}j_2^2j_6^2 + \frac{22}{27}j_2^3j_5^2 + \frac{5}{27}j_3^2j_5^2 - \frac{14}{9}j_2j_3j_5j_6, \\ I_{12} &= \frac{4400}{243}j_2^3j_6^3 - \frac{11}{243}j_3^2j_6^3 - \frac{242}{9}j_2^2j_3j_5j_6^2 + \frac{2479}{81}j_2^4j_5^2j_6 + \frac{692}{81}j_2j_3^2j_5^2j_6 \\ &\quad + \frac{7156}{243}j_2^3j_3j_5^3 - \frac{92}{243}j_3^3j_5^3, \\ I_{18} &= -\frac{625}{729}j_2^6j_5^3j_9 - \frac{512}{729}j_2^3j_3^2j_5^3j_9 - \frac{4}{729}j_2^3j_3j_6^3j_9 + \frac{1}{729}j_3^3j_6^3j_9 + \frac{1}{3}j_2^4j_3j_5^2j_6j_9 \\ &\quad + \frac{4}{243}j_2^5j_5^2j_6j_9 - \frac{1}{243}j_2^2j_3^2j_5^2j_6j_9; \end{aligned}$$

see the computations in (2). For Δ and H , we use (6) and (7).

Finally, we summarize the invariants used in Theorems 1, 2 and 3, and their respective degrees as follows:

Invariant	I_4	I_8	I_{12}	I_{18}	Δ	H
Degree	4	8	12	18	8	12

Invariant	j_2	j_3	j_5	j_6	j_9
Degree	2	3	5	6	9

TABLE 1. Invariants of binary quintics (on the left) and $(4, 1)$ -forms (on the right) together with their degrees.

2.2. Picard curves. Let K denote a field of characteristic 0. A Picard curve X over K is a smooth projective curve of genus 3 in \mathbb{P}^2 given by an equation of the form

$$y^3\ell(x, z) = q(x, z),$$

where q and ℓ are homogeneous polynomials of degrees 4 and 1 respectively in $K[x, z]$. Here the smoothness of X is equivalent to the separability of the quintic $q(x, z) \cdot \ell(x, z)$. For a Picard curve, the five distinct roots of this quintic are the branch points of the $\mathbb{Z}/3\mathbb{Z}$ -covering of \mathbb{P}^1 induced by the rational map

$$\pi: X \dashrightarrow \mathbb{P}^1, \quad [x : y : z] \mapsto [x : z].$$

Either using the fact that X is smooth or an explicit computation, we find that it gives a morphism, so that it is defined everywhere. The Galois group of π is $G = \mathbb{Z}/3\mathbb{Z}$ and it acts as follows on X . Let $\zeta \in K$ be a primitive third root of unity. There is then an automorphism

$$\alpha : [x : y : z] \mapsto [x : \zeta y : z]$$

of order three and the quotient map $X \rightarrow X/\langle \alpha \rangle = \mathbb{P}^1$ is π . For each of the five points P_i in the zero set of the quintic $q(x, z) \cdot \ell(x, z)$, there is a unique point Q_i lying over P_i . The four points Q_1, \dots, Q_4 corresponding to the quartic $q(x, z)$ differ from the remaining point Q_5 in the sense that α acts by ζ on the tangent spaces of Q_1, \dots, Q_4 and as ζ^2 on the tangent space of Q_5 . This gives a second, and equivalent, definition of a Picard curve as a $\mathbb{Z}/3\mathbb{Z}$ -covering of \mathbb{P}^1 branched over five points with a specified action of $\mathbb{Z}/3\mathbb{Z}$ on the tangent spaces of the ramification points; see [2]. This rigidification can also be given in terms of differential forms, see [6, Section 5].

3. TROPICAL INVARIANTS FOR GENERAL GROUP ACTIONS

In this section, we define the notion of a set of tropical invariants for general group actions. The main underlying idea is that they separate orbits that have the same limit points in some toric compactification. For binary forms, this gives us the notion of a set of tropical invariants. For Picard curves, there are two interesting types of binary forms: quintics and $(4, 1)$ -forms. We will see in Section 4 that we can give a tropical set of invariants for these forms.

We assume throughout this section that K is a complete, non-archimedean and algebraically closed field K of characteristic 0 with valuation ring R and residue field k .

3.1. Compactifications and tropicalizations. Let U be an irreducible variety over K with Berkovich analytification U^{an} . We can define tropical separating sets for partitions of subsets of U^{an} as follows.

Definition 14. [Separating set] Let $U^{\text{an}} \supset S = \bigsqcup S_i$ be a partition of a subset S of U^{an} into disjoint subsets and let $\phi : U \rightarrow X(\Delta)$ be an embedding of U into a toric variety $X(\Delta)$ with fan Δ . We say that ϕ is *separating* for the partition if $\text{trop}(\phi(S_i)) \cap \text{trop}(\phi(S_j)) = \emptyset$ for $i \neq j$, where $\text{trop}(\cdot)$ is the natural tropicalization map associated to the toric variety $X(\Delta)$. If the embedding is given by the global sections of a line bundle on U and ϕ is separating for a partition, then we call the sections a *tropical separating set* for the partition. If the sections of the line bundle give a morphism to $X(\Delta) = \mathbb{P}(a_1, \dots, a_n)$, then we say that they are a *projective tropical separating set*.

Example 15. If we take $S = \{P_1\} \cup \{P_2\}$ for two points $P_1, P_2 \in U^{\text{an}}$, then the proof of [18, Theorem 1.1] shows that we can find an embedding that separates these points.

The partitions of U^{an} we are interested in are given using compactifications of U as follows. Let $\mathcal{X} \rightarrow \text{Spec}(R)$ be a proper flat R -scheme and let $U \rightarrow \mathcal{X}$ be an open immersion. Consider the reduction map

$$(10) \quad \text{red} : U^{\text{an}} \rightarrow \mathcal{X}_s$$

as in [9, Section 4], where \mathcal{X}_s is the special fiber of \mathcal{X} . We now obtain a partition of U^{an} by taking the inverse image under $\text{red}(\cdot)$ of a partition of \mathcal{X}_s .

3.2. Tropical invariants. We start with the definition of tropical weighted projective space.

Definition 16. [Tropical weighted projective space] Let $(\mathbb{T}\mathbb{A}^n)^* = \overline{\mathbb{R}}^n \setminus \{(\infty, \dots, \infty)\}$, the punctured tropical affine plane. For a fixed n -tuple of natural numbers (a_1, \dots, a_n) , we define an action of \mathbb{R} on $(\mathbb{T}\mathbb{A}^n)^*$ as follows:

$$\lambda \odot (x_1, \dots, x_n) := (x_1, \dots, x_n) + \lambda \cdot (a_1, \dots, a_n).$$

The (set-theoretic) quotient of $(\mathbb{T}\mathbb{A}^n)^*$ by this action is called the *tropical weighted projective space*, and is denoted by $\text{TP}(a_1, \dots, a_n)$.

Now, let U/K be an irreducible variety and let G/K be a group scheme acting on U . Suppose that there exists a geometric quotient $V = U/G$ for U and G . The notion of a tropical separating from Section 3.1 now allows us to define a set of tropical invariants.

