

Zero-free half-planes of the ζ -function via spaces of analytic functions

Aditya Ghosh
Kobi Kremnizer
S. Waleed Noor
Charles F. Santos

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Abstract

In this article we introduce a general approach for deriving zero-free half-planes for the Riemann zeta function ζ by identifying topological vector spaces of analytic functions with specific properties. This approach is applied to weighted ℓ^2 spaces and the classical Hardy spaces H^p ($0 < p \leq 2$). As a consequence precise conditions are obtained for the existence of zero-free half planes for the ζ -function.

0 Introduction

The Riemann Hypothesis (RH) is equivalent to a completeness problem in $L^2(0, 1)$. This was first stated in the 1950's by Nyman [15] and Beurling [10]:

Theorem 0.1 Define $\rho_\alpha(x) := \rho\left(\frac{x}{\alpha}\right) - \alpha\rho\left(\frac{1}{x}\right)$ for $0 < \alpha < 1$ (where $\rho(x)$ denotes the fractional part of x). If we denote $\nu := \text{span}\{\rho_\alpha \mid 0 < \alpha < 1\}$ and $\mathbb{1}_{(0,1)}$ the characteristic function of $(0, 1)$, then the following statements are equivalent:

1. The RH holds true,
2. $\mathbb{1}_{(0,1)}$ belongs to the closure of ν in $L^2(0, 1)$,
3. ν is dense in $L^2(0, 1)$.

A remarkable strengthening by Báez-Duarte [2] in 2003 showed it is enough to restrict ν to the countable index set $\alpha = 1/\ell$ for $\ell \in \mathbb{N}$ in condition 2 (whereby the density of ν in condition 3 no longer holds). The reader is directed to an article by Bagchi [3] that collects these results of Nyman, Beurling and Báez-Duarte and their proofs in one place. Recently in [14] these ideas have been transferred to the Hardy-Hilbert Space $H^2(\mathbb{D})$. Here, the RH is equivalent to the constant function $\mathbf{1}$ being in the closed linear span of certain elements $\{h_k \mid k \geq 2\}$ in $H^2(\mathbb{D})$. In [14] it is also proved that

$$\sum_{k=2}^n \mu(k)(I - S)h_k \rightarrow \mathbf{1} - z \quad \text{as } n \rightarrow \infty$$

in $H^2(\mathbb{D})$, where S is the shift operator and μ the Möbius function. As a consequence, this proves the density of $\text{span}\{h_k \mid k \geq 2\}$ in $H^2(\mathbb{D})$ in the compact-open topology (weaker than the H^2 -topology) by relating it with invertibility of $(I - S)$. The goal of this paper is to generalize these ideas to other spaces of analytic functions and establish criteria that would guarantee zero-free regions for the ζ -function.

The plan of the paper is the following. After a section of preliminaries, our general framework for obtaining zero-free half-planes for ζ is introduced in Section 2. This approach entails finding topological vector spaces of analytic functions X that satisfy a *checklist* of conditions. This general framework is then applied to the weighted Hardy spaces (unitarily equivalent to weighted

ℓ^2 spaces) in Section 3. After considering some concrete examples in Subsection 3.1, we obtain numerical conditions on the weights that guarantee zero-free regions for ζ in Subsection 3.2. In Subsection 3.3, we see that for non-trivial zero free half-planes the required weights can exhibit very extreme behavior. In Section 4, we apply this approach to the classical Hardy spaces H^p for $0 < p \leq 2$. In Subsection 4.1, we show that the checklist is completely satisfied for H^p with $0 < p < 1$. In particular we prove that $\text{span}\{h_k \mid k \in \mathbb{N}, k \geq 2\}$ is dense in H^p (see Corollary 4.6) for $0 < p < 1$. Since convergence in H^p implies convergence in the compact-open topology, this strengthens one of the main results of [14, Theorem 10]. In Subsection 4.2 all but one condition is satisfied for H^p with $1 \leq p \leq 2$. More precisely, we show that

$$1 \in \overline{\text{span}}_{H^p} \{h_k \mid k \geq 2\} \implies \zeta(s) \neq 0 \text{ for } \Re(s) > 1/p$$

(see Theorem 4.8) where the h_k are defined in Section 1. This is an H^p analogue of Beurling's results from [10].

Before we begin, it is worthwhile to highlight some results in the literature, with the hope of making evident the continuity of ideas and techniques after relocating the RH and questions surrounding zero-free regions of the ζ -function from the L^p spaces to analytic function spaces. Beurling in [10] proved that for every $1 < p < \infty$, the non-vanishing of $\zeta(s)$ for $\Re(s) > 1/p$ is equivalent to the density of ν in $L^p(0, 1)$, hence generalizing Theorem 0.1 considerably. Balazard and Saias [5] continued the study of the relation between zero-free half-planes and approximation problems in L^p spaces. Bercovici and Foias [9] proved that the $L^2(0, 1)$ -closure of ν equals

$$\left\{ f \in L^2(0, 1) : \frac{\mathcal{F}f(s)}{\zeta(s)} \text{ is holomorphic for } \Re(s) > 1/2 \right\}$$

where \mathcal{F} denotes the Mellin transform which is an isometric isomorphism of $L^2(0, 1)$ onto the Hardy space $H^2(\mathbb{C}_{1/2})$ of the half-plane $\Re(s) > 1/2$. This formula may be viewed as an unconditional version of Theorem 0.1. See [6] for interesting discussions around this formula and its possible generalizations to other $L^p(0, 1)$ spaces. The transform \mathcal{F} has played an important role in this theory due to the identity

$$\mathcal{F}\rho_\alpha(s) = \frac{\zeta(s)}{s}(\alpha - \alpha^s) \quad (0 < \alpha < 1, \Re(s) > 0).$$

The starting point of our approach is to introduce functionals $\Lambda^{(s)}$ on spaces of analytic functions on \mathbb{D} which replicate the role of \mathcal{F} (see Section 2). The

two families of spaces we apply $\Lambda^{(s)}$ to are the weighted sequence spaces ℓ_w^2 (Section 3) and the Hardy spaces H^p for $p > 0$ (Section 4). In particular we obtain H^p analogues of Beurling's results from [10]. The Hardy space $H^{1/3}$ was employed by Balazard [8] to prove the formula

$$\frac{1}{2\pi} \int_{\Re(s)=\frac{1}{2}} \frac{\log |\zeta(s)|}{|s|^2} |ds| = \sum_{\Re(\rho) > 1/2} \log \left| \frac{\rho}{1-\rho} \right|$$

where the sum is taken over the zeros of ζ (counting multiplicities) to the right of the critical line. In conclusion, spaces of analytic functions and Hardy spaces in particular have played an important role within the Nyman-Beurling approach to the RH and the theory it has inspired. Our goal is to explore further these connections. Balazard's bibliographical survey [4] contains detailed discussions on numerous works throughout the 20th century regarding completeness problems and a functional approach to the RH.

