Relative Unification in Intuitionistic Logic: Towards provability logic of HA

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Abstract

This paper studies relative unification and admissibility in the intuitionistic logic. We generalize results of [Ghilardi, 1999; Iemhoff, 2001a] and prove them relative in $\mathsf{NNIL(par)}$ propositions, the class of propositions with No Nested Implications in the Left made up from parameters. The main application of such generalization is to characterize provability logic of Heyting Arithmetic HA and prove its decidability [Mojtahedi, 2022].

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1 Introduction

Silvio Ghilardi [Ghilardi, 1999, 2000] studies unification in propositional logics. More precisely, he describes all solutions for $A(x_1, \ldots, x_n) \leftrightarrow \top$ within a background logic like intuitionistic logic IPC or a modal logic containing K4. By a solution we mean a substitution θ such that $\theta(A \leftrightarrow \top)$ holds.

On the other hand, we have a related question for decidability/characterization of admissible rules of IPC. A rule A/B is admissible to a logic L if $L \vdash \theta(A)$ implies $L \vdash \theta(B)$ for every substitution θ. Despite classical logic, in which every admissible rule is also derivable, the case of modal logic and intuitionistic logic are not trivial. Probably the first such underivable admissible rule for IPC is the following [Harrop, 1960]:

$$\frac{\neg A \to (B \vee C)}{(\neg A \to B) \vee (\neg A \to C)}$$

Using the tools and results in [Ghilardi, 1999], Rosalie Iemhoff proves the completeness of a base for *all* admissible rules of IPC [Iemhoff, 2001b,a], which previously conjectured by de Jongh and Visser. Decidability of admissibility for IPC was already known [Rybakov, 1987b, 1992, 1997]. There are similar results in some modal logics extending K4 both for unification [Ghilardi, 2000] and admissibility [Ježrábek, 2005; Iemhoff and Metcalfe, 2009].

There is yet another related notion, *preservativity*, an intuitionistic alternate for the classical notion of interpretability or conservativity [Iemhoff, 2003; Visser, 2002]. Preservativity is a binary relation $A \underset{\Gamma}{\overset{\top}{\approx}} B$ defined as "$\Gamma \vdash_\top A$ implies $\Gamma \vdash_\top B$". Albert Visser in [Visser, 2002] shows that NNIL-preservativity and admissibility are tightly related, in which NNIL, is the class of No Nested Implications in the Left, introduced in [Visser et al., 1995] and more elaborated in [Visser, 2002]. This class of propositions are proved to be helpful in the realm of intuitionistic logic. A crucial result concerning NNIL appeared in [Visser, 2002] is to provide an algorithm that takes $A \in \mathcal{L}_0$ and returns its best NNIL approximation A^* from below, i.e., $\vdash A^* \to A$ and for all NNIL formulae B such that $\vdash B \to A$, we have $\vdash B \to A^*$. Later in section 4.5 we also provide an algorithm which computes $A^\star$, the best NNIL(par)-approximation of A from below.

The main work of current paper is to extend [Ghilardi, 1999; Iemhoff, 2001a] and prove their results relative in NNIL(par) propositions, the class of No Nested Implications in the Left [Visser et al., 1995] which are made up from set of atomic parameters par. First we imitate [Ghilardi, 1999] and study projectivity and extendibility relative in NNIL(par)-propositions (theorem 3.12). This will lead us to a relativised version of projective approximations (theorem 3.27). Then we take a route similar to [Iemhoff, 2001a] and provide a base called $\mathsf{AR_{par}}$, for NNIL(par)-admissibility of IPC and prove its completeness (theorem 4.15). This last result together with [Ardeshir and Mojtahedi, 2018; Mojtahedi, 2021], lead us to the characterization and decidability of provability logic of Heyting arithmetic HA, which is splitted to another manuscript [Mojtahedi, 2022].

Finally we axiomatize two interesting preservativity predicates $\underset{\Gamma}{\overset{\mathsf{IPC}}{\approx}}$: first when Γ is considered as the set of NNIL(par)-projective propositions (this is same as projectivity relative in NNIL(par), as defined in section 3.1), and second when $\Gamma := \mathsf{NNIL(par)}$.

2 Preliminary definitions and facts

This section is devoted to preliminaries and conventions. Among other well-known notions, we define NNIL propositions, admissibility, preservativity and greatest lower bounds.

2.1 propositional language

The propositional language $\mathcal{L}_0$ includes connectives $\vee$, $\wedge$, $\to$ and $\bot$. Negation $\neg$ is defined as $\neg A := A \to \bot$ and $\top := \neg\bot$. By default we assume that $\mathcal{L}_0$ includes finite set of atomic variables var and also finite set of atomic parameters par. The union var $\cup$ par is annotated as atom, the set

of atomics. We use $\vec{p}$ and $\vec{q}$ as a finite set or list of parameters and $\vec{x}$ and $\vec{y}$ for a finite set or list of variables. Finite lists or sets of atomics are annotated by $\vec{a}$ and $\vec{b}$. We use x, y and z (possibly with subscripts) as meta-variables for variables and also p, q and r (possibly with subscripts) for parameters. Also a, b and c (again possibly with subscripts) are used for both atomic variables and parameters.

Let $\vec{a} = a_1, \ldots, a_n$ be a list of atomics and $\vec{B} = B_1, \ldots, B_n$. Then $A[\vec{a} : \vec{B}]$ indicate the simultaneous substitution of B_i for a_i in A.

We also use the notation $\mathcal{L}_0(X)$ to indicate the language of all boolean combinations of propositions in X. We consider IPC as the intuitionistic propositional logic [Troelstra and van Dalen, 1988] and $\vdash$ indicates derivability in IPC. All propositional logics considered in this paper are assumed to be closed under (1) modus ponens and (2) substitutions

2.2 Substitutions

A substitution θ is a function on propositional language $\mathcal{L}_0$ which commutes with all connectives, i.e.

- $\theta(B \circ C) = \theta(B) \circ \theta(C)$ for every $\circ \in \{\vee, \wedge, \rightarrow\}$.

- $\theta(\bot) = \bot$.

By default we assume that all substitutions are identity on the set par *of parameters.* We say that a substitution is *general*, if we relax this condition on par and allow the parameters to be substituted as well.

2.3 Kripke models for intuitionistic logic

A Kripke model for intuitionistic logic, is a triple $\mathcal{K} = (W, \prec, V)$ with following properties:

- $W \neq \emptyset$.

- $(W, \prec)$ is a partial order (transitive and irreflexive). We write $\preceq$ for the reflexive closure of $\prec$.

- V is the valuation on atomics, i.e. $V \subseteq W \times$ atom.

- $w \preceq u$ and $w \, V \, a$ implies $u \, V \, a$ for every $w, u \in W$ and $a \in$ atom.

The valuation V may be extended to include all propositions as follows:

- $\mathcal{K}, w \Vdash a$ iff $w \, V \, a$, for $a \in$ atom.

- $\mathcal{K}, w \Vdash A \wedge B$ iff $\mathcal{K}, w \Vdash A$ and $\mathcal{K}, w \Vdash B$.

- $\mathcal{K}, w \Vdash A \vee B$ iff $\mathcal{K}, w \Vdash A$ or $\mathcal{K}, w \Vdash B$.

- $\mathcal{K}, w \Vdash A \rightarrow B$ iff for every $u \succeq w$ if we have $\mathcal{K}, w \Vdash A$ then $\mathcal{K}, w \Vdash B$.

We also define the following notions for Kripke models:

- *Finite:* if W is a finite set.

- *Rooted:* if there is some node $w_0 \in W$ such that $w_0 \preceq w$ for every $w \in W$.

- *Tree:* if for every $w \in W$ the set $\{u \in W : u \preceq w\}$ is finite linearly ordered (by $\preceq$) set.

By default we assume that *all Kripke models of* IPC *in this paper are finite rooted and tree.* As we will see in section 4.2, some other sort of Kripke semantics are used, called AR$_{\text{par}}$-models, which might not be finite or tree. Given $A \in \mathcal{L}_0$, we define $\text{Mod}(A)$ as the class of all (finite rooted tree) Kripke models of A.

2.4 NNIL propositions

The class of *No Nested Implications in the Left*, NNIL formulae, was discovered by Albert Visser and first published in [Visser et al., 1995], and more explored in [Visser, 2002; Ilin et al., 2020]. For simplicity of notations, we may write N for NNIL. The crucial result of [Visser, 2002] is to provide an algorithm that takes $A \in \mathcal{L}_0$ and returns its best NNIL approximation A^* from below, i.e., $\vdash A^* \to A$ and for all NNIL formulae B such that $\vdash B \to A$, we have $\vdash B \to A^*$. Later in this paper we define another algorithm $A^\star$ which calculates the best NNIL(par)-approximation of A from below (section 4.5). The classes NNIL and NI of propositions in $\mathcal{L}_0$ are defined inductively:

- $a \in$ NNIL and $a \in$ NI for every $a \in$ atom.

- $B \circ C \in$ NNIL if $B, C \in$ NNIL. Also $B \circ C \in$ NI if $B, C \in$ NI. ($\circ \in \{\vee, \wedge\}$)

- $B \to C \in$ NNIL if $B \in$ NI and $C \in$ NNIL.

2.5 Notations on sets of propositions

In rest of the paper we deal with several sets of propositions and following notations make life easier. Given $A \in \mathcal{L}_0$, let $\mathsf{sub}(A)$ be the set of all subformulas of A. For simplicity of notations, we write $\mathsf{X}_1 \ldots \mathsf{X}_n$ for $\mathsf{X}_1 \cap \ldots \cap \mathsf{X}_n$, when X_i are sets of propositions. For a set Γ of propositions define

- $\Gamma^\vee := \{\bigvee \Delta : \Delta \subseteq_{\mathrm{fin}} \Gamma \text{ and } \Delta \neq \emptyset\}$. ($X \subseteq_{\mathrm{fin}} Y$ indicates that X is a finite subset of Y)

- $\Gamma(X)$ indicates the set $\Gamma \cap \mathcal{L}_0(X)$.

- $\downarrow^{\mathsf{T}}\Gamma :=$ the class of all Γ-projective propositions in T. We say that a proposition A is Γ-projective in T, if there is some substitution θ and $B \in \Gamma$ such that $\mathsf{T} \vdash \theta(A) \leftrightarrow B$ and $A \vdash_{\mathsf{T}} x \leftrightarrow \theta(x)$ for every $x \in$ var (see section 3.1). Whenever $\mathsf{T} = \mathsf{IPC}$, we may omit the superscript T and simply write $\downarrow\Gamma$.

Also define

- $\mathsf{N} := \mathsf{NNIL} :=$ as defined in section 2.4.

- $\mathsf{P}^\mathsf{T} := \mathsf{Prime}^\mathsf{T} :=$ the set of all T-prime propositions, i.e. the set of propositions A such that for every B, C with $\mathsf{T} \vdash A \to (B \vee C)$ either we have $\mathsf{T} \vdash A \to B$ or $\mathsf{T} \vdash A \to C$. Whenever $\mathsf{T} = \mathsf{IPC}$, we may omit the T-superscript from notations.

And finally we assume that $(.)^\vee$ has the lowest precedence after $\downarrow(.)$. This means that

$$\downarrow XY^\vee := (\downarrow(XY))^\vee .$$

2.6 Admissibility and preservativity

Given a Logic T, the binary relation $\vdash\!\!\!\sim^{\mathsf{T}}$ is defined to hold for those pairs A and B such that the inference rule A/B is admissible. More precisely $A \vdash\!\!\!\sim^{\mathsf{T}} B$ iff for every substitution θ, $\mathsf{T} \vdash \theta(A)$ implies $\mathsf{T} \vdash \theta(B)$. The admissibility relationship is trivial when one considers the classical propositional logic, since every admissible A/B is also derivable. However this relationship is highly nontrivial when one considers a modal logic or intuitionistic logic. Probably the first known non-derivable admissible rule is the following rule [Harrop, 1960]:

$$\frac{\neg A \to (B \vee C)}{(\neg A \to B) \vee (\neg A \to C)}$$

Harvey Friedman asked in 1975 for decidability of admissibility in the intuitionistic propositional logic. Then [Rybakov, 1987b, 1992, 1997] answers to this question positively. Although it was shown that no finite base exists for all admissible rules of the intuitionistic logic IPC [Rybakov, 1987a], de Jongh and Visser introduced a recursive base and conjectured it to generate all admissible rules of IPC. Then Iemhoff proved this conjecture [Iemhoff, 2001a,b].

Here in this paper, we consider a relativised version of admissibility. Given a logic T and a set Γ of propositions define the Γ-admissibility relation in T as follows

$$A \mathrel{\vdash^{\mathsf{T}}_{\Gamma}} B \text{ iff for every substitution } \theta \text{ and } C \in \Gamma \colon \mathsf{T} \vdash \theta(C \to A) \text{ implies } \mathsf{T} \vdash \theta(C \to B).$$

Note that there is a hidden role for the language $\mathcal{L}_0$ in the definition of $\mathrel{\vdash^{\mathsf{T}}_{\Gamma}}$, when we consider substitution θ. However since almost everywhere in the paper we fix the language $\mathcal{L}_0$, by default we assume substitutions over this fixed language and we do not explicitly mention $\mathcal{L}_0$.

There is also another binary relation on propositions, called preservativity, which is known . The Γ-preservativity relation in T is defined as follows:

$$A \mathrel{\vDash^{\mathsf{T}}_{\Gamma}} B \quad \text{iff} \quad \forall E \in \Gamma(\mathsf{T} \vdash E \to A \Rightarrow \mathsf{T} \vdash E \to B).$$

Preservativity could be considered as intuitionistic analogue of classical interpretability or conservativity. This notion as a propositional logic, well studied in [Visser, 2002] and [Iemhoff, 2003] provided Kripke semantics for it. [Zhou, 2003; Iemhoff et al., 2005] include some more elaboration on preservativity and provability, including fixed-point theorem and Beth property.

Following theorem says that $\mathrel{\vdash^{\mathsf{T}}_{\Gamma}}$ and $\mathrel{\vDash^{\mathsf{T}}_{\Gamma}}$ are ascending on Γ. All over this paper we may use this fact without mentioning.

Theorem 2.1. *If $\Gamma \subseteq \Gamma'$ then $\mathrel{\vDash^{\mathsf{T}}_{\Gamma'}} \subseteq \mathrel{\vDash^{\mathsf{T}}_{\Gamma}}$ and $\mathrel{\vdash^{\mathsf{T}}_{\Gamma'}} \subseteq \mathrel{\vdash^{\mathsf{T}}_{\Gamma}}$.*

Proof. Left to the reader. ❏

Theorem 2.2. *$A \mathrel{\vdash^{\mathsf{T}}_{\Gamma}} B$ implies $A \mathrel{\vDash^{\mathsf{T}}_{\downarrow\Gamma}} B$.*

Proof. Let $A \mathrel{\vdash^{\mathsf{T}}_{\Gamma}} B$ and $E \in \downarrow\Gamma$ such that $\mathsf{T} \vdash E \to A$. Since $E \in \downarrow\Gamma$ there is some θ and $E^{\dagger} \in \Gamma$ such that $E \vdash_{\mathsf{T}} \theta(a) \leftrightarrow a$ for every $a \in \mathsf{atom}$ and $\mathsf{T} \vdash \theta(E) \leftrightarrow E^{\dagger}$. Hence $\mathsf{T} \vdash E^{\dagger} \to \theta(A)$. Then by $A \mathrel{\vdash^{\mathsf{T}}_{\Gamma}} B$ we get $\mathsf{T} \vdash E^{\dagger} \to \theta(B)$ and thus $E \vdash_{\mathsf{T}} \theta(E \to B)$. Since θ is E-projective, we may conclude $E \vdash_{\mathsf{T}} E \to B$ and thus $\mathsf{T} \vdash E \to B$. ❏

Question 1. *What can be said about the other direction of theorem 2.2?*

Remark 2.3. *By theorems 2.1 and 2.2, $A \mathrel{\vdash^{\mathsf{T}}_{\Gamma}} B$ implies $A \mathrel{\vDash^{\mathsf{T}}_{\Gamma}} B$, however the converse may not hold. As a counterexample let A and B are two different variables and $\top \in \Gamma$ and $\mathsf{T} = \mathsf{IPC}$. Then we have $A \mathrel{\vDash^{\mathsf{T}}_{\Gamma}} B$ and not $A \mathrel{\vdash^{\mathsf{T}}_{\Gamma}} B$.*

Later in this paper we axiomatize $\mathrel{\vDash^{\mathsf{T}}_{\Gamma}}$ and $\mathrel{\vdash^{\mathsf{T}}_{\Gamma}}$ for several pairs (T, Γ). Before we continue with this, let us see some basic axioms.

Let T be a logic. The logic [T] proves statements $A \triangleright B$ for $A, B \in \mathcal{L}_0$ and has the following axioms and rules:

Aximos

Ax : $\quad A \triangleright B$, for every $\mathsf{T} \vdash A \to B$.

Rules

$$\frac{A \triangleright B \qquad A \triangleright C}{A \triangleright B \wedge C} \text{ Conj} \qquad\qquad \frac{A \triangleright B \qquad B \triangleright C}{A \triangleright C} \text{ Cut}$$

The above mentioned axiom and rules are not interesting, because $[\mathsf{T}] \vdash A \triangleright B$ iff $\mathsf{T} \vdash A \to B$. However we define several interesting additional rules:

$$\frac{B \rhd A \qquad C \rhd A}{B \vee C \rhd A} \; \text{Disj} \qquad\qquad \frac{A \rhd B \qquad (C \in \Delta)}{C \to A \rhd C \to B} \; \text{Mont}(\Delta)$$

Theorem 2.4 (Soundness). *If* T *is closed under substitutions, then* $[\mathsf{T}]$ *is sound for relative admissibility interpretations, i.e.* $[\mathsf{T}] \vdash A \rhd B$ *implies* $A \mathrel{\vert\!\approx^{\mathsf{T}}_{\Gamma}} B$ *and* $A \mathrel{\vert\!\approx^{\mathsf{T}}_{\Gamma}} B$ *for every set* Γ *of propositions and every logic* T. *Moreover*

1. *if* Γ *is* T-*prime i.e.* $\mathsf{T} \vdash A \to (B \vee C)$ *implies either* $\mathsf{T} \vdash A \to B$ *or* $\mathsf{T} \vdash A \to C$ *for every* $A \in \Gamma$ *and arbitrary* B, C, *and* Γ *is closed under substitutions, then Disj is also sound,*

2. *if* Γ *is closed under* Δ-*conjunctions, i.e.* $A \in \Gamma$ *and* $B \in \Delta$ *implies* $A \wedge B \in \Gamma$ (*up to* T-*provable equivalence relation*), *then* $\text{Mont}(\Delta)$ *is sound.*

Proof. Easy induction on the complexity of proof $[\mathsf{T}] \vdash A \rhd B$ and left to the reader. $\qquad\Box$

Theorem 2.5. $\mathrel{\vert\!\approx^{\mathsf{T}}_{\Gamma}} = \mathrel{\vert\!\approx^{\mathsf{T}}_{\Gamma^{\vee}}}$ *and* $\mathrel{\vert\!\sim^{\mathsf{T}}_{\Gamma}} = \mathrel{\vert\!\sim^{\mathsf{T}}_{\Gamma^{\vee}}}$.