Definition 17. [Tropical invariants] Let $V^{\text{an}} \supset S = \sqcup S_i$ be a partition of a subset S of V^{an} . A set of *tropical invariants* for the partition is a tropical separating set for the partition. If the tropical invariants give an embedding into a weighted projective space $\mathbb{P}(a_1, \dots, a_n)$, then we call this a set of *tropical projective invariants*. The image of the tropicalization is the space $\text{TP}(a_1, \dots, a_n)$.

Since points of V^{an} correspond to orbits in U^{an} , we find that a set of tropical invariants separates a partition of a subset of U^{an} that is stable under the action of G .

Example 18. Consider the ring $K[x, y]$. The group $S_2 \simeq \mathbb{Z}/2\mathbb{Z}$ acts on $K[x, y]$ by $x \mapsto y$ and $y \mapsto x$. The invariant ring is $K[xy, x + y]$ and, on the level of schemes, the map $\text{Spec}(K[x, y]) \rightarrow \text{Spec}(K[xy, x + y])$ gives the geometric quotient. Consider the standard embedding of $\mathbb{A}^2 = \text{Spec}(K[x, y])$ into \mathbb{P}^2 . The group action then extends to \mathbb{P}^2 . The boundary is isomorphic to \mathbb{P}^1 and we decompose \mathbb{P}^2 into

$$\overline{S}_0 = \{[x : y : z] \mid z \neq 0\}, \quad \overline{S}_1 = \{[1 : 1 : 0]\} \quad \text{and} \quad \overline{S}_2 = \{[x : y : 0] \mid x \neq y\}.$$

This induces a partition

$$S_0 \sqcup S_1 \sqcup S_2 = \text{Spec}(K[xy, x + y])^{\text{an}}$$

since the group action preserves the \overline{S}_i . We now note that the standard embedding

$$\text{Spec}(K[xy, x + y]) = \mathbb{A}^2 \rightarrow \mathbb{P}^2$$

is not tropically separating since $\text{char}(K) \neq 2$. If we however consider the embedding into \mathbb{P}^3 given by $\{xy, x + y, (x - y)^2, 1\}$, then this does form a tropical set of invariants.

We will see many more examples of tropical invariants in Sections 3.3 and 3.4.

3.3. Moduli of tropical curves. We now review moduli of stable curves of genus zero. The functor $\mathcal{M}_{0,n} : \mathbf{Sch} \rightarrow \mathbf{Sets}$ from the category of schemes to the category of sets, defined by

$$(11) \quad \mathcal{M}_{0,n}(S) = \{\text{smooth curves } C \rightarrow S \text{ of genus zero with } n \text{ marked points}\} / \sim$$

for $n \geq 3$ is representable by a scheme $M_{0,n}$. Note that there are indeed no non-trivial elements of $\text{Aut}(\mathbb{P}^1) = \text{PGL}_2$ that fix three or more elements of \mathbb{P}^1 . We refer to $M_{0,n}$ as the moduli space of smooth n -marked curves of genus zero. There is a natural compactification of this space, given by replacing smooth curves by stable curves in (11). We denote the corresponding scheme by $\overline{M}_{0,n}$. It is proper and flat over $\text{Spec}(\mathbb{Z})$. The boundary locus $\Delta = \overline{M}_{0,n} \setminus M_{0,n}$ corresponds to non-smooth stable curves.

The group S_n acts on the above moduli spaces by permuting the marked points. Since S_n is finite and $M_{0,n}$ and $\overline{M}_{0,n}$ are quasi-projective, there exist geometric quotients $M_{0,n}/S_n$ and $\overline{M}_{0,n}/S_n$ which fit into a commutative diagram

$$\begin{array}{ccc} M_{0,n} & \longrightarrow & \overline{M}_{0,n} \\ \downarrow & & \downarrow \\ M_{0,n}/S_n & \longrightarrow & \overline{M}_{0,n}/S_n \end{array}$$

The scheme $\overline{M}_{0,n}/S_n$ is again proper over \mathbb{Z} . The geometric points of $M_{0,n}/S_n$ (resp. $\overline{M}_{0,n}/S_n$) can be identified with smooth (resp. stable) curves of genus 0 with n unordered points.

We now describe their abstract tropicalizations as in [1, Section 4]. We start with the original set-up without quotients. Consider phylogenetic trees with n marked leaves, together with a length function ℓ on the non-leaves. These are the points of the tropical moduli space $M_{0,n}^{\text{trop}}$. It can be given the structure of a generalized cone complex as in [1, Section 2]. It can also be given as the tropicalization of the Plücker map as in [15, Theorem 6.4.12]. We obtain an abstract tropicalization map

$$\text{trop} : M_{0,n}^{\text{an}} \rightarrow M_{0,n}^{\text{trop}}$$

by the following procedure. A point $P \in M_{0,n}^{\text{an}}$ can be represented by an L -valued point of $M_{0,n}$, and we use the natural map

$$(12) \quad M_{0,n}(L) \rightarrow \overline{M}_{0,n}(L) = \overline{M}_{0,n}(R_L)$$

to obtain a stable model $\mathcal{X} \rightarrow \text{Spec}(R_L)$ with special fiber $\mathcal{X}_s \rightarrow \text{Spec}(k_L)$. Here, L is a complete valued field extension of K with valuation ring R_L and residue field k_L . Let T be the marked dual intersection graph of \mathcal{X}_s , together with the natural edge length function ℓ induced from \mathcal{X} . Note that there is a unique leaf for every marked point. We set $\text{trop}(P) = (T, \ell)$.

We can also express the tropicalization of a point in $M_{0,n}^{\text{an}}$ in terms of Berkovich skeleta as in [4]. Namely, a point $P \in M_{0,n}^{\text{an}}$ corresponds to a marked curve $(\mathbb{P}^1, (P_1, \dots, P_n))$, where $P_i \in \mathbb{P}^1(L)$ are distinct L -valued points. The minimal skeleton Σ of this marked curve as in [4, Section 2.6]² is then a marked metric tree whose leaves correspond to the points P_i . The underlying graph of this marked metric tree is the same as T and it gives rise to these same edge length function ℓ . By abuse of notation, we will write

²In [4], these are referred to as punctured curves.

$\Sigma = (T, \ell)$, where we either view (T, ℓ) as a marked finite graph with a length function, or a marked metric tree.

Definition 19. Let $P \in M_{0,n}^{\text{an}}$ for $n \geq 3$ with induced marked curve $(\mathbb{P}^1, (P_1, \dots, P_n))$. We call $\Sigma = (T, \ell)$ the marked metric tree associated to P .

Remark 20. We note that the edge lengths of the marked metric tree Σ are independent of the chosen representative of the isomorphism class corresponding to P . That is, if we have an isomorphism of marked curves $(\mathbb{P}^1, (P_1, \dots, P_n)) \rightarrow (\mathbb{P}^1, (\sigma(P_1), \dots, \sigma(P_n)))$, then this induces a morphism of marked metric trees $\Sigma \rightarrow \sigma(\Sigma)$ that is an isometry outside the leaves. This, for instance, follows from the results in [4, Section 2].