1 Preliminaries

Definition 1.1 The Hardy-Hilbert space $H^2(\mathbb{D})$ consists of all holomorphic function on the unit disk \mathbb{D} that satisfy

$$\|f\| := \left(\sup_{0 < r < 1} \int_{\mathbb{T}} |f(rz)|^2 dm(z) \right)^{1/2} < \infty \quad (1)$$

where m is the normalized Lebesgue measure on \mathbb{T} . The space $H^2(\mathbb{D})$ is a Hilbert space which inherits its inner product from the sequence space ℓ^2 :

$$H^2(\mathbb{D}) = \left\{ f = \sum_{n=0}^{\infty} a_n z^n \mid (a_n)_{n \in \mathbb{N}} \in \ell^2 \right\}. \quad (2)$$

The functions h_k in $H^2(\mathbb{D})$ were defined in [14] as

$$h_k(z) := \frac{1}{k} \frac{1}{1-z} \log \left(\frac{1+z+\dots+z^{k-1}}{k} \right), \quad k \geq 2.$$

Actually this definition of h_k differs from that of [14] by the factor $1/k$. The $H^2(\mathbb{D})$ version of Báez-Duarte's result in [14] plays a central role in this work.

Theorem 1.2 The following statements are equivalent:

1. RH holds true,
2. $\mathbf{1}$ belongs to the closure of $\text{span}\{h_k \mid k \geq 2\}$ in $H^2(\mathbb{D})$, and
3. $\text{span}\{h_k \mid k \geq 2\}$ is dense in $H^2(\mathbb{D})$.

where $\mathbf{1}$ is the constant function.

Let $S = M_z$ denote the shift operator on $H^2(\mathbb{D})$ of multiplication by z . Then in [14, Lemma 11] it is also proved that

$$\left\| \sum_{k=2}^n \mu(k)(I - S)h_k - (1 - z) \right\|_{H^2(\mathbb{D})} \longrightarrow 0 \quad \text{as } n \rightarrow \infty \quad (3)$$

and as a consequence it was established that

$$\overline{\text{span}}_{H^2(\mathbb{D})} \{(I - S)h_k \mid k \geq 2\} = H^2(\mathbb{D}). \quad (4)$$

It is important to note that $I - S$ is not an invertible operator on $H^2(\mathbb{D})$ (but has dense range). If it were invertible, then $(I - S)^{-1}$ applied to (3) or (4) would prove the RH by Theorem 1.2. It is worth remarking that a result similar to (3) appears in [7, Section 13], but in the latter article the shift operator S is defined on the weighted ℓ^2 space with weights $(1/(\zeta(2)k^2))_k$ which is equivalent to multiplication by z on a weighted Bergman space \mathcal{A} (defined below). The shifts on $H^2(\mathbb{D})$ and \mathcal{A} are not equivalent, and hence (3) is not an immediate consequence of results in [7]. In fact, unlike the shift on $H^2(\mathbb{D})$, even basic questions such as a characterization of the closed invariant subspaces of the Bergman shifts remain open problems (see [12]).

It is useful to locate the $H^2(\mathbb{D})$ version of Báez-Duarte's theorem among the cornucopia of spaces where it appears in the literature (see Figure 1). The space \mathcal{A} is the Hilbert space of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ defined on \mathbb{D} for which the inner product is given by

$$\langle f, g \rangle := \sum_{n=0}^{\infty} \frac{a_n \bar{b}_n}{(n+1)(n+2)}.$$

Then the maps $T : H^2(\mathbb{D}) \rightarrow \mathcal{A}$ (defined in Figure 1) and $\Psi : \ell_{\omega}^2 \rightarrow \mathcal{A}$

$$\Psi : (x(1), x(2), \dots) \longmapsto \sum_{n=0}^{\infty} x(n+1)z^n$$

are isometric isomorphisms, where ℓ_ω^2 is the weighted ℓ^2 -space with weights $\omega_n = \frac{1}{(n+1)(n+2)}$ corresponding to the coefficients of functions in \mathcal{A} . See [14] for more details on T and Ψ . Let \mathcal{M} be the closed subspace of $L^2(0, 1)$ consisting of functions almost everywhere constant on the intervals $[\frac{1}{n+1}, \frac{1}{n})$ for $n \geq 1$, and $H^2(\mathbb{C}_{\frac{1}{2}})$ the Hardy space of analytic functions F on the half-plane $\mathbb{C}_{\frac{1}{2}} := \{s \in \mathbb{C} : \Re(s) > 1/2\}$ such that

$$\|F\|^2 := \sup_{\sigma > \frac{1}{2}} \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\sigma + it)|^2 dt < \infty.$$

The maps $U : \mathcal{M} \rightarrow \mathcal{H}$ and the Mellin transform $\mathcal{F} : L^2(0, 1) \rightarrow H^2(\mathbb{C}_{\frac{1}{2}})$ are also isometric isomorphisms. See [3] for more details on U and \mathcal{F} .

