BOUNDARY POINTS, MINIMAL L 2 INTEGRALS AND CONCAVITY PROPERTY V—VECTOR BUNDLES

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Abstract. In this article, for singular hermitian metrics on holomorphic vector bundles, we consider minimal L^2 integrals on sublevel sets of plurisubharmonic functions on weakly pseudoconvex Kähler manifolds related to modules at boundary points of the sublevel sets, and establish a concavity property of the minimal L^2 integrals. As applications, we present a necessary condition for the concavity degenerating to linearity, a strong openness property of the modules and a twisted version, an effectiveness result of the strong openness property of the modules, and an optimal support function related to the modules.

1. INTRODUCTION

The strong openness property of multiplier ideal sheaves [\[36\]](#page-72-0) (Demailly's strong openness conjecture [\[14\]](#page-72-1): $\mathcal{I}(\varphi) = \mathcal{I}_+(\varphi) := \bigcup_{\epsilon > 0} \mathcal{I}((1+\epsilon)\varphi)$ is an important feature of multiplier ideal sheaves, which was called "opened the door to new types of approximation techniques" (see e.g. [\[36,](#page-72-0) 47, [44,](#page-73-0) [4,](#page-71-0) [5,](#page-71-1) [20,](#page-72-2) [8,](#page-71-2) [54,](#page-73-1) [39,](#page-72-3) [55,](#page-73-2) [56,](#page-73-3) [21,](#page-72-4) [45,](#page-73-4) 9), where the multiplier ideal sheaf $\mathcal{I}(\varphi)$ was defined as the sheaf of germs of holomorphic functions f such that $|f|^2 e^{-\varphi}$ is locally integrable (see e.g. [\[53,](#page-73-5) [48,](#page-73-6) [49,](#page-73-7) [16,](#page-72-5) [17,](#page-72-6) [14,](#page-72-1) [18,](#page-72-7) [46,](#page-73-8) [51,](#page-73-9) [52,](#page-73-10) [15,](#page-72-8) [40\]](#page-72-9)), and φ is a plurisubharmonic function on a complex manifold M (see [\[13\]](#page-72-10)).

Guan-Zhou [\[36\]](#page-72-0) proved the strong openness property (the 2-dimensional case was proved by Jonsson-Mustată $[42]$. After that, using the strong openness property, Guan-Zhou [\[37\]](#page-72-11) proved a conjecture about volumes growth of the sublevel sets of quasi-plurisubharmonic functions which was posed by Jonsson-Mustată (Conjecture J-M for short, see [\[42\]](#page-73-11)).

Considering the minimal L^2 integrals on sublevel sets of a plurisubharmonic function with respect to a module at a boundary point of the sublevel sets, Bao-Guan-Yuan [\[2\]](#page-71-4) (see also [\[29\]](#page-72-12)) established a concavity property of the minimal L^2 integrals, which deduces an approach to Conjecture J-M independent of the strong openness property.

In this article, for singular hermitian metrics on holomorphic vector bundles, we consider minimal L^2 integrals on sublevel sets of plurisubharmonic functions on weakly pseudoconvex Kähler manifolds related to modules at boundary points of the sublevel sets, and obtain a concavity property of minimal L^2 integrals. As applications, we present a necessary condition for the concavity degenerating to linearity, a strong openness property of the modules and a twisted version, an

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effectiveness result of the strong openness property of the modules, and an optimal support function related to the modules.

1.1. Singular hermitian metrics on vector bundles. Let M be an n -dimensional complex manifold. Let E be a rank r holomorphic vector bundle over M and E the conjugate of E. Let h be a section of the vector bundle $E^* \otimes \overline{E}^*$ with measurable coefficients, such that h is an almost everywhere positive definite hermitian form on E ; we call such an h a measurable metric on E .

We would like to use the following definition for singular hermitian metrics on vector bundles in this article which is a modified version of the definition in [\[7\]](#page-71-5).

Definition 1.1. Let M, E and h be as above and $\Sigma \subset M$ be a closed set of measure zero. Let ${M_j}_{j=1}^{+\infty}$ be a sequence of relatively compact subsets of M such that $M_1 \in M_2 \in \dots \in M_j \in M_{j+1} \in \dots$ and $\cup_{j=1}^{+\infty} M_j = M$. Assume that for each M_j , there exists a sequence of hermitian metrics $\{h_{j,s}\}_{s=1}^{+\infty}$ on M_j of class C^2 such that

 $\lim_{s \to +\infty} h_{j,s} = h$ point-wisely on $M_j \backslash \Sigma$.

We call the collection of data $(M, E, \Sigma, M_j, h, h_{j,s})$ a singular hermitian metric $(s.h.m.$ for short) on E .

Remark 1.2 (see [\[7\]](#page-71-5)). Let M, E, Σ , h be as in Definition [1.1.](#page-1-0) Assume that there exists a sequence of hermitian metrics \tilde{h}_s of class C^2 such that $\lim_{s \to +\infty} \tilde{h}_s = h$ in the C^2 -topology on $M \backslash \Sigma$.

The authors of [\[7\]](#page-71-5) called such a collection of data $(X, E, \Sigma, h, \tilde{h}_s)$ a singular hermitian metric on E. They called $\Theta_h(E_{X\setminus\Sigma})$ the curvature of $(X, E, \Sigma, h, \tilde{h}_s)$ and denoted it by $\Theta_h(E)$. $\Theta_h(E)$ has continuous coefficients and values in $Herm_h(E)$ away from Σ ; they denoted the a.e.-defined associated hermitian form on $TX \otimes E$ by the same symbol $\Theta_h(E)$.

We use the following definition of singular version of Nakano positivity in this article. Let ω be a hermitian metric on M, θ be a hermitian form on TM with continuous coefficients and $(M, E, \Sigma, M_j, h, h_{j,s})$ be a s.h.m in the sense of Definition [1.1.](#page-1-0)

Definition 1.3. Let things be as above. We write:

$$
\Theta_h(E) \geq^s_{Nak} \theta \otimes Id_E
$$

if the following requirements are met.

For each M_j , there exist a sequence of continuous functions $\lambda_{j,s}$ on $\overline{M_j}$ and a continuous function λ_j on $\overline{M_j}$ subject to the following requirements:

(1.2.1) for any $x \in \Omega$: $|e_x|_{h_{j,s}} \leq |e_x|_{h_{j,s+1}}$, for any $s \in \mathbb{N}$ and any $e_x \in E_x$; (1.2.2) $\Theta_{h_{j,s}}(E) \geq_{Nak} \theta - \lambda_{j,s} \omega \otimes Id_E$ on M_j ; $(1.2.3)$ $\lambda_{j,s} \rightarrow 0$ a.e. on M_j ; $(1.2.4)$ $0 \leq \lambda_{j,s} \leq \lambda_j$ on M_j , for any s.

We would also like to recall the following notation of singular version of Nakano positivity in [\[7\]](#page-71-5). Let ω be a hermitian metric on $M, \tilde{\theta}$ be a hermitian form on TM with continuous coefficients and $(X, E, \Sigma, h, \tilde{h}_s)$ be a s.h.m in the sense of Remark [1.2.](#page-1-1)

Remark 1.4 (see [\[7\]](#page-71-5)). Let things be as above. In [\[7\]](#page-71-5), the authors wrote

$$
\Theta_h(E) \geq^s_{Nak} \tilde{\theta} \otimes Id_E
$$

if the following requirements are met.

There exist a sequence of hermitian forms $\tilde{\theta}_s$ on $TM \otimes E$ with continuous coefficients, a sequence of continuous functions $\tilde{\lambda}_s$ on M and a continuous function $\tilde{\lambda}$ on M subject to the following requirements:

(1.2.1) for any $x \in X: \|e_x\|_{\tilde{h}_s} \leq |e_x|_{\tilde{h}_{s+1}}$, for any $s \in \mathbb{N}$ and any $e_x \in E_x$;

(1.2.2) $\tilde{\theta}_s \geq_{Nak} \tilde{\theta} \otimes Id_E;$ $(1.2.3) \Theta_{\tilde{h}_s}(E) \geq_{Nak} \tilde{\theta}_s - \tilde{\lambda}_s \omega \otimes Id_E;$ $(1.2.4) \tilde{\theta}_s \rightarrow \Theta_h(E)$ a.e on M; $(1.2.5)\ \tilde{\lambda}_s \rightarrow 0\ a.e\ on\ M;$ $(1.2.6)$ $0 \le \tilde{\lambda}_s \le \tilde{\lambda}$, for any s.

Remark 1.5. Let M be a weakly pseudoconvex Kähler manifold. Let φ be a plurisubharmonic function on M. Using regularization of quasi-plurisubharmonic function (see Theorem [9.11\)](#page-54-0), we know that $h := e^{-\varphi}$ is a singular metric on $E := M \times \mathbb{C}$ in the sense of Definition [1.1](#page-1-0) and h satisfies $\Theta_h(E) \geq_{Nak}^s 0$ in the sense of Definition [1.3.](#page-1-2) We will prove Remark [1.5](#page-2-0) in appendix (see Remark [9.13\)](#page-54-1).

We recall the following definitions which can be referred to [\[7\]](#page-71-5).

Definition 1.6 (see [\[7\]](#page-71-5)). Let h be a measurable metric on E. Let $\mathcal{I}(h)$ be the analytic sheaf of germs of holomorphic functions on M defined as follows: $\mathcal{I}(h)_x := \{f_x \in \mathcal{O}_{X,x} : |f_x e_x|_h^2 \text{ is integrable in some neighborhood of } x, \ \forall e_x \in$ $\mathcal{O}(E)_x$.

Analogously, we define an analytic sheaf $\mathcal{E}(h)$ by setting:

 $\mathcal{E}(h)_x := \{e_x \in \mathcal{O}(E)_x : |e_x|_h^2 \text{ is integrable in some neighborhood of } x\}.$

1.2. Main result: minimal L^2 integrals and concavity property. Let M be a complex manifold. Let X and Z be closed subsets of M . We call that a triple (M, X, Z) satisfies condition (A) , if the following two statements hold:

I. X is a closed subset of M and X is locally negligible with respect to L^2 holomorphic functions; i.e., for any local coordinated neighborhood $U \subset M$ and for any L^2 holomorphic function f on $U\backslash X$, there exists an L^2 holomorphic function \tilde{f} on U such that $\tilde{f}|_{U\setminus X} = f$ with the same L^2 norm;

II. Z is an analytic subset of M and $M\setminus (X\cup Z)$ is a weakly pseudoconvex Kähler manifold.

Let M be an n-dimensional complex manifold. Assume that (M, X, Z) satisfies condition (A). Let K_M be the canonical line bundle on M. Let dV_M be a continuous volume form on M. Let $F \neq 0$ be a holomorphic function on M. Let ψ be a plurisubharmonic function on M . Let E be a holomorphic vector bundle on M with rank r. Let h be a smooth metric on E. Let h be a measurable metric on E. Denote $\tilde{h} := he^{-\psi}$. Let $(M, E, \Sigma, M_j, \tilde{h}, \tilde{h}_{j,s})$ be a singular metric on E. Assume that $\Theta_{\tilde{h}}(E) \geq_{Nak}^s 0$.

Let (V, z) be a local coordinate near a point p of M and $E|_V$ is trivial. Let $g \in H^0(V, \mathcal{O}(K_M \otimes E))$ and $g = \hat{g} \otimes e$ locally, where \hat{g} is a holomorphic $(n, 0)$ form on V and e is a local section of E on V. We define $|g|^2_{h_0}|_V = \sqrt{-1}^{n^2} g \wedge \bar{g} \langle e, e \rangle_{h_0}$,

where h_0 is any (smooth or singular) metric on E. Note that $|g|^2_{h_0}|_V$ is invariant under the coordinate change and $|g|^2_{h_0}$ is a globally defined (n, n) form on M.

Let $T \in [-\infty, +\infty)$. Denote that

$$
\Psi := \min\{\psi - 2\log|F|, -T\}.
$$

For any $z \in M$ satisfying $F(z) = 0$, we set $\Psi(z) = -T$. Note that for any $t > T$, the holomorphic function F has no zero points on the set $\{\Psi < -t\}$. Hence $\Psi = \psi - 2 \log |F| = \psi + 2 \log \left| \frac{1}{F} \right|$ is a plurisubharmonic function on $\{\Psi < -t\}.$

Definition 1.7. We call that a positive measurable function c (so-called "gain") on $(T, +\infty)$ is in class $\tilde{P}_{T,M,\Psi,h}$ if the following two statements hold:

(1) $c(t)e^{-t}$ is decreasing with respect to t;

(2) For any $t_0 > T$, there exists a closed subset E_0 of M such that $E_0 \subset Z \cap$ $\{\Psi(z) = -\infty\}$ and for any compact subset $K \subset M \setminus E_0$, $|e_x|_h^2 c(-\psi) \geq C_K |e_x|_h^2$ for any $x \in K \cap {\Psi < -t_0}$, where $C_K > 0$ is a constant and $e_x \in E_x$.

Let z_0 be a point in M. Denote that $\tilde{J}(E,\Psi)_{z_0} := \{f \in H^0(\{\Psi < -t\} \cap$ $V, \mathcal{O}(E)$: $t \in \mathbb{R}$ and V is a neighborhood of z_0 . We define an equivalence relation \sim on $\tilde{J}(E,\Psi)_{z_0}$ as follows: for any $f,g \in \tilde{J}(\Psi)_{z_0}$, we call $f \sim g$ if $f = g$ holds on $\{\Psi < -t\} \cap V$ for some $t \gg T$ and open neighborhood $V \ni z_0$. Denote $\tilde{J}(E, \Psi)_{z_0}/\sim$ by $J(E, \Psi)_{z_0}$, and denote the equivalence class including $f \in \tilde{J}(E, \Psi)_{z_0}$ by f_{z_0} .

If $z_0 \in \bigcap_{t>T} {\{\Psi < -t\}}$, then $J(E, \Psi)_{z_0} = \mathcal{O}(E)_{z_0}$ (the stalk of the sheaf $\mathcal{O}(E)$ at (z_0) , and f_{z_0} is the germ (f, z_0) of holomorphic section f of E. If $z_0 \notin \bigcap_{t>T} {\{\Psi < -t\}}$, then $J(E, \Psi)_{z_0}$ is trivial.

Let $f_{z_0}, g_{z_0} \in J(E, \Psi)_{z_0}$ and $(q, z_0) \in \mathcal{O}_{M, z_0}$. We define $f_{z_0} + g_{z_0} := (f+g)_{z_0}$ and $(q, z_0) \cdot f_{z_0} := (qf)_{z_0}$. Note that $(f+g)_{z_0}$ and $(qf)_{z_0}$ ($\in J(E, \Psi)_{z_0}$) are independent of the choices of the representatives of f, g and q. Hence $J(E, \Psi)_{z_0}$ is an \mathcal{O}_{M,z_0} module.

Let dV_M be a continuous volume form on M. Recall that h is a measurable metric on E. For $f_{z_0} \in J(E, \Psi)_{z_0}$ and $a \geq 0$, we call $f_{z_0} \in I(h, a\Psi)_{z_0}$ if there exist $t \gg T$ and a neighborhood V of z_0 , such that $\int_{\{\Psi<-t\}\cap V} |f|^2_h e^{-a\Psi} dV_M < +\infty$. Note that $I(h, a\Psi)_{z_0}$ is an \mathcal{O}_{M, z_0} -submodule of $J(E, \Psi)_{z_0}$. If $z_0 \in \bigcap_{t > T} {\Psi < -t}$, then $I_{z_0} = \mathcal{O}(E)_{z_0}$, where $I_{z_0} := I(\hat{h}_1, 0\Psi)_{z_0}$ and \hat{h}_1 is a smooth metric on E.

Let Z_0 be a subset of $\bigcap_{t>T} \overline{\{\Psi < -t\}}$. Let f be an E-valued holomorphic $(n, 0)$ form on $\{\Psi < -t_0\} \cap V$, where $V \supset Z_0$ is an open subset of M and $t_0 \geq T$ is a real number. Let J_{z_0} be an \mathcal{O}_{M,z_0} -submodule of $\overline{J}(E,\Psi)_{z_0}$ such that $I(h,\Psi)_{z_0} \subset J_{z_0}$, where $z_0 \in Z_0$. Denote $J := \cup_{z_0 \in Z_0} J_{z_0}$. Denote the **minimal** L^2 **integral** related to J

$$
\inf \left\{ \int_{\{\Psi<-t\}} |\tilde{f}|_{h}^{2} c(-\Psi) : \tilde{f} \in H^{0}(\{\Psi<-t\}, \mathcal{O}(K_{M} \otimes E)) \right\}
$$
\n
$$
\& (\tilde{f}-f)_{z_{0}} \in \mathcal{O}(K_{M})_{z_{0}} \otimes J_{z_{0}}, \text{for any } z_{0} \in Z_{0} \right\}
$$
\n(1.1)

by $G(t; c, \Psi, h, J, f)$, where $t \in [T, +\infty)$, c is a nonnegative function on $(T, +\infty)$. Without misunderstanding, we denote $G(t; c, \Psi, h, J, f)$ by $G(t)$ for simplicity. For various $c(t)$, we denote $G(t; c, \Psi, h, J, f)$ by $G(t; c)$ respectively for simplicity.

In this article, we obtain the following concavity property of $G(t)$.

Theorem 1.8. Let $c \in \tilde{P}_{T,M,\Psi,h}$. If there exists $t \in [T,+\infty)$ satisfying that $G(t)$ $+\infty$, then $G(h^{-1}(r))$ is concave with respect to $r \in \left(\int_{T_1}^T c(t)e^{-t}dt, \int_{T_1}^{+\infty} c(t)e^{-t}dt\right)$, $\lim_{t \to T+0} G(t) = G(T)$ and $\lim_{t \to +\infty} G(t) = 0$, where $h(t) = \int_{T_1}^t c(t_1) e^{-t_1} dt_1$ and $T_1 \in$ $(T, +\infty)$.

Remark 1.9. Let $c \in \tilde{P}_{T,M,\Psi,h}$. If $\int_{T_1}^{+\infty} c(t)e^{-t}dt = +\infty$ and $f_{z_0} \notin \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$ for some $z_0 \in Z_0$, then $G(t) = +\infty$ for any $t \geq T$. Thus, when there exists $t \in [T, +\infty)$ satisfying that $G(t) \in (0, +\infty)$, we have $\int_{T_1}^{+\infty} c(t)e^{-t}dt < +\infty$ and $G(\hat{h}^{-1}(r))$ is concave with respect to $r \in (0, \int_T^{+\infty} c(t)e^{-t}dt)$, where $\hat{h}(t) =$ $\int_{t}^{+\infty} c(l)e^{-l}dl$.

For any $t \geq T$, denote

$$
\mathcal{H}^{2}(t;c,f) := \left\{ \tilde{f} : \int_{\{\Psi<-t\}} |\tilde{f}|_{h}^{2} c(-\Psi) < +\infty, \ \tilde{f} \in H^{0}(\{\Psi<-t\}, \mathcal{O}(K_{M} \otimes E)) \right\}
$$

$$
\& (\tilde{f}-f)_{z_{0}} \in \mathcal{O}(K_{M})_{z_{0}} \otimes J_{z_{0}}, \text{for any } z_{0} \in Z_{0} \right\},
$$

where f is an E-valued holomorphic $(n, 0)$ form on $\{\Psi < -t_0\} \cap V$ for some $V \supset Z_0$ is an open subset of M and some $t_0 \geq T$ and $c(t)$ is a positive measurable function on $(T, +\infty)$.

As a corollary of Theorem [1.8,](#page-4-0) we give a necessary condition for the concavity property degenerating to linearity.

Corollary 1.10. Let $c \in \tilde{P}_{T,M,\Psi,h}$. Assume that $G(t) \in (0,+\infty)$ for some $t \geq$ T, and $G(\hat{h}^{-1}(r))$ is linear with respect to $r \in [0, \int_T^{+\infty} c(s)e^{-s}ds)$, where $\hat{h}(t) =$ $\int_{t}^{+\infty} c(l)e^{-l}dl$.

Then there exists a unique E-valued holomorphic $(n,0)$ form \tilde{F} on $\{\Psi < -T\}$ $such that \ (\tilde{F}-f)_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0} \ holds \ for \ any \ z_0 \in Z_0, \ and \ G(t) = \int_{\{\Psi<-t\}} |\tilde{F}|_h^2 c(-\Psi)$ holds for any $t \geq T$.

Furthermore

$$
\int_{\{-t_1 \le \Psi < -t_2\}} |\tilde{F}|_h^2 a(-\Psi) = \frac{G(T_1; c)}{\int_{T_1}^{+\infty} c(t)e^{-t}dt} \int_{t_2}^{t_1} a(t)e^{-t}dt \tag{1.2}
$$

holds for any nonnegative measurable function a on $(T, +\infty)$, where $T \le t_2 < t_1 \le$ $+\infty$ and $T_1 \in (T, +\infty)$.

Remark 1.11. If $\mathcal{H}^2(t_0; \tilde{c}, f) \subset \mathcal{H}^2(t_0; c, f)$ for some $t_0 \geq T$, we have

$$
G(t_0; \tilde{c}) = \int_{\{\Psi < -t_0\}} |\tilde{F}|_h^2 \tilde{c}(-\Psi) = \frac{G(T_1; c)}{\int_{T_1}^{+\infty} c(t)e^{-t}dt} \int_{t_0}^{+\infty} \tilde{c}(s)e^{-s}ds,\tag{1.3}
$$

where \tilde{c} is a nonnegative measurable function on $(T, +\infty)$ and $T_1 \in (T, +\infty)$. Thus, if $\mathcal{H}^2(t;\tilde{c})$ ⊂ $\mathcal{H}^2(t;c)$ for any $t > T$, then $G(\hat{h}^{-1}(r);\tilde{c})$ is linear with respect to $r \in [0, \int_T^{+\infty} c(s)e^{-s}ds).$

1.3. Applications. In this section, we give some applications of Theorem [1.8.](#page-4-0)

1.3.1. Strong openness property of $I(h, a\Psi)_{z_0}$. In this section, we give an estimate of $|f|^2_h$ on sublevel sets of Ψ , which implies the strong openness property of $I(h, \Psi)_{z_0}$.

Let M be an n-dimensional weakly pseudoconvex Kähler manifold, and let dV_M be a continuous volume form on M . Let K_M be the canonical line bundle on M . Let $F \neq 0$ be a holomorphic function on M. Let ψ be a plurisubharmonic function on M. Let E be a holomorphic vector bundle on M with rank r. We call a measurable metric h on E has a positive locally lower bound if for any compact subset K of M, there exists a constant $C_K > 0$ such that $h \geq C_K h_1$ on K, where h_1 is a smooth metric on E . Let h be a measurable metric on E satisfying that h has a positive locally lower bound.

Denote that

$$
\Psi := \min\{\psi - 2\log|F|, 0\}.
$$

Let $z_0 \in M$. Recall that $\hat{f}_{z_0} \in I(h, a\Psi)_{z_0}$ if and only if there exist $t \gg 0$ and a neighborhood V of z_0 such that $\int_{\{\Psi<-t\}\cap V} |\hat{f}|_h^2 e^{-a\Psi} dV_M < +\infty$, where $a \geq 0$. Denote that

$$
I_+(h,a\Psi)z_0 := \cup_{s>a} I(h,s\Psi)_{z_0}.
$$

Let f be an E-valued holomorphic $(n,0)$ form on $\{\Psi < -t_0\}$ such that $f_{z_0} \in$ $\mathcal{O}(K_M)_{z_0} \otimes I(h,0\Psi))_{z_0}$. Denote that

$$
a_{z_0}^f(\Psi; h) := \sup\{a \ge 0 : f_{z_0} \in (\mathcal{O}(K_M) \otimes I(h, 2a\Psi))_{z_0}\}.
$$

Especially, $a_{z_0}^f(\Psi; h)$ is the jumping number $c_{z_0}^f(\psi)$ (see [\[43\]](#page-73-12)), when $F \equiv 1, \psi(z_0) =$ $-\infty$, *E* is the trivial line bundle and $h \equiv 1$.

Theorem 1.12. Assume that $a_{z_0}^f(\Psi; h) < +\infty$ and $\Theta_{\tilde{h}}(E) \geq_{Nak}^s 0$, where $\tilde{h} :=$ $he^{-2a_{z_0}^f(\Psi;h)\psi}$. Then we have $a_{z_0}^f(\Psi;h) > 0$ and

$$
\frac{1}{r^2} \int_{\{a_{z_0}^f(\Psi;h) \Psi < \log r\}} |f|^2_h \geq G(0;c\equiv 1, \Psi, h, I_+(h, 2a^f_{z_0}(\Psi;h) \Psi)_{z_0}, f)>0
$$

holds for any $r \in (0, e^{-a_{z_0}^f(\Psi;h)t_0}]$, where the definition of $G(0; c \equiv 1, \Psi, h, I_+(h, 2a_{z_0}^f(\Psi;h)\Psi)_{z_0}, f)$ can be found in Section [1.2.](#page-2-1)

Theorem [1.12](#page-5-0) implies the following strong openness property of $I(h, a\Psi)_{z_0}$.

Corollary 1.13. $I(h, a\Psi)_{z_0} = I_+(h, a\Psi)_{z_0}$ holds for any $a \geq 0$ satisfying $\Theta_{he^{-2a\psi}} \geq_{Nak}^s$ 0.

When E is the trivial line bundle and $h = e^{-\varphi}$, where φ is a plurisubharmonic function on M , Theorem [1.12](#page-5-0) and Corollary [1.13](#page-5-1) can be referred to [\[29\]](#page-72-12).

Remark 1.14. Let $F \equiv 1$ and $\psi(z_0) = -\infty$. Note that $z_0 \in \bigcap_{t \geq T} {\Psi < -t}$ and $I(h, a\Psi)_{z_0} = \mathcal{E}(he^{-a\psi})_{z_0}$, then Corollary [1.13](#page-5-1) is a vector bundle version of the strong openness property of multiplier ideal sheaves [\[36\]](#page-72-0).

1.3.2. Effectiveness of the strong openness property of $I(h, \Psi)_{z_0}$. In this section, we give an effectiveness result of the strong openness property of $I(h, \Psi)_{z_0}$ (Corollary [1.13\)](#page-5-1). We follow the notations and assumptions in Section [1.3.1.](#page-5-2) Let f be an E-valued holomorphic $(n, 0)$ form on $\{\Psi < 0\}$, and denote that

$$
\frac{1}{K_{\Psi,f,h,a}(z_0)} := \inf \left\{ \int_{\{\Psi < 0\}} |\tilde{f}|_h^2 e^{-(1-a)\Psi} : \tilde{f} \in H^0(\{\Psi < 0\}, \mathcal{O}(K_M \otimes E)) \right\}
$$
\n
$$
\& (\tilde{f} - f)_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes I_+(h, 2a_{z_0}^f(\Psi; h)\Psi)_{z_0} \right\},
$$

where $a \in (0, +\infty)$.

We present the following effectiveness result of the strong openness property of $I(h,\Psi)_{z_0}.$

Theorem 1.15. Assume that $\Theta_{\tilde{h}}(E) \geq_{Nak}^{s} 0$, where $\tilde{h} := he^{-2a_{z_0}^f(\Psi,h)\psi}$. Let C_1 and C_2 be two positive constants. If there exists $a > 0$, such that

(1) $\int_{\{\Psi < 0\}} |f|^2 h e^{-\Psi} \leq C_1;$ (2) $\frac{1}{K_{\Psi, f, h, a}(z_0)} \geq C_2.$ Then for any $q > 1$ satisfying

$$
\theta_a(q) > \frac{C_1}{C_2},
$$

we have $f_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes I(h,q\Psi)_{z_0}$, where $\theta_a(q) = \frac{q+a-1}{q-1}$.

1.3.3. A twisted version of the strong openness property of $I(h, a\Psi)_{o}$. Let $D \subseteq \mathbb{C}^n$ be a pseudoconvex domain containing the origin o , and let ψ be a plurisubharmonic function on D. Let $F \neq 0$ be a holomorphic function on D. Denote that

$$
\Psi := \min\{\psi - 2\log|F|, 0\}.
$$

For any $z \in M$ satisfying $F(z) = 0$, we set $\Psi(z) = 0$. Let E be a holomorphic vector bundle on D with rank r , and let h be a measurable metric on E satisfying that h has a positive locally lower bound.

It is clear that the following two statements are equivalent:

(1) The strong openness property of $I(h, a\Psi)$ _o (Corollary [1.13\)](#page-5-1): $I(h, a\Psi)$ _o = $I_+(h, a\Psi)_o$ for any $a \geq 0$ satisfying $\Theta_{he^{-a\psi}} \geq_{Nak}^s 0;$

(2) $f_o \notin I(h, 2a_o^f(\Psi; h)\Psi)_o$ for any $f_o \in I(h, 0\Psi)_o$ satisfying $a_o^f(\Psi; h) < +\infty$ and $\Theta_{he^{-a_o^f(\Psi;h)\psi}} \geq^s_{Nak} 0.$

We present a twisted version of the strong openness property of $I(h, a\Psi)_{o}$.

Theorem 1.16. Let a(t) be a positive measurable function on $(-\infty, +\infty)$. If one of the following conditions holds:

(1) $a(t)$ is decreasing near $+\infty$;

(2) $a(t)e^t$ is increasing near $+\infty$,

then the following two statements are equivalent:

(A) $a(t)$ is not integrable near $+\infty$;

(B) for any Ψ , h and $f_o \in I(h, 0\Psi)_o$ satisfying $a_o^f(\Psi; h) < +\infty$ and $\Theta_{he^{-a_o^f(\Psi; h)\psi}} \geq_{Nak}^s$ 0, we have

$$
|f|^2_h e^{-2a_o^f(\Psi;h)\Psi}a(-2a_o^f(\Psi;h)\Psi)\not\in L^1(U\cap\{\Psi<-t\})
$$

for any neighborhood U of σ and any $t > 0$.

When E is the trivial line bundle, Theorem [1.16](#page-6-0) can be referred to [\[29\]](#page-72-12). When $F \equiv 1, \psi(o) = -\infty$ and E is the trivial line bundle, Theorem [1.16](#page-6-0) is a twisted version of the strong openness property of multiplier ideal sheaves (some related results can be referred to [\[38\]](#page-72-13), [\[6\]](#page-71-6) and [\[34\]](#page-72-14)).

1.3.4. An optimal support function related to $I(h, \Psi)$. Let M be an n-dimensional complex manifold. Let X and Z be closed subsets of M such that (M, X, Z) satisfies condition (A). Let K_M be the canonical line bundle on M. Let $F \neq 0$ be a holomorphic function on M. Let ψ be a plurisubharmonic function on M. Let E be a holomorphic vector bundle on M with rank r , and let h be a measurable metric on E satisfying that $\Theta_{he^{-\psi}} \geq_{Nak}^s 0$ and h has a positive locally lower bound.

Denote that

$$
\Psi:=\min\{\psi-2\log|F|,0\}
$$

and $M_t := \{z \in M : -t \leq \Psi(z) < 0\}$. Let Z_0 be a subset of M, and let f be an E-valued holomorphic $(n, 0)$ form on $\{\Psi < 0\}$. Denote

$$
\inf \left\{ \int_{M_t} |\tilde{f}|_h^2 : f \in H^0(\{\Psi < 0\}, \mathcal{O}(K_M \otimes E)) \right\}
$$
\n
$$
\& (\tilde{f} - f)_{z_0} \in \mathcal{O}(K_M) \otimes I(h, \Psi)_{z_0} \text{ for any } z_0 \in Z_0 \right\}
$$

by $C_{\Psi,f,h,t}(Z_0)$ for any $t \geq 0$. When $C_{\Psi,f,h,t}(Z_0) = 0$ or $+\infty$, we set $\frac{\int_{M_t} |f|^2_{h} e^{-\Psi}}{C_{\Psi,f,h,t}(Z_0)}$ $+\infty$.

We obtain the following optimal support function of $\frac{\int_{M_t} |f|^2_h e^{-\Psi}}{\int_{M_t} |f|^2_h e^{-\Psi}}$ $\frac{J_{M_t}+J_{H_t}}{C_{\Psi,f,h,t}(Z_0)}$.

Proposition 1.17. Assume that $\int_{\{\Psi<-l\}} |f|^2_h < +\infty$ holds for any $l > 0$. Then the inequality

$$
\frac{\int_{M_t} |f|_h^2 e^{-\Psi}}{C_{\Psi, f, h, t}(Z_0)} \ge \frac{t}{1 - e^{-t}}\n\tag{1.4}
$$

holds for any $t \geq 0$, where $\frac{t}{1-e^{-t}}$ is the optimal support function.

When E is the trivial line bundle and $h \equiv 1$, Proposition [1.17](#page-7-0) can be referred to [\[29\]](#page-72-12).

Take $M = \Delta \subset \mathbb{C}$, $Z_0 = o$ the origin of \mathbb{C} , $F \equiv 1$ and $\psi = 2 \log |z|$. Let E is the trivial line bundle, $h \equiv 1$ and $f \equiv dz$. It is clear that $\int_M |f|^2 \leq +\infty$. By direct calculations, we have $C_{\Psi,f,h,t}(Z_0) = 2\pi(1 - e^{-t})$ and $\int_{M_t}^{\infty} |f|^2_h e^{-\Psi} = 2t\pi$. Then $\frac{\int_{M_t} |f|^2 e^{-\Psi}}{C_{f,\Psi,t}(Z_0)} = \frac{t}{1-e^{-t}}$, which shows the optimality of the support function $\frac{t}{1-e^{-t}}$.

2. Preparations

2.1. L^2 methods. Let X be an n-dimensional weakly pseudoconvex Kähler manifolds. Let ψ be a plurisubharmonic function on M. Let F be a holomorphic function on X. We assume that F is not identically zero. Let E be a rank r holomorphic vector bundle over X. Let \hat{h} be a smooth metric on E. Let $(X, E, \Sigma, M_j, h, h_{j,s})$ be a singular hermitian metric on E. Assume that $\Theta_h(E) \geq_{Nak}^s 0$.

Let δ be a positive integer. Let T be a real number. Denote

$$
\tilde{M} := \max\{\psi + T, 2\log|F|\}
$$

and

$$
\Psi := \min\{\psi - 2\log|F|, -T\}.
$$

If $F(z) = 0$ for some $z \in M$, we set $\Psi(z) = -T$.

Let $c(t)$ be a positive measurable function on $[T, +\infty)$ such that $c(t)e^{-t}$ is decreasing with respect t . We have the following lemma.

Lemma 2.1. Let $B \in (0, +\infty)$ and $t_0 > T$ be arbitrarily given. Let f be an E-valued holomorphic $(n, 0)$ form on $\{\Psi < -t_0\}$ such that

$$
\int_{\{\Psi<-t_0\}\cap K} |f|^2_{\hat{h}} < +\infty,
$$
\n(2.1)

for any compact subset $K \subset X$, and

$$
\int_{M} \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \Psi < -t_0\}} |fF|_{h}^2 < +\infty. \tag{2.2}
$$

Then there exists an E-valued holomorphic $(n, 0)$ form \tilde{F} on X such that

$$
\int_{X} |\tilde{F} - (1 - b_{t_0, B}(\Psi)) f F^{1+\delta}|_{h}^{2} e^{v_{t_0, B}(\Psi) - \delta \tilde{M}} c(-v_{t_0, B}(\Psi))
$$
\n
$$
\leq \left(\frac{1}{\delta} c(T) e^{-T} + \int_{T}^{t_0 + B} c(s) e^{-s} ds \right) \int_{X} \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \Psi < -t_0\}} |f F|_{h}^{2},
$$
\n
$$
\int_{\mathcal{F}} |f F|^2 \, dv \leq \int_{\mathcal{F}} |f F|^2
$$

where $b_{t_0,B}(t) = \int_{-\infty}^{t}$ $\frac{1}{B} \mathbb{I}_{\{-t_0 - B < s < -t_0\}} ds, \ v_{t_0, B}(t) = \int_{-t_0}^t b_{t_0, B}(s) ds - t_0.$

We would like to recall the following notations in section [1.2.](#page-2-1) Let M be an n−dimensional complex manifold. Assume that (M, X, Z) satisfies condition (A) . Let K_M be the canonical line bundle on M. Let dV_M be a continuous volume form on M. Let $F \neq 0$ be a holomorphic function on M. Let ψ be a plurisubharmonic function on M. Let E be a holomorphic vector bundle on M with rank r. Let \hat{h} be a smooth metric on E. Let h be a measurable metric on E. Denote $h := he^{-\psi}$. Let $(M, E, \Sigma, M_j, \tilde{h}, \tilde{h}_{j,s})$ be a singular metric on E. Assume that $\Theta_{\tilde{h}}(E) \geq_{Nak}^{s} 0$. Let $c(t) \in \tilde{P}_{T,M,\Psi,h}$.

Let $T \in [-\infty, +\infty)$. Denote

$$
\Psi := \min\{\psi - 2\log|F|, -T\}.
$$

If $F(z) = 0$ for some $z \in M$, we set $\Psi(z) = -T$. Let $T_1 > T$ be a real number. Denote $\tilde{M} := \max\{\psi + T_1, 2 \log |F|\}.$ Denote

$$
\Psi_1 := \min{\psi - 2\log|F|, -T_1}.
$$

If $F(z) = 0$ for some $z \in M$, we set $\Psi_1(z) = -T_1$.

It follows from Lemma [2.1](#page-8-0) that we have the following lemma.

Lemma 2.2. Let (M, X, Z) satisfy condition (A) . Let $B \in (0, +\infty)$ and $t_0 > T_1 >$ T be arbitrarily given. Let f be an E-valued holomorphic $(n, 0)$ form on $\{\Psi < -t_0\}$ such that

$$
\int_{\{\Psi<-t_0\}} |f|^2_h c(-\Psi) < +\infty,
$$
\n(2.3)

Then there exists an E-valued holomorphic $(n,0)$ form \tilde{F} on M such that

$$
\int_{M} |\tilde{F} - (1 - b_{t_0, B}(\Psi_1)) f F^{1+\delta}|_{\tilde{h}}^2 e^{v_{t_0, B}(\Psi_1) - \delta \tilde{M}} c(-v_{t_0, B}(\Psi_1))
$$
\n
$$
\leq \left(\frac{1}{\delta} c(T_1) e^{-T_1} + \int_{T_1}^{t_0 + B} c(t) e^{-t} dt\right) \int_{M} \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \Psi_1 < -t_0\}} |f F|_{\tilde{h}}^2,
$$
\n
$$
\leq (c) e^{-t} \int_{T_1}^{t_0 + B} c(t) e^{-t} dt \int_{M} \int_{M} \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \Psi_1 < -t_0\}} |f F|_{\tilde{h}}^2,
$$
\n
$$
\leq (d) e^{-t} \int_{T_1}^{t_0 + B} c(t) e^{-t} dt \int_{M} \int_{M} \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \Psi_1 < -t_0\}} |f F|_{\tilde{h}}^2,
$$
\n
$$
\leq (e^{-t} \int_{T_1}^{T_1} \mathbb{I}_{\{-t_0 - B < \Psi_1 < -t_0\}} |f F|_{\tilde{h}}^2,
$$
\n
$$
\leq (f e^{-t} \int_{T_1}^{T_1} \mathbb{I}_{\{-t_0 - B < \Psi_1 < -t_0\}} |f F|_{\tilde{h}}^2).
$$
\n
$$
\leq (f e^{-t} \int_{T_1}^{T_1} \mathbb{I}_{\{-t_0 - B < \Psi_1 < -t_0\}} |f F|_{\tilde{h}}^2,
$$
\n
$$
\leq (f e^{-t} \int_{T_1}^{T_1} \mathbb{I}_{\{-t_0 - B < \Psi_1 < -t_0\}} |f F|_{\tilde{h}}^2).
$$

where $b_{t_0,B}(t) = \int_{-\infty}^{t}$ $\frac{1}{B} \mathbb{I}_{\{-t_0 - B < s < -t_0\}} ds$ and $v_{t_0, B}(t) = \int_{-t_0}^t b_{t_0, B}(s) ds - t_0$. *Proof.* We note that $\{\Psi < -t_0\} = \{\Psi_1 < -t_0\}$ and $\Psi_1 = \Psi = \psi - 2 \log |F|$ on $\{\Psi < -t_0\}$. It follows from inequality [\(2.3\)](#page-8-1), $\tilde{h} = he^{-\psi}$ and $c(t)e^{-t}$ is decreasing with respect to t that

$$
\int_M \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \Psi_1 < -t_0\}} |fF|_{\tilde{h}}^2 < +\infty.
$$

As $c(t) \in \tilde{P}_{T,M,\Psi,h}$, $\{\Psi \le -t_0\} = \{\Psi_1 < -t_0\}$ and $\Psi_1 = \Psi$ on $\{\Psi < -t_0\}$, there exists a closed subset $E_0 \subset Z \cap {\Psi = -\infty}$ such that for any compact subset $K \subset M \backslash E_0$, $|e|_h^2 c(-\Psi) \ge C_K |e|_h^2$ on $K \cap {\Psi_1 < -t_0}$, where $C_K > 0$ is a constant and e is any E-valued holomorphic $(n,0)$ form on $\{\Psi_1 < -t_0\}$. It follows from inequality [\(2.3\)](#page-8-1) that we have

$$
\int_{K\cap\{\Psi_1<-t_0\}}|f|^2_{\hat{h}}<+\infty.
$$

As (M, X, Z) satisfies condition $(A), M\setminus (Z\cup X)$ is a weakly pseudoconvex Kähler manifold. It follows from Lemma [2.1](#page-8-0) that there exists an E-valued holomorphic $(n, 0)$ form F_Z on $M\backslash (Z \cup X)$ such that

$$
\int_{M\backslash (Z\cup X)} |\tilde{F}_Z - (1 - b_{t_0,B}(\Psi_1)) f F^{1+\delta}|^2_{\tilde{h}} e^{v_{t_0,B}(\Psi_1) - \delta \tilde{M}} c(-v_{t_0,B}(\Psi_1))
$$
\n
$$
\leq \left(\frac{1}{\delta} c(T_1) e^{-T_1} + \int_{T_1}^{t_0+B} c(s) e^{-s} ds\right) \int_M \frac{1}{B} \mathbb{I}_{\{-t_0-B<\Psi_1<-t_0\}} |f F|^2_{\tilde{h}} < +\infty.
$$

For any $z \in ((Z \cup X)\backslash E_0)$, there exists an open neighborhood V_z of z such that $V_z \in M \backslash E_0$.