Let $\mathcal{P}([n])$ be the power set of $[n] = \{1, \dots, n\}$ and let $\Xi \subset \mathcal{P}([n])$ be a partition. We write $S_\Xi \subset S_n$ for the subgroup of permutations that preserve Ξ . Note that S_Ξ is isomorphic to a product of symmetry groups S_k for $k \leq n$. We now define an action of S_Ξ on $M_{0,n}^{\text{trop}}$. We view an element of $M_{0,n}^{\text{trop}}$ as a metric tree $\Sigma = (T, \ell)$, together with an injection $i : \{1, \dots, n\} \rightarrow L(T)$, where $L(T)$ is the set of (infinite) leaves of T . By permuting the leaves indexed by the partition Ξ , we obtain an action of S_Ξ on $M_{0,n}^{\text{trop}}$, where the latter is viewed as an object in the category of generalized cone complexes. We now note that categorical quotients in the category of generalized cone complex exist since arbitrary finite colimits exist, see [1, Remark 2.6.1]. From this, we obtain the following definition of $M_{0,n}^{\text{trop}}/S_\Xi$.

Definition 21. Let $\Xi \subset \mathcal{P}([n])$ be a partition of $[n] = \{1, \dots, n\}$ and consider the induced action of S_Ξ on $M_{0,n}^{\text{trop}}$. The space $M_{0,n}^{\text{trop}}/S_\Xi$ is the categorical quotient of $M_{0,n}^{\text{trop}}$ under the action of S_Ξ . A Ξ -tree-type is an equivalence class of a tree type in $M_{0,n}^{\text{trop}}$ under the action of S_Ξ .

Example 22. Suppose that $\Xi = \{\{1, \dots, n\}\}$. The corresponding moduli space $M_{0,n}^{\text{trop}}/S_n$ parametrizes tropical trees with n unmarked points.

Example 23. There are three non-trivial marked tree types for $n = 4$, giving three cones $\mathbb{R}_{\geq 0}$. The types with edge lengths zero are all identified, so we glue these cones together to obtain $M_{0,4}^{\text{trop}}$, which is a standard tropical line. The three cones are permuted by S_4 , giving the quotient $M_{0,4}^{\text{trop}}/S_4 = \mathbb{R}_{\geq 0}$.

Example 24. Let $n = 5$. Then, there are exactly three unmarked types: I, II and III; they are depicted in Figure 1. Type III corresponds to a folded positive orthant of dimension 2, since the automorphism group of the underlying graph is $\mathbb{Z}/2\mathbb{Z}$. Figure 4 represents the space $M_{0,5}^{\text{trop}}/S_5$.

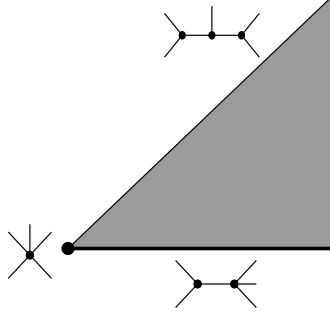


FIGURE 4. The space $M_{0,5}^{\text{trop}}/S_5$

Example 25. Suppose that $\Xi = \{\{1, 2, \dots, n-1\}, \{n\}\}$. In this case, we will write $S_\Xi = S_{n-1}$ for brevity. The corresponding tropical moduli space $M_{0,n}^{\text{trop}}/S_{n-1}$ parametrizes tropical trees with one marked point.

For instance, consider the case where $n = 5$ and $\Xi = \{\{1, 2, \dots, 4\}, \{5\}\}$, giving rise to an action of S_4 on $M_{0,5}^{\text{trop}}$. The quotient $M_{0,5}^{\text{trop}}/S_4$ is the moduli space of phylogenetic trees with 4 unmarked leaves and 1 marked leaf, see Figure 5. There are five different corresponding phylogenetic tree types: I, II.1, II.2, III.1 and III.2; see Figure 2.

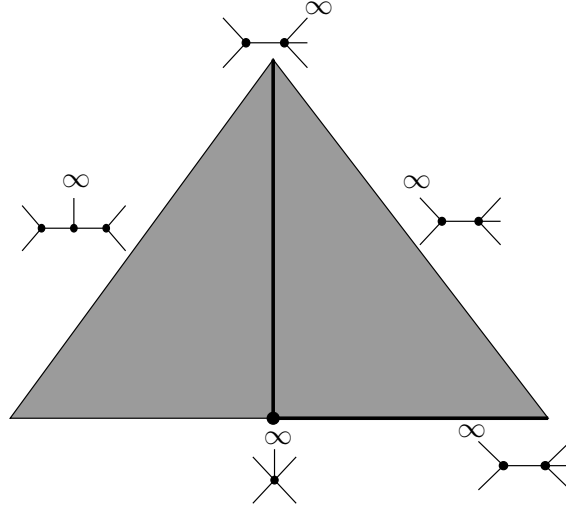


FIGURE 5. The space $M_{0,5}^{\text{trop}}/S_4$

In Figures 4 and 5, the (possibly folded) positive orthants are glued with respect to degeneration of the corresponding tree types, which is shown in Figure 6. We refer to [16, Section 2] and [5, Section 4] for more details.

For the symmetrized moduli space $M_{0,n}/S_\Xi$, we can again introduce an abstract tropicalization map as follows. Let $\Xi \subset \mathcal{P}[n]$ be a partition and let P be an L -valued point of $M_{0,n}^{\text{an}}/S_\Xi$. This again gives a phylogenetic tree (T, ℓ) with a marking $\{1, \dots, n\} \rightarrow L(T)$, but we identify two markings $\phi_1, \phi_2 : \{1, \dots, n\} \rightarrow L(T)$ if they are related by an element of S_Ξ . This gives a map

$$(13) \quad \text{trop} : M_{0,n}^{\text{an}}/S_\Xi \rightarrow M_{0,n}^{\text{trop}}/S_\Xi.$$

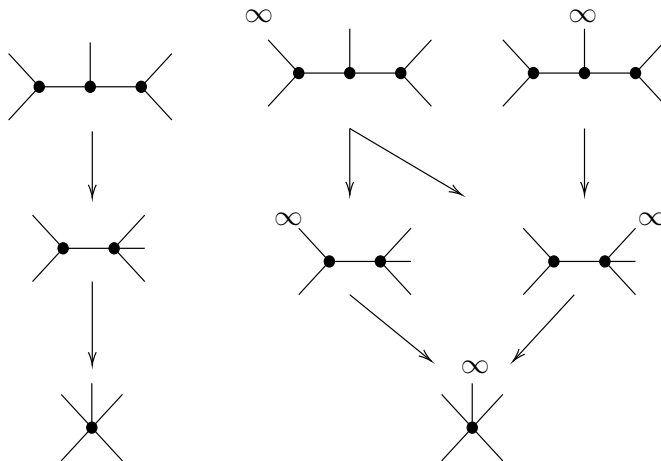


FIGURE 6. Degenerations of trees

Remark 26. The map in (13) in particular gives a partition of $M_{0,n}^{\text{an}}/S_{\Xi}$ by considering the inverse images of the loci of Ξ -marked tree types. We will use this partition to obtain a partition of the space of binary forms in Section 3.4. We note that the partition of $M_{0,n}^{\text{an}}/S_{\Xi}$ is a special instance of the partitions we considered in Section 3.1 since the construction in (12) gives the reduction map from (10).

