MDS and AMDS symbol-pair codes constructed from repeated-root codes

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Abstract

Symbol-pair codes introduced by Cassuto and Blaum in 2010 are designed to protect against the pair errors in symbol-pair read channels. One of the central themes in symbol-error correction is the construction of maximal distance separable (MDS) symbol-pair codes that possess the largest possible pair-error correcting performance. Based on repeated-root cyclic codes, we construct two classes of MDS symbol-pair codes for more general generator polynomials and also give a new class of almost MDS (AMDS) symbol-pair codes with the length lp. In addition, we derive all MDS and AMDS symbol-pair codes with length $3p$, when the degree of the generator polynomials is no more than 10. The main results are obtained by determining the solutions of certain equations over finite fields.

Keywords: MDS symbol-pair codes, AMDS symbol-pair codes, Minimum symbol-pair distance, Repeated-root cyclic codes

1. Introduction

Cassuto and Blaum (2010) proposed a new coding framework called symbol-pair codes to combat symbol-pair errors over symbol-pair read channels in [1] with the development of high-density data storage technologies. For example, Blu-ray disc is a high-density data storage symbol-pair read channels for the practical application. The seminal works [1]–[3] established relationships between an error-correcting code's minimum Hamming distance and the minimum symbol-pair distance, discovered methods for code construction and decoding, and obtained lower and upper bounds on code size. If a code $\mathcal{C}$ over $\mathbb{F}_p^n$ with length n contains M elements and has the minimum symbol-pair distance d_p, then $\mathcal{C}$ is referred as an $(n, M, d_p)_p$ symbol-pair code. Finding symbol-pair codes with high symbol-pair error correcting performance has become a significant theoretical challenge.

In 2012, Chee et al. [4] established the Singleton-type bound on symbol-pair codes. Similar to the classical codes, the symbol-pair codes meeting the Singleton-type bound are called MDS symbol-pair codes and almost MDS symbl-pair codes are denoted AMDS symbol-pair codes. MDS symbol-pair codes are the most useful and interesting symbol-pair codes due to their optimality. Many researchers used various mathematical tools to try to obtain MDS symbol-pair

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codes. Constructing MDS symbol-pair codes is thus important both in theory and in practice. However, because determining the exact values of symbol-pair distances of constacyclic codes is a very complicated and difficult task in general, little work has been done on it.

In 2013, Chee et al. [5] obtained some MDS symbol-pair codes from classical MDS codes. Between 2015 and 2018, researchers [6][7][8] constructed some MDS symbol-pair codes with minimum symbol-pair distance 5 and 6 from constacyclic codes. In 2017, Chen et al. [9] proposed to construct MDS symbol-pair codes by repeated-cyclic codes and obtained some length $3p$ MDS symbol-pair codes with symbol-pair distance 6 to 8 and MDS $(lp,5)_p$ symbol-pair codes. In the next few years, MDS symbol-pair codes with some new parameters were found by using repeated-root codes over $\mathbb{F}_p$. In 2018, Dinh et al. [10] presented all MDS symbol-pair codes with prime power lengths by repeated-root constacyclic codes. Two years later, Dinh et al. [11] also constructed a families of MDS sympol-pair codes with length $2p^s$. In the next 2019 to 2021, Zhao [12] [13] constructed some MDS symbol-pair codes using repeated-root constacyclic codes. In 2022, Ma et al. [14] obtained MDS $(3p,10)_p$ and MDS $(3p,12)_p$ sympol-pair codes.

Inspired by these works, in order to obtain longer and more flexible code length, as well as a larger minimum symbol-pair distance, this paper proves that there are more general generator polynomials for MDS $(lp,6)_p$ and MDS $(lp,5)_p$ symbol-pair codes by using repeated-cyclic codes. Furthermore, the parameter AMDS $(lp,7)_p$ symbol-pair codes are obtained by using repeated-cyclic codes. For length $n=3p$, this paper gives all MDS and AMDS symbol-pair codes from repeated-root cyclic codes $\mathcal{C}_{(r_1,r_2,r_3)}$, when the degree of the generator polynomial $g_{(r_1,r_2,r_3)}(x)$ is no more than 10, i.e., $\deg(g_{(r_1,r_2,r_3)}(x)) \leq 10$.