Proof. We only show $A \mathrel{\vert\!\approx^{\mathsf{T}}_{\Gamma}} B$ iff $A \mathrel{\vert\!\approx^{\mathsf{T}}_{\Gamma^{\vee}}} B$ and leave the similar argument for $A \mathrel{\vert\!\sim^{\mathsf{T}}_{\Gamma}} B$ iff $A \mathrel{\vert\!\sim^{\mathsf{T}}_{\Gamma^{\vee}}} B$ to the reader. The right-to-left direction holds since $\Gamma \subseteq \Gamma^{\vee}$. For the other direction assume that $A \mathrel{\vert\!\approx^{\mathsf{T}}_{\Gamma}} B$ and let $E \in \Gamma^{\vee}$ such that $\mathsf{T} \vdash E \to A$. Then $E = \bigvee_i E_i$ with $E_i \in \Gamma$. Hence for every i we have $\mathsf{T} \vdash E_i \to A$. Then $A \mathrel{\vert\!\approx^{\mathsf{T}}_{\Gamma}} B$ implies $\mathsf{T} \vdash E_i \to B$. Thus $\mathsf{T} \vdash E \to B$, as desired. $\qquad\Box$

Notation. Whenever $\mathsf{T} = \mathsf{IPC}$ we may omit the T form notations $\mathrel{\vert\!\sim^{\mathsf{T}}_{\Gamma}}$ and $\mathrel{\vert\!\approx^{\mathsf{T}}_{\Gamma}}$ and simply write $\mathrel{\vert\!\sim_{\Gamma}}$ and $\mathrel{\vert\!\approx_{\Gamma}}$ for them. Also if $\Gamma := \{\top, \bot\}$ we may omit Γ from notations.

2.7 Greatest lower bounds

Given a set $\Gamma \cup \{A\}$ of propositions, and a logic T, we say that B is a lower bound for A w.r.t. (Γ, T), if the following conditions met:

1. $B \in \Gamma$,

2. $\mathsf{T} \vdash B \to A$.

Moreover we say that B is the greatest lower bound (glb) for A w.r.t. (Γ, T), if for every lower bound B' for A w.r.t. (Γ, T) we have $\mathsf{T} \vdash B' \to B$. Note that up to T-provable equivalence relation, such glb is unique and we annotate it as $\lfloor A \rfloor^{\mathsf{T}}_{\Gamma}$.
We say that (Γ, T) is downward compact, if every $A \in \mathcal{L}_0$ has glb w.r.t. (Γ, T).

Question 2. *One may similarly define the notion of least upper bounds and upward compactness. Does downward compactness imply upward compactness?*

Theorem 2.6. *B is the glb for A w.r.t.* (Γ, T), *iff*

- $B \in \Gamma$,

- $\mathsf{T} \vdash B \to A$,

- $A \mathrel{\vert\!\approx^{\mathsf{T}}_{\Gamma}} B$.

Hence we have $A \mathrel{\vert\!\approx^{\mathsf{T}}_{\Gamma}} \lfloor A \rfloor^{\mathsf{T}}_{\Gamma}$.

Proof. Left to the reader. $\qquad\Box$

Question 3. *As we saw in theorem 2.6, the glb may be expressed via preservativity relation $\mathrel{\vert\!\approx^{\mathsf{T}}_{\Gamma}}$. One may think of its adjoint relation which best suites for lub's:*

$$A \mathrel{{}^{*}\!\vert\!\approx^{\mathsf{T}}_{\Gamma}} B \quad \textit{iff} \quad \forall E \in \Gamma(\mathsf{T} \vdash A \to E \Rightarrow \mathsf{T} \vdash B \to E).$$

[Visser, 2002, Corollary 7.2] axiomatizes $\mathrel{\vert\!\approx^{\mathsf{T}}_{\Gamma}}$ for $\mathsf{T} = \mathsf{IPC}$ and $\Gamma = \mathsf{NNIL}$. We ask for an axiomatization for $\mathrel{{}^{*}\!\vert\!\approx^{\mathsf{T}}_{\Gamma}}$ when we let $\mathsf{T} = \mathsf{IPC}$ and $\Gamma = \mathsf{NNIL}$.

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Corollary 2.7. *If $\lfloor A \rfloor_\Gamma^\mathsf{T}$ exists, then for every $B \in \mathcal{L}_0$ we have*

$$\mathsf{T} \vdash \lfloor A \rfloor_\Gamma^\mathsf{T} \to B \quad \textit{iff} \quad A \underset{\Gamma}{\overset{\mathsf{T}}{\approx}} B.$$

Proof. First assume that $\mathsf{T} \vdash \lfloor A \rfloor_\Gamma^\mathsf{T}$. Also let $E \in \Gamma$ such that $\mathsf{T} \vdash E \to A$. Theorem 2.6 implies $A \underset{\Gamma}{\overset{\mathsf{T}}{\approx}} \lfloor A \rfloor_\Gamma^\mathsf{T}$ and hence $\mathsf{T} \vdash E \to \lfloor A \rfloor_\Gamma^\mathsf{T}$. Then by $\mathsf{T} \vdash \lfloor A \rfloor_\Gamma^\mathsf{T} \to B$ we get $\mathsf{T} \vdash E \to B$, as desired.
For the other direction let $A \underset{\Gamma}{\overset{\mathsf{T}}{\approx}} B$. By definition we have $\lfloor A \rfloor_\Gamma^\mathsf{T} \in \Gamma$ and $\mathsf{T} \vdash \lfloor A \rfloor_\Gamma^\mathsf{T} \to A$. Hence by $A \underset{\Gamma}{\overset{\mathsf{T}}{\approx}} B$ we get $\mathsf{T} \vdash \lfloor A \rfloor_\Gamma^\mathsf{T} \to B$, as desired. ❑

3 NNIL(par)-fication: unification to NNIL(par)

Silvio Ghilardi, in [Ghilardi, 1999] characterizes projective propositions in the language $\mathcal{L}_0(\mathsf{var})$ with the aid of Kripke semantics. Then he uses this characterization to prove that the unification type of IPC is finitary. Afterwards, Rosalie Iemhoff [Iemhoff, 2001b,a] uses this result together with a special sort of Kripke models, called AR-models, to characterize the admissible rules of IPC. In this section we consider a relativised version for those results. The difference from previous version is that we are not allowed to substitute parameters (a reserved set of atomics), and also instead of unification, we expect to simplify the proposition to a NNIL(par) proposition, called NNIL(par)-fication. In fact, previous results will be an special case of ours when $\mathsf{par} = \emptyset$ and hence $\mathsf{NNIL(par)} = \{\top, \bot\}$. The methods of our proof follows main roads took in [Ghilardi, 1999; Iemhoff, 2001a].
We start with relativised version of projective unification (section 3.1) and extension property (section 3.2). Then (section 3.3) we prove a correspondence between relativised projectivity and extendibility. Having such Kripke semantical charcterization in hand, then we prove that every proposition has a finitary projective approximation (section 3.4). Actually we prove something more: every proposition has a finitary projective resolution (definition 3.15). Finally at the end of this section (section 3.5), we prove that in the specific case, when $A \in \mathsf{NNIL}$, this finitary projective resolution takes an elegant form.

3.1 Relative Projectivity

Given $A \in \mathcal{L}_0$, a substitution θ is called *A-projective* (in IPC) if

(3.1)   For all atomic a we have $A \vdash a \leftrightarrow \theta(a)$.

When one considers unification for propositional logics, projectivity is proved to be of great help [Ghilardi, 1997]. As we will see, our study is not an exception.
If $\Gamma \subseteq \mathcal{L}_0(\mathsf{par})$, a substitution θ is a Γ-*fier* (as a generalization for uni-fier) for A, if

$$\vdash \theta(A) \in \Gamma \quad \text{i.e. } \theta(A) \text{ is IPC-equivalent to some } A' \in \Gamma.$$

In this case we use the notation $A \overset{\theta}{\twoheadrightarrow} \Gamma$. If Γ is a singleton $\{A'\}$ we write $A \overset{\theta}{\twoheadrightarrow} A'$ instead of $A \overset{\theta}{\twoheadrightarrow} \{A'\}$. θ is a unifier for A if it is $\{\top\}$-fier for A. We say that a substitution θ projects A to Γ (notation: $A \overset{\theta}{\rightarrowtail\twoheadrightarrow} \Gamma$) if θ is A-projective and Γ-fier. We say that A is Γ-projective (notation $A \rightarrowtail\twoheadrightarrow \Gamma$) if there is some θ such that $A \overset{\theta}{\rightarrowtail\twoheadrightarrow} \Gamma$. We say that A is projective, if it is $\{\top\}$-projective. Also $\downarrow\Gamma$ indicates the set of all propositions which are Γ-projective.

Uniqueness of Γ-projections

Let $A \overset{\theta}{\rightarrowtail\twoheadrightarrow} A'$ and $A \overset{\tau}{\rightarrowtail\twoheadrightarrow} A''$ and $A', A'' \in \Gamma \subseteq \mathcal{L}_0(\mathsf{par})$. From the A-projectivity of θ and τ, for every atomic a we have $A \vdash \theta(a) \leftrightarrow \tau(a)$. Hence $A \vdash \theta(A) \leftrightarrow \tau(A)$ and then $A \vdash A' \leftrightarrow A''$. By applying θ to both sides of this derivation, we have $\theta(A) \vdash \theta(A') \leftrightarrow \theta(A'')$. Since θ is identity over

parameters and $A', A'' \in \mathcal{L}_0(\mathsf{par})$, we have $A' \vdash A' \leftrightarrow A''$. Hence $\vdash A' \to A''$. Similarly we have $\vdash A'' \to A'$, and hence $\vdash A' \leftrightarrow A''$. This argument shows that for every Γ-projective A, there is a unique (modulo IPC-provable equivalence) $A^\dagger \in \Gamma$ such that for some A-projective substitution θ we have $\vdash \theta(A) \leftrightarrow A^\dagger$. Such unique $A^\dagger$ is called the Γ-*projection* of A.

Lemma 3.1. *Let A be Γ-projective and $A^\dagger \in \Gamma$ its projection. Then $\vdash A \to A^\dagger$.*

Proof. Let θ be the A-projective Γ-fier for A, i.e. $A \vdash B \leftrightarrow \theta(B)$ for every B, and $\vdash \theta(A) \leftrightarrow A^\dagger$. Hence we have $A \vdash A \leftrightarrow \theta(A)$ and thus $A \vdash A \leftrightarrow A^\dagger$. This implies $\vdash A \to A^\dagger$, as desired. ❏

Lemma 3.2. *Let $\Gamma \subseteq \mathcal{L}_0(\mathsf{par})$. If $A \xrightarrow{\theta} A^\dagger \in \Gamma$ and B is an arbitrary proposition, then we have*

$$\vdash A \to B \quad iff \quad \vdash \theta(A^\dagger \to B).$$

Proof. The left-to-right direction is obvious. For other direction, let $\vdash \theta(A^\dagger \to B)$. Hence $A \vdash \theta(A^\dagger \to B)$ and then $A \vdash A^\dagger \to B$. Lemma 3.1 implies $\vdash A \to A^\dagger$ and thus $\vdash A \to B$. ❏

3.2 Relative Extendibility

Given a Kripke model $\mathcal{K} = (W, \preccurlyeq, V)$ and $w \in W$, $\mathcal{K}_w$ indicates the restriction of $\mathcal{K}$ to the nodes $u \succcurlyeq w$. For a set $\vec{a} \subseteq \mathsf{atom}$, we say that $\mathcal{K}' = (W', \preccurlyeq', V')$ is an $\vec{a}$-submodel of $\mathcal{K} = (W, \preccurlyeq, V)$, annotated as $\mathcal{K}' \preceq^{\vec{a}} \mathcal{K}$, if there exists a relation $R \subseteq W' \times W$ such that

- $\mathcal{K}', w' \Vdash a$ iff $\mathcal{K}, w \Vdash a$, for every $a \in \vec{a}$ and $(w', w) \in R$.

- $v' \succcurlyeq' w' \, R \, w$ implies $\exists v \in W \, (v' \, R \, v \succcurlyeq w)$, for every $w \in W$ and $w', v' \in W'$.

- $\forall w' \in W' \exists w \in W (w' \, R \, w)$.

Also we say that $\mathcal{K}' \preceq_1^{\vec{a}} \mathcal{K}$ if the above relation R, is function. In this case the second condition takes a more readable face:

- $w' \preccurlyeq' v'$ implies $f(w') \preccurlyeq f(v')$.

Moreover define $\mathcal{K}' \subseteq^{\vec{a}} \mathcal{K}$ iff we have

- $W' \subseteq W$,

- $\preccurlyeq'$ is the restriction of $\preccurlyeq$ to W',

- $w \, V' \, a$ iff $w \, V \, a$ for every $w \in W'$ and $a \in \vec{a}$,

Also $\mathcal{K} \preceq^{\vec{a}} \mathcal{K}$ for a class $\mathcal{K}$ of Kripke models and a Kripke model $\mathcal{K}$ indicates that for every $\mathcal{K}' \in \mathcal{K}$ we have $\mathcal{K}' \preceq^{\vec{a}} \mathcal{K}$. We have similar notations for $\mathcal{K} \subseteq^{\vec{a}} \mathcal{K}$ and $\mathcal{K} \preceq_1^{\vec{a}} \mathcal{K}$.

Since we only consider Kripke models with finite rooted tree frames, we have the equivalency of $\preceq_1^{\vec{a}}$ and $\preceq^{\vec{a}}$:

Lemma 3.3. *$\mathcal{K}' \preceq_1^{\vec{a}} \mathcal{K}$ is equivalent to $\mathcal{K}' \preceq^{\vec{a}} \mathcal{K}$.*

Proof. Let $\mathcal{K}' \preceq^{\vec{a}} \mathcal{K}$ and R is a relation with above mentioned properties. It is enough to define a function $f \subseteq R$ such that f and R share the same domain W'. For every $\preccurlyeq'$-minimal node $w' \in W'$, define $f(w')$ an arbitrary node w with $w' \, R \, w$. Note that such w always exists. Since $\mathcal{K}'$ is tree, for every $w_1' \in W'$ which is not minimal, there is a unique predecessor $w_0' \preccurlyeq' w_1'$. Then by definition for every $w_1' \in W'$ there is some $w_1 \in W$ such that $f(w_0') \preccurlyeq w_1$. Define $f(w_1') := w_1$ for some such w_1. Then it is not difficult to observe that this f satisfies the condition "$w' \preccurlyeq' v'$ implies $f(w') \preccurlyeq f(v')$". ❏

Although $\subseteq^{\vec{a}}$ is not equivalent to $\preceq_1^{\vec{a}}$, we have the following partial equivalency:

Lemma 3.4. *If $\mathcal{K}_0 \preceq_1^{\vec{a}} \mathcal{K}_1 \Vdash A$ then $\mathcal{K}_0 \subseteq^{\vec{a}} \mathcal{K}_2 \Vdash A$ for some $\mathcal{K}_2$.*

Proof. The proof is almost identical to the proof of theorem 6.9 in [Visser et al., 1995] and we refer the reader to it for more details. Let $\mathcal{K}_i = (W_i, \preccurlyeq_i, V_i)$ for $i \in \{0,1\}$ and $f : W_0 \to W_1$ be the embedding of $\mathcal{K}_0$ in $\mathcal{K}_1$. Moreover we may assume that f is surjective, lest we add a copy of $\mathcal{K}_1$ to $\mathcal{K}_0$ with a fresh root in beneath of them and then extend the embedding on the new nodes.

Define $\mathcal{K}_2 := (W_2, \preccurlyeq_2, V_2)$ as follows:

- $W_2 := \{(w_0, f(w_0), w_1) : w_0 \in W_0 \text{ and } f(w_0) \preccurlyeq_1 w_1 \in W_1\}$.

- $(w_0, f(w_0), w_1) \preccurlyeq_2 (w'_0, f(w'_0), w'_1)$ iff either of the following holds:

    - $w_0 \preccurlyeq_0 w'_0$ and $w_1 = f(w_0)$,
    - $w_0 = w'_0$ and $w_1 \preccurlyeq_1 w'_1$.

- $(w_0, f(w_0), w_1) V_2 a$ iff $w_1 V_1 a$.

It is straightforward to show that $\mathcal{K}_2$ is a finite rooted tree-frame Kripke model with the root $(\rho, f(\rho), f(\rho))$ in which ρ is the root of $\mathcal{K}_0$. Also $\mathcal{K}_1$ and $\mathcal{K}_2$ are bisimilar and hence prove the same set of propositions including A. Moreover one may easily show that g as defined in the following, is a 1-1 embedding of $\mathcal{K}_0$ into $\mathcal{K}_2$: $g(w_0) := (w_0, f(w_0), f(w_0))$. $\qquad\square$

Remember that $\mathsf{NNIL}(\vec{a})$ indicate $\mathsf{NNIL} \cap \mathcal{L}_0(\vec{a})$. The following theorem is a Kripke semantical characterization of NNIL propositions [Visser et al., 1995].

Theorem 3.5. *Given $\vec{a} \subseteq \mathsf{atom}$, we have $A \in \mathsf{NNIL}(\vec{a})$ iff the class of Kripke models of A is closed under $\subseteq^{\vec{a}}$.*

Proof. See [Visser et al., 1995] or [Visser, 2002]. $\qquad\square$

Remark 3.6. *Modulo IPC-provable equivalence, NNIL is finite.*

Proof. Observe that each proposition can be written as $\bigvee \bigwedge C$ in which C is an atomic or implication, which we call it a component. Observe that the number of propositions in n atomics, $f(n)$, is less than or equal to $2^{2^{g(n)}}$, in which $g(n)$ is the number of components in n atomics. Then observe that $g(n+1) \le (n+1)f(n) + n + 1$, because one may assume that each component is either of the form $p \to A$ for some atomic p and some A in n variables, or it is of the form p for some atomic p. Hence the following recursive function is an upperbound for the number of all formulas in n atomics:

$$ f(0) := 2 \quad , \quad f(n+1) := 2^{2^{(n+1)(f(n)+1)}} \qquad\square $$

Define $[\![\mathcal{K}]\!]_\Gamma := \{A \in \Gamma : \mathcal{K} \Vdash A\}$.