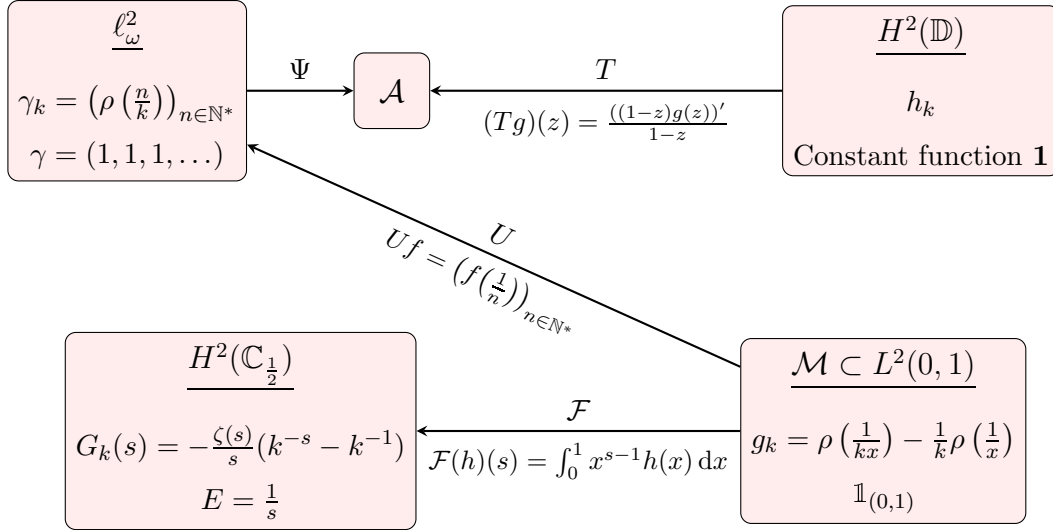


Figure 1: Spaces and isometries between them.

2 General framework

In this section we outline our approach to finding zero free half-planes for ζ using more general spaces of analytic functions, but that contain $H^2(\mathbb{D})$. We begin by looking more carefully at the situation in $H^2(\mathbb{D})$.

From Figure 1, we see that there is a chain of isometries:

$$H^2(\mathbb{D}) \xrightarrow[\cong]{T} \mathcal{A} \xrightarrow[\cong]{\Psi^{-1}} \ell_\omega^2 \xrightarrow[\cong]{U^{-1}} \mathcal{M} \xrightarrow[\text{not onto}]{\mathcal{F}} H^2(\mathbb{C}_{\frac{1}{2}}). \quad (5)$$

We denote by $\Lambda: H^2(\mathbb{D}) \rightarrow H^2(\mathbb{C}_{\frac{1}{2}})$ the composition of these isometries. Since $H^2(\mathbb{C}_{\frac{1}{2}})$ is a reproducing kernel Hilbert space, the evaluation functionals $E_s: H^2(\mathbb{C}_{\frac{1}{2}}) \rightarrow \mathbb{C}$ for each $s \in \mathbb{C}_{\frac{1}{2}}$ are bounded. So if we define the functionals $\Lambda^{(s)} := E_s \circ \Lambda: H^2(\mathbb{D}) \rightarrow \mathbb{C}$, then we get

Lemma 2.1 $\Lambda^{(s)}$ is bounded on $H^2(\mathbb{D})$ for $\Re(s) > 1/2$.

The principal feature of these functionals is the property that

$$\Lambda^{(s)}(h_k) = G_k(s) = -\frac{\zeta(s)}{s}(k^{-s} - k^{-1}) \quad (6)$$

which can be seen from Figure 1. By (5) one can also check that

$$\Lambda^{(s)}(1) = -\frac{1}{s}, \quad \Lambda^{(s)}(z^k) = f_k(s) := -\frac{1}{s}((k+1)^{1-s} - k^{1-s}). \quad (7)$$

Indeed, since $T(z^k) = kz^{k-1} - \frac{z^k}{1-z}$ we have the sequence

$$s_k := \Psi^{-1}T(z^k) = (0, \dots, 0, k, -1, -1, \dots) \in \mathcal{H}.$$

where the k -th term of s_k is k . Now $p_k := U^{-1}s_k$ belongs to \mathcal{M} such that $p_k(x) = s_k(n)$ for all $x \in [\frac{1}{n+1}, \frac{1}{n})$ and its Mellin transform is

$$\mathcal{F}(p_k)(s) = \int_{\frac{1}{k+1}}^{\frac{1}{k}} kx^{s-1} dx - \int_0^{\frac{1}{k+1}} x^{s-1} dx = f_k(s).$$

Therefore $\Lambda^{(s)}(z^k) = f_k(s)$ and $\Lambda^{(s)}$ is uniquely determined by these values.

The following growth estimate for $f_k(s)$ will be used frequently.

Proposition 2.2 For each $\Re(s) > 0$, we have $|f_k(s)| \asymp k^{-\Re(s)}$ for all $k \geq 1$.

Proof. By the Fundamental Theorem of Calculus we have

$$|(k+1)^{1-s} - k^{1-s}| = |1-s| \left| \int_k^{k+1} y^{-s} dy \right| \asymp |1-s| k^{-\Re(s)} \quad (8)$$

and the growth estimate easily follows. \square

The following *checklist* of conditions summarizes our abstract approach. Denote by \mathbb{C}_r the half-plane $\{s \in \mathbb{C} \mid \Re(s) > r\}$.

Suppose a topological vector space X of analytic functions satisfies the following conditions:

- (C1) $z^k \in X$ for $k \in \mathbb{N}$ form a Schauder basis of X .
(C2) $H^2(\mathbb{D}) \subseteq X$ with the relative topology weaker than that of $H^2(\mathbb{D})$.
(C3) 1 belongs to the closure of $\text{span}\{h_k \mid k \geq 2\}$ in X .
(C4) There exists $r \in \mathbb{R}$ such that the functionals $\Lambda^{(s)} : X \rightarrow \mathbb{C}$ defined by

$$\begin{aligned} z^k &\mapsto f_k(s) = -\frac{1}{s} \left((k+1)^{1-s} - k^{1-s} \right) & (k \in \mathbb{N}^*) \\ 1 &\mapsto -\frac{1}{s} \end{aligned}$$

are bounded on X for all $s \in \mathbb{C}_r$.

The following result provides the justification for our general approach.

Proposition 2.3 If there exists a space of analytic functions X satisfying the conditions above for some $r \in \mathbb{R}$, then $\zeta(s) \neq 0$ for all $s \in \mathbb{C}_r$.