The rest of this paper is structured as follows. In Section 2, we introduce some basic notations and results on symbol-pair codes. In Section 3, we derive some new classes of MDS symbol-pair codes and AMDS symbol-pair codes from repeated-root cyclic codes. In section 4, we conclude the paper.

2. Preliminaries

In this section, we introduce some notations and auxiliary tools on symbol-pair codes, which will be used to prove our main results in the sequel. We donote that $\mathbb{F}_p$ and $\mathbb{F}_q$ are finite fields, where p is an odd prime and $q=p^m$. Then we denote $\mathbb{F}_p^*$ is the cyclic group $\mathbb{F}_p \backslash \{0\}$. Let

$$\boldsymbol{x} = (x_0, x_1, \cdots, x_{n-1})$$

be a vector in $\mathbb{F}_p^n$. Then the symbol-pair read vector of $\boldsymbol{x}$ is

$$\pi(\boldsymbol{x}) = [(x_0,x_1),(x_1,x_2),\cdots,(x_{n-1},x_0)].$$

Similar to the Hamming weight $\omega_H(\boldsymbol{x})$ and Hamming distance $D_H(\boldsymbol{x},\boldsymbol{y})$. The symbol-pair weight $\omega_p(\boldsymbol{x})$ of the symbol-pair vector $\boldsymbol{x}$ is defined as

$$\omega_p(\boldsymbol{x}) = |\{i \,|(x_i,x_{i+1}) \neq (0,0)\}|.$$

The symbol-pair distance $D_p(\boldsymbol{x},\boldsymbol{y})$ between any two vectors $\boldsymbol{x},\boldsymbol{y}$ is

$$D_p(\boldsymbol{x},\boldsymbol{y}) = |\{\, i |\,(x_i,x_{i+1}) \neq (y_i,y_{i+1})\}|.$$

The minimum symbol-pair distance of a code $\mathcal{C}$ is

$$d_p = \min\{D_p(\boldsymbol{x},\boldsymbol{y}) \,|\, \boldsymbol{x},\boldsymbol{y} \in \mathcal{C}\},$$

and we denote $(n,k,d_p)_p$ a symbol-pair code with length n, dimension k and minimum symbol-pair distance d_p over $\mathbb{F}_p$. For any code $\mathcal{C}$ of length n with $0 < d_H(\mathcal{C}) < n$, there is an important inequality between $d_H(\mathcal{C})$ and $d_p(\mathcal{C})$ in [2]:

$$d_H(\mathcal{C}) + 1 \leq d_p(\mathcal{C}) \leq 2d_H(\mathcal{C}).$$

In this paper, we always regard the codeword $\mathbf{c}$ in $\mathcal{C}$ as the corresponding polynomial $c(x)$. The following lemmas will be applied in our later proofs.

Similar to classical error-correcting codes, the size of symbol-pair codes satisfies the following Singleton bound. The symbol-pair code achieving the Singleton bound is called a maximum distance separable (MDS) symbol-pair code.

Lemma 2.1. *([4])If $\mathcal{C}$ is a symbol-pair code with length n and minimum symbol-pair distance d_p over $\mathbb{F}_q$, we call an $(n,d_p)_p$ symbol-pair code of size q^{n-d_p+2} maximum distance separable (MDS) and an $(n,d_p)_p$ symbol-pair code of size q^{n-d_p+1} almost maximum distance separable (AMDS) for $q \geqslant 2$.*

Lemma 2.2. *([5]) Let $q \geqslant 2$ and $2 \leqslant d_p \leqslant n$. If $\mathcal{C}$ is a symbol-pair code with length n and minimum symbol-pair distance d_p over $\mathbb{F}_q$, then $|\mathcal{C}| \leqslant q^{n-d_p+2}$.*

In some cases, the bound of minimum symbol-pair distance can be improved.

Lemma 2.3. *([9]) Let $\mathcal{C}$ be an $[n,k,d_H]$ constacyclic code over $\mathbb{F}_q$ with $2 \leqslant d_H \leqslant n$. Then we have the following $d_p(\mathcal{C}) \geqslant d_H + 2$ if and only if $\mathcal{C}$ is not an MDS code.*

The next lemma will be used by the later proof of Proposition 3.14.