3.4. Trees and tropical binary forms. Let $V = V_{k_1} \oplus \cdots \oplus V_{k_r}$ be the standard SL_2 -module and write A^G for the ring of invariants. We consider $\text{Proj}(A^G)$, where the grading is induced by the degree of an invariant. We are interested in the affine open $U = D_+(\Delta)$ for Δ the discriminant. The set of L -valued points $U(L)$ of U can be identified with tuples of binary forms (f_1, \dots, f_r) over L without common zeros and up to GL_2 -equivalence. Here $\deg(f_i) = k_i$.

We now consider an L -valued point of U . This gives a set of binary forms (f_1, \dots, f_r) over L . We write $f = \prod_{i=1}^r f_i$ for the resulting $(k_1 + \cdots + k_r)$ -binary form. Let $Z(f)$ and $Z(f_i)$ denote the zero sets of f and f_i respectively, for $1 \leq i \leq r$. Note that $Z(f) = \sqcup_{i=1}^r Z(f_i)$ gives a well-defined L -valued point of the moduli space $M_{0,n}/S_{\Xi}$, where $n = k_1 + \cdots + k_r$ and $\Xi = \{\{1, \dots, k_1\}, \{k_1 + 1, \dots, k_1 + k_2\}, \dots, \{k_1 + \cdots + k_{r-1} + 1, \dots, n\}\}$. Indeed, taking a different equivalent binary form changes the zero sets by the action of GL_2 , so that the induced map

$$\left(\mathbb{P}^1, \bigsqcup_{i=1}^r Z(f_i) \right) \rightarrow \left(\mathbb{P}^1, \bigsqcup_{i=1}^r Z(f_i^{\sigma}) \right)$$

is an isomorphism, which means that we obtain the same point in $M_{0,n}/S_{\Xi}$.

Definition 27. [Metric tree of a binary form] Let $f = (f_1, \dots, f_r)$ be a separable binary form over a valued field L and consider a set of markings of the $Z(f_i)$. Let $\{1, \dots, n\} \rightarrow Z(f)$ be the induced marking of $Z(f)$ with marked metric tree (T, ℓ) , see Definition 19. Then the *metric tree* associated to f is the image of (T, ℓ) in $M_{0,n}^{\text{trop}}/S_{\Xi}$ under the map $M_{0,n}^{\text{trop}} \rightarrow M_{0,n}^{\text{trop}}/S_{\Xi}$. For any $\sigma \in \text{GL}_2$ with binary form $\sigma(f) = (\sigma(f_1), \dots, \sigma(f_r))$, the induced point of $M_{0,n}^{\text{trop}}/S_{\Xi}$ is the same.

The partition on the Berkovich analytification of $M_{0,n}/S_{\Xi}$ given in Remark 26 induces a partition of U^{an} . We use this as our definition of a tropical invariant of a binary form.

Definition 28. [**Tropical invariants of binary forms**] A set of *tropical invariants* of a set of binary forms of degree n is a tropical separating set for the partition of U^{an} induced from the maps $U^{\text{an}} \rightarrow M_{0,n}^{\text{an}}/S_{\Xi} \rightarrow M_{0,n}^{\text{trop}}/S_{\Xi}$, see (13).

For a given set of invariants h_i , we obtain a rational map $U \rightarrow \mathbb{P}(a_0, \dots, a_n)$, where $\deg(h_i) = a_i$. If the h_i include a set of generators of the ring of invariants, then this is automatically a morphism since the set of generators generate the nullcone. If the set of generators is a projective tropical separating set, then we call this a set of *projective tropical invariants*.

Example 29. For residue characteristics not equal to 2, the tropical invariants of a binary quartic are given by c_4 , c_6 and Δ ; see [20, Chapter VII]. These are furthermore weighted projective invariants, giving a tropicalization map to $\mathbb{TP}(4, 6, 12)$.

Example 30. For residue characteristics not equal to 2, the tropical invariants of a binary sextic are given by the Igusa invariants from [10].

Example 31. In Section 4, we will show that the invariants introduced in the introduction form a set of tropical invariants for binary quintics and $(4, 1)$ -forms. These then also give the reduction types of Picard curves when the residue characteristic is not 3.

As promised in Section 2.1.2, we now show that the invariants I_4 , I_8 , I_{12} , I_{18} and Δ are not enough to distinguish between the unmarked trees of a quintic.

Example 32. Let $K = \mathbb{C}\{\{t\}\}$ be the field of Puiseux series over the complex numbers with $v(t) = 1$. Consider the two quintics given by

$$\begin{aligned} f_2 &= xz(x-z)(x-t^2z)(x-2z), \\ f_3 &= xz(x-z)(x-tz)(x-(1+2t)z). \end{aligned}$$

The tree of f_2 is of Type II and the tree of f_3 is of Type III. However, in both cases, the tropical invariants are the same:

$$[v(I_4) : v(I_8) : v(I_{12}) : v(I_{18}) : v(\Delta)] = [0 : 0 : 0 : 2 : 4].$$

We thus see that we cannot distinguish between the two tree types using these invariants. This also implies that we cannot distinguish the various reduction types of Picard curves using these invariants. They are however enough to decide whether a tree is of Type I or not.

The results in this paper show that there exists a set of projective tropical invariants for quintics and $(4, 1)$ -forms. In general, we conjecture that there exists a finite set of projective tropical invariants for any set of binary forms. Moreover, there should be a practical algorithm that can calculate these tropical invariants.

4. PROOFS OF THE MAIN RESULTS

In this final section, we prove the main results stated in the introduction, namely Theorem 1, Theorem 2, Theorem 3 and Corollary 4. Some parts of the proofs rely on computing the invariants explicitly. The computations are made available in (2).

Recall that our base field K is a non-archimedean, complete and algebraically closed valued field of characteristic 0 with associated data (v, R, \mathfrak{m}, k) . Moreover, the residue characteristic p is different from 2, 3 and 11.

Remark 33. If the residue characteristic of K is 11, then the second condition in Theorem 1 does not characterize trees of Type II. In this case, to obtain condition for Type II, we simply need to check that the conditions for Type I and Type III are not satisfied.

4.1. Universal families. In this section, we explain the general idea of the proofs. We begin by recalling the setup in Sections 3.3 and 3.4 in the case of binary quintics and $(4, 1)$ -forms.

Let f be a separable binary quintic or $(4, 1)$ -form over K . The zero set $Z(f)$ of f consists of five points in \mathbb{P}^1 , which we also consider as points of the corresponding Berkovich analytification $\mathbb{P}^{1,\text{an}}$. These are often called points of type 1 in the literature. These five points define a natural metric tree $\Sigma_f \subset \mathbb{P}^{1,\text{an}}$ that contains $Z(f)$. If we choose a marking $\{1, \dots, 5\} \rightarrow Z(f)$, then this is the *minimal Berkovich skeleton* of the marked curve $(\mathbb{P}^1, P_1, \dots, P_5)$ as defined in [4, Definition 4.20] in terms of semistable vertex sets. If $f = (f_4, f_1)$ is a $(4, 1)$ -form, then we assume that the marking is chosen so that $P_1, \dots, P_4 \in Z(f_4)$.