Proof. By (C1) and (C2) it is clear that $\Lambda^{(s)}$ is determined by its values on $H^2(\mathbb{D})$. Therefore by (C3), (C4) and (6) it follows that $\Lambda^{(s)}(1) = -1/s$ can be approximated pointwise by linear combinations of

$$\Lambda^{(s)}(h_k) = -\frac{\zeta(s)}{s} (k^{-s} - k^{-1})$$

for all $s \in \mathbb{C}_r$. Since $1/s$ has no zeros for $s \in \mathbb{C}_r$, the same must be true for $\Lambda^{(s)}(h_k)$ and hence for $\zeta(s)$. \square

In the remainder of this article we apply this approach to the weighted sequence spaces ℓ_w^2 and the Hardy spaces H^p . In particular, we investigate the extent to which these spaces satisfy conditions (C1) to (C4). It will become evident in the following sections that condition (C3) poses the main challenge here. For instance Example 3.2 and Theorem 4.8 show that (C3) would imply $\zeta(s) \neq 0$ for \mathbb{C}_r with $1/2 < r < 1$. An alternative route to proving (C3) is to show that the operator $I - S$ is invertible on X . This is because the approximation (3) holds in X by (C2), that is

$$\left\| \sum_{k=2}^n \mu(k)(I - S)h_k - (1 - z) \right\|_X \longrightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (9)$$

We therefore call the invertibility of $I - S$ on X the **Easy (C3)** condition.

We want to be clear that no new zero-free half-planes for ζ are obtained in this work. But rather, we hope it will lead to a deeper understanding of the challenges posed en route to such results, when viewed through the lens of spaces of analytic functions. It is reasonable to ask why $H^2(\mathbb{D})$ was chosen over $H^2(\mathbb{C}_{\frac{1}{2}})$ (see [3]) or \mathcal{A} (equivalently ℓ_w^2 as in [7]) to formulate our approach. Compared to the latter two spaces, the theory of $H^2(\mathbb{D})$ is the most complete of any reproducing kernel Hilbert space, with a vast array of tools and techniques developed over many decades. The article [14] contains some applications of these tools to the Nyman-Beurling approach to the RH.

3 Weighted ℓ^2 sequence spaces

In this section our goal is to apply the fundamental principle of Section 2 to spaces X of analytic functions with Taylor series coefficients in some weighted ℓ^2 space. We begin with some illustrative examples.

3.1 Some examples

Example 3.1 (Smaller disks) Recall the definition 1.1 of $H^2(\mathbb{D})$. We can restrict the supremum to $0 < r < \epsilon$ to get a Hardy Space on the smaller disk $\mathbb{D}_\epsilon := B(0; \epsilon)$, where $0 < \epsilon < 1$, defined by

$$H^2(\mathbb{D}_\epsilon) := \{f \in \text{Hol}(\mathbb{D}_\epsilon) \mid \sup_{0 < r < \epsilon} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty\}. \quad (10)$$

It is a quick check to see this is equivalent to the weighted ℓ^2 definition:

$$H^2(\mathbb{D}_\epsilon) = \left\{ \sum_{n=0}^{\infty} a_n z^n \mid (a_n \epsilon^n)_{n \in \mathbb{N}} \in \ell^2 \right\}. \quad (11)$$

Comparing with the Checklist, **(C1)** certainly holds as $(z/\epsilon)^k$ form an orthonormal basis. From (10), we see that $H^2(\mathbb{D}) \subseteq H^2(\mathbb{D}_\epsilon)$ and **(C2)** holds. The **Easy (C3)** condition also follows by definition 10:

$$\left\| \frac{1}{1-z} f(z) \right\|_{H^2(\mathbb{D}_\epsilon)} \leq \frac{1}{1-\epsilon} \|f(z)\|_{H^2(\mathbb{D}_\epsilon)}.$$

The problem arises in **(C4)**. That is because $\Lambda^{(s)}$ is bounded on $H^2(\mathbb{D}_\epsilon)$ if and only if $\Lambda^{(s)}(z^k/\epsilon^k) = f_k(s)/\epsilon^k$ forms an ℓ^2 sequence. Since $|f_k(s)| \asymp k^{-\Re(s)}$ by Proposition 2.2, this requires that

$$\left((1/\epsilon)^k k^{-\Re(s)} \right)_{k \in \mathbb{N}^*} \in \ell^2 \quad \text{where } (1/\epsilon) > 1.$$

However, there are no values of $s \in \mathbb{C}$ for which this is true as the sequence above is unbounded. \square

In the next example we see that **(C4)** does hold.

Example 3.2 For $\alpha > 0$, consider the space of analytic functions

$$X_\alpha := \left\{ \sum_{n=0}^{\infty} a_n z^n \mid (a_n n^{-\alpha})_{n \in \mathbb{N}^*} \in \ell^2 \right\} \quad (12)$$

with the inner product $\langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \rangle = a_0 \bar{b}_0 + \sum_{n=1}^{\infty} a_n \bar{b}_n n^{-2\alpha}$. Condition **(C1)** holds as $(n^\alpha z^n)_{n \in \mathbb{N}^*} \cup \{1\}$ forms an orthonormal basis for X_α . Also, **(C2)** holds as $H^2(\mathbb{D}) \subseteq X_\alpha$ and the norm on X_α is dominated by the $H^2(\mathbb{D})$ norm. Condition **(C4)** also holds: $\Lambda^{(s)}$ is bounded on X_α if and only if $\Lambda^{(s)}(n^\alpha z^n) = n^\alpha f_n(s) \in \ell^2$. Since $|f_n(s)| \asymp n^{-\Re(s)}$, we require that

$$(n^\alpha n^{-\Re(s)})_{n \in \mathbb{N}^*} \in \ell^2 \quad \text{where } \alpha > 0.$$

This holds for $\Re(s) > \frac{1}{2} + \alpha$. It follows that if **(C3)** holds for any $0 < \alpha < 1/2$, then the checklist is satisfied by X_α and we obtain a non-trivial zero free half-plane for ζ . But verifying **(C3)** for any such α is not easy. So let's consider **Easy (C3)** instead. Define the functions $f_\delta(z) = \sum_{m=1}^{\infty} m^\delta z^m$ for some real $\delta > 0$. Then $f_\delta \in X_\alpha$ if and only if $\delta < \alpha - \frac{1}{2}$. Since $(I - S)^{-1}$ is formally the operator of multiplication by $\frac{1}{1-z}$, we have

$$(I - S)^{-1} f_\delta = \sum_{n=0}^{\infty} z^n \cdot \sum_{m=1}^{\infty} m^\delta z^m.$$