Lemma 2.4. *([9]) Let $n = 3p$ with $p \equiv 1 \pmod 3$. If $\mathcal{C}_{(3,2,1)}$ is the cyclic code in $\mathbb{F}_p[x]/\langle x^n - 1\rangle$ generated by*

$$g_{(3,2,1)}(x) = (x-1)^3(x-\omega)^2(x-\omega^2),$$

then $\mathcal{C}_{(3,2,1)}$ is an MDS $(3p,8)_p$ symbol-pair code, where ω is a primitive 3-th root of unity in $\mathbb{F}_p$.

Researchers constructed some MDS symbol-pair codes with minimum symbol-pair distance 6 by repeated-root cyclic codes. The next lemma will be used by several parts of Theorem 3.1.

Lemma 2.5.

- *[10] $\mathcal{C} = \left\langle (x-1)^4 \right\rangle$ is an MDS symbol-pair code.*
- *[11] $\mathcal{C} = \left\langle (x-1)^i(x+1)^j \right\rangle$ is an MDS symbol-pair code with $d_p = 6$ over $\mathbb{F}_p$, where $|i-j| \leqslant 2$ and $i, j \leq p-1$.*
- *[12] $\mathcal{C} = \left\langle (x-1)^3(x-\omega) \right\rangle$ is an MDS $(lp,\ 6)_p$ symbol-pair code over $\mathbb{F}_p$, where ω is a primitive l-th root of unity in $\mathbb{F}_p$.*

The method for calculating the minimum Hamming distance about repeated-cyclic codes is given in the following lemma.

Lemma 2.6. *([15]) Let $\mathcal{C}$ be a repeated-root cyclic code with length lp^e over $\mathbb{F}_q$ generated by $g(x) = \prod m_i^{e_i}(x)$, where l and e are positive integers with $\gcd(l,p) = 1$. Then we have*

$$d_H(\mathcal{C}) = \min\left\{P_t \cdot d_H\left(\overline{\mathcal{C}}_t\right) | 0 \leq t \leq lp^e\right\},$$

where $P_t = \omega_H\left((x-1)^t\right)$ and $\overline{\mathcal{C}}_t = \left\langle \prod_{e_i > t} m_i(x) \right\rangle$.

3. Constructions of MDS and AMDS Symbol-Pair Codes

In this section, we propose some MDS and AMDS symbol-pair codes by repeated-root cyclic codes over $\mathbb{F}_p$, where p is an odd prime and $\mathbb{F}_p$ is a p-ary finite field.

3.1. MDS and AMDS Symbol-Pair Codes with length lp

In this subsection, we prove that there exist more general generator polynomials about MDS $(lp,6)_p$ and MDS $(lp,5)_p$ symbol-pair codes. Furthermore, the parameter AMDS $(lp,7)_p$ symbol-pair codes are obtained by using repeated-cyclic codes.

Let $\mathcal{C}_{(r_1,r_2,r_3)}$ be the repeated-root cyclic code over $\mathbb{F}_p$ and the generator ploynomial of $\mathcal{C}_{(r_1,r_2,r_3)}$ is

$$g_{(r_1,r_2,r_3)}(x) = (x-1)^{r_1}(x+1)^{r_2}(x-\omega)^{r_3},$$

where ω is a primitive l-th root of unity in $\mathbb{F}_p$ and $r_1+r_2+r_3=4$. Dinh [10] [11] discussed all cases of $l=1$ and $l=2$, here we focus on the case of $l>2$.

Theorem 3.1. *Let $\mathcal{C}_{(r_1,r_2,r_3)}$ be an MDS symbol-pair code with $d_p=6$, if r_1, r_2 and r_3 meet the conditions in Table 1.*

Table 1: MDS symbol-pair codes of Theorem 3.1

r_1	r_2	r_3	$(n,d_p)_p$	Reference or Proposition
4	0	0	$(p,6)_p$	Reference [10]
3	0	1	$(lp,6)_p$	Reference [12]
2	1	1	$(klp,6)_p$	Proposition 3.2
2	0	2	$(lp,6)_p$	Proposition 3.4

* When l even, $(klp,6)_p=(lp,6)_p$; When l odd, $(klp,6)_p=(2lp,6)_p$.
* Let l be an odd number or $l\equiv 0 \pmod 4$, if $(r_1,r_1,r_1)\in S$ and $S=\{(0,1,3),(0,2,2),(0,3,1)\}$.

The proof of Theorem 3.1 needs Propositions 3.2 to Proposition 3.4. For the case of generator polynomials with three factors $(x-1),(x+1)$ and $(x-\omega)$, we have the following proposition.