The group $\text{GL}_2(K)$ acts on $\mathbb{P}^{1,\text{an}}$ through Möbius transformations. The individual Σ_f are not invariant under this action, since $Z(f)$ is not invariant. The abstract marked metric tree Σ_f however is invariant, up to suitable permutations of the markings. More precisely, if f is a binary quintic, then we consider Σ_f as an element of the moduli space $M_{0,5}^{\text{trop}}/S_5$ of unmarked metric trees on five leaves, and if $f = (f_4, f_1)$ is a $(4, 1)$ -form, then we consider Σ_f as an element of the moduli space $M_{0,5}^{\text{trop}}/S_4$ of metric trees with five leaves and one marked point. For every separable binary form in the same GL_2 -orbit, this gives the same metric tree, so that we can associate an element of $M_{0,5}^{\text{trop}}/S_5$ or $M_{0,5}^{\text{trop}}/S_4$ to any GL_2 -equivalence class of separable binary quintics or $(4, 1)$ -forms. Note that if two separable binary forms lie in the same GL_2 -orbit, then they also have the same projective invariants since a transformation σ scales an invariant I by $\det(\sigma)^{\deg(I)}$; see Remark 9. To summarize, applying a transformation in GL_2 to a binary quintic or $(4, 1)$ -form does not change

- (1) the metric tree of the binary form,
- (2) the projective invariants of the binary form.

For a given binary quintic or $(4, 1)$ -form defined over a valued field, we can now apply a projective transformation so that the resulting binary form is

$$(14) \quad f(x, z) = xz(x - z)(x - \lambda_1 z)(x - \lambda_2 z).$$

Here, we send the marked point to ∞ . By the discussion above, this does not change the associated metric tree, nor the projective invariants. We now view the binary form in (14) as being defined over the ring $A_0 = K[\lambda_1, \lambda_2]$. The latter is the coordinate ring of a two-dimensional affine space $Y_0 = \mathbb{A}_K^2$ over K . For specializations of the λ_i outside the vanishing locus $Z(\Delta_\lambda)$ of $\Delta_\lambda := \lambda_1 \lambda_2 (\lambda_1 - 1)(\lambda_2 - 1)(\lambda_1 - \lambda_2)$, we again obtain a separable binary form. We write Y for this open subspace of Y_0 and U for the space of separable binary forms as in Section 3.4. We then have a natural map

$$Y \rightarrow U$$

sending (λ_1, λ_2) to the separable binary quintic in (14). This map is surjective by construction.

We will write down parametrizations of the strata of Y^{an} that correspond to the different tree types in U^{an} . This will be done in terms of *universal algebras*. For each tree type, we will give a free algebra $A = R[t_i, \mu_i]$ with an injective map $R[\lambda_1, \lambda_2] \rightarrow A$. We will view this as an inclusion, so that $\lambda_i \in A$. Note that the binary form in (14) automatically becomes a binary form over A through this map. Let R_L be the valuation ring of a valued field extension $K \subset L$. We then consider R_L -valued points $\psi : A \rightarrow R_L$ with $v(\psi(t_i)) > 0$. Such an R_L -valued point corresponds to a specific binary form of the given type. We will also write $v(t_i) > 0$ if ψ is understood. The universal algebras are constructed in such a way that every separable binary form of the given type occurs as an R_L -valued point with $v(t_i) > 0$.

To explicitly construct these universal families, we fix a tree type and write down the implied conditions on the valuations of the λ_i . We then parametrize these conditions using additional parameters t_i and μ_i . Our choices for these universal families can be found in Table 2; see Figure 7 for the corresponding trees.

Example 34. To give an example, let us explain the universal family for Type III.2 in more detail. By assumption, we have the conditions

$$v(\lambda_1) > 0 \quad \text{and} \quad v(\lambda_2 - 1) > 0.$$

Note that we can assume by permuting the λ_i that $v(\lambda_1) \leq v(\lambda_2 - 1)$. To parametrize the different edge lengths, we then write

$$\lambda_1 = t_1\mu_1 \quad \text{and} \quad \lambda_2 = 1 + t_2\mu_2.$$

Here we specialize the μ_i to elements with valuation zero and the t_i to elements in R , so that $v(t_1) = v(\lambda_1)$ and $v(t_2) = v(\lambda_2 - 1)$. In particular, note that the two non-trivial edge lengths are given exactly by $v(t_i)$. By our assumption on the edge lengths, we then have $v(t_1) \leq v(t_2)$.

Type	λ_1	λ_2	Conditions
I	μ_1	μ_2	$\bar{\mu}_i \neq 0, 1$ and $\bar{\mu}_1 \neq \bar{\mu}_2$
II.1	$t_1\mu_1$	$t_1\mu_2$	$v(t_1) > 0$, $\bar{\mu}_i \neq 0$ and $\bar{\mu}_1 \neq \bar{\mu}_2$
II.2	$t_1\mu_1$	μ_2	$v(t_1) > 0$, $\bar{\mu}_i \neq 0$ and $\bar{\mu}_2 \neq 1$
III.1	$t_1\mu_1$	$t_1t_2\mu_2$	$v(t_i) > 0$ and $\bar{\mu}_i \neq 0$
III.2	$t_1\mu_1$	$1 + t_2\mu_2$	$v(t_i) > 0$, $\bar{\mu}_i \neq 0$ and $v(t_1) \leq v(t_2)$

TABLE 2. The chosen universal families. The t_i parametrize the different edge lengths and the μ_i the different algebraic residue classes.

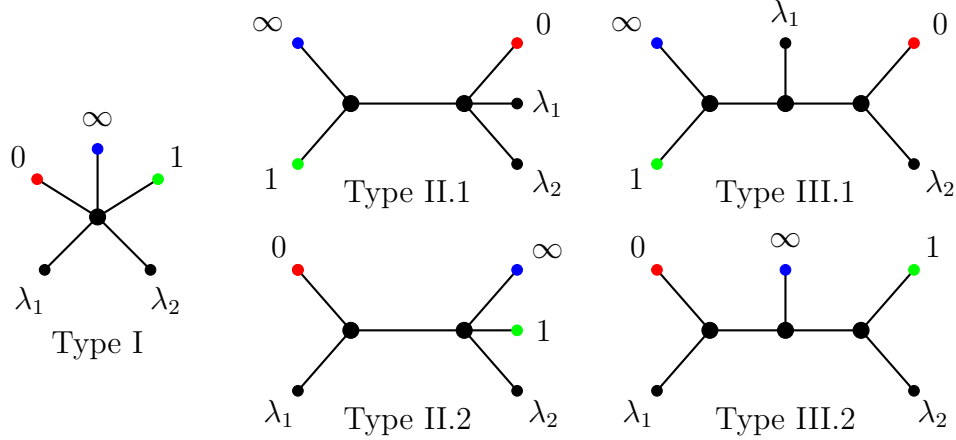


FIGURE 7. Trees corresponding to the universal families in Table 2

In these universal algebras, we have various natural reduction maps. We will be particularly interested in reducing elements modulo $I = \mathfrak{m} + (t_1, t_2)$. Here we think of the t_i as elements of \mathfrak{m} whose valuations we can freely alter. In this vein, we will also refer to working modulo I as working modulo \mathfrak{m} . For a given tree type with universal algebra A , we can consider the universal binary form f from (14) as being defined over A . The invariants of this binary form are thus elements of A .