As $(M, E, \Sigma, M_j, \tilde{h}, \tilde{h}_{j,s})$ is a singular metric on E and $\Theta_{\tilde{h}}(E) \geq_{Nak}^s 0$, there exist a relatively compact subset $M_{j'} \subset M$ containing V_z and a C^2 smooth metric $\tilde{h}_{j',1} \leq \tilde{h}$ on $V_z \subset M_{j'}$. Note that $\delta \tilde{M}$ is a plurisubharmonic function on M. As $c(t)e^{-t}$ is decreasing with respect to t and $v_{t_0,B}(\Psi_1) \geq -t_0 - \frac{B}{2}$, we have $c(-v_{t_0,B}(\Psi_1))e^{v_{t_0,B}(\Psi_1)} \ge c(t_0+\frac{B}{2})e^{-t_0-\frac{B}{2}} > 0.$ Denote $C := \inf_{V_z} e^{v_{t_0,B}(\Psi_1)-\delta M}c(-v_{t_0,B}(\Psi_1)),$

we know $C > 0$. On V_z , as both \hat{h} and $\tilde{h}_{j',1}$ are continuous, we have $\tilde{h}_{j',1} \leq \tilde{C}\hat{h}$ for some $\tilde{C} > 0$. Then we have

$$
\begin{split} &\int_{V_z \backslash (Z \cup X)} |\tilde{F}_Z|^2_{\tilde{h}_{j',1}} \\ \leq & 2 \int_{V_z \backslash (Z \cup X)} |\tilde{F}_Z - (1-b_{t_0,B}(\Psi_1)) f F^{1+\delta}|^2_{\tilde{h}_{j',1}} + 2 \int_{V_z \backslash (Z \cup X)} |(1-b_{t_0,B}(\Psi_1)) f F^{1+\delta}|^2_{\tilde{h}_{j',1}} \\ \leq & 2 \int_{V_z \backslash (Z \cup X)} |\tilde{F}_Z - (1-b_{t_0,B}(\Psi_1)) f F^{1+\delta}|^2_{\tilde{h}} + 2 \sup_{V_z} |F^{1+\delta}|^2 \int_{\{\Psi_1 < -t_0\} \cap V_z} |f|^2_{\tilde{h}_{j',1}} \\ \leq & \frac{2}{C} \bigg(\int_{M \backslash (Z \cup X)} |\tilde{F}_Z - (1-b_{t_0,B}(\Psi_1)) f F^{1+\delta}|^2_{\tilde{h}} e^{v_{t_0,B}(\Psi_1) - \delta M} c(-v_{t_0,B}(\Psi_1)) \bigg) \\ & + \tilde{C} \sup_{V_z} |F^{1+\delta}|^2 \int_{\{\Psi_1 < -t_0\} \cap V_z} |f|^2_{\tilde{h}} \\ < & + \infty. \end{split}
$$

As $Z \cup X$ is locally negligible with respect to L^2 holomorphic function, we can find an E-valued holomorphic extension F_{E_0} of F_Z from $M\setminus (Z\cup X)$ to $M\setminus E_0$ such that

$$
\int_{M\setminus E_0} |\tilde{F}_{E_0} - (1 - b_{t_0, B}(\Psi_1)) f F^{1+\delta}|^2_{\tilde{h}} e^{v_{t_0, B}(\Psi_1) - \delta \tilde{M}} c(-v_{t_0, B}(\Psi_1))
$$
\n
$$
\leq \left(\frac{1}{\delta} c(T_1) e^{-T_1} + \int_{T_1}^{t_0 + B} c(s) e^{-s} ds\right) \int_M \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \Psi_1 < -t_0\}} |f F|^2_{\tilde{h}}.
$$

Note that $E_0 \subset {\Psi = -\infty} \subset {\Psi < -t_0}$ and ${\Psi < -t_0}$ is open, then for any $z \in E_0$, there exists an open neighborhood U_z of z such that $U_z \in \{\Psi \leq \Psi\}$ $-t_0$ } = { Ψ_1 < -t₀}. As $(M, E, \Sigma, M_j, \tilde{h}, \tilde{h}_{j,s})$ is a singular metric on E and $\Theta_{\tilde{h}}(E) \geq_{Nak}^{s} 0$, there exist a relatively compact subset $M_{j''} \subset M$ containing U_z and a C^2 smooth metric $\tilde{h}_{j'',1} \leq \tilde{h}$ on $U_z \subset M_{j''}$. As $v_{t_0,B}(t) \geq -t_0 - \frac{B}{2}$, we have $c(-v_{t_0,B}(\Psi_1))e^{v_{t_0,B}(\Psi_1)} \geq c(t_0+\frac{B}{2})e^{-t_0-\frac{B}{2}} > 0$. Note that $\delta \tilde{M}$ is plurisubharmonic on M. Thus we have

$$
\begin{split} & \int_{U_z \backslash E_0} |\tilde{F}_{E_0} - (1 - b_{t_0, B}(\Psi)) f F^{1+\delta}|^2_{\tilde{h}_{j'',1}} \\ & \leq \int_{U_z \backslash E_0} |\tilde{F}_{E_0} - (1 - b_{t_0, B}(\Psi)) f F^{1+\delta}|^2_{\tilde{h}} \\ & \leq \frac{1}{C_1} \int_{U_z \backslash E_0} |\tilde{F}_{E_0} - (1 - b_{t_0, B}(\Psi_1)) f F^{1+\delta}|^2_{\tilde{h}} e^{v_{t_0, B}(\Psi_1) - \delta \tilde{M}} c(-v_{t_0, B}(\Psi_1)) < +\infty, \end{split}
$$

where C_1 is some positive number.

As $U_z \in {\Psi < -t_0}$, we have

$$
\int_{U_z\backslash E_0}|(1-b_{t_0,B}(\Psi))fF^{1+\delta}|^2_{\tilde{h}_{j'',1}}\leq \left(\sup_{U_z}|F^{1+\delta}|^2\right)\int_{U_z}|f|^2_{\tilde{h}_{j'',1}}<+\infty.
$$

Hence we have

$$
\int_{U_z\backslash E_0}|\tilde{F}_{E_0}|^2_{\tilde{h}_{j'',1}}<+\infty.
$$

As E_0 is contained in some analytic subset of M , we can find a holomorphic extension \tilde{F} of \tilde{F}_{E_0} from $M\backslash E_0$ to M such that

$$
\int_{M} |\tilde{F} - (1 - b_{t_0,B}(\Psi_1)) f F^{1+\delta}|_{\tilde{h}}^2 e^{v_{t_0,B}(\Psi_1) - \delta \tilde{M}} c(-v_{t_0,B}(\Psi_1))
$$
\n
$$
\leq \left(\frac{1}{\delta} c(T_1) e^{-T_1} + \int_{T_1}^{t_0+B} c(t) e^{-t} dt\right) \int_{M} \frac{1}{B} \mathbb{I}_{\{-t_0-B<\Psi_1<-t_0\}} |fF|_{\tilde{h}}^2.
$$
\n(2.5)

Lemma [2.2](#page-8-2) is proved. \square

Let $T \in [-\infty, +\infty)$. Let $c(t) \in \tilde{P}_{T,M,\Psi,h}$. Following the notations in Lemma [2.2](#page-8-2) and using the result of Lemma [2.2,](#page-8-2) we have the following lemma, which will be used to prove Theorem [1.8.](#page-4-0)

Lemma 2.3. Let (M, X, Z) satisfy condition (A) . Let $B \in (0, +\infty)$ and $t_0 > t_1 >$ T be arbitrarily given. Let f be a holomorphic $(n,0)$ form on $\{\Psi < -t_0\}$ such that

$$
\int_{\{\Psi<-t_0\}} |f|^2_h c(-\Psi) < +\infty,
$$
\n(2.6)

Then there exists an E-valued holomorphic $(n, 0)$ form \tilde{F} on $\{\Psi < -t_1\}$ such that

$$
\int_{\{\Psi<-t_1\}} |\tilde{F} - (1 - b_{t_0,B}(\Psi))f|_h^2 e^{v_{t_0,B}(\Psi) - \Psi} c(-v_{t_0,B}(\Psi))
$$

$$
\leq \left(\int_{t_1}^{t_0+B} c(s)e^{-s}ds\right) \int_M \frac{1}{B} \mathbb{I}_{\{-t_0-B<\Psi<-t_0\}} |f|_h^2 e^{-\Psi},
$$

where $b_{t_0,B}(t) = \int_{-\infty}^{t}$ $\frac{1}{B} \mathbb{I}_{\{-t_0 - B < s < -t_0\}} ds, \ v_{t_0, B}(t) = \int_{-t_0}^t b_{t_0, B}(s) ds - t_0.$ Proof of Lemma [2.3.](#page-10-0) Denote that

$$
\tilde{\Psi} := \min\{\psi - 2\log|F|, -t_1\}.
$$

As $t_0 > t_1 > T$, we have $\{\tilde{\Psi} < -t_0\} = {\Psi < -t_0}$. It follows from inequality [\(2.6\)](#page-10-1) and Lemma [2.2](#page-8-2) that there exists an E-valued holomorphic $(n,0)$ form \tilde{F}_δ on M such that

$$
\int_{M} |\tilde{F}_{\delta} - (1 - b_{t_0, B}(\tilde{\Psi})) f F^{1+\delta}|_{\tilde{h}}^2 e^{v_{t_0, B}(\tilde{\Psi}) - \delta \tilde{M}} c(-v_{t_0, B}(\tilde{\Psi}))
$$
\n
$$
\leq \left(\frac{1}{\delta} c(t_1) e^{-t_1} + \int_{t_1}^{t_0 + B} c(s) e^{-s} ds\right) \int_{M} \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \tilde{\Psi} < -t_0\}} |f F|_{\tilde{h}}^2.
$$

Note that on $\{\Psi < -t_1\}$, we have $\Psi = \tilde{\Psi} = \psi - 2 \log |F|$. Hence

$$
\int_{\{\Psi<-t_{1}\}}|\tilde{F}_{\delta} - (1-b_{t_{0},B}(\Psi))fF^{1+\delta}|_{\tilde{h}}^{2}e^{v_{t_{0},B}(\Psi)-\delta\tilde{M}}c(-v_{t_{0},B}(\Psi))\n= \int_{\{\Psi<-t_{1}\}}|\tilde{F}_{\delta} - (1-b_{t_{0},B}(\tilde{\Psi}))fF^{1+\delta}|_{\tilde{h}}^{2}e^{v_{t_{0},B}(\tilde{\Psi})-\delta\tilde{M}}c(-v_{t_{0},B}(\tilde{\Psi}))\n\leq \int_{M}|\tilde{F}_{\delta} - (1-b_{t_{0},B}(\tilde{\Psi}))fF^{1+\delta}|_{\tilde{h}}^{2}e^{v_{t_{0},B}(\tilde{\Psi})-\delta\tilde{M}}c(-v_{t_{0},B}(\tilde{\Psi}))\n\leq \left(\frac{1}{\delta}c(t_{1})e^{-t_{1}} + \int_{t_{1}}^{t_{0}+B}c(s)e^{-s}ds\right)\int_{M}\frac{1}{B}\mathbb{I}_{\{-t_{0}-B<\tilde{\Psi}<-t_{0}\}}|fF|_{\tilde{h}}^{2}\n= \left(\frac{1}{\delta}c(t_{1})e^{-t_{1}} + \int_{t_{1}}^{t_{0}+B}c(s)e^{-s}ds\right)\int_{M}\frac{1}{B}\mathbb{I}_{\{-t_{0}-B<\Psi<-t_{0}\}}|fF|_{\tilde{h}}^{2}<+\infty.
$$

Let $F_{\delta} := \frac{\tilde{F}_{\delta}}{F^{\delta}}$ be an E-valued holomorphic $(n,0)$ form on $\{\Psi < -t_1\}$. Then it follows from (2.7) that

$$
\int_{\{\Psi<-t_1\}} |F_{\delta} - (1 - b_{t_0,B}(\Psi))fF|_{\tilde{h}}^2 e^{v_{t_0,B}(\Psi)} c(-v_{t_0,B}(\Psi))
$$
\n
$$
\leq \left(\frac{1}{\delta}c(t_1)e^{-t_1} + \int_{t_1}^{t_0+B} c(s)e^{-s}ds\right) \int_M \frac{1}{B} \mathbb{I}_{\{-t_0-B<\Psi<-t_0\}} |fF|_{\tilde{h}}^2.
$$
\n(2.8)

Note that $e^{v_{t_0,B}(\Psi)}c(-v_{t_0,B}(\Psi)) \geq (c(t_0 + \frac{2}{B})e^{-t_0-\frac{2}{B}}) > 0$. As $c(t) \in \tilde{P}_{T,M,\Psi,h}$, there exists a closed subset E_0 of M such that $E_0 \subset Z \cap {\Psi(z) = -\infty}$ (where Z is an analytic subset of M) and for any compact subset $K \subset M \setminus E_0$, $|e_x|_h^2 c(-\Psi) \ge$ $C_K|e_x|^2_{\hat{h}}$ for any $x \in K \cap {\Psi < -t_0}$, where $C_K > 0$ is a constant and $e_x \in E_x$. Let K be any compact subset of $M\backslash E_0$. As $(M, E, \Sigma, M_j, \tilde{h}, \tilde{h}_{j,s})$ is a singular metric on E and $\Theta_{\tilde{h}}(E) \geq_{Nak}^s 0$, there exist a relatively compact subset $M_{j_K} \subset M$ containing

K and a C^2 smooth metric $\tilde{h}_{j_K,1} \leq \tilde{h}$ on $K \subset M_{j_K}$. It follows from inequality [\(2.8\)](#page-11-1) that we have

$$
\sup_{\delta} \int_{\{\Psi<-t_1\}\cap K} |F_{\delta} - (1 - b_{t_0,B}(\Psi))fF|_{\tilde{h}_{j_K,1}}^2 < +\infty.
$$

We also note that

$$
\int_{\{\Psi<-t_1\}\cap K}|(1-b_{t_0,B}(\Psi))fF|^2_{\tilde{h}_{j_K,1}}\le \left(\sup_K|F|^2\right)\int_{\{\Psi<-t_0\}\cap K}|f|^2_{\tilde{h}_{j_K,1}}<+\infty.
$$

Then we know that

$$
\sup_{\delta}\int_{\{\Psi<-t_1\}\cap K}|F_\delta|^2_{\tilde{h}_{j_K,1}}<+\infty.
$$

By Montel theorem and diagonal method, there exists a subsequence of $\{F_\delta\}$ (also denoted by F_δ) compactly convergent to a holomorphic $(n,0)$ form \tilde{F}_1 on $\{\Psi \leq$ $-t_1\}\E_0$. It follows from Fatou's Lemma and inequality [\(2.8\)](#page-11-1) that we have

$$
\int_{\{\Psi<-t_{1}\}\backslash E_{0}} |\tilde{F}_{1} - (1 - b_{t_{0},B}(\Psi))fF|_{\tilde{h}}^{2} e^{v_{t_{0},B}(\Psi)} c(-v_{t_{0},B}(\Psi))
$$
\n
$$
\leq \liminf_{\delta \to +\infty} \int_{\{\Psi<-t_{1}\}\backslash E_{0}} |F_{\delta} - (1 - b_{t_{0},B}(\Psi))fF|_{\tilde{h}}^{2} e^{v_{t_{0},B}(\Psi)} c(-v_{t_{0},B}(\Psi))
$$
\n
$$
\leq \liminf_{\delta \to +\infty} \int_{\{\Psi<-t_{1}\}} |F_{\delta} - (1 - b_{t_{0},B}(\Psi))fF|_{\tilde{h}}^{2} e^{v_{t_{0},B}(\Psi)} c(-v_{t_{0},B}(\Psi))
$$
\n
$$
\leq \liminf_{\delta \to +\infty} \left(\frac{1}{\delta} c(t_{1})e^{-t_{1}} + \int_{t_{1}}^{t_{0}+B} c(s)e^{-s}ds\right) \int_{M} \frac{1}{B} \mathbb{I}_{\{-t_{0}-B<\Psi<-t_{0}\}} |fF|_{\tilde{h}}^{2}
$$
\n
$$
\leq \left(\int_{t_{1}}^{t_{0}+B} c(s)e^{-s}ds\right) \int_{M} \frac{1}{B} \mathbb{I}_{\{-t_{0}-B<\Psi<-t_{0}\}} |fF|_{\tilde{h}}^{2}.
$$
\n(2.9)

Note that $E_0 \subset {\Psi = -\infty} \subset {\Psi < -t_1}$ and ${\Psi < -t_1}$ is open, then for any $z \in E_0$, there exists an open neighborhood U_z of z such that $U_z \in {\Psi < -t_1}$. As $(M, E, \Sigma, M_j, \tilde{h}, \tilde{h}_{j,s})$ is a singular metric on E and $\Theta_{\tilde{h}}(E) \geq_{Nak}^{s} 0$, there exist a relatively compact subset $M_{j''} \subset M$ containing U_z and a C^2 smooth metric $\tilde{h}_{j'',1} \leq \tilde{h}$ on $V_z \subset M_{j''}$. As $v_{t_0,B}(t) \geq -t_0 - \frac{B}{2}$, we have $c(-v_{t_0,B}(\Psi_1))e^{v_{t_0,B}(\Psi_1)} \geq$ $c(t_0 + \frac{B}{2})e^{-t_0 - \frac{B}{2}} > 0$. Thus we have

$$
\int_{U_z \backslash E_0} |\tilde{F}_1 - (1 - b_{t_0, B}(\Psi)) f F|_{\tilde{h}_{j'',1}}^2
$$
\n
$$
\leq \int_{U_z \backslash E_0} |\tilde{F}_1 - (1 - b_{t_0, B}(\Psi)) f F|_{\tilde{h}}^2
$$
\n
$$
\leq \frac{1}{C_1} \int_{U_z \backslash E_0} |\tilde{F}_1 - (1 - b_{t_0, B}(\Psi)) f F|_{\tilde{h}}^2 e^{v_{t_0, B}(\Psi)} c(-v_{t_0, B}(\Psi)) < +\infty,
$$

where C_1 is some positive number.

As $U_z \in {\Psi < -t_1}$, we have

$$
\int_{U_z \setminus E_0} |(1 - b_{t_0, B}(\Psi)) f F|_{\tilde{h}_{j'',1}}^2 \leq \left(\sup_{U_z} |F|^2\right) \int_{U_z} |f|_{\tilde{h}_{j'',1}}^2 < +\infty.
$$

Hence we have

$$
\int_{U_z\backslash E_0}|\tilde{F}_1|^2_{\tilde{h}_{j'',1}}<+\infty.
$$

As E_0 is contained in some analytic subset of M , we can find a holomorphic extension \tilde{F}_0 of \tilde{F}_1 from $\{\Psi < -t_1\} \setminus E_0$ to $\{\Psi < -t_1\}$ such that

$$
\int_{\{\Psi<-t_1\}} |\tilde{F}_1 - (1 - b_{t_0,B}(\Psi))fF|_{\tilde{h}}^2 e^{v_{t_0,B}(\Psi)} c(-v_{t_0,B}(\Psi))
$$
\n
$$
\leq \left(\int_{t_1}^{t_0+B} c(s)e^{-s}ds\right) \int_M \frac{1}{B} \mathbb{I}_{\{-t_0-B<\Psi<-t_0\}} |fF|_{\tilde{h}}^2.
$$
\n(2.10)

Denote $\tilde{F} := \frac{\tilde{F}_0}{F}$. Note that $\tilde{h} = he^{-\psi}$ and on $\{\Psi < -t_1\}$, we have $\Psi =$ $\psi - 2 \log |F|$. It follows from inequality [\(2.10\)](#page-13-0) that we have

$$
\int_{\{\Psi<-t_1\}} |\tilde{F} - (1 - b_{t_0,B}(\Psi))f|_{h}^{2} e^{v_{t_0,B}(\Psi) - \Psi} c(-v_{t_0,B}(\Psi))
$$

$$
\leq \left(\int_{t_1}^{t_0+B} c(s)e^{-s}ds\right) \int_M \frac{1}{B} \mathbb{I}_{\{-t_0-B<\Psi<-t_0\}} |f|_{h}^{2} e^{-\Psi}.
$$

Lemma [2.3](#page-10-0) is proved.

 \Box

2.2. Properties of \mathcal{O}_{M,z_0} -module J_{z_0} . In this section, we present some properties of \mathcal{O}_{M,z_0} -module J_{z_0} .

We recall the following property of closedness of holomorphic functions on a neighborhood of o.

Lemma 2.4 (Closedness of Submodules, see [\[22\]](#page-72-15)). Let N be a submodule of $\mathcal{O}^q_{\mathbb{C}^n,0}$, $1 \leq q < +\infty$, let $f_j \in \mathcal{O}_{\mathbb{C}^n}^q(U)$ be a sequence of q-tuples holomorphic in an open neighborhood U of the origin. Assume that the f_j converge uniformly in U towards a q-tuple $f \in \mathcal{O}_{\mathbb{C}^n}^q(U)$, assume furthermore that all germs $f_{j,0}$ belong to N. Then $f_0 \in N$.

We recall the following lemma which will be used in the proof of Lemma [2.9.](#page-19-0)

Lemma 2.5. Let M be a complex manifold. Let dV_M be a continuous volume form on M . Let S be an analytic subset of M . Let E be a holomorphic vector bundle on M with rank r. Let \hat{h} be a smooth metric on E. Let h be a measurable metric on E.

Let ${g_j}_{j=1,2,...}$ be a sequence of nonnegative Lebesgue measurable functions on M, which satisfies that g_j are almost everywhere convergent to g on M when $j \rightarrow$ $+\infty$, where g is a nonnegative Lebesgue measurable function on M. Assume that for any compact subset K of $M \backslash S$, we have $|e_x|^2_h g_j \geq C_K |e_x|^2_h$ for any $x \in K$ and any $j \in \mathbb{Z}_+$, where $C_K > 0$ is a constant and e_x is any section of E_x .

Let ${F_j}_{j=1,2,...}$ be a sequence of E-valued holomorphic $(n,0)$ forms on M. Assume that $\liminf_{j\to+\infty} \int_M |F_j|^2_{h} g_j \leq C$, where C is a positive constant. Then there exists a subsequence $\{F_{j_l}\}_{l=1,2,\ldots}$, which satisfies that $\{F_{j_l}\}$ is uniformly convergent to an E-valued holomorphic $(n, 0)$ form F on M on any compact subset of M when $l \rightarrow +\infty$, such that

$$
\int_M |F|_h^2 g \le C.
$$

Proof. Let $(U \subset\subset M, \theta)$ be a local trivialization of E, where E is a holomorphic vector bundle on M. For any $f = (f_1, \ldots, f_r) \in H^0(U, \mathcal{O}(E))$, denote $|f|^2$ vector bundle on M. For any $f = (f_1, \ldots, f_r) \in H^0(U, \mathcal{O}(E))$, denote $|f|^2$:=
 $\sum_{i=1}^r |f_i|^2$. Then there exists a constant $\lambda > 0$ such that $\frac{1}{\lambda} |f|^2_1 \leq |f|^2_{\hat{h}} \leq \lambda |f|^2_1$ on U. Let \tilde{K} be any compact subset of U and \tilde{S} be an analytic subset of M. By Local Parametrization Theorem (see [\[13\]](#page-72-10)) and Maximum Principle, there exists a compact subset $\tilde{K}_1 \subset U \backslash \tilde{S}$ such that

$$
\sup_{z \in \tilde{K}} |f(z)|_1^2 \leq \tilde{C}_1 \sup_{z \in \tilde{K}_1} |f(z)|_1^2.
$$

Hence we have

$$
\sup_{z \in \tilde{K}} |f(z)|_{\hat{h}}^2 \leq \lambda \sup_{z \in \tilde{K}} |f(z)|_1^2 \leq \lambda \tilde{C}_1 \sup_{z \in \tilde{K}_1} |f(z)|_1^2 \leq \lambda^2 \tilde{C}_1 \sup_{z \in \tilde{K}_1} |f(z)|_{\hat{h}}^2.
$$

Recall that S is an analytic subset of M . By the argument above, for any compact set $K \subset\subset M$, there exists $K_1 \subset M\backslash S$ such that for any $j \geq 0$, we have

$$
\sup_{z \in K} \frac{|F_j(z)|_{\hat{h}}^2}{dV_M} \le C_1 \sup_{z \in K_1} \frac{|F_j(z)|_{\hat{h}}^2}{dV_M},\tag{2.11}
$$

where $C_1 > 0$ (depends on K and \hat{h}) is a real number. Then there exists a compact subset $K_2 \subset M \backslash S$ such that $K_1 \subset K_2$ and for any $z \in K_1$ and $j \geq 0$,

$$
\frac{|F_j(z)|^2_{\hat{h}}}{dV_M} \le C_2 \int_{K_2} |F_j(z)|^2_{\hat{h}} \le \frac{C_2}{C_{K_2}} \int_{K_2} |F_j(z)|^2_{h} g_j \le \frac{C_2}{C_{K_2}} C < +\infty.
$$

Hence we know that $\sup_{K_1} \frac{|F_j(z)|^2_{\hat{h}}}{dV_M}$ is uniformly bounded with respect to j. Then it follows from inequality [\(2.11\)](#page-14-0), Montel theorem and diagonal method that we have a subsequence of $\{F_i\}$ (still denoted by $\{F_i\}$) uniformly converges to an E-valued holomorphic $(n, 0)$ form F on any compact subset of M. It follows from Fatou's Lemma and $\liminf_{j\to+\infty} \int_M |F_j|^2_h g_j \leq C$ that we have

$$
\int_M |F|_h^2 g \le \liminf_{j \to +\infty} \int_M |F_j|_h^2 g_j \le C.
$$

Lemma [2.5](#page-13-1) has been proved. \square

Since the properties of J_{z_0} is local, we assume that D is a pseudoconvex domain in \mathbb{C}^n containing the origin $o \in \mathbb{C}^n$. Let F be a holomorphic function on D. Let $f = (f_1, f_2, \ldots, f_r)$ be a holomorphic section of $D \times \mathbb{C}^r$. Let ψ be a plurisubharmonic function on D. Let h be a *measurable metric* on $D \times \mathbb{C}^r$. Denote $\tilde{h} := he^{-\psi}$. Let $(D, D \times \mathbb{C}^r, \Sigma, D_j, \tilde{h}, \tilde{h}_{j,s})$ be a singular metric on $E := D \times \mathbb{C}^r$ which satisfies $\Theta_{\tilde{h}}(E) \geq_{Nak}^{s} 0$. Let $T \in [-\infty, +\infty)$. Denote

$$
\Psi := \min\{\psi - 2\log|F|, -T\}.
$$

If $F(z) = 0$ for some $z \in M$, we set $\Psi(z) = -T$. Let $T_1 > T$ be a real number. Denote

$$
\tilde{M} := \max{\psi + T, 2 \log |F|},
$$

$$
\varphi_1 := 2 \max{\psi + T_1, 2 \log |F|},
$$

and

$$
\Psi_1 := \min{\psi - 2\log|F|, -T_1}.
$$

If $F(z) = 0$ for some $z \in M$, we set $\Psi_1(z) = -T_1$. We also note that by definition $I(h, \Psi_1)_o = I(h, \Psi)_o.$

Let $c(t)$ be a positive measurable function on $(T, +\infty)$ such that $c(t) \in \tilde{P}_{T,D,\Psi,h}$. Let dV_D be a continuous volume form on D. Denote that $H_o := \{f_o \in J(E, \Psi)_o :$

 $\int_{\{\Psi<-t\}\cap V_0}|f|^2_hc(-\Psi)dV_D<+\infty$ for some $t>T$ and V_0 is an open neighborhood of o $J\{\Psi<-t\}\cap V_0$ $J\left|h^c\right\rangle = \{ (F,o) \in \mathcal{O}_{\mathbb{C}^n,o}^r : \int_{U_0} |F|^2_h e^{-\varphi_1} c(-\Psi_1) dV_D < +\infty$ for some open neighborhood U_0 of $o \}.$

As $c(t) \in \tilde{P}_{T,D,\Psi,h}$, $c(t)e^{-t}$ is decreasing with respect to t and we have $I(h,\Psi_1)$ _o = $I(h,\Psi)_o \subset H_o$. We also note that \mathcal{H}_o is an submodule of $\mathcal{O}_{\mathbb{C}^n,o}^r$.

Lemma 2.6. For any $f_o \in H_o$, there exist a pseudoconvex domain $D_0 \subset D$ containing o and a holomorphic section \tilde{F} of $D \times \mathbb{C}^r$ on D_0 such that $(\tilde{F}, o) \in \mathcal{H}_o$ and

$$
\int_{\{\Psi_1<-t_1\}\cap D_0}|\tilde{F}-fF^2|^2_h e^{-\varphi_1-\Psi_1}<+\infty,
$$

for some $t_1 > T_1$.

Proof. It follows from $f_o \in H_o$ that there exist $t_0 > T_1 > T$ and a pseudoconvex domain $D_0 \in D$ containing o such that

$$
\int_{\{\Psi<-t_0\}\cap D_0} |f|^2_h c(-\Psi) < +\infty.
$$
\n(2.12)

Then it follows from Lemma [2.2](#page-8-2) that there exists a holomorphic section \tilde{F} of $D \times \mathbb{C}^r$ on D_0 such that

$$
\int_{D_0} |\tilde{F} - (1 - b_{t_0}(\Psi_1)) f F^2|_{\tilde{h}}^2 e^{v_{t_0}(\Psi_1) - \tilde{M}} c(-v_{t_0}(\Psi_1))
$$
\n
$$
\leq \left(c(T_1) e^{-T_1} + \int_{T_1}^{t_0 + 1} c(s) e^{-s} ds \right) \int_{D_0} \mathbb{I}_{\{-t_0 - 1 < \Psi_1 < -t_0\}} |f F|_{\tilde{h}}^2,
$$
\n
$$
(2.13)
$$

where $b_{t_0}(t) = \int_{-\infty}^t \mathbb{I}_{\{-t_0-1 \leq s \leq -t_0\}} ds$, $v_{t_0}(t) = \int_{-t_0}^t b_{t_0}(s) ds - t_0$. Note that $\tilde{h} =$ $he^{-\psi}$, $\psi + \tilde{M} = \varphi_1 + \Psi_1$ and $\Psi = \Psi_1 = \psi - 2 \log |F|$ on $\{\Psi < -t_0\}$. Hence, by [\(2.13\)](#page-15-0), we have

$$
\int_{D_0} |\tilde{F} - (1 - b_{t_0}(\Psi_1)) f F^2|_h^2 e^{-\varphi_1 - \Psi_1 + v_{t_0}(\Psi_1)} c(-v_{t_0}(\Psi_1))
$$
\n
$$
\leq \left(c(T_1) e^{-T_1} + \int_{T_1}^{t_0 + 1} c(s) e^{-s} ds \right) \int_{D_0} \mathbb{I}_{\{-t_0 - 1 < \Psi_1 < -t_0\}} |f|_h^2 e^{-\Psi_1}.
$$

Denote $C := c(T_1)e^{-T_1} + \int_{T_1}^{t_0+B} c(s)e^{-s}ds$, we note that C is a positive number.

As $v_{t_0}(t) > -t_0 - 1$, we have $e^{v_{t_0}(\Psi)}c(-v_{t_0}(\Psi)) \ge c(t_0 + 1)e^{-(t_0+1)} > 0$. As $b_{t_0}(t) \equiv 0$ on $(-\infty, -t_0 - 1)$, we have

$$
\int_{D_0 \cap \{\Psi_1 < -t_0 - 1\}} |\tilde{F} - fF^2|_h^2 e^{-\varphi_1 - \Psi_1} \le \frac{1}{c(t_0 + 1)e^{-(t_0 + 1)}} \int_{D_0} |\tilde{F} - (1 - b_{t_0}(\Psi_1)) fF^2|_h^2 e^{-\varphi_1 - \Psi_1 + v_{t_0}(\Psi_1)} c(-v_{t_0}(\Psi_1)) \le \frac{C}{c(t_0 + 1)e^{-(t_0 + 1)}} \int_{D_0} \mathbb{I}_{\{-t_0 - 1 < \Psi_1 < -t_0\}} |f|_h^2 e^{-\Psi_1} < +\infty. \tag{2.14}
$$

Note that on $\{\Psi_1 < -t_0\}$, $|F|^4 e^{-\varphi_1} = 1$. As $v_{t_0}(\Psi_1) \ge \Psi_1$, we have $c(-v_{t_0}(\Psi_1))e^{v_{t_0}(\Psi_1)} \ge$ $c(-\Psi_1)e^{-\Psi_1}$. Hence we have

$$
\int_{D_0} |\tilde{F}|_h^2 e^{-\varphi_1} c(-\Psi_1)
$$
\n
$$
\leq 2 \int_{D_0} |\tilde{F} - (1 - b_{t_0}(\Psi_1)) f F^2|_h^2 e^{-\varphi_1} c(-\Psi_1)
$$
\n
$$
+ 2 \int_{D_0} |(1 - b_{t_0}(\Psi_1)) f F^2|_h^2 e^{-\varphi_1} c(-\Psi_1)
$$
\n
$$
\leq 2 \int_{D_0} |\tilde{F} - (1 - b_{t_0}(\Psi_1)) f F^2|_h^2 e^{-\varphi_1 - \Psi_1 + v_{t_0}(\Psi_1)} c(-v_{t_0}(\Psi_1))
$$
\n
$$
+ 2 \int_{D_0 \cap \{\Psi < -t_0\}} |f|_h^2 c(-\Psi)
$$
\n
$$
< + \infty.
$$

Hence we know that $(\tilde{F}, o) \in \mathcal{H}_o$.

For any $(\tilde{F}, o) \in \mathcal{H}_o$ and $(\tilde{F}_1, o) \in \mathcal{H}_o$ such that $\int_{D_1 \cap {\{\Psi_1 < -t_1\}}} |\tilde{F} - fF^2|_h^2 e^{-\varphi_1 - \Psi_1} <$ $+\infty$ and $\int_{D_1 \cap {\Psi_1} \leq -t_1} |\tilde{F}_1 - fF^2|^2_h e^{-\varphi_1 - \Psi_1} < +\infty$, for some open neighborhood D_1 of *o* and $t_1 > T_1$, we have

$$
\int_{D_1\cap \{\Psi_1<-t_1\}}|\tilde{F}_1-\tilde{F}|^2_h e^{-\varphi_1-\Psi_1}<+\infty.
$$

As $(\tilde{F}, o) \in \mathcal{H}_o$ and $(\tilde{F}_1, o) \in \mathcal{H}_o$, there exists a neighborhood D_2 of o such that

$$
\int_{D_2} |\tilde{F}_1 - \tilde{F}|_h^2 e^{-\varphi_1} c(-\Psi_1) < +\infty.
$$
\n(2.15)

Note that we have $c(-\Psi_1)e^{\Psi_1} \geq c(t_1)e^{-t_1}$ on $\{\Psi \geq -t_1\}$. It follows from inequality [\(2.15\)](#page-16-0) that we have

$$
\int_{D_2 \cap \{\Psi \geq -t_1\}} |\tilde{F}_1-\tilde{F}|^2_h e^{-\varphi_1-\Psi_1}<+\infty.
$$

Hence we have $(\tilde{F} - \tilde{F}_1, o) \in \mathcal{E}(he^{-\varphi_1 - \Psi_1})_o$.

Thus it follows from Lemma [2.6](#page-15-1) that there exists a map $\tilde{P}: H_o \to \mathcal{H}_o/\mathcal{E}(he^{-\varphi_1 - \Psi_1})_o$ given by

$$
\tilde{P}(f_o) = [(\tilde{F}, o)]
$$

for any $f_o \in H_o$, where (\tilde{F}, o) satisfies $(\tilde{F}, o) \in \mathcal{H}_o$ and $\int_{D_1 \cap {\{\Psi_1 < -t_1\}}} |\tilde{F} - fF^2|_h^2 e^{-\varphi_1 - \Psi_1}$ +∞, for some $t_1 > T_1$ and some open neighborhood D_1 of o, and $[(F, o)]$ is the equivalence class of (\tilde{F}, o) in $\mathcal{H}_o/\mathcal{E}(he^{-\varphi_1-\Psi_1})_o$.

Proposition 2.7. \tilde{P} is an $\mathcal{O}_{\mathbb{C}^n,o}$ -module homomorphism and $Ker(\tilde{P}) = I(h,\Psi_1)o$.

Proof. For any $f_o, g_o \in H_o$. Denote that $\tilde{P}(f_o) = [(\tilde{F}, o)], \tilde{P}(g_o) = [(\tilde{G}, o)]$ and $P(f_o + g_o) = [(H, o)].$

Note that there exist an open neighborhood D_1 of o and $t > T_1$ such that $\int_{D_1 \cap \{\Psi_1 < -t\}} |\tilde{F} - fF^2|^2_h e^{-\varphi_1 - \Psi_1} < +\infty, \int_{D_1 \cap \{\Psi_1 < -t\}} |\tilde{G} - gF^2|^2_h e^{-\varphi_1 - \Psi_1} < +\infty,$ and $\int_{D_1 \cap {\Psi_1} < -t} |\tilde{H} - (f+g)F^2|_h^2 e^{-\varphi_1 - \Psi_1} < +\infty$. Hence we have

$$
\int_{D_1 \cap \{\Psi_1 < -t\}} |\tilde{H} - (\tilde{F} + \tilde{G})|_{h}^{2} e^{-\varphi_1 - \Psi_1} < +\infty.
$$

As $(\tilde{F}, o), (\tilde{G}, o)$ and (\tilde{H}, o) belong to \mathcal{H}_o , there exists an open neighborhood $\tilde{D}_1 \subset$ D_1 of o such that $\int_{\tilde{D}_1} |\tilde{H} - (\tilde{F} + \tilde{G})|_h^2 e^{-\varphi_1} c(-\Psi_1) < +\infty$. As $c(t)e^{-t}$ is decreasing with respect to t, we have $c(-\Psi_1)e^{\Psi_1} \geq c(t)e^{-t}$ on $\{\Psi_1 \geq -t\}$. Hence we have

$$
\int_{\tilde{D}_1\cap \{\Psi_1\geq -t\}} |\tilde{H}-(\tilde{F}+\tilde{G})|_h^2 e^{-\varphi_1-\Psi_1} \leq \frac{1}{c(t)e^{-t}}\int_{\tilde{D}_1\cap \{\Psi_1\geq -t\}} |\tilde{H}-(\tilde{F}+\tilde{G})|_h^2 e^{-\varphi_1}c(-\Psi_1)<+\infty.
$$

Thus we have $\int_{\tilde{D}_1} |\tilde{H} - (\tilde{F} + \tilde{G})|_h^2 e^{-\varphi_1 - \Psi_1} < +\infty$, which implies that $\tilde{P}(f_o + g_o) =$ $\tilde{P}(f_o) + \tilde{P}(q_o).$

For any $(q, o) \in \mathcal{O}_{\mathbb{C}^n, o}$. Denote $\tilde{P}((qf)_{o}) = [(\tilde{F}_q, o)]$. Note that there exist an open neighborhood D_2 of o and $t > T_1$ such that $\int_{D_2 \cap \{\Psi_1 < -t\}} |\tilde{F}_q - (qf) F^2|^2_h e^{-\varphi_1 - \Psi_1} <$ $+\infty$. It follows from $\int_{D_2 \cap {\{\Psi_1 < -t\}}} |\tilde{F} - fF^2|^2_h e^{-\varphi_1 - \Psi_1} < +\infty$ and q is holomorphic on $\overline{D_2}$ (shrink D_2 if necessary) that $\int_{D_2 \cap {\{\Psi_1 < -t\}}} |q\tilde{F} - qfF^2|_h^2 e^{-\varphi_1 - \Psi_1} < +\infty$. Then we have

$$
\int_{D_2\cap\{\Psi_1<-t\}}|\tilde F_q-q\tilde F|^2he^{-\varphi_1-\Psi_1}<+\infty.
$$

Note that $(q\tilde{F}, o)$ and (\tilde{F}_q, o) belong to \mathcal{H}_o , we have $\int_{D_2} |\tilde{F}_q - q\tilde{F}|_h^2 e^{-\varphi_1} c(-\Psi_1)$ $+\infty$. As $c(t)e^{-t}$ is decreasing with respect to t, we have $c(-\Psi_1)e^{\Psi_1} \geq c(t)e^{-t}$ on $\{\Psi_1 \geq -t\}$. Hence we have

$$
\int_{D_2\cap\{\Psi_1\geq -t\}}|\tilde{F}_q-q\tilde{F}|^2_he^{-\varphi_1-\Psi_1}\leq \frac{1}{c(t)e^{-t}}\int_{D_2\cap\{\Psi_1\geq -t\}}|\tilde{F}_q-q\tilde{F}|^2_he^{-\varphi_1}c(-\Psi_1)<+\infty.
$$

Thus we have $\int_{D_2} |\tilde{F}_q - q\tilde{F}|_h^2 e^{-\varphi_1 - \Psi_1} < +\infty$, which implies that $\tilde{P}(qf_o) = (q, o)\tilde{P}(f_o)$. We have proved that \tilde{P} is an $\mathcal{O}_{\mathbb{C}^n,o}$ -module homomorphism.

Next, we prove $Ker(\tilde{P}) = I(h, \Psi_1)_o$.

If $f_o \in I(h, \Psi_1)_o$. Denote $\tilde{P}(f_o) = [(\tilde{F}, o)]$. It follows from Lemma [2.6](#page-15-1) that $(F, o) \in \mathcal{H}_o$ and there exist an open neighborhood D_3 of o and a real number $t_1 > T_1$ such that

$$
\int_{\{\Psi_1<-t_1\}\cap D_3} |\tilde{F}-fF^2|^2_h e^{-\varphi_1-\Psi_1} < +\infty.
$$

As $f_0 \in I(h, \Psi_1)_0$, shrink D_3 and t_1 if necessary, we have

$$
\int_{\{\Psi_1<-t_1\}\cap D_3} |\tilde{F}|_h^2 e^{-\varphi_1-\Psi_1}
$$
\n
$$
\leq 2 \int_{\{\Psi_1<-t_1\}\cap D_3} |\tilde{F}-fF^2|_h^2 e^{-\varphi_1-\Psi_1} + 2 \int_{\{\Psi_1<-t_1\}\cap D_3} |fF^2|_h^2 e^{-\varphi_1-\Psi_1}
$$
\n
$$
\leq 2 \int_{\{\Psi_1<-t_1\}\cap D_3} |\tilde{F}-fF^2|_h^2 e^{-\varphi_1-\Psi_1} + 2 \int_{\{\Psi_1<-t_1\}\cap D_3} |f|_h^2 e^{-\Psi_1}
$$
\n
$$
< +\infty.
$$
\n(2.16)

As $c(t)e^{-t}$ is decreasing with respect to t, $c(-\Psi_1)e^{\Psi_1} \geq C_0 > 0$ for some positive number C_0 on $\{\Psi_1 \geq -t_1\}$. Then we have

$$
\int_{\{\Psi_1 \ge -t_1\} \cap D_3} |\tilde{F}|^2_h e^{-\varphi_1 - \Psi_1} \le \frac{1}{C_0} \int_{\{\Psi_1 \ge -t_1\} \cap D_3} |\tilde{F}|^2_h e^{-\varphi_1} c(-\Psi_1) < +\infty. \tag{2.17}
$$

Combining inequality [\(2.16\)](#page-17-0) and inequality [\(2.17\)](#page-18-0), we know that $\tilde{F} \in \mathcal{E}(he^{-\varphi_1 - \Psi_1})_o$, which means $\tilde{P}(\underline{f}_o) = 0$ in $\mathcal{H}_o/\mathcal{E}(he^{-\varphi_1 - \Psi_1})_o$. Hence we know $I(h, \Psi_1)_o \subset Ker(\tilde{P})$.

If $f_o \in Ker(\tilde{P})$, we know $\tilde{F} \in \mathcal{E}(he^{-\varphi_1 - \Psi_1})_o$. We can assume that \tilde{F} satisfies $\int_{D_4} |\tilde{F}|_h^2 e^{-\varphi_1 - \Psi_1} < +\infty$ for some open neighborhood D_4 of o . Then we have

$$
\int_{\{\Psi_1<-t_1\}\cap D_4} |f|_h^2 e^{-\Psi_1}
$$
\n
$$
= \int_{\{\Psi_1<-t_1\}\cap D_4} |fF^2|_h^2 e^{-\varphi_1-\Psi_1}
$$
\n
$$
\leq \int_{\{\Psi_1<-t_1\}\cap D_4} |\tilde{F}|_h^2 e^{-\varphi_1-\Psi_1} + \int_{\{\Psi_1<-t_1\}\cap D_4} |\tilde{F}-fF^2|_h^2 e^{-\varphi_1-\Psi_1}
$$
\n
$$
< +\infty.
$$
\n(2.18)

By definition, we know $f_o \in I(h, \Psi_1)_o$. Hence $Ker(\tilde{P}) \subset I(h, \Psi_1)_o$. $Ker(\tilde{P}) = I(h, \Psi_1)_o$ is proved.