Proposition 3.2. *Let $\mathcal{C}_{(r_1,r_2,r_3)}$ be an MDS symbol-pair code with $d_p=6$, if $(r_1,r_2,r_3)\in S_1$ and $S_1=\{(2,1,1),(1,2,1),(1,1,2)\}$.*

Proof. When $(r_1,r_2,r_3)=(2,1,1)$, let $\mathcal{C}_{(2,1,1)}$ be the cyclic code over $\mathbb{F}_p$ and generated by

$$g_{(2,1,1)}(x) = (x-1)^2(x+1)(x-\omega).$$

By Lemma 2.6 , let $\overline{g}_t(x)$ be the generator ploynomials of $\overline{\mathcal{C}}_t$.

- If $t=0$, then we have
$$\overline{g}_0(x) = (x-1)(x+1)(x-\omega).$$
It is easy to verify that the minimum Hamming diatance is 3 in $\overline{\mathcal{C}}_0$ and $P_0=1$. Therefore, this indicates $P_0\cdot d_H(\overline{\mathcal{C}}_0)=3$.

- If $t=1$, then we have $\overline{g}_1(x)=(x-1)$ and $P_1=2$. Thus, one can derive that $P_1\cdot d_H(\overline{\mathcal{C}}_1)=4$.
- If $2\le t\le p-1$, then we have $\overline{g}_t(x)=1$ and $P_t\ge 2$. This implies that $P_t\cdot d_H(\overline{\mathcal{C}}_t)\ge 3$.

  Therefore, it can be verified that $\mathcal{C}$ is an $[lp,\ lp-4,\ 3]$ repeated-root cyclic code over $\mathbb{F}_p$. Lemma 2.3 yields that $d_p\geqslant 5$, since $\mathcal{C}$ is not an MDS cyclic code.

If $c\in\mathcal{C}_{(2,1,1)}$ has $\omega_p=5$ with $\omega_H=4$, then its certain cyclic shift must have the form

$$(\star,\ \star,\ \star,\ \star,0_s),$$

where each $\star$ denotes an element in $\mathbb{F}_p^*$ and 0_s is all-zero vector with length s. Without loss of generality, suppose that the constant term of $c(x)$ is 1. We denote that

$$c(x)=1+a_1x+a_2x^2+a_3x^3.$$

This is a contradiction for $\deg(c(x))\geq\deg\left(g_{(2,1,1)}(x)\right)$.

If $c\in\mathcal{C}_{(2,1,1)}$ has $\omega_p=5$ with $\omega_H=3$, then its certain cyclic shift must have the form

$$(\star,\ \star,\ 0_{s_1},\ \star,\ 0_{s_2}),$$

where each $\star$ denotes an element in $\mathbb{F}_p^*$ and 0_{s_1}, 0_{s_2} are all-zero vectors with lengths s_1 and s_2 respectively. Without loss of generality, suppose that the constant term of $c(x)$ is 1. We denote that

$$c(x)=1+a_1x+a_2x^t.$$

When t is even, it can be verified that

$$\begin{cases}1+a_1+a_2=0,\\ 1-a_1+a_2=0,\end{cases}$$

since $c(1)=c(-1)=0$. This is impossible, since $a_1\neq 0$ and $p\neq 2$.

Similarly, if t is odd, one can obtain that

$$\begin{cases}1+a_1+a_2=0,\\ 1-a_1-a_2=0,\end{cases}$$

which contradicts p odd.

Let $y=-x, z=\frac{x}{\omega}$, we can deduce the following results by deforming it,

$$\begin{aligned}g_{(2,1,1)}(x)&=(x-1)^2(x+1)(x-\omega)\\&=(y+1)^2(y-1)(y+\omega)\\&=g_{(1,2,1)}(y)\\&=\omega^4(z-\frac{1}{\omega})^2(z+\frac{\omega}{\omega})(z-\frac{\omega}{\omega})\\&=\omega^4g_{(1,1,2)}(z).\end{aligned}$$

Thus, we have $\mathcal{C}_{(2,1,1)}=\mathcal{C}_{(1,2,1)}=\mathcal{C}_{(1,1,2)}$.

□

We have the following two propositions, when the generator polynomials only have two of these three factors $(x-1),(x+1)$ and $(x-\omega)$.