In particular, we now see that for any specialization $\psi : A \rightarrow R_L$, we have that $\psi(I) \in R_L$. We write $I \in A^\times$ if for every specialization ψ as above, we have that $v(\psi(I)) = 0$.

4.2. Proof of Theorem 1. In this subsection, we prove Theorem 1. In the proofs, we are free to choose the universal family of either II.1 or II.2, and similarly for III.1 and III.2. We do the necessity of the statement first and finish with the sufficiency in Section 4.2.4.

4.2.1. *Type I.* Computing the reduction modulo \mathfrak{m} of the discriminant Δ , we obtain

$$\overline{\Delta} = \overline{\lambda_1}^2 \overline{\lambda_2}^2 (\overline{\lambda_1} - 1)^2 (\overline{\lambda_2} - 1)^2 (\overline{\lambda_1} - \overline{\lambda_2})^2.$$

So we deduce that $v(\Delta) = 0$, hence the condition $8v(I) - \deg(I)v(\Delta) \geq 0$ is satisfied for any $I \in S$.

4.2.2. *Type II.* Consider the universal family for Type II.1 in Table 2. We calculate the invariants $I \in S$ and find that they are divisible by $t_1^{\deg I/2}$. The reduction of $H/t_1^{\deg(H)/2}$ modulo \mathfrak{m} is

$$\overline{H/t_1^{\deg(H)/2}} = -22\overline{\mu_1}^2\overline{\mu_2}^2(\overline{\mu_1} - \overline{\mu_2})^2.$$

Since $p \neq 2, 11$, we find that the latter is non-zero. We thus obtain

$$v(H) = \deg(H)v(t_1)/2 = 6v(t_1), \quad \text{and} \quad v(I) \geq \deg(I)v(t_1)/2 \quad \text{for } I \in S \setminus \{H\}.$$

Therefore,

$$12v(I) - \deg(I)v(H) \geq 0 \quad \text{for all } I \in S.$$

On the other hand, computing the reduction modulo \mathfrak{m} of I_4/t_1^2 and I_{18}/t_1^9 , we find that

$$\begin{aligned}\overline{I_4/t_1^2} &= -2(\overline{\mu_1^2} - \overline{\mu_1\mu_2} + \overline{\mu_2^2}), \\ \overline{I_{18}/t_1^9} &= -\overline{\mu_1^2\mu_2^2}(\overline{\mu_1} - 2\overline{\mu_2})(\overline{\mu_1} + \overline{\mu_2})(2\overline{\mu_1} - \overline{\mu_2})(\overline{\mu_1} - \overline{\mu_2})^2.\end{aligned}$$

It is not so hard to check that, since $p \neq 3$, the quantities $\overline{I_4/t_1^2}$ and $\overline{I_{18}/t_1^9}$ cannot be simultaneously zero. We thus find that

$$v(I_4) = 2v(t_1) \quad \text{or} \quad v(I_{18}) = 9v(t_1).$$

Our computations give $v(\Delta/t_1^6) \geq 0$, so we deduce that $v(\Delta) > 4v(t_1)$ and thus

$$v(\Delta) - 2v(I_4) > 0 \quad \text{or} \quad 9v(\Delta) - 4v(I_{18}) > 0.$$

Combining these, we obtain the statement of the theorem.

4.2.3. *Type III.* Consider the universal family for Type III.2 in Table 2. We compute the reduction of I_4 modulo \mathfrak{m} to obtain

$$\overline{I_4} = -2.$$

So, since $p \neq 2$, we deduce that $v(I_4) = 0$. Computing Δ and H , we obtain that Δ is divisible by $t_1^2 t_2^2$ and $\overline{H} = 0$. So we deduce that

$$v(\Delta) - 2v(I_4) > 0 \quad \text{and} \quad v(H) - 3v(I_4) > 0.$$

4.2.4. *Finishing the proof.* Finally, we check sufficiency. Suppose that the condition in (I) is satisfied. In particular,

$$8v(I_4) - 4v(\Delta) \geq 0 \quad \text{and} \quad 8v(I_{18}) - 18v(\Delta) \geq 0.$$

But these imply that

$$v(\Delta) - 2v(I_4) \leq 0 \quad \text{and} \quad 9v(\Delta) - 4v(I_{18}) \leq 0,$$

so the conditions in (II) and (III) can not be satisfied. Now suppose that the conditions in (II) are satisfied. The condition

$$v(\Delta) - 2v(I_4) > 0 \quad \text{or} \quad 9v(\Delta) - 4v(I_{18}) > 0$$

gives

$$8v(I_4) - \deg(I_4)v(\Delta) < 0 \quad \text{or} \quad 8v(I_{18}) - \deg(I_{18})v(\Delta) < 0,$$

and therefore, the condition in (I) can not be satisfied. On the other hand, the inequality $12v(I_4) - 4v(H) \geq 0$ implies $v(H) - 3v(I_4) \leq 0$, which means that the conditions in (III) can not be satisfied either. Finally, suppose that the conditions in (III) are satisfied. Hence,

$$8v(I_4) - \deg(I_4)v(\Delta) < 0 \quad \text{and} \quad 12v(I_4) - \deg(I_4)v(H) < 0,$$

so that the conditions in (I) and (II) can not be satisfied. This finishes the proof.

4.3. **Proof of Theorem 2.** In this subsection, we prove Theorem 2. The strategy is the same as in Section 4.2: we first calculate the invariants for each universal family and show that the given inequalities hold. This gives the necessity of the conditions. The sufficiency in this case is trivial, so this concludes the proof. We start with trees of Type II.1 since the marking does not matter for trees of Type I.

4.3.1. *Type II.1.* We calculate j_2 and find that it is divisible by t_1^2 . Computing the reductions modulo \mathfrak{m} of j_5 we obtain $\overline{j_5} = 1$. We then deduce that $5v(j_2) - 2v(j_5) > 0$.

4.3.2. *Type II.2.* We calculate the reductions of j_2 and j_5 modulo \mathfrak{m} and find $\bar{j}_2 = \bar{\mu}_2^2$ and $\bar{j}_5 = 1$. So we deduce that $5v(j_2) - 2v(j_5) = 0$.

4.3.3. *Type III.1.* We find that j_2 is divisible by t_1^2 . Calculating the reduction of j_5 modulo \mathfrak{m} , we obtain $\bar{j}_5 = 1$. This implies $5v(j_2) - 2v(j_5) > 0$.

4.3.4. *Type III.2.* We compute the reductions of j_2 and j_5 modulo \mathfrak{m} and find $\bar{j}_2 = \bar{j}_5 = 1$. This implies $5v(j_2) - 2v(j_5) = 0$.

4.4. **Proof of Theorem 3.** For trees of Type I, there is nothing to prove.

4.4.1. *Type II.* We assume that the quintic f has tree Type II and use the universal family for Type II.1 in Table 2. Recall that the length $L(e_1)$ of the unique edge in the tree in this case is simply the valuation of t_1 .