Theorem 3.7. *Let $\mathcal{K}, \mathcal{K}'$ be two Kripke models and $\vec{a} \subseteq \mathsf{atom}$. Then $\mathcal{K}' \Vdash [\![\mathcal{K}]\!]_{\mathsf{N}(\vec{a})}$ iff $\mathcal{K}' \preceq^{\vec{a}} \mathcal{K}$.*

Proof. See [Visser et al., 1995, theorem 7.1.2]. $\qquad\square$

Given a substitution θ and a Kripke model $\mathcal{K} = (W, \preccurlyeq, V)$, we define $\theta(\mathcal{K}) := (W, \preccurlyeq, V')$ as follows. For every atomic a, define $w V' a$ iff $\mathcal{K}, w \Vdash \theta(a)$.

Lemma 3.8. *Given a general substitution θ, Kripke model $\mathcal{K}$ and $A \in \mathcal{L}_0$, we have*

$$ \mathcal{K}, w \Vdash \theta(A) \qquad \text{iff} \qquad \theta(\mathcal{K}), w \Vdash A. $$

Proof. Use induction on the complexity of A. $\qquad\square$

Remark 3.9. *Let θ and τ be two general substitutions and $(\theta\tau)$ is their composition. Above lemma, implies that $(\theta\tau)(\mathcal{K}) = \tau(\theta(\mathcal{K}))$. This conflicts with our standard notation for the composition of functions. This confliction could be resolved by choosing another name, e.g. θ^* for the operation on Kripke models corresponding to θ. However, for the simplicity of notations, we prefer not to do so.*

Given Kripke models $\mathcal{K}$ and $\mathcal{K}'$, we say that $\mathcal{K}'$ is an $\vec{a}$-*variant* of $\mathcal{K}$, if $\mathcal{K}'$ and $\mathcal{K}$ share the same frame and the same atomic valuations, except possibly at the root and only for atomics not in $\vec{a}$, for which we may have different valuations. We also say that $\mathcal{K}'$ is a variant of $\mathcal{K}$, if it is $\emptyset$-variant of $\mathcal{K}$. Also for a Kripke model $\mathcal{K}$ with the root w_0 we define $\mathcal{K} \Vdash^- A$ iff $\mathcal{K}, w \Vdash A$ for every $w \neq w_0$. We say that a proposition A is $\vec{a}$-*extendible* if for every Kripke model $\mathcal{K} \Vdash A$ and every $\mathcal{K}' \subseteq^{\vec{a}} \mathcal{K}$ with $\mathcal{K}' \Vdash^- A$ there is an $\vec{a}$-variant $\mathcal{K}''$ of $\mathcal{K}'$ such that $\mathcal{K}'' \Vdash A$. Also we say that A is extendible if it is $\emptyset$-extendible.

For later applications in this paper it is helpful to define par-extendibility also for a class of Kripke models. Let $\mathscr{K}$ is a class of Kripke models. Define the Kripke model $\sum(\mathscr{K})$ as the disjoint union of all Kripke models in $\mathscr{K}$ with a fresh root w_0 such that for every atomic a we have $\sum(\mathscr{K}), w_0 \Vdash a$ iff $\mathscr{K} \Vdash a$. Also for a Kripke model $\mathcal{K}$ with $\mathscr{K} \subseteq^{\mathcal{K}}$ define $\sum(\mathscr{K}, \mathcal{K})$ as disjoint union of the Kripke models in $\mathscr{K}$ with a fresh root w_0 and following valuation for atomics $a \in \vec{a}$:

$$\sum(\mathscr{K}, \mathcal{K}), w_0 \Vdash a \quad \text{iff} \quad \mathcal{K} \Vdash a.$$

We say that $\mathscr{K}$ is $\vec{a}$-*extendible*, if for every finite $\mathscr{K}' \subseteq \mathscr{K}$ with $\mathscr{K}' \subseteq^{\vec{a}} \mathcal{K} \in \mathscr{K}$, there is an $\vec{a}$-variant of $\sum(\mathscr{K}', \mathcal{K})$ which belongs to $\mathscr{K}$. We say that $\mathscr{K}$ is *extendible* if it is $\emptyset$-extendible. One may easily observe that A is par-extendible iff $\text{Mod}(A)$ is so.

3.3   NNIL(par)-projectivity and par-extendibility

In this section we will prove theorem 3.12, an extension of Ghilardi's characterization of projective propositions via the notion of extendibility (see theorem 3.10).
For a proposition A and a set $\vec{x} \subseteq \mathsf{var}$, define the substitution $\theta_A^{\vec{x}} : \mathcal{L}_0 \longrightarrow \mathcal{L}_0$ as follows:

$$\theta_A^{\vec{x}}(x) := \begin{cases} A \to x & : x \in \mathsf{var} \cap \vec{x} \\ A \wedge x & : x \in \mathsf{var} \setminus \vec{x} \end{cases}$$

Let $\vec{x}_1, \ldots, \vec{x}_s$ be a list of all subsets of var such that $\vec{x}_i \subseteq \vec{x}_j$ implies $i \leq j$. Finally define

$$\theta_A := \theta_A^{\vec{x}_s} \theta_A^{\vec{x}_{s-1}} \ldots \theta_A^{\vec{x}_1}$$

The following theorem, is the main preliminary tool provided in [Ghilardi, 1999] to characterize the unification type of IPC. We refer the reader to [Ghilardi, 1999, theorem 5] for its proof. We will prove a generalization of this in theorem 3.12.

Theorem 3.10. *For $A \in \mathcal{L}_0$, the following conditions are equivalent:*

1. *θ_A is a unifier for A, i.e. $\vdash \theta_A(A)$,*

2. *A is projective,*

3. *A is extendible.*

Before we continue with a generalization of above theorem, let us give another definition. Let $\mathcal{K}$ be a Kripke model and $\vec{p} \subseteq \mathsf{par}$. Then define $A^\dagger$ as follows: (remember that previously we defined $A^\dagger$ for NNIL(par)-projective A as the unique $A' \in \mathsf{NNIL(par)}$ such that $A \rightarrowtail A'$. As we will see in next theorem, these two definitions are the same up to IPC-provable equivalence relation.)

$$A^\dagger := \bigwedge_{\vec{p} \subseteq \mathsf{par}} \left(\bigwedge \vec{p} \to \bigvee_{\mathcal{K} \Vdash \vec{p}, A} \bigwedge [\![\mathcal{K}]\!]_{\mathsf{N(par)}} \right)$$

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Note that since by remark 3.6 the set $\mathsf{NNIL}(\mathsf{par})$ is finite and $[\![\mathcal{K}]\!]_{\mathsf{N}(\vec{p})} \subseteq \mathsf{NNIL}(\mathsf{par})$, the conjunction $\bigwedge [\![\mathcal{K}]\!]_{\mathsf{N}(\mathsf{par})}$ is a proposition and also the disjunction may considered as a finite disjunction.

Remark 3.11. *Note that by above definition, if $\vdash A \to B$ then $\vdash A^\dagger \to B^\dagger$.*

Theorem 3.12. *For $A \in \mathcal{L}_0$, the following conditions are equivalent:*

1. *$A \xrightarrow{\theta_A} A^\dagger$,*

2. *$A \rightarrowtail \mathsf{NNIL}(\mathsf{par})$,*

3. *A is par-extendible.*

Proof. $1 \to 2$: From definitions of $A^\dagger$, evidently $A^\dagger \in \mathsf{NNIL}$. Also observe that $\theta_A^{\vec{x}}$ is A-projective. Then since A-projective substitutions are closed under compositions, θ_A are A-projective.

$2 \to 3$: Let $A \xrightarrow{\theta} A' \in \mathsf{NNIL}(\mathsf{par})$ and $\mathcal{K} \Vdash A$ and $\mathcal{K}' \subseteq^{\mathsf{par}} \mathcal{K}$ and $\mathcal{K}' \Vdash^- A$ seeking some variant $\mathcal{K}''$ of $\mathcal{K}'$ such that $\mathcal{K}'' \Vdash A$. Let $\mathcal{K}'' = \theta(\mathcal{K}')$. First note that by A-projectivity of θ, $\mathcal{K}''$ is a par-variant of $\mathcal{K}'$. Since $\mathcal{K} \Vdash A'$, $A' \in \mathsf{NNIL}(\mathsf{par})$ and $\mathcal{K}' \subseteq^{\mathsf{par}} \mathcal{K}$, theorem 3.5 implies that $\mathcal{K}' \Vdash A'$. Hence $\mathcal{K}' \Vdash \theta(A)$, and by lemma 3.8 we have $\mathcal{K}'' \Vdash A$.

$3 \to 1$: Let A is par-extendible and show $\vdash A^\dagger \leftrightarrow \theta_A(A)$. We use induction on the height of Kripke model $\mathcal{K}$ and show $\mathcal{K} \Vdash A^\dagger \leftrightarrow \theta_A(A)$. Suppose that w_0 is the root of $\mathcal{K}$. By induction hypothesis, we have $\mathcal{K} \Vdash^- A^\dagger \leftrightarrow \theta_A(A)$ and hence by lemma 3.8, $\theta_A(\mathcal{K}) \Vdash^- A^\dagger \leftrightarrow A$. We show $\theta_A(\mathcal{K}) \Vdash A^\dagger \leftrightarrow A$. If $\theta_A(\mathcal{K}) \nVdash^- A^\dagger$, then $\theta_A(\mathcal{K}) \nVdash^- A$ and hence $\theta_A(\mathcal{K}), w_0 \nVdash A^\dagger$ and $\theta_A(\mathcal{K}), w_0 \nVdash A$. Then $\theta_A(\mathcal{K}), w_0 \Vdash A^\dagger \leftrightarrow A$ and we are done. So assume that $\theta_A(\mathcal{K}) \Vdash^- A \wedge A^\dagger$. It is enough to show the following items:

- $\theta_A(\mathcal{K}), w_0 \Vdash A$ implies $\theta_A(\mathcal{K}), w_0 \Vdash A^\dagger$. By lemma 3.1 we have $\vdash A \to A^\dagger$ and hence we have desired result.

- $\theta_A(\mathcal{K}), w_0 \Vdash A^\dagger$ implies $\theta_A(\mathcal{K}), w_0 \Vdash A$. Let $\theta_A(\mathcal{K}), w_0 \Vdash A^\dagger$. Also assume that $\theta_A(\mathcal{K})$ is $\vec{p}$-model, i.e. $\theta_A(\mathcal{K}), w_0 \Vdash \vec{p}$ and $\theta_A(\mathcal{K}), w_0 \nVdash \bigvee(\mathsf{par} \setminus \vec{p})$. Since $\theta_A(\mathcal{K}), w_0 \Vdash A^\dagger$, for some $\mathcal{K}_1 \in \mathrm{Mod}(A)$ with $\mathcal{K}_1 \Vdash \vec{p}$ we have $\theta_A(\mathcal{K}) \Vdash [\![\mathcal{K}_1]\!]_{\mathsf{N}(\mathsf{par})}$. Theorem 3.7 implies that $\theta_A(\mathcal{K}) \preceq^{\mathsf{par}} \mathcal{K}_1$. Then lemma 3.3 implies $\theta_A(\mathcal{K}) \preceq_1^{\mathsf{par}} \mathcal{K}_1$ and thus by lemma 3.4 there is some $\mathcal{K}_2 \Vdash A$ such that $\theta_A(\mathcal{K}) \subseteq^{\mathsf{par}} \mathcal{K}_2$. Since A is par-extendible, there is a par-variant $\mathcal{K}'$ of $\theta_A(\mathcal{K})$ such that $\mathcal{K}' \Vdash A$. Thus lemma 3.14 implies $\theta_A(\mathcal{K}) \Vdash A$. ❑

Corollary 3.13. $\mathsf{N}(\mathsf{par})$-*projectivity is decidable. In other words, given $A \in \mathcal{L}_0$, one may algorithmically decide $A \in \downarrow\mathsf{N}(\mathsf{par})$.*

Proof. Given A, by theorem 3.12 it is enough to decide $\mathsf{IPC} \vdash \theta_A(A) \leftrightarrow A^\dagger$, which is decidable since IPC is decidable. ❑

Lemma 3.14. *If $\theta_A(\mathcal{K}) \Vdash^- A$ and A is valid in a par-variant of $\theta_A(\mathcal{K})$ then $\theta_A(\mathcal{K}) \Vdash A$.*

Proof. See proof of the theorem 5 in [Ghilardi, 1999]. ❑

3.4 Projective resolution

The main result in [Ghilardi, 1999] is that the unification type of IPC is finitary. It means that for every $A \in \mathcal{L}_0(\mathsf{var})$, there exists a finite *complete set of unifiers* for A, i.e. a finite set Θ of unifiers for A such that every unifier of A is less general than some $\theta \in \Theta$. We say that θ is less general than γ if there is some substitution λ such that for every $x \in \mathsf{var}$ we have

$$\vdash \theta(x) \leftrightarrow \lambda(\gamma(x)).$$

The proof of above mentioned fact is based on projective approximations which later [Ghilardi, 2002] provides a resolution/tableaux method for its computation. The aim for this subsection is to prove a relativised version of projective approximations in theorem 3.27.

Definition 3.15. *Given $\Gamma, \Pi \subseteq \mathcal{L}_0$ and $A \in \mathcal{L}_0$, we say that Π is Γ-projective resolution for A if*

- *Π is a set of independent propositions, i.e. for $B, C \in \Pi$, $\vdash B \to C$ implies $B = C$.*

- *Every $B \in \Pi$ is Γ-projective.*

- *$A \vDash_{\Gamma} \bigvee \Pi$.*

- *$\vdash \bigvee \Pi \to A$.*

A $\{\top\}$-projective resolution is also called projective resolution.

Note that $\emptyset$ is a projective resolution for a proposition which is not unifiable. The greatest lower bound (glb) for a proposition A is defined in section 2.7. Intuitively a glb for A w.r.t. (Γ, T) is the best Γ-approximation from below inside the logic T.

Remark 3.16. *If Π is Γ-projective resolution of A then $\bigvee \Pi$ is a glb for A w.r.t. $(\downarrow\Gamma^\vee, \mathsf{IPC})$.*

Proof. By theorem 2.2 we have $A \vDash_{\downarrow\Gamma}^{\mathsf{T}} \bigvee \Pi$ and hence by theorem 2.5 we have $A \vDash_{\downarrow\Gamma^\vee}^{\mathsf{T}} \bigvee \Pi$. Thus theorem 2.6 implies desired result. ❏

Theorem 3.17. *Whenever $\mathsf{par} = \emptyset$, every $A \in \mathcal{L}_0$ has projective resolution.*

Proof. See [Ghilardi, 1999, theorem 5]. We will also prove a generalization of this result in theorem 3.27. ❏

First some preliminary definitions. We refer the reader to [Ghilardi, 1999] for more information on these notions.

Let $\mathcal{K} = (W, \preccurlyeq, V)$ and $\mathcal{K}' = (W', \preccurlyeq', V')$ are two Kripke models with the roots w_0 and w_0'. Also let $\mathcal{K}(w) := \{a \in \mathsf{atom} : \mathcal{K}, w \Vdash a\}$ and $\mathcal{L}_0(\mathcal{K})$ be defined as $\mathcal{L}_0(\bigcup_{w \in W} \mathcal{K}(w))$. We say that $\mathcal{K}$ is with finite valuations if for every $w \in W$ we have $\mathcal{K}(w)$ is finite. Also define:

$$\begin{aligned}
\mathcal{K} \sim_0 \mathcal{K}' &\quad \text{iff} \quad \mathcal{K}(w_0) = \mathcal{K}'(w_0') \\
\mathcal{K} \sim_{n+1} \mathcal{K}' &\quad \text{iff} \quad \forall w \in W \, \exists w' \in W'(\mathcal{K}_w \sim_n \mathcal{K}'_{w'}) \text{ and vice versa} \\
\mathcal{K} \leq_0 \mathcal{K}' &\quad \text{iff} \quad \mathcal{K}(w_0) \supseteq \mathcal{K}'(w_0') \\
\mathcal{K} \leq_{n+1} \mathcal{K}' &\quad \text{iff} \quad \forall w \in W \, \exists w' \in W'(\mathcal{K}_w \sim_n \mathcal{K}'_{w'})
\end{aligned}$$

Evidently $\sim_n$ is an equivalence relation and $\leq_n$ is reflexive transitive. One may easily observe by induction on n that $\mathcal{K} \sim_{n+1} \mathcal{K}'$ implies $\mathcal{K} \sim_n \mathcal{K}'$. Hence $\mathcal{K} \sim_n \mathcal{K}'$ ($\mathcal{K} \leq_n \mathcal{K}'$) implies $\mathcal{K} \sim_m \mathcal{K}'$ ($\mathcal{K} \leq_m \mathcal{K}'$) for every $m \leq n$.

Let $c_\to(A)$ indicate the maximum number of nested implications in A:

- $c_\to(a) = c_\to(\top) = c_\to(\bot) = 0$ for atomic a.

- $c_\to(A \circ B) := \max\{c_\to(A), c_\to(B)\}$, for $\circ \in \{\vee, \wedge\}$.

- $c_\to(A \to B) := 1 + \max\{c_\to(A), c_\to(B)\}$.

Remember that by default we assume the set atom to be a finite set.

Remark 3.18. *Modulo IPC-provable equivalence relation, there are finitely many propositions $A \in \mathcal{L}_0$ with $c_\to(A) \leq n$.*

Proof. By induction on n, we define an upper bound $f(n)$ for the number of propositions $A \in \mathcal{L}_0$ with $c_\to(A) \leq n$.

1. $f(0)$: Observe that any A with $c_\to(A) = 0$ is IPC-equivalent to a disjunction of conjunctions of atomics. Hence $f(0) = 2^{2^m}$ is an obvious upper bound, in which m is the number of atomics in atom.