Now we can define an $\mathcal{O}_{\mathbb{C}^n,\alpha}$ -module homomorphism $P: H_o/I(h,\Psi_1)_o \to \mathcal{H}_o/\mathcal{E}(he^{-\varphi_1-\Psi_1})_o$ as follows,

$$
P([f_o]) = \tilde{P}(f_o)
$$

for any $[f_o] \in H_o/I(h, \Psi_1)_o$, where $f_o \in H_o$ is any representative of $[f_o]$. It follows from Proposition [2.7](#page-16-1) that $P([f_o])$ is independent of the choices of the representatives of $[f_o]$.

Let $(\tilde{F}, o) \in \mathcal{H}_o$, i.e. $\int_U |\tilde{F}|_h^2 e^{-\varphi_1} c(-\Psi_1) < +\infty$ for some neighborhood U of o. Note that $|F|^4 e^{-\varphi_1} \equiv 1$ on $\{\Psi_1 < -T\}$. Hence we have $\int_{U \cap \{\Psi_1 < -t\}} \left| \frac{\tilde{F}}{F^2} \right|^2_h c(-\Psi_1)$ $+\infty$ for some $t > T$, i.e. $(\frac{\tilde{F}}{F^2})_o \in H_o$. And if $(\tilde{F}, o) \in \mathcal{E}(he^{-\varphi_1 - \Psi_1})_o$, it is easy to verify that $(\frac{\tilde{F}}{F^2})_o \in I(h, \Psi_1)_o$. Hence we have an $\mathcal{O}_{\mathbb{C}^n,o}$ -module homomorphism $Q: \mathcal{H}_o/\mathcal{E}(he^{-\varphi_1-\Psi_1})_o \to H_o/I(h, \Psi_1)_o$ defined as follows,

$$
Q([(\tilde{F}, o)]) = [(\frac{\tilde{F}}{F^2})_o].
$$

The above discussion shows that Q is independent of the choices of the representatives of $[(F, o)]$ and hence Q is well defined.

Proposition 2.8. $P: H_o/I(h, \Psi_1)_o \to H_o/\mathcal{E}(he^{-\varphi_1 - \Psi_1})_o$ is an $\mathcal{O}_{\mathbb{C}^n,o}$ -module isomorphism and $P^{-1} = Q$.

Proof. It follows from Proposition [2.7](#page-16-1) that we know P is injective.

Now we prove P is surjective.

For any $\left[\left(\tilde{F}, o\right)\right]$ in $\mathcal{H}_o/\mathcal{E}(he^{-\varphi_1-\Psi_1})_o$. Let (\tilde{F}, o) be any representatives of $\left[\left(\tilde{F}, o\right)\right]$ in \mathcal{H}_o . Denote that $[(f_1)_o] := [(\frac{\tilde{F}}{F^2})_o] = Q([(\tilde{F}, o)])$. Let $(f_1)_o := (\frac{\tilde{F}}{F^2})_o \in H_o$ be the representative of $[(f_1)_o]$. Denote $[(\tilde{F}_1, o)] := \tilde{P}((f_1)_o) = P([(f_1)_o])$. By the construction of \tilde{P} , we know that $(\tilde{F}_1, o) \in \mathcal{H}_o$ and

$$
\int_{D_1 \cap \{\Psi_1 < -t\}} |\tilde{F}_1 - f_1 F^2|_h^2 e^{-\varphi_1 - \Psi_1} < +\infty,
$$

where $t > T$ and D_1 is some neighborhood of *o*. Note that $(f_1)_o := (\frac{\tilde{F}}{F^2})_o$. Hence we have

$$
\int_{D_1\cap\{\Psi_1<-t\}}|\tilde{F}_1-\tilde{F}|^2_h e^{-\varphi_1-\Psi_1}<+\infty.
$$

It follows from $(\tilde{F}, o) \in \mathcal{H}_o$ and $(\tilde{F}_1, o) \in \mathcal{H}_o$ that there exists a neighborhood $D_2 \subset D_1$ of o such that

$$
\int_{D_2} |\tilde{F}-\tilde{F}_1|^2_h e^{-\varphi_1} c(-\Psi_1) < +\infty.
$$

Note that on $\{\Psi_1 \ge -t\}$, we have $c(-\Psi_1)e^{\Psi_1} \ge c(t)e^{-t} > 0$. Hence we have

$$
\int_{D_2\cap\{\Psi_1\geq -t\}}|\tilde{F}-\tilde{F}_1|^2he^{-\varphi_1-\Psi_1}<+\infty.
$$

Thus we know that $(\tilde{F}_1 - \tilde{F}, o) \in \mathcal{E}(he^{-\varphi_1 - \Psi_1})_o$, i.e. $[(\tilde{F}, o)] = [(\tilde{F}_1, o)]$ in $\mathcal{H}_o/\mathcal{E}(he^{-\varphi_1 - \Psi_1})_o$. Hence we have $P \circ Q([(\tilde{F}, o)]) = [(\tilde{F}, o)]$, which implies that P is surjective.

We have proved that $P: H_o/I(h, \Psi_1)_o \to H_o/\mathcal{E}(he^{-\varphi_1-\Psi_1})_o$ is an $\mathcal{O}_{\mathbb{C}^n,o}$ -module morphism and $P^{-1} = O$. isomorphism and $P^{-1} = Q$.

The following lemma shows the closedness of submodules of H_o .

Recall that D is a pseudoconvex domain in \mathbb{C}^n containing the origin $o \in \mathbb{C}^n$, F is a holomorphic function on D and $f = (f_1, f_2, \ldots, f_r)$ be a holomorphic section of $E := D \times \mathbb{C}^r$. Let ψ be a plurisubharmonic function on D. Let h be a measurable metric on $D \times \mathbb{C}^r$ and $\tilde{h} := he^{-\psi}$. Let $(D, D \times \mathbb{C}^r, \Sigma, D_j, \tilde{h}, \tilde{h}_{j,s})$ be a singular metric on $E := D \times \mathbb{C}^r$ which satisfies $\Theta_{\tilde{h}}(E) \geq_{Nak}^s 0$. Let $c(t) \in \tilde{P}_{T,D,\Psi,h}$.

Lemma 2.9. Let $U_0 \n\subseteq D$ be a Stein neighborhood of o. Let J_0 be an $\mathcal{O}_{\mathbb{C}^n,o}$ submodule of H_o such that $I(h, \Psi)_o \subset J_o$. Assume that $f_o \in J(\Psi)_o$. Let $\{f_j\}_{j\geq 1}$ be a sequence of E-valued holomorphic $(n, 0)$ forms on $U_0 \cap {\Psi < -t_j}$ for any $j \geq 1$, where $t_j > T$. Assume that $t_0 := \lim_{j \to +\infty} t_j \in [T, +\infty)$,

$$
\limsup_{j \to +\infty} \int_{U_0 \cap \{\Psi < -t_j\}} |f_j|^2 h^{-1}(\Psi) \le C < +\infty,\tag{2.19}
$$

and $(f_j-f)_o \in J_o$. Then there exists a subsequence of $\{f_j\}_{j\geq 1}$ compactly convergent to an E-valued holomorphic $(n, 0)$ form f_0 on $\{\Psi < -t_0\} \cap U_0$ which satisfies

$$
\int_{U_0\cap \{\Psi<-t_0\}}|f_0|^2_h c(-\Psi)\leq C,
$$

and $(f_0 - f)_o \in J_o$.

Proof. It follows from $c(t) \in \tilde{P}_{T,D,\Psi,h}$ that there exists an analytic subset Z of D and for any compact subset $K \subset D\backslash Z$, $|e_x|^2_R c(-\psi) \geq C_K |e_x|^2_{\hat{h}}$ for any $x \in K \cap \{\Psi \leq \Psi\}$ $-t_0$, where $C_K > 0$ is a constant and $e_x \in E_x$.

It follows from inequality [\(2.19\)](#page-19-1), Lemma [2.5](#page-13-1) and diagonal method that there exists a subsequence of $\{f_j\}_{j\geq 1}$ (also denoted by $\{f_j\}_{j\geq 1}$) compactly convergent to an E-valued holomorphic $(n,0)$ form f_0 on $\{\Psi < -t_0\} \cap U_0$. It follows from Fatou's Lemma that

$$
\int_{U_0 \cap \{\Psi<-t_0\}} |f_0|^2 h^{-1}(\Psi) \le \liminf_{j \to +\infty} \int_{U_0 \cap \{\Psi<-t_j\}} |f_j|^2 h^{-1}(\Psi) \le C.
$$

Now we prove $(f_0 - f)$ _o ∈ J_o. We firstly recall some constructions in Lemma [2.6.](#page-15-1)

As $t_0 := \lim_{j \to +\infty} t_j \in [T, +\infty)$. We can assume that $\{t_j\}_{j \geq 0}$ is upper bounded by some real number $T_1 + 1$. Denote $\Psi_1 := \min{\lbrace \psi - 2 \log |F|, -T_1 \rbrace}$, and if $F(z) = 0$ for some $z \in M$, we set $\Psi_1(z) = -T_1$. We note that

$$
\limsup_{j \to +\infty} \int_{U_0 \cap \{\Psi < -T_1 - 1\}} |f_j|^2 h^{-1}(\Psi) \le C < +\infty.
$$

It follows from $c(t) \in \tilde{P}_{T,D,\Psi,h}$ and Lemma [2.2](#page-8-2) that there exists an E-valued holomorphic $(n,0)$ form \tilde{F}_j on U_0 such that

$$
\int_{U_0} |\tilde{F}_j - (1 - b_1(\Psi_1)) f_j F^2|_h^2 e^{-\varphi_1 + v_1(\Psi_1) - \Psi_1} c(-v_1(\Psi_1))
$$
\n
$$
\leq \left(c(T_1) e^{-T_1} + \int_{T_1}^{T_1 + 2} c(s) e^{-s} ds \right) \int_{U_0} \mathbb{I}_{\{-T_1 - 2 < \Psi_1 < -T_1 - 1\}} |f_j|_h^2 e^{-\Psi_1},\tag{2.20}
$$

where $b_1(t) = \int_{-\infty}^t \mathbb{I}_{\{-T_1 - 2 < s < -T_1 - 1\}} ds$, $v_1(t) = \int_{-T_1 - 1}^t b_1(s) ds - (T_1 + 1)$. Denote $C_1 := c(T_1)e^{-T_1} + \int_{T_1}^{T_1+1} c(s)e^{-s}ds.$

Note that $v_1(t) > -T_1 - 2$. We have $e^{v_1(\Psi_1)}c(-v_1(\Psi)) \ge c(T_1+2)e^{-(T_1+2)} > 0$. As $b_1(t) \equiv 0$ on $(-\infty, -T_1 - 2)$, we have

$$
\int_{U_0 \cap \{\Psi < -T_1 - 2\}} |\tilde{F}_j - f_j F^2|_h^2 e^{-\varphi_1 - \Psi_1}
$$
\n
$$
\leq \frac{1}{c(T_1 + 2)e^{-(T_1 + 2)}} \int_{U_0} |\tilde{F}_j - (1 - b_1(\Psi_1))f_j F^2|_h^2 e^{-\varphi_1 - \Psi_1 + v_1(\Psi_1)} c(-v_{t_j}(\Psi_1))
$$
\n
$$
\leq \frac{C_1}{c(T_1 + 2)e^{-(T_1 + 2)}} \int_{U_0} \mathbb{I}_{\{-T_1 - 2 < \Psi_1 < -T_1 - 1\}} |f_j|_h^2 e^{-\Psi_1} < +\infty. \tag{2.21}
$$

Note that $|F^2|^2 e^{-\varphi_1} = 1$ on $\{\Psi_1 < -T_1 - 1\}$. As $v_{t_j}(\Psi_1) \geq \Psi_1$, we have $c(-v_{t_j}(\Psi_1))e^{v_{t_j}(\Psi_1)} \geq c(-\Psi_1)e^{-\Psi_1}$. Hence we have

$$
\int_{U_0} |\tilde{F}_j|_{h}^{2} e^{-\varphi_1} c(-\Psi_1)
$$
\n
$$
\leq 2 \int_{U_0} |\tilde{F}_j - (1 - b_1(\Psi_1)) f_j F^2|_{h}^{2} e^{-\varphi_1} c(-\Psi_1)
$$
\n
$$
+ 2 \int_{U_0} |(1 - b_1(\Psi_1)) f_j F^2|_{h}^{2} e^{-\varphi_1} c(-\Psi_1)
$$
\n
$$
\leq 2 \int_{U_0} |\tilde{F}_j - (1 - b_1(\Psi_1)) f_j F^2|_{h}^{2} e^{-\varphi_1 - \Psi_1 + v_1(\Psi_1)} c(-v_1(\Psi_1))
$$
\n
$$
+ 2 \int_{U_0 \cap {\Psi_1} < -T_1 - 1} |f_j|_{h}^{2} c(-\Psi_1)
$$
\n
$$
< + \infty.
$$
\n(2.22)

Hence we know that $(\tilde{F}_j, o) \in \mathcal{H}_o$.

It follows from inequality [\(2.19\)](#page-19-1), $\sup_{j\geq 1} \left(\int_{U_0} \mathbb{I}_{\{-T_1-2<\Psi<-T_1-1\}} |f_j|^2_h e^{-\Psi} \right)$ $+\infty$ and inequality [\(2.22\)](#page-20-0) that we actually have

$$
\sup_{j} \left(\int_{U_0} |\tilde{F}_j|_{h}^{2} e^{-\varphi_1} c(-\Psi_1) \right) < +\infty.
$$
 (2.23)

Note that $c(t)e^{-t}$ is decreasing with respect to t and there exists an analytic subset S of D and for any compact subset $K \subset D\backslash S$, $|e_x|_h^2 c(-\psi) \geq C_K |e_x|_h^2$ for any $x \in K \cap {\Psi < -t_0}$, where $C_K > 0$ is a constant and $e_x \in E_x$.

Let $K \subset U_0 \backslash S \subset D \backslash S$ be any compact set, then for any f being an E-valued holomorphic $(n, 0)$ form, we have

$$
|f_x|_h^2 e^{-\varphi_1} c(-\Psi_1) \ge \tilde{C}_K |f_x|_{\hat{h}}^2
$$

for any $x \in K \cap {\Psi_1 < -T_1}$ and

$$
|f_x|_h^2 e^{-\varphi_1} c(-\Psi_1) \ge C_1 |f_x|_h^2 e^{-\varphi_1 - \Psi_1} = |f_x|_h^2 e^{-\delta \tilde{M}} \ge C_1 C_2 |f_x|_h^2
$$

for any $x \in K \cap {\Psi_1 \geq -T_1}$, where $C_K, C_1, C_2 > 0$ are constants. Hence

$$
|f_x|_h^2 e^{-\varphi_1} c(-\Psi_1) \ge \min\{\tilde{C}_K, C_1 C_2\} |f_x|_{\hat{h}}^2,\tag{2.24}
$$

for any $x \in K \cap {\Psi_1 \geq -T_1}$. It follows from inequality [\(2.23\)](#page-20-1), inequality [\(2.24\)](#page-21-0) and Lemma [2.5](#page-13-1) that there exists a subsequence of $\{\tilde{F}_j\}_{j\geq 1}$ (also denoted by $\{\tilde{F}_j\}_{j\geq 1}$) compactly convergent to an E-valued holomorphic $(n, 0)$ form \tilde{F}_0 on U_0 and

$$
\int_{U_0} |\tilde{F}_0|_{h}^2 e^{-\varphi_1} c(-\Psi_1) \le \liminf_{j \to +\infty} \int_{U_0} |\tilde{F}_j|_{h}^2 e^{-\varphi_1} c(-\Psi_1) < +\infty. \tag{2.25}
$$

As f_j converges to f_0 , it follows from Fatou's Lemma and inequality [\(2.20\)](#page-20-2) that

$$
\int_{U_0} |\tilde{F}_0 - (1 - b_1(\Psi)) f_0 F^2|_h^2 e^{-\varphi_1 + v_1(\Psi_1) - \Psi_1} c(-v_1(\Psi_1))
$$
\n
$$
\leq \liminf_{j \to +\infty} \int_{U_0} |\tilde{F}_j - (1 - b_1(\Psi)) f_j F^2|_h^2 e^{-\varphi_1 + v_1(\Psi) - \Psi} c(-v_1(\Psi_1))
$$
\n
$$
< +\infty,
$$

which implies that

$$
\int_{U_0 \cap \{\Psi<-T_1-2\}} |\tilde{F}_0 - f_0 F^2|_h^2 e^{-\varphi_1-\Psi_1} < +\infty.
$$
\n(2.26)

It follows from inequality (2.21) , inequality (2.22) , inequality (2.25) , inequality [\(2.26\)](#page-21-2) and definition of $P: H_o/I(h, \Psi_1)_o \to \mathcal{H}_o/\mathcal{E}(he^{-\varphi_1-\Psi_1})_o$ that for any $j \geq 0$, we have

$$
P([(f_j)_o]) = [(\tilde{F}_j, o)].
$$

Note that $I(h, \Psi_1)_o = I(h, \Psi)_o \subset J_o$. As $(f_j - f)_o \in J_o$ for any $j \ge 1$, we have $(f_j - f_1)_o \in J_o$ for any $j \geq 1$. It follows from Proposition [2.8](#page-18-1) that there exists a submodule \tilde{J} of $\mathcal{O}_{\mathbb{C}^n,o}^r$ such that $\mathcal{E}(he^{-\varphi_1-\Psi_1})_o \subset \tilde{J} \subset \mathcal{H}_o$ and $\tilde{J}/\mathcal{E}(he^{-\varphi_1-\Psi_1})_o =$ $\text{Im}(P|_{J_o/I(h,\Psi_1_o})$. It follows from $(f_j - f_1)_o \in J_o$ and $P([(f_j)_o]) = [(F_j, o)]$ for any $j \geq 1$ that we have

$$
(\tilde{F}_j-\tilde{F}_1)\in \tilde{J},
$$

for any $j \geq 1$.

As \tilde{F}_j compactly converges to \tilde{F}_0 , using Lemma [2.4,](#page-13-2) we obtain that $(\tilde{F}_0 (\tilde{F}_1, o) \in \tilde{J}$. Note that P is an $\mathcal{O}_{\mathbb{C}^n,o}$ -module isomorphism and $\tilde{J}/\mathcal{E}(he^{-\varphi_1-\Psi_1})_o =$ $\text{Im}(P|_{J_o/I(h,\Psi_1)_o})$. We have $(f_0 - f_1)_o \in J_o$, which implies that $(f_0 - f)_o \in J_o$. Lemma [2.9](#page-19-0) is proved.

Let $c \equiv 1$, and note that $H_o = I(h, 0\Psi)_o$ and $\mathcal{H}_o = \mathcal{E}(he^{-\varphi_1})_o$. It is clear that $I(h, a\Psi)_{o} \subset I(h, a'\Psi)_{o}$ for any $0 \le a' < a < +\infty$. Denote that $I_{+}(h, a\Psi)_{o} :=$ $\bigcup_{p>a}I(h,p\Psi)_o$ is an $\mathcal{O}_{\mathbb{C}^n,o}$ -submodule of H_o , where $a\geq 0$.

Lemma 2.10. There exists $a' > a$ such that $I(h, a'\Psi)_o = I_+(h, a\Psi)_o$ for any $a \ge 0$.

Proof. The definition of $I_+(h, a\Psi)_o$ shows $I(h, p\Psi)_o \subset I_+(h, a\Psi)_o$ for any $p > a$. It suffices to prove that there exists $a' > a$ such that $I_+(h, a\Psi)_o \subset I(h, a'\Psi)_o$.

Denote that $\tilde{\varphi}_1 := k\varphi_1 = 2 \max\{k\psi + kT, 2 \log |F^k|\}$ and $\tilde{\Psi} := k\Psi = \min\{k\psi - kT, 2 \log |F^k|\}$ $2\log|F^k|, -kT\}$, where $k > a$ is an integer. As $he^{-\psi} \geq_{Nak}^s 0$ and ψ is plurisub-harmonic on M, it follows from Remark [1.5](#page-2-0) that $he^{-k\psi} \geq_{Nak}^{s} 0$. Proposition [2.8](#page-18-1) shows that there exists an $\mathcal{O}_{\mathbb{C}^n,o}$ -module isomorphism P from $I(h,0\Psi)_{o}/I(h,\tilde{\Psi})_{o} \to$ $\mathcal{E}(he^{-\varphi_1})_o/\mathcal{E}(he^{-\tilde{\varphi}_1-\tilde{\Psi}})_o$, which implies that for any $p \in (0,k)$, there exists an $\mathcal{O}_{\mathbb{C}^n,o}$ -submodule K_p of $\mathcal{O}_{\mathbb{C}^n,o}^r$ such that

$$
P(I(h, p\Psi)_o/I(h, \tilde{\Psi})_o) = K_p \langle \mathcal{E}(he^{-\tilde{\varphi}_1 - \tilde{\Psi}})_o.
$$

Denote that

$$
L := \cup_{a < p < k} K_p
$$

be an $\mathcal{O}_{\mathbb{C}^n,o}$ -submodule K_p of $\mathcal{O}_{\mathbb{C}^n,o}^r$. Hence $P|_{I+(h,a\Psi)o/I(h,\tilde{\Psi})_o}$ is an $\mathcal{O}_{\mathbb{C}^n,o}$ -module isomorphism from $I_+(h, a\Psi)_o/I(h, \tilde{\Psi})_o$ to $L/\mathcal{E}(he^{-\tilde{\varphi}_1-\tilde{\Psi}})_o$. As $\mathcal{O}_{\mathbb{C}^n,o}$ is a Noe-therian ring (see [\[41\]](#page-72-16)), we know that $\mathcal{O}_{\mathbb{C}^n,o}^r$ is a Noetherian $\mathcal{O}_{\mathbb{C}^n,o}$ -module, which implies that L is finitely generated. Thus, we have a finite set $\{(f_1)_o, \ldots, (f_m)_o\} \subset$ $I_+(h, a\Psi)_o$, which satisfies that for any $f_o \in I_+(h, a\Psi)_o$, there exists $(h_j, o) \in \mathcal{O}_{\mathbb{C}^n, o}$ for $1 \leq j \leq m$ such that

$$
f_o - \sum_{j=1}^m (h_j, o) \cdot (f_j)_o \in I(h, \tilde{\Psi})_o.
$$

By the definition of $I_+(h, a\Psi)_{\rho}$, there exists $a' \in (a, k)$ such that $\{(f_1)_o, \ldots, (f_m)_o\} \subset$ $I(h, a'\Psi)_{o}$. Note that $I(h, \tilde{\Psi})_{o} = I(h, k\Psi)_{o} \subset I(h, a'\Psi)_{o}$. Then we obtain that $I_+(h, a\Psi)_o \subset I(h, a'\Psi)_o.$

Thus, Lemma [2.10](#page-22-0) holds.

3. PROPERTIES OF $G(t)$

Following the notations in Section [1.2,](#page-2-1) we present some properties of the function $G(t)$ in this section.

For any $t \geq T$, denote

$$
\mathcal{H}^2(t;c,f,H) := \left\{ \tilde{f} : \int_{\{\Psi<-t\}} |\tilde{f}|_h^2 c(-\Psi) < +\infty, \ \tilde{f} \in H^0(\{\Psi<-t\}, \mathcal{O}(K_M \otimes E)) \right\}
$$

$$
\& (\tilde{f}-f)_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes (J_{z_0} \cap H_{z_0}), \text{ for any } z_0 \in Z_0 \right\},\
$$

where f is an E-valued holomorphic $(n, 0)$ form on $\{\Psi < -t_0\} \cap V$ for some $V \supset Z_0$ is an open subset of M and some $t_0 \geq T$, $c(t)$ is a positive measurable function on $(T, +\infty)$ and $H_{z_0} = \{f_o \in J(\Psi)_o : \int_{\{\Psi < -t\} \cap V_0} |f|^2 e^{-\varphi} c(-\Psi) < +\infty \text{ for some } t >$ T_0 and V_0 is an open neighborhood of z_0 (the definition of H_{z_0} can be referred to Section [2.2\)](#page-13-3).

If $G(t_1; c, \Psi, \varphi, J, f) < +\infty$, then there exists an E-valued holomorphic $(n, 0)$ form \tilde{f}_0 on $\{\Psi < -t_1\}$ such that $(\tilde{f}_0 - f)_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$, for any $z_0 \in Z_0$ and

$$
\int_{\{\Psi<-t_1\}}|\tilde{f}_0|^2_{h}c(-\Psi)<+\infty.
$$

Lemma 3.1. If $G(t_1; c, \Psi, \varphi, J, f) < +\infty$ for some $t_1 \geq T$, we have $\mathcal{H}^2(t; c, f) =$ $\mathcal{H}^2(t; c, \tilde{f}_0) = \mathcal{H}^2(t; c, \tilde{f}_0, H)$ for any $t \geq T$.

Proof. As $({\tilde f}_0 - f)_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$, for any $z_0 \in Z_0$, we have $\mathcal{H}^2(t; c, f) =$ $\mathcal{H}^2(t; c, \tilde{f}_0)$ for any $t \geq T$.

Now we prove $\mathcal{H}^2(t; c, \tilde{f}_0) = \mathcal{H}^2(t; c, \tilde{f}_0, H)$ for any $t \geq T$. It is obviously that $\mathcal{H}^2(t; c, \tilde{f}_0) \supset \mathcal{H}^2(t; c, \tilde{f}_0, H)$. We only need to show $\mathcal{H}^2(t; c, \tilde{f}_0) \subset \mathcal{H}^2(t; c, \tilde{f}_0, H)$.

Let $\tilde{f}_1 \in \mathcal{H}^2(t_2; c, \tilde{f}_0)$ for some $t_2 \geq T$. As $\int_{\{\Psi<-t_2\}} |\tilde{f}_1|^2_h c(-\Psi) < +\infty$, denote $t = \max\{t_1, t_2\}$, we know that

$$
\int_{\{\Psi<-t\}}|\widetilde{f}_1-\widetilde{f}_0|^2_hc(-\Psi)<+\infty,
$$

which implies that $({\tilde f}_1 - {\tilde f}_0)_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes H_{z_0}$, for any $z_0 \in Z_0$. Hence $({\tilde f}_1 \tilde{f}_0|_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes (J_{z_0} \cap H_{z_0}),$ for any $z_0 \in Z_0$, which implies that $\tilde{f}_1 \in \mathcal{H}^2(t; c, \tilde{f}_0, H)$. Hence $\mathcal{H}^2(t; c, \tilde{f}_0) = \mathcal{H}^2(t; c, \tilde{f}_0, H).$

Remark 3.2. If $G(t_1; c, \Psi, \varphi, J, f) < +\infty$ for some $t_1 \geq T$, we can always assume that J_{z_0} is an \mathcal{O}_{M,z_0} -submodule of H_{z_0} such that $I(h, \Psi)_{z_0} \subset J_{z_0}$, for any $z_0 \in Z_0$ in the definition of $G(t; c, \Psi, h, J, f)$, where $t \in [T, +\infty)$.

Proof. If $G(t_1; c, \Psi, \varphi, J, f) < +\infty$ for some $t_1 \geq T$, it follows from Lemma [3.1](#page-23-0) that $\mathcal{H}^2(t; c, f) = \mathcal{H}^2(t; c, \tilde{f}_0) = \mathcal{H}^2(t; c, \tilde{f}_0, H)$ for any $t \geq T$. By definition, we have $G(t; c, \Psi, h, J, f) = G(t; c, \Psi, h, J, \hat{f}_0) = G(t; c, \Psi, h, J \cap H, \hat{f}_0).$

Hence we can always assume that J_{z_0} is an \mathcal{O}_{M,z_0} -submodule of H_{z_0} such that $I(h, \Psi)_{z_0} \subset J_{z_0}$, for any $z_0 \in Z_0$.

In the following discussion, we assume that J_{z_0} is an \mathcal{O}_{M,z_0} -submodule of H_{z_0} such that $I(h, \Psi)_{z_0} \subset J_{z_0}$, for any $z_0 \in Z_0$.

Let $c(t) \in \tilde{P}_{T,M,\Psi,h}$. The following lemma will be used to discuss the convergence property of E-valued holomorphic forms on $\{\Psi < -t\}.$

Lemma 3.3. Let f be an E-valued holomorphic $(n,0)$ form on $\{\Psi < -\hat{t}_0\} \cap V$, where $V \supset Z_0$ is an open subset of M and $\hat{t}_0 > T$ is a real number. For any $z_0 \in Z_0$, let J_{z_0} be an \mathcal{O}_{M,z_0} -submodule of H_{z_0} such that $I(h, \Psi)_{z_0} \subset J_{z_0}$.

Let $\{f_j\}_{j\geq 1}$ be a sequence of E-valued holomorphic $(n,0)$ forms on $\{\Psi < -t_j\}$. Assume that $t_0 := \lim_{j \to +\infty} t_j \in [T, +\infty)$,

$$
\limsup_{j \to +\infty} \int_{\{\Psi < -t_j\}} |f_j|^2 h^{-1}(\Psi) \le C < +\infty,\tag{3.1}
$$

and $(f_j - f)_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$ for any $z_0 \in Z_0$. Then there exists a subsequence of ${f_j}_{j \in \mathbb{N}^+}$ compactly convergent to an E-valued holomorphic $(n, 0)$ form f_0 on $\{\Psi < -t_0\}$ which satisfies

$$
\int_{\{\Psi<-t_0\}} |f_0|^2_h c(-\Psi) \leq C,
$$

and $(f_0 - f)_{z_0}$ ∈ $\mathcal{O}(K_M)_{z_0}$ ⊗ J_{z_0} for any $z_0 \in Z_0$.

Proof. It follows from $c(t) \in \tilde{P}_{T,M,\Psi,h}$ that there exists an analytic subset Z of D and for any compact subset $K \subset D\backslash Z$, $|e|_h^2 c(-\psi) \geq C_K |e|_h^2$ on $K \cap {\Psi < -t_0}$, where $C_K > 0$ is a constant and e is any E-valued holomorphic $(n, 0)$ form. It follows from inequality [\(3.1\)](#page-23-1), Lemma [2.5](#page-13-1) and diagonal method that there exists a subsequence of $\{f_j\}_{j\geq 1}$ (also denoted by $\{f_j\}_{j\geq 1}$) compactly convergent to an Evalued holomorphic $(n, 0)$ form f_0 on $\{\Psi < -t_0\}$. It follows from Fatou's Lemma that

$$
\int_{\{\Psi<-t_0\}} |f_0|^2 h^{-1}(\Psi) \le \liminf_{j \to +\infty} \int_{\{\Psi<-t_j\}} |f_j|^2 h^{-1}(\Psi) \le C.
$$

Next we prove $(f_0 - f)_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$ for any $z_0 \in Z_0$.

Let $z_0 \in Z_0$ be a point. As $\limsup_{j \to +\infty}$ $\int_{\{\Psi<-t_j\}} |f_j|^2_{h} c(-\Psi) \leq C < +\infty$, there exists an open Stein neighborhood $U_{z_0} \in M$ of z_0 such that

$$
\limsup_{j\rightarrow +\infty} \int_{U_{z_0}\cap \{\Psi<-t_j\}} |f_j|^2_h c(-\Psi)\leq C<+\infty.
$$

Note that we also have $(f_j - f)_{z_0} \in J_{z_0}$. It follows from Lemma [2.9](#page-19-0) and the uniqueness of limit function that $(f_0 - f)_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$ for any $z_0 \in Z_0$. Lemma [3.3](#page-23-2) is proved. \square

Lemma 3.4. Let $t_0 > T$. The following two statements are equivalent, (1) $G(t_0) = 0$; (2) $f_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$, for any $z_0 \in Z_0$.

Proof. If $f_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$, for any $z_0 \in Z_0$, then take $\tilde{f} \equiv 0$ in the definition of $G(t)$ and we get $G(t_0) \equiv 0$.

If $G(t_0) = 0$, by definition, there exists a sequence of E-valued holomorphic $(n, 0)$ forms $\{f_j\}_{j\in\mathbb{Z}^+}$ on $\{\Psi<-t_0\}$ such that

$$
\lim_{j \to +\infty} \int_{\{\Psi < -t_0\}} |f_j|_h^2 c(-\Psi) = 0,
$$
\n(3.2)

and $(f_j - f)_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$, for any $z_0 \in Z_0$ and $j \ge 1$. It follows from Lemma [3.3](#page-23-2) that there exists a subsequence of $\{f_j\}_{j\in\mathbb{N}^+}$ compactly convergent to an E-valued holomorphic $(n, 0)$ form f_0 on $\{\Psi < -t_0\}$ which satisfies

$$
\int_{\{\Psi<-t_0\}}|f_0|^2_{h}c(-\Psi)=0
$$

and $(f_0 - f)_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$ for any $z_0 \in Z_0$. It follows from $\int_{\{\Psi<-t_0\}} |f_0|^2_{h} c(-\Psi) =$ 0 that we know $f_0 \equiv 0$. Hence we have $f_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$ for any $z_0 \in Z_0$. Statement (2) is proved.

The following lemma shows the existence and uniqueness of the E-valued holomorphic $(n, 0)$ form related to $G(t)$.

Lemma 3.5. Assume that $G(t) < +\infty$ for some $t \in [T, +\infty)$. Then there exists a unique E-valued holomorphic $(n, 0)$ form F_t on $\{\Psi < -t\}$ satisfying

$$
\int_{\{\Psi<-t\}} |F_t|^2_{h} c(-\Psi) = G(t)
$$

and $(F_t - f) \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$, for any $z_0 \in Z_0$.

Furthermore, for any E-valued holomorphic $(n,0)$ form \hat{F} on $\{\Psi < -t\}$ satisfying

$$
\int_{\{\Psi<-t\}}|\hat{F}|^2_{h}c(-\Psi)<+\infty
$$

and $(\hat{F} - f) \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$, for any $z_0 \in Z_0$. We have the following equality

$$
\int_{\{\Psi<-t\}} |F_t|_h^2 c(-\Psi) + \int_{\{\Psi<-t\}} |\hat{F} - F_t|_h^2 c(-\Psi)
$$
\n
$$
= \int_{\{\Psi<-t\}} |\hat{F}|_h^2 c(-\Psi). \tag{3.3}
$$

Proof. We firstly show the existence of F_t . As $G(t) < +\infty$, then there exists a sequence of E-valued holomorphic $(n, 0)$ forms $\{f_j\}_{j \in \mathbb{N}^+}$ on $\{\Psi \langle -t \rangle\}$ such that

$$
\lim_{j \to +\infty} \int_{\{\Psi < -t\}} |f_j|_h^2 c(-\Psi) = G(t)
$$

and $(f_j - f) \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$, for any $z_0 \in Z_0$ and any $j \ge 1$. It follows from Lemma [3.3](#page-23-2) that there exists a subsequence of $\{f_j\}_{j\in\mathbb{N}^+}$ compactly convergent to an E-valued holomorphic $(n,0)$ form F on $\{\Psi < -t\}$ which satisfies

$$
\int_{\{\Psi<-t\}}|F|^2_hc(-\Psi)\leq G(t)
$$

and $(F - f)_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$ for any $z_0 \in Z_0$. By the definition of $G(t)$, we have $\int_{\{\Psi<-t\}} |F|_h^2 c(-\Psi) = G(t)$. Then we obtain the existence of $F_t(=F)$.

We prove the uniqueness of F_t by contradiction: if not, there exist two different holomorphic $(n, 0)$ forms f_1 and f_2 on $\{\Psi < -t\}$ satisfying $\int_{\{\Psi < -t\}} |f_1|^2_R c(-\Psi) =$ $\int_{\{\Psi<-t\}}|f_2|^2_{h}c(-\Psi) = G(t), (f_1-f)_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$ for any $z_0 \in Z_0$ and $(f_2 - f)_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$ for any $z_0 \in Z_0$. Note that

$$
\begin{split} &\int_{\{\Psi<-t\}}|\frac{f_1+f_2}{2}|^2_{h}c(-\Psi)+\int_{\{\Psi<-t\}}|\frac{f_1-f_2}{2}|^2_{h}c(-\Psi)\\ &=\frac{1}{2}(\int_{\{\Psi<-t\}}|f_1|^2_{h}c(-\Psi)+\int_{\{\Psi<-t\}}|f_1|^2_{h}c(-\Psi))=G(t), \end{split}
$$

then we obtain that

$$
\int_{\{\Psi<-t\}}|\frac{f_1+f_2}{2}|^2_{h}c(-\Psi)
$$

and $(\frac{f_1+f_2}{2}-f)_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$ for any $z_0 \in Z_0$, which contradicts to the definition of $G(t)$.

Now we prove equality [\(3.3\)](#page-25-0). Let q be an E-valued holomorphic $(n,0)$ form on $\{\Psi<-t\}$ such that $\int_{\{\Psi<-t\}}|q|^2_h c(-\Psi)<+\infty$ and $q\in\mathcal{O}(K_M)_{z_0}\otimes J_{z_0}$ for any $z_0\in$ Z₀. It is clear that for any complex number α , $F_t + \alpha q$ satisfying $((F_t + \alpha q) - f) \in$ $\mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$ for any $z_0 \in Z_0$ and $\int_{\{\Psi<-t\}} |F_t|_h^2 c(-\Psi) \leq \int_{\{\Psi<-t\}} |F_t + \alpha q|^2_h c(-\Psi)$. Note that

$$
\int_{\{\Psi<-t\}}|F_t+\alpha q|^2_h c(-\Psi)-\int_{\{\psi<-t\}}|F_t|^2_h c(-\Psi)\geq 0
$$

(By considering $\alpha \to 0$) implies

$$
\Re\int_{\{\Psi<-t\}}\langle F_t,\bar{q}\rangle_h c(-\Psi)=0,
$$

then we have

$$
\int_{\{\Psi<-t\}} |F_t + q|_h^2 c(-\Psi) = \int_{\{\Psi<-t\}} (|F_t|_h^2 + |q|_h^2) c(-\Psi).
$$

Letting $q = \hat{F} - F_t$, we obtain equality [\(3.3\)](#page-25-0).

.

The following lemma shows the lower semicontinuity property of $G(t)$.

Lemma 3.6. $G(t)$ is decreasing with respect to $t \in [T, +\infty)$, such that $\lim_{t \to t_0+0} G(t) =$ $G(t_0)$ for any $t_0 \in [T, +\infty)$, and if $G(t) < +\infty$ for some $t > T$, then $\lim_{t \to +\infty} G(t) = 0$. Especially, $G(t)$ is lower semicontinuous on $[T, +\infty)$.

Proof. By the definition of $G(t)$, it is clear that $G(t)$ is decreasing on $[T, +\infty)$. If $G(t) < +\infty$ for some $t > T$, by the dominated convergence theorem, we know $\lim_{t\to+\infty} G(t) = 0$. It suffices to prove $\lim_{t\to t_0+0} G(t) = G(t_0)$. We prove it by contradiction: if not, then $\lim_{t \to t_0+0} G(t) < G(t_0)$.

By using Lemma [3.5,](#page-24-0) for any $t > t_0$, there exists a unique E-valued holomorphic $(n, 0)$ form F_t on $\{\Psi < -t\}$ satisfying $\int_{\{\Psi<-t\}} |F_t|_h^2 c(-\Psi) = G(t)$ and $(F_t - f) \in$ $\mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$ for any $z_0 \in Z_0$. Note that $G(t)$ is decreasing with respect to t. We have $\int_{\{\Psi<-t\}}|F_t|_h^2 c(-\Psi) \le \lim_{t\to t_0+0} G(t)$ for any $t > t_0$. If $\lim_{t\to t_0+0} G(t) = +\infty$, the equality $\lim_{t\to t_0+0} G(t) = G(t_0)$ obviously holds, thus it suffices to prove the case $\lim_{t\to t_0+0} G(t) < +\infty$. It follows from $\int_{\{\Psi<-t\}} |F_t|_h^2 c(-\Psi) \leq \lim_{t\to t_0+0} G(t) < +\infty$ holds for any $t \in (t_0, t_1]$ (where $t_1 > t_0$ is a fixed number) and Lemma [3.3](#page-23-2) that there exists a subsequence of ${F_t}$ (denoted by ${F_{t_i}}$) compactly convergent to an Evalued holomorphic $(n,0)$ form \hat{F}_{t_0} on $\{\Psi < -t_0\}$ satisfying

$$
\int_{\{\Psi<-t_0\}}|\hat{F}_{t_0}|^2_{h}c(-\Psi)\leq \lim_{t\to t_0+0}G(t)<+\infty
$$

and $(\hat{F}_{t_0} - f)_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$ for any $z_0 \in Z_0$.

Then we obtain that $G(t_0) \leq \int_{\{\Psi<-t_0\}} |\hat{F}_{t_0}|^2_{h} c(-\Psi) \leq \lim_{t \to t_0+0} G(t)$, which contradicts $\lim_{t\to t_0+0} G(t) < G(t_0)$. Thus we have $\lim_{t\to t_0+0} G(t) = G(t_0)$.

We consider the derivatives of $G(t)$ in the following lemma.

Lemma 3.7. Assume that $G(t_1) < +\infty$, where $t_1 \in (T, +\infty)$. Then for any $t_0 > t_1$, we have

$$
\frac{G(t_1) - G(t_0)}{\int_{t_1}^{t_0} c(t)e^{-t}dt} \le \liminf_{B \to 0+0} \frac{G(t_0) - G(t_0 + B)}{\int_{t_0}^{t_0 + B} c(t)e^{-t}dt},
$$

i.e.

$$
\frac{G(t_0) - G(t_1)}{\int_{T_1}^{t_0} c(t)e^{-t}dt - \int_{T_1}^{t_1} c(t)e^{-t}dt} \ge \limsup_{B \to 0+0} \frac{G(t_0 + B) - G(t_0)}{\int_{T_1}^{t_0 + B} c(t)e^{-t}dt - \int_{T_1}^{t_0} c(t)e^{-t}dt}
$$

Proof. It follows from Lemma [3.6](#page-26-0) that $G(t) < +\infty$ for any $t > t_1$. By Lemma [3.5,](#page-24-0) there exists an E-valued holomorphic $(n, 0)$ form F_{t_0} on $\{\Psi < -t_0\}$, such that $(F_{t_0}-f)_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$ for any $z_0 \in Z_0$ and $G(t_0) = \int_{\{\Psi<-t_0\}} |F_{t_0}|^2_{h} c(-\Psi)$.