We compute the invariants Δ, I_4 and I_{18} , and find that $\Delta \in t_1^6 A$, $I_4 \in t_1^2 A$ and $I_{18} \in t_1^9 A$, where A is the universal algebra in Section 4.1. Computing the reductions, we see

$$\begin{aligned}\overline{\Delta/t_1^6} &= \bar{\mu}_1^2 \bar{\mu}_2^2 (\bar{\mu}_1 - \bar{\mu}_2)^2, \\ \overline{I_4/t_1^2} &= -2 (\bar{\mu}_1^2 - \bar{\mu}_1 \bar{\mu}_2 + \bar{\mu}_2^2), \\ \overline{I_{18}/t_1^9} &= -\bar{\mu}_1^2 \bar{\mu}_2^2 (\bar{\mu}_1 + \bar{\mu}_2) (\bar{\mu}_1 - 2\bar{\mu}_2) (2\bar{\mu}_1 - \bar{\mu}_2) (\bar{\mu}_1 - \bar{\mu}_2)^2.\end{aligned}$$

Notice that, since $p \neq 3$, the two quantities $\overline{I_4/t_1^2}$ and $\overline{I_{18}/t_1^9}$ cannot vanish simultaneously. So we obtain

$$v(\Delta) = 6v(t_1) \quad \text{and} \quad (v(I_4) = 2v(t_1) \quad \text{or} \quad v(I_{18}) = 9v(t_1)).$$

From this, we deduce that

$$v(t_1) = \frac{1}{2}(v(\Delta) - 2v(I_4)) \quad \text{or} \quad v(t_1) = \frac{1}{3}(2v(\Delta) - v(I_{18})),$$

and the valuation of t_1 , the length of the unique non-trivial edge, is then the maximum of these two quantities, i.e.,

$$L(e_1) = \max \left(\frac{1}{2}(v(\Delta) - 2v(I_4)), \frac{1}{3}(2v(\Delta) - v(I_{18})) \right).$$

4.4.2. *Type III.* Before we give the proof of Theorem 3 for trees of Type III, we shortly discuss the various formulas occurring in that theorem. For a tree of Type III, we can only recover the edge lengths from the quintic invariants up to a permutation of the edges, see Example 24. A set of representatives of the edges here is given by (e_1, e_2) with $L(e_1) \leq L(e_2)$. For trees of Type III.2, this symmetry continues to hold, so the formulas do not change. For trees of Type III.1, there is no such symmetry, meaning that we can single out the edge next to the marked point. See Section 4.4.3 for the formulas in this case.

Now suppose that f has tree Type III. Recall that the lengths $L(e_1)$ and $L(e_2)$ of the edges e_1 and e_2 are respectively $v(t_1)$ and $v(t_2)$. Computing the invariants Δ, I_4 and I_{18} , we obtain³

$$\Delta = \mathbf{t}_1^2 \mathbf{t}_2^2 \mu_1^2 \mu_2^2 (t_2 \mu_2 + 1)^2 (-t_1 \mu_1 + t_2 \mu_2 + 1)^2 (t_1 \mu_1 - 1)^2$$

³The bold-faced factors have positive valuation, the remaining factors have valuation 0.

$$\begin{aligned}
I_{18} = & (\mu_1 t_1 - \mu_2 t_2)(\mu_1 t_1 + \mu_2 t_2)(-\mu_1 t_1 + \mu_2 t_2 + 2)(\mu_2^2 t_2^2 - \mu_1 t_1 + 2\mu_2 t_2 + 1) \\
& (\mu_2^2 t_2^2 + \mu_1 t_1 - 1)(-2\mu_1 \mu_2 t_1 t_2 + \mu_2^2 t_2^2 - \mu_1 t_1 + 2\mu_2 t_2 + 1)(\mu_1 \mu_2 t_1 t_2 - \mu_2 t_2 - 1) \\
& (\mu_1 \mu_2 t_1 t_2 + \mu_2 t_2 + 1)(\mu_1 \mu_2 t_1 t_2 - \mu_1 t_1 + 1)(\mu_1 \mu_2 t_1 t_2 + \mu_1 t_1 - 1) \\
& (\mu_1 \mu_2 t_1 t_2 + \mu_1 t_1 - 2\mu_2 t_2 - 1)(\mu_1 \mu_2 t_1 t_2 + 2\mu_1 t_1 - \mu_2 t_2 - 1)(-\mu_1^2 t_1^2 + \mu_2 t_2 + 1) \\
& (-\mu_1^2 t_1^2 + 2\mu_1 \mu_2 t_1 t_2 + 2\mu_1 t_1 - \mu_2 t_2 - 1)(\mu_1^2 t_1^2 - 2\mu_1 t_1 + \mu_2 t_2 t_2 + 1)
\end{aligned}$$

and $I_4 \in A$ with $\bar{I}_4 = -2$. So we deduce that

$$v(\Delta) = 2v(t_1) + 2v(t_2) \quad \text{and} \quad v(I_4) = 0.$$

If $v(t_1) < v(t_2)$, then we find $v(I_{18}) = 2v(t_1)$ and this gives

$$L(e_1) = v(t_1) = \frac{1}{2} \left(v(I_{18}) - \frac{9}{2}v(I_4) \right).$$

If, on the other hand, $v(t_1) = v(t_2)$, then we have

$$L(e_1) = v(t_1) = \frac{1}{4} (v(\Delta) - 2v(I_4)).$$

Therefore, in both cases, we obtain

$$L(e_1) = v(t_1) = \min \left(\frac{1}{2} \left(v(I_{18}) - \frac{9}{2}v(I_4) \right), \frac{1}{4} (v(\Delta) - 2v(I_4)) \right).$$

The length of the second edge is $v(t_2)$ and can be computed using Δ as

$$L(e_2) = v(t_2) = \frac{1}{2} (v(\Delta) - 2v(I_4)) - v(t_1).$$

Hence we deduce that

$$\begin{aligned}
L(e_1) &= \min \left(\frac{1}{2} \left(v(I_{18}) - \frac{9}{2}v(I_4) \right), \frac{1}{4} (v(\Delta) - 2v(I_4)) \right), \\
L(e_2) &= \frac{1}{2} (v(\Delta) - 2v(I_4)) - L(e_1).
\end{aligned}$$

4.4.3. *Type III.1.* Now let (q, ℓ) be a $(4, 1)$ -form with tree Type III.1. Let e_1 be the edge adjacent to ∞ and e_2 the second edge in the tree Type III.1, see Figure 2. Using the universal family for Type III.1 and computing the invariants we find $j_2 \in t_1^2 A^\times$, $j_5 = 1$, $\Delta \in t_1^6 t_2^2 A$ and $I_4 \in t_1^2 A^\times$ and

$$\overline{j_2/t_1^2} = \bar{\mu}_1^2, \quad \overline{\Delta/(t_1^6 t_2^2)} = \bar{\mu}_1^4 \bar{\mu}_2^2, \quad \text{and} \quad \overline{I_4/t_1^2} = -2\bar{\mu}_1^2.$$

So we deduce that

$$L(e_1) = v(t_1) = \frac{1}{10} (5v(j_2) - 2v(j_5)),$$

and

$$L(e_2) = v(t_2) = \frac{1}{2} (v(\Delta) - 2v(I_4)) - L(e_1).$$

4.5. Proof of Corollary 4. We now recall from [11] how the reduction type of a Picard curve $y^3\ell(x, z) = q(x, z)$ can be recovered from the $(4, 1)$ -marked tree of (q, ℓ) . We refer the reader to [11, Section 1.2] for the definition of the reduction type of a curve. We note here that our assumption that K is algebraically closed is not restrictive. Namely, if we are interested in the reduction type of a curve over a complete discretely valued field K , then its reduction type is completely determined by the reduction type of the base change over \overline{K} , see [11, Remark 3.6]. Finally, we note that the notion of an edge length used here is the same as the notion of *thickness* of the nodal point corresponding to the edge in question, which is used in other sources.