By the ratio test we see that the infinite sum has radius of convergence 1. Collecting the coefficient of z^k we see that $(I - S)^{-1} f_\delta = \sum_{k=0}^{\infty} c_k z^k$ where $c_k = \sum_{l=1}^k l^\delta$. Since $\delta > 0$, we can bound this sum from below as follows

$$c_k = \sum_{l=1}^k l^\delta \geq \int_0^k x^\delta dx = \frac{k^{\delta+1}}{\delta+1}.$$

Hence for $(I - S)^{-1}f_\delta \in X_\alpha$ we must necessarily have

$$\sum_{k=1}^{\infty} k^{2(\delta+1)} k^{-2\alpha} < \infty$$

or equivalently $\delta < \alpha - \frac{3}{2}$. Hence if we choose δ such that $\alpha - \frac{3}{2} < \delta < \alpha - \frac{1}{2}$, then $f_\delta \in X_\alpha$ but $(I - S)^{-1}f_\delta \notin X_\alpha$. Therefore **Easy (C3)** fails for X_α . \square

3.2 Analysis of weights

We now consider the general setting of weighted ℓ^2 sequence spaces.

Definition 3.3 For $w_n \geq 1$ ($n \in \mathbb{N}$), we define the following Hilbert space:

$$X = \left\{ \sum_{n=0}^{\infty} a_n z^n \mid (a_n/w_n)_{n \in \mathbb{N}} \in \ell^2 \right\} \simeq \ell_w^2 \quad (13)$$

where X is equipped with the inner product

$$\left\langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \right\rangle = \sum_{n=0}^{\infty} a_n \overline{b_n} / w_n^2.$$

Clearly X is isometrically isomorphic to a weighted ℓ^2 -space.

It follows from the definition that $(z^k w_k)_{k \in \mathbb{N}}$ forms an orthonormal basis for X . As $w_n \geq 1$, the norm of X is dominated by that of $H^2(\mathbb{D})$ and $H^2(\mathbb{D}) \subset X$. Therefore **(C1)** and **(C2)** both hold. We shall therefore focus on **Easy (C3)** and **(C4)**.

The examples we considered in Subsection 3.1 are:

1. $w_n = 1$: Corresponds to $X = H^2(\mathbb{D})$. Here $(I - S)^{-1}$ is not invertible, but $\Lambda^{(s)}$ is bounded for $\Re(s) > 1/2$.
2. $w_n = (1/\epsilon)^n$ for $0 < \epsilon < 1$: Corresponds to $H^2(\mathbb{D}_\epsilon)$ in Example 3.1. Here $I - S$ is invertible, but $\Lambda^{(s)}$ is not bounded for any s .
3. $w_n = n^\alpha$, $\alpha > 0$: Corresponds to X in Example 3.2. Here $I - S$ is not invertible, but $\Lambda^{(s)}$ is bounded for $\Re(s) > 1/2 + \alpha$.

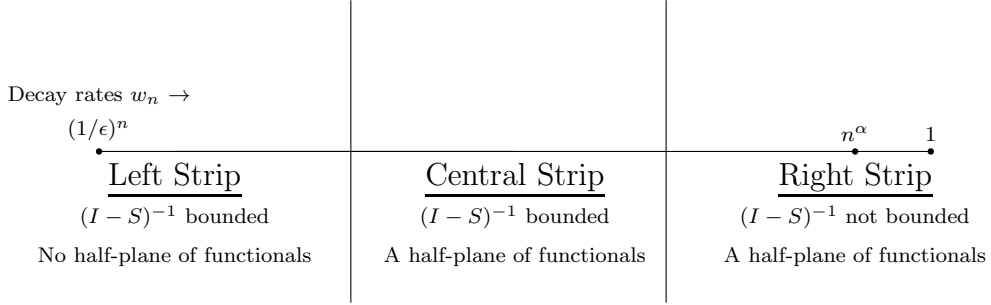


Figure 2: Weights and Conditions

A general tendency can be observed as depicted by Figure 2. If the decay rates of weights are too fast (left strip), we do not have a half-plane where $\Lambda^{(s)}$ is bounded. If the decay rates are too slow (right strip), then we do not have $(I - S)^{-1}$ as a bounded operator. Ideally we would like to obtain weights that belong between these extremes (central strip).

The next result characterizes the weights required for X to satisfy **(C4)**.

Proposition 3.4 Given a weighted Hardy space X as in (13) and $s \in \mathbb{C}$, the functional $\Lambda^{(s)}$ is bounded on X if and only if

$$\left(\frac{w_k}{k^{\Re(s)}} \right)_{k \in \mathbb{N}^*} \in \ell^2. \quad (14)$$

Proof. The monomials $(z^k w_k)_{k \in \mathbb{N}}$ form an orthonormal basis for X . Hence $\Lambda^{(s)}$ is bounded on X precisely when $\Lambda^{(s)}(z^k w_k) = w_k f_k(s)$ forms an ℓ^2 sequence. Therefore the estimate $|f_k(s)| \asymp k^{-\Re(s)}$ (Proposition 2.2) implies that this is equivalent to

$$\left(\frac{w_k}{k^{\Re(s)}} \right)_{k \in \mathbb{N}^*} \in \ell^2.$$

This concludes the result. \square

The following is a necessary condition for **Easy (C3)** to hold.

Proposition 3.5 Given a Hilbert Space X as in (13), let

$$r_m := \sum_{n=m}^{\infty} \frac{w_n^2}{w_m^2}.$$

If $I - S$ is invertible on X , then $(r_m)_{m \in \mathbb{N}}$ is a bounded sequence.

Proof. Suppose the operator $(I - S)^{-1}$ is well defined and bounded, with operator norm $C > 0$. For each $m \in \mathbb{N}$, consider $(I - S)^{-1}z^m$ in the equation above. Then

$$\|(I - S)^{-1}z^m\|_X^2 = r_m/w_m^2 = r_m\|z^m\|_X^2.$$

By definition of the operator norm of $(I - S)^{-1}$, $r_m \leq C^2$ for all $m \in \mathbb{N}$ \square

Table 1 includes a collection of weights that have been tested.