It suffices to consider that $\liminf_{B\to 0+0}$ $\frac{G(t_0)-G(t_0+B)}{f_0^{t_0+B}c(t)e^{-t}dt} \in [0,+\infty)$ because of the decreasing property of $G(t)$. Then there exists $1 \geq B_j \to 0 + 0$ (as $j \to +\infty$) such that

$$
\lim_{j \to +\infty} \frac{G(t_0) - G(t_0 + B_j)}{\int_{t_0}^{t_0 + B_j} c(t) e^{-t} dt} = \liminf_{B \to 0+0} \frac{G(t_0) - G(t_0 + B)}{\int_{t_0}^{t_0 + B} c(t) e^{-t} dt}
$$
(3.4)

and $\{\frac{G(t_0)-G(t_0+B_j)}{f_{t_0}^{t_0+B_j}c(t)e^{-t}dt}\}_{j\in\mathbb{N}^+}$ is bounded. As $c(t)e^{-t}$ is decreasing and positive on $(t, +\infty)$, then

$$
\lim_{j \to +\infty} \frac{G(t_0) - G(t_0 + B_j)}{\int_{t_0}^{t_0 + B_j} c(t)e^{-t}dt} = \left(\lim_{j \to +\infty} \frac{G(t_0) - G(t_0 + B_j)}{B_j}\right) \left(\frac{1}{\lim_{t \to t_0 + 0} c(t)e^{-t}}\right)
$$

$$
= \left(\lim_{j \to +\infty} \frac{G(t_0) - G(t_0 + B_j)}{B_j}\right) \left(\frac{e^{t_0}}{\lim_{t \to t_0 + 0} c(t)}\right).
$$
(3.5)

Hence $\{\frac{G(t_0)-G(t_0+B_j)}{B_j}\}_{j\in\mathbb{N}^+}$ is uniformly bounded with respect to j. As $t \leq v_{t_0,j}(t)$, the decreasing property of $c(t)e^{-t}$ shows that

$$
e^{-\Psi + v_{t_0,B_j}(\Psi)}c(-v_{t_0,B_j}(\Psi)) \ge c(-\Psi).
$$

It follows from Lemma [2.3](#page-10-0) that, for any B_j , there exists an E-valued holomorphic $(n, 0)$ form \tilde{F}_j on $\{\Psi < -t_1\}$ such that

$$
\int_{\{\Psi<-t_{1}\}}|\tilde{F}_{j} - (1 - b_{t_{0},B_{j}}(\Psi))F_{t_{0}}|_{h}^{2}c(-\Psi) \n\leq \int_{\{\Psi<-t_{1}\}}|\tilde{F}_{j} - (1 - b_{t_{0},B_{j}}(\Psi))F_{t_{0}}|_{h}^{2}e^{-\Psi+v_{t_{0},B_{j}}(\Psi)}c(-v_{t_{0},B_{j}}(\Psi)) \n\leq \int_{t_{1}}^{t_{0}+B_{j}}c(t)e^{-t}dt \int_{\{\Psi<-t_{1}\}}\frac{1}{B_{j}}\mathbb{I}_{\{-t_{0}-B_{j}<\Psi<-t_{0}\}}|F_{t_{0}}|_{h}^{2}e^{-\Psi} \n\leq \frac{e^{t_{0}+B_{j}}\int_{t_{1}}^{t_{0}+B_{j}}c(t)e^{-t}dt}{\inf_{t\in(t_{0},t_{0}+B_{j})}(t)}\int_{\{\Psi<-t_{1}\}}\frac{1}{B_{j}}\mathbb{I}_{\{-t_{0}-B_{j}<\Psi<-t_{0}\}}|F_{t_{0}}|_{h}^{2}c(-\Psi) \n=\frac{e^{t_{0}+B_{j}}\int_{t_{1}}^{t_{0}+B_{j}}c(t)e^{-t}dt}{\inf_{t\in(t_{0},t_{0}+B_{j})}(t)}\times \left(\int_{\{\Psi<-t_{1}\}}\frac{1}{B_{j}}\mathbb{I}_{\{\Psi<-t_{0}\}}|F_{t_{0}}|_{h}^{2}c(-\Psi) \right) \n-\int_{\{\Psi<-t_{1}\}}\frac{1}{B_{j}}\mathbb{I}_{\{\Psi<-t_{0}-B_{j}\}}|F_{t_{0}}|_{h}^{2}c(-\Psi)\right) \n\leq \frac{e^{t_{0}+B_{j}}\int_{t_{1}}^{t_{0}+B_{j}}c(t)e^{-t}dt}{\inf_{t\in(t_{0},t_{0}+B_{j})}(t)}\times \frac{G(t_{0})-G(t_{0}+B_{j})}{B_{j}}<+\infty.
$$
\n(3.6)

Note that $b_{t_0,B_j}(t) = 0$ for $t \le -t_0 - B_j$, $b_{t_0,B_j}(t) = 1$ for $t \ge t_0$, $v_{t_0,B_j}(t) >$ $-t_0 - B_j$ and $c(t)e^{-t}$ is decreasing with respect to t. It follows from inequality [\(3.6\)](#page-27-0) that $(F_j - F_{t_0})_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes I(h, \Psi)_{z_0} \subset \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$ for any $z_0 \in Z_0$. Note that

$$
\int_{\{\Psi<-t_1\}} |\tilde{F}_j|_h^2 c(-\Psi) \n\leq 2 \int_{\{\Psi<-t_1\}} |\tilde{F}_j - (1 - b_{t_0, B_j}(\Psi)) F_{t_0}|_h^2 c(-\Psi) + 2 \int_{\{\Psi<-t_1\}} |(1 - b_{t_0, B_j}(\Psi)) F_{t_0}|_h^2 c(-\Psi) \n\leq 2 \frac{e^{t_0 + B_j} \int_{t_1}^{t_0 + B_j} c(t) e^{-t} dt}{\inf_{t \in (t_0, t_0 + B_j)} c(t)} \times \frac{G(t_0) - G(t_0 + B_j)}{B_j} + 2 \int_{\{\Psi<-t_0\}} |F_{t_0}|_h^2 c(-\Psi). \tag{3.7}
$$

We also note that $B_j \leq 1$, $\frac{G(t_0) - G(t_0 + B_j)}{B_j}$ is uniformly bounded with respect to j and $G(t_0) = \int_{\{\Psi<-t_0\}} |F_{t_0}|^2_{h} c(-\Psi)$. It follows from inequality [\(3.7\)](#page-28-0) that we know $\int_{\{\Psi<-t_1\}} |\tilde{F}_j|^2 e^{-\varphi} c(-\Psi)$ is uniformly bounded with respect to j.

It follows from Lemma [3.3](#page-23-2) that there exists a subsequence of $\{\tilde{F}_j\}_{j\in\mathbb{N}^+}$ compactly convergent to an E-valued holomorphic $(n, 0)$ form \tilde{F}_{t_1} on $\{\Psi < -t_1\}$ which satisfies

$$
\int_{\{\Psi<-t_1\}} |\tilde{F}_{t_1}|^2_{h} c(-\Psi) \le \liminf_{j \to +\infty} \int_{\{\Psi<-t_1\}} |\tilde{F}_j|^2_{h} c(-\Psi) < +\infty,
$$

and $(\tilde{F}_{t_1} - F_{t_0})_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$ for any $z_0 \in Z_0$. Note that $\lim_{j\to+\infty} b_{t_0,B_j}(t) = \mathbb{I}_{\{t\geq -t_0\}}$ and

$$
v_{t_0}(t) := \lim_{j \to +\infty} v_{t_0, B_j}(t) = \begin{cases} -t_0 & \text{if } x < -t_0, \\ t & \text{if } x \ge t_0. \end{cases}
$$

It follows from inequality [\(3.6\)](#page-27-0) and Fatou's lemma that

$$
\int_{\{\Psi<-t_0\}} |\tilde{F}_{t_1} - F_{t_0}|_h^2 c(-\Psi) + \int_{\{-t_0 \le \Psi<-t_1\}} |\tilde{F}_{t_1}|_h^2 c(-\Psi)
$$
\n
$$
\le \int_{\{\Psi<-t_1\}} |\tilde{F}_{t_1} - \mathbb{I}_{\{\Psi<-t_0\}} F_{t_0}|_h^2 e^{-\Psi+v_{t_0}(\Psi)} c(-v_{t_0}(\Psi))
$$
\n
$$
\le \liminf_{j \to +\infty} \int_{\{\Psi<-t_1\}} |\tilde{F}_j - (1 - b_{t_0, B_j}(\Psi)) F_{t_0}|_h^2 c(-\Psi)
$$
\n
$$
\le \liminf_{j \to +\infty} \left(\frac{e^{t_0 + B_j} \int_{t_1}^{t_0 + B_j} c(t) e^{-t} dt}{\inf_{t \in (t_0, t_0 + B_j)} c(t)} \times \frac{G(t_0) - G(t_0 + B_j)}{B_j} \right). \tag{3.8}
$$

It follows from Lemma [3.5,](#page-24-0) equality [\(3.4\)](#page-27-1), equality [\(3.5\)](#page-27-2) and inequality [\(3.8\)](#page-28-1) that we have

$$
\int_{\{\Psi<-t_{1}\}}|\tilde{F}_{t_{1}}|_{h}^{2}c(-\Psi)-\int_{\{\Psi<-t_{0}\}}|F_{t_{0}}|_{h}^{2}c(-\Psi)
$$
\n
$$
\leq \int_{\{\Psi<-t_{0}\}}|\tilde{F}_{t_{1}}-F_{t_{0}}|_{h}^{2}c(-\Psi)+\int_{\{-t_{0}\leq \Psi<-t_{1}\}}|\tilde{F}_{t_{1}}|_{h}^{2}c(-\Psi)
$$
\n
$$
\leq \int_{\{\Psi<-t_{1}\}}|\tilde{F}_{t_{1}}-\mathbb{I}_{\{\Psi<-t_{0}\}}F_{t_{0}}|_{h}^{2}e^{-\Psi+v_{t_{0}}(\Psi)}c(-v_{t_{0}}(\Psi))
$$
\n
$$
\leq \liminf_{j\to+\infty}\int_{\{\Psi<-t_{1}\}}|\tilde{F}_{j}-(1-b_{t_{0},B_{j}}(\Psi))F_{t_{0}}|_{h}^{2}c(-\Psi)
$$
\n
$$
\leq \liminf_{j\to+\infty}\left(\frac{e^{t_{0}+B_{j}}\int_{t_{1}}^{t_{0}+B_{j}}c(t)e^{-t}dt}{\inf_{t\in(t_{0},t_{0}+B_{j})}c(t)}\times\frac{G(t_{0})-G(t_{0}+B_{j})}{B_{j}}\right)
$$
\n
$$
\leq \left(\int_{t_{1}}^{t_{0}}c(t)e^{-t}dt\right)\liminf_{B\to0+0}\frac{G(t_{0})-G(t_{0}+B)}{\int_{t_{0}}^{t_{0}+B}c(t)e^{-t}dt}.
$$
\n(3.9)

Note that $(\tilde{F}_{t_1} - F_{t_0})_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$ for any $z_0 \in Z_0$. It follows from the definition of $G(t)$ and inequality (3.9) that we have

$$
G(t_{1}) - G(t_{0})
$$
\n
$$
\leq \int_{\{\Psi<-t_{1}\}} |\tilde{F}_{t_{1}}|_{h}^{2} c(-\Psi) - \int_{\{\Psi<-t_{0}\}} |F_{t_{0}}|_{h}^{2} c(-\Psi)
$$
\n
$$
\leq \int_{\{\Psi<-t_{1}\}} |\tilde{F}_{t_{1}} - \mathbb{I}_{\{\Psi<-t_{0}\}} F_{t_{0}}|_{h}^{2} c(-\Psi)
$$
\n
$$
\leq \int_{\{\Psi<-t_{1}\}} |\tilde{F}_{t_{1}} - \mathbb{I}_{\{\Psi<-t_{0}\}} F_{t_{0}}|_{h}^{2} e^{-\Psi+v_{t_{0}}(\Psi)} c(-v_{t_{0}}(\Psi))
$$
\n
$$
\leq (\int_{t_{1}}^{t_{0}} c(t) e^{-t} dt) \liminf_{B \to 0+0} \frac{G(t_{0}) - G(t_{0} + B)}{\int_{t_{0}}^{t_{0} + B} c(t) e^{-t} dt}.
$$
\n(3.10)

Lemma [3.7](#page-26-1) is proved. \square

The following property of concave functions will be used in the proof of Theorem [1.8.](#page-4-0)

Lemma 3.8 (see [\[24\]](#page-72-17)). Let $H(r)$ be a lower semicontinuous function on $(0, R]$. Then $H(r)$ is concave if and only if

$$
\frac{H(r_1) - H(r_2)}{r_1 - r_2} \le \liminf_{r_3 \to r_2 \to 0} \frac{H(r_3) - H(r_2)}{r_3 - r_2}
$$

holds for any $0 < r_2 < r_1 \leq R$.

4. Proof of Theorem [1.8,](#page-4-0) Remark [1.9,](#page-4-1) Corollary [1.10](#page-4-2) and Remark [1.11](#page-4-3)

We firstly prove Theorem [1.8.](#page-4-0)

Proof. We firstly show that if $G(t_0) < +\infty$ for some $t_0 > T$, then $G(t_1) < +\infty$ for any $T < t_1 < t_0$. As $G(t_0) < +\infty$, it follows from Lemma [3.5](#page-24-0) that there exists an unique E-valued holomorphic $(n, 0)$ form F_{t_0} on ${\Psi < -t}$ satisfying

$$
\int_{\{\Psi<-t_0\}} |F_{t_0}|_h^2 c(-\Psi) = G(t_0) < +\infty
$$

and $(F_{t_0} - f)_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$, for any $z_0 \in Z_0$.

It follows from Lemma [2.3](#page-10-0) that there exists an E -valued holomorphic $(n, 0)$ form \tilde{F}_1 on $\{\Psi < -t_1\}$ such that

$$
\int_{\{\Psi<-t_1\}} |\tilde{F}_1 - (1 - b_{t_0,B}(\Psi)) F_{t_0}|_h^2 e^{v_{t_0,B}(\Psi) - \Psi} c(-v_{t_0,B}(\Psi))
$$
\n
$$
\leq (\int_{t_1}^{t_0+B} c(s) e^{-s} ds) \int_M \frac{1}{B} \mathbb{I}_{\{-t_0-B<\Psi<-t_0\}} |F_{t_0}|_h^2 e^{-\Psi} < +\infty.
$$
\n
$$
(4.1)
$$

Note that $b_{t_0,B}(t) = 0$ on $\{\Psi < -t_0 - B\}$ and $v_{t_0,B}(\Psi) > -t_0 - B$. We have $e^{v_{t_0,B}(\Psi)}c(-v_{t_0,\underline{B}}(\Psi))$ has a positive lower bound. It follows from inequality [\(4.1\)](#page-30-0) that we have $(\tilde{F}_1 - F_{t_0})_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes I(h, \Psi)_{z_0} \subset \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$ for any $z_0 \in Z_0$, which implies that $(\tilde{F}_1 - f)_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$, for any $z_0 \in Z_0$. As $v_{t_0,B}(\Psi) \ge \Psi$ and $c(t)e^{-t}$ is decreasing with respect to t, it follows from inequality [\(4.1\)](#page-30-0) that we have

$$
\int_{\{\Psi<-t_1\}} |\tilde{F}_1 - (1 - b_{t_0,B}(\Psi)) F_{t_0}|_h^2 c(-\Psi)
$$
\n
$$
\leq \int_{\{\Psi<-t_1\}} |\tilde{F}_1 - (1 - b_{t_0,B}(\Psi)) F_{t_0}|_h^2 e^{v_{t_0,B}(\Psi) - \Psi} c(-v_{t_0,B}(\Psi))
$$
\n
$$
\leq (\int_{t_1}^{t_0+B} c(s) e^{-s} ds) \int_M \frac{1}{B} \mathbb{I}_{\{-t_0-B<\Psi<-t_0\}} |F_{t_0}|_h^2 e^{-\Psi} < +\infty.
$$
\n(4.2)

Then we have

$$
\int_{\{\Psi<-t_1\}} |\tilde{F}_1|^2_{h} c(-\Psi)
$$
\n
$$
\leq 2 \int_{\{\Psi<-t_1\}} |\tilde{F}_1 - (1 - b_{t_0,B}(\Psi)) F_{t_0}|_{h}^2 c(-\Psi) + 2 \int_{\{\Psi<-t_1\}} |(1 - b_{t_0,B}(\Psi)) F_{t_0}|_{h}^2 c(-\Psi)
$$
\n
$$
\leq 2 \Big(\int_{t_1}^{t_0+B} c(s) e^{-s} ds \Big) \int_M \frac{1}{B} \mathbb{I}_{\{-t_0-B<\Psi<-t_0\}} |F_{t_0}|_{h}^2 e^{-\Psi} + 2 \int_{\{\Psi<-t_0\}} |F_{t_0}|_{h}^2 c(-\Psi)
$$
\n
$$
< + \infty.
$$
\n(4.3)

Hence we have $G(t_1) \leq \int_{\{\Psi<-t_1\}} |\tilde{F}_1|^2 h c(-\Psi) < +\infty$.

Now, it follows from Lemma [3.6,](#page-26-0) Lemma [3.7](#page-26-1) and Lemma [3.8](#page-29-1) that we know $G(h^{-1}(r))$ is concave with respect to r. It follows from Lemma [3.6](#page-26-0) that $\lim_{t \to T+0} G(t) =$ $G(T)$ and $\lim_{t\to+\infty} G(t) = 0.$

Theorem [1.8](#page-4-0) is proved.

Now we prove Remark [1.9.](#page-4-1)

Proof. Note that if there exists a positive decreasing concave function $g(t)$ on $(a, b) \subset \mathbb{R}$ and $g(t)$ is not a constant function, then $b < +\infty$.

Assume that $G(t_0) < +\infty$ for some $t_0 \geq T$. As $f_{z_0} \notin \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$ for some $z_0 \in Z_0$, Lemma [3.4](#page-24-1) shows that $G(t_0) \in (0, +\infty)$. Following from Theorem [1.8](#page-4-0) we know $G(h^{-1}(r))$ is concave with respect to $r \in \left(\int_{T_1}^T c(t)e^{-t}dt, \int_{T_1}^{+\infty} c(t)e^{-t}dt\right)$ and $G(h^{-1}(r))$ is not a constant function, therefore we obtain $\int_{T_1}^{+\infty} c(t)e^{-t}dt < +\infty$, which contradicts to $\int_{T_1}^{+\infty} c(t)e^{-t}dt = +\infty$. Thus we have $G(t) \equiv +\infty$.

When $G(t_2) \in (0, +\infty)$ for some $t_2 \in [T, +\infty)$, Lemma [3.4](#page-24-1) shows that $f_{z_0} \notin$ $\mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$, for any $z_0 \in Z_0$. Combining the above discussion, we know $\int_{T_1}^{+\infty} c(t)e^{-t}dt < +\infty$. Using Theorem [1.8,](#page-4-0) we obtain that $G(\hat{h}^{-1}(r))$ is concave with respect to $r \in (0, \int_T^{+\infty} c(t)e^{-t}dt)$, where $\hat{h}(t) = \int_t^{+\infty} c(t)e^{-t}dt$. Thus, Remark [1.9](#page-4-1) holds.

Now we prove Corollary [1.10.](#page-4-2)

Proof. As $G(h^{-1}(r))$ is linear with respect to $r \in [0, \int_T^{+\infty} c(s)e^{-s}ds)$, we have $G(t) = \frac{G(T_1)}{\int_{T_1}^{+\infty} c(s)e^{-s}ds} \int_t^{+\infty} c(s)e^{-s}ds$ for any $t \in [T, +\infty)$ and $T_1 \in (T, +\infty)$.

We follow the notation and the construction in Lemma [3.7.](#page-26-1) Let $t_0 > t_1 > T$ be given. It follows from $G(h^{-1}(r))$ is linear with respect to $r \in [0, \int_T^{+\infty} c(s)e^{-s}ds)$ that we know that all inequalities in [\(3.10\)](#page-29-2) should be equalities, i.e., we have

$$
G(t_1) - G(t_0)
$$

=
$$
\int_{\{\Psi<-t_1\}} |\tilde{F}_{t_1}|_h^2 c(-\Psi) - \int_{\{\Psi<-t_0\}} |F_{t_0}|_h^2 c(-\Psi)
$$

=
$$
\int_{\{\Psi<-t_1\}} |\tilde{F}_{t_1} - \mathbb{I}_{\{\Psi<-t_0\}} F_{t_0}|_h^2 c(-\Psi)
$$

=
$$
\int_{\{\Psi<-t_1\}} |\tilde{F}_{t_1} - \mathbb{I}_{\{\Psi<-t_0\}} F_{t_0}|_h^2 e^{-\Psi+v_{t_0}(\Psi)} c(-v_{t_0}(\Psi))
$$

=
$$
(\int_{t_1}^{t_0} c(t) e^{-t} dt) \liminf_{B \to 0+0} \frac{G(t_0) - G(t_0 + B)}{\int_{t_0}^{t_0 + B} c(t) e^{-t} dt}.
$$
 (4.4)

Note that $G(t_0) = \int_{\{\Psi<-t_0\}} |F_{t_0}|^2_{h} c(-\Psi)$. Equality [\(4.4\)](#page-31-0) shows that $G(t_1) =$ $\int_{\{\Psi<-t_1\}} |\tilde{F}_{t_1}|^2_{h} c(-\Psi).$

Note that on $\{\Psi \geq -t_0\}$, we have $e^{-\Psi + v_{t_0}(\Psi)}c(-v_{t_0}(\Psi)) = c(-\Psi)$. It follows from

$$
\int_{\{\Psi<-t_1\}} |\tilde{F}_{t_1} - \mathbb{I}_{\{\Psi<-t_0\}} F_{t_0}|^2_{h} c(-\Psi) \n= \int_{\{\Psi<-t_1\}} |\tilde{F}_{t_1} - \mathbb{I}_{\{\Psi<-t_0\}} F_{t_0}|^2_{h} e^{-\Psi+v_{t_0}(\Psi)} c(-v_{t_0}(\Psi))
$$

that we have (note that $v_{t_0}(\Psi) = -t_0$ on $\{\Psi < -t_0\}$)

$$
\int_{\{\Psi<-t_0\}} |\tilde{F}_{t_1} - F_{t_0}|_h^2 c(-\Psi)
$$
\n
$$
= \int_{\{\Psi<-t_0\}} |\tilde{F}_{t_1} - F_{t_0}|_h^2 e^{-\Psi-t_0} c(t_0).
$$
\n(4.5)

As $\int_T^{+\infty} c(t)e^{-t}dt < +\infty$ and $c(t)e^{-t}$ is decreasing with respect to t, we know that there exists $t_2 > t_0$ such that $c(t)e^{-t} < c(t_0)e^{-t_0} - \epsilon$ for any $t \ge t_2$, where $\epsilon > 0$ is a constant. Then equality [\(4.5\)](#page-31-1) implies that

$$
\epsilon \int_{\{\Psi<-t_2\}} |\tilde{F}_{t_1} - F_{t_0}|_h^2 e^{-\Psi} \n\leq \int_{\{\Psi<-t_2\}} |\tilde{F}_{t_1} - F_{t_0}|_h^2 (e^{-\Psi-t_0} c(t_0) - c(-\Psi)) \n\leq \int_{\{\Psi<-t_0\}} |\tilde{F}_{t_1} - F_{t_0}|_h^2 (e^{-\Psi-t_0} c(t_0) - c(-\Psi)) \n= 0.
$$
\n(4.6)

Note that for any relatively compact subset $K \subset {\Psi < -t_2}, |\tilde{F}_{t_1} - F_{t_0}|_h^2 e^{-\Psi} =$ $|(\tilde{F}_{t_1} - F_{t_0})F|^2_h e^{-\psi} = |\tilde{F}_{t_1}F - F_{t_0}F|^2_{\tilde{h}} \geq |\tilde{F}_{t_1}F - F_{t_0}F|^2_{\tilde{h}_{K,1}}$ on K, and the integrand in [\(4.6\)](#page-32-0) is nonnegative, we must have $\tilde{F}_{t_1}|_{\{\Psi<-t_0\}}=F_{t_0}$.

It follows from Lemma [3.5](#page-24-0) that for any $t > T$, there exists an unique E-valued holomorphic $(n, 0)$ form F_t on $\{\Psi < -t\}$ satisfying

$$
\int_{\{\Psi<-t\}}|F_t|_h^2 c(-\Psi)=G(t)
$$

and $(F_t - f)_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$, for any $z_0 \in Z_0$. By the above discussion, we know $F_t = F_{t'}$ on $\{\Psi \leq -\max\{t, t'\}\}\$ for any $t \in (T, +\infty)$ and $t' \in (T, +\infty)$. Hence combining $\lim_{t\to T+0} G(t) = G(T)$, we obtain that there exists an unique E-valued holomorphic $(n,0)$ form \tilde{F} on $\{\Psi \leq -T\}$ satisfying $(\tilde{F} - f)_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$ for any $z_0 \in Z_0$ and $G(t) = \int_{\{\Psi<-t\}} |\tilde{F}|_h^2 c(-\Psi)$ for any $t \geq T$.

Secondly, we prove equality (1.2) .

As $a(t)$ is a nonnegative measurable function on $(T, +\infty)$, then there exists a sequence of functions $\left\{\sum_{i=1}^{n_i}\right\}$ $\sum_{j=1}^{n} a_{ij} \mathbb{I}_{E_{ij}} \}_{i \in \mathbb{N}^{+}}$ ($n_i < +\infty$ for any $i \in \mathbb{N}^{+}$) satisfying that $\sum_{i=1}^{n}$ $\sum_{j=1}^{n} a_{ij} \mathbb{I}_{E_{ij}}$ is increasing with respect to i and $\lim_{i \to +\infty}$ $\frac{n_i}{\sum}$ $\sum_{j=1}^{\infty} a_{ij} \mathbb{I}_{E_{ij}} = a(t)$ for any $t \in (T, +\infty)$, where E_{ij} is a Lebesgue measurable subset of $(T, +\infty)$ and $a_{ij} \geq 0$ is a constant for any i, j . It follows from Levi's Theorem that it suffices to prove the case that $a(t) = \mathbb{I}_E(t)$, where $E \subset \subset (T, +\infty)$ is a Lebesgue measurable set.

Note that $G(t) = \int_{\{\Psi<-t\}} |\tilde{F}|^2_h c(-\Psi) = \frac{G(T_1)}{\int_{T_1}^{+\infty} c(s)e^{-s}ds} \int_t^{+\infty} c(s)e^{-s}ds$ where $T_1 \in$ $(T, +\infty)$, then

$$
\int_{\{-t_1 \le \Psi < -t_2\}} |\tilde{F}|_h^2 c(-\Psi) = \frac{G(T_1)}{\int_{T_1}^{+\infty} c(s)e^{-s}ds} \int_{t_2}^{t_1} c(s)e^{-s}ds \tag{4.7}
$$

holds for any $T \leq t_2 < t_1 < +\infty$. It follows from the dominated convergence theorem and equality [\(4.7\)](#page-32-1) that

$$
\int_{\{z \in M : -\Psi(z) \in N\}} |\tilde{F}|_h^2 = 0
$$
\n(4.8)

holds for any $N \subset \subset (T, +\infty)$ such that $\mu(N) = 0$, where μ is the Lebesgue measure on R.

As $c(t)e^{-t}$ is decreasing on $(T, +\infty)$, there are at most countable points denoted by $\{s_j\}_{j\in\mathbb{N}^+}$ such that $c(t)$ is not continuous at s_j . Then there is a decreasing sequence of open sets $\{U_k\}$, such that $\{s_j\}_{j\in\mathbb{N}^+}\subset U_k\subset (T, +\infty)$ for any k, and $\lim_{k \to +\infty} \mu(U_k) = 0$. Choosing any closed interval $[t'_2, t'_1] \subset (T, +\infty)$, then we have

$$
\int_{\{-t'_1 \le \Psi < -t'_2\}} |\tilde{F}|_h^2
$$
\n
$$
= \int_{\{z \in M: -\Psi(z) \in (t'_2, t'_1] \backslash U_k\}} |\tilde{F}|_h^2 + \int_{\{z \in M: -\Psi(z) \in [t'_2, t'_1] \cap U_k\}} |\tilde{F}|_h^2
$$
\n
$$
= \lim_{n \to +\infty} \sum_{i=0}^{n-1} \int_{\{z \in M: -\Psi(z) \in I_{i,n} \backslash U_k\}} |\tilde{F}|_h^2 + \int_{\{z \in M: -\Psi(z) \in [t'_2, t'_1] \cap U_k\}} |\tilde{F}|_h^2,
$$
\n(4.9)

where $I_{i,n} = (t'_1 - (i+1)\alpha_n, t'_1 - i\alpha_n]$ and $\alpha_n = \frac{t'_1 - t'_2}{n}$. Note that

$$
\lim_{n \to +\infty} \sum_{i=0}^{n-1} \int_{\{z \in M : -\Psi(z) \in I_{i,n} \setminus U_k\}} |\tilde{F}|_h^2
$$
\n
$$
\leq \limsup_{n \to +\infty} \sum_{i=0}^{n-1} \frac{1}{\inf_{I_{i,n} \setminus U_k} c(t)} \int_{\{z \in M : -\Psi(z) \in I_{i,n} \setminus U_k\}} |\tilde{F}|_h^2 c(-\Psi).
$$
\n(4.10)

It follows from equality [\(4.7\)](#page-32-1) that inequality [\(4.10\)](#page-33-0) becomes

$$
\lim_{n \to +\infty} \sum_{i=0}^{n-1} \int_{\{z \in M : -\Psi(z) \in I_{i,n} \setminus U_k\}} |\tilde{F}|_h^2
$$
\n
$$
\leq \frac{G(T_1)}{\int_{T_1}^{+\infty} c(s)e^{-s}ds} \limsup_{n \to +\infty} \sum_{i=0}^{n-1} \frac{1}{\inf_{I_{i,n} \setminus U_k} c(t)} \int_{I_{i,n} \setminus U_k} c(s)e^{-s}ds.
$$
\n(4.11)

It is clear that $c(t)$ is uniformly continuous and has positive lower bound and upper bound on $[t'_2, t'_1] \backslash U_k$. Then we have

$$
\limsup_{n \to +\infty} \sum_{i=0}^{n-1} \frac{1}{\inf_{I_{i,n} \setminus U_k} c(t)} \int_{I_{i,n} \setminus U_k} c(s) e^{-s} ds
$$
\n
$$
\leq \limsup_{n \to +\infty} \sum_{i=0}^{n-1} \frac{\sup_{I_{i,n} \setminus U_k} c(t)}{\inf_{I_{i,n} \setminus U_k} c(t)} \int_{I_{i,n} \setminus U_k} e^{-s} ds
$$
\n
$$
= \int_{(t'_2, t'_1] \setminus U_k} e^{-s} ds.
$$
\n(4.12)

Combining inequality (4.9) , (4.11) and (4.12) , we have

$$
\int_{\{-t'_1 \le \Psi < -t'_2\}} |\tilde{F}|_h^2
$$
\n
$$
= \int_{\{z \in M: -\Psi(z) \in (t'_2, t'_1] \backslash U_k\}} |\tilde{F}|_h^2 + \int_{\{z \in M: -\Psi(z) \in [t'_2, t'_1] \cap U_k\}} |\tilde{F}|_h^2 \qquad (4.13)
$$
\n
$$
\le \frac{G(T_1)}{\int_{T_1}^{+\infty} c(s)e^{-s}ds} \int_{(t'_2, t'_1] \backslash U_k} e^{-s}ds + \int_{\{z \in M: -\Psi(z) \in [t'_2, t'_1] \cap U_k\}} |\tilde{F}|_h^2.
$$

Let $k \to +\infty$, following from equality [\(4.8\)](#page-32-2) and inequality [\(4.13\)](#page-33-4), then we obtain that

$$
\int_{\{-t'_1 \le \Psi < -t'_2\}} |\tilde{F}|_h^2 \le \frac{G(T_1)}{\int_{T_1}^{+\infty} c(s)e^{-s}ds} \int_{t'_2}^{t'_1} e^{-s}ds.
$$
\n(4.14)

Following from a similar discussion we can obtain that

$$
\int_{\{-t'_1 \le \Psi < -t'_2\}} |\tilde{F}|^2_h \ge \frac{G(T_1)}{\int_{T_1}^{+\infty} c(s)e^{-s}ds} \int_{t'_2}^{t'_1} e^{-s}ds.
$$

Then combining inequality [\(4.14\)](#page-34-0), we know

$$
\int_{\{-t'_1 \le \Psi < -t'_2\}} |\tilde{F}|_h^2 = \frac{G(T_1)}{\int_{T_1}^{+\infty} c(s)e^{-s}ds} \int_{t'_2}^{t'_1} e^{-s}ds.
$$
\n(4.15)

Then it is clear that for any open set $U \subset (T, +\infty)$ and compact set $V \subset (T, +\infty)$,

$$
\int_{\{z \in M; -\Psi(z) \in U\}} |\tilde{F}|_h^2 = \frac{G(T_1)}{\int_{T_1}^{+\infty} c(s)e^{-s}ds} \int_U e^{-s}ds,
$$

and

$$
\int_{\{z \in M; -\Psi(z) \in V\}} |\tilde{F}|_{h}^{2} = \frac{G(T_{1})}{\int_{T_{1}}^{+\infty} c(s)e^{-s}ds} \int_{V} e^{-s}ds.
$$

As $E \subset\subset (T, +\infty)$, then $E \cap (t_2, t_1]$ is a Lebesgue measurable subset of $(T + \frac{1}{n}, n)$ for some large n, where $T \leq t_2 < t_1 \leq +\infty$. Then there exists a sequence of compact sets $\{V_j\}$ and a sequence of open subsets $\{V'_j\}$ satisfying $V_1 \subset \ldots \subset$ $V_j \subset V_{j+1} \subset \ldots \subset E \cap (t_2, t_1] \subset \ldots \subset V'_{j+1} \subset V'_j \subset \ldots \subset V'_1 \subset \subset (T, +\infty)$ and $\lim_{j \to +\infty} \mu(V'_j - V_j) = 0$, where μ is the Lebesgue measure on R. Then we have

$$
\int_{\{-t'_1 \le \Psi < -t'_2\}} |\tilde{F}|_h^2 \mathbb{I}_E(-\Psi) = \int_{z \in M: -\Psi(z) \in E \cap (t_2, t_1]} |\tilde{F}|_h^2
$$
\n
$$
\le \liminf_{j \to +\infty} \int_{\{z \in M: -\Psi(z) \in V'_j\}} |\tilde{F}|_h^2
$$
\n
$$
\le \liminf_{j \to +\infty} \frac{G(T_1)}{\int_{T_1}^{+\infty} c(s) e^{-s} ds} \int_{V'_j} e^{-s} ds
$$
\n
$$
\le \frac{G(T_1)}{\int_{T_1}^{+\infty} c(s) e^{-s} ds} \int_{E \cap (t_2, t_1]} e^{-s} ds
$$
\n
$$
= \frac{G(T_1)}{\int_{T_1}^{+\infty} c(s) e^{-s} ds} \int_{t_2}^{t_1} e^{-s} \mathbb{I}_E(s) ds,
$$

and

$$
\begin{aligned} \int_{\{-t'_1\leq \Psi<-t'_2\}} |\tilde{F}|^2_h \mathbb{I}_E(-\Psi) & \geq \liminf_{j\to +\infty} \int_{\{z\in M:-\Psi(z)\in V_j\}} |\tilde{F}|^2_h \\ & \geq \liminf_{j\to +\infty} \frac{G(T_1)}{\int_{T_1}^{+\infty} c(s)e^{-s}ds} \int_{V_j} e^{-s}ds \\ & = \frac{G(T_1)}{\int_{T_1}^{+\infty} c(s)e^{-s}ds} \int_{t_2}^{t_1} e^{-s} \mathbb{I}_E(s)ds, \end{aligned}
$$

which implies that

$$
\int_{\{-t'_1 \le \Psi < -t'_2\}} |\tilde{F}|_h^2 \mathbb{I}_E(-\Psi) = \frac{G(T_1)}{\int_{T_1}^{+\infty} c(s)e^{-s}ds} \int_{t_2}^{t_1} e^{-s} \mathbb{I}_E(s)ds.
$$

Hence we know that equality [\(1.2\)](#page-4-4) holds.

Corollary [1.10](#page-4-2) is proved. \Box

Now we prove Remark [1.11.](#page-4-3)

Proof of Remark [1.11.](#page-4-3) By the definition of $G(t; \tilde{c})$, we have $G(t_0; \tilde{c}) \leq \int_{\{\Psi<-t_0\}} |\tilde{F}|_h^2 \tilde{c}(-\Psi)$, where \tilde{F} is the holomorphic $(n, 0)$ form on $\{\Psi < -T\}$ such that $G(t) = \int_{\{\Psi < -t\}} |\tilde{F}|_h^2 c(-\Psi)$ for any $t \geq T$. Hence we only consider the case $G(t_0; \tilde{c}) < +\infty$.

By the definition of $G(t; \tilde{c})$, we can choose an E-valued holomorphic $(n, 0)$ form $F_{t_0,\tilde{c}}$ on $\{\Psi \leq -t_0\}$ satisfying $(F_{t_0,\tilde{c}}-f)_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes J_{z_0}$, for any $z_0 \in Z_0$ and $\int_{\{\Psi<-t_0\}} |F_{t_0,\tilde{c}}|^2_h \tilde{c}(-\Psi) < +\infty$. As $\mathcal{H}^2(\tilde{c},t_0) \subset \mathcal{H}^2(c,t_0)$, we have $\int_{\{\Psi<-t_0\}} |F_{t_0,\tilde{c}}|^2 \hat{c}(-\Psi) < +\infty$. Using Lemma [3.5,](#page-24-0) we obtain that

$$
\begin{aligned} \int_{\{\Psi<-t\}}|F_{t_0,\tilde{c}}|^2_hc(-\Psi)=&\int_{\{\Psi<-t\}}|\tilde{F}|^2_hc(-\Psi)\\ &+\int_{\{\Psi<-t\}}|F_{t_0,\tilde{c}}-\tilde{F}|^2_hc(-\Psi) \end{aligned}
$$

for any $t \geq t_0$, then

$$
\int_{\{-t_3 \le \Psi < -t_4\}} |F_{t_0,\tilde{c}}|_{h}^{2} c(-\Psi) = \int_{\{-t_3 \le \Psi < -t_4\}} |\tilde{F}|_{h}^{2} c(-\Psi) + \int_{\{-t_3 \le \Psi < -t_4\}} |F_{t_0,\tilde{c}} - \tilde{F}|_{h}^{2} c(-\Psi)
$$
\n(4.16)

holds for any $t_3 > t_4 \geq t_0$. It follows from the dominated convergence theorem, equality [\(4.16\)](#page-35-0), [\(4.8\)](#page-32-2) and $c(t) > 0$ for any $t > T$, that

$$
\int_{\{z \in M : -\Psi(z) = t\}} |F_{t_0, \tilde{c}}|_{h}^{2} = \int_{\{z \in M : -\Psi(z) = t\}} |F_{t_0, \tilde{c}} - \tilde{F}|_{h}^{2}
$$
(4.17)

holds for any $t > t_0$.