2. $f(n+1)$: For every implication $B \to C$ with $c_{\to}(B \to C) \leq n+1$, we have $c_{\to}(B), c_{\to}(C) \leq n$, and hence $f(n)^2$ is an upper bound for the number of inequivalent such propositions. Then since modulo IPC-provable equivalence every proposition is a disjunction of conjunctions of atomics or implications, the following definition is an upperbound:

$$f(n+1) := 2^{2^{[m+f(n)^2]}}. \qquad \qquad \square$$

Lemma 3.19. *Every $A \in$ NNIL has an IPC-provable equivalent $A' \in$ NNIL with $c_{\to}(A') \leq \#$atom.*

Proof. Observe that every $A \in$ NNIL has an IPC-equivalent $B \in$ NNIL such that $B = \bigwedge_i \bigvee_j C_i^j$ and every implication in C_i^j is of the form $a \to E$, with $a \in$ atom and E does not contain a. Then one may easily prove the statement of this lemma by induction on the number of elements in atom. $\qquad \square$

Lemma 3.20. *For every Kripke model $\mathcal{K}$, there exists a proposition $[\![\mathcal{K}]\!]_n \in \mathcal{L}_0$ with the following properties:*

- $\mathcal{K}' \Vdash [\![\mathcal{K}]\!]_n$ *iff* $\mathcal{K}' \leq_n \mathcal{K}$.

- $c_{\to}([\![\mathcal{K}]\!]_n) \leq n$.

Proof. We only give the definition of A here, and refer the reader to [Ghilardi, 1999, proposition 1] for its proof.
Let $\mathcal{K} = (W, \preccurlyeq, V)$ and define $[\![\mathcal{K}]\!]_0 := \bigwedge \mathcal{K}(w_0)$ and

$$[\![\mathcal{K}]\!]_{n+1} := \bigwedge_{\{\mathcal{K}' : \forall w \in W(\mathcal{K}' \nsim_n \mathcal{K}_w)\}} \left([\![\mathcal{K}']\!]_n \to \bigvee_{\{\mathcal{K}'' : \mathcal{K}' \nleq_n \mathcal{K}''\}} [\![\mathcal{K}'']\!]_n \right). \qquad \square$$

Corollary 3.21. *For every Kripke models $\mathcal{K}$ and $\mathcal{K}'$, we have $\mathcal{K}' \leq_n \mathcal{K}$ iff for every $A \in \mathcal{L}_0$ with $c_{\to}(A) \leq n$ we have $\mathcal{K} \Vdash A$ implies $\mathcal{K}' \Vdash A$.*

Proof. For the left to right direction, use induction on n. For the right to left, if $\mathcal{K}$ is with finite valuations, one may easily use lemma 3.20 and have desired result. Otherwise One may restrict $\mathcal{K}$ and $\mathcal{K}'$ to arbitrary finite atomics and then apply previous argument. $\qquad \square$

Corollary 3.22. $\mathcal{K}' \sim_n \mathcal{K}$ *iff for every A with $c_{\to}(A) \leq n$ we have*

$$\mathcal{K} \Vdash A \text{ iff } \mathcal{K}' \Vdash A.$$

Proof. First observe that $\mathcal{K} \sim_n \mathcal{K}'$ is equivalent to $\mathcal{K} \leq_n \mathcal{K}' \leq_n \mathcal{K}$ and then use corollary 3.21. $\qquad \square$

Lemma 3.23. *A class $\mathscr{K}$ of Kripke models is of the form $\mathrm{Mod}(A)$ with $c_{\to}(A) \leq n$, iff $\mathscr{K}$ is $\leq_n$-downward closed, i.e. for every Kripke model $\mathcal{K}'$ with $\mathcal{K}' \leq_n \mathcal{K} \in \mathscr{K}$ we have $\mathcal{K}' \in \mathscr{K}$.*

Proof. For the left-to-right direction, let $\mathcal{K}' \leq_n \mathcal{K} \in \mathrm{Mod}(A)$ for some A with $c_{\to}(A) \leq n$. Since $\mathcal{K} \Vdash A$, corollary 3.21 implies $\mathcal{K}' \Vdash A$ and hence $\mathcal{K}' \in \mathrm{Mod}(A)$.
For the other direction, let $\mathscr{K}$ is $\leq_n$-downward closed and define

$$A := \bigvee_{\mathcal{K} \in \mathscr{K}} [\![\mathcal{K}]\!]_n.$$

By remark 3.18, the disjunction is finite and hence A is indeed a proposition. One may easily observe that lemma 3.20 implies that $c_{\to}(A) \leq n$ and $\mathscr{K} = \mathrm{Mod}(A)$. $\qquad \square$

Lemma 3.24. *If a class of Kripke models $\mathscr{K}$ is par-extendible and θ is a substitution, then $\theta(\mathscr{K})$ is also par-extendible.*

Proof. Easy and left to the reader. ❏

We say that a class $\mathscr{K}$ of Kripke models is stable, if for every $\mathcal{K} \in \mathscr{K}$ and every node w in $\mathcal{K}$ we have $\mathcal{K}_w \in \mathscr{K}$.

Remark 3.25. *A class $\mathscr{K}$ of Kripke models is* par*-extendible iff for every finite stable class of models $\mathscr{K}'$ which is* par*-submodel of some $\mathcal{K} \in \mathscr{K}$, a* par*-variant of $\sum(\mathscr{K}', \mathcal{K})$ belongs to $\mathscr{K}$.*

Proof. Easy and left to the reader. ❏

Define $\langle \mathscr{K} \rangle_n := \{\mathcal{K} : \exists \mathcal{K}' \in \mathscr{K} \ (\mathcal{K} \leq_n \mathcal{K}')$ and $\mathcal{K}$ is a Kripke model$\}$.

Lemma 3.26. *If $\mathscr{K}$ is* par*-extendible stable class of Kripke models, then so is $\langle \mathscr{K} \rangle_n$, for every $n > \#$par. ($\#$par indicates the number of elements in* par*.)*

Proof. We only prove here the par-extendibility of $\langle \mathscr{K} \rangle_n$ and leave other properties to the reader. Let $\mathscr{F}' = \{\mathcal{K}'_i\}_i$ is a finite set of models in $\langle \mathscr{K} \rangle_n$, which are par-submodels of some $\mathcal{K}' \in \langle \mathscr{K} \rangle_n$. By remark 3.25 we may also assume that $\mathscr{F}'$ is stable. We must show that a par-variant of $\sum(\mathscr{F}', \mathcal{K}')$ belongs to $\langle \mathscr{K} \rangle_n$. Since $\mathcal{K}' \in \langle \mathscr{K} \rangle_n$ and $\mathscr{K}$ is stable, there is some $\mathcal{K} \in \mathscr{K}$ such that $\mathcal{K}' \sim_{n-1} \mathcal{K}$. Similarly, since $\mathcal{K}'_i \in \langle \mathscr{K} \rangle_n$, there is some $\mathcal{K}_i \in \mathscr{K}$ such that $\mathcal{K}'_i \sim_{n-1} \mathcal{K}_i$. Let $\mathscr{F} := \{\mathcal{K}_i\}_i$.
First we show that $\mathscr{F}$ is a par-submodel of $\mathcal{K}$. Since $\mathscr{F}'$ is a par-submodel of $\mathcal{K}'$, by theorem 3.7 we have $\mathcal{K}'_i \Vdash [\![\mathcal{K}']\!]_{\mathsf{N(par)}}$. From $\mathcal{K}'_i \sim_{n-1} \mathcal{K}_i$, lemma 3.19 and corollary 3.22 we get $\mathcal{K}_i \Vdash [\![\mathcal{K}']\!]_{\mathsf{N(par)}}$. Also since $\mathcal{K}' \sim_{n-1} \mathcal{K}$, by lemma 3.19 and corollary 3.22 we have $[\![\mathcal{K}']\!]_{\mathsf{N(par)}} = [\![\mathcal{K}]\!]_{\mathsf{N(par)}}$. Hence $\mathcal{K}_i \Vdash [\![\mathcal{K}]\!]_{\mathsf{N(par)}}$, and by theorem 3.7 we have $\mathcal{K}_i$ is a par-submodel of $\mathcal{K}$. Hence $\mathscr{F}$ is a par-submodel of $\mathcal{K}$.
We go back to the main proof. Since $\mathscr{F}$ is par-submodel of $\mathcal{K}$, by extendibility of $\mathscr{K}$, there exist a par-variant $\hat{\mathcal{K}}$ of $\sum(\mathscr{F}, \mathcal{K})$ in $\mathscr{K}$. Let w_0 is the root of $\mathcal{K}$ which is also the root of $\hat{\mathcal{K}}$ and w'_0 is the root of $\mathcal{K}'$. Define the par-variant $\hat{\mathcal{K}}'$ of $\sum(\mathscr{F}', \mathcal{K}')$ for atomic $x \notin$ par as follows:

$$\hat{\mathcal{K}}', w'_0 \Vdash x \iff \hat{\mathcal{K}}, w_0 \Vdash x.$$

It is enough to show that $\hat{\mathcal{K}}' \in \langle \mathscr{K} \rangle_n$. For this aim it is enough to show $\hat{\mathcal{K}}' \leq_n \hat{\mathcal{K}}$. From definition of $\hat{\mathcal{K}}$ and $\hat{\mathcal{K}}'$, it is clear that it is enough to show $\hat{\mathcal{K}}' \sim_{n-1} \hat{\mathcal{K}}$. We use induction on $k \leq n-1$ and show $\hat{\mathcal{K}}' \sim_k \hat{\mathcal{K}}$.
If $k = 0$, we must show that for every atomic a we have

$$\hat{\mathcal{K}}', w'_0 \Vdash a \iff \hat{\mathcal{K}}, w_0 \Vdash a.$$

For atomic variables x, by definition of $\hat{\mathcal{K}}'$, we already have this. Also since $\hat{\mathcal{K}}$ is a par-variant of $\sum(\mathscr{F}, \mathcal{K})$, $\hat{\mathcal{K}}'$ is a par-variant of $\sum(\mathscr{F}', \mathcal{K}')$ and $\mathcal{K} \sim_0 \mathcal{K}'$, for every $p \in$ par we also have

$$\hat{\mathcal{K}}', w'_0 \Vdash p \iff \hat{\mathcal{K}}, w_0 \Vdash p.$$

Then let $0 < k < n$ and show $\hat{\mathcal{K}}' \sim_k \hat{\mathcal{K}}$. We have the following items to prove:

- For every node w' in $\hat{\mathcal{K}}'$, there is some w in $\hat{\mathcal{K}}$ such that $\hat{\mathcal{K}}'_{w'} \sim_{k-1} \hat{\mathcal{K}}_w$. If w' is the root of $\hat{\mathcal{K}}'$, take w also the root of $\hat{\mathcal{K}}$ and we have desired result by induction hypothesis. If w' is not the root of $\hat{\mathcal{K}}'$, since $\mathscr{F}'$ is stable, we may let w' as a root w'_i of some $\mathcal{K}'_i$. Take $w = w_i$. Then by definition of $\mathcal{K}_i$, we have

$$\hat{\mathcal{K}}'_{w'} = \mathcal{K}'_i \sim_{n-1} \mathcal{K}_i = \hat{\mathcal{K}}_w$$

  Since $k - 1 \leq n - 1$, we have the desired result.

- For every node w in $\hat{\mathcal{K}}$, there is some w' in $\hat{\mathcal{K}}'$ such that $\hat{\mathcal{K}}'_{w'} \sim_{k-1} \hat{\mathcal{K}}_w$. Again if w is the root, take w' also the root and we are done by induction hypothesis. If w is not the root, there is some i such that w is a node of $\mathcal{K}_i$. Since $\mathcal{K}_i \sim_{n-1} \mathcal{K}'_i$, there is some w' in $\mathcal{K}'_i$ such that $(\mathcal{K}'_i)_{w'} \sim_{n-2} (\mathcal{K}_i)_w$. Since $k - 1 \leq n - 2$, we have

$$\hat{\mathcal{K}}'_{w'} = (\mathcal{K}'_i)_{w'} \sim_{k-1} (\mathcal{K}_i)_w = \hat{\mathcal{K}}_w,$$

  as desired. ❏

Theorem 3.27. *Every $A \in \mathcal{L}_0$ has* NNIL(par)*-projective resolution* Π. *Moreover for every $B \in \Pi$ we have $c_{\rightarrow}(B) \leq \max\{c_{\rightarrow}(A), 1 + \#\mathsf{par}\}$ and Π is a computable function of A.*

Proof. Given a substitution θ and $A' \in$ NNIL(par) such that $\vdash A' \rightarrow \theta(A)$, we will find some $B_\theta^{A'} \in \mathcal{L}_0$, with the following properties:

1. $B_\theta^{A'}$ is NNIL(par)-projective.

2. $\vdash A' \leftrightarrow \theta(B_\theta^{A'})$.

3. $\vdash B_\theta^{A'} \rightarrow A$.

4. $c_{\rightarrow}(B_\theta^{A'}) \leq n$ for $n := \max\{c_{\rightarrow}(A), 1 + \#\mathsf{par}\}$ (#par indicates the number of atomics in par).

Then by items 1-3 (an independent subset of) the following set is a NNIL(par)-projective resolution for A:
$$\Pi := \{B_\theta^{A'} : A' \in \mathsf{NNIL(par)} \text{ and } \theta \text{ a substitution such that } \vdash A' \rightarrow \theta(A)\}.$$

Moreover remark 3.18 and item (4) implies that Π is finite, as desired. So it remains to find $B_\theta^{A'}$ with mentioned properties. Define

$$\mathscr{K} := \theta(\mathrm{Mod}(A')) := \{\theta(\mathcal{K}) : \mathcal{K} \in \mathrm{Mod}(A')\}$$

Since $\langle \mathscr{K} \rangle_n$ has downward $\leq_n$-closure condition, we may apply lemma 3.23 and find some proposition, e.g. $B_\theta^{A'}$, such that $c_{\rightarrow}(B_\theta^{A'}) \leq n$ (so item 4 is satisfied) and $\langle \mathscr{K} \rangle_n = \mathrm{Mod}(B_\theta^{A'})$. Since $A' \in$ NNIL(par), evidently it is NNIL(par)-projective. Hence by theorem 3.12, A' is par-extendible. Hence by lemma 3.24, $\mathscr{K}$ is par-extendible. Since $\mathscr{K}$ is stable and $n > \#\mathsf{par}$, Lemma 3.26 implies that $\langle \mathscr{K} \rangle_n$ is also par-extendible. Hence $B_\theta^{A'}$ is par-extendible and by theorem 3.12, $B_\theta^{A'}$ is NNIL(par)-projective. So item (1) is satisfied.

To show item 3 for $B_\theta^{A'}$, it is enough to show $\mathcal{K} \Vdash B_\theta^{A'} \rightarrow A$ for every finite rooted model $\mathcal{K}$. If $\mathcal{K} \Vdash B_\theta^{A'}$, we have $\mathcal{K} \in \langle \mathscr{K} \rangle_n$. Hence $\mathcal{K} \leq_n \mathcal{K}'$ for some $\mathcal{K}' \in \mathscr{K}$. Then $\mathcal{K}' = \theta(\mathcal{K}'')$ for some finite rooted $\mathcal{K}''$ such that $\mathcal{K}'' \Vdash A'$. Since $\vdash A' \rightarrow \theta(A)$, we have $\mathcal{K}'' \Vdash \theta(A)$, and by lemma 3.8 we get $\theta(\mathcal{K}'') \Vdash A$, whence $\mathcal{K}' \Vdash A$. Since $c_{\rightarrow}(A) \leq n$ and $\mathcal{K} \leq_n \mathcal{K}'$, corollary 3.21 implies that $\mathcal{K} \Vdash A$, as desired.

It remains to show that item 2 holds. It is enough to show $\mathcal{K} \Vdash A' \leftrightarrow \theta(B_\theta^{A'})$ for arbitrary finite rooted $\mathcal{K}$. If $\mathcal{K} \Vdash A'$, then $\theta(\mathcal{K}) \Vdash A'$ and hence $\theta(\mathcal{K}) \in \mathscr{K} \subseteq \langle \mathscr{K} \rangle_n = \mathrm{Mod}(B_\theta^{A'})$. Then $\theta(\mathcal{K}) \Vdash B_\theta^{A'}$ and hence $\mathcal{K} \Vdash \theta(B_\theta^{A'})$. For the other direction, let $\mathcal{K} \Vdash \theta(B_\theta^{A'})$. Hence $\theta(\mathcal{K}) \Vdash B_\theta^{A'}$ and then $\theta(\mathcal{K}) \in \mathrm{Mod}(B_\theta^{A'}) = \langle \mathscr{K} \rangle_n$. So there is some $\mathcal{K}' \in \mathscr{K}$ such that $\theta(\mathcal{K}) \leq_n \mathcal{K}'$. Since $\mathcal{K}' \in \mathscr{K}$, there is some $\mathcal{K}''$ such that $\mathcal{K}' = \theta(\mathcal{K}'')$ and $\mathcal{K}'' \Vdash A'$. Since $A' = \theta(A')$, we have $\mathcal{K}'' \Vdash \theta(A')$ and hence $\mathcal{K}' \Vdash A'$. By lemma 3.19 $c_{\rightarrow}(A') < n$ and the corollary 3.21 implies $\mathcal{K} \Vdash A'$.

Finally we provide an algorithm which computes Π. Given A, compute the finite set

$$\Pi' := \{B \in \mathcal{L}_0 : c_{\rightarrow}(B) \leq \max\{c_{\rightarrow}(A), 1 + \#\mathsf{par}\} \text{ and } \vdash B \rightarrow A \text{ and } B \in \downarrow\mathsf{N(par)}\}.$$

Note that Π' is computable since IPC is decidable and by corollary 3.13 we can decide $B \in \downarrow\mathsf{N(par)}$. Finally one may easily find $\Pi \subseteq \Pi'$ which includes pairwise IPC-independent propositions, as required for projective resolutions. ❏

3.5 Projective resolution for NNIL

In this subsection, we will see that projective resolution of a NNIL-proposition gets a more elegant form. We will use this form later for characterization of NNIL(par)-admissible rules of IPC, specifically for the validity of disjunction rule. By theorem 3.17 or equivalently theorem 3.27 with empty par, there is a finite projective resolution for every proposition A, i.e. a set $\{A_1, \ldots, A_n\}$, with the following properties:

- Every unifier of A, is also a unifier of some A_i, in other words $A \vdash\!\!\!\sim \bigvee A_i$.

- $\vdash \bigvee A_i \to A$.

- A_i is projective for every $i \leq n$.

We will prove here that if $A \in \mathsf{NNIL}$, the projective resolution can be chosen such that every A_i is NNIL and moreover $\vdash A \leftrightarrow \bigvee A_i$.