Table 1: Weights w_n and their classifications

Weight w_n	$I - S$ invertible	$\Lambda^{(s)}$ bounded	Strip
1	\times	\checkmark	Right
n^α	\times	\checkmark	Right
$n^\alpha + (\log n)^\beta$	\times	\checkmark	Right
$\exp((\log n)^{1+\alpha}), \alpha > 0$	\times	\times	-
$\exp(n^\alpha), 0 < \alpha < 1$	\times	\times	-
$(1/\epsilon)^n$	\checkmark	\times	Left
$\exp(n^\alpha), \alpha > 1$	\checkmark	\times	Left

3.3 Extremal behavior of weights

Suppose we have weights w_n for a space X that satisfies both conditions **Easy (C3)** and **(C4)** (thus giving a zero-free half plane $\Re(s) > r$). We are only interested in $\frac{1}{2} < r < 1$ since $\zeta(s)$ has no zeroes for $r = \Re(s) \geq 1$. The following result shows that such weights w_n necessarily exhibit extremely divergent behavior.

Proposition 3.6 Let w_n be the weights for a sequence space X satisfying conditions **Easy (C3)** and **(C4)** with $\frac{1}{2} < r < 1$. If $(n_i) \subseteq \mathbb{N}$ is a subsequence with $\sum \frac{1}{n_i} = \infty$, then

$$\liminf_{i \rightarrow \infty} \frac{w_{n_i}}{n_i^{r-\frac{1}{2}}} = 0 \quad \text{and} \quad \limsup_{i \rightarrow \infty} \frac{w_{n_i}}{n_i^{r-\frac{1}{2}}} = \infty. \quad (15)$$

In particular, $\lim_{i \rightarrow \infty} \frac{w_{n_i}}{n_i^{r-\frac{1}{2}}}$ does not exist for any such subsequence (n_i) .

Proof. We prove by contradiction for each half of the result.

Suppose, $\liminf_{i \rightarrow \infty} \frac{w_{n_i}}{n_i^{r-\frac{1}{2}}} = C > 0$. Then for any $C' < C$, there exists $N \in \mathbb{N}$ such that for all $i \geq N$, $\frac{w_{n_i}}{n_i^{r-\frac{1}{2}}} > C'$. Thus,

$$\sum_{i=N}^{\infty} \left(\frac{w_{n_i}}{n_i^r} \right)^2 > C'^2 \sum_{i=N}^{\infty} \left(\frac{n_i^{r-\frac{1}{2}}}{n_i^r} \right)^2 = C'^2 \sum_{i=N}^{\infty} \frac{1}{n_i} = \infty.$$

Hence we contradict Proposition 3.4 since $(w_n/n^r)_{n>0} \in \ell^2$.

Now suppose, $\limsup_{i \rightarrow \infty} \frac{w_{n_i}}{n_i^{r-\frac{1}{2}}} < +\infty$. Then, there is $C'' > 0$ and $M \in \mathbb{N}$ such that for all $i > M$, $\frac{w_{n_i}}{n_i^{r-\frac{1}{2}}} < C''$. This gives,

$$\sum_{i=N}^{\infty} \frac{1}{w_{n_i}^2} > \frac{1}{C''^2} \sum_{i=N}^{\infty} \frac{1}{n_i^{2r-1}} > \frac{1}{C''^2} \sum_{i=N}^{\infty} \frac{1}{n_i} = \infty.$$

Hence we contradict Proposition 3.5 since r_N is finite. □

We note that Proposition 3.6 highlights a tension between **Easy (C3)** and **(C4)** which is supported by Figure 2 and Table 1.

4 The Classical Hardy spaces H^p

In this section we consider the spaces H^p ($p > 0$) consisting of functions f holomorphic in \mathbb{D} for which

$$\|f\|_p^p := \sup_{0 < r < 1} \int_{\mathbb{T}} |f(rz)|^p dm(z) < \infty$$

where m is normalized Lebesgue measure on \mathbb{T} . The text of Duren [11] is a classical reference. The H^p spaces are Banach spaces for $p \geq 1$ and complete metric spaces for $0 < p < 1$. By Fatou's theorem, any $f \in H^p$ has radial limits a.e. on \mathbb{T} with respect to m . Using f to also denote the radial limit function, we have

$$\|f\|_p^p = \int_{\mathbb{T}} |f(z)|^p dm(z).$$

The case $p = 2$ gives us the Hardy-Hilbert space $H^2(\mathbb{D})$ as defined in 1.1. In this section we will focus on $0 < p \leq 2$. It is well-known that the monomials form a basis for H^p , that $H^p \subset H^q$ for $p > q$ and that the topology of H^p weakens as p decreases. Therefore the conditions **(C1)** and **(C2)** in the *checklist* are satisfied for $X = H^p$ with $0 < p \leq 2$. We shall see that **(C4)** also holds for all $0 < p \leq 2$. As for **(C3)**, the next subsection uses the invertibility of $I - S$ between distinct H^p spaces to prove **(C3)** when $0 < p < 1$. In Subsection 4.2 we show that **(C3)** for some $1 < p \leq 2$ would imply $\zeta(s) \neq 0$ for $\Re(s) > 1/p$. This is an H^p analogue of Beurling's result [10].

4.1 The H^p spaces for $0 < p < 1$

We first show that **(C4)** holds in this case.

Proposition 4.1 $\Lambda^{(s)}$ is bounded on H^p for $0 < p < 1$ if $\Re(s) > \frac{1}{p}$.

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^p$ for $0 < p < 1$. Then $|a_n| \leq C n^{1/p-1} \|f\|_{H^p}$ for some constant $C > 0$ by [11, Theorem 6.4]. Hence by Lemma 2.2

$$|\Lambda^{(s)} f| \leq \sum_{n=0}^{\infty} |a_n| |f_n(s)| \leq C \sum_{n=0}^{\infty} n^{1/p-1-\Re(s)} \|f\|_{H^p}.$$

So, $\Lambda^{(s)}$ is bounded on H^p if $\Re(s) > \frac{1}{p}$. □

We now move to the proof of **(C3)**. We shall need the following result from Duren [11, Theorem 6.1].