Choosing any closed interval $[t'_4, t'_3] \subset (t_0, +\infty) \subset (T, +\infty)$. Note that $c(t)$ is uniformly continuous and have positive lower bound and upper bound on $[t'_4, t'_3] \setminus U_k$, where $\{U_k\}$ is the decreasing sequence of open subsets of $(T, +\infty)$, such that c is continuous on $(T, +\infty) \setminus U_k$ and $\lim_{k \to +\infty} \mu(U_k) = 0$. Take $N = \bigcap_{k=1}^{+\infty} U_k$. Note that

$$
\int_{\{-t'_3 \le \Psi < -t'_4\}} |F_{t_0,\tilde{c}}|_h^2
$$
\n
$$
= \lim_{n \to +\infty} \sum_{i=0}^{n-1} \int_{\{z \in M: -\Psi(z) \in S_{i,n} \setminus U_k\}} |F_{t_0,\tilde{c}}|_h^2 + \int_{\{z \in M: -\Psi(z) \in (t'_4, t'_3] \cap U_k\}} |F_{t_0,\tilde{c}}|_h^2
$$
\n
$$
\le \limsup_{n \to +\infty} \sum_{i=0}^{n-1} \frac{1}{\inf_{S_{i,n}} c(t)} \int_{\{z \in M: -\Psi(z) \in S_{i,n} \setminus U_k\}} |F_{t_0,\tilde{c}}|_h^2 c(-\Psi)
$$
\n
$$
+ \int_{\{z \in M: -\Psi(z) \in (t'_4, t'_3] \cap U_k\}} |F_{t_0,\tilde{c}}|_h^2,
$$
\n(4.18)
where $S_{i,n} = (t'_4 - (i+1)\alpha_n, t'_3 - i\alpha_n]$ and $\alpha_n = \frac{t'_3 - t'_4}{n}$. It follows from equality $(4.16), (4.17), (4.8)$ $(4.16), (4.17), (4.8)$ $(4.16), (4.17), (4.8)$ $(4.16), (4.17), (4.8)$ $(4.16), (4.17), (4.8)$ and the dominated convergence theorem that

$$
\int_{\{z \in M: -\Psi(z) \in S_{i,n} \setminus U_k\}} |F_{t_0,\tilde{c}}|^2_h c(-\Psi) \n= \int_{\{z \in M: -\Psi(z) \in S_{i,n} \setminus U_k\}} |\tilde{F}|^2_h c(-\Psi) + \int_{\{z \in M: -\Psi(z) \in S_{i,n} \setminus U_k\}} |F_{t_0,\tilde{c}} - \tilde{F}|^2_h c(-\Psi).
$$
\n(4.19)

As $c(t)$ is uniformly continuous and have positive lower bound and upper bound on $[t'_3, t'_4] \backslash U_k$, combining equality [\(4.19\)](#page-36-0), we have

$$
\limsup_{n \to +\infty} \sum_{i=0}^{n-1} \frac{1}{\inf_{S_{i,n} \setminus U_k} c(t)} \int_{\{z \in M : -\Psi(z) \in S_{i,n} \setminus U_k\}} |F_{t_0,\tilde{c}}|^2_h c(-\Psi) \n= \limsup_{n \to +\infty} \sum_{i=0}^{n-1} \frac{1}{\inf_{S_{i,n} \setminus U_k} c(t)} \left(\int_{\{z \in M : -\Psi(z) \in S_{i,n} \setminus U_k\}} |\tilde{F}|^2_h c(-\Psi) \right) \n+ \int_{\{z \in M : -\Psi(z) \in S_{i,n} \setminus U_k\}} |F_{t_0,\tilde{c}} - \tilde{F}|^2_h c(-\Psi) \right) \n\leq \limsup_{n \to +\infty} \sum_{i=0}^{n-1} \frac{\sup_{S_{i,n} \setminus U_k} c(t)}{\inf_{S_{i,n} \setminus U_k} c(t)} \left(\int_{\{z \in M : -\Psi(z) \in S_{i,n} \setminus U_k\}} |\tilde{F}|^2_h \right) \n+ \int_{\{z \in M : -\Psi(z) \in S_{i,n} \setminus U_k\}} |F_{t_0,\tilde{c}} - \tilde{F}|^2_h \right) \n= \int_{\{z \in M : -\Psi(z) \in (t'_4, t'_3] \setminus U_k\}} |\tilde{F}|^2_h + \int_{\{z \in M : -\Psi(z) \in (t'_4, t'_3] \setminus U_k\}} |F_{t_0,\tilde{c}} - \tilde{F}|^2_h.
$$
\n(4.20)

If follows from inequality [\(4.18\)](#page-35-2) and [\(4.20\)](#page-36-1) that

$$
\int_{\{-t'_3 \le \Psi < -t'_4\}} |F_{t_0,\tilde{c}}|^2_h
$$
\n
$$
\le \int_{\{z \in M: -\Psi(z) \in (t'_4, t'_3] \setminus U_k\}} |\tilde{F}|_h^2 + \int_{\{z \in M: -\Psi(z) \in (t'_4, t'_3] \setminus U_k\}} |F_{t_0,\tilde{c}} - \tilde{F}|_h^2 \qquad (4.21)
$$
\n
$$
+ \int_{\{z \in M: -\Psi(z) \in (t'_4, t'_3] \cap U_k\}} |F_{t_0,\tilde{c}}|_h^2.
$$

It follows from $F_{t_0,\tilde{c}} \in \mathcal{H}^2(c,t_0)$ that $\int_{\{-t'_3 \le \Psi < -t'_4\}} |F_{t_0,\tilde{c}}|^2 \, \leq +\infty$. Let $k \to +\infty$, by equality [\(4.8\)](#page-32-0), inequality [\(4.21\)](#page-36-2) and the dominated theorem, we have

$$
\int_{\{-t'_3 \le \Psi < -t'_4\}} |F_{t_0, \tilde{c}}|^2 e^{-\varphi} \n\le \int_{\{z \in M: -\Psi(z) \in (t'_4, t'_3]\}} |\tilde{F}|_h^2 + \int_{\{z \in M: -\Psi(z) \in (t'_4, t'_3]\} \backslash N\}} |F_{t_0, \tilde{c}} - \tilde{F}|_h^2 \n+ \int_{\{z \in M: -\Psi(z) \in (t'_4, t'_3] \cap N\}} |F_{t_0, \tilde{c}}|_h^2.
$$
\n(4.22)

By similar discussion, we also have that

Z

$$
\int_{\{-t'_3 \le \Psi < -t'_4\}} |F_{t_0,\tilde{c}}|_h^2
$$
\n
$$
\ge \int_{\{z \in M: -\Psi(z) \in (t'_4, t'_3]\}} |\tilde{F}|_h^2 + \int_{\{z \in M: -\Psi(z) \in (t'_4, t'_3]\} \vee N\}} |F_{t_0,\tilde{c}} - \tilde{F}|_h^2
$$
\n
$$
+ \int_{\{z \in M: -\Psi(z) \in (t'_4, t'_3] \cap N\}} |F_{t_0,\tilde{c}}|_h^2.
$$

then combining inequality [\(4.22\)](#page-36-3), we have

$$
\int_{\{-t'_3 \le \Psi < -t'_4\}} |F_{t_0, \tilde{c}}|_h^2
$$
\n
$$
= \int_{\{z \in M: -\Psi(z) \in (t'_4, t'_3]\}} |\tilde{F}|_h^2 + \int_{\{z \in M: -\Psi(z) \in (t'_4, t'_3]\} \backslash N\}} |F_{t_0, \tilde{c}} - \tilde{F}|_h^2 \qquad (4.23)
$$
\n
$$
+ \int_{\{z \in M: -\Psi(z) \in (t'_4, t'_3] \cap N\}} |F_{t_0, \tilde{c}}|_h^2.
$$

Using equality [\(4.8\)](#page-32-0), [\(4.17\)](#page-35-1) and Levi's Theorem, we have

$$
\int_{\{z \in M: -\Psi(z) \in U\}} |F_{t_0, \tilde{c}}|_h^2
$$
\n
$$
= \int_{\{z \in M: -\Psi(z) \in U\}} |\tilde{F}|_h^2 + \int_{\{z \in M: -\Psi(z) \in U \setminus N\}} |F_{t_0, \tilde{c}} - \tilde{F}|_h^2
$$
\n
$$
+ \int_{\{z \in M: -\Psi(z) \in U \cap N\}} |F_{t_0, \tilde{c}}|_h^2
$$
\n(4.24)

holds for any open set $U \subset\subset (t_0, +\infty)$, and

$$
\int_{\{z \in M: -\Psi(z) \in V\}} |F_{t_0, \tilde{c}}|_h^2
$$
\n
$$
= \int_{\{z \in M: -\Psi(z) \in V\}} |\tilde{F}|_h^2 + \int_{\{z \in M: -\Psi(z) \in V \setminus N\}} |F_{t_0, \tilde{c}} - \tilde{F}|_h^2
$$
\n
$$
+ \int_{\{z \in M: -\Psi(z) \in V \cap N\}} |F_{t_0, \tilde{c}}|_h^2
$$
\n(4.25)

holds for any compact set $V \subset (t_0, +\infty)$. For any measurable set $E \subset \subset (t_0, +\infty)$, there exists a sequence of compact set $\{V_l\}$, such that $V_l \subset V_{l+1} \subset E$ for any l and $\lim_{l \to +\infty} \mu(V_l) = \mu(E)$, hence by equality [\(4.25\)](#page-37-0), we have

$$
\int_{\{\Psi<-t_0\}} |F_{t_0,\tilde{c}}|_{h}^2 \mathbb{I}_E(-\Psi) \ge \lim_{l \to +\infty} \int_{\{\Psi<-t_0\}} |F_{t_0,\tilde{c}}|_{h}^2 \mathbb{I}_{V_j}(-\Psi)
$$
\n
$$
\ge \lim_{l \to +\infty} \int_{\{\Psi<-t_0\}} |\tilde{F}|_{h}^2 \mathbb{I}_{V_j}(-\Psi) \tag{4.26}
$$
\n
$$
= \int_{\{\Psi<-t_0\}} |\tilde{F}|_{h}^2 \mathbb{I}_{V_j}(-\Psi).
$$

It is clear that for any $t > t_0$, there exists a sequence of functions $\{\sum_{j=1}^{n_i} \mathbb{I}_{E_{i,j}}\}_{i=1}^{+\infty}$
defined on $(t, +\infty)$, satisfying $E_{i,j} \subset\subset (t, +\infty)$, $\sum_{j=1}^{n_{i+1}} \mathbb{I}_{E_{i+1,j}}(s) \ge \sum_{j=1}^{n_i} \mathbb{I}_{E_{i,j}}(s)$

and $\lim_{i \to +\infty}$ $\sum_{j=1}^{n_i} \mathbb{I}_{E_{i,j}}(s) = \tilde{c}(s)$ for any $s > t$. Combining Levi's Theorem and inequality [\(4.26\)](#page-37-1), we have

$$
\int_{\{\Psi<-t_0\}} |F_{t_0,\tilde{c}}|^2_h \tilde{c}(-\Psi) \ge \int_{\{\Psi<-t_0\}} |\tilde{F}|^2_h \tilde{c}(-\Psi). \tag{4.27}
$$

By the definition of $G(t_0, \tilde{c})$, we have $G(t_0, \tilde{c}) = \int_{\{\Psi<-t_0\}} |\tilde{F}|^2_h \tilde{c}(-\Psi)$. Equality [\(1.3\)](#page-4-0) is proved.

$$
\Box
$$

5. Proofs of Theorem [1.12](#page-5-0) and Corollary [1.13](#page-5-1)

In this section, we prove Theorem [1.12](#page-5-0) and Corollary [1.13.](#page-5-1)

5.1. **Proof of Theorem [1.12.](#page-5-0)** Lemma [2.10](#page-22-0) tells us that there exists $p_0 > 2a_{z_0}^f(\Psi; h)$ such that $I(h, p_0\Psi)_{z_0} = I_+(h, 2a_{z_0}^f(\Psi; h)\Psi)_{z_0}$. Following from the definition of $a_{z_0}^f(\Psi; h)$ and Lemma [3.4,](#page-24-0) we obtain that

$$
G(0; c \equiv 1, \Psi, h, I_{+}(h, 2a_{z_{0}}^{f}(\Psi; h)\Psi)_{z_{0}}, f) > 0.
$$
\n
$$
(5.1)
$$

Without loss of generality, assume that there exists $t > t_0$ such that $\int_{\{\Psi<-t\}} |f|^2_h$ +∞. Denote that $t_1 := \inf\{t \ge t_0 : \int_{\{\Psi < -t\}} |f|^2_h < +\infty\}$. Denote

$$
\inf \left\{ \int_{\{p\Psi<-t\}} |\tilde{f}|_h^2 : \tilde{f} \in H^0(\{p\Psi<-t\}, \mathcal{O}(K_M \otimes E)) \right\}
$$

$$
\& (\tilde{f}-f)_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes I(h, p\Psi)_{z_0} \right\}
$$

by $G_p(t)$, where $t \in [0, +\infty)$ and $p > 2a_{z_0}^f(\Psi; h)$. Then we know that $G_p(0) \ge$ $G(0; c \equiv 1, \Psi, h, I_+(h, 2a_{z_0}^f(\Psi; h)\Psi)_{z_0}, f)$ for any $p > 2a_{z_0}^f(\Psi; h)$. Note that

$$
p\Psi = \min\{p\psi + (2\lceil p \rceil - 2p) \log |F| - 2 \log |F^{[p]}|, 0\},\
$$

where $[p] = \min\{n \in \mathbb{Z} : n \geq p\}$, and

$$
G_p(pt) \le \int_{\{\Psi<-t\}} |f|^2_h < +\infty
$$

for any $t > t_1$. Note that $\Theta_{\tilde{h}}(E) \geq_{Nak}^s 0$ and $(p-2a_{z_0}^f(\Psi; h))\psi + (2[p]-2p)\log|F|$ is plurisubharmonic on M. Remark [1.5](#page-2-0) implies that $\Theta_{he^{-(p\psi+(2\lceil p\rceil-2p)\log|F|)}}(E) \geq_{Nak}^{s}$
0. Note that h has a positive locally lower bound. Theorem [1.8](#page-4-1) tells us that $G_p(-\log r)$ is concave with respect to $r \in (0,1]$ and $\lim_{t\to+\infty} G_p(t) = 0$, which implies that

$$
\frac{1}{r_1^2} \int_{\{p\bar{\Psi} < 2\log r_1\}} |f|_h^2 \ge \frac{1}{r_1^2} G_p(-2\log r_1) \\
\ge G_p(0) \\
\ge G(0; c \equiv 1, \Psi, h, I_+(h, 2a_{z_0}^f(\Psi; h)\Psi)_{z_0}, f),
$$
\n
$$
(5.2)
$$

where $0 < r_1 \leq e^{-\frac{pt_0}{2}}$.

We prove $a_{z_0}^f(\Psi; h) > 0$ by contradiction: if $a_{z_0}^f(\Psi; h) = 0$, as $\int_{\{\Psi < -t_1-1\}} |f|^2_h$ $+\infty$, it follows from the dominated convergence theorem and inequality [\(5.2\)](#page-38-0) that

$$
\frac{1}{r_1^2} \int_{\{\Psi = -\infty\}} |f|_h^2 = \lim_{p \to 0+0} \frac{1}{r_1^2} \int_{\{p\Psi < 2\log r_1\}} |f|_h^2
$$
\n
$$
\ge G(0; c \equiv 1, \Psi, h, I_+(h, 2a_{z_0}^f(\Psi; h)\Psi)_{z_0}, f).
$$
\n
$$
(5.3)
$$

Note that $\mu({\Psi = -\infty}) = \mu({\psi = -\infty}) = 0$, where μ is the Lebesgue measure on *M*. Inequality [\(5.3\)](#page-39-0) implies that $G(0; c \equiv 1, \Psi, h, I_{+}(h, 2a_{z_0}^f(\Psi; h)\Psi)_{z_0}, f) = 0$, which contradicts inequality [\(5.1\)](#page-38-1). Thus, we get that $a_{z_0}^f(\Psi; h) > 0$.

For any $r_2 \in (0, e^{-a_{z_0}^f(\Psi; h)t_1})$, note that $\frac{2\log r_2}{p} < -t_1$ for any $p \in (2a_0^f(\Psi; h), -\frac{2\log r_2}{t_1})$, which implies that $\int_{\{p\Psi < 2\log r_2\}} |f|^2_h < +\infty$ for any $p \in (2a_0^f(\Psi; h), -\frac{2\log r_2}{t_1})$. Then it follows from the dominated convergence theorem and inequality [\(5.2\)](#page-38-0) that

$$
\frac{1}{r_2^2} \int_{\{2a_0^f(\Psi;h)\Psi \le 2\log r_2\}} |f|^2_h = \lim_{p \to 2a_0^f(\Psi;h)+0} \frac{1}{r_2^2} \int_{\{p\Psi < 2\log r_2\}} |f|^2_h
$$
\n
$$
\ge G(0; c \equiv 1, \Psi, h, I_+(h, 2a_{z_0}^f(\Psi;h)\Psi)_{z_0}, f).
$$
\n
$$
(5.4)
$$

For any $r \in (0, e^{-a_{z_0}^f(\Psi;h)t_0}],$ if $r > e^{-a_{z_0}^f(\Psi;h)t_1}$, we have $\int_{\{a_0^f(\Psi;h)\Psi < \log r\}} |f|^2_h =$ $+\infty > G(0; c \equiv 1, \Psi, h, I_+(h, 2a_{z_0}^f(\Psi; h)\Psi)_{z_0}, f),$ and if $r \in (0, e^{-a_{z_0}^f(\Psi; h)t_1}],$ it follows from $\{a_{z_0}^f(\Psi;h)\Psi < \log r\} = \bigcup_{0 \leq r_2 \leq r} \{a_{z_0}^f(\Psi;h)\Psi < \log r_2\}$ and inequality [\(5.4\)](#page-39-1) that

$$
\int_{\{a_0^f(\Psi;h)\Psi<\log r\}} |f|^2_h = \sup_{r_2\in(0,r)} \int_{\{2a_0^f(\Psi;h)\Psi\leq 2\log r_2\}} |f|^2_h
$$
\n
$$
\geq \sup_{r_2\in(0,r)} r_2^2 G(0;c \equiv 1, \Psi, h, I_+(h, 2a_{z_0}^f(\Psi;h)\Psi)_{z_0}, f)
$$
\n
$$
= r^2 G(0;c \equiv 1, \Psi, h, I_+(h, 2a_{z_0}^f(\Psi;h)\Psi)_{z_0}, f).
$$

Thus, Theorem [1.12](#page-5-0) holds.

5.2. **Proof of Corollary [1.13.](#page-5-1)** It is clear that $I_+(h, a\Psi)_{z_0} \subset I(h, a\Psi)_{z_0}$, hence it suffices to prove that $I(h, a\Psi)_{z_0} \subset I_+(h, a\Psi)_{z_0}$.

If there exists $f_{z_0} \in I(h, a\Psi)_{z_0}$ such that $f_{z_0} \notin I_+(h, a\Psi)_{z_0}$, then $a_{z_0}^f(\Psi; h)_{z_0} =$ $\frac{a}{2}$ < + ∞ . Theorem [1.12](#page-5-0) shows that $a > 0$. Without loss of generality, assume that $\overline{M} = D$ is a domain in \mathbb{C}^n and $f \in H^0(\{\Psi < -t_0\} \cap D, \mathcal{O}(E)),$ where $t_0 > 0$. For any neighborhood $U \subset D$ of z_0 , it follows from Proposition [1.12](#page-5-0) that there exists $C_U > 0$ such that

$$
\frac{1}{r^2} \int_{\{a\Psi < 2\log r\} \cap U} |f|^2_h \ge C_U \tag{5.5}
$$

for any $r \in (0, e^{-\frac{at_0}{2}}]$. For any $t > at_0$, it follows from Fubini's Theorem and inequality [\(5.5\)](#page-39-2) that

$$
\int_{\{a\Psi<-t\}\cap U} |f|^2_h e^{-a\Psi} = \int_{\{a\Psi<-t\}\cap U} \left(|f|^2_h \int_0^{e^{-a\Psi}} dl\right)
$$
\n
$$
= \int_0^{+\infty} \left(\int_{\{l\n
$$
\geq \int_{e^t}^{+\infty} \left(\int_{\{a\Psi<- \log l\}\cap U} |f|^2_h\right) dl
$$
\n
$$
\geq C_U \int_{e^t}^{+\infty} \frac{1}{l} dl
$$
\n
$$
= +\infty,
$$
$$

which contradicts $f_{z_0} \in I(h, a\Psi)_{z_0}$. Thus, we have $I(h, a\Psi)_{z_0} \setminus I_+(h, a\Psi)_{z_0} = \emptyset$ for any $a \geq 0$, which shows that $I(h, a\Psi)_{z_0} = I_+(h, a\Psi)_{z_0}$ for any $a \geq 0$.

6. Proof of Theorem [1.15](#page-6-0)

In this section, we prove Theorem [1.15](#page-6-0) by using Theorem [1.8.](#page-4-1)

For any Lebesgue measurable function c on $(0, +\infty)$ and any $q' > 2a_{z_0}^f(\Psi; h) \ge 1$, denote that

$$
G_{c,q'}(t) := \inf \Bigg\{ \int_{\{q'\Psi<-t\}} |\tilde{f}|^2_h c(-q'\Psi) : (\tilde{f} - f)_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes I(h, q'\Psi)_{z_0}
$$

$$
\& \tilde{f} \in H^0(\{q'\Psi<-t\}, \mathcal{O}(K_M \otimes E)) \Bigg\}.
$$

Note that there exist a plurisubharmonic function $\psi_1 = q' \psi + (2k - 2q') \log |F|$ and a holomorphic function $F_1 = F^k$ on M such that

$$
\psi_1 - 2\log|F_1| = q'(\psi - 2\log|F|)
$$

on M, where $k > q'$ is a integer. Denote that $\Psi_1 := \min{\{\psi_1 - 2\log |F_1|, 0\}} = q'\Psi$ on M.

Firstly, we prove inequality

$$
\int_{\{\Psi<-\frac{l}{a}\}}|f|^2_{h}e^{-(1-a)\Psi} \ge e^{-\frac{a-1+q'}{a}l}\frac{1}{K_{\Psi,f,h,a}(z_0)}\tag{6.1}
$$

in two case $a \in (0,1]$ and $a > 1$, where $l \geq 0$ and $q' > 2a_{z_0}^f(\Psi; h)$.

We prove inequality [\(6.1\)](#page-40-0) for the case $a \in (0,1]$. Let $c_1(t) = e^{\frac{1-a}{q'}t}$ on $(0, +\infty)$, hence $c_1(t)e^{-t}$ is decreasing on $(0, +\infty)$ and $c_1(-q'\Psi) = e^{-(1-a)\Psi} \ge 1$ on M. Note that h has a positive locally lower bound. As $\Theta_{\tilde{h}} \geq_{Nak}^{s} 0$ and ψ is plurisubharmonic, where $\tilde{h} = he^{-2a_{z_0}^f(\Psi,h)\psi}$, then we have

$$
\Theta_{he^{-\psi_1}} \geq^s_{Nak} 0.
$$

Theorem [1.8](#page-4-1) (replace Ψ and c by Ψ_1 and c₁, respectively) shows that $G_{c_1,q'}(h^{-1}(r))$ is concave with respect to r, where $h(t) = \int_{t}^{+\infty} c_1(s)e^{-s}ds$. Note that

$$
G_{c_1,q'}(0) \ge \frac{1}{K_{\Psi,f,h,a}(z_0)}
$$

for any $q' > 2a_{z_0}^f(\Psi; h)$. Hence we have

$$
\int_{\{\Psi<-\frac{l}{a}\}} |f|_{h}^{2} e^{-(1-a)\Psi} \geq G_{c_{1},q'} \left(\frac{q'l}{a}\right)
$$
\n
$$
\geq \frac{\int_{\frac{q'l}{a}}^{\frac{l}{a'}} c_{1}(s) e^{-s} ds}{\int_{0}^{+\infty} c_{1}(s) e^{-s} ds} G_{c_{1},q'}(0)
$$
\n
$$
\geq e^{-\frac{a-1+q'}{a}} \frac{1}{K_{\Psi,f,h,a}(z_{0})}.
$$

We prove inequality [\(6.1\)](#page-40-0) for the case $a > 1$. Take $\tilde{c}_m(t) = e^{\frac{1-a}{q^t}t}$ on $(0, m)$ and $\tilde{c}_m(t) = e^{\frac{1-a}{q'}m}$ on $(m, +\infty)$, then $\tilde{c}_m(t)$ is a continuous function on $(0, +\infty)$ and $c_1(t)e^{-t}$ is decreasing on $(0, +\infty)$, where m is any positive integer. Note that $c(t) \geq e^{\frac{1-a}{q'}m}$ on $(0, +\infty)$ and h has a positive locally lower bound. Theorem [1.8](#page-4-1) (replace Ψ and c by Ψ_1 and \tilde{c}_m , respectively) shows that $G_{\tilde{c}_m,q'}(h_m^{-1}(r))$ is concave with respect to r, where $h_m(t) = \int_t^{+\infty} \tilde{c}_m(s) e^{-s} ds$. Note that

$$
G_{\tilde{c}_m,q'}(0) \ge \frac{1}{K_{\Psi,f,h,a}(z_0)}
$$

for any $q' > 2a_{z_0}^f(\Psi; h)$. Hence we have

$$
\int_{\{\Psi<-\frac{l}{a}\}} |f|_{h}^{2} \tilde{c}_{m}(-q'\Psi) \geq G_{\tilde{c}_{m},q'}\left(\frac{q'l}{a}\right)
$$
\n
$$
\geq \frac{\int_{\frac{q'}{a}}^{\frac{t}{a'}c} \tilde{c}_{m}(s)e^{-s}ds}{\int_{0}^{+\infty} \tilde{c}_{m}(s)e^{-s}ds} G_{\tilde{c}_{m},q'}(0)
$$
\n
$$
\geq \frac{\int_{\frac{q'}{a}}^{\frac{t}{a'}c} \tilde{c}_{m}(s)e^{-s}ds}{\int_{0}^{+\infty} \tilde{c}_{m}(s)e^{-s}ds} \frac{1}{K_{\Psi,f,h,a}(z_{0})}.
$$
\n(6.2)

As $\int_{\{\Psi<0\}}|f|^2_h e^{-\Psi} \leq C_1 < +\infty$, it follows from $\tilde{c}_m(-q'\Psi) \leq e^{-\Psi}$, the dominated convergence theorem and inequality [\(6.2\)](#page-41-0) that

$$
\int_{\{\Psi<-\frac{l}{a}\}} |f|^2_h e^{-(1-a)\Psi} = \lim_{m \to +\infty} \int_{\{\Psi<- \frac{l}{a}\}} |f|^2_h \tilde{c}_m(-q'\Psi)
$$
\n
$$
\geq \lim_{m \to +\infty} \frac{\int_{\frac{q'}{a}}^{+\infty} \tilde{c}_m(s) e^{-s} ds}{\int_0^{+\infty} \tilde{c}_m(s) e^{-s} ds} \frac{1}{K_{\Psi,f,h,a}(z_0)}
$$
\n
$$
= e^{-\frac{a-1+q'}{a}l} \frac{1}{K_{\Psi,f,h,a}(z_0)}.
$$

Next, we complete the proof. Following from Fubini's Theorem, we have

$$
\int_{\{\Psi<0\}} |f|_{h}^{2} e^{-\Psi} \n= \int_{\{\Psi<0\}} \left(|f|_{h}^{2} e^{-\Psi + a\Psi} \int_{0}^{e^{-a\Psi}} ds \right) \n= \int_{0}^{+\infty} \left(\int_{\{\Psi<0\} \cap \{s < e^{-a\Psi}\}} |f|_{h}^{2} e^{-\Psi + a\Psi} \right) ds \n= \int_{-\infty}^{+\infty} \left(\int_{\{\Psi<-\frac{l}{a}\} \cap \{\Psi<0\}} |f|_{h}^{2} e^{-\Psi + a\Psi} \right) e^{l} dl \n= \int_{\{\Psi<0\}} |f|_{h}^{2} e^{-\Psi + a\Psi} + \int_{0}^{+\infty} \left(\int_{\{\Psi<-\frac{l}{a}\}} |f|_{h}^{2} e^{-\Psi + a\Psi} \right) e^{l} dl.
$$

Using inequality [\(6.1\)](#page-40-0) and the definition of $K_{\Psi,f,h,a}(z_0)$, we obtain that

$$
\int_{\{\Psi<0\}} |f|_{h}^{2} e^{-\Psi}
$$
\n
$$
= \int_{\{\Psi<0\}} |f|_{h}^{2} e^{-\Psi+a\Psi} + \int_{0}^{+\infty} \left(\int_{\{\Psi<-\frac{l}{a}\}} |f|_{h}^{2} e^{-\Psi+a\Psi}\right) e^{l} dl
$$
\n
$$
\geq \left(1 + \int_{0}^{+\infty} e^{-\frac{-1+q'}{a}l} dl\right) \frac{1}{K_{\Psi,f,h,a}(z_0)}
$$
\n
$$
= \frac{a+q'-1}{q'-1} \cdot \frac{1}{K_{\Psi,f,h,a}(z_0)}
$$
\n(6.3)

for any $q' > 2a_{z_0}^f(\Psi; h)$. Let $q' \to 2a_{z_0}^f(\Psi; h)$, we get that inequality [\(6.3\)](#page-42-0) also holds when $q' = 2a_{z_0}^f(\Psi; h)$. Thus, if $q > 1$ satisfies

$$
\frac{q+a-1}{q-1} > \frac{C_1}{C_2} \ge K_{\Psi, f, h, a}(z_0) \int_{\{\Psi < 0\}} |f|^2_h e^{-\Psi},
$$

we have $p < 2a_{z_0}^f(\Psi; h)$, i.e. $f_{z_0} \in \mathcal{O}(K_M)_{z_0} \otimes I(h, p\Psi)_{z_0}$.

7. Proof of Theorem [1.16](#page-6-1)

In this section, we prove Theorem [1.16](#page-6-1) by using Remark [1.9](#page-4-2) and Theorem [1.12.](#page-5-0) Firstly, we recall two basic lemmas, which will be used in the proof of Theorem [1.16.](#page-6-1)

Lemma 7.1 (see [\[34\]](#page-72-0)). Let a(t) be a positive measurable function on $(-\infty, +\infty)$, such that $a(t)e^t$ is increasing near $+\infty$, and $a(t)$ is not integrable near $+\infty$. Then there exists a positive measurable function $\tilde{a}(t)$ on $(-\infty, +\infty)$ satisfying the following statements:

- (1) there exists $T < +\infty$ such that $\tilde{a}(t) \leq a(t)$ for any $t > T$;
- (2) $\tilde{a}(t)e^t$ is strictly increasing and continuous near $+\infty$;
- (3) $\tilde{a}(t)$ is not integrable near $+\infty$.

Lemma 7.2 (see [\[38\]](#page-72-1)). For any two measurable spaces (X_i, μ_i) and two measurable functions g_i on X_i respectively $(i \in \{1,2\})$, if $\mu_1(\{g_1 \ge s^{-1}\}) \ge \mu_2(\{g_2 \ge s^{-1}\})$ for any $s \in (0, s_0]$, then $\int_{\{g_1 \ge s_0^{-1}\}} g_1 d\mu_1 \ge \int_{\{g_2 \ge s_0^{-1}\}} g_2 d\mu_2$.

Proof of Theorem [1.16.](#page-6-1) We prove Theorem [1.16](#page-6-1) in two cases, that $a(t)$ satisfies condition (1) or condition (2).

Case (1). $a(t)$ is decreasing near $+\infty$.

Firstly, we prove $(B) \Rightarrow (A)$. Consider $F \equiv 1, f = (f_1, f_2, \ldots, f_r) = (1, 1, \ldots, 1)$, $h \equiv 1$ and $\psi = \log |z_1|$ on the unit polydisc $\Delta^n \subset \mathbb{C}^n$. Note that $a_o^f(\log |z_1|; h) = 1$ and

$$
\int_{\Delta_{s_0}^n} |f_h^2 e^{-2a_o^f(\log|z_1|;h)\Psi} a(-2a_o^f(\log|z_1|;h)\Psi)
$$
\n
$$
= r \int_{\Delta_{s_0}^n} a(-2\log|z_1|) \frac{1}{|z_1|^2}
$$
\n
$$
= r(\pi s_0^2)^{n-1} \int_{\Delta_{s_0}} a(-2\log|z_1|) \frac{1}{|z_1|^2}
$$
\n
$$
= r(\pi s_0^2)^{n-1} 2\pi \int_0^{s_0} a(-2\log r)r^{-1} dr
$$
\n
$$
= r(\pi s_0^2)^{n-1} \pi \int_{-2\log s_0}^{+\infty} a(t) dt
$$

for $s_0 \in (0,1)$, hence we obtain $(B) \Rightarrow (A)$.

Then, we prove $(A) \Rightarrow (B)$. Corollary [1.13](#page-5-1) shows that $f_o \notin I(h, 2a_o^f(\Psi; h)\Psi)_o$ and $a_o^f(\Psi; h) > 0$. Now we assume that there exist $t_0 > 0$ and a pseudoconvex domain $D_0 \subset D$ containing o such that $\int_{\{\Psi<-t_0\}\cap D_0} |f|^2_h e^{-2a^f_o(\Psi;h)\Psi} a(-2a^f_o(\Psi;h)\Psi)$ +∞ to get a contradiction. As $f_o \in I(h, 0\Psi)_o$, there exist $t_1 > t_0$ and a pseudoconvex domain $D_1 \subset D_0$ containing o such that $\int_{D_1 \cap {\Psi}(-t_1)} |f|^2 \, d \infty$. Set $c(t) = a(t)e^{t} + 1$, then we have

$$
\int_{D_1 \cap \{\Psi < -t_1\}} |f|^2_h c(-2a_o^f(\Psi; h)\Psi) < +\infty.
$$
\n(7.1)

Without loss of generality, assume that $a(t)$ is decreasing on $(2a_o^f(\Psi; h)t_1, +\infty)$. Note that $c(t)e^{-t} = a(t)+e^{-t}$ is decreasing on $(2a^f_{\rho}(\Psi;h)t_1, +\infty)$ and $\liminf_{t\to+\infty} c(t)$ 0. As $a(t)$ is not integrable near $+\infty$, so is $c(t)e^{-t}$. Note that there exist a plurisubharmonic function $\psi_1 = 2a_o^f(\Psi; h)\psi + 2(k - a_o^f(\Psi; h)) \log |F|$ and a holomorphic function $F_1 = F^k$ on D_1 such that

$$
\psi_1 - 2\log|F_1| = 2a_o^f(\Psi; h)(\psi - 2\log|F|)
$$

on D_1 , where $k > a_0^f(\Psi; h)$ is a integer. Denote that $\Psi_1 := \min{\{\psi_1 - 2\log |F_1|, -2a_0^f(\Psi; h)t_1\}}$ on D_1 . Note that h has a positive locally lower bound, $c(t) \geq 1$ on $(2a_o^f(\Psi; h)t_1, +\infty)$ and $\Theta_{he^{-\psi_1}} \geq_{Nak}^s 0$. Using Remark [1.9](#page-4-2) (replacing M, Ψ and T by D_1 , Ψ_1 and $2a_o^f(\Psi;h)t_1$ respectively), as $f_o \notin I(h, 2a_o^f(\Psi;h)\Psi)_o = I(h, \Psi_1)_o$, then we have $G(2a_o^f(\Psi;h)t_1;c,\Psi_1,h,I(\Psi_1+\varphi)_o,f)=+\infty$, which contradicts to inequality [\(7.1\)](#page-43-0). Thus, we obtain $(A) \Rightarrow (B)$.

Case (2). $a(t)e^t$ is increasing near $+\infty$.

In this case, the proof of $(B) \Rightarrow (A)$ is the same as the case (1) , hence it suffices to prove $(A) \Rightarrow (B)$.

Assume that statement (A) holds. Lemma [7.1](#page-42-1) shows that there exists a positive function $\tilde{a}(t)$ on $(-\infty, +\infty)$ satisfying that: $\tilde{a}(t) \leq a(t)$ near $+\infty$; $\tilde{a}(t)e^t$ is strictly increasing and continuous near $+\infty$; $\tilde{a}(t)$ is not integrable near $+\infty$. Thus, it suffices to prove that for any Ψ , h and $f_o \in I(h, 0\Psi)_o$ satisfying $a_o^f(\Psi; h) < +\infty$, $|f|_h^2 e^{-2a_o^f(\Psi;h)\Psi} a(-2a_o^f(\Psi;h)\Psi) \notin L^1(U \cap {\Psi < -t})$ for any neighborhood U of o and any $t > 0$.

Take any $t_0 \gg 0$ and any pseudoconvex domain $D_0 \subset D$ containing the origin o such that $f \in \mathcal{O}(D_0 \cap {\Psi < -t_0})$. Let $\mu_1(X) = \int_X |f|_h^2$, where X is a Lebesgue measurable subset of $D_0 \cap {\Psi < -t_0}$, and let μ_2 be the Lebeague measure on $(0, 1]$. Denote that $Y_s = \{-2a_o^f(\Psi; h)\Psi \ge -\log s\}$. Theorem [1.12](#page-5-0) shows that there exists a positive constant C such that $\mu_1(Y_s) \geq Cs$ holds for any $s \in (0, e^{-2a_o^f(\Psi; h)t_0}).$

Let $g_1 = \tilde{a}(-2a_0^f(\Psi; h)\Psi) \exp(-2a_0^f(\Psi; h)\Psi)$ and $g_2(x) = \tilde{a}(-\log x + \log C)Cx^{-1}$. As $\tilde{a}(t)e^{-t}$ is strictly increasing near $+\infty$, then $g_1 \geq \tilde{a}(-\log s)s^{-1}$ on Y_s implies that

$$
\mu_1(\{g_1 \ge \tilde{a}(-\log s)s^{-1}\}) \ge \mu_1(Y_s) \ge Cs \tag{7.2}
$$

holds for any $s > 0$ small enough. As $\tilde{a}(t)e^t$ is strictly increasing near $+\infty$, then there exists $s_0 \in (0, e^{-2a_o^f(\Psi)t_0})$ such that

$$
\mu_2(\{x \in (0, s_0] : g_2(x) \ge \tilde{a}(-\log s)s^{-1}\}) = \mu_2(\{0 < x \le Cs\}) = Cs \tag{7.3}
$$

for any $s \in (0, s_0]$. As $\tilde{a}(-\log s)s^{-1}$ converges to $+\infty$ (when $s \to 0+0$) and $\tilde{a}(t)$ is continuous near $+\infty$, we obtain that

$$
\mu_1(\{g_1 \ge s^{-1}\}) \ge \mu_2(\{x \in (0, s_0] : g_2(x) \ge s^{-1}\})
$$

holds for any $s > 0$ small enough. Following from Lemma [7.2](#page-42-2) and $\tilde{a}(t)$ is not integrable near $+\infty$, we obtain $|f|^2_h e^{-2a_o^f(\Psi;h)\Psi} a(-2a_o^f(\Psi;h)\Psi) \notin L^1(U \cap {\Psi \leq \Psi)}$ $-t\}$).

Thus, Theorem [1.16](#page-6-1) holds. \square

8. Proof of Proposition [1.17](#page-7-0)

In this section, we prove Proposition [1.17](#page-7-0) by using Theorem [1.8.](#page-4-1)

Proof. Let

$$
h(x) = \begin{cases} e^{-\frac{1}{1 - (x - 1)^2}} & \text{if } |x - 1| < 1\\ 0 & \text{if } |x - 1| \ge 1 \end{cases}
$$

be a real function defined on R, and let $g_n(x) = \frac{n}{(n+1)d} \int_0^{nx} h(s)ds$, where $d =$ $\int_{\mathbb{R}} h(s)ds$. Note that $h(x) \in C_0^{\infty}(\mathbb{R})$ and $h(x) \geq 0$ for any $x \in \mathbb{R}$. Then we get that $g_n(x)$ is increasing with respect to $x, g_n(x) \leq g_{n+1}(x)$ for any $n \in \mathbb{N}$ and $x \in \mathbb{R}$, and $\lim_{n \to +\infty} g_n(x) = \mathbb{I}_{\{s \in \mathbb{R} : s > 0\}}(x)$ for any $x \in \mathbb{R}$. Setting $c_t^n(x) = 1 - g_n(x - t)$, where t is the given positive number in Proposition [1.17,](#page-7-0) it follows from the properties of $\{g_n(x)\}_{n\in\mathbb{N}}$ that $c_t^n(x)$ is decreasing with respect to x, $c_t^n(x) \geq c_t^{n+1}(x)$ for any of $\{y_n(x)\}_{n\in\mathbb{N}}$ and $t_t(x)$ is decreasing with respect to x , $c_t(x) \geq c_t$
 $n \in \mathbb{N}$ and $x \in \mathbb{R}$, and $\lim_{n\to+\infty} c_t^n(x) = \mathbb{I}_{\{s \in \mathbb{R}:s \leq t\}}(x)$ for any $x \in \mathbb{R}$. Denote

$$
\inf \left\{ \int_{\{\Psi<-t\}} |\tilde{f}|_{h}^{2} c_{t}^{n}(-\Psi) : \tilde{f} \in H^{0}(\{\Psi<-t\}, \mathcal{O}(K_{M} \otimes E)) \right\}
$$

$$
\& (\tilde{f}-f)_{z_{0}} \in \mathcal{O}(K_{M})_{z_{0}} \otimes I(h, \Psi)_{z_{0}} \text{ for any } z_{0} \in Z_{0} \right\}
$$

by $G_{t,n}(s)$. Note that $\Theta_{he^{-\psi}} \geq_{Nak}^{s} 0$, h has a positive locally lower bound and

$$
c_t^n(x) \in [\frac{1}{n+1}, 1]
$$

on $(0, +\infty)$. By using Theorem [1.8,](#page-4-1) we have

$$
\int_{\{\Psi<-l\}} |f|^2_h c_t^n(-\Psi) \ge G_{t,n}(l) \ge \frac{\int_l^{+\infty} c_t^n(s) e^{-s} ds}{\int_0^{+\infty} c_t^n(s) e^{-s} ds} G_{t,n}(0) \tag{8.1}
$$

for any $l > 0$. Following from $\int_{\{\Psi<-l\}}|f|_h^2 < +\infty$ for any $l > 0$, the properties of $\{c_t^n\}_{n\in\mathbb{N}}$ and the dominated convergence theorem, we obtain that

$$
\lim_{n \to +\infty} \int_{\{\Psi < -l\}} |f|^2_h c_t^n(-\Psi) = \int_{\{-t \le \Psi < -l\}} |f|^2_h. \tag{8.2}
$$

As $c_t^n(x) \geq \mathbb{I}_{\{s \in \mathbb{R}: s \leq t\}}(x)$ for any $x > 0$ and $n \in \mathbb{N}$, then it follows from the definitions of $G_{t,n}(0)$ and $C_{\Psi,f,h,t}(Z_0)$ that

$$
G_{t,n}(0) \ge C_{\Psi,f,h,t}(Z_0). \tag{8.3}
$$

Combining inequality (8.1) , equality (8.2) , and inequality (8.3) , we obtain that

$$
\int_{\{-t \le \Psi < -l\}} |f|_{h}^{2} = \lim_{n \to +\infty} \int_{\{\Psi < -l\}} |f|_{h}^{2} c_{t}^{n}(-\Psi)
$$
\n
$$
\ge \lim_{n \to +\infty} \int_{0}^{+\infty} c_{t}^{n}(s) e^{-s} ds
$$
\n
$$
= \frac{e^{-l} - e^{-t}}{1 - e^{-t}} C_{\Psi, f, h, t}(Z_{0})
$$

for any $l \in (0, t)$. Following from the definition of $C_{\Psi, f, h, t}(Z_0)$, we have $\int_{\{-t \le \Psi < 0\}} |f|^2_h \ge$ $C_{\Psi,f,h,t}(Z_0)$. Thus, we have

$$
\int_{\{-t \le \Psi < -l\}} |f|_{h}^{2} \ge \frac{e^{-l} - e^{-t}}{1 - e^{-t}} C_{\Psi, f, h, t}(Z_{0})
$$
\n(8.4)

for any $l \in [0, t)$. Following from Fubini's Theorem and inequality [\(8.4\)](#page-45-3), we obtain that

$$
\int_{M_t} |f|_h^2 e^{-\Psi} = \int_{M_t} (|f|_h^2 \int_0^{e^{-\Psi}} dr)
$$
\n
$$
= \int_0^{+\infty} \left(\int_{M_t \cap \{r < e^{-\Psi}\}} |f|_h^2 \right) dr
$$
\n
$$
= \int_{-\infty}^t \left(\int_{\{-t \le \Psi < \min\{-l, 0\}\}} |f|_h^2 \right) e^l dl
$$
\n
$$
= \int_{-\infty}^0 \left(\int_{\{-t \le \Psi < \min\{-l, 0\}\}} |f|_h^2 \right) e^l dl + \int_0^t \left(\int_{\{-t \le \Psi < -l\}} |f|_h^2 \right) e^l dl
$$
\n
$$
\ge C_{\Psi, f, h, t}(Z_0) \left(\int_{-\infty}^0 e^l dl + \int_0^t \frac{1 - e^{l-t}}{1 - e^{-t}} dl \right)
$$
\n
$$
= \frac{t}{1 - e^{-t}} C_{\Psi, f, h, t}(Z_0).
$$

Then Proposition [1.17](#page-7-0) has thus been proved.

9. Appendix: Proof of Lemma [2.1](#page-8-0)

In this section, we prove Lemma [2.1.](#page-8-0)

9.1. Some results used in the proof of Lemma [2.1.](#page-8-0) In this section, we do some preparations for the proof of Lemma [2.1.](#page-8-0)

Let M be a complex manifold. Let ω be a continuous hermitian metric on M. Let dV_M be a continuous volume form on M. We denote by $L_{p,q}^2(M, \omega, dV_M)$ the spaces of L^2 integrable (p, q) forms over M with respect to ω and dV_M . It is known that $L_{p,q}^2(M, \omega, dV_M)$ is a Hilbert space.

Lemma 9.1. Let $\{u_n\}_{n=1}^{+\infty}$ be a sequence of (p,q) forms in $L^2_{p,q}(M,\omega,dV_M)$ which is weakly convergent to u. Let $\{v_n\}_{n=1}^{+\infty}$ be a sequence of Lebesgue measurable real functions on M which converges pointwisely to v. We assume that there exists a constant $C > 0$ such that $|v_n| \leq C$ for any n. Then $\{v_n u_n\}_{n=1}^{+\infty}$ weakly converges to vu in $L^2_{p,q}(M,\omega,dV_M)$.

Proof. Let $g \in L^2_{p,q}(M, \omega, dV_M)$. Consider

$$
I = |\langle v_n u_n, g \rangle - \langle vu, g \rangle|
$$

\n
$$
= |\int_M (v_n u_n, g)_{\omega} dV_M - \int_M (vu, g)_{\omega} dV_M|
$$

\n
$$
\leq |\int_M (v_n u_n - v u_n, g)_{\omega} dV_M| + |\int_M (v u_n - v u, g)_{\omega} dV_M|
$$

\n
$$
= |\int_M (u_n, v_n g - v g)_{\omega} dV_M| + |\int_M (u_n - u, v g)_{\omega} dV_M|
$$

\n
$$
\leq ||u_n|| \cdot ||v_n g - v g|| + |\int_M (u_n - u, v g)_{\omega} dV_M|.
$$

Denote $I_1 := ||u_n|| \cdot ||v_n g - v g||$ and $I_2 := ||\int_M (u_n - u, v g)_{\omega} dV_M||$. It follows from ${u_n}_{n=1}^{+\infty}$ weakly converges to u that $||u_n||$ is uniformly bounded with respect to n. Note that $|v_n|$ is uniformly bounded with respect to n. We know that $|v| < C$ and then $vg \in L^2_{p,q}(M,\omega,dV_M)$. Hence we have $I_2 \to 0$ as $n \to +\infty$. It follows from Lebesgue dominated convergence theorem that we have $\lim_{n\to+\infty} I_1 = 0$.

Hence $\lim_{n\to+\infty} I = 0$ and we know $\{v_n u_n\}_{n=1}^{+\infty}$ weakly converges to vu in L^2_n $P_{p,q}(M,\omega,dV_M).$

Lemma 9.2 (see [\[12\]](#page-71-0)). Let Q be a Hermitian vector bundle on a Kähler manifold M of dimension n with a Kähler metric ω . Assume that $\eta, g > 0$ are smooth functions on M. Then for every form $v \in D(M, \wedge^{n,q}T^*M \otimes Q)$ with compact support we have

$$
\int_{M} (\eta + g^{-1}) |D''^* v|_{Q}^2 dV_M + \int_{M} \eta |D'' v|_{Q}^2 dV_M
$$
\n
$$
\geq \int_{M} \langle [\eta \sqrt{-1} \Theta_Q - \sqrt{-1} \partial \bar{\partial} \eta - \sqrt{-1} g \partial \eta \wedge \bar{\partial} \eta, \Lambda_\omega] v, v \rangle_{Q} dV_M.
$$
\n(9.1)

The following approximation result can be referred to [\[13\]](#page-72-2). Let (X, ω) be a hermitian manifold. Let Q be a holomorphic vector bundle on X and h be a hermitian metric on Q. Denote $D(M, \wedge^{n,q}T^*M \otimes Q)$ be the space of Q-valued smooth (n, q) forms with compact support for any $q \geq 0$. Let $D'' : L^2(X, \wedge^{n,q} T^*M \otimes Q) \rightarrow$

 $L^2(X, \wedge^{n,q+1}T^*M \otimes Q)$ be the extension of $\overline{\partial}$ -operator in the sense of distribution. Let D''^* be the adjoint operator of D'' in the Von-Neumann sense.