Let X be a Picard curve over K . The branch locus B of the covering

$$X \rightarrow \mathbb{P}^1$$

given by $[x : y : z] \mapsto [x : z]$ is the zero locus of $q \cdot \ell$. The minimal skeleton of the marked curve $(\mathbb{P}^{1, \text{an}}, B)$ is then the $(4, 1)$ -marked tree of (q, ℓ) . By applying a projective transformation, we can assume that the zero of ℓ is ∞ .

For tame coverings, we have that the inverse image of a skeleton is a skeleton (see [11, Theorem 3.1] or [12, Theorem 1.1]), so we obtain a map

$$\Sigma' \rightarrow \Sigma,$$

where Σ is the $(4, 1)$ -marked tree and Σ' is its inverse image under the morphism of Berkovich analytifications $X^{\text{an}} \rightarrow \mathbb{P}^{1, \text{an}}$. Consider the dehomogenized polynomial $q = q(x, 1)$. The criteria in [11, Section 3.1] allow us to reconstruct Σ' explicitly in terms of the piecewise-linear function $-\log|q|$ on $\mathbb{P}^{1, \text{an}} \setminus B$. This function can in turn be obtained from potential theory. We then find that over an edge in Σ , there are three edges if and only if the slope of $-\log|q|$ is divisible by three. If it is not divisible by three, then the length of an edge has to be divided by three. That is, the expansion factor $d_{e'/e}$ in this case is three. This data is enough to determine the skeleton for Picard curves, as the weights of the vertices are determined by the Riemann–Hurwitz conditions. The resulting graphs can be found in Figure 3.

Proof of Corollary 4. By Theorem 2, the $(4, 1)$ -marked tree type of (q, ℓ) is determined by the tropical invariants. The edge lengths of the $(4, 1)$ -marked tree type are then given by Theorem 3. To obtain the edge lengths for the curve X , we use the formula

$$d_{e'/e}L(e') = L(e),$$

where $d_{e'/e}$ is the expansion factor, see [3, Definition 2.4, Theorem 4.23]. □

REFERENCES

- [1] ABRAMOVICH, D., CAPORASO, L., AND PAYNE, S. The tropicalization of the moduli space of curves. *Ann. Sci. Éc. Norm. Supér. (4)* 48, 4 (2015), 765–809.
- [2] ACHTER, J. D., AND PRIES, R. The integral monodromy of hyperelliptic and trielliptic curves. *Math. Ann.* 338, 1 (2007), 187–206.
- [3] AMINI, O., BAKER, M., BRUGALLÉ, E., AND RABINOFF, J. Lifting harmonic morphisms I: metrized complexes and Berkovich skeleta. *Res. Math. Sci.* 2 (2015), Art. 7, 67.
- [4] BAKER, M., PAYNE, S., AND RABINOFF, J. On the structure of non-Archimedean analytic curves. In *Tropical and non-Archimedean geometry*, vol. 605 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 2013, pp. 93–121.
- [5] CHAN, M. Lectures on tropical curves and their moduli spaces. In *Moduli of curves*, vol. 21 of *Lect. Notes Unione Mat. Ital.* Springer, Cham, 2017, pp. 1–26.

- [6] CLÉRY, F., AND VAN DER GEER, G. Generating Picard modular forms by means of invariant theory. *Pure Appl. Math. Q.* 19, 1 (2023), 95–147.
- [7] DERKSEN, H., AND KEMPER, G. *Computational invariant theory*, vol. 130. Springer, Heidelberg, 2015. Invariant Theory and Algebraic Transformation Groups, VIII.
- [8] GORDAN, P. Beweis, dass jede covariante und invariante einer binären form eine ganze function mit numerischen coefficienten einer endlichen anzahl solcher formen ist. *J. Reine Angew. Math.* 69 (1868), 323–354.
- [9] GUBLER, W. A guide to tropicalizations. In *Algebraic and combinatorial aspects of tropical geometry*, vol. 589 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 2013, pp. 125–189.
- [10] HELMINCK, P. A. Tropical Igusa Invariants, 2021. preprint available at <https://arxiv.org/abs/1604.03987>.
- [11] HELMINCK, P. A. Invariants for trees of non-archimedean polynomials and skeleta of superelliptic curves. *Mathematische Zeitschrift* (2022).
- [12] HELMINCK, P. A. Skeletal filtrations of the fundamental group of a non-archimedean curve. *Advances in Mathematics* 431 (2023), 109242.
- [13] KAVEH, K., AND MANON, C. Gröbner theory and tropical geometry on spherical varieties. *Transform. Groups* 24, 4 (2019), 1095–1145.
- [14] LIU, Q. Courbes stables de genre 2 et leur schéma de modules. *Math. Ann.* 295, 2 (1993), 201–222.
- [15] MACLAGAN, D., AND STURMFELS, B. *Introduction to tropical geometry*, vol. 161 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2015.
- [16] MARKWIG, H. Tropical curves and covers and their moduli spaces. *Jahresber. Dtsch. Math.-Ver.* 122, 3 (2020), 139–166.
- [17] MUMFORD, D., FOGARTY, J., AND KIRWAN, F. *Geometric invariant theory*, third ed., vol. 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, 1994.
- [18] PAYNE, S. Analytification is the limit of all tropicalizations. *Math. Res. Lett.* 16, 3 (2009), 543–556.
- [19] POPOVICIU DRAISMA, M. I. Invariants of binary forms. *Doctoral Thesis, University of Basel, Faculty of Science* (2014).
- [20] SILVERMAN, J. H. *The arithmetic of elliptic curves*, second ed., vol. 106 of *Graduate Texts in Mathematics*. Springer, Dordrecht, 2009.
- [21] STURMFELS, B. *Algorithms in invariant theory*, second ed. Texts and Monographs in Symbolic Computation. SpringerWienNewYork, Vienna, 2008.
- [22] THE SAGE DEVELOPERS. *SageMath, the Sage Mathematics Software System (Version 9.5)*, 2022.
- [23] VAN DER GEER, G. Siegel modular forms of degree two and three and invariant theory. In *The art of doing algebraic geometry*, Trends Math. Birkhäuser/Springer, Cham, [2023] ©2023, pp. 217–239.

DEPARTMENT OF MATHEMATICAL SCIENCES DURHAM UNIVERSITY, UPPER MOUNTJOY CAMPUS, STOCKTON ROAD, DURHAM DH1 3LE.

Email address: paul.a.helminck@durham.ac.uk

U.C. BERKELEY, DEPARTMENT OF STATISTICS, 335 EVANS HALL # 3860, BERKELEY CA 94720, U.S.A.

Email address: yassine.el-maazouz@berkeley.edu

DEPARTMENT OF MATHEMATICS, KU LEUVEN, CELESTIJNENLAAN 200B, 3001 HEVERLEE, BELGIUM

Email address: enis.kaya@kuleuven.be