Theorem 4.2 Let $1 \leq q \leq 2$ and p satisfying $1/p + 1/q = 1$. If $(a_n)_{n \in \mathbb{N}}$ is a sequence in ℓ^q , then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defines a function in H^p satisfying

$$\|f\|_p \leq \|(a_n)_{n \in \mathbb{N}}\|_q.$$

We first extend the validity of equation (3) to all H^p spaces with $0 < p < \infty$. Denote the ℓ^q norm of $f(z) = \sum_n a_n z^n$ by $\|f\|_{\ell^q} = (\sum_n |a_n|^q)^{1/q}$.

Lemma 4.3 For all $0 < p < \infty$, we have

$$\sum_{k=2}^n \mu(k)(I - S)h_k \rightarrow 1 - z \text{ in } H^p. \quad (16)$$

Proof. First note that we already have the result for $0 < p \leq 2$ by (3) which corresponds to $2 \leq q < \infty$. Therefore by Theorem 4.2 it is enough to prove that $\sum_{k=2}^n \mu(k)(I - S)h_k \rightarrow 1 - z$ in the ℓ^q sense for $1 < q < 2$. We have

$$\sum_{k=2}^n \mu(k)(I - S)h_k(z) = \sum_{k=1}^n \frac{\mu(k)}{k} [\log(1 - z^k) - \log(1 - z) - \log k]. \quad (17)$$

The Taylor coefficients of $\log(1 - z) = -\sum_{j=1}^{\infty} z^j/j$ belong to ℓ^q for $1 < q < 2$, which implies the same for $\log(1 - z^k) = -\sum_{j=1}^{\infty} z^{jk}/j$. Therefore the Taylor coefficients of (17) belong to ℓ^q for each $n \geq 2$. We shall need the following relations involving the Möbius function

$$\sum_{k=1}^{\infty} \frac{\mu(k)}{k} = 0 \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{\mu(k) \log k}{k} = -1 \quad (18)$$

(see [1, Thm. 4.16] and [13, p. 185, Exercise 16]). This implies that

$$\sum_{k=1}^n \frac{\mu(k)}{k} [-\log(1 - z) - \log k] \rightarrow 1$$

in ℓ^q norm as $n \rightarrow \infty$. Hence by (17) it suffices to prove that

$$\sum_{k=1}^n \frac{\mu(k)}{k} \log(1 - z^k) \rightarrow -z \quad \text{in the } \ell^q \text{ sense.} \quad (19)$$

By [14, eq. (4.6)] we have

$$\sum_{k=1}^n \frac{\mu(k)}{k} \log(1 - z^k) + z = - \sum_{j=n+1}^{\infty} \frac{z^j}{j} \sum_{\substack{d|j \\ 1 \leq d \leq n}} \mu(d)$$

with

$$\left| \sum_{\substack{d|j \\ 1 \leq d \leq n}} \mu(d) \right| \leq \sum_{\substack{d|j \\ 1 \leq d \leq n}} |\mu(d)| \leq \sum_{d|j} 1 = \tau(j),$$

where $\tau(n)$ denotes the number of divisors of n . Since $\tau(n) = o(n^\epsilon)$ for every $\epsilon > 0$ [1, p. 296], we get

$$\left\| \sum_{k=1}^n \frac{\mu(k)}{k} \log(1 - z^k) + z \right\|_{\ell^q}^q = \sum_{j=n+1}^{\infty} \frac{1}{j^q} \left| \sum_{\substack{d|j \\ 1 \leq d \leq n}} \mu(d) \right|^q \leq \sum_{j=n+1}^{\infty} \frac{\sigma(j)^q}{j^q} \rightarrow 0$$

since $\sigma(j) \lesssim j^\epsilon$ where $\epsilon > 0$ can be chosen small enough so that $q - \epsilon q > 1$. This proves (19) and hence the result. \square

Although the $I - S$ is not invertible on any H^p space, $(I - S)^{-1} = M_{\frac{1}{1-z}}$ may still be bounded between different H^p spaces.

Lemma 4.4 For each $0 < q < 1$ there exists $p > 0$ sufficiently large such that the operator $M_{\frac{1}{1-z}}$ from H^p to H^q is bounded.

Proof. We shall need the reverse Hölder's inequality: Let $0 < r < 1$ and s satisfying $\frac{1}{r} + \frac{1}{s} = 1$ (so that $s < 0$). For any non-negative $f \in L^r(\mathbb{T})$ and $g \in L^s(\mathbb{T})$ with $\int g^s dm > 0$,

$$\int fg dm \geq \left(\int f^r dm \right)^{1/r} \left(\int g^s dm \right)^{1/s}.$$

Choose any $0 < q < p$ (not necessarily conjugate exponents) and let $h \in H^p$. Define $r := q/p < 1$ (hence $s < 0$), $f(z) := |h(z)/(z-1)|^p$ and $g(z) := |1-z|^p$. So we have

$$\int f^r dm = \int_{\mathbb{T}} \left| \frac{h(z)}{z-1} \right|^q dm, \quad \int g^s dm = \int_{\mathbb{T}} |1-z|^{ps} dm, \quad \int fg dm = \|h\|_p^p.$$

Therefore we get

$$\left(\int_{\mathbb{T}} \left| \frac{h(z)}{z-1} \right|^q dm \right)^{1/r} \leq \|h\|_p^p \left(\int_{\mathbb{T}} |1-z|^{ps} dm \right)^{-1/s}. \quad (20)$$

For the right side of (20) to be finite, we need $\int_{\mathbb{T}} |1-z|^{ps} dm < \infty$ keeping in mind that $s < 0$. This occurs precisely when

$$ps > -1 \iff -\frac{1}{s} > p \iff \frac{1}{r} > 1+p \iff \frac{q}{p} < \frac{1}{1+p} \iff q < \frac{p}{1+p}.$$

So we necessarily have $0 < q < 1$. Hence we conclude that for any $0 < q < 1$ there exists $p > 0$ large enough satisfying $q < \frac{p}{1+p}$ for which (20) gives

$$\left\| \frac{h}{1-z} \right\|_q \leq C_{p,q} \|h\|_p \quad \forall h \in H^p \quad (21)$$

where constant $C_{p,q} > 0$ depends only on p and q . This proves the lemma. \square

We are now ready to prove **(C3)** for $X = H^q$ with $0 < q < 1$.

Theorem 4.5 $\sum_{k=2}^n \mu(k)h_k \rightarrow 1$ in H^q for $0 < q < 1$.