Lemma 9.3 (see [\[13\]](#page-72-2)). Assume that (X, ω) is complete. Then $D(M, \wedge^n \cdot T^*M \otimes Q)$ is dense in DomD'', $DomD''^*$ and $DomD'' \cap DomD''^*$ respectively for the graph norms

 $u \to ||u|| + ||D''u||, u \to ||u|| + ||D''^*u||, u \to ||u|| + ||D''u|| + ||D''^*u||.$

Lemma 9.4 (Lemma 4.2 in [\[35\]](#page-72-3)). Let Q be a Hermitian vector bundle on a Kähler manifold M of dimension n with a Kähler metric ω . Let θ be a continuous (1,0) form on M. Then we have

$$
[\sqrt{-1}\theta \wedge \bar{\theta}, \Lambda_{\omega}]\alpha = \bar{\theta} \wedge (\alpha_{\mathsf{L}}(\bar{\theta})^{\sharp}), \qquad (9.2)
$$

for any $(n,1)$ form α with value in Q. Moreover, for any positive (1,1) form β , we have $[\beta, \Lambda_{\omega}]$ is semipositive.

We need the following propositions of positive definite hermitian matrices.

Let $\mathcal{M} := \{ M \in M_n(\mathbb{C}) : M \text{ is a positive definite hermitian matrix} \}.$ Note that $M_n(\mathbb{C})$ is an $2n^2$ -dimensional real manifold. Then M is an n^2 -dimensional real sub-manifold of $M_n(\mathbb{C})$. Denote $F: M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by $F(X) = X^2$ for any $X \in M_n(\mathbb{C})$. Denote $F|_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$. We have the following property of $F|_{\mathcal{M}}$.

Lemma 9.5. $F|_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is a smooth diffeomorphism.

Proof. It is easy to see that $F|_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is a smooth injection.

Let $M \in \mathcal{M} \subset M_n(\mathbb{C})$ be any positive definite hermitian matrix. Then M can be viewed as a self-adjoint positive definite linear map on \mathbb{C}^n . Then we can find a unitary matrix P such that $M = P^{-1} \tilde{M} P$, where $\tilde{M} = diag(\lambda_1, \lambda_2, \dots, \lambda_n)$ is a diagonal matrix and all $\lambda_i \in \mathbb{R}_{>0}$. Denote $\tilde{N} := diag(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n})$. Then we have $M = P^{-1} \tilde{N} P P^{-1} \tilde{N} P = N^2$, where $N := P^{-1} \tilde{N} P$ is a positive definite hermitian matrix. Then we have $M = N^2$. By the theory of positive linear operator, we know that N is unique. Hence we know that $F|_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is surjective and the inverse mapping $(F|_{\mathcal{M}})^{-1} : \mathcal{M} \to \mathcal{M}$ of $F|_{\mathcal{M}}$ exists.

Assume that X is a positive definite hermitian matrix. Let dF_X be the tangent map induced by F at point $X \in M_n(\mathbb{C})$. Then for any matrix $Y \in T_X(M_n(\mathbb{C})) \cong$ $M_n(\mathbb{C})$, $dF(Y) = \lim_{t\to 0} \frac{F(X+tY) - F(X)}{t} = XY + YX$. As X is a positive definite hermitian matrix. We can find a unitary matrix Q such that $X = Q^{-1} \tilde{X} Q$, where \tilde{X} is a diagonal matrix and denote \tilde{Y} by the equation $Y = Q^{-1} \tilde{Y} Q$. Then $XY +$ $YX = 0$ if and only if $\tilde{X}\tilde{Y} + \tilde{Y}\tilde{X} = 0$. As \tilde{X} is a diagonal matrix, we know that $\tilde{X}\tilde{Y} + \tilde{Y}\tilde{X} = 0$ if and only if $\tilde{Y} = 0$ which implies that $XY + YX = 0$ if and only if $Y = 0$. Hence we know that dF_X is non-degenerate at X when X is a positive definite hermitian matrix. Hence we know that F^{-1} exists locally near X and F^{-1} is smooth.

By the uniqueness of inverse map, we know that $(F|_{\mathcal{M}})^{-1} = F^{-1}|_{\mathcal{M}}$, hence $(F|_{\mathcal{M}})^{-1}$ is a smooth map from $\mathcal{M} \to \mathcal{M}$. We have proved that $F|_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is a smooth diffeomorphism.

Remark 9.6. Let $A_k = (a_{ij}^k) \in M_n(\mathbb{C})$ and $A = (a_{ij}) \in M_n(\mathbb{C})$ be a family of $n \times n$ positive definite hermitian matrices such that $\lim_{k\to+\infty} A_k = A$ (which means for any $i, j \in \{1, 2, \ldots n\}$, $\lim_{k \to +\infty} a_{ij}^k = a_{ij}$. Then there exists a unique family of $n \times n$ positive definite hermitian matrices $B_k = (b_{ij}^k)$ and $B = (b_{ij})$ such that $B_k^2 = A_k$ and $B^2 = A$. More over, we have $\lim_{k \to +\infty} B_k = B$.

Proof. Denote $B_k := (F|\mathcal{M})^{-1}(A_k)$ and $B := (F|\mathcal{M})^{-1}(A)$. Then we have the existence and uniqueness of B_k and B. As $\lim_{k \to +\infty} A_k = A$, by the smoothness of $(F|_{\mathcal{M}})^{-1}$, we know that $\lim_{k \to +\infty} B_k = B$. Hence we have remark 9.6. $(F|_{\mathcal{M}})^{-1}$, we know that $\lim_{k\to+\infty} B_k = B$. Hence we have remark [9.6.](#page-47-0)

Lemma 9.7. Let A and B be two $n \times n$ positive definite hermitian matrices. Then there exists a unique matrix C with positive eigenvalue such that $A = CB\overline{C}^T$ and $CB = B\overline{C}^T$. The matrix C depends smoothly on A and B in $\mathcal{M} \times \mathcal{M}$. Especially, if $\lim_{i\to\infty} A_i = A_0$ and $\lim_{i\to\infty} B_i = B_0$, then we have $\lim_{i\to\infty} C_i = C_0$.

Proof. It follows from Remark [9.6](#page-47-0) that there exists a unique positive definite hermitian matrix b such that $B = b^2$ and the matrix b depends smoothly on B in M. As $b = \overline{b}^T$, we know that $b^{-1}Ab^{-1}$ is a positive definite hermitian matrices. It follows from Remark [9.6](#page-47-0) that there exists a unique positive definite hermitian matrix a such that $b^{-1}Ab^{-1} = a^2$ and we note that the matrix a depends smoothly on A and B in $\mathcal{M} \times \mathcal{M}$. Denote $C := bab^{-1}$. Then C depends smoothly on A and B in $\mathcal{M} \times \mathcal{M}$. We note that all eigenvalues of C are positive and $\overline{C}^T = b^{-1}ab$. We have

$$
CB\overline{C}^T = bab^{-1}b^2b^{-1}ab = ba^2b = A,
$$

and

$$
CB = bab^{-1}b^2 = bab = b^2b^{-1}ab = B\overline{C}^T.
$$

Now we prove the uniqueness of $C.$ Assume that there exists another \tilde{C} satisfies $\tilde{C}B\overline{\tilde{C}}^T = A$ and $\tilde{C}B = B\overline{\tilde{C}}^T$. It follows from $\tilde{C}B = B\overline{\tilde{C}}^T$ and $B = b^2$ that we have $b^{-1}\tilde{C}b = b\overline{\tilde{C}}^T b^{-1}$, which shows that $b^{-1}\tilde{C}b$ is a hermitian matrix. We note that

$$
(b^{-1}\tilde{C}b)^2 = b^{-1}\tilde{C}bb\overline{\tilde{C}}^T b^{-1} = b^{-1}\tilde{C}B\overline{\tilde{C}}^T b^{-1} = b^{-1}Ab^{-1}.
$$

By the uniqueness of a such that $b^{-1}Ab^{-1} = a^2$, we know that $b^{-1}\tilde{C}b = a$ and then we have $C = \tilde{C} = bab^{-1}$.

If ${A_i}_{i=0}^{+\infty}$ and ${B_i}_{i=0}^{+\infty}$ satisfy $\lim_{i\to+\infty} A_i = A_0$ and $\lim_{i\to+\infty} B_i = B_0$, then we have C_i such that $A_i = C_i B \overline{C_i}^T$ and $C_i B = B \overline{C_i}^T$, for any $i \ge 0$. As C_i depends smoothly on A_i and B_i in $\mathcal{M} \times \mathcal{M}$ for any $i \geq 0$, we know that $\lim_{i \to +\infty} C_i = C_0$. \Box

Let X be an n-dimensional complex manifold and ω be a hermitian metric on X. Let Q be a vector bundle on X with rank r. Let $\{h_i\}_{i=1}^{+\infty}$ be a family of $C²$ smooth hermitian metric on Q and h be a measurable metric on Q such that $\lim_{i\to+\infty} h_i = h$ almost everywhere on X. We assume that $\{h_i\}_{i=1}^{+\infty}$ and h satisfy one of the following conditions,

(A) h_i is increasingly convergent to h as $i \to +\infty$;

(B) there exists a continuous metric \hat{h} on Q and a constant $C > 0$ such that for any $i \geq 0$, $\frac{1}{C}\hat{h} \leq h_i \leq C\hat{h}$ and $\frac{1}{C}\hat{h} \leq h \leq C\hat{h}$.

Denote $\mathcal{H}_i := L^2(X, K_X \otimes \overline{Q}, h_i, dV_\omega)$ and $\mathcal{H} := L^2(X, K_X \otimes \overline{Q}, h, dV_\omega)$. Note that $\mathcal{H} \subset \mathcal{H}_i \subset \mathcal{H}_1$ for any $i \in \mathbb{Z}_{>0}$.

Lemma 9.8. There exists a linear isomorphism $H_i: \mathcal{H}_i \to \mathcal{H}_1$ (and $H: \mathcal{H} \to \mathcal{H}_1$) which preserves inner product, i.e., for any $\alpha, \beta \in \mathcal{H}_i$ (or $\tilde{\alpha}, \tilde{\beta} \in \mathcal{H}$),

$$
\langle \alpha, \beta \rangle_{h_i} = \langle H_i(\alpha), H_i(\beta) \rangle_{h_1} \, (and \, \langle \tilde{\alpha}, \tilde{\beta} \rangle_h = \langle H(\tilde{\alpha}), H(\tilde{\beta}) \rangle_{h_1}),
$$

and satisfies $H_i^{-1}: \mathcal{H}_1 \to \mathcal{H}_i \subset \mathcal{H}_1$ (and $H^{-1}: \mathcal{H}_1 \to \mathcal{H} \subset \mathcal{H}_1$) is self-adjoint. Moreover, $H_i^{-1}(\gamma)$ converges to $H^{-1}(\gamma)$ point-wisely for any $\gamma \in \mathcal{H}_1$.

Proof. We firstly consider the local case.

Let $X = \Omega \subset \mathbb{C}^n$ be a bounded domain and $Q = \Omega \times \mathbb{C}^r$. In the local case, every metric h_i (or h) on Q can be viewed as a positive definite hermitian matrix $h_i = (h_{kl}^i(x))$ (or $h = (h_{kl}(x))$) where all $\{h_{kl}^i(x)\}_{k,l=1}^r$ are C^2 smooth functions on Ω (all $\{h_{kl}(x)\}_{k,l=1}^r$ are measurable functions on Ω). It follows from Lemma [9.7](#page-48-0) that there exists C_i (or C) such that $h_i = C_i h_1 \overline{C_i}^T$ and $C_i h_1 = h_1 \overline{C_i}^T$ ($h = C h_1 \overline{C}^T$ and $Ch_1 = h_1 \overline{C}^T$). By Lemma [9.7,](#page-48-0) we also know that $C_i := (C_{k,l}^i(x))_{k,l=1}^r$ is C^2 smooth matrix functions, $C := (C_{k,l}(x))_{k,l=1}^r$ is measurable matrix function and $\lim_{i\to+\infty} C_i(x) = C(x)$ almost everywhere. Then for any measurable section $f = (f_1, f_2, \ldots, f_r)$ of $Q = \Omega \times \mathbb{C}^r$, denote $H_i(f) := (f_1, f_2, \ldots, f_r)C_i$ and $H(f) :=$ $(f_1, f_2, \ldots, f_r)C$. Then for any $\alpha, \beta \in \mathcal{H}_i$,

$$
\langle \alpha, \beta \rangle_{h_i} = \alpha(h_{rl}^i) \overline{\beta}^T = \alpha C_i h_1 \overline{C_i}^T \overline{\beta}^T = \langle H_i(\alpha), H_i(\beta) \rangle_{h_1},
$$

As $C_i h_1 = h_1 \overline{C_i}^T$, we know that $H_i^{-1} : \mathcal{H}_1 \to \mathcal{H}_i \subset \mathcal{H}_1$ is self-adjoint. Similar discussion shows that for any $\tilde{\alpha}, \tilde{\beta} \in \mathcal{H}$,

$$
\langle \tilde{\alpha}, \tilde{\beta} \rangle_h = \langle H(\tilde{\alpha}), H(\tilde{\beta}) \rangle_{h_1},
$$

and $H^{-1}: \mathcal{H}_1 \to \mathcal{H} \subset \mathcal{H}_1$ is self-adjoint.

It follows from $\lim_{i\to+\infty} C_i = C$ that we know that $H_i^{-1}(f)$ converges to $H^{-1}(f)$ almost everywhere on Ω for any measurable section $f = (f_1, f_2, \ldots, f_r)$ of $Q =$ $\Omega \times \mathbb{C}^r$. C_i and C are obviously linear isomorphisms. Hence in the local case, we have found linear isomorphism satisfying all the requirements in Lemma [9.8.](#page-48-1)

Now we prove the existences of H_i and H in the global case. Let U_α and U_β be two open subsets of X such that $U_{\alpha} \cap U_{\beta} \neq \emptyset$. Let $G_{\alpha\beta}: U_{\alpha} \cap U_{\beta} \to GL_r(\mathbb{C})$ be the transition functions of Q. Then we know that the representative matrices of metric H_i^{α} and H_i^{β} under different basis are congruent, i.e. $H_i^{\alpha} = G_{\alpha\beta}^T H_i^{\beta} \overline{G_{\alpha\beta}}$, for all $i \in \mathbb{Z}_{\geq 0}$. On U_{α} , we have

$$
H_i^{\alpha} = C_i^{\alpha} H_1^{\alpha} (\overline{C_i^{\alpha}}^T)
$$
 and
$$
C_i^{\alpha} H_1^{\alpha} = H_1^{\alpha} (\overline{C_i^{\alpha}}^T).
$$

Similarly, on U_β , we have

$$
H_i^{\beta} = C_i^{\beta} H_1^{\beta} (\overline{C_i^{\beta}}^T) \text{ and } C_i^{\beta} H_1^{\beta} = H_1^{\beta} (\overline{C_i^{\beta}}^T).
$$

On $U_{\alpha} \cap U_{\beta}$, we have

$$
G_{\alpha\beta}^T H_i^{\beta} \overline{G_{\alpha\beta}} = H_i^{\alpha}
$$

= $C_i^{\alpha} H_1^{\alpha} (\overline{C_i^{\alpha}}^T) = C_i^{\alpha} G_{\alpha\beta}^T H_1^{\beta} \overline{G_{\alpha\beta}} (\overline{C_i^{\alpha}}^T).$ (9.3)

On $U_{\alpha} \cap U_{\beta}$, it follows from $C_i^{\alpha} H_1^{\alpha} = H_1^{\alpha}(\overline{C_i^{\alpha}})$ T) that

$$
C_i^{\alpha} G_{\alpha\beta}^T H_1^{\beta} \overline{G_{\alpha\beta}} = G_{\alpha\beta}^T H_1^{\beta} \overline{G_{\alpha\beta}} (\overline{C_i^{\alpha}}^T)
$$
\n(9.4)

Denote $\hat{C}_{\alpha} := (G_{\alpha\beta}^T)^{-1} C_i^{\alpha} G_{\alpha\beta}^T$. Then we have \hat{C}_{α} $T = (\overline{G_{\alpha\beta}})\overline{C_i^{\alpha}}$ $T(\overline{G_{\alpha\beta}})^{-1}$. It follows from equalities (9.3) , (9.4) that we have

$$
H_i^{\beta} = \hat{C}_{\alpha} H_1^{\beta} \overline{\hat{C}_{\alpha}}^T \text{ and } \overline{\hat{C}_{\alpha}}^T H_1^{\beta} = H_1^{\beta} \overline{\hat{C}_{\alpha}}^T.
$$

It follows from the uniqueness of C_i^{β} that we have $C_i^{\beta} = \hat{C}_{\alpha} := (G_{\alpha\beta}^T)^{-1} C_i^{\alpha} G_{\alpha\beta}^T$, which shows that $H_i(f) = (f_1, \ldots, f_r)C_i$ can be defined globally. Similar discussion shows that $H(f) = (f_1, \ldots, f_r)C$ can be defined globally.

Lemma [9.8](#page-48-1) has been proved.

$$
\qquad \qquad \Box
$$

Let X, Q, $\{h_i\}_{i=1}^{+\infty}$ and h be as in Lemma [9.8.](#page-48-1) Denote $P_i := \mathcal{H}_i \to \text{Ker}D''$ and $P := \mathcal{H} \to \text{Ker}D''$ be the orthogonal projections with respect to h_i and h respectively.

Lemma 9.9. For any sequence of Q-valued $(n,0)$ -forms $\{f_i\}_{i=1}^{+\infty}$ which satisfies $f_i \in \mathcal{H}_i$ and $||f_i||_{h_i} \leq C_1$ for some constant $C_1 > 0$, there exists a Q-valued $(n,0)$ -form $f_0 \in \mathcal{H}$ such that there exists a subsequence of $\{f_i\}_{i=1}^{+\infty}$ (also denoted by ${f_i}_{i=1}^{+\infty}$ weakly converges to f_0 in \mathcal{H}_1 and $P_i(f_i)$ weakly converges to $P(f_0)$ in \mathcal{H}_1 .

Proof. Denote $a_i = P_i(f_i)$ and $b_i := f_i - P_i(f_i)$. We note that $b_i \in (\text{Ker}D'')_i^{\perp} \subset \mathcal{H}_i$, where $(\text{Ker}D'')_i^{\perp}$ is the orthogonal complement space of $\text{Ker}D''$ in \mathcal{H}_i with respect to h_i . It follows from $||f_i||_{h_i} \leq C_1$ that we have $||a_i||_{h_i} \leq C_1$ and $||b_i||_{h_i} \leq C_1$. Denote

 $F_i := H_i(f_i), \ A_i := H_i(a_i) \text{ and } B_i := H_i(b_i).$

Then we know that $||A_i||_{h_1} = ||a_i||_{h_i} \leq C_1$ is uniformly bounded. Since the closed unit ball of the Hilbert space is weakly compact, we can extract a subsequence of $\{A_i\}_{i=1}^{+\infty}$ (still denoted by A_i) weakly convergent to some A_0 in \mathcal{H}_1 . For similar reason, we know that ${B_i}_{i=1}^{+\infty}$ weakly converges to some B_0 in \mathcal{H}_1 and ${F_i}_{i=1}^{+\infty}$ weakly converges to some F_0 in \mathcal{H}_1 .

Let $\beta \in \mathcal{H}_1$. When $\{h_i\}_{i=1}^{+\infty}$ and h satisfy condition (A) , it follows from dominated convergence theorem,

 $||H^{-1}(\beta)||_{h_1} \leq ||H^{-1}(\beta)||_h = ||\beta||_{h_1}$ and $||H_i^{-1}(\beta)||_{h_1} \leq ||H_i^{-1}(\beta)||_{h_i} = ||\beta||_{h_1}$, that we have $||H_i^{-1}(\beta) - H^{-1}(\beta)||_{h_1} \to 0$ as $i \to +\infty$. When $\{h_i\}_{i=1}^{+\infty}$ and h satisfy condition (B) , it follows from dominated convergence theorem,

 $||H^{-1}(\beta)||_{h_1} \leq C^2 ||H^{-1}(\beta)||_h = C^2 ||\beta||_{h_1}$ and $||H_i^{-1}(\beta)||_{h_1} \leq C^2 ||H_i^{-1}(\beta)||_{h_i} = C^2 ||\beta||_{h_1}$, that we have $||H_i^{-1}(\beta) - H^{-1}(\beta)||_{h_1} \to 0$ as $i \to +\infty$.

Then when $\{h_i\}_{i=1}^{+\infty}$ and h satisfy condition (A) or (B) , we have

$$
\lim_{i \to +\infty} \int_X \langle a_i, \beta \rangle_{h_1} dV_\omega
$$
\n
$$
= \lim_{i \to +\infty} \int_X \langle H_i(a_i), H_i^{-1}(\beta) \rangle_{h_1} dV_\omega
$$
\n
$$
= \lim_{i \to +\infty} \int_X \langle H_i(a_i), H^{-1}(\beta) \rangle_{h_1} dV_\omega + \lim_{i \to +\infty} \int_X \langle H_i(a_i), H_i^{-1}(\beta) - H^{-1}(\beta) \rangle_{h_1} dV_\omega
$$
\n
$$
= \int_X \langle A_0, H^{-1}(\beta) \rangle_{h_1} dV_\omega
$$
\n
$$
= \int_X \langle H^{-1}(A_0), \beta \rangle_{h_1} dV_\omega,
$$

where the first equality holds because of H_i^{-1} is self-adjoint and the third equality holds because of

$$
\int_M \langle H_i(a_i), H_i^{-1}(\beta) - H^{-1}(\beta) \rangle_{h_1} dV_\omega \le ||H_i(a_i)||_{h_1} ||H_i^{-1}(\beta) - H^{-1}(\beta)||_{h_1},
$$

 $||H_i(a_i)||_{h_1} = ||a_i||_{h_i} \leq C_1$, $||H_i^{-1}(\beta) - H^{-1}(\beta)||_{h_1} \to 0$ as $i \to +\infty$. Denote $a_0 := H^{-1}(A_0)$. Then $a_0 \in \mathcal{H} \subset \mathcal{H}_1$ and we know that a_i weakly converges to a_0 in \mathcal{H}_1 . It follows from $D''(a_i) = 0$ that we have $D''(a_0) = 0$. Denote $b_0 := H^{-1}(B_0) \in \mathcal{H} \subset \mathcal{H}_1$ and $f_0 := H^{-1}(F_0) \in \mathcal{H} \subset \mathcal{H}_1$. Using similar discussion, we know that b_i weakly converges to b_0 in \mathcal{H}_1 and f_i weakly converges to f_0 in \mathcal{H}_1 .

It follows from the uniqueness of weak limit and $f_i = a_i + b_i$ that we have $f_0 = a_0 + b_0$ in H. Now we prove that $b_0 \in (\text{Ker} D'')^{\perp} \subset H$, where $(\text{Ker} D'')^{\perp}$ is the orthogonal complement space of KerD'' in H with respect to h. Let $\gamma \in \text{Ker}D'' \subset$ H. We have

$$
\int_{X} \langle b_{0}, \gamma \rangle_{h} dV_{\omega}
$$
\n
$$
= \int_{X} \langle H(b_{0}), H(\gamma) \rangle_{h_{1}} dV_{\omega}
$$
\n
$$
= \lim_{i \to +\infty} \int_{X} \langle H_{i}(b_{i}), H_{i}(\gamma) \rangle_{h_{1}} dV_{\omega} + \lim_{i \to +\infty} \int_{X} \langle H_{i}(b_{i}), H(\gamma) - H_{i}(\gamma) \rangle_{h_{1}} dV_{\omega} \quad (9.5)
$$
\n
$$
\leq \lim_{i \to +\infty} \int_{X} \langle H_{i}(b_{i}), H_{i}(\gamma) \rangle_{h_{1}} dV_{\omega} + \lim_{i \to +\infty} ||b_{i}||_{h_{i}} ||H(\gamma) - H_{i}(\gamma)||_{h_{1}}
$$
\n
$$
= \lim_{i \to +\infty} \int_{X} \langle b_{i}, \gamma \rangle_{h_{i}} dV_{\omega} + \lim_{i \to +\infty} ||b_{i}||_{h_{i}} ||H(\gamma) - H_{i}(\gamma)||_{h_{1}}
$$

When $\{h_i\}_{i=1}^{+\infty}$ and h satisfy condition (A) , it follows from dominated convergence theorem and

$$
||H(\gamma)||_{h_1} = ||\gamma||_h \text{ and } ||H_i(\gamma)||_{h_1} = ||\gamma||_{h_i} \le ||\gamma||_h,
$$

that we have $||H(\gamma) - H_i(\gamma)||_{h_1} \to 0$ as $i \to +\infty$. When $\{h_i\}_{i=1}^{+\infty}$ and h satisfy condition (B) , it follows from dominated convergence theorem and

$$
||H(\gamma)||_{h_1} = ||\gamma||_h \text{ and } ||H_i(\gamma)||_{h_1} = ||\gamma||_{h_i} \leq C^2 ||\gamma||_h,
$$

that we have $||H(\gamma) - H_i(\gamma)||_{h_1} \to 0$ as $i \to +\infty$. Note that $||b_i||_{h_i}$ is uniformly bounded with respect to *i*. It follows from above discussion, $b_i \in (\text{Ker}D'')_i^{\perp}$ and inequality [\(9.5\)](#page-51-0) that we have

$$
\int_X \langle b_0, \gamma \rangle_h dV_\omega = 0.
$$

Hence we know $b_0 \in (\text{Ker} D'')^{\perp} \subset \mathcal{H}$. Hence we have $P(f_0) = a_0$ and we have showed that $a_i = P_i(f_i)$ weakly converges to $a_0 = P(f_0)$ in \mathcal{H}_1 .

Lemma [9.9](#page-50-0) has been proved.

 \Box

Lemma 9.10. Let (M, ω) be a complete Kähler manifold equipped with a (nonnecessarily complete) Kähler metric ω , and let (Q, h) be a hermitian vector bundle over M. Assume that η and g are smooth bounded positive functions on M such that $\eta + g^{-1}$ is a smooth bounded positive function on M and let B := $[\eta\sqrt{-1}\Theta_Q - \sqrt{-1}\partial\bar{\partial}\eta - \sqrt{-1}g\partial\eta \wedge \bar{\partial}\eta, \Lambda_\omega]$. Assume that $\lambda \geq 0$ is a bounded continuous functions on M such that $B + \lambda I$ is positive definite everywhere on $\wedge^{n,q}T^*M \otimes Q$ for some $q \geq 1$. Then given a form $v \in L^2(M, \wedge^{n,q}T^*M \otimes Q)$ such that $D''v = 0$ and $\int_M \langle (B + \lambda I)^{-1}v, v \rangle_{Q,\omega} dV_\omega < +\infty$, there exists an approximate solution $u \in L^2(M, \wedge^{n,q-1}T^*M \otimes Q)$ and a correcting term $\tau \in L^2(M, \wedge^{n,q}T^*M \otimes Q)$

such that $D''u + P_h(\sqrt{\lambda}\tau) = v$, where $P_h: L^2(M, \wedge^{n,q}T^*M \otimes Q) \to Ker D''$ is the orthogonal projection and

$$
\int_M (\eta + g^{-1})^{-1} |u|^2_{Q,\omega} dV_\omega + \int_M |\tau|^2_{Q,\omega} dV_\omega \le \int_M \langle (B + \lambda I)^{-1} v, v \rangle_{Q,\omega} dV_\omega. \tag{9.6}
$$

Proof. Let $\tilde{\omega}$ be a complete Kähler metric on M. Denote $\omega_{\epsilon} = \omega + \epsilon \tilde{\omega}$, where $\epsilon \in \epsilon$ [0, 1]. Then ω_{ϵ} is a complete Kähler metric on M for any $\epsilon > 0$. For any Q-valued smooth (n, q) form α with compact support, we have $\alpha = \alpha_1 + \alpha_2$ where $\alpha_1 \in \text{Ker}D''$ and $\alpha_2 \in (\text{Ker } D'')^{\perp} = \text{Im } D''^* \subset \text{Ker } D''^*$. It follows from $\alpha \in \text{Dom } D'' \cap \text{Dom } D''^*$ and $\alpha_2 \in \text{Dom}D''^*$ that we have $\alpha_1 \in \text{Dom}D''^*$. For similar reason, we know that $\alpha_2 \in \text{Dom}D''$. Then it follows from Lemma [9.3](#page-47-1) and η , g and $\eta + g^{-1}$ are smooth bounded positive functions on M that inequality [\(9.1\)](#page-46-0) in Lemma [9.2](#page-46-1) also holds for α_1 and α_2 . By using Cauchy-Schwarz inequality, inequality [\(9.1\)](#page-46-0) and $\alpha_2 \in \text{Ker}D''^*$, we have

$$
\begin{split}\n&|\langle v, \alpha \rangle|_{\omega_{\epsilon},h}^{2} \\
&= |\langle v, \alpha_{1} \rangle|_{\omega_{\epsilon},h}^{2} \\
&\leq (\int_{M} \langle (B + \lambda I)^{-1} v, v \rangle_{\omega_{\epsilon},h} dV_{\omega_{\epsilon}}) \left(\int_{M} \langle B\alpha_{1}, \alpha_{1} \rangle_{\omega_{\epsilon},h} dV_{\omega_{\epsilon}} + \int_{M} \langle \lambda \alpha_{1}, \alpha_{1} \rangle_{\omega_{\epsilon},h} dV_{\omega_{\epsilon}} \right) \\
&= (\int_{M} \langle (B + \lambda I)^{-1} v, v \rangle_{\omega_{\epsilon},h} dV_{\omega_{\epsilon}}) \left(||(\eta + g^{-1})^{\frac{1}{2}} D''^{*} \alpha_{1} ||_{\omega_{\epsilon},h}^{2} + ||\sqrt{\lambda} \alpha_{1} ||_{\omega_{\epsilon},h}^{2} \right) \\
&= (\int_{M} \langle (B + \lambda I)^{-1} v, v \rangle_{\omega_{\epsilon},h} dV_{\omega_{\epsilon}}) \left(||(\eta + g^{-1})^{\frac{1}{2}} D''^{*} \alpha ||_{\omega_{\epsilon},h}^{2} + ||\sqrt{\lambda} P_{\omega_{\epsilon},h} \alpha ||_{\omega_{\epsilon},h}^{2} \right),\n\end{split} \tag{9.7}
$$

where $P_{\omega_{\epsilon},h}: L^2(M,\wedge^{n,q}T^*M\otimes Q,\omega_{\epsilon},h)\to \text{Ker}D''$ is the projection map. Denote $H_{1,\epsilon} := L^2(M, \wedge^{n,q-1}T^*M \otimes Q, \omega_{\epsilon}, h)$ and $H_{2,\epsilon} := L^2(M, \wedge^{n,q}T^*M \otimes Q, \omega_{\epsilon}, h)$. Then it follows from Hahn-Banach theorem and inequality [\(9.7\)](#page-52-0) that we have a bounded linear map $H_{1,\epsilon} \oplus H_{2,\epsilon} \to \mathbb{C}$, which is an extension of the following linear map

$$
((\eta + g^{-1})^{\frac{1}{2}} D''^* \alpha, P_{\omega_{\epsilon},h}(\alpha)) \to \langle v, \alpha \rangle_{\omega_{\epsilon},h}.
$$

Then there exist \tilde{u}_{ϵ} and τ_{ϵ} such that

$$
\langle v, \alpha \rangle_{\omega_{\epsilon},h} = \langle \tilde{u}_{\epsilon}, (\eta + g^{-1})^{\frac{1}{2}} D^{\prime\prime*} \alpha \rangle_{\omega_{\epsilon},h} + \langle \tau_{\epsilon}, \sqrt{\lambda} P_{\omega_{\epsilon},h}(\alpha) \rangle_{\omega_{\epsilon},h},
$$

and

$$
||\tilde{u}_{\epsilon}||_{\omega_{\epsilon},h} + ||\tau_{\epsilon}||_{\omega_{\epsilon},h} \leq \int_{M} \langle (B + \lambda I)^{-1}v,v \rangle_{\omega_{\epsilon},h} dV_{\omega_{\epsilon}}.
$$

Denote $u_{\epsilon} := (\eta + g^{-1})^{\frac{1}{2}} \tilde{u}_{\epsilon}$, then we have

$$
v = D''u_{\epsilon} + P_{\omega_{\epsilon},h}(\sqrt{\lambda}\tau_{\epsilon})
$$
\n(9.8)

and

$$
||(\eta + g^{-1})^{-\frac{1}{2}}u_{\epsilon}||_{\omega_{\epsilon},h} + ||\tau_{\epsilon}||_{\omega_{\epsilon},h} \leq \int_{M} \langle (B + \lambda I)^{-1}v, v \rangle_{\omega_{\epsilon},h} dV_{\omega_{\epsilon}}.
$$
 (9.9)

Note that $\int_M \langle (B + \lambda I)^{-1}v, v \rangle_{\omega_{\epsilon},h} dV_{\omega_{\epsilon}} \leq \int_M \langle (B + \lambda I)^{-1}v, v \rangle_{\omega,h} dV_{\omega}$ for any $\epsilon > 0$. Then we know that $||(\eta + g^{-1})^{-\frac{1}{2}}u_{\epsilon}||_{\omega_{\epsilon},h}$, $||\tau_{\epsilon}||_{\omega_{\epsilon},h}$ and $||\sqrt{\lambda}\tau_{\epsilon}||_{\omega_{\epsilon},h}$ is uniformly bounded with respect to ϵ .

Note that on any compact subset $K \subset M$, we have $\omega \leq \omega_{\epsilon} \leq \omega_1 \leq C_K \omega$ for some $C_K > 0$. It follows from $||(\eta + g^{-1})^{-\frac{1}{2}}u_{\epsilon}||_{\omega_{\epsilon},h}$ is uniformly bounded with respect to ϵ and $\eta + g^{-1}$ is a smooth bounded positive function on any compact subset K of M

that we know that u_{ϵ} weakly converges to some u_0 in $L^2(M, \wedge^{n,q-1}T^*M\otimes Q, \text{loc})$. It follows from Lemma [9.1](#page-46-2) that $(\eta + g^{-1})^{-\frac{1}{2}}u_{\epsilon}$ weakly converges to some $(\eta + g^{-1})^{-\frac{1}{2}}u_0$ in $L^2(M, \wedge^{n,q-1}T^*M \otimes Q, \text{loc})$. Let $\epsilon_1 > 0$ be given. Then we have

$$
\int_{K} ||(\eta + g^{-1})^{\frac{1}{2}} u_0||_{\omega_{\epsilon_1}} \le \liminf_{\epsilon \to 0} \int_{K} ||(\eta + g^{-1})^{\frac{1}{2}} u_{\epsilon}||_{\omega_{\epsilon_1}}\n\n\le \liminf_{\epsilon \to 0} \int_{K} ||(\eta + g^{-1})^{\frac{1}{2}} u_{\epsilon}||_{\omega_{\epsilon}}\n\n\le \liminf_{\epsilon \to 0} \int_{M} ||(\eta + g^{-1})^{\frac{1}{2}} u_{\epsilon}||_{\omega_{\epsilon}}.
$$
\n(9.10)

It follows from Lemma [9.9](#page-50-0) that we know that τ_{ϵ} weakly converges to τ_0 in $\mathcal{H}_{2,1}$, $\overline{\lambda} \tau_{\epsilon}$ weakly converges to $\tilde{\tau}_0$ in $\mathcal{H}_{2,1}$ and $P_{\omega_{\epsilon},h}(\sqrt{\lambda}\tau_{\epsilon})$ weakly converges to $P_{\omega,h}(\tilde{\tau}_0)$ in $\mathcal{H}_{2,1}$. Lemma [9.1](#page-46-2) shows that $\sqrt{\lambda}\tau_{\epsilon}$ weakly converges to $\sqrt{\lambda}\tau_0$ in $\mathcal{H}_{2,1}$ and we know that $\tilde{\tau}_0 = \sqrt{\lambda} \tau_0$ in $\mathcal{H}_{2,1}$ since the weak limit is unique. Hence we have $P_{\omega_{\epsilon},h}(\sqrt{\lambda}\tau_{\epsilon})$ weakly converges to $P_{\omega,h}(\sqrt{\lambda}\tau_0)$ in $\mathcal{H}_{2,1}$.

It follows from τ_{ϵ} weakly converges to τ_0 and $P_{\omega_{\epsilon},h}(\sqrt{\lambda}\tau_{\epsilon})$ weakly converges to $P_{\omega,h}(\sqrt{\lambda}\tau_0)$ in $\mathcal{H}_{2,1}$ that we have τ_{ϵ} weakly converges to τ_0 in $L^2(M, \wedge^{n,q}T^*M \otimes$ $Q,$ loc) and $P_{\omega_e,h}(\sqrt{\lambda}\tau_e)$ weakly converges to $P_{\omega,h}(\sqrt{\lambda}\tau_0)$ in $L^2(M, \wedge^{n,q}T^*M \otimes$ Q, loc . Let K be any compact subset of M. We have

$$
\int_{K} ||\tau_{0}||_{\omega_{\epsilon_{1}}} \leq \liminf_{\epsilon \to 0} \int_{K} ||\tau_{\epsilon}||_{\omega_{\epsilon_{1}}}
$$
\n
$$
\leq \liminf_{\epsilon \to 0} \int_{K} ||\tau_{\epsilon}||_{\omega_{\epsilon}} \tag{9.11}
$$
\n
$$
\leq \liminf_{\epsilon \to 0} \int_{M} ||\tau_{\epsilon}||_{\omega_{\epsilon}}.
$$

Note that u_{ϵ} weakly converges to u_0 in $L^2(M, \wedge^{n,q-1}T^*M\otimes Q, \text{loc})$ and $P_{\omega_{\epsilon},h}(\sqrt{\lambda}\tau_{\epsilon})$ weakly converges to $P_{\omega,h}(\sqrt{\lambda}\tau_0)$ in $L^2(M, \wedge^{n,q}T^*M\otimes Q, \text{loc})$. Letting $\epsilon \to 0$ in [\(9.8\)](#page-52-1), then we have

$$
v = D''u_0 + P_{\omega,h}(\sqrt{\lambda}\tau_0).
$$

It follows from inequalities (9.9) , (9.10) and (9.11) that we have

J

$$
\int_{K} ||(\eta + g^{-1})^{\frac{1}{2}} u_{0}||_{\omega_{\epsilon_{1}}} dV_{\omega_{\epsilon_{1}}} + \int_{K} ||\tau_{0}||_{\omega_{\epsilon_{1}}} dV_{\omega_{\epsilon_{1}}}
$$
\n
$$
\leq \liminf_{\epsilon \to 0} \left(\int_{M} ||(\eta + g^{-1})^{\frac{1}{2}} u_{\epsilon}||_{\omega_{\epsilon}} dV_{\omega_{\epsilon}} + \int_{M} ||\tau_{\epsilon}||_{\omega_{\epsilon}} dV_{\omega_{\epsilon}} \right)
$$
\n
$$
\leq \liminf_{\epsilon \to 0} \int_{M} \langle (B + \lambda I)^{-1} v, v \rangle_{\omega_{\epsilon}, h} dV_{\omega_{\epsilon}}
$$
\n
$$
\leq \int_{M} \langle (B + \lambda I)^{-1} v, v \rangle_{\omega, h} dV_{\omega}.
$$
\n(9.12)

When $\epsilon_1 \rightarrow 0$ in [\(9.12\)](#page-53-2), by monotone convergence theorem, we have

$$
\int_K ||(\eta + g^{-1})^{\frac{1}{2}} u_0||_{\omega} + \int_K ||\tau_0||_{\omega} \le \int_M \langle (B + \lambda I)^{-1} v, v \rangle_{\omega, h} dV_{\omega}.
$$

As K is any compact subset of M , we have

$$
\int_M ||(\eta + g^{-1})^{\frac{1}{2}} u_0||_{\omega} + \int_M ||\tau_0||_{\omega} \le \int_M \langle (B + \lambda I)^{-1} v, v \rangle_{\omega, h} dV_{\omega}.
$$

Lemma [9.10](#page-51-1) has been proved.

Lemma 9.11 (Theorem 6.1 in [\[11\]](#page-71-1), see also Theorem 2.2 in [\[55\]](#page-73-0)). Let (M, ω) be a complex manifold equipped with a hermitian metric ω , and $\Omega \subset\subset M$ be an open set. Assume that $T = \tilde{T} + \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \varphi$ is a closed (1,1)-current on M, where \tilde{T} is a smooth real (1,1)-form and φ is a quasi-plurisubharmonic function. Let γ be a continuous real (1,1)-form such that $T \ge \gamma$. Suppose that the Chern curvature tensor of TM satisfies

$$
(\sqrt{-1}\Theta_{TM} + \varpi \otimes Id_{TM})(\kappa_1 \otimes \kappa_2, \kappa_1 \otimes \kappa_2) \ge 0
$$

$$
\forall \kappa_1, \kappa_2 \in TM \quad with \quad \langle \kappa_1, \kappa_2 \rangle = 0
$$
 (9.13)

 \Box

for some continuous nonnegative $(1,1)$ -form ϖ on M. Then there is a family of closed (1,1)-currents $T_{\zeta,\rho} = \widetilde{T} + \frac{\sqrt{-1}}{\pi} \partial \overline{\partial} \varphi_{\zeta,\rho}$ on M ($\zeta \in (0, +\infty)$ and $\rho \in (0, \rho_1)$ for some positive number ρ_1) independent of γ , such that

(i) $\varphi_{\zeta,\rho}$ is quasi-plurisubharmonic on a neighborhood of $\overline{\Omega}$, smooth on $M\backslash E_{\zeta}(T)$, increasing with respect to ζ and ρ on Ω and converges to φ on Ω as $\rho \to 0$.

(*ii*) $T_{\zeta,\rho} \geq \gamma - \zeta \varpi - \delta_{\rho} \omega$ on Ω .

where $E_{\zeta}(T) := \{x \in M : v(T, x) \geq \zeta\} \; (\zeta > 0)$ is the ζ -upper level set of Lelong numbers and $\{\delta_\rho\}$ is an increasing family of positive numbers such that $\lim_{\rho \to 0} \delta_\rho = 0$.

Remark 9.12 (see Remark 2.1 in [\[55\]](#page-73-0)). Lemma [9.11](#page-54-0) is stated in [\[11\]](#page-71-1) in the case M is a compact complex manifold. The similar proof as in [\[11\]](#page-71-1) shows that Lemma [9.11](#page-54-0) on noncompact complex manifold still holds where the uniform estimate (i) and (ii) are obtained only on a relatively compact subset Ω .

Remark 9.13. Let M be a weakly pseudoconvex Kähler manifold. Let φ be a plurisubharmonic function on M. Then $h := e^{-\varphi}$ is a singular metric on $E :=$ $M \times \mathbb{C}$ in the sense of Definition [1.1](#page-1-0) and h satisfies $\Theta_h(E) \geq_{Nak}^s 0$ in the sense of Definition [1.3.](#page-1-1)

Proof. As M is weakly pseudoconvex, there exists a smooth plurisubharmonic exhaustion function P on M. Let $M_j := \{P < j\} \ (k = 1, 2, \ldots)$. We choose P such that $M_1 \neq \emptyset$. Then M_j satisfies $M_1 \in M_2 \in \dots \in M_j \in M_{j+1} \in \dots$ and $\cup_{j=1}^{n} M_j = M$. Each M_j is weakly pseudoconvex Kähler manifold.