Given $A \in \mathsf{NNIL}$, we say that A is a T-component if $A = \bigwedge \Gamma \wedge \bigwedge \Delta$ with the following properties:

- Every $B \in \Gamma$ is atomic.

- Every $B \in \Delta$ is an implication $C \to D$ for some atomic C such that $\mathsf{T} \nvdash \bigwedge \Gamma \to C$.

Lemma 3.28. *Let T be a logic extending IPC. Every $A \in \mathsf{NNIL}$ can be decomposed to T-components, i.e. there is a finite set of T-components Γ_A such that $\mathsf{T} \vdash A \leftrightarrow \bigvee \Gamma_A$. Moreover, if $A \in \mathsf{NNIL}(\vec{a})$ then $\Gamma_A \subseteq \mathsf{NNIL}(\vec{a})$.*

Proof. We use induction on $\mathsf{sub}^{\mathsf{atom}}(A)$ (the set of atomic formulas in A) ordered by $\supset$ and find some finite set Γ_A of T-components with $\mathsf{sub}^{\mathsf{atom}}(\Gamma_A) \subseteq \mathsf{sub}^{\mathsf{atom}}(A)$ and $\mathsf{T} \vdash \bigvee \Gamma_A \leftrightarrow A$. As induction hypothesis assume that for every T and $B \in \mathsf{NNIL}$ with $\mathsf{sub}^{\mathsf{atom}}(B) \subset \mathsf{sub}^{\mathsf{atom}}(A)$ there is a finite set Γ_B of T-components such that $\mathsf{T} \vdash B \leftrightarrow \bigvee \Gamma$ and $\mathsf{sub}^{\mathsf{atom}}(\Gamma_B) \subseteq \mathsf{sub}^{\mathsf{atom}}(B)$. For the induction step, assume that $A \in \mathsf{NNIL}$ is given. Using derivation in IPC one may easily find finite sets Γ_i and Δ_i for $1 \leq i \leq n$ such that

- $\mathsf{IPC} \vdash A \leftrightarrow \bigvee_{i=1}^n A_i$, in which $A_i := \bigwedge \Gamma_i \wedge \bigwedge \Delta_i$.

- Δ_i includes only atomic propositions.

- Γ_i includes implications with atomic antecedents.

- $\mathsf{sub}^{\mathsf{atom}}(\Gamma_i \cup \Delta_i) \subseteq \mathsf{sub}^{\mathsf{atom}}(A)$.

It is enough to decompose every A_i to T-components. If $\mathsf{T} \nvdash \bigwedge \Delta_i \to E$ for every antecedent E of an implication in Γ_i, then A_i already is a T-component and we are done. Otherwise, there is some $E \to F \in \Gamma_i$ such that $\mathsf{T} \vdash \bigwedge \Delta_i \to E$. Then let $A_i' := A_i[E : \top]$, i.e. the replacement of every occurrences of E in A_i with $\top$. Also let $\mathsf{T}' := \mathsf{T} + E$. Hence $\mathsf{sub}^{\mathsf{atom}}(A_i') \subsetneq \mathsf{sub}^{\mathsf{atom}}(A)$ and by induction hypothesis we may decompose A_i' to T'-components:

$$\mathsf{T}, E \vdash A_i' \leftrightarrow \bigvee_j B_j$$

It is not difficult to observe that if B_j is a T'-component then $B_j' := E \wedge B_j$ is a T-component. Moreover $\mathsf{T} \vdash E \wedge A_i' \leftrightarrow \bigvee_j B_j'$ and since $\mathsf{IPC} \vdash (E \wedge A_i') \leftrightarrow (E \wedge A_i)$ and $\mathsf{T} \vdash A_i \to E$, we get

$$\mathsf{T} \vdash A_i \leftrightarrow \bigvee_j B_j'$$

Hence we have decomposed A_i to T-components B_j' with $\mathsf{sub}^{\mathsf{atom}}(B_j') \subseteq \mathsf{sub}^{\mathsf{atom}}(A)$, as desired. ❏

Lemma 3.29. *Every IPC-component is extendible.*

Proof. Let $B = \bigwedge B_i$ is an IPC-component and $\mathscr{K}$ be a finite class of finite rooted Kripke models for B. We must show that a variant of $\sum(\mathscr{K})$ is a model of B. Let w_0 be the root of $\sum(\mathscr{K})$ and define a variant $\mathcal{K}$ of $\sum(\mathscr{K})$ as follows. $\mathcal{K}, w_0 \Vdash a$ iff $a = B_i$ for some i. Then it is easy to observe that $\mathcal{K}, w_0 \Vdash B$. ❏

Corollary 3.30. *For $A \in$ NNIL there is a finite set Δ of projective and NNIL propositions with*

$$\vdash A \leftrightarrow \bigvee \Delta.$$

Proof. Use lemmas 3.28 and 3.29 and theorem 3.10. ❏

Lemma 3.31. *Every extendible A is IPC-prime, i.e. if $\vdash A \to (B \vee C)$, then either $\vdash A \to B$ or $\vdash A \to C$.*

Proof. We prove this by contraposition. Let $\nvdash A \to B$ and $\nvdash A \to C$. Then there are some Kripke models $\mathcal{K}_1$ and $\mathcal{K}_2$ such that $\mathcal{K}_1 \Vdash A$, $\mathcal{K}_1 \nVdash B$, $\mathcal{K}_2 \Vdash A$ and $\mathcal{K}_2 \nVdash B$. Since A is extendible, there is some variant $\mathcal{K}'$ of $\sum(\{\mathcal{K}_1, \mathcal{K}_2\})$ such that $\mathcal{K} \Vdash A$. Since $\mathcal{K}_1 \nVdash B$ we have $\mathcal{K} \nVdash B$. Similarly $\mathcal{K} \nVdash C$. Hence $\mathcal{K} \nVdash B \vee C$ and then $\mathcal{K} \nVdash A \to (B \vee C)$. ❏

Theorem 3.32. *Given $A \in \mathcal{L}_0$, the following are equivalent:*

1. *A is an IPC-component, (modulo IPC-provable equivalence relation)*

2. *A is extendible,*

3. *A is IPC-prime.*

Proof. $1 \Rightarrow 2$: lemma 3.29. $2 \Rightarrow 3$: lemma 3.31. $3 \Rightarrow 1$: Let A is IPC-prime. By lemma 3.28 it can be decomposed to IPC-components Γ_A. Thus $\vdash A \leftrightarrow \bigvee \Gamma_A$ and by IPC-primality of A we have $\vdash A \to B$ for some $B \in \Gamma_A$. Then $\vdash A \leftrightarrow B$ and hence A is IPC-equivalent to some IPC-component. ❏

Remember that PNNIL(par) indicates the set of IPC-prime and NNIL(par)-propositions.

Corollary 3.33. *Up to IPC-provable equivalence relation, we have $\mathsf{NNIL} = \mathsf{PNNIL}^{\vee}$ and $\mathsf{NNIL(par)} = \mathsf{PNNIL(par)}^{\vee}$.*

Proof. By lemma 3.28 every $A \in$ NNIL can be decomposed to IPC-components Γ_A such that $A \in$ NNIL(par) implies $\Gamma_A \subseteq$ NNIL(par). Then theorem 3.32 implies that every $E \in \Gamma_A$ is IPC-prime. Hence $\bigvee \Gamma_A \in \mathsf{PNNIL}^{\vee}$ and moreover $A \in$ NNIL(par) implies $\bigvee \Gamma_A \in \mathsf{PNNIL(par)}^{\vee}$. ❏

Corollary 3.34. $\vdash^{\mathsf{IPC}}_{\mathsf{PN(par)}} = \vdash^{\mathsf{IPC}}_{\mathsf{N(par)}}$.

Proof. Corollary 3.33 and theorem 2.5. ❏

A consequence of the results in this subsection is that now we have uniqueness of the projective resolutions:

Theorem 3.35 (Projective Resolution). *Every $A \in \mathcal{L}_0$ has a PNNIL(par)-projective resolution. Moreover this resolution is computable and unique up to IPC-provable equivalency, i.e. for every two PNNIL(par)-projective resolutions $\Delta = \{B_1, \ldots, B_m\}$ and $\Delta' = \{C_1, \ldots, C_n\}$ for A, we have $m = n$ and there is some permutation σ such that for every i, $\vdash B_i \leftrightarrow C_{\sigma(i)}$.*

Proof. Given A, by theorem 3.27 there is a NNIL(par)-projective resolution Δ for A. Then define

$$\Pi_0 := \{E \wedge E' : E \in \Delta \text{ and } E' \in \Gamma_{E^{\dagger}}\}$$

in which $E^{\dagger} \in$ NNIL(par) is the NNIL(par)-projection of E and $\Gamma_{E^{\dagger}}$ is the decomposition of $E^{\dagger}$ to IPC-components, as provided by lemma 3.28. Finally let $\Pi \subseteq \Pi_0$ be some $\subseteq$-minimal set with $\vdash \bigvee \Pi_0 \leftrightarrow \bigvee \Pi$. Then by corollary 3.34 and the following fact one may easily observe that Π is a PNNIL(par)-projective resolution for A: if $A \xrightarrow{\theta} A' \in$ NNIL(par) and $E \in$ PNNIL(par), then $(A \wedge E) \xrightarrow{\theta} (A' \wedge E)$.

For the uniqueness, it is enough to show that for every PNNIL(par)-projective E, if $E \vdash_{\mathsf{PN(par)}} \bigvee_i F_i$ then for some i we have $\vdash E \to F_i$. Let $E \xrightarrow{\theta} E^{\dagger}$. Then by $E \vdash_{\mathsf{PN(par)}} \bigvee_i F_i$ we have $\vdash \theta(E^{\dagger} \to \bigvee_i F_i)$. Hence $\vdash E^{\dagger} \to \bigvee_i \theta(F_i)$ and since $E^{\dagger}$ is IPC-prime, we have $\vdash E^{\dagger} \to \theta(F_i)$ for some i. Thus lemma 3.2 implies $\vdash E \to F_i$, as desired. ❏

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4 NNIL(par)-admissible rules of IPC

In [Iemhoff, 2001b], the admissibility relation $\succ\!\!\sim$ is characterized by means of preservation relation $\vartriangleright$ and its Kripke semantics, called AR-models. In this section we will characterize and prove the decidability of $\succ\!\!\sim_{\mathsf{N(par)}}$, the NNIL(par)-*admissible rules of* IPC (see section 2.6). For this end, we imitate the route in [Iemhoff, 2001b], i.e. we define a system $\mathsf{AR_{par}}$ for the NNIL(par)-admissible rules of IPC and also introduce a Kripke semantic for it and prove the soundness and completeness. Finally using this and the results in section 3 we prove that $\mathsf{AR_{par}}$ is sound and complete for both NNIL(par)-admissibility and ↓NNIL(par)-preservativity, i.e. $\mathsf{AR_{par}} \vdash A \vartriangleright B$ iff $A \succ\!\!\sim_{\mathsf{N(par)}} B$ iff $A \approx_{\downarrow\mathsf{N(par)}} B$.

4.1 The system $\mathsf{AR_{par}}$

$\mathsf{AR_{par}}$ is a system which proves propositions in the form $A \vartriangleright B$, and $A, B \in \mathcal{L}_0$. Before we continue with the axioms and rules of the system $\mathsf{AR_{par}}$, let us first define a notation.

$$\{A\}_{\mathsf{par}}(B) := \begin{cases} B & : B \in \mathsf{par} \cup \{\bot\} \\ A \to B & : \text{otherwise} \end{cases}$$

Then $\mathsf{AR_{par}}$ is defined as [IPC] (as defined in section 2.6) plus the following axiom and rule:

$\mathsf{V_{AR}^{par}}: \quad B \to C \vartriangleright \bigvee_{i=1}^{n+m} \{B\}_{\mathsf{par}}(E_i)$, in which $B = \bigwedge_{i=1}^{n}(E_i \to F_i)$ and $C = \bigvee_{i=n+1}^{n+m} E_i$.

$$(p \in \mathsf{par}) \ \frac{A \vartriangleright B}{p \to A \vartriangleright p \to B} \ \text{Mont}(\mathsf{par})$$

Remark 4.1. *The system* AR, *as defined in [Iemhoff, 2001b], is* $\mathsf{AR_{par}}$ *with* $\mathsf{par} = \emptyset$. *The Visser rule* $\mathsf{V_{AR}^{par}}$ *in this case is proved to be of central importance [Iemhoff, 2005].*

Remark 4.2. *As we will see in corollary 4.16, the following extension of the Montagna's rule is admissible in* $\mathsf{AR_{par}}$*:*

$$(E \in \mathsf{NNIL(par)}) \ \frac{A \vartriangleright B}{E \to A \vartriangleright E \to B} \ .$$

Remark 4.3. $\mathsf{AR_{par}}$ *is closed under general substitutions* θ *with* $\theta(p) \in \{\top, \bot\} \cup \mathsf{par}$ *for every* $p \in \mathsf{par}$, *i.e.* $\mathsf{AR_{par}} \vdash A \vartriangleright B$ *implies* $\mathsf{AR_{par}} \vdash \theta(A) \vartriangleright \theta(B)$.

Proof. Use induction on the complexity of proof $\mathsf{AR_{par}} \vdash A \vartriangleright B$. All cases are easy and left to the reader. ❏

The following theorem is from [Iemhoff, 2001b].

Theorem 4.4. $A \succ\!\!\sim B$ *iff* $\mathsf{AR} \vdash A \vartriangleright B$.

Lemma 4.5. $\mathsf{AR_{par}} \vdash A \vartriangleright B$ *implies* $A \succ\!\!\sim_{\mathsf{N(par)}} B$.

Proof. We use induction on the complexity of the proof $\mathsf{AR_{par}} \vdash A \vartriangleright B$. All cases are easy except for the axiom $\mathsf{V_{AR}^{par}}$ and the rules Mont(par) and Disj.

- $\mathsf{V_{AR}^{par}}$: Let $C = \bigwedge_{i=1}^{n}(E_i \to F_i)$ and $D = \bigvee_{i=n+1}^{n+m} E_i$. We show $C \to D \succ\!\!\sim_{\mathsf{N(par)}} \bigvee_{i=1}^{n+m} \{C\}_{\mathsf{par}}(E_i)$. So assume that θ is a substitution and $G \in \mathsf{NNIL}(\vec{p})$ and show that $\vdash G \to \theta(C)$ implies $\vdash G \to \theta(\bigvee_{i=1}^{n+m} \{C\}_{\mathsf{par}}(E_i))$. We reason by contraposition. Let $\nvdash G \to \theta(\bigvee_{i=1}^{n+m} \{C\}_{\mathsf{par}}(E_i))$. Hence for every $i \leq n + m$ we have $\nvdash \theta(G \to \{C\}_{\mathsf{par}}(E_i))$. Then for every i there is some Kripke model $\mathcal{K}_i$ with the root w_i such that $\mathcal{K}_i \Vdash G$ and $\mathcal{K}_i \nVdash \theta(E_i)$ and moreover for every i with $E_i \notin \mathsf{par} \cup \{\bot\}$ we have $\mathcal{K}_i \Vdash C$. Since G is projective, by theorem 3.10 it is extendible. Let $\mathcal{K}$ be a variant of

$\sum(\{\mathcal{K}_i\}_i)$ such that $\mathcal{K} \Vdash G$ and w_0 be its root. Also define $\mathcal{K}'$ as follows: for every i such that $E_i \in \mathsf{par} \cup \{\bot\}$, eliminate $\mathcal{K}_i$ (we mean its nodes) from $\mathcal{K}$. Then evidently $\mathcal{K}'$ is a submodel of $\mathcal{K}$. Since G is NNIL, by theorem 3.5, we have $\mathcal{K}' \Vdash G$. It is enough to show that $\mathcal{K}' \Vdash \theta(C)$ and $\mathcal{K}' \nVdash \theta(D)$. Since for every node w in $\mathcal{K}'$ other than w_0, we have $\mathcal{K}', w \Vdash \theta(C)$, if we show $\mathcal{K}', w_0 \nVdash \theta(E_i)$ for every i, we have both $\mathcal{K}' \Vdash \theta(C)$ and $\mathcal{K}' \nVdash \theta(D)$ and we are done. We have two cases. (1) $E_i \in \mathsf{par} \cup \{\bot\}$. In this case we have $\theta(E_i) = E_i$, and since $\mathcal{K}, w_i \nVdash E_i$ we have $\mathcal{K}, w_0 \nVdash E_i$ and hence $\mathcal{K}', w_0 \nVdash E_i$. (2) $E_i \notin \mathsf{par} \cup \{\bot\}$. Since w_i in this case is a node of $\mathcal{K}'$ and $\mathcal{K}', w_i \nVdash \theta(E_i)$, we have $\mathcal{K}', w_0 \nVdash \theta(E_i)$.

- Mont(par): Let $A \mathrel{\vdash\mkern-9mu\sim}_{\mathsf{N(par)}} B$ and show $p \to A \mathrel{\vdash\mkern-9mu\sim}_{\mathsf{N(par)}} p \to B$ for every $p \in \mathsf{par}$. Let θ be a substitution and $E \in \mathsf{NNIL}(\vec{p})$ such that $\vdash E \to \theta(p \to A)$. Hence $\vdash (E \wedge p) \to \theta(A)$. Then by $A \mathrel{\vdash\mkern-9mu\sim}_{\mathsf{N(par)}} B$ we have $\vdash (E \wedge p) \to \theta(B)$ and hence $\vdash E \to \theta(p \to B)$, as desired.

- *Disj:* Let $B \mathrel{\vdash\mkern-9mu\sim}_{\mathsf{N(par)}} A$ and $C \mathrel{\vdash\mkern-9mu\sim}_{\mathsf{N(par)}} A$ and show $B \vee C \mathrel{\vdash\mkern-9mu\sim}_{\mathsf{N(par)}} A$. Corollary 3.34 and $B \mathrel{\vdash\mkern-9mu\sim}_{\mathsf{N(par)}} A$ and $C \mathrel{\vdash\mkern-9mu\sim}_{\mathsf{N(par)}} A$ imply $B \mathrel{\vdash\mkern-9mu\sim}_{\mathsf{PN(par)}} A$ and $C \mathrel{\vdash\mkern-9mu\sim}_{\mathsf{PN(par)}} A$. Let $E \in \mathsf{PNNIL}(\vec{p})$ and θ a substitution such that $\vdash E \to \theta(B \vee C)$. Since E is IPC-prime, either we have $\vdash E \to \theta(B)$ or $\vdash E \to \theta(C)$. In either of the cases, by $B \mathrel{\vdash\mkern-9mu\sim}_{\mathsf{PN(par)}} A$ and $C \mathrel{\vdash\mkern-9mu\sim}_{\mathsf{PN(par)}} A$ we have $\vdash E \to \theta(A)$. So by this argument we may conclude that $(B \vee C) \mathrel{\vdash\mkern-9mu\sim}_{\mathsf{PN(par)}} A$ and then by Corollary 3.34 we have $(B \vee C) \mathrel{\vdash\mkern-9mu\sim}_{\mathsf{N(par)}} A$. ❏

Corollary 4.6. $A \rhd B$ *implies* $A \mathrel{\vdash\mkern-9mu\sim}_{\mathsf{PN(par)}} B$.