Proof. For any $0 < q < 1$ there exists a $p > 0$ large enough so that $M_{\frac{1}{1-z}}$ is bounded from H^p to H^q by Lemma 4.4. Therefore applying $M_{\frac{1}{1-z}}$ to the approximation $\sum_{k=2}^n \mu(k)(I-S)h_k \rightarrow 1-z$ in H^p from Lemma 4.3 gives the result. \square

Therefore the checklist is completely satisfied for H^p with $0 < p < 1$ giving the zero-free half-plane $\Re(s) > 1$. As an immediate corollary we get

Corollary 4.6 $\text{span}(h_k)_{k \geq 2}$ is dense in H^p for all $0 < p < 1$.

Proof. We need to show that the weighted composition operators

$$W_n f(z) = (1 + z + \cdots + z^{n-1})f(z^n), \quad n \geq 1$$

introduced in [14] are bounded in H^p . The Littlewood Subordination Theorem [11, Theorem 1.7] states that if $\varphi \in \text{Hol}(\mathbb{D})$, then

$$|\varphi(z)| \leq |z| \quad \forall z \in \mathbb{D} \implies \int_{\mathbb{T}} |f \circ \varphi|^p dm \leq \int_{\mathbb{T}} |f|^p dm$$

for all $p \in (0, \infty]$. So the operator $f \mapsto f \circ \varphi$ with $\varphi(z) = z^n$ is bounded on H^p as is the multiplication operator $f \mapsto \psi f$ where $\psi(z) = 1 + z + \cdots + z^{n-1}$. Therefore similar to [14, Section 3], the bounded semigroup $(W_n)_{n \in \mathbb{N}}$ leaves $\overline{\text{span}}_{H^p} \{h_k \mid k \geq 2\}$ invariant and it contains the constant 1 by Theorem 4.5 for $0 < p < 1$. But 1 is a cyclic vector for $(W_n)_{n \in \mathbb{N}}$ since $\text{span}(W_n 1)_{n \in \mathbb{N}}$ contains all analytic polynomials and is hence dense in H^p for $0 < p < 1$. \square

4.2 Zero free half-planes via H^p spaces

Our main goal here is to show that condition **(C4)** holds for H^p with $1 \leq p \leq 2$ and therefore that proving **(C3)** immediately provides nontrivial zero free half-planes for ζ . Recall that for each $s \in \mathbb{C}$, the linear functionals $\Lambda^{(s)} : X \rightarrow \mathbb{C}$ are formally defined by

$$\Lambda^{(s)}(z^n) = f_n(s) = -\frac{1}{s}((n+1)^{1-s} - n^{1-s})$$

where $|f_n(s)| \asymp n^{-\Re(s)}$ for $n \in \mathbb{N}$ by Proposition 2.2.

Proposition 4.7 If $1 \leq p \leq 2$, then $\Lambda^{(s)}$ is bounded on H^p for $\Re(s) > \frac{1}{p}$ and is bounded on H^1 for $\Re(s) \geq 1$.

To prove this we need the following results from Duren's book [11]:

(a) ([11, Theorem 3.15]) If $f(z) = \sum a_n z^n \in H^1$, then

$$\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} \leq \pi \|f\|_1.$$

(b) ([11, Theorem 6.3]) If (a_n) is a sequence such that

$$\sum_{n=0}^{\infty} n^{q-2} |a_n|^q < \infty$$

for some $2 \leq q < \infty$, then $f(z) = \sum a_n z^n \in H^q$.

(c) ([11, Theorem 7.3]) For $1 < p < \infty$, each $\phi \in (H^p)^*$ is representable in the form

$$\phi(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} d\theta \quad \text{for } f \in H^p$$

by a unique $g \in H^q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $f(z) = \sum a_n z^n \in H^p$ and let $p \in (1, 2]$. Define the functions

$$k_s(z) = \sum_{n=0}^{\infty} f_n(s) z^n \quad \text{for each } s \in \mathbb{C}.$$

Then $q = \frac{p}{p-1} \geq 2$ and we get

$$\sum_{n=1}^{\infty} n^{q-2} |f_n(s)|^q \leq C \sum_{n=1}^{\infty} n^{q-2-q\Re(s)} < \infty$$

if $q - 2 - q\Re(s) < -1$ or equivalently if $\Re(s) > \frac{q-1}{q} = \frac{1}{p}$. So (b) implies that $k_s \in H^q$ for $\Re(s) > \frac{1}{p}$. Therefore the functional ϕ_s defined by

$$\phi_s(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{k_s(e^{i\theta})} d\theta$$

is bounded on H^p for $\Re(s) > \frac{1}{p}$ by **(c)**. Now since $k_s \in H^q \subset H^2$ for $q \geq 2$, we see that

$$\phi_s(z^n) = \langle z^n, k_{\bar{s}} \rangle = \overline{f_n(\bar{s})} = f_n(s) = \Lambda^{(s)}(z^n)$$

for all $n \in \mathbb{N}$ and hence $\Lambda^{(s)} = \phi_s$ and $\Lambda^{(s)} \in (H^p)^*$ for $\Re(s) > \frac{1}{p}$ when $p \in (1, 2]$. For $p = 1$ and $\Re(s) \geq 1$, **(a)** gives

$$|\Lambda^{(s)} f| \leq \sum_{n=0}^{\infty} |a_n| |f_n(s)| \leq \frac{|a_0|}{|s|} + \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\Re(s)}} \leq \sum_{n=0}^{\infty} \frac{2|a_n|}{n+1} \leq 2\pi \|f\|_1$$

and hence $\Lambda^{(s)}$ is bounded on H^1 for $\Re(s) \geq 1$. \square

Therefore proving condition **(C3)** for H^p with $1 < p \leq 2$ will lead to nontrivial zero free half-planes for ζ .

Theorem 4.8 For any $1 < p \leq 2$, we have

$$1 \in \overline{\text{span}}_{H^p} \{h_k \mid k \geq 2\} \implies \zeta(s) \neq 0 \text{ for } \Re(s) > 1/p.$$

Note that the case $p = 1$ gives the known zero-free half-plane $\Re(s) \geq 1$ and the hypothesis above is equivalent to the density of $\text{span}(h_k)_{k \geq 2}$ in H^p by Theorem 4.6. It is unclear whether the converse of Theorem 4.8 holds.

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