Let $\delta > 0$ be a real number. Denote $\varphi_l := \max\{\varphi, -l\} + \frac{\delta}{l}$. Note that $\varphi_{l+1} - \varphi_l \leq$ $-\frac{1}{l(l+1)}\delta$. We also note that φ_l is a plurisubharmonic function on M and $v(T, x) = 0$ for any $x \in M$.

Let $M = M_{j+1}$, $\Omega = M_j$, $T = \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \psi$, $\gamma = 0$ in Lemma [9.11,](#page-54-0) then there exists a family of functions $\varphi_{j,l,\zeta,\rho}$ $(\zeta \in (0, +\infty)$ and $\rho \in (0, \rho_1)$ for some positive ρ_1) on M_{i+1} such that

(1) $\varphi_{j,l,\zeta,\rho}$ is a quasi-plurisubharmonic function on a neighborhood of $\overline{M_j}$, smooth on M_{j+1} , increasing with respect to ζ and ρ on M_j and converges to φ_l on M_j as $\rho \rightarrow 0$,

 $\sqrt{(2)} \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \varphi_{j,l,\zeta,\rho} \geq -\zeta \varpi - \delta_{\rho} \omega \text{ on } M_j,$ where $\{\delta_{\rho}\}\$ is an increasing family of positive numbers such that $\lim_{\rho\to 0} \delta_{\rho} = 0$.

Let $\rho = \frac{1}{m}$. Let $\tilde{\delta}_m := \delta_{\frac{1}{m}}$ and $\zeta = \tilde{\delta}_m$. Denote $\varphi_{j,l,m} := \varphi_{j,l,\tilde{\delta}_m,\frac{1}{m}}$. Then we have a sequence of functions $\{\varphi_{j,l,m}\}$ satisfying

(1') $\varphi_{j,l,m}$ is quasi-plurisubharmonic function on $\overline{M_j}$, smooth on M_{j+1} , decreasing

with respect to m and converges to φ_l on M_i as $m \to +\infty$, (2') $\frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \varphi_{j,l,m} \geq -\tilde{\delta}_m \varpi - \tilde{\delta}_m \omega$ on M_j ,

where $\{\tilde{\delta}_m\}$ is an decreasing family of positive numbers such that $\lim_{m\to+\infty}\tilde{\delta}_m=0$. As M_i is relatively compact in M, there exists a positive number $b \geq 1$ such that $b\omega \geq \infty$ on M_j . Then condition (2') becomes

 $(2") \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \varphi_{j,l,m} \geq -\tilde{\delta}_m \varpi - \tilde{\delta}_m \omega \geq -2b \tilde{\delta}_m \omega \text{ on } M_j.$

Now, for each $l \geq 1$, we choose $m_l \in \mathbb{Z}_{\geq 1}$ such that $\varphi_{j,l} := \varphi_{j,l,m_l}$ is decreasing with respect to l and converges to φ when $l \to +\infty$. Note that $M_{j-1} \subset\subset M_j \subset\subset$ M_{j+1} . Let m_1 be any positive integer. Now we assume that $\{m_1, m_2, \ldots, m_l\}$ has been chosen. Now we choose m_{l+1} .

Denote $E_{j,l+1,m} := \{x \in M_j | \varphi_{j,l+1,m}(x) - \varphi_{j,l,m_l}(x) < 0\}$ and denote $E_{j,l+1} :=$ $\{x \in M_j | \varphi_{l+1}(x) - \varphi_{j,l,m_l}(x) < 0\}$. Note that $\varphi_{j,l+1,m}$ and φ_{j,l,m_l} is smooth on M_{j+1} , we know that $E_{j,2,m}$ is open subset of M_j for any $m \geq 1$. As $\varphi_{j,l+1,m+1} \leq$ $\varphi_{j,l+1,m}$ on M_j and $\varphi_{l+1} < \varphi_l \leq \varphi_{j,l,m_l}$, we have $E_{j,l+1,1} \subset E_{j,l+1,2} \subset \cdots \subset$ $E_{j,l+1,m} \subset E_{j,l+1,m+1} \subset \cdots \subset E_{j,l+1} = M_j$ for any $m \geq 1$. Hence we know that $\cup_{m=1}^{+\infty} E_{j,l+1,m}$ is an open cover of M_j and then an open cover of $\overline{M_{j-1}}$. By the relatively compactness of M_{i-1} , we know that there exists a positive integer m_{l+1} such that $M_{j-1} \subset E_{j,l+1,m_{l+1}}$. Let $\varphi_{j,l+1} := \varphi_{2,l+1,m_{l+1}}$ and we have $\varphi_{j,l+1}$ $\varphi_{j,l} = \varphi_{1,l,m_l}$ on M_{j-1} .

Hence we can find a sequence of smooth plurisubharmonic functions $\varphi_{j,l}$:= φ_{j,l,m_l} on M_{j-1} which is decreasing with respect to l and converges to φ when $l \rightarrow +\infty$. We note that

$$
\frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \varphi_{j,l} \geq -2b \tilde \delta_{m_l} \omega
$$

on M_{j-1} . Then we know that $(M, M \times \mathbb{C}, \emptyset, M_j, e^{-\varphi}, e^{-\varphi_{j,l}})$ is a singular metric on $M \times \mathbb{C}$ and $\Theta_{\sigma-\varphi}(E) \geq_{j=1}^s$, 0. $M \times \mathbb{C}$ and $\Theta_{e^{-\varphi}}(E) \geq_{Nak}^{s} 0$. $N_{\alpha k}$ 0.

Lemma 9.14 (Theorem 1.5 in [\[10\]](#page-71-2)). Let M be a Kähler manifold, and Z be an analytic subset of M. Assume that Ω is a relatively compact open subset of M possessing a complete Kähler metric. Then $\Omega \backslash Z$ carries a complete Kähler metric.

Lemma 9.15 (Lemma 6.9 in [\[10\]](#page-71-2)). Let Ω be an open subset of \mathbb{C}^n and Z be a complex analytic subset of Ω . Assume that v is a $(p,q-1)$ -form with L^2_{loc} coefficients and h is a (p,q)-form with L_{loc}^1 coefficients such that $\overline{\partial}v = h$ on $\Omega \backslash \overline{Z}$ (in the sense of distribution theory). Then $\partial v = h$ on Ω .

The following notations can be referred to [3].

Let X be a complex manifold. An upper semi-continuous function $u : X \rightarrow$ $[-\infty, +\infty)$ is quasi-plurisubharmonic if it is locally of the form $u = \varphi + g$ where φ is plurisubharmonic and q is smooth. Let θ be a closed real $(1,1)$ form on X. By Poincaré lemma, θ is locally of the form $\theta = dd^c f$ for a smooth real-valued function f which is called a local potential of θ . We call a quasi-plurisubharmonic function u is θ -plurisubharmonic if $\theta + dd^c u \geq 0$ in the sense of currents.

Lemma 9.16 (see [\[13\]](#page-72-2), see also [3]). For arbitrary $\eta = (\eta_1, \dots, \eta_p) \in (0, +\infty)^p$, the function

$$
M_{\eta}(t_1,\ldots,t_p) = \int_{\mathbb{R}^p} \max\{t_1 + h_1,\ldots,t_p + h_p\} \prod_{1 \leq j \leq p} \theta(\frac{h_j}{\eta_j}) dh_1 \ldots dh_p
$$

possesses the following properties:

(1) $M_{\eta}(t_1,\ldots,t_p)$ is non decreasing in all variables, smooth and convex on \mathbb{R}^p ; (2) $\max\{t_1, \ldots, t_p\} \leq M_{\eta}(t_1, \ldots, t_p) \leq \max\{t_1 + \eta_1, \ldots, t_p + \eta_p\};$

$$
(3) M_{\eta}(t_1,\ldots,t_p) = M_{\eta_1,\ldots,\hat{\eta}_j,\ldots,\eta_p}(t_1,\ldots,\hat{t}_j,\ldots,t_p) \text{ if } t_j + \eta_j \leq \max_{k \neq j} \{t_k - \eta_k\};
$$

(4) $M_{\eta}(t_1 + a, ..., t_p + a) = M_{\eta}(t_1, ..., t_p) + a$ for any $a \in \mathbb{R}$;

(5) if u_1, \ldots, u_p are plurisubharmonic functions, then $u = M_n(u_1, \ldots, u_p)$ is plurisubharmonic;

(6) if u_1, \ldots, u_p are θ -plurisubharmonic functions, then $u = M_n(u_1, \ldots, u_p)$ is θ-plurisubharmonic function.

Proof. The proof of $(1)-(5)$ can be referred to $|13|$ and the proof of (6) can be referred to [3]. For the convenience of the readers, we recall the proof of (6).

Let f be a local potential of θ . We know $f + u_i$ is plurisubharmonic function. It follows from (4) and (5) that $M_{\eta}(u_1 + f, \ldots, u_p + f) = M_{\eta}(u_1, \ldots, u_p) + f$ is plurisubharmonic. Hence $u = M_n(u_1, \ldots, u_p)$ is θ -plurisubharmonic function. \square

9.2. Proof of Lemma [2.1.](#page-8-0) Note that $X\backslash\{F = 0\}$ is a weakly pseudoconvex Kähler manifold. The following remark shows that we can assume that F has no zero points on M.

Remark 9.17. As $(X, E, \Sigma, X_j, h, h_s)$ is a singular hermitian metric on E and $\Theta_h(E) \geq_{Nak}^s 0$. We know that for any compact subset K, there exist a relatively compact subset $X_{j_K} \subset X$ containing K and a C^2 smooth hermitian metric $h_{j_K,1}$ on X_{j_K} such that $h_{j_K,1} \leq h$ on $K \subset X_{j_K}$.

Assume that there exists a holomorphic E-valued $(n,0)$ form \hat{F} on $X\backslash\{F=0\}$ such that

$$
\int_{X \setminus \{F=0\}} |\hat{F} - (1 - b_{t_0, B}(\Psi)) f F^{1+\delta}|^2_h e^{v_{t_0, B}(\Psi) - \delta \tilde{M}} c(-v_{t_0, B}(\Psi))
$$
\n
$$
\leq \left(\frac{1}{\delta} c(T) e^{-T} + \int_T^{t_0 + B} c(s) e^{-s} ds\right) \int_{X \setminus \{F=0\}} \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \Psi < -t_0\}} |f F|^2_h.
$$

Let K be any compact subset of X. Both $h_{j_K,1}$ and \hat{h} are C^2 smooth hermitian metrics on E, then there exists a constant $c_K > 0$, such that $h_{j_K,1} \leq c_K \hat{h}$ on K. Note that $M_k := \inf_{K} e^{v_{t_0,B}(\Psi) - \delta M} c(-v_{t_0,B}(\Psi)) > 0$ and $h_{j_K,1} \leq h$. Then we have

$$
\int_{(X \setminus \{F=0\}) \cap K} |\hat{F}|_{h_{j_K,1}}^2
$$
\n
$$
\leq 2 \int_{(X \setminus \{F=0\}) \cap K} |\hat{F} - (1 - b_{t_0, B}(\Psi)) f F^{1+\delta}|_{h_{j_K,1}}^2 + 2 \int_{(X \setminus \{F=0\}) \cap K} |(1 - b_{t_0, B}(\Psi)) f F^{1+\delta}|_{h_{j_K,1}}^2
$$
\n
$$
\leq \frac{2}{M_K} \int_{X \setminus \{F=0\}} |\hat{F} - (1 - b_{t_0, B}(\Psi)) f F^{1+\delta}|_{h^2}^2 e^{v_{t_0, B}(\Psi) - \delta \tilde{M}} c(-v_{t_0, B}(\Psi))
$$
\n
$$
+ 2 \left(\sup_K |F^{1+\delta}|^2 \right) \int_{\{\Psi < -t_0\} \cap K} |f|_{\hat{h}}^2 < +\infty.
$$
\n(9.14)

As K is arbitrarily chosen, by Montel theorem and diagonal method, we know that there exists a holomorphic E-valued $(n,0)$ form \tilde{F} on X such that $\tilde{F} = \hat{F}$ on $X\backslash\{F=0\}$. And we have

$$
\int_X |\hat{F} - (1 - b_{t_0, B}(\Psi)) f F^{1+\delta}|^2_h e^{v_{t_0, B}(\Psi) - \delta \tilde{M}} c(-v_{t_0, B}(\Psi))
$$
\n
$$
\leq \left(\frac{1}{\delta} c(T) e^{-T} + \int_T^{t_0 + B} c(s) e^{-s} ds\right) \int_X \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \Psi < -t_0\}} |f F|^2_h
$$

The following remark shows that we can assume that $c(t)$ is a smooth function.

.

Remark 9.18. We firstly introduce the regularization process of $c(t)$.

Let $f(x) = 2\mathbb{I}_{(-\frac{1}{2},\frac{1}{2})} * \rho(x)$ be a smooth function on \mathbb{R} , where ρ is the kernel of convolution satisfying supp $(\rho) \subset (-\frac{1}{3}, \frac{1}{3})$ and $\rho > 0$.

Let $g_i(x) = \begin{cases} if(ix) & if x \leq 0 \\ if (i^2x) & if x > 0 \end{cases}$ $if(i²x)$ if $x > 0$, then ${g_i}_{i \in \mathbb{N}^+}$ is a family of smooth functions on R satisfying:

(1) supp(g) ⊂ $[-\frac{1}{i}, \frac{1}{i}], g_i(x) \ge 0$ for any $x \in \mathbb{R}$, (2) $\int_{-\frac{1}{i}}^{0} g_i(x) dx = 1$, $\int_{0}^{\frac{1}{i}} g_i(x) dx \leq \frac{1}{i}$ for any $i \in \mathbb{N}^+$. Let $\tilde{h}(t)$ be an extension of the function $c(t)e^{-t}$ from $[T, +\infty)$ to $\mathbb R$ such that (1) $\tilde{h}(t) = h(t) := c(t)e^{-t}$ on $[T, +\infty);$ (2) $\tilde{h}(t)$ is decreasing with respect to t; (3) $\lim_{t \to T - 0} \tilde{h}(t) = c(T) e^{-T}.$

Denote $c_i(t) := e^t \int_{\mathbb{R}} \tilde{h}(t+y) g_i(y) dy$. By the construction of convolution, we know $c_i(t) \in C^\infty(-\infty, +\infty)$. For any $t \geq T$, we have

$$
c_i(t) - c(t) \ge e^t \left(\int_{-\frac{1}{i}}^0 (\tilde{h}(t+y) - \tilde{h}(t)) g_i(y) dy \right) \ge 0.
$$

As $\tilde{h}(t)$ is decreasing with respect to t, we know that $c_i(t)e^{-t}$ is also decreasing with respect to t. Hence $c_i(t)e^{-t}$ is locally L^1 integrable on $\mathbb R$.

As $\tilde{h}(t)$ is decreasing with respect to t, then set $\tilde{h}^-(t) = \lim_{s \to t-0} \tilde{h}(s) \ge h(t)$ for

any $t \in \mathbb{R}$. Note that $c^-(t) := \lim_{s \to t-0} \tilde{h}(s)e^t \ge c(t)$ for any $t \ge T$. Now we prove $\lim_{i \to +\infty} c_i(t)e^{-t} = \tilde{h}^{-}(t)$. In fact, we have

$$
|c_i(t)e^{-t} - \tilde{h}^{-}(t)| \le \int_{-\frac{1}{t}}^{0} |\tilde{h}(t+y) - h^{-}(t)|g_i(y)dy
$$

+
$$
\int_{0}^{\frac{1}{t}} \tilde{h}(t+y)g_i(y)dy.
$$
 (9.15)

For any $\epsilon > 0$, there exists $\delta > 0$ such that $|h(t - \delta) - h^{-}(t)| < \epsilon$. Then $\exists N > 0$, such that for any $n > N$, $t \ge t + y > t - \delta$ for all $y \in [-\frac{1}{i}, 0)$ and $\frac{1}{i} < \epsilon$. It follows from (9.15) that

$$
|c_i(t)e^{-t}-\tilde{h}^-(t)| \leq \epsilon + \epsilon \tilde{h}(t),
$$

hence $\lim_{i \to +\infty} c_i(t)e^{-t} = \tilde{h}^{-}(t)$ for any $t \in \mathbb{R}$. Especially, we have $\lim_{i \to +\infty} c_i(T)e^{-T} =$ $\tilde{h}^{-}(T) = c(T)e^{-T}.$

Assume that for each i, we have an E-valued holomorphic $(n,0)$ form \tilde{F}_i on X such that

$$
\int_{X} |\tilde{F}_{i} - (1 - b_{t_{0},B}(\Psi)) f F^{1+\delta}|_{h}^{2} e^{v_{t_{0},B}(\Psi) - \delta \tilde{M}} c(-v_{t_{0},B}(\Psi))
$$
\n
$$
\leq (\frac{1}{\delta} c_{i}(T) e^{-T} + \int_{T}^{t_{0}+B} c_{i}(s) e^{-s} ds) \int_{X} \frac{1}{B} \mathbb{I}_{\{-t_{0} - B < \Psi < -t_{0}\}} |f F|_{h}^{2},
$$
\n
$$
(9.16)
$$

By construction of $c_i(t)$, we have

$$
\int_{T}^{t_{0}+B} c_{i}(t_{1})e^{-t_{1}}dt_{1}
$$
\n
$$
= \int_{T}^{t_{0}+B} \int_{\mathbb{R}} \tilde{h}(t_{1}+y)g_{i}(y)dydt_{1}
$$
\n
$$
= \int_{\mathbb{R}} g_{i}(y) \left(\int_{T}^{t_{0}+B} \tilde{h}(t_{1}+y)dt_{1} \right) dy
$$
\n
$$
= \int_{\mathbb{R}} g_{i}(y) \left(\int_{T+y}^{t_{0}+B+y} \tilde{h}(s)ds \right) dy
$$
\n
$$
= \int_{\mathbb{R}} g_{i}(y) \left(\int_{T}^{t_{0}+B} \tilde{h}(s)ds + \int_{t_{0}+B}^{t_{0}+B+y} \tilde{h}(s)ds - \int_{T}^{T+y} \tilde{h}(s)ds \right) dy,
$$
\n(9.17)

then it follows from the construction of $g_i(t)$, $\tilde{h}(t)$ is decreasing with respect to t, inequality [\(9.17\)](#page-58-0) and $\tilde{h}(t) = c(t)e^{-t}$ on $[T, +\infty)$ that we have

$$
\lim_{i \to +\infty} \int_{T}^{t_0+B} c_i(t_1) e^{-t_1} dt_1 = \int_{T}^{t_0+B} c(t_1) e^{-t_1} dt_1.
$$
 (9.18)

As $(X, E, \Sigma, X_j, h, h_s)$ is a singular hermitian metric on E and $\Theta_h(E) \geq_{Nak}^s 0$. We know that for any compact subset K , there exist a relatively compact subset $M_{j_K} \subset M$ containing K and a C^2 smooth hermitian metric $h_{j_K,1}$ on M_{j_K} such that $h_{j_K,1} \leq h$ on $K \subset M_{j_K}$. For any compact subset K of M, we have

$$
\inf_{i} \inf_{K} e^{v_{t_0,B}(\Psi) - \delta \tilde{M}} c_i(-v_{t_0,B}(\Psi)) \ge \inf_{K} e^{v_{t_0,B}(\Psi) - \delta \tilde{M}} c(-v_{t_0,B}(\Psi)),
$$

then

$$
\sup_{i} \int_{K} |\tilde{F}_{i} - (1 - b_{t_{0},B}(\Psi)) f F^{1+\delta}|_{h}^{2} < +\infty.
$$

Hence we have

$$
\sup_{i}\int_{K}|\tilde{F}_{i}-(1-b_{t_{0},B}(\Psi))fF^{1+\delta}|^{2}_{h_{j_{K},1}}<+\infty.
$$

Note that there exists a constant $C_K > 0$ such that $h_{j_K,1} \leq C_K \hat{h}$ on K. We have

$$
\int_K |(1-b_{t_0,B}(\Psi))fF^{1+\delta}|^2_{h_{j_K,1}}\leq C_K (\sup_K |F^{1+\delta}|^2)\int_{K\cap \{\Psi<-t_0\}} |f|^2_{\hat{h}}<+\infty,
$$

then \sup_i $\int_K |\tilde{F}_i|^2_{h_{j_K,1}} < +\infty$, by Montel theorem and diagonal method, we know that there exists a subsequence of $\{\tilde{F}_i\}$ (also denoted by $\{\tilde{F}_i\}$), which is compactly convergent to an E-valued holomorphic $(n, 0)$ form \overline{F} on X. Then it follows from inequality [\(9.16\)](#page-58-1) and Fatou's Lemma that

$$
\int_{X} |\tilde{F} - (1 - b_{t_0, B}(\Psi)) f F^{1+\delta}|_{h}^{2} e^{v_{t_0, B}(\Psi) - \delta \tilde{M}} c(-v_{t_0, B}(\Psi))
$$
\n
$$
\leq \int_{X} |\tilde{F} - (1 - b_{t_0, B}(\Psi)) f F^{1+\delta}|_{h}^{2} e^{v_{t_0, B}(\Psi) - \delta \tilde{M}} c^{-}(-v_{t_0, B}(\Psi))
$$
\n
$$
\leq \liminf_{i \to +\infty} \int_{X} |\tilde{F}_i - (1 - b_{t_0, B}(\Psi)) f F^{1+\delta}|_{h}^{2} e^{v_{t_0, B}(\Psi) - \delta \tilde{M}} c_i(-v_{t_0, B}(\Psi))
$$
\n
$$
\leq \liminf_{i \to +\infty} \left(\frac{1}{\delta} c_i(T) e^{-T} + \int_{T}^{t_0 + B} c_i(s) e^{-s} ds \right) \int_{X} \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \Psi < -t_0\}} |f F|_{h}^{2}
$$
\n
$$
= \left(\frac{1}{\delta} c(T) e^{-T} + \int_{T}^{t_0 + B} c(s) e^{-s} ds \right) \int_{X} \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \Psi < -t_0\}} |f F|_{h}^{2}.
$$

In the following discussion, we assume that F has no zero points on X and $c(t)$ is smooth.

As X is weakly pseudoconvex, there exists a smooth plurisubharmonic exhaustion function P on X. Let $X_j := \{P < j\}$ $(k = 1, 2, ...,)$. We choose P such that $X_1 \neq \emptyset$.

Then X_j satisfies $X_1 \subseteq X_2 \subseteq \ldots \subseteq X_j \subseteq X_{j+1} \subseteq \ldots$ and $\bigcup_{j=1}^n X_j = X$. Each X_j is weakly pseudoconvex Kähler manifold with exhaustion plurisubharmonic function $P_i = 1/(j - P).$

We will fix j during our discussion until step 7.

Step 1: Regularization of Ψ.

We note that there must exists a continuous nonnegative $(1, 1)$ -form ϖ on X_{j+1} satisfying

$$
(\sqrt{-1}\Theta_{TM} + \varpi \otimes Id_{TM})(\kappa_1 \otimes \kappa_2, \kappa_1 \otimes \kappa_2) \geq 0,
$$

for $\forall \kappa_1, \kappa_2 \in TM$ on M_{i+1} .

Let $M = X_{j+1}$, $\Omega = X_j$, $T = \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \psi$, $\gamma = 0$ in Lemma [9.11,](#page-54-0) then there exists a family of functions $\psi_{\zeta,\rho}$ ($\zeta \in (0, +\infty)$ and $\rho \in (0, \rho_1)$ for some positive ρ_1) on X_{i+1} such that

(1) $\psi_{\zeta,\rho}$ is a quasi-plurisubharmonic function on a neighborhood of $\overline{X_j}$, smooth on $X_{j+1}\backslash E_{\zeta}(\psi)$, increasing with respect to ζ and ρ on X_j and converges to ψ on X_j as $\rho \to 0$,

 $(2) \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \psi_{\zeta,\rho} \geq -\zeta \varpi - \delta \omega \text{ on } X_j,$

where $E_{\zeta}(\psi) := \{x \in X : v(\psi, x) \geq \zeta\}$ is the upper-level set of Lelong number and $\{\delta_\rho\}$ is an increasing family of positive numbers such that $\lim_{\rho\to 0} \delta_\rho = 0$.

Let $\rho = \frac{1}{m}$. Let $\tilde{\delta}_m := \delta_{\frac{1}{m}}$ and $\zeta = \tilde{\delta}_m$. Denote $\psi_m := \psi_{\tilde{\delta}_m, \frac{1}{m}}$. Then we have a sequence of functions $\{\psi_m\}$ satisfying

(1') ψ_m is quasi-plurisubharmonic function on $\overline{X_j}$, smooth on $X_{j+1}\backslash E_m(\psi)$, decreasing with respect to m and converges to ψ on X_j as $m \to +\infty$,

(2') $\frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \psi_m \geq -\tilde{\delta}_m \varpi - \tilde{\delta}_m \omega$ on X_j ,

where $E_m(\psi) = \{x \in X : v(\psi, x) \geq \frac{1}{m}\}\$ is the upper level set of Lelong number and $\{\tilde{\delta}_m\}$ is an decreasing family of positive numbers such that $\lim_{m\to+\infty}\tilde{\delta}_m=0$.

As X_i is relatively compact in X, there exists a positive number $b \ge 1$ such that $b\omega \geq \varpi$ on X_j . Then condition (2') becomes

 $(2^{\nu}) \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \psi_m \geq -\tilde{\delta}_m \overline{\omega} - \tilde{\delta}_m \omega \geq -2b \tilde{\delta}_m \omega \text{ on } X_j.$

Let $\eta_m = \{\frac{t_0 - T}{3m}, \frac{t_0 - T}{3m}\}\$ and we have the function $M_{\eta_m}(\psi_m + T, 2\log|F|)$. Denote $M_{\eta_m} := M_{\eta_m}(\psi_m + T, 2\log|F|)$ for simplicity. Note that $\psi_m + T$ is a $2b\tilde{\delta}_m\omega$ plurisubharmonic function. As F is a holomorphic function, ω is a Kähler form and $b\tilde{\delta}_m > 0$, we know that $2\log|F|$ is a $2b\tilde{\delta}_m\omega$ -plurisubharmonic function. It follows from Lemma [9.16](#page-55-0) that M_{η_m} is a $2b\tilde{\delta}_m\omega$ -plurisubharmonic function, i.e.,

$$
\frac{\sqrt{-1}}{\pi} \partial \bar{\partial} M_{\eta_m} \geq -2\pi b \tilde{\delta}_m \omega.
$$

Denote $\Psi_m := \psi_m - M_{\eta_m}(\psi_m + T, 2 \log |F|)$. Then Ψ_m is smooth on $X_j \backslash E_m$. It is easy to verify that when $m \to +\infty$, $\Psi_m \to \Psi$. It follows from Lemma [9.16](#page-55-0) that we know we know

(1) if $\psi_m + T \leq 2 \log |F| - \frac{2(t_0-T)}{s^3 m_m}$ holds, we have $\Psi_m = \psi_m - 2 \log |F|$;

(2) if $\psi_m + T \geq 2 \log |F| + \frac{2(t_0 - T)}{3m}$ holds, we have $\Psi_m = -T$;

(3) if $2\log|F| - \frac{2(t_0-T)}{3m} < \psi_m + T < 2\log|F| + \frac{2(t_0-T)}{3m}$ holds, we have $\max\{\psi_m +$ $T, 2\log |F|\} \le M_{\eta_m} \le (\psi_m + T + \frac{t_0 - T}{m})$ and hence $-T - \frac{t_0 - T}{m} \le \Psi_m \le -T$.

Thus we have $\{\Psi_m < -t_0\} = \{\psi_m - 2\log|F| < -t_0\} \subset \{\psi - 2\log|F| < -t_0\} =$ $\{\Psi < -t_0\}$. We also note that $\Psi_m \leq -T$ on M_{j+1} .

Step 2: Recall some constructions.

To simplify our notations, we denote $b_{t_0,B}(t)$ by $b(t)$ and $v_{t_0,B}(t)$ by $v(t)$.

Let $\epsilon \in (0, \frac{1}{8}B)$. Let $\{v_{\epsilon}\}_{{\epsilon \in (0, \frac{1}{8}B)}}$ be a family of smooth increasing convex functions on R, such that:

(1) $v_{\epsilon}(t) = t$ for $t \ge -t_0 - \epsilon$, $v_{\epsilon}(t) = constant$ for $t < -t_0 - B + \epsilon$;

(2) $v_{\epsilon}^{\prime\prime}(t)$ are convergence pointwisely to $\frac{1}{B}\mathbb{I}_{(-t_0-B,-t_0)}$, when $\epsilon \to 0$, and $0 \leq$ $v_{\epsilon}^{\prime\prime}(t) \leq \frac{2}{\beta} \mathbb{I}_{(-t_0 - B + \epsilon, -t_0 - \epsilon)}$ for ant $t \in \mathbb{R}$;

(3) $v_{\epsilon}'(t)$ are convergence pointwisely to $b(t)$ which is a continuous function on $\mathbb R$ when $\epsilon \to 0$ and $0 \le v_{\epsilon}'(t) \le 1$ for any $t \in \mathbb R$.

One can construct the family $\{v_{\epsilon}\}_{{\epsilon \in (0, \frac{1}{8}B)}}$ by setting

$$
v_{\epsilon}(t) := \int_{-\infty}^{t} \left(\int_{-\infty}^{t_1} \left(\frac{1}{B - 4\epsilon} \mathbb{I}_{(-t_0 - B + 2\epsilon, -t_0 - 2\epsilon)} * \rho_{\frac{1}{4}\epsilon} \right)(s) ds \right) dt_1
$$

$$
- \int_{-\infty}^{-t_0} \left(\int_{-\infty}^{t_1} \left(\frac{1}{B - 4\epsilon} \mathbb{I}_{(-t_0 - B + 2\epsilon, -t_0 - 2\epsilon)} * \rho_{\frac{1}{4}\epsilon} \right)(s) ds \right) dt_1 - t_0,
$$

where $\rho_{\frac{1}{4}\epsilon}$ is the kernel of convolution satisfying $\text{supp}(\rho_{\frac{1}{4}\epsilon}) \subset (-\frac{1}{4}\epsilon, \frac{1}{4}\epsilon)$. Then it follows that

$$
v_{\epsilon}^{\prime\prime}(t) = \frac{1}{B - 4\epsilon} \mathbb{I}_{(-t_0 - B + 2\epsilon, -t_0 - 2\epsilon)} * \rho_{\frac{1}{4}\epsilon}(t),
$$

and

$$
v_{\epsilon}'(t) = \int_{-\infty}^{t} \left(\frac{1}{B - 4\epsilon} \mathbb{I}_{(-t_0 - B + 2\epsilon, -t_0 - 2\epsilon)} * \rho_{\frac{1}{4}\epsilon}\right)(s)ds.
$$

Let $\eta = s(-v_{\epsilon}(\Psi_m))$ and $\phi = u(-v_{\epsilon}(\Psi_m))$, where $s \in C^{\infty}([T, +\infty))$ satisfies $s \geq$ $\frac{1}{\delta}$ and $u \in C^{\infty}([T, +\infty))$, such that $s'(t) \neq 0$ for any $t, u''s - s'' > 0$ and $s' - u's = 1$. Let $\Phi_m = \phi + \delta M_{\eta_m}$. Recall that $(X, E, \Sigma, X_j, h, h_{j,m'})$ is a singular hermitian metric on E and $\Theta_h(E) \geq_{Nak}^{m'} 0$. Then there exists a sequence of hermitian metrics ${h_{j,m'}}_{m'=1}^n$ on X_j of class C^2 such that $\lim_{m'\to+\infty} h_{j,m'} = h$ almost everywhere on X_j and $\{h_{j,m'}\}_{m'=1}^n$ satisfies the conditions of Definition [1.3.](#page-1-1) Since j is fixed until the last step, we simply denote $\{h_{j,m'}\}_{m'=1}^{+\infty}$ by $h_{m'}$ and denote $\tilde{h} := h_{m'}e^{-\Phi_m}$.

Step 3: Solving $\bar{\partial}$ -equation with error term.

Set $B = [\eta \sqrt{-1} \Theta_{\tilde{h}} - \sqrt{-1} \partial \bar{\partial} \eta \otimes \mathrm{Id}_E - \sqrt{-1} g \partial \eta \wedge \bar{\partial} \eta \otimes \mathrm{Id}_E, \Lambda_\omega],$ where g is a positive function. We will determine g by calculations. On $X_j\backslash E_m$, direct calculation shows that

$$
\partial \bar{\partial} \eta = -s'(-v_{\epsilon}(\Psi_m))\partial \bar{\partial}(v_{\epsilon}(\Psi_m)) + s''(-v_{\epsilon}(\Psi_m))\partial(v_{\epsilon}(\Psi_m)) \wedge \bar{\partial}(v_{\epsilon}(\Psi_m)),
$$

\n
$$
\eta \Theta_{\tilde{h}} = \eta \partial \bar{\partial} \phi \otimes \text{Id}_E + \eta \Theta_{h_{m'}} + \eta \partial \bar{\partial}(\delta M_{\eta_m}) \otimes \text{Id}_E
$$

\n
$$
= su''(-v_{\epsilon}(\Psi_m))\partial(v_{\epsilon}(\Psi_m)) \wedge \bar{\partial}(v_{\epsilon}(\Psi_m)) \otimes \text{Id}_E - su'(-v_{\epsilon}(\Psi_m))\partial \bar{\partial}(v_{\epsilon}(\Psi_m)) \otimes \text{Id}_E
$$

\n
$$
+ s\Theta_{h_{m'}} + s\partial \bar{\partial}(\delta M_{\eta_m}) \otimes \text{Id}_E.
$$

Hence

$$
\eta \sqrt{-1} \Theta_{\tilde{h}} - \sqrt{-1} \partial \bar{\partial} \eta \otimes \mathrm{Id}_{E} - \sqrt{-1} g \partial \eta \wedge \bar{\partial} \eta \otimes \mathrm{Id}_{E}
$$

= $s \Theta_{h_{m'}} + s \sqrt{-1} \partial \bar{\partial} (\delta M_{\eta_m}) \otimes \mathrm{Id}_{E}$
+ $(s' - su') (v'_{\epsilon}(\Psi_m) \sqrt{-1} \partial \bar{\partial} (\Psi_m) + v''_{\epsilon}(\psi_m) \sqrt{-1} \partial (\Psi_m) \wedge \bar{\partial} (\Psi_m)) \otimes \mathrm{Id}_{E}$
+ $[(u''s - s'') - gs'^{2}] \sqrt{-1} \partial (v_{\epsilon}(\Psi_m)) \wedge \bar{\partial} (v_{\epsilon}(\Psi_m)) \otimes \mathrm{Id}_{E},$

where we omit the term $-v_{\epsilon}(\Psi_m)$ in $(s'-su')(-v_{\epsilon}(\Psi_m))$ and $[(u''s-s'')-gs'^2](-v_{\epsilon}(\Psi_m))$ for simplicity. Let $g = \frac{u''s - s''}{s'^2}$ $\frac{s-s^{\prime\prime}}{s^{\prime\prime}}(-v_{\epsilon}(\Psi_m))$ and note that $s'-su'=1, 0 \le v'_{\epsilon}(\Psi_m) \le 1$. Then

$$
\eta \sqrt{-1} \Theta_{\tilde{h}} - \sqrt{-1} \partial \bar{\partial} \eta \otimes \mathrm{Id}_{E} - \sqrt{-1} g \partial \eta \wedge \bar{\partial} \eta \otimes \mathrm{Id}_{E}
$$

\n
$$
= s \sqrt{-1} \Theta_{h_{m'}} + s \sqrt{-1} \partial \bar{\partial} (\delta M_{\eta_{m}}) \otimes \mathrm{Id}_{E}
$$

\n
$$
+ v'_{\epsilon} (\Psi_{m}) \sqrt{-1} \partial \bar{\partial} (\Psi_{m}) \otimes \mathrm{Id}_{E} + v''_{\epsilon} (\psi_{m}) \sqrt{-1} \partial (\Psi_{m}) \wedge \bar{\partial} (\Psi_{m}) \otimes \mathrm{Id}_{E}
$$

\n
$$
= v''_{\epsilon} (\psi_{m}) \sqrt{-1} \partial (\Psi_{m}) \wedge \bar{\partial} (\Psi_{m}) \otimes \mathrm{Id}_{E} + v'_{\epsilon} (\Psi_{m}) \sqrt{-1} \partial \bar{\partial} (\Psi_{m}) \otimes \mathrm{Id}_{E}
$$

\n
$$
+ s (\sqrt{-1} \Theta_{h_{m'}} + \lambda_{m'} \omega \otimes \mathrm{Id}_{E}) - s \lambda_{m'} \omega \otimes \mathrm{Id}_{E}
$$

\n
$$
+ s (\sqrt{-1} \partial \bar{\partial} (\delta M_{\eta_{m}}) + 2 \pi b \delta \tilde{\delta}_{m} \omega) \otimes \mathrm{Id}_{E} - 2 \pi b s \delta \tilde{\delta}_{m} \omega \otimes \mathrm{Id}_{E}
$$

\n
$$
\geq v''_{\epsilon} (\psi_{m}) \sqrt{-1} \partial (\Psi_{m}) \wedge \bar{\partial} (\Psi_{m}) \otimes \mathrm{Id}_{E} + v'_{\epsilon} (\Psi_{m}) \sqrt{-1} \partial \bar{\partial} (\Psi_{m}) \otimes \mathrm{Id}_{E}
$$

\n
$$
+ \frac{1}{\delta} (\sqrt{-1} \Theta_{h_{m'}} + \lambda_{m'} \omega \otimes \mathrm{Id}_{E}) + \frac{1}{\delta} (\sqrt{-1} \partial \bar{\partial} (\delta M_{\eta_{m}}) + 2 \pi b \delta \tilde{\delta}_{m} \omega) \otimes \mathrm{Id}_{E}
$$

\n
$$
- s \lambda_{m'} \omega \otimes \mathrm{Id}_{E} - 2 \pi b s \delta \tilde{\delta}_{m} \omega \otimes \mathrm{Id}_{E}.
$$

\n(9.19)

Note that
\n
$$
\delta v'_{\epsilon}(\Psi_m)\sqrt{-1}\partial\bar{\partial}(\Psi_m)\otimes \mathrm{Id}_E + (\sqrt{-1}\Theta_{h_{m'}} + \lambda_{m'}\omega \otimes \mathrm{Id}_E)
$$
\n
$$
+ (\sqrt{-1}\partial\bar{\partial}(\delta M_{\eta_m}) + 2\pi b\delta\tilde{\delta}_m\omega) \otimes \mathrm{Id}_E
$$
\n
$$
= (1 - v'_{\epsilon}(\Psi_m))(\sqrt{-1}\Theta_{h_{m'}} + \lambda_{m'}\omega \otimes \mathrm{Id}_E + \sqrt{-1}\partial\bar{\partial}(\delta M_{\eta_m}) \otimes \mathrm{Id}_E + 2\pi b\delta\tilde{\delta}_m\omega \otimes \mathrm{Id}_E)
$$
\n
$$
+ v'_{\epsilon}(\Psi_m)(\sqrt{-1}\Theta_{h_{m'}} + \lambda_{m'}\omega \otimes \mathrm{Id}_E + \sqrt{-1}\partial\bar{\partial}(\delta M_{\eta_m}) \otimes \mathrm{Id}_E + 2\pi b\delta\tilde{\delta}_m\omega \otimes \mathrm{Id}_E)
$$
\n
$$
+ v'_{\epsilon}(\Psi_m)(\partial\bar{\partial}(\delta\psi_m)\otimes \mathrm{Id}_E - \partial\bar{\partial}(\delta M_{\eta_m}) \otimes \mathrm{Id}_E)
$$
\n
$$
= (1 - v'_{\epsilon}(\Psi_m))(\sqrt{-1}\Theta_{h_{m'}} + \lambda_{m'}\omega \otimes \mathrm{Id}_E + \sqrt{-1}\partial\bar{\partial}(\delta M_{\eta_m}) \otimes \mathrm{Id}_E + 2\pi b\delta\tilde{\delta}_m\omega \otimes \mathrm{Id}_E)
$$
\n
$$
+ v'_{\epsilon}(\Psi_m)(\sqrt{-1}\Theta_{h_{m'}} + \lambda_{m'}\omega \otimes \mathrm{Id}_E + \sqrt{-1}\partial\bar{\partial}(\delta\psi_m) \otimes \mathrm{Id}_E + 2\pi b\delta\tilde{\delta}_m\omega \otimes \mathrm{Id}_E)
$$
\n
$$
\geq 0.
$$
\n(9.20)

It follows from inequality [\(9.19\)](#page-61-0) and inequality [\(9.20\)](#page-62-0) that

$$
\begin{aligned} &\eta\sqrt{-1}\Theta_{\tilde{h}}-\sqrt{-1}\partial\bar{\partial}\eta\otimes \mathrm{Id}_{E}-\sqrt{-1}g\partial\eta\wedge\bar{\partial}\eta\otimes \mathrm{Id}_{E}\\ \geq& v''_{\epsilon}(\Psi_{m})\sqrt{-1}\partial(\Psi_{m})\wedge\bar{\partial}(\Psi_{m})\otimes \mathrm{Id}_{E}-2\pi bs\delta\tilde{\delta}_{m}\omega\otimes \mathrm{Id}_{E}-s\lambda_{m'}\omega\otimes \mathrm{Id}_{E}. \end{aligned}
$$

By the constructions of $s(t)$, $v_{\epsilon}(t)$ and $\sup_m \sup_{X_j} \Psi_m \leq -T$, we have $s(-v_{\epsilon}(\Psi_m))$ is uniformly bounded on X_j with respect to ϵ and m. Let N_1 be the uniformly upper bound of $s(-v_{\epsilon}(\Psi_m))$ on X_j . Then on $X_j\backslash E_m$, we have

$$
\begin{aligned} &\eta\sqrt{-1}\Theta_{\tilde{h}}-\sqrt{-1}\partial\bar{\partial}\eta\otimes \mathrm{Id}_{E}-\sqrt{-1}g\partial\eta\wedge\bar{\partial}\eta\otimes \mathrm{Id}_{E}\\ \geq& v''_{\epsilon}(\Psi_{m})\sqrt{-1}\partial(\Psi_{m})\wedge\bar{\partial}(\Psi_{m})\otimes \mathrm{Id}_{E}-2\pi bN_{1}\delta\tilde{\delta}_{m}\omega\otimes \mathrm{Id}_{E}-N_{1}\lambda_{m'}\omega\otimes \mathrm{Id}_{E}. \end{aligned}
$$

Hence, for any E-valued $(n, 1)$ form α , we have

$$
\langle (B + (2\pi bN_1 \delta\tilde{\delta}_m + N_1 \lambda_{m'})\mathrm{Id}_E)\alpha, \alpha \rangle_{\tilde{h}}
$$

\n
$$
\geq \langle [v''_{\epsilon}(\Psi_m)\partial(\Psi_m) \wedge \bar{\partial}(\Psi_m) \otimes \mathrm{Id}_E, \Lambda_{\omega}]\alpha, \alpha \rangle_{\tilde{h}}
$$

\n
$$
= \langle (v''_{\epsilon}(\Psi_m)\bar{\partial}(\Psi_m) \wedge (\alpha_{\mathsf{L}}(\bar{\partial}\Psi_m)^{\sharp})), \alpha \rangle_{\tilde{h}}.
$$
\n(9.21)

It follows from Lemma [9.4](#page-47-2) that $B + (2\pi b N_1 \delta \tilde{\delta}_m + N_1 \lambda_{m'}) \mathrm{Id}_E$ is semi-positive. Denote $\tilde{\lambda}_{m'} := \lambda_{m'} + \frac{1}{m'}$, then $B + (2\pi b N_1 \delta \tilde{\delta}_m + N_1 \tilde{\lambda}_{m'}) \mathbf{Id}_E$ is positive. Using the definition of contraction, Cauchy-Schwarz inequality and inequality [\(9.21\)](#page-62-1), we have

$$
\begin{split} |\langle v''_{\epsilon}(\Psi_m)\bar{\partial}\Psi_m \wedge \gamma, \tilde{\alpha}\rangle_{\tilde{h}}|^2 &= |\langle v''_{\epsilon}(\Psi_m)\gamma, \tilde{\alpha}\llcorner(\bar{\partial}\Psi_m)^{\sharp}\rangle_{\tilde{h}}|^2 \\ &\leq \langle (v''_{\epsilon}(\Psi_m)\gamma, \gamma)\rangle_{\tilde{h}}(v''_{\epsilon}(\Psi_m))|\tilde{\alpha}\llcorner(\bar{\partial}\Psi_m)^{\sharp}|_{\tilde{h}}^2 \\ &= \langle (v''_{\epsilon}(\Psi_m)\gamma, \gamma)\rangle_{\tilde{h}} \langle (v''_{\epsilon}(\Psi_m))\bar{\partial}\Psi_m \wedge (\tilde{\alpha}\llcorner(\bar{\partial}\Psi_m)^{\sharp}), \tilde{\alpha}\rangle_{\tilde{h}} \\ &\leq \langle (v''_{\epsilon}(\Psi_m)\gamma, \gamma)\rangle_{\tilde{h}} \langle (B + (2\pi bN_1\delta\tilde{\delta}_m + N_1\tilde{\lambda}_m\prime)\mathrm{Id}_E)\tilde{\alpha}, \tilde{\alpha})\rangle_{\tilde{h}} \end{split} \tag{9.22}
$$

for any E-valued $(n, 0)$ form γ and E-valued $(n, 1)$ form $\tilde{\alpha}$.