Proof. Use lemma 4.5 and corollary 3.34. ❏

Corollary 4.7. *For every* $A \in \mathsf{NNIL}(\mathsf{par})$ *and* $B \in \mathcal{L}_0$, *if* $\mathsf{AR_{par}} \vdash A \rhd B$ *then* $\vdash A \to B$.

Proof. Let $\mathsf{AR_{par}} \vdash A \rhd B$. Then by lemma 4.5, we have $A \mathrel{\vdash\mkern-9mu\sim}_{\mathsf{N(par)}} B$. Let θ be identity substitution. Then we have $\vdash A \to \theta(A)$. Hence $\vdash A \to \theta(B)$, which implies $\vdash A \to B$, as desired. ❏

4.2 $\mathsf{AR_{par}}$-models

Before we define $\mathsf{AR_{par}}$-models, the Kripke models for which $\mathsf{AR_{par}}$ is sound and complete, let us present some definitions. Let $\mathcal{K} = (W, \preccurlyeq, V)$ is a Kripke model, possibly infinite or not tree. All over the rest of this subsection we assume that in general a Kripke model might be infinite or not tree. Given a set Γ of propositions, two nodes $v, w \in W$ are called Γ-*similar*, notation $v \equiv_\Gamma w$, if for every $A \in \Gamma$ we have $\mathcal{K}, v \Vdash A$ iff $\mathcal{K}, w \Vdash A$. Let $W' \subseteq W$ is a set of nodes and $w \in W$. The notation $w \preccurlyeq W'$, means $w \preccurlyeq w'$ for every $w' \in W'$. We say that $w \in W$ is a *tight predecessor* of W', if $w \preccurlyeq W'$ and for every $u \succcurlyeq w$, either $u = w$ or $u \succcurlyeq v$ for some $v \in W'$. A node w is called a *base*, if for every finite set $W' \subseteq W$ such that $w \preccurlyeq W'$, there is some $w' \in W$ such that: $w \preccurlyeq w' \preccurlyeq W'$ and $w \equiv_{\mathsf{par}} w'$ and w' is a tight predecessor of W'. And finally, a Kripke model $\mathcal{K} = (W, \preccurlyeq, V)$ is an $\mathsf{AR_{par}}$-model if it is rooted (let w_0 be its root) and there is some set $W_b \subseteq W$ with the following properties:

- $w_0 \in W_b$,

- every $w \in W_b$ is a base,

- for every $w' \in W_b$ and $w \succcurlyeq w'$, there is some $v \in W_b$ such that $v \equiv_{\mathsf{par}} w$ and $w' \preccurlyeq v \preccurlyeq w$.

Such W_b is called a *base-set* for $\mathcal{K}$.
We say that $\mathcal{K}$ is *good*, if for every finite set of nodes W', and every $X \subseteq \mathsf{par}$ such that $\mathcal{K}, W' \Vdash X$, there is some $w' \in W_b$ such that $w' \preccurlyeq W'$ and $\mathcal{K}(w') \cap \mathsf{par} = X$.

Remark 4.8. *Let* $\mathcal{K} = (W, \preccurlyeq, V)$ *is an* $\mathsf{AR_{par}}$-*model with a base-set* W_b, *and* $w \in W_b$. *Then* $\mathcal{K}_w$ *is also an* $\mathsf{AR_{par}}$-*model with the base-set* $W_b' := \{v \in W_b : v \succcurlyeq w\}$.

Theorem 4.9. (**Soundness**) $\mathsf{AR_{par}} \vdash A \rhd B$ *implies* $\mathcal{K} \Vdash B$, *for every* $\mathsf{AR_{par}}$-*model* $\mathcal{K}$ *with* $\mathcal{K} \Vdash A$.

Proof. We use induction on the proof $\mathsf{AR_{par}} \vdash A \rhd B$. All cases are trivial except for the axiom $\mathsf{V_{AR}^{par}}$ and the rule Mont. First we treat the Montagna's rule. As induction hypothesis, let $\mathcal{K} \Vdash A$ implies $\mathcal{K} \Vdash B$, for every $\mathsf{AR_{par}}$-model $\mathcal{K}$. Also let $\mathcal{K} \Vdash p \to A$ for some $p \in \mathsf{par}$ and $\mathsf{AR_{par}}$-model $\mathcal{K} = (W, \preccurlyeq, V)$ with the base-set W_b. We will show $\mathcal{K} \Vdash p \to B$. Let $w \in W$ such that $\mathcal{K}, w \Vdash p$. Since $\mathcal{K}$ is an $\mathsf{AR_{par}}$-model, there is some $w' \in W_b$ such that $w \equiv_{\mathsf{par}} w'$ and $w' \preccurlyeq w$. Then $\mathcal{K}, w' \Vdash p$ and hence $\mathcal{K}, w' \Vdash A$. Observe that $\mathcal{K}_{w'}$ is also an $\mathsf{AR_{par}}$-model and $\mathcal{K}_{w'} \Vdash A$. Hence by induction hypothesis $\mathcal{K}_{w'} \Vdash B$, which implies $\mathcal{K}, w \Vdash B$, as desired.

Next we show $\mathcal{K} \Vdash \mathsf{V_{AR}^{par}}$ for every $\mathsf{AR_{par}}$-model $\mathcal{K} = (W, \preccurlyeq, V)$ with the root w_0. Let $B = \bigwedge_{i=1}^{n}(E_i \to F_i)$ and $C = \bigvee_{i=n+1}^{n+m} E_i$. Also assume that $\mathcal{K}, w_0 \nVdash \bigvee_{i=1}^{n+m}\{B\}_{\mathsf{par}}(E_i)$. We show that $\mathcal{K}, w_0 \nVdash B \to C$. By definition of $\{B\}_{\mathsf{par}}(E_i)$, for every $E_i \in \mathsf{par} \cup \{\bot\}$, we have $\mathcal{K}, w_0 \nVdash E_i$, and for every $E_i \notin \mathsf{par} \cup \{\bot\}$, there is some $w_i \succcurlyeq w_0$ such that $\mathcal{K}, w_i \Vdash B$ and $\mathcal{K}, w_i \nVdash E_i$. Let $W' := \{w_i : E_i \notin \mathsf{par} \cup \{\bot\}\}$. There is some $w' \in W_b$ which is a tight predecessor of W' and $w' \equiv_{\mathsf{par}} w_0$. We show that $\mathcal{K}, w' \nVdash B \to C$ by showing $\mathcal{K}, w' \Vdash B$ and $\mathcal{K}, w' \nVdash C$. Let E_i be some disjunct in C. If $E_i \in \mathsf{par} \cup \{\bot\}$, then since $w' \equiv_{\mathsf{par}} w_0$ and $\mathcal{K}, w_0 \nVdash E_i$, we have $\mathcal{K}, w' \nVdash E_i$. Otherwise, since $\mathcal{K}, w_i \nVdash E_i$ and $w' \preccurlyeq w_i$, we have $\mathcal{K}, w' \nVdash E_i$. This finishes showing $\mathcal{K}, w' \nVdash C$. Then we show $\mathcal{K}, w' \Vdash B$. Let $E_i \to F_i$ be a conjunct in B. Consider some $w \succcurlyeq w'$ such that $\mathcal{K}, w \Vdash E_i$. Since w' is a tight predecessor of W', either we have $w = w'$ or $w \succcurlyeq w_j$ for some $w_j \in W'$. If $w \succcurlyeq w_j$, since $\mathcal{K}, w_j \Vdash B$, we have $\mathcal{K}, w \Vdash B$ and then $\mathcal{K}, w \Vdash E_i \to F_i$, whence $\mathcal{K}, w \Vdash F_i$. Also if $w = w'$, then by the following argument, we have $\mathcal{K}, w' \nVdash E_i$, a contradiction with our first assumption $\mathcal{K}, w \Vdash E_i$. Finally, the argument for $\mathcal{K}, w' \nVdash E_i$: if $E_i \in \mathsf{par} \cup \{\bot\}$, then since $w' \equiv_{\mathsf{par}} w_0$ and $\mathcal{K}, w_0 \nVdash E_i$, we have $\mathcal{K}, w' \nVdash E_i$. Otherwise, since $\mathcal{K}, w_i \nVdash E_i$ and $w' \preccurlyeq w_i$, we have $\mathcal{K}, w' \nVdash E_i$. ❏

We follow the methods in [Iemhoff, 2001b] to prove the completeness theorem. This proof is almost identical to the one for [Iemhoff, 2001b, proposition 7.2.2]. First some definitions and lemmas. A set w of propositions is IPC-saturated if

- $w \vdash A$ implies $A \in w$,

- $\bot \notin w$,

- $A \lor B \in w$ implies either $A \in w$ or $B \in w$.

Also w is called $\mathsf{AR_{par}}$-saturated if it is IPC-saturated and

- If $\mathsf{AR_{par}} \vdash A \rhd B$ and $A \in w$, then $B \in w$.

Let $*(.)$ is a property on sets of propositions. We say that $*(.)$ is an *extendible* property if the following conditions hold:

- If $*(w)$ and $w \vdash A$, then $*(w \cup \{A\})$.

- If $*(w \cup \{A \lor B\})$ then either $*(w \cup \{A\})$ or $*(w \cup \{B\})$ hold.

If also the following condition holds, we say that $*(.)$ is $\mathsf{AR_{par}}$-extendible property.

- If $*(w)$ and $\mathsf{AR_{par}} \vdash w \rhd A$, then $*(w \cup \{A\})$.

In the above expression, $\mathsf{AR_{par}} \vdash w \rhd A$ is a shorthand for $\mathsf{AR_{par}} \vdash (\bigwedge_i B_i) \rhd A$ for some finite set $\{B_i\}_i \subseteq w$.

Lemma 4.10. *For every extendible property $*(.)$, if $*(w)$ for some set w of propositions holds, there is some maximal IPC-saturated $w' \supseteq w$ such that $*(w')$. Moreover if $*(.)$ is $\mathsf{AR_{par}}$-extendible, then w' is also $\mathsf{AR_{par}}$-saturated.*

Proof. Let $A_1, A_2, \ldots$ be a list of all propositions such that each proposition occurs infinitely often. We define a sequence $w = w_0 \subseteq w_1 \subseteq w_2 \subseteq \ldots$ and then define $w' := \bigcup_i w_i$.

$$w_{n+1} := \begin{cases} w_n \cup \{A_n\} & : *(w_n \cup \{A_n\}) \\ w_n & : \text{otherwise} \end{cases}$$

It can be easily proved that this w' satisfies all required conditions. $\square$

Corollary 4.11. *If* $\mathsf{AR_{par}} \nvdash A \rhd B$, *then there is some* $\mathsf{AR_{par}}$*-saturated* w *such that* $A \in w$ *and* $B \notin w$.

Proof. Define the property $*(.)$ as follows:

$$*(y) : \mathsf{AR_{par}} \nvdash y \rhd B$$

Then it is straightforward to observe that $*(.)$ is $\mathsf{AR_{par}}$-extendible and $*(\{A\})$ holds. Hence lemma 4.10 implies the desired result. $\square$

Theorem 4.12. (**Completeness**) $\mathsf{AR_{par}}$ *is complete for good* $\mathsf{AR_{par}}$*-models, i.e. if for every good* $\mathsf{AR_{par}}$*-model* $\mathcal{K}$, *we have* $\mathcal{K} \Vdash A$ *implies* $\mathcal{K} \Vdash B$, *then* $\mathsf{AR_{par}} \vdash A \rhd B$.

Proof. As usual, we reason corapositively. Let $\mathsf{AR_{par}} \nvdash A \rhd B$. Define the Kripke model $\mathcal{K} = (W, \preccurlyeq, V)$ as follows. Since $\mathsf{AR_{par}} \nvdash A \rhd B$, by corollary 4.11 there is some $\mathsf{AR_{par}}$-saturated set w_0 such that $A \in w_0$ and $B \notin w_0$. Then define

$$W := \{w \supseteq w_0 : w \text{ is a } \mathsf{IPC}\text{-saturated set of propositions}\}$$

Also define $u \preccurlyeq v$ iff $u \subseteq v$. Finally define $w \, V \, a$ iff $a \in w$ for atomic a. We will show that this model is a good $\mathsf{AR_{par}}$-model such that $\mathcal{K} \Vdash A$ and $\mathcal{K} \nVdash B$. First note that by a standard argument, one may easily prove by induction on the complexity of A that $A \in w$ iff $\mathcal{K}, w \Vdash A$. Then since $A \in w_0$ and $B \notin w_0$, we have $\mathcal{K} \Vdash A$ and $\mathcal{K} \nVdash B$. So it remains to show that $\mathcal{K}$ is a good $\mathsf{AR_{par}}$-model. Let W_b as follows:

$$W_b := \{w \in W : w \text{ is } \mathsf{AR_{par}}\text{-saturated}\}$$

We will show that W_b is a base-set, i.e. has the following properties:

- $w_0 \in W_b$,

- every $w \in W_b$ is a base,

- for every $w' \in W_b$ and $w \succcurlyeq w'$, there is some $v \in W_b$ such that $v \equiv_{\mathsf{par}} w$ and $w' \preccurlyeq v \preccurlyeq w$.

The first property is obvious. For the second property we will need $\mathsf{V_{AR}^{par}}$ and for the third one we will use Mont's rule.

Let $w \in W_b$ and $w \preccurlyeq \{w_1, \ldots, w_n\}$. We find some tight predecessor u such that $u \equiv_{\mathsf{par}} w$ and $w \preccurlyeq u \preccurlyeq \{w_1, \ldots, w_n\}$. Let $\hat{w} := \bigcap_i w_i$ and define

$$\Delta := \{E \to F : E \to F \in \hat{w} \text{ and } (E \notin \hat{w} \lor E \in \mathsf{par} \setminus w)\}$$

Define the property $*(.)$ as follows:

$$*(y) : y \vdash \bigvee_i A_i \lor \bigvee_i p_i \text{ and } \forall i \, (p_i \in \mathsf{par}) \text{ implies } \exists i \, (A_i \in \hat{w}) \lor \exists i \, (p_i \in w).$$

Note that by letting the second disjunction as empty, from $*(y)$ we have $y \vdash \bigvee_i A_i$ implies $\exists i \, (A_i \in \hat{w})$. Similarly and by considering the first disjunction as empty disjunction, from $*(y)$ we get $y \vdash \bigvee_i p_i$ implies $\exists i \, (p_i \in w)$. It is not difficult to observe that $*(.)$ is an extendible property. Then we show

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$*(w \cup \Delta)$. Let $w \cup \Delta \vdash \bigvee_i C_i \vee \bigvee_i p_i$ and $p_i \in \mathsf{par}$. Then $w \vdash G \to (\bigvee_i C_i \vee \bigvee_i p_i)$ in which $G = \bigwedge_i (E_i \to F_i)$ and $E_i \to F_i \in \Delta$. Since $w \in W_b$, and

$$\mathsf{AR_{par}} \vdash \left(G \to (\bigvee_i C_i \vee \bigvee_i p_i) \right) \triangleright \left(\bigvee_i \{G\}_{\mathsf{par}}(E_i) \vee \bigvee_i \{G\}_{\mathsf{par}}(C_i) \vee \bigvee_i \{G\}_{\mathsf{par}}(p_i) \right),$$

we have $w \vdash \bigvee_i \{G\}_{\mathsf{par}}(E_i) \vee \bigvee_i \{G\}_{\mathsf{par}}(C_i) \vee \bigvee_i \{G\}_{\mathsf{par}}(p_i)$. Since w is IPC-saturated, either $w \vdash \{G\}_{\mathsf{par}}(E_i)$ or $w \vdash \{G\}_{\mathsf{par}}(C_i)$ or $w \vdash \{G\}_{\mathsf{par}}(p_i)$, for some i. If $w \vdash \{G\}_{\mathsf{par}}(E_i)$, then $w, \Delta \vdash E_i$ and since $w \cup \Delta \subseteq \hat{w}$, we have $E_i \in \hat{w}$, a contradiction. So either we have $w \vdash \{G\}_{\mathsf{par}}(C_i)$ or $w \vdash \{G\}_{\mathsf{par}}(D)$, in which we have $w, \Delta \vdash C_i$ or $w \vdash p_i$. Hence either $C_i \in \hat{w}$ or $p_i \in w$. This finishes showing $*(w \cup \Delta)$. Now let $u \supseteq (w \cup \Delta)$ be a maximal IPC-saturated set such that $*(u)$, as provided by lemma 4.10. Then we show that u satisfies all required conditions:

- $w \preccurlyeq u \preccurlyeq \{w_1, \ldots, w_n\}$. Since $w \subseteq u$, we have $w \preccurlyeq u$. Also from $*(u)$, we get $u \subseteq \hat{w}$ and hence for every i we have $u \preccurlyeq w_i$.

- $w \equiv_{\mathsf{par}} u$. Since $w \subseteq u$, we have $w \cap \mathsf{par} \subseteq u \cap \mathsf{par}$. For the other direction, let $p \in \mathsf{par} \cap u$. Then $u \vdash p$ and from $*(u)$ we have $p \in w$.

- u is a tight predecessor of $\{w_1, \ldots, w_n\}$. We reason by contraposition. Let $v \supsetneqq u$ such that for every i, $w_i \nsubseteq v$. Then for every i there is some $C_i \in w_i \setminus v$ and hence $\bigvee C_i \in \hat{w} \setminus v$. On the other hand, since u is a maximal saturated set with $*(u)$ and $v \supsetneqq u$ and v is IPC-saturated, we have $\neg * (v)$. Hence $v \vdash \bigvee_i A_i \vee \bigvee_i p_i$ and for every i we have $p_i \in \mathsf{par}$ and $A_i \notin \hat{w}$ and $p_i \notin w$. From $v \vdash \bigvee_i A_i \vee \bigvee_i p_i$, there is some E such that either we have $E \in v \setminus \hat{w}$ or $E \in \mathsf{par} \setminus w$. In either of the cases, by definition of Δ we have $E \to \bigvee C_i \in \Delta$. Hence $E \to \bigvee C_i \in v$ and then $\bigvee C_i \in v$, a contradiction.