Proposition 3.3. *Let $\mathcal{C}_{(r_1,r_2,r_3)}$ be an MDS symbol-pair code with $d_p=6$, if $(r_1,r_2,r_3)\in S_2$ and $S_2=\{(1,0,3),(0,3,1),(0,1,3)\}$.*

Proof. When $(r_1,r_2,r_3)=(1,0,3)$, let $\mathcal{C}_{(1,0,3)}$ be a repeated-root cyclic code over $\mathbb{F}_p$ generated by

$$g_{(1,0,3)}(x)=(x-1)(x-\omega)^3.$$

By Lemma 2.6, we can derive $d_H=3$ and Lemma 2.3 implies that $d_p\geq 5$. With arguments similar to the proof of Proposition 3.2, there are no codewords of $\mathcal{C}_{(1,0,3)}$ with $\omega_H=4$ such that the 4 nonzero terms appear with consecutive coordinates. Next we show that there are no codewords of $\mathcal{C}_{(1,0,3)}$ with $\omega_H=3$ in the form

$$(\star,\ \star,\ 0_{s_1},\ \star,\ 0_{s_2}),$$

where each $\star$ denotes an element in $\mathbb{F}_p^*$ and 0_{s_1}, 0_{s_2} are all-zero vectors with lengths s_1 and s_2 respectively. Without loss of generality, suppose that the constant term of $c(x)$ is 1. We denote that

$$c(x)=1+a_1x+a_2x^t.$$

Then $c^{(1)}(\omega)=c^{(2)}(\omega)=0$ induces that $t-1=kp$ for some positive integers $k\leq l-2$, together with $c(\omega)=0$, one can immediately get

$$\begin{cases} 1+a_1\omega+a_2\omega^{k+1}=0,\\ a_1+a_2\omega^k=0. \end{cases}$$

By solving the system, we can derive a contradiction, since p is an odd prime.

For the case of $(r_1,r_2,r_3)\in S_2$, with similar to the proof of Proposition 3.2, we can deduce $\mathcal{C}_{(1,0,3)}=\mathcal{C}_{(0,3,1)}=\mathcal{C}_{(0,1,3)}$.

□

Proposition 3.4. *$\mathcal{C}_{(r_1,r_2,r_3)}$ is an MDS symbol-pair code with $d_p=6$, when $(r_1,r_2,r_3)\in S_3$ and $S_3=\{(2,0,2),(0,2,2)\}$.*

Proof. When $(r_1,r_2,r_3)=(2,0,2)$, let $\mathcal{C}_{(2,0,2)}$ be a cyclic code over $\mathbb{F}_p$ generated by

$$g_{(2,0,2)}(x)=(x-1)^2(x-\omega)^2.$$

By Lemma 2.6, we can derive that the minimum Hamming distance of $\mathcal{C}_{(2,0,2)}$ is $d_H=3$ and Lemma 2.3 implies $d_p\geq 5$. With arguments similar to the proof of Proposition 3.2, there are no codewords of $\mathcal{C}_{(2,0,2)}$ with $\omega_H=4$ such that the 4 nonzero terms appear with consecutive coordinates. Next, we show that there are no codewords of $\mathcal{C}_{(2,0,2)}$ with $\omega_H=3$ in the form

$$(\star,\ \star,\ 0_{s_1},\ \star,\ 0_{s_2}),$$

where each $\star$ denotes an element in $\mathbb{F}_p^*$ and 0_{s_1}, 0_{s_2} are all-zero vectors with lengths s_1 and s_2 respectively. Without loss of generality, suppose that the constant term of $c(x)$ is 1. We denote that

$$c(x)=1+a_1x+a_2x^t.$$

However, by $c^{(1)}(1) = c^{(1)}(\omega) = 0$, we can deduce that l is a divisor of $t-1$. Then combined with $c(1) = c(\omega) = 0$, we have

$$\begin{cases} 1 + a_1 + a_2 = 0, \\ 1 + a_1\omega + a_2\omega = 0. \end{cases}$$

This implies $\omega = 1$, which is impossible, since $\omega^l = 1$ and $l > 2$.

When $(r_1, r_2, r_3) = (0,2,2)$, with similar to the proof of Proposition 3.2, we can deduce $\mathcal{C}_{(2,0,2)} = \mathcal{C}_{(0,2,2)}$.