As $fF^{1+\delta}$ is holomorphic on $\{\Psi < -t_0\}$ and $\{\Psi_m < -t_0 - \epsilon\} \subset \{\Psi_m < -t_0\}$ $\{\Psi < -t_0\}$, then $\lambda := \overline{\partial}((1 - v'_{\epsilon}(\Psi_m))fF^{1+\delta})$ is well defined and smooth on $X_j\backslash E_m$. Taking $\gamma = fF^{1+\delta}$, $\tilde{\alpha} = (B + (2\pi bN_1\delta\tilde{\delta}_m + N_1\tilde{\lambda}_{m'})\mathrm{Id}_E)^{-1}(\bar{\partial}v'_{\epsilon}(\Psi_m)) \wedge fF^{1+\delta}$. Then it follows from inequality [\(9.22\)](#page-62-2) that

$$
\langle (B+(2\pi bN_1\delta\tilde{\delta}_{m}+N_1\tilde{\lambda}_{m'})\text{Id}_E)^{-1}\lambda,\lambda\rangle_{\tilde{h}}\leq v''_{\epsilon}(\Psi_m)|fF^{1+\delta}|^2_{\tilde{h}}.
$$

Thus we have

$$
\int_{X_j\setminus E_m} \langle (B + (2\pi bN_1\delta\tilde{\delta}_m + N_1\tilde{\lambda}_{m'})\mathrm{Id}_E)^{-1}\lambda, \lambda \rangle_{\tilde{h}} \le \int_{X_j\setminus E_m} v''_{\epsilon}(\Psi_m)|fF^{1+\delta}|_{\tilde{h}}^2.
$$

Recall that $\tilde{h} = h_{m'} e^{-\Phi_m}$ and $\Phi_m = \phi + \delta M_{\eta_m}$. As $h_{m'}$ is C^2 smooth, on $X_j \subset\subset$ X, there exists a constant $C_{j,m'} > 0$ such that $C_{j,m'}^{-1} |e_x|_{\hat{h}} \leq |e_x|_{h_{m'}} \leq C_{j,m'} |e_x|_{\hat{h}}$, for any $e_x \in E_x$. Note that $0 \le v''_{\epsilon}(t) \le \frac{2}{B} \mathbb{I}_{(-t_0 - B + \epsilon, -t_0 - \epsilon)}, e^{-\phi}$ is smooth function on X_j and $\delta M_{\eta_m} \geq \delta 2 \log |F|$. It follows from [\(2.1\)](#page-8-1) that

$$
\int_{X_j \setminus E_m} v''_{\epsilon}(\Psi_m)|fF^{1+\delta}|^2_{\tilde{h}} \le C_{j,m'} \sup_{X_k} (|F|^2 e^{-\phi}) \int_{\overline{X_j} \cap {\{\Psi < -t_0\}}} |f|^2_{\hat{h}} < +\infty.
$$

By Lemma [9.14,](#page-55-1) $X_i \backslash E_m$ carries a complete Kähler metric. Then it follows from Lemma [9.10](#page-51-1) that there exists

$$
u_{m,m',\epsilon,j} \in L^2(X_j \backslash E_m, K_X \otimes E, h_{m'}e^{-\Phi_m}),
$$

$$
h_{m,m',\epsilon,j} \in L^2(X_j \backslash E_m, \wedge^{n,1}T^*X \otimes E, h_{m'}e^{-\Phi_m}),
$$

such that $\bar{\partial}u_{m,m',\epsilon,j} + P_{m,m'}(\sqrt{2\pi bN_1\delta\tilde{\delta}_m + N_1\tilde{\lambda}_{m'}}h_{m,m',\epsilon,j}) = \lambda$ holds on $X_j\backslash E_m$ where $P_{m,m'}: L^2(X_j \backslash E_m, \wedge^{n,1} T^* X \otimes E, h_{m'}e^{-\Phi_m}) \to \text{Ker}D''$ is the orthogonal projection, and

$$
\begin{aligned} &\int_{X_j\backslash E_m}\frac{1}{\eta+g^{-1}}|u_{m,m',\epsilon,j}|^2_{h_{m'}}e^{-\Phi_m}+\int_{X_j\backslash E_m}|h_{m,m',\epsilon,j}|^2_{h_{m'}}e^{-\Phi_m}\\ \leq &\int_{X_j\backslash E_m}\langle(B+(2\pi bN_1\tilde{\delta}\delta_m+N_1\tilde{\lambda}'_m)\mathrm{Id}_E)^{-1}\lambda,\lambda\rangle_{\tilde{h}}\\ \leq &\int_{X_j\backslash E_m}v''_{\epsilon}(\Psi_m)|fF^{1+\delta}|^2_{h_{m'}}e^{-\Phi_m}<+\infty. \end{aligned}
$$

Assume that we can choose η and ϕ such that $(\eta + g^{-1})^{-1} = e^{v_{\epsilon}(\Psi_m)} e^{\phi} c(-v_{\epsilon}(\Psi_m))$. Then we have

$$
\int_{X_j \backslash E_m} |u_{m,m',\epsilon,j}|_{h_{m'}}^2 e^{v_{\epsilon}(\Psi_m) - \delta M_{\eta_m}} c(-v_{\epsilon}(\Psi_m)) + \int_{X_j \backslash E_m} |h_{m,m',\epsilon,j}|_{h_{m'}}^2 e^{-\phi - \delta M_{\eta_m}}
$$
\n
$$
\leq \int_{X_j \backslash E_m} v''_{\epsilon}(\Psi_m) |f F^{1+\delta}|_{h_{m'}}^2 e^{-\phi - \delta M_{\eta_m}} < +\infty.
$$
\n(9.23)

By the construction of $v_{\epsilon}(t)$ and $c(t)e^{-t}$ is decreasing with respect to t, we know $c(-v_{\epsilon}(\Psi_m))e^{v_{\epsilon}(\Psi_m)}$ has a positive lower bound on $X_j \in X$. By the constructions of $v_{\epsilon}(t)$ and u, we know $e^{-\phi} = e^{-u(-v_{\epsilon}(\Psi_m))}$ has a positive lower bound on $X_j \in X$. By the upper semi-continuity of M_{η_m} , we know $e^{-\delta M_{\eta_m}}$ has a positive lower bound on $X_j \in M$. Note that $h_{m'}$ is C^2 smooth on $X_j \in X$. Hence it follows from inequality [\(9.23\)](#page-63-0) that

$$
u_{m,m',\epsilon,j} \in L^2(X_j, K_M \otimes E, h_{m'}e^{-\Phi_m}),
$$

$$
h_{m,m',\epsilon,j} \in L^2(X_j, \wedge^{n,1}T^*M \otimes E, h_{m'}e^{-\Phi_m}).
$$

It follows from Lemma [9.15](#page-55-2) that we know

$$
D''u_{m,m',\epsilon,j} + P_{m,m'}\left(\sqrt{2\pi bN_1\delta\tilde{\delta}_m + N_1\tilde{\lambda}_{m'}}h_{m,m',\epsilon,j}\right) = \lambda \tag{9.24}
$$

holds on X_i . And we have

$$
\int_{X_j} |u_{m,m',\epsilon,j}|_{h_{m'}}^2 e^{v_{\epsilon}(\Psi_m) - \delta M_{\eta_m}} c(-v_{\epsilon}(\Psi_m)) + \int_{X_j} |h_{m,m',\epsilon,j}|_{h_{m'}}^2 e^{-\phi - \delta M_{\eta_m}}
$$
\n
$$
\leq \int_{X_j} v_{\epsilon}''(\Psi_m) |f F^{1+\delta}|_{h_{m'}}^2 e^{-\phi - \delta M_{\eta_m}} < +\infty.
$$
\n(9.25)

Step 4: Letting $m \to +\infty$.

In the Step 4, we note that m' is fixed.

Note that $\sup_m \sup_{X_j} e^{-\phi} = \sup_m \sup_{X_j} e^{-u(-v_{\epsilon}(\Psi_m))} < +\infty$ and $e^{-\delta M_{\eta_m}} \leq$ $e^{-\delta 2 \log |F|}$. As $\{\Psi_m < -t_0 - \epsilon\} \subset \{\Psi_m < -t_0\} \subset \{\Psi < -t_0\}$, we have

$$
v''_{\epsilon}(\Psi_m)|fF^{1+\delta}|^2_{h_{m'}} e^{-\phi-\delta M_{\eta_m}}\leq \frac{2}{B}C_{j,m'}\left(\sup_m\sup_{X_j}e^{-\phi}\right)\left(\sup_{X_j}|F|^2\right)\mathbb{I}_{\{\Psi<-t_0\}}|f|^2_{\hat{h}}
$$

holds on M_j . It follows from $\int_{\{\Psi<-t_0\}\cap\overline{M_j}}|f|^2_{\hat{h}} < +\infty$ and dominated convergence theorem that

$$
\begin{split} & \lim_{m\to +\infty} \int_{X_j} v''_{\epsilon}(\Psi_m)|fF^{1+\delta}|^2_{h_{m'}} e^{-\phi-\delta M_{\eta_m}} \\ = & \int_{X_j} v''_{\epsilon}(\Psi)|fF^{1+\delta}|^2_{h_{m'}} e^{-u(-v_{\epsilon}(\Psi))-\delta \max{\{\psi+T,2\log|F|\}}} \end{split}
$$

Note that $\inf_m \inf_{X_j} c(-v_\epsilon(\Psi_m))e^{-v_\epsilon(\Psi_m)} > 0$. It follows from Lemma [9.16](#page-55-0) that $M_{\eta_m} \le \max{\psi_m + T, 2\log|F|} + \frac{t_0 - T}{3m} \le \max{\psi_m + T, 2\log|F|} + t_0 - \frac{T}{s}$ $\max \{\psi_1 + T, 2 \log |F|\} + t_0 - T$. As ψ_1 is a quasi-plurisubharmonic function on X_j , we know max $\{\psi_1 + T, 2 \log |F|\}$ is upper semi-continuous function on X_j . Hence

$$
\inf_{m} \inf_{X_j} e^{-M_{\eta_m}} \ge \inf_{X_j} e^{-\max\{\psi_1 + T, 2\log|F|\} - t_0} > 0.
$$
\n(9.26)

.

Then it follows from inequality [\(9.25\)](#page-64-0) that

$$
\sup_m \int_{X_j} |u_{m,m',\epsilon,j}|^2_{h_{m'}}<+\infty.
$$

Therefore the solutions $u_{m,m',l,\epsilon,j}$ are uniformly bounded with respect to m in $L^2(X_j, K_M, h_{m'})$. Since the closed unit ball of the Hilbert space is weakly compact, we can extract a subsequence of $\{u_{m,m',\epsilon,j}\}$ (also denoted by $\{u_{m,m',\epsilon,j}\}\$) weakly convergent to $u_{m',\epsilon,j}$ in $L^2(X_j, K_M, h_{m'})$ as $m \to +\infty$.

Note that $\sup_m \sup_{M_j} e^{v_{\epsilon}(\Psi_m)} c(-v_{\epsilon}(\Psi_m)) < +\infty$. As $M_{\eta_m} \geq \max\{ \psi_m + T, 2 \log |F| \} \geq$ $2 \log |F|$ and F has no zero points on M, we have $\sup_m \sup_{M_j} e^{-M_{\eta_m}} \leq \sup_{M_j} \frac{1}{|F|^2}$ +∞. Hence we know

$$
\sup_m \sup_{M_j} e^{v_{\epsilon}(\Psi_m)} c(-v_{\epsilon}(\Psi_m)) e^{-\delta M_{\eta_m}} < +\infty.
$$

It follows from Lemma [9.1](#page-46-2) that we know $u_{m,m',\epsilon,j}\sqrt{e^{v_{\epsilon}(\Psi_{m_1})}c(-v_{\epsilon}(\Psi_m))e^{-\delta M_{\eta_m}}}$ weakly converges to $u_{m',\epsilon,j}\sqrt{e^{v_\epsilon(\Psi)}c(-v_\epsilon(\Psi))e^{-\delta \max\{\psi+T,2\log|F|\}}}$ in $L^2(X_j,K_M,h_{m'})$ as $m\to +\infty$. Hence we have

$$
\int_{X_j} |u_{m',\epsilon,j}|_{h_{m'}}^2 e^{v_{\epsilon}(\Psi) - \delta \max\{\psi + T, 2 \log |F|\}} c(-v_{\epsilon}(\Psi))
$$
\n
$$
\leq \liminf_{m \to +\infty} \int_{X_j} |u_{m,m',\epsilon,j}|_{h_{m'}}^2 e^{v_{\epsilon}(\Psi_m) - \delta M_{\eta_m}} c(-v_{\epsilon}(\Psi_m))
$$
\n
$$
\leq \liminf_{m \to +\infty} \int_{X_j} v''_{\epsilon}(\Psi_m) |f F^{1+\delta}|_{h_{m'}}^2 e^{-u(-v_{\epsilon}(\Psi_m)) - \delta M_{\eta_m}}
$$
\n
$$
\leq \int_{X_j} v''_{\epsilon}(\Psi) |f F^{1+\delta}|_{h_{m'}}^2 e^{-u(-v_{\epsilon}(\Psi)) - \delta \max\{\psi + T, 2 \log |F|\}} < +\infty.
$$
\n(9.27)

It follows from Lemma [9.9](#page-50-0) that we know that $h_{m,m',\epsilon,j}$ weakly converges to $h_{m',\epsilon,j}$ in $L^2(X_j, \wedge^{n,1}T^*M \otimes E, h_{m'}e^{-\Phi_1})$ and then $\sqrt{2\pi bN_1\delta\tilde{\delta}_m + N_1\tilde{\lambda}_{m'}}h_{m,m',\epsilon,j}$ weakly converges to $\sqrt{N_1\lambda_{m'}}h_{m',\epsilon,j}$ in $L^2(X_j,\wedge^{n,1}T^*M\otimes E,h_{m'}e^{-\Phi_1})$. Hence by Lemma [9.9](#page-50-0) and the uniqueness of weak limit, we know that $P_{m,m'}(\sqrt{2\pi bN_1\delta\tilde{\delta}_m+N_1\tilde{\lambda}_{m'}}h_{m,m',\epsilon,j})$ weakly converges to $P_{m'}\left(\sqrt{N_1\widetilde{\lambda}_{m'}}h_{m',\epsilon,j}\right)$ in $L^2(X_j,\wedge^{n,1}T^*M\otimes E,h_{m'}e^{-\Phi_1})$.

It follows from $\inf_{X_j} e^{-\Phi_1} = \inf_{X_j} e^{-u(-v_{\epsilon}(\Psi_1))} > 0$ and inequality [\(9.26\)](#page-64-1) that we have $h_{m,m',\epsilon,j}$ weakly converges to $h_{m',\epsilon,j}$ in $L^2(X_j, \wedge^{n,1}T^*M \otimes E, h_{m'})$ and $P_{m,m'}(\sqrt{2\pi bN_1\delta\tilde{\delta}_m+N_1\tilde{\lambda}_{m'}}h_{m,m',\epsilon,j})$ weakly converges to $P_{m'}(\sqrt{N_1\tilde{\lambda}_{m'}}h_{m',\epsilon,j})$ in $L^2(X_j, \wedge^{n,1} T^*M \otimes E, h_{m'})$.

Note that $\sup_m \sup_{X_j} e^{-u(-v_\epsilon(\Psi_m))} < +\infty$ and $\sup_m \sup_{X_j} e^{-M_{\eta_m}} \leq \sup_{M_j} \frac{1}{|F|^2} <$ $+\infty$. We know

$$
\sup_m \sup_{X_j} e^{-u(-v_{\epsilon}(\Psi_m)) - \delta M_{\eta_m}} < +\infty.
$$

It follows from Lemma [9.1](#page-46-2) that we have $h_{m,m',l,\epsilon,j}\sqrt{e^{-u(-v_{\epsilon}(\Psi_m))-\delta M_{\eta_m}}}\;$ is weakly convergent to $h_{m',l,\epsilon,j} \sqrt{e^{-u(-v_{\epsilon}(\Psi)) - \delta \max{\{\psi + T,2\log|F|\}}}}$ in $L^2(X_j, \wedge^{n,1} T^* M \otimes E, h_{m'})$ as $m \to +\infty$. Hence we have

$$
\int_{X_j} |h_{m',\epsilon,j}|_{h_{m'}}^2 e^{-u(-v_{\epsilon}(\Psi)) - \delta \max \{\psi + T, 2 \log |F|\}}\n\leq \liminf_{m \to +\infty} \int_{M_j} |h_{m,m',\epsilon,j}|_{h_{m'}}^2 e^{-u(-v_{\epsilon}(\Psi_m)) - \delta M_{\eta_m}}\n\leq \liminf_{m \to +\infty} \int_{X_j} v''_{\epsilon}(\Psi_m) |f F^{1+\delta}|_{h_{m'}}^2 e^{-u(-v_{\epsilon}(\Psi_m)) - \delta M_{\eta_m}}\n\leq \int_{X_j} v''_{\epsilon}(\Psi) |f F^{1+\delta}|_{h_{m'}}^2 e^{-u(-v_{\epsilon}(\Psi)) - \delta \max \{\psi + T, 2 \log |F|\}} < +\infty.
$$
\n(9.28)

Letting $m \to +\infty$ in [\(9.24\)](#page-63-1), we have

$$
D''u_{m',\epsilon,j} + P_{m'}(\sqrt{N_1\tilde{\lambda}_{m'}}h_{m',\epsilon,j}) = D''((1 - v'_{\epsilon}(\Psi))fF^{1+\delta}).
$$
 (9.29)

Step 5: Letting $m' \rightarrow +\infty$.

When $\Psi < -t_0 - \epsilon < -t_0$, we have $\psi - 2 \log |F| < -T$ and then max $\{\psi + T, 2 \log |F|\}$ $2 \log |F|$. Hence

$$
\int_{X_j} v''_{\epsilon}(\Psi) |fF^{1+\delta}|^2_{h_{m'}} e^{-u(-v_{\epsilon}(\Psi)) - \delta \max \{ \psi + T, 2 \log |F| \}} = \int_{X_j} v''_{\epsilon}(\Psi) |fF|^2_{h_{m'}} e^{-u(-v_{\epsilon}(\Psi))}.
$$

Note that $\sup_{X_j}(e^{-u(-v_{\epsilon}(\Psi))}) < +\infty$, $0 \le v_{\epsilon}''(t) \le \frac{2}{B}\mathbb{I}_{(-t_0-B+\epsilon,-t_0-\epsilon)}$ and $|e_x|_{h_{m'}} \leq |e_x|_{h_{m'+1}} \leq |e_x|_h$ for any $m' \in \mathbb{Z}_{\geq 0}$. We have

$$
v''_{\epsilon}(\Psi)|fF|_{h_{m'}}^2 e^{-u(-v_{\epsilon}(\Psi))} \le \sup_{X_j} (e^{-u(-v_{\epsilon}(\Psi))}) \frac{2}{B} \mathbb{I}_{\{-t_0 - B + \epsilon < \Psi < -t_0 - \epsilon\}} |fF|_{h}^2. \tag{9.30}
$$

It follows from inequality [\(2.2\)](#page-8-2) and dominated convergence theorem that we have

$$
\begin{split} &\lim_{m'\to+\infty}\int_{X_j}v''_{\epsilon}(\Psi)|fF|^2_{h_{m'}}e^{-u(-v_{\epsilon}(\Psi))}\\ =&\int_{X_j}v''_{\epsilon}(\Psi)|fF|^2_{h}e^{-u(-v_{\epsilon}(\Psi))}<+\infty. \end{split}
$$

Let $C_j := \inf_{X_j} e^{v_{\epsilon}(\Psi) - \delta \max\{\psi + T, 2\log |F|\}} c(-v_{\epsilon}(\Psi))$ be a real number and we note that $C_j > 0$. Then it follows from $C_j > 0$, inequalities [\(9.27\)](#page-65-0), [\(9.30\)](#page-66-0) and [\(2.2\)](#page-8-2) that we have

$$
\sup_{m'} \int_{X_j} |u_{m',\epsilon,j}|^2_{h_{m'}} < +\infty.
$$

As $|e_x|_{h_{m'}} \leq |e_x|_{h_{m'+1}}$ for any $m' \in \mathbb{Z}_{\geq 0}$, for any fixed *i*, we have

$$
\sup_{m'}\int_{X_j}|u_{m',\epsilon,j}|_{h_i}^2<+\infty.
$$

Especially letting $h_i = h_1$, since the closed unit ball of the Hilbert space is weakly compact, we can extract a subsequence $u_{m'',\epsilon,j}$ weakly convergent to $u_{\epsilon,j}$ in $L^2(M_j,K_M\otimes$ E, h_1) as $m'' \to +\infty$. Note that

$$
\sup_{X_j} e^{v_{\epsilon}(\Psi) - \delta \max\{\psi + T, 2\log|F|\}} c(-v_{\epsilon}(\Psi)) \leq \sup_{X_j} \frac{e^{v_{\epsilon}(\Psi)}c(-v_{\epsilon}(\Psi))}{|F|^{2\delta}} < +\infty.
$$

It follows from Lemma [9.1](#page-46-2) that $u_{m'',\epsilon,j}\sqrt{e^{v_{\epsilon}(\Psi)}c(-v_{\epsilon}(\Psi))}e^{-\delta \max\{\psi+T,2\log|F|\}}$ weakly converges to $u_{\epsilon,j}\sqrt{e^{v_{\epsilon}(\Psi)}c(-v_{\epsilon}(\Psi))e^{-\delta \max\{\psi+T,2\log|F|\}}}$ in $L^2(X_j, K_X \otimes E, h_1)$ as $m'' \to +\infty$.

For fixed $i \in \mathbb{Z}_{\geq 0}$, as h_1 and h_i are both C^2 smooth hermitian metrics on X_k and $X_j \subset \subset X$, we know that the two norms in $L^2(X_j, K_X \otimes E, h_1)$ and $L^2(X_j, K_X \otimes E, h_i)$ are equivalent. Note that $\sup_{m''} \int_{X_j} |u_{m'', \epsilon, j}|^2_{h_i} < +\infty$. Hence we know that $u_{m'',\epsilon,j}\sqrt{e^{v_{\epsilon}(\Psi)}c(-v_{\epsilon}(\Psi))e^{-\delta \max\{\psi+T,2\log|F|\}}}$ also weakly converges to $u_{\epsilon,j}\sqrt{e^{v_{\epsilon}(\Psi)}c(-v_{\epsilon}(\Psi))e^{-\delta \max\{\psi+T,2\log|F|\}}}\$ in $L^2(X_j, K_X \otimes E, h_i)$ as $m'' \to +\infty$. Then we have

$$
\begin{aligned} &\int_{X_j} |u_{\epsilon,j}|^2_{h_i} e^{v_{\epsilon}(\Psi)-\delta \max\{\psi+T,2\log|F|\}} c\bigl(-v_{\epsilon}(\Psi)\bigr) \\ \leq & \liminf_{m'' \to +\infty} \int_{X_j} |u_{m'',\epsilon,j}|^2_{h_i} e^{v_{\epsilon}(\Psi)-\delta \max\{\psi+T,2\log|F|\}} c\bigl(-v_{\epsilon}(\Psi)\bigr) \\ \leq & \liminf_{m'' \to +\infty} \int_{X_j} v''_{\epsilon}(\Psi) |f F^{1+\delta}|^2_{h_{m''}} e^{-u(-v_{\epsilon}(\Psi))-\delta \max\{\psi+T,2\log|F|\}} \\ \leq & \int_{X_j} v''_{\epsilon}(\Psi) |f F|^2_{h} e^{-u(-v_{\epsilon}(\Psi))} < +\infty. \end{aligned}
$$

Letting $i \to +\infty$, by monotone convergence theorem, we have

$$
\int_{X_j} |u_{\epsilon,j}|_h^2 e^{v_\epsilon(\Psi) - \delta \max\{\psi + T, 2\log|F|\}} c(-v_\epsilon(\Psi)) \le \int_{X_j} v''_\epsilon(\Psi) |fF|_h^2 e^{-u(-v_\epsilon(\Psi))} < +\infty.
$$
\n(9.31)

Let $\tilde{C}_j := \inf_{X_j} e^{-u(-v_{\epsilon}(\Psi)) - \delta \max\{\psi + T, 2\log|F|\}}$ and note that \tilde{C}_j is a positive number. Then it follows from $\tilde{C}_j > 0$, inequalities [\(9.28\)](#page-65-1), [\(9.30\)](#page-66-0) and [\(2.2\)](#page-8-2) that we have

$$
\sup_{m''}\int_{X_j}|h_{m'',\epsilon,j}|^2_{h_{m''}}<+\infty.
$$

As $|e_x|_{h_{m'}} \leq |e_x|_{h_{m'+1}}$ for any $m' \in \mathbb{Z}_{\geq 0}$, for h_1 , we have

$$
\sup_{m''}\int_{X_j}|h_{m'',\epsilon,j}|_{h_1}^2<+\infty.
$$

Since the closed unit ball of the Hilbert space is weakly compact, we can extract a subsequence of $\{h_{m'',\epsilon,j}\}$ (also denote by $h_{m'',\epsilon,j}$) weakly convergent to $h_{\epsilon,j}$ in $L^2(X_j, \wedge^{n,1}T^*M \otimes E, h_1)$ as $m'' \to +\infty$. As $0 \leq \tilde{\lambda}_{m''} \leq \lambda + 1$ and X_j is relatively compact in X , we know that

$$
\sup_{m''}\int_{X_j} N_1\tilde{\lambda}_{m''}|h_{m'',\epsilon,j}|^2_{h_{m''}} <+\infty.
$$

It follows from Lemma [9.9](#page-50-0) that we know that $\sqrt{N_1 \tilde{\lambda}_{m''}} h_{m'', \epsilon, j}$ weakly converges to $\text{some } \tilde{h}_{\epsilon,j} \text{ and } P_{m'}\big(\sqrt{N_1\tilde{\lambda}_{m'}}h_{m',\epsilon,j}\big) \text{ weakly converges to } P\big(\tilde{h}_{\epsilon,j}\big) \text{ in } L^2(X_j,\wedge^{n,1} T^\ast M\otimes \mathbb{C})$ E, h_1).

It follows from $0 \leq \tilde{\lambda}_{m''} \leq \lambda + 1$, X_j is relatively compact in X and Lemma [9.1](#page-46-2) that we know $\sqrt{N_1 \tilde{\lambda}_{m''}} h_{m'', \epsilon, j}$ weakly convergent to 0 in $L^2(X_j, \wedge^{n,1} T^* M \otimes E, h_1)$. It follows from the uniqueness of weak limit that we know $\tilde{h}_{\epsilon,j} = 0$. Then we have $P_{m'}\left(\sqrt{N_1\tilde{\lambda}_{m'}}h_{m',\epsilon,j}\right)$ weakly converges to $0=P\left(\tilde{h}_{\epsilon,j}\right)$ in $L^2(X_j,\wedge^{n,1}T^*M\otimes E,h_1)$

Replace m' by m'' in [\(9.29\)](#page-65-2) and let m'' go to $+\infty$, we have

$$
D''u_{\epsilon,j} = D''((1 - v'_{\epsilon}(\Psi))fF^{1+\delta}).
$$
\n(9.32)

Denote $F_{\epsilon,j} := -u_{\epsilon,j} + (1 - v'_{\epsilon}(\Psi)) f F^{1+\delta}$. It follows from [\(9.32\)](#page-67-0) and inequality (9.31) that we know $F_{\epsilon,j}$ is an E-valued holomorphic $(n,0)$ form on X_j and

$$
\int_{X_j} |F_{\epsilon,j} - (1 - v'_{\epsilon}(\Psi)) f F^{1+\delta}|_{h}^2 e^{v_{\epsilon}(\Psi) - \delta \max\{\psi + T, 2\log|F|\}} c(-v_{\epsilon}(\Psi))
$$
\n
$$
\leq \int_{X_j} v''_{\epsilon}(\Psi) |f F|_{h}^2 e^{-u(-v_{\epsilon}(\Psi))} < +\infty.
$$
\n(9.33)

Step 6: Letting $\epsilon \to 0$.

Note that $\sup_{\epsilon} \sup_{X_j} (e^{-u(-v_{\epsilon}(\Psi))}) < +\infty$, $0 \le v_{\epsilon}''(t) \le \frac{2}{B} \mathbb{I}_{(-t_0 - B + \epsilon, -t_0 - \epsilon)}$. We have

$$
v''_{\epsilon}(\Psi)|fF|_{h}^{2}e^{-u(-v_{\epsilon}(\Psi))} \le \sup_{\epsilon} \sup_{X_{j}} (e^{-u(-v_{\epsilon}(\Psi))})\frac{2}{B}\mathbb{I}_{\{-t_{0}-B<\Psi<-t_{0}\}}|fF|_{h}^{2}.
$$
 (9.34)

It follows from inequality [\(2.2\)](#page-8-2) and dominated convergence theorem that we have

$$
\lim_{\epsilon \to 0} \int_{X_j} v''_{\epsilon}(\Psi) |fF|_{h}^{2} e^{-u(-v_{\epsilon}(\Psi))}
$$
\n
$$
= \int_{X_j} v''(\Psi) |fF|_{h}^{2} e^{-u(-v(\Psi))}
$$
\n
$$
\leq \left(\sup_{X_j} e^{-u(-v(\Psi))} \right) \int_{X_j} \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \Psi < -t_0\}} |fF|_{h}^{2}.
$$

Combining with

$$
\inf_{\epsilon}\inf_{X_j}e^{v_{\epsilon}(\Psi)-\delta \max\{\psi+T,2\log|F|\}}c\big({-}v_{\epsilon}(\Psi)\big) > 0,
$$

we have

$$
\sup_{\epsilon} \int_{X_j} |F_{\epsilon,j} - (1 - v'_{\epsilon}(\Psi)) f F^{1+\delta}|^2_h < +\infty.
$$

For any $i \in \mathbb{Z}_{\geq 0}$, as $h_i \leq h$, we have

$$
\sup_{\epsilon}\int_{X_j}|F_{\epsilon,j}-(1-v'_{\epsilon}(\Psi))fF^{1+\delta}|^2_{h_i} <+\infty.
$$

For any fixed $i \in \mathbb{Z}_{\geq 0}$, note that $\overline{X_j}$ is compact and both h_i and \hat{h} are C^2 smooth hermitian metrics on E, then there exists a constant $c_i > 0$, such that $h_i \leq c_i \hat{h}$ on X_k . By (2.1) , we have

$$
\sup_{\epsilon} \int_{X_j} |(1 - v'_{\epsilon}(\Psi)) f F^{1+\delta}|^2_{h_i} \le c_i (\sup_{X_j} |F|^{2+2\delta}) \int_{X_j} \mathbb{I}_{\{\Psi < -t_0\}} |f|^2_{\hat{h}} < +\infty,
$$

one can obtain that $\sup_{\epsilon} \int_{X_j} |F_{\epsilon,j}|_{h_i}^2 < +\infty$.

Especially, we know $\sup_{\epsilon} \int_{X_k} |F_{\epsilon,j}|_{h_1}^2 < +\infty$. Note that h_1 is a C^2 hermitian metric on E, $X_j \subset \subset X$ and $F_{\epsilon,j}$ is E-valued holomorphic $(n,0)$ form on X_j , there exists a subsequence of ${F_{\epsilon}}$, j_{ϵ} (also denoted by ${F_{\epsilon}}$, j_{ϵ}) compactly convergent to an E-valued holomorphic $(n, 0)$ form F_j on X_j .

It follows from Fatou's lemma that we have

$$
\int_{K} |F_j - (1 - b(\Psi)) f F^{1+\delta}|_{h_i}^2 e^{v(\Psi) - \delta \max\{\psi + T, 2 \log |F|\}} c(-v(\Psi))
$$
\n
$$
= \liminf_{\epsilon \to 0} \int_{K} |F_{\epsilon,j} - (1 - v'_{\epsilon}(\Psi)) f F^{1+\delta}|_{h_i}^2 e^{v_{\epsilon}(\Psi) - \delta \max\{\psi + T, 2 \log |F|\}} c(-v_{\epsilon}(\Psi))
$$
\n
$$
\leq \limsup_{\epsilon \to 0} \int_{K} |F_{\epsilon,j} - (1 - v'_{\epsilon}(\Psi)) f F^{1+\delta}|_{h_i}^2 e^{v_{\epsilon}(\Psi) - \delta \max\{\psi + T, 2 \log |F|\}} c(-v_{\epsilon}(\Psi))
$$
\n
$$
\leq \limsup_{\epsilon \to 0} \int_{X_j} v''_{\epsilon}(\Psi) |f F|_{h_i}^2 e^{-u(-v_{\epsilon}(\Psi))}
$$
\n
$$
\leq \left(\sup_{X_j} e^{-u(-v(\Psi))} \right) \int_{X_j} \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \Psi < -t_0\}} |f F|_{h_i}^2.
$$
\n(9.35)

Letting $i \to +\infty$ in inequality [\(9.35\)](#page-68-0) and by monotone convergence Theorem, we have

$$
\int_{K} |F_j - (1 - b(\Psi)) f F^{1+\delta}|^2_h e^{v(\Psi) - \delta \max\{\psi + T, 2\log|F|\}} c(-v(\Psi))
$$

$$
\leq \left(\sup_{X_j} e^{-u(-v(\Psi))}\right) \int_{X_j} \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \Psi < -t_0\}} |f F|^2_h.
$$

As K is any compact subset of X_j and by monotone convergence Theorem, we know

$$
\int_{X_j} |F_j - (1 - b(\Psi)) f F^{1+\delta}|_h^2 e^{v(\Psi) - \delta \max\{\psi + T, 2 \log |F|\}} c(-v(\Psi))
$$
\n
$$
\leq \left(\sup_{X_j} e^{-u(-v(\Psi))} \right) \int_{X_j} \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \Psi < -t_0\}} |f F|_h^2.
$$
\n
$$
(9.36)
$$

Step 7: Letting $j \rightarrow +\infty$. It is easy to see that

$$
\begin{aligned} & \left(\sup_{X_j} e^{-u(-v(\Psi))} \right) \int_{X_j} \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \Psi < -t_0\}} |fF|_h^2 \\ \le & \left(\sup_{X} e^{-u(-v(\Psi))} \right) \int_X \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \Psi < -t_0\}} |fF|_h^2 < +\infty. \end{aligned}
$$

For fixed j, as $e^{v(\Psi)-\delta \max\{\psi+T,2\log|F|\}}c(-v(\Psi))$ has a positive lower bound on any $\overline{X_j}$, we have for $j_1 > j$,

$$
\sup_{j_1>j}\int_{X_j}|F_{j_1}-(1-b(\Psi))fF^{1+\delta}|^2_h<+\infty.
$$

For any $i \in \mathbb{Z}_{\geq 0}$, as $h_i \leq h$, we have

$$
\sup_{j_1>j}\int_{X_j}|F_{j_1}-(1-b(\Psi))fF^{1+\delta}|^2_{h_i}<+\infty.
$$

Note that $\overline{X_j}$ is compact and both h_i and \hat{h} are C^2 smooth hermitian metrics on E, then there exists a constant $c_i > 0$, such that $h_i \leq c_i \hat{h}$ on X_k . By [\(2.1\)](#page-8-1), we have

$$
\int_{X_j} |(1-b(\Psi))fF^{1+\delta}|^2_{h_i} \leq c_i (\sup_{X_j}|F|^{2+2\delta}) \int_{X_j} \mathbb{I}_{\{\Psi<-t_0\}}|f|^2_{\hat{h}} < +\infty,
$$

one can obtain that $\sup_{j_1 > j} \int_{X_j} |F_{j_1}|_{h_i}^2 < +\infty$. Especially

$$
\sup_{j_1 > j} \int_{X_j} |F_{j_1}|_{h_1}^2 < +\infty.
$$

By diagonal method, there exists a subsequence $F_{j''}$ uniformly convergent on any $\overline{X_j}$ to an E-valued holomorphic $(n, 0)$ -form on X denoted by \tilde{F} . It follows from Fatou's lemma that we have

$$
\int_{X_j} |\tilde{F} - (1 - b(\Psi)) f F^{1+\delta}|_{h_i}^2 e^{v(\Psi) - \delta \max\{\psi + T, 2 \log |F|\}} c(-v(\Psi))
$$
\n
$$
\leq \liminf_{j'' \to +\infty} \int_{X_j} |F_{j''} - (1 - b(\Psi)) f F^{1+\delta}|_{h_i}^2 e^{v(\Psi) - \delta \max\{\psi + T, 2 \log |F|\}} c(-v(\Psi))
$$
\n
$$
\leq \limsup_{j'' \to +\infty} \int_{X_j} |F_{j''} - (1 - b(\Psi)) f F^{1+\delta}|_{h_i}^2 e^{v(\Psi) - \delta \max\{\psi + T, 2 \log |F|\}} c(-v(\Psi))
$$
\n
$$
\leq \limsup_{j'' \to +\infty} \int_{X_{j''}} |F_{j''} - (1 - b(\Psi)) f F^{1+\delta}|_{h_i}^2 e^{v(\Psi) - \delta \max\{\psi + T, 2 \log |F|\}} c(-v(\Psi))
$$
\n
$$
\leq \limsup_{j'' \to +\infty} \left(\sup_{X_{j''}} e^{-u(-v(\Psi))} \right) \int_{X_{j''}} \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \Psi < -t_0\}} |f F|_{h_i}^2
$$
\n
$$
\leq \left(\sup_{X} e^{-u(-v(\Psi))} \right) \int_X \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \Psi < -t_0\}} |f F|_{h_i}^2 < +\infty.
$$
\n(9.37)

Letting $i \to +\infty$ in inequality [\(9.37\)](#page-70-0), and by monotone convergence theorem, we have

$$
\int_{X_j} |\tilde{F} - (1 - b(\Psi)) f F^{1+\delta}|^2_h e^{v(\Psi) - \delta \max\{\psi + T, 2 \log |F|\}} c(-v(\Psi))
$$
\n
$$
\leq \left(\sup_X e^{-u(-v(\Psi))} \right) \int_X \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \Psi < -t_0\}} |f F|^2_h < +\infty. \tag{9.38}
$$

Letting $j \to +\infty$ in inequality [\(9.38\)](#page-70-1), and by monotone convergence theorem, we have

$$
\int_{X} |\tilde{F} - (1 - b(\Psi)) f F^{1+\delta}|_{h}^{2} e^{v(\Psi) - \delta \max\{\psi + T, 2\log|F|\}} c(-v(\Psi))
$$
\n
$$
\leq \left(\sup_{X} e^{-u(-v(\Psi))} \right) \int_{X} \frac{1}{B} \mathbb{I}_{\{-t_{0} - B < \Psi < -t_{0}\}} |f F|_{h}^{2} < +\infty. \tag{9.39}
$$

Step 9: ODE System.

Now we want to find η and ϕ such that $(\eta + g^{-1}) = e^{-v_{\epsilon}(\Psi_m)} e^{-\phi} \frac{1}{c(-v_{\epsilon}(\Psi_m))}$. As $\eta = s(-v_{\epsilon}(\Psi_m))$ and $\phi = u(-v_{\epsilon}(\Psi_m))$, we have $(\eta + g^{-1})e^{v_{\epsilon}(\Psi_m)}e^{\phi} = \left((s + \frac{s'^2}{u'^2 s})e^{v_{\epsilon}(\Psi_m)}\right)$ $\frac{s'^2}{u''s-s''}$) $e^{-t}e^u$) ◦ $(-v_{\epsilon}(\Psi_m)).$

Summarizing the above discussion about s and u , we are naturally led to a system of ODEs:

$$
1)(s + \frac{s'^2}{u''s - s''})e^{u-t} = \frac{1}{c(t)},
$$

\n
$$
2)s' - su' = 1,
$$
\n(9.40)

when $t \in (T, +\infty)$.

We solve the ODE system [\(9.40\)](#page-70-2) and get $u(t) = -\log(\frac{1}{\delta}c(T)e^{-T} + \int_T^t c(t_1)e^{-t_1}dt_1)$ and $s(t) = \frac{\int_T^t (\frac{1}{\delta}c(T)e^{-T} + \int_T^{t_2}c(t_1)e^{-t_1}dt_1)dt_2 + \frac{1}{\delta^2}c(T)e^{-T}}{\frac{1}{\delta^2}c(T)e^{-T} + \int_T^{t}c(t_1)e^{-t_1}dt_1}$ $\frac{1}{\delta}c(T)e^{-T} + \int_T^t c(t_1)e^{-t_1} dt_1$.

It follows that $s \in C^{\infty}([T, +\infty))$ satisfies $s \geq \frac{1}{\delta}$ and $u \in C^{\infty}([T, +\infty))$ satisfies $u''s - s'' > 0.$

As $u(t) = -\log(\frac{1}{\delta}c(T)e^{-T} + \int_T^t c(t_1)e^{-t_1}dt_1)$ is decreasing with respect to t, then it follows from $-T \ge v(t) \ge \max\{t, -t_0 - B_0\} \ge -t_0 - B_0$, for any $t \le 0$ that

$$
\sup_{X} e^{-u(-v(\Psi))} \le \sup_{t \in [T, t_0 + B]} e^{-u(t)} = \frac{1}{\delta} c(T) e^{-T} + \int_{T}^{t_0 + B} c(t_1) e^{-t_1} dt_1. \tag{9.41}
$$

Combining with inequality [\(9.39\)](#page-70-3), we have

$$
\begin{aligned} &\int_X |\tilde{F} - (1-b(\Psi)) f F^{1+\delta}|_h^2 e^{v(\Psi)-\delta \max\{\psi+T,2\log|F|\}} c(-v(\Psi)) \\ \leq & \left(\frac{1}{\delta} c(T) e^{-T} + \int_T^{t_0+B} c(t_1) e^{-t_1} dt_1 \right) \int_X \frac{1}{B} \mathbb{I}_{\{-t_0-B<\Psi<-t_0\}} |f F|_h^2 < +\infty. \end{aligned}
$$

We get Lemma [2.1.](#page-8-0)

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