It finishes showing the second property of base-set W_b. Next we show that W_b satisfies the third condition. Let $w' \in W_b$ and $w' \preccurlyeq w$. Define the property $*(.)$ as follows.

$$*(y): \text{ for every } C, \text{ if } \mathsf{AR_{par}} \vdash y \triangleright C, \text{ then } C \in w.$$

We show that $*(.)$ is an $\mathsf{AR_{par}}$-extendible property and $*(w' \cup w_{\mathsf{par}})$, in which $w_{\mathsf{par}} := w \cap \mathsf{par}$. First let us show why this finishes the proof. From lemma 4.10 we get some $\mathsf{AR_{par}}$-saturated $v \supseteq (w' \cup w_{\mathsf{par}})$ such that $*(v)$. Hence by definition $v \in W_b$. Since $v \supseteq w'$, we have $w' \preccurlyeq v$. Then we show $v \preccurlyeq w$. Let $C \in v$. From $*(v)$ and $\mathsf{AR_{par}} \vdash v \triangleright C$, we have $C \in w$, as desired. So we have $v \preccurlyeq w$. Finally we show $v \equiv_{\mathsf{par}} w$. We must show $v_{\mathsf{par}} = w_{\mathsf{par}}$, which holds because $v \supseteq w_{\mathsf{par}}$ and $v \subseteq w$.
So it remains to show that $*(.)$ is an $\mathsf{AR_{par}}$-extendible property and $*(w' \cup w_{\mathsf{par}})$. First we show that $*(.)$ satisfies all required conditions for $\mathsf{AR_{par}}$-extendibility:

- If $*(y)$ and $y \vdash E$. We must show $*(y \cup \{E\})$. Let $\mathsf{AR_{par}} \vdash y \cup \{E\} \triangleright C$. Hence $\mathsf{AR_{par}} \vdash E \wedge \bigwedge_i F_i \triangleright C$ for some finite set $\{F_i\}_i \subseteq y$. Then since $y \vdash E$, we have $\mathsf{AR_{par}} \vdash y \triangleright E \wedge \bigwedge_i F_i$. Hence $\mathsf{AR_{par}} \vdash y \triangleright C$. Then from $*(y)$ we have $C \in w$, as desired.

- If neither $*(y \cup \{E\})$ nor $*(y \cup \{F\})$ hold, then we show that $*(y \cup \{E \vee F\})$ does not hold. Let C, D such that $\mathsf{AR_{par}} \vdash y \cup \{E\} \triangleright C$ and $\mathsf{AR_{par}} \vdash y \cup \{F\} \triangleright D$ and $C \notin w$ and $D \notin w$. Hence by disjunction rule, we have $\mathsf{AR_{par}} \vdash y \cup \{E \vee F\} \triangleright C \vee D$. Since w is IPC-saturated, we also have $C \vee D \notin w$. Hence $*(y \cup \{E \vee F\})$ does not hold.

- Let $*(y)$ and $\mathsf{AR_{par}} \vdash y \triangleright E$. We must show that $*(y \cup \{E\})$. Let $\mathsf{AR_{par}} \vdash y \cup \{E\} \triangleright C$. Then from $\mathsf{AR_{par}} \vdash y \triangleright E$ we have $\mathsf{AR_{par}} \vdash y \triangleright C$. Then from $*(y)$ we have $C \in w$.

Finally we show that $*(w' \cup w_{\mathsf{par}})$. Let $\mathsf{AR_{par}} \vdash w' \cup w_{\mathsf{par}} \triangleright C$. Hence $\mathsf{AR_{par}} \vdash \bigwedge w_{\mathsf{par}} \wedge E \triangleright C$, for some $E \in w'$. Then by Mont's rule we have $\mathsf{AR_{par}} \vdash \bigwedge w_{\mathsf{par}} \to E \triangleright \bigwedge w_{\mathsf{par}} \to C$. Since $E \in w'$, we have $\bigwedge w_{\mathsf{par}} \to E \in w'$ and hence by $\mathsf{AR_{par}}$-saturatedness of w' we have $\bigwedge w_{\mathsf{par}} \to C \in w'$. Since

$w' \subseteq w$, we have $\bigwedge w_{\mathsf{par}} \to C \in w$ and hence $C \in w$.

It only remains to show that $\mathcal{K}$ is good. Let $w_1, \ldots, w_n \in W$ and $\hat{w} := \bigcap_i w_i$. Also assume that $X \subseteq \hat{w} \cap \mathsf{par}$. We find some $w \in W_b$ such that $w \subseteq \hat{w}$ and $w \cap \mathsf{par} = X$. Define

$$*(y) : \text{For every } C_i \text{ and } p_i \in \mathsf{par}, \text{ if } \mathsf{AR}_{\mathsf{par}} \vdash y \rhd \bigvee_i C_i \vee \bigvee_i p_i, \text{ then } \exists i \ C_i \in \hat{w} \vee \exists i \ p_i \in X.$$

We show that $*(.)$ is an $\mathsf{AR}_{\mathsf{par}}$-extendible property and $*(X)$. Then by lemma 4.10 we have some $\mathsf{AR}_{\mathsf{par}}$-saturated w such that $X \subseteq w$ and $*(w)$ holds. From $*(w)$ it is clear that $w \subseteq \hat{w}$. Also if $p \in \mathsf{par} \cap w$, then by $*(w)$ we have $p \in X$ and hence $w' \cap \mathsf{par} = X$. Hence w satisfies all required conditions. It remains only to show that $*(.)$ is $\mathsf{AR}_{\mathsf{par}}$-extendible property and $*(X)$. First the $\mathsf{AR}_{\mathsf{par}}$-extendibility of $*(.)$:

- If $*(y)$ and $y \vdash E$. We must show $*(y \cup \{E\})$. Let $C = \bigvee_i C_i \bigvee_i p_i$ and $p_i \in \mathsf{par}$ and $\mathsf{AR}_{\mathsf{par}} \vdash y \cup \{E\} \rhd C$. Hence $\mathsf{AR}_{\mathsf{par}} \vdash E \wedge \bigwedge_i F_i \rhd C$ for some finite set $\{F_i\}_i \subseteq y$. Then since $y \vdash E$, we have $\mathsf{AR}_{\mathsf{par}} \vdash y \rhd E \wedge \bigwedge_i F_i$. Hence $\mathsf{AR}_{\mathsf{par}} \vdash y \rhd C$. Then from $*(y)$ we have $C_i \in \hat{w}$ or $p_i \in X$, for some i.

- If neither $*(y \cup \{E\})$ nor $*(y \cup \{F\})$ hold, then we show that $*(y \cup \{E \vee F\})$ does not hold. Let $C = \bigvee_i C_i \vee \bigvee_i p_i$ and $D = \bigvee_i D_i \vee \bigvee_i q_i$ and $p_i, q_i \in \mathsf{par}$ such that $\mathsf{AR}_{\mathsf{par}} \vdash y \cup \{E\} \rhd C$ and $\mathsf{AR}_{\mathsf{par}} \vdash y \cup \{F\} \rhd D$ and for all i we have $C_i, D_i \notin \hat{w}$ and $p_i, q_i \notin X$. Hence by disjunction rule, we have $\mathsf{AR}_{\mathsf{par}} \vdash y \cup \{E \vee F\} \rhd C \vee D$, while for all i, $C_i, D_i \notin \hat{w}$ and $p_i, q_i \notin X$. Hence $*(y \cup \{E \vee F\})$ does not hold.

- Let $*(y)$ and $\mathsf{AR}_{\mathsf{par}} \vdash y \rhd E$. We must show that $*(y \cup \{E\})$. Let $C = \bigvee_i C_i \vee \bigvee_i p_i$ and $p_i \in \mathsf{par}$ and $\mathsf{AR}_{\mathsf{par}} \vdash y \cup \{E\} \rhd C$. Then from $\mathsf{AR}_{\mathsf{par}} \vdash y \rhd E$ we have $\mathsf{AR}_{\mathsf{par}} \vdash y \rhd C$. Then from $*(y)$ we have $C_i \in \hat{w}$ or $p_i \in X$ for some i.

It finishes showing that $*(.)$ is an $\mathsf{AR}_{\mathsf{par}}$-extendible property. Then we show $*(X)$. Let $C = \bigvee_i C_i \vee \bigvee_i p_i$ and $p_i \in \mathsf{par}$ and $\mathsf{AR}_{\mathsf{par}} \vdash \mathsf{par} \cap w \rhd C$. Then by corollary 4.7 we have $X \vdash C$. Since $\bigwedge(X)$ is extendible, by lemma 3.31 for some i we have $X \vdash C_i$ or $X \vdash p_i$. Since $X \subseteq \hat{w}$, for some i either we have $C_i \in \hat{w}$ or $p_i \in X$, as desired. ❑

4.3 NNIL(par)-admissibility

Lemma 4.13. *For every good $\mathsf{AR}_{\mathsf{par}}$-model $\mathcal{K}$ and $n \in \mathbb{N}$, there is some par-extendible stable class of finite rooted models $\mathscr{K}$ such that for every proposition A with $c(A) \leq n$ we have $\mathcal{K} \Vdash A$ iff $\mathscr{K} \Vdash A$.*

Proof. Given a good $\mathsf{AR}_{\mathsf{par}}$-model $\mathcal{K} = (W, \preccurlyeq, V)$ with $W_b \subseteq W$ as its base-set, we define a stable par-extendible class $\mathscr{K}$ of finite rooted Kripke models as follows. $\mathscr{K}$ includes all Kripke models $\mathcal{K}' = (W', \preccurlyeq', V')$ with the following properties:

- $\mathcal{K}'$ is finite rooted with tree frame.

- $\mathcal{K}'$ is embeddable in $\mathcal{K}$, i.e. there is a function $f : W' \longrightarrow W$ such that $w' V' a$ iff $f(w') V a$; and $w' \preccurlyeq' v'$ implies $f(w') \preccurlyeq f(v')$.

- For all A with $c(A) \leq n$ and for every $w' \in W'$ we have $\mathcal{K}', w' \Vdash A$ iff $\mathcal{K}, f(w') \Vdash A$.

Obviously $\mathscr{K}$ is stable and $\mathcal{K} \Vdash A$ implies $\mathscr{K} \Vdash A$ for every A with $c(A) \leq n$. It remains to show:

1. $\mathscr{K} \Vdash A$ implies $\mathcal{K} \Vdash A$ for every A with $c(A) \leq n$. It is enough to show that for a given n and $w_0 \in W$, there is a finite rooted (with the root w_0') tree-frame Kripke model $\mathcal{K}' = (W', \preccurlyeq', V')$ which is embeddable in $\mathcal{K}$ with the embedding f such that $f(w_0') = w_0$ and for every $w' \in W'$ and A with $c(A) \leq n$ we have $\mathcal{K}', w' \Vdash A$ iff $\mathcal{K}, w \Vdash A$. First we inductively define sets W_i of sequences of implications $B \to C$ with $c(B \to C) \leq n$, for $0 \leq i \leq n$ and the function f from

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W_i to W. Then let $W' := \bigcup_{i=0}^{n} W_i$. Let $W_0 := \{\langle\rangle\}$ and $f(\langle\rangle) := w_0$. Assume that we already defined W_i and define W_{i+1} as follows. For every sequence $\sigma \in W_i$ and implication $B \to C$ with $c(B \to C) \leq n$ such that $\mathcal{K}, f(\sigma) \nVdash B \vee (B \to C)$, add the new node $\sigma * \langle B \to C \rangle$ to W_{i+1} and define $f(\sigma * \langle B \to C \rangle) = u$ for some u such that $u \succcurlyeq f(\sigma)$ and $\mathcal{K}, u \Vdash B$ and $\mathcal{K}, u \nVdash C$. This finishes definition of W_i and W' and the embedding $f : W' \longrightarrow W$. Finally define $\sigma \preccurlyeq' \gamma$ iff σ is an initial segment of γ. Since there are only finitely many inequivalent propositions A with $c(A) \leq n$, one may easily observe that $\mathcal{K}'$ is finite. The other required properties for $\mathcal{K}'$ are easy and left to the reader.

2. $\mathscr{K}$ is par-extendible. Let $\mathscr{K}' := \{\mathcal{K}_1, \ldots, \mathcal{K}_n\} \subseteq \mathscr{K}$ be finite such that $\mathscr{K}'$ is a $\vec{p}$-submodel of some $\mathcal{K}_0 \in \mathscr{K}$ and w_i' be the root of $\mathcal{K}_i$. Let f_i be the embedding of $\mathcal{K}_i$ in $\mathcal{K}$ and $w_i := f_i(w_i')$. Since $\mathcal{K}$ is good, there is some $u \in W$ such that $u \equiv_{\text{par}} w_0$ and $u \preccurlyeq w_1, \ldots, w_n$ and $u \in W_b$. Since u is a base, there is some tight predecessor $v \in W$ for the set $\{w_1, \ldots, w_n\}$ such that $u \equiv_{\text{par}} v$ and $u \preccurlyeq v \preccurlyeq w_1, \ldots, w_n$. Define a $\vec{p}$-variant $\mathcal{K}''$ of $\mathcal{K}' := \sum(\mathscr{K}', \mathcal{K}_0)$ in this way: $\mathcal{K}'', w_0' \Vdash a$ iff $\mathcal{K}, v \Vdash a$, for every atomic a. Then it is not difficult to observe that $\mathcal{K}'' \in \mathscr{K}$.

❑

Lemma 4.14. *If* $A \mathrel{\vrule height 1.6ex depth 0pt width 0pt}{\sim}_{\mathsf{N(par)}} B$ *and* $\mathscr{K} \subseteq \mathrm{Mod}(A)$ *is* par*-extendible and stable, then* $\mathscr{K} \Vdash B$.

Proof. Let $A \mathrel{\vrule}{\sim}_{\mathsf{N(par)}} B$ and $\mathscr{K}$ is a stable class of finite rooted models with tree frames. Let $\mathscr{K}'$ be the restriction of $\mathscr{K}$ to the atomics appeared in A, B, par. Obviously $\mathscr{K}' \subseteq \mathrm{Mod}(A)$ also is a par-extendible stable class. Let $n := \max\{c(A), \#\mathsf{par}\}$. Then lemma 3.26 implies that $\langle \mathscr{K}' \rangle_n$ is also a par-extendible stable class of finite rooted models with tree frames. Lemma 3.23 implies $\langle \mathscr{K}' \rangle_n = \mathrm{Mod}(C)$ for some C with $c(C) \leq n$. Moreover, by theorem 3.12, there is a substitution θ and $C' \in \mathsf{NNIL(par)}$ such that $\vdash C' \leftrightarrow \theta(C)$ and $C \vdash E \leftrightarrow \theta(E)$ for every proposition E. On the other hand, corollary 3.21 implies $\langle \mathscr{K}' \rangle_n \Vdash A$. Hence A is valid in $\mathrm{Mod}(C)$, which implies $\vdash C \to A$. Hence $\vdash \theta(C) \to \theta(A)$ and then $\vdash C' \to \theta(A)$. From $A \mathrel{\vrule}{\sim}_{\mathsf{N(par)}} B$ infer $\vdash C' \to \theta(B)$, or equivalently $\vdash \theta(C \to B)$. Hence for every $\mathcal{K}$, and of course for every $\mathcal{K} \in \langle \mathscr{K}' \rangle_n$ we have $\mathcal{K} \Vdash \theta(C \to B)$. Since $\mathcal{K} \Vdash C$ and θ is C-projective, we have $\mathcal{K} \Vdash C \to B$, and hence $\mathcal{K} \Vdash B$. So we may deduce $\mathcal{K} \in \langle \mathscr{K}' \rangle_n \Vdash B$. Since $\mathscr{K}' \subseteq \mathcal{K} \in \langle \mathscr{K}' \rangle_n$, we also have $\mathscr{K}' \Vdash B$. Whence $\mathscr{K} \Vdash B$, as desired. ❑

Theorem 4.15. *The following statements are equivalent:*

1. $\mathsf{AR_{par}} \vdash A \rhd B$.

2. $A \mathrel{\vrule}{\sim}_{\mathsf{N(par)}} B$.

3. B *is valid in every* par*-extendible stable class of Kripke models of* A.

4. B *is valid in every good* $\mathsf{AR_{par}}$*-model of* A.

Proof. $1 \to 2$: lemma 4.5.
$2 \to 3$: lemma 4.14.
$3 \to 4$: lemma 4.13.
$4 \to 1$: theorem 4.12. ❑

Corollary 4.16. *The following rule is admissible in* $\mathsf{AR_{par}}$: $(E \in \mathsf{NNIL(par)})$ $\dfrac{A \rhd B}{E \to A \rhd E \to B}$.

Proof. Since $\rhd = \mathrel{\vrule}{\sim}_{\mathsf{N(par)}}$ and $\dfrac{A \mathrel{\vrule}{\sim}_{\mathsf{N(par)}} B}{E \to A \mathrel{\vrule}{\sim}_{\mathsf{N(par)}} E \to B}$, we have the desired result. ❑

4.4 $\downarrow$NNIL(par)-preservativity logic

In the following theorem we show that the other direction of theorem 2.2 holds when $\Gamma = $ NNIL(par):

Theorem 4.17. $\underset{\downarrow\mathsf{N(par)}}{\approx} = \underset{\mathsf{N(par)}}{\vdash\!\!\!\sim}$.