□

This completes the proof of Theorem 3.1 from Proposition 3.2 to Proposition 3.4, Remark 3.5 to Remark 3.6 and Lemma 2.5. Therefore, we find all MDS symbol-pair codes containing these three factors $(x-1)$, $(x+1)$ and $(x-\omega)$ with minimum symbol-pair distance 6. In fact, Lemma 2.5 is a special form of Proposition 3.3 and Proposition 3.4 for $d_p = 6$, where ω is a $\frac{p-1}{2}$-th primitive element in $\mathbb{F}_p$.

Remark 3.5. *When l even, factor $(x+1)$ is a divisor of $x^l - 1$; when l odd, factor $(x+1)$ is a divisor of $x^{2l} - 1$.*

Remark 3.6. *If $l \equiv 2 \pmod 4$ and $(r_1, r_1, r_1) \in S$ for $S = \{(0,1,3),(0,2,2),(0,3,1)\}$, the minimum Hamming distance of $\mathcal{C}_{(r_1,r_2,r_3)}$ is 2.*

In what follows, we obtain more general generator polynomials for symbol-pair codes with length $n = lp$ and minimum symbol-pair distance 5 or 6.

If m_1 and m_2 are two positive integers, then $\mathrm{lcm}[m_1, m_2]$ is the lowest common multiple of m_1 and m_2, as well as $\gcd(m_1, m_2)$ is the greatest common divisor of m_1 and m_2. Let $\mathcal{C}_a$ and $\mathcal{C}_b$ be the repeated-root cyclic codes over $\mathbb{F}_p$ and the generator ploynomial of $\mathcal{C}_a$ and $\mathcal{C}_b$ are

$$g_a(x) = \left(x - {\omega_0}^{t_1}\right)^{r_1} \left(x - {\omega_0}^{t_2}\right)^{r_2}$$

and

$$g_b(x) = (x-1)^2 \left(x - {\omega_0}^{t_1}\right) \left(x - {\omega_0}^{t_2}\right)$$

respectively, where $t_1 \geq t_2$, $\mathrm{ord}({\omega_0}^{t_1}) = m_1$, $\mathrm{ord}({\omega_0}^{t_2}) = m_2$, $\mathrm{lcm}[m_1, m_2] = l$, $\gcd(t_1 - t_2, l) = 1$, $3 \leq r_1 + r_2 \leq 4$ and ω_0 is the primitive element in $\mathbb{F}_p$.

Corollary 3.7. *Let $\mathcal{C}_a$ be an MDS symbol-pair code, if $r_1 \neq 0$ and $r_2 \neq 0$.*

Proof. There are three cases that need to be discussed, $(r_1, r_2) = (2,1)$, $(1,3)$ and $(2,2)$. When $(r_1, r_2) = (1,2)$ and $(3,1)$ is satisfied, it is similar to $(r_1, r_2) = (2,1)$ and $(1,3)$.

Case I. For the case of $(r_1, r_2) = (2,1)$, if there exsits a nonzero codeword with $\omega_H = 2$ in $\mathcal{C}_a$, without loss of generality, suppose that the constant term of $c(x)$ is 1. We denote that

$$c(x) = 1 + a_1 x^t,$$

where $a_1 \neq 0$ and $t \neq 0$. It follows from $c({\omega_0}^{t_1}) = c({\omega_0}^{t_2}) = 0$ that

$$\begin{cases} 1 + a_1 {\omega_0}^{tt_1} = 0, \\ 1 + a_1 {\omega_0}^{tt_2} = 0. \end{cases}$$

By solving the system, we have ${\omega_0}^{t(t_1-t_2)}=1$. Together with $c^{(1)}({\omega_0}^{t_1})=0$ and $\gcd(t_1-t_2,l)=1$, one can immediately get lp is a divisor of t, which contradicts with the code length lp.

Thus, combined with Lemma 2.6, the minimum Hamming distance of $\mathcal{C}_a$ is $d_H=3$. By Lemma 2.3, we have $\mathcal{C}_a$ is an MDS $(lp,5)_p$ symbol-pair code.

Case II. For the case of $(t_1,t_2)=(3,1)$, we have the generator polynomial

$$g_a(x)=\left(x-{\omega_0}^{t_1}\right)^3\left(x-{\omega_0}^{t_2}\right).$$

By the proof of **Case I**, since **Case II** is a subcode of **Case I**, we can draw the conclusion that the minimum Hamming distance of $\mathcal{C}_a$ is 3 in **Case II**, when $(t_1,t_2)=(3,1)$.

If there is a codeword with Hamming weight 3 and symbol-pair weight 5. Then its certain cyclic shift must be the following form

$$(\star,\ \star,\ 0_{s_1},\ \star,\ 0_{s_2}),$$

where each $\star$ denotes an element in $\mathbb{F}_p^*$ and 0_{s_1}, 0_{s_2} are all-zero vectors with lengths s_1 and s_2 respectively. Then we have a codeword polynomial

$$c(x)=1+a_1x+a_2x^t.$$

However, it follows from $c^{(1)}({\omega_0}^{t_1})=c^{(2)}({\omega_0}^{t_1})=0$ that

$$\begin{cases} a_1+ta_2{\omega_0}^{(t-1)t_1}=0,\\ t(t-1)a_2{\omega_0}^{(t-2)t_1}=0.\end{cases}$$

By solving the system, we have $p\,|\,t-1$. Then $c^{(1)}({\omega_0}^{t_1})=c({\omega_0}^{t_1})=0$ indicates

$$\begin{cases} 1+a_1{\omega_0}^{t_1}+a_2{\omega_0}^{t_1t}=0,\\ a_1+a_2{\omega_0}^{t_1(t-1)}=0.\end{cases}$$

Then, we can derive a contradiction, since p is an odd prime.

Therefore, there does not exsit a nonzero codeword with Hamming weight 3 and symbol-pair weight 5. Then, $\mathcal{C}_a$ is an MDS $(lp,6)_p$ symbol-pair code, when $(r_1,r_2)=(3,1)$.

Case III. For the case of $(t_1,t_2)=(2,2)$, similarly, we have the generator polynomial

$$g_a(x)=\left(x-{\omega_0}^{t_1}\right)^2\left(x-{\omega_0}^{t_2}\right)^2,$$

by the proof of **Case I**, we can draw the conclusion that the minimum Hamming distance of $\mathcal{C}_a$ is $d_H=3$. Similar to **Case II**, there is no codeword with Hamming weight 4 and symbol-pair weight 5.

If there exsits a codeword with Hamming weight 3 and symbol-pair weight 5, the codeword certain cyclic shift must have a form

$$(\star,\ \star,\ 0_{s1},\star,0_{s2}),$$

where each $\star$ denotes an element in $\mathbb{F}_p^*$ and 0_{s1}, 0_{s2} are all-zero vectors with lengths s_1 and s_2 respectively. Without loss of generality, suppose that the constant term of $c(x)$ is 1. We denote that

$$c(x)=1+a_1x+a_2x^t.$$

However, $c^{(1)}(\omega_0{}^{t_1}) = c^{(1)}(\omega_0{}^{t_2}) = 0$ induces that

$$\begin{cases} a_1 + ta_2\omega_0{}^{(t-1)t_1} = 0, \\ a_1 + ta_2\omega_0{}^{(t-1)t_2} = 0. \end{cases}$$

By solving the system, we have $\omega_0{}^{(t-1)(t_1-t_2)} = 1$, since $t \neq kp$, otherwise $a_1 = 0$. Together with $\gcd(t_1 - t_2, l) = 1$, one can immediately get $l \,|\, t-1$ and $a_1 + ta_2 = 0$. Combined with

$$c\left(\omega_0{}^{t_1}\right) = c\left(\omega_0{}^{t_2}\right) = 0,$$

we have

$$\begin{cases} 1 + a_1\omega_0{}^{t_1} + a_2\omega_0{}^{t_1} = 0, \\ 1 + a_1\omega_0{}^{t_2} + a_2\omega_0{}^{t_2} = 0, \end{cases}$$

which implies $a_1 + a_2 = 0$. Thus, we have $p \,|\, t-1$, which contradicts the code length lp.

As a consequence, we prove that there no exsits a codeword with Hamming weight 3 and symbol-pair weight 5. Then, $\mathcal{C}_a$ is an MDS $(lp, 6)_p$ symbol-pair code, when $(r_1, r_2) = (2, 2)$ and $\gcd(t_1 - t_2, l) = 1$.