Proof. Theorem 2.2 implies that if $A \underset{\mathsf{N(par)}}{\vdash\!\!\!\sim} B$ then $A \underset{\downarrow\mathsf{N(par)}}{\approx} B$. For the other direction, assume that $A \underset{\downarrow\mathsf{N(par)}}{\approx} B$ seeking to show $A \underset{\mathsf{N(par)}}{\vdash\!\!\!\sim} B$. By corollary 3.34 it is enough to show $A \underset{\mathsf{PN(par)}}{\vdash\!\!\!\sim} B$. Let $E \in$ PNNIL(par) and substitution θ such that $\vdash E \to \theta(A)$. Let Π_A be the PNNIL(par)-projective resolution for A, as guarantied by theorem 3.35. Since $A \underset{\mathsf{PN(par)}}{\vdash\!\!\!\sim} \bigvee \Pi_A$ we have $\vdash E \to \bigvee \theta(\Pi_A)$ and hence by primality of E, for some $F \in \Pi_A$ we have $\vdash E \to \theta(F)$. On the other hand, since Π_A is a projective resolution for A we have $\vdash F \to A$. Then by $A \underset{\downarrow\mathsf{N(par)}}{\approx} B$ we get $\vdash F \to B$. Hence $\vdash \theta(F) \to \theta(B)$, which implies $\vdash E \to \theta(B)$, as desired. ❑

Remark 4.18. *For every Γ and a logic $\mathsf{T} \supseteq$ IPC which admits Γ-projective resolutions, i.e. every $A \in \mathcal{L}_0$ has a Γ-projective resolution in T, the above proof works and we have $\vdash\!\!\!\sim^{\mathsf{T}}_{\Gamma} = \approx^{\mathsf{T}}_{\Gamma}$. Hence we have $\underset{\mathsf{PN(par)}}{\vdash\!\!\!\sim} = \underset{\downarrow\mathsf{N(par)}}{\approx}$.*

Remark 4.19. *By theorems 4.15 and 4.17 and remark 4.18 we may conclude that:*

$$\mathsf{AR_{par}} \vdash A \triangleright B \quad \textit{iff} \quad A \underset{\mathsf{N(par)}}{\vdash\!\!\!\sim} B \quad \textit{iff} \quad A \underset{\mathsf{PN(par)}}{\vdash\!\!\!\sim} B \quad \textit{iff} \quad A \underset{\downarrow\mathsf{N(par)}}{\approx} B \quad \textit{iff} \quad A \underset{\downarrow\mathsf{PN(par)}}{\approx} B.$$

4.5 NNIL(par)-preservativity logic

In this subsection we axiomatize the N(par)-preservativity and show $\underset{\mathsf{N(par)}}{\approx} = [\![\mathsf{IPC, par}]\!]A$ in which $[\![\mathsf{IPC, par}]\!]A$ is defined as $\mathsf{AR_{par}}$ plus the following axiom schema (the substitution axiom):

$$\mathsf{sub} : A \triangleright \theta(A) \text{ for every substitution } \theta \text{ (which by default is identity on parameters)}$$

The main point of the axiom sub is that we may annihilate occurrences of atomic variables, and together with other axioms of $\mathsf{AR_{par}}$ we may simplify propositions to NNIL(par)-propositions. Before we continue with providing such simplifying algorithm, let us define $\{A\}'(B)$ and $\{A\}''(B)$, two variants of $\{A\}_{\mathsf{parb}}(B)$:

$$\{A\}'(B) := \begin{cases} B & : B \text{ is } \bot \text{ or parameter} \\ A \to B & : B \in \mathsf{var} \\ \{A\}'(C) \circ \{A\}'(D) & : B = C \circ D \text{ and } \circ \in \{\vee, \wedge\} \\ (A{\downarrow}C) \to B & : B = C \to D \end{cases}$$

$$\{A\}''(B) := \begin{cases} A \to \bot & : B \in \mathsf{var} \\ \{A\}'(B) & : \text{otherwise} \end{cases}$$

In which $A{\downarrow}C$ indicates the replacement of D for every occurrence of an implication $C \to D$ in A. Then define the following variant of Visser rule:

$\mathsf{V'_{AR}} :\quad B \to C \triangleright \bigvee_{i=1}^{n+m} \{B\}''(E_i)$, in which $B = \bigwedge_{i=1}^{n}(E_i \to F_i)$ and $C = \bigvee_{i=n+1}^{n+m} E_i$.

Note that $\mathsf{AR_{par}} \vdash (B \to C) \triangleright \bigvee_{i=1}^{n+m} \{B\}'(E_i)$ for B and C as defined in above lines.

Lemma 4.20. $[\![\mathsf{IPC, par}]\!]A \vdash \mathsf{V'_{AR}}$.

Proof. Let B and C as in $\mathsf{V'_{AR}}$. By $\mathsf{V(parb)}$ we have $(B \to C) \triangleright \bigvee_{i=1}^{n+m} \{B\}_{\mathsf{parb}}(E_i)$. Also Since for every A and B we have $\vdash \{A\}'(B) \to \{A\}_{\mathsf{parb}}(B)$, then $\mathsf{AR_{par}} \vdash (B \to C) \triangleright \bigvee_{i=1}^{n+m} \{B\}'(E_i)$. So it is enough to show for every $1 \le j \le n + m$:

$$[\![\mathsf{IPC, par}]\!]A \vdash \{B\}'(E_j) \triangleright \{B\}''(E_j) \vee \bigvee_{\substack{j \neq i=1}}^{n+m} \{B\}'(E_i).$$

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If $E_j \notin \mathsf{var}$ we are trivially done. So assume that $E_j = x \in \mathsf{var}$ and hence $\{B\}'(E_j) = B \to x$. Then by the substitution axiom sub we have $[\![\mathsf{IPC}, \mathsf{par}]\!]\mathsf{A} \vdash (B \to x) \rhd \hat{\theta}(B \to x)$, in which θ is the substitution with $\theta(x) = y \vee z$ and identity elsewhere and $y, z \in \mathsf{var}$ are fresh variables, i.e. variables not appeared in B and C. Let $E_i' := \hat{\theta}(E_i)$ and $F_i' := \hat{\theta}(F_i)$ and $B' := \hat{\theta}(B)$. Hence $[\![\mathsf{IPC}, \mathsf{par}]\!]\mathsf{A} \vdash (B \to x) \rhd (B' \to (y \vee z))$. On the other hand by $\mathsf{V}(\mathsf{parb})$ we have $\mathsf{AR}_{\mathsf{par}} \vdash (B' \to (y \vee z)) \rhd \bigvee_{j \neq i=1}^{n} \{B'\}'(E_i') \vee (B' \to y) \vee (B' \to z)$. Let α, β and γ be substitutions that

$$\alpha(y) = \alpha(z) = x, \quad \beta(y) = \gamma(z) = \bot, \quad \beta(z) = \gamma(y) = x$$

and identity elsewhere. Then by sub we have

$$[\![\mathsf{IPC}, \mathsf{par}]\!]\mathsf{A} \vdash (B' \to z) \rhd \hat{\gamma}(B' \to z) \quad \text{and hence} \quad [\![\mathsf{IPC}, \mathsf{par}]\!]\mathsf{A} \vdash (B' \to z) \rhd (B \to \bot),$$

$$[\![\mathsf{IPC}, \mathsf{par}]\!]\mathsf{A} \vdash (B' \to y) \rhd \hat{\beta}(B' \to y) \quad \text{and hence} \quad [\![\mathsf{IPC}, \mathsf{par}]\!]\mathsf{A} \vdash (B' \to z) \rhd (B \to \bot),$$

$$[\![\mathsf{IPC}, \mathsf{par}]\!]\mathsf{A} \vdash \{B'\}'(E_i') \rhd \hat{\alpha}(\{B'\}'(E_i')) \quad \text{and hence} \quad [\![\mathsf{IPC}, \mathsf{par}]\!]\mathsf{A} \vdash \{B'\}'(E_i') \rhd \{B\}'(E_i).$$

Hence $[\![\mathsf{IPC}, \mathsf{par}]\!]\mathsf{A} \vdash (B' \to (y \vee z)) \rhd \bigvee_{j \neq i=1}^{n} \{B\}'(E_i) \vee (B \to \bot)$ and thus $[\![\mathsf{IPC}, \mathsf{par}]\!]\mathsf{A} \vdash \{B\}'(E_j) \rhd \bigvee_{j \neq i=1}^{n} \{B\}'(E_i) \vee \{B\}''(E_j)$. ❑

Lemma 4.21. *For every $A \in \mathcal{L}_0$ one may effectively compute $A^{\star} \in \mathsf{NNIL}(\mathsf{par})$ such that:*

1. $\mathsf{IPC} \vdash A^{\star} \to A$,

2. $[\![\mathsf{IPC}, \mathsf{par}]\!]\mathsf{A} \vdash A \rhd A^{\star}$,

3. $\mathsf{sub}^{\mathsf{atom}}(A^{\star}) \subseteq \mathsf{sub}^{\mathsf{atom}}(A)$.

Proof. By induction on $\mathfrak{o}(A)$ we define $A^{\star}$ with required properties. So let us first define the complexity number $\mathfrak{o}(A) \in \mathbb{N}^3$.

- $I(A) := \{E \to F : E \to F \in \mathsf{sub}(A)\}$.

- $\mathfrak{i}(A) := \max\{\#I(B) : B \in I(A)\}$. ($\#B$ indicates the number of elements in B)

- $\mathfrak{c}(A)$ is defined as the number of connectives occurring in A.

- $\mathfrak{v}(A)$ is defined as the number of occurrences of variables occurring in A.

- $\mathfrak{o}(A) := (\mathfrak{i}(A), \mathfrak{d}(A), \mathfrak{v}(A))$. Finally we order triples in $\mathbb{N}^3$ lexicographically.

Then by induction on $\mathfrak{o}(A)$ define $A^{\star}$ fulfilling the required conditions in the statement of lemma.

- $A = B \wedge C$: Define $A^{\star} := B^{\star} \wedge C^{\star}$.

- $A = B \vee C$: Define $A^{\star} := B^{\star} \vee C^{\star}$.

- $A \in \mathsf{var}$: Define $A^{\star} := \bot$. Note that to show $[\![\mathsf{IPC}, \mathsf{par}]\!]\mathsf{A} \vdash A \rhd A^{\star}$, here we need the substitution axiom sub.

- $A \in \mathsf{par}$: Define $A^{\star} := A$.

- $A = B \to C$: We have several sub-cases:

  - B has outer disjunction, i.e. a disjunction $E \vee F$ which is not in the scope of $\to$. Then there is some proposition $B_0(x)$ with the following properties: (1) x is a variable not appearing in B, (2) x occurs only once in B, (3) x has outer occurrence in B_0, i.e. x is not in the scope of arrows, (4) $B = B_0[x : E \vee F]$. Then define $B_1 := B_0[x : E]$ and $B_2 := B_0[x : F]$ and let

    $$A^{\star} := (B_1 \to C)^{\star} \wedge (B_2 \to C)^{\star}.$$

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– C has outer conjunction, i.e. a conjunction $E \wedge F$ which is not in the scope of $\rightarrow$. Then there is some proposition $C_0(x)$ with the following properties: (1) x is a variable not appearing in C, (2) x occurs only once in C, (3) x has outer occurrence in C_0, i.e. x is not in the scope of arrows, (4) $C = C_0[x : E \wedge F]$. Then define $C_1 := C_0[x : E]$ and $C_2 := C_0[x : F]$ and let

$$A^\star := (B \rightarrow C_1)^\star \wedge (B \rightarrow C_2)^\star.$$

– $B = \bigwedge_{i=1}^{n} B_i$ and $C = \bigvee_{i=n+1}^{n+m} E_i$ in which every B_i and E_i is either atomic or implication. Again we have several sub-cases:

* $B_i = \top$ for some i. Remove B_i from the conjunction B and let B_0 be the result. Then define $A^\star := (B_0 \rightarrow C)^\star$.

* $B_i = \bot$ for some i. Then define $A^\star := \top$.

* $B_i \in \mathsf{var}$ for some i. Let θ be a substitution such that $\theta(B_i) := \top$ and θ is identity elsewhere. Then define $A^\star := (\hat{\theta}(A))^\star$. Note that $\mathfrak{v}(\hat{\theta}(A)) < \mathfrak{v}(A)$.

* $B_i \in \mathsf{par}$ for some i. Let B_0 results in by removing B_i from the conjunction B and define

$$A^\star := B_i \rightarrow (B_0 \rightarrow C)^\star.$$

* $B_i = E_i \rightarrow F_i$, for every $1 \leq i \leq n$. Then define

$$A_1 := \bigwedge_{i=1}^{n} ((B{\downarrow}E_i) \rightarrow C) \quad \text{and} \quad A^\star := (A_1 \wedge \bigvee_{i=1}^{n+m} \{B\}''(E_i))^\star.$$

For last case, we reason for the following facts:

• $\mathfrak{o}((B{\downarrow}E_i) \rightarrow C) < \mathfrak{o}(A)$ for every $0 \leq i \leq n$. Note that $\mathfrak{i}((B{\downarrow}E_i) \rightarrow C) \leq \mathfrak{i}(A)$ and $\mathfrak{c}((B{\downarrow}E_i) \rightarrow C) < \mathfrak{c}(A)$.

• $\mathfrak{o}(\{B\}''(E_i)) < \mathfrak{o}(A)$ for every $1 \leq i \leq n+m$. If $E_i \in \mathsf{var}$, we have this inequality because $\{B\}''(E_i) = B \rightarrow \bot$ and hence $\mathfrak{v}(\{B\}''(E_i)) < \mathfrak{v}(A)$. For every other case we may show $\mathfrak{i}(\{B\}''(E_i)) < \mathfrak{i}(A)$. We refer the reader to [Visser, 2002, sec. 7].

• $\llbracket \mathsf{IPC}, \mathsf{par} \rrbracket \mathsf{A} \vdash A \rhd A^\star$: Use induction hypothesis and lemma 4.20.

• $\mathsf{IPC} \vdash A^\star \rightarrow A$: By induction hypothesis, it is enough to show $\mathsf{IPC} \vdash (A_1 \wedge \{B\}''(E_i)) \rightarrow A$ for every $1 \leq i \leq n+m$. So we reason inside IPC. Assume A_1 and $\{B\}''(E_i)$ and B seeking to derive C. If $i > n$, then by definition we have $\{B\}''(E_i) \rightarrow C$ and we are done. Otherwise, by A_1 we have $(B{\downarrow}E_i) \rightarrow C$ and hence it is enough to show $B{\downarrow}E_i$. Hence by B it is enough to show E_i, which holds by B and $\{B\}''(E_i)$. ❑

Theorem 4.22. *For every A, B, following items are equivalent:*

1. $\llbracket \mathsf{IPC}, \mathsf{par} \rrbracket \mathsf{A} \vdash A \rhd B$,

2. $A \underset{\mathsf{N}(\mathsf{par})}{\approx\!\!\!|} B$,

3. $\vdash A^\star \rightarrow B$.

Proof. We show $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$:

• $1 \Rightarrow 2$: By lemma 4.24.

• $2 \Rightarrow 3$: Let $A \underset{\mathsf{N}(\mathsf{par})}{\approx\!\!\!|} B$. By lemma 4.21 we have $A^\star \in \mathsf{NNIL}(\mathsf{par})$ and $\vdash A^\star \rightarrow A$. Then by $A \underset{\mathsf{N}(\mathsf{par})}{\approx\!\!\!|} B$ we get $\vdash A^\star \rightarrow B$.

- $3 \Rightarrow 1$: From $\vdash A^\star \to B$ we get $[\![\mathsf{IPC}, \mathsf{par}]\!]\mathsf{A} \vdash A^\star \rhd B$. Also by lemma 4.21 we have $[\![\mathsf{IPC}, \mathsf{par}]\!]\mathsf{A} \vdash A \rhd A^\star$ and then Cut implies desired result. ❑

Lemma 4.23. $\mathrel{\approx\mkern-14mu\mid}_{\mathsf{N(par)}} = \mathrel{\approx\mkern-14mu\mid}_{\mathsf{PN(par)}}$.

Proof. Corollary 3.33 and theorem 2.5. ❑

Lemma 4.24. $[\![\mathsf{IPC}, \mathsf{par}]\!]\mathsf{A} \vdash A \rhd B$ *implies* $A \mathrel{\approx\mkern-14mu\mid}_{\mathsf{N(par)}} B$.

Proof. We use induction on complexity of the proof $[\![\mathsf{IPC}, \mathsf{par}]\!]\mathsf{A} \vdash A \rhd B$. All steps trivially hold except:

- sub: This axiom holds because IPC is closed under substitutions and $\theta(E) = E$ for every $E \in \mathsf{NNIL(par)}$.

- $\mathsf{V(atomb)}$: Lemma 4.25.

- Disj: Let $A \mathrel{\approx\mkern-14mu\mid}_{\mathsf{N(par)}} C$ and $B \mathrel{\approx\mkern-14mu\mid}_{\mathsf{N(par)}} C$ seeking to show $A \vee B \mathrel{\approx\mkern-14mu\mid}_{\mathsf{N(par)}} C$. By lemma 4.23 it is enough to show $A \vee B \mathrel{\approx\mkern-14mu\mid}_{\mathsf{PN(par)}} C$. Let $E \in \mathsf{PNNIL(par)}$ such that $\vdash E \to (A \vee B)$. Since E is IPC-prime, either we have $\vdash E \to A$ or $\vdash E \to B$. Then by $A \mathrel{\approx\mkern-14mu\mid}_{\mathsf{N(par)}} C$ and $B \mathrel{\approx\mkern-14mu\mid}_{\mathsf{N(par)}} C$, in either of the cases we have $\vdash E \to C$. ❑

Lemma 4.25. $B \to C \mathrel{\approx\mkern-14mu\mid}_{\mathsf{N(par)}} \bigvee_{i=1}^{n+m} \{B\}_{\mathsf{par}}(E_i)$, *in which* $B = \bigwedge_{i=1}^{n} (E_i \to F_i)$ *and* $C = \bigvee_{i=n+1}^{n+m} E_i$.

Proof. We reason by contraposition. Let $E \in \mathsf{NNIL(par)}$ be such that $\nvdash E \to (\bigvee_{i=1}^{n+m} \{B\}_{\mathsf{par}}(E_i))$. Hence there is some finite rooted $\mathcal{K} = (W, \preccurlyeq, V)$ such that $\mathcal{K}, w_0 \Vdash E$ and $\mathcal{K}, w_0 \nVdash \bigvee_{i=1}^{n+m} \{B\}_{\mathsf{par}}(E_i)$. Let I be the set of indexes i such that $E_i \in \mathsf{par}$ or $E_i = \bot$. Also let J be the complement of I. Thus for every $i \in I$ we have $\mathcal{K}, w_0 \nVdash E_i$ and for every $j \in J$, there is some $w_j \succcurlyeq w_0$ such that $\mathcal{K}, w_j \Vdash B$ and $\mathcal{K}, w_j \nVdash E_j$. Let W' defined as follows:

$$W' := W \setminus \{v \in W : \neg \exists j \in J (w_j \preccurlyeq v)\}$$

and define $\mathcal{K}' := (W', \preccurlyeq, V)$. Then since $E \in \mathsf{NNIL}$, theorem 3.5 implies $\mathcal{K}', w_0 \Vdash E$. Moreover, it is not difficult to observe that $\mathcal{K}', w_0 \Vdash B$ and $\mathcal{K}', w_0 \nVdash C$. Thus $\mathcal{K}', w_0 \nVdash E \to (B \to C))$ and then $\nvdash E \to (B \to C)$. ❑

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