□

Corollary 3.8. *$\mathcal{C}_b$ is an MDS $(lp, 6)_6$ symbol-pair code.*

Proof. By Lemma 2.6 and Lemma 2.3 the minimum Hamming distance of $\mathcal{C}_b$ is 3 and the minimum symbol-pair distance $d_p(\mathcal{C}_b) \geq 5$, for the generator ploynomial $\mathcal{C}_b$ is

$$g_b(x) = (x-1)^2\left(x - \omega_0{}^{t_1}\right)\left(x - \omega_0{}^{t_2}\right).$$

Using techniques similar to those used in the proof of Proposition 3.2, we see that there are no codewords of $\mathcal{C}_b$ with Hamming weight 4 such that the 4 nonzero terms appear with consecutive coordinates.

If there exsits a codeword with Hamming weight 3 and symbol-pair weight 5, the codeword certain cyclic shift must have a form

$$(\star,\ \star,\ 0_{s1}, \star, 0_{s2}),$$

where each $\star$ denotes an element in $\mathbb{F}_p^*$ and 0_{s1}, 0_{s2} are all-zero vectors with lengths s_1 and s_2 respectively. Without loss of generality, suppose that the constant term of $c(x)$ is 1. We denote that

$$c(x) = 1 + a_1x + a_2x^t.$$

It follows from $c(1) = c^{(1)}(1) = 0$ that

$$\begin{cases} 1 + a_1 + a_2 = 0, \\ a_1 + ta_2 = 0, \end{cases}$$

By solving the system, we have $a_1 = \frac{-t}{t-1}$ and $a_2 = \frac{1}{t-1}$. Then combined with $c(1) = c(\omega_0{}^{t_1}) = 0$, we have

$$t = \omega_0{}^{t_1(t-1)} + \omega_0{}^{t_1(t-2)} + \cdots + \omega_0{}^{t_1} + 1.$$

By $a_1 = \frac{-t}{t-1}$, $a_2 = \frac{1}{t-1}$ and $c(\omega_0{}^{t_1}) = 0$, one can obtain that

$$t - 1 - t\omega_0{}^{t_1} + \omega_0{}^{t_1 t} = 0.$$

This implies that

$$\begin{aligned} &{\omega_0}^{t_1(t-1)} + {\omega_0}^{t_1(t-2)} + \cdots + {\omega_0}^{t_1} + {\omega_0}^{t_1 t} + {\omega_0}^{t_1(t-1)} + {\omega_0}^{t_1(t-2)} + \cdots + {\omega_0}^{t_1} + {\omega_0}^{t_1 t} \\ = &2{\omega_0}^{t_1}\left({\omega_0}^{t_1(t-1)} + {\omega_0}^{t_1(t-2)} + \cdots + {\omega_0}^{t_1}\right) = 2t{\omega_0}^{t_1} = 0, \end{aligned}$$

which is a contradiction for $c^{(1)}(1) = 0$.

This completes the proof of the Corollary 3.8. □

Remark 3.9. *Let ω_0 be the primitive l-th root of unity in $\mathbb{F}_p$, we can deduce*

- $\mathcal{C}_a = \left\langle (x-{\omega_0}^{t_1})^2 (x-{\omega_0}^{t_2}) \right\rangle = \langle (x-1)^2(x-\omega) \rangle$*;*
- $\mathcal{C}_a = \left\langle (x-{\omega_0}^{t_1})^2 (x-{\omega_0}^{t_2})^2 \right\rangle = \langle (x-1)^2(x-\omega)^2 \rangle$*;*
- $\mathcal{C}_a = \left\langle (x-{\omega_0}^{t_1})^3 (x-{\omega_0}^{t_2}) \right\rangle = \langle (x-1)^3(x-\omega) \rangle$*,*

Repeated-root cyclic code $\mathcal{C} = \langle (x-1)^2(x-\omega) \rangle$ is proposed in Chen [9]. In this paper, $\mathcal{C} = \langle (x-1)^3(x-\omega) \rangle$ and $\mathcal{C} = \langle (x-1)^2(x-\omega)^2 \rangle$ are are two cases in Theorem 3.1.

We use an example to illustrate that the repeated-root cyclic codes of the generator polynomials with the same forms in the above Corollary 3.7 are not all MDS symbol-pair codes.

Example 3.10. *Let $\mathcal{C}$ and be a repeated-root cyclic code over $\mathbb{F}_5$ and the generator ploynomial of $\mathcal{C}$ is*

$$g(x) = (x-2)^2(x-3),$$

where $\omega = 3$ is a primitive element in $\mathbb{F}_5$ and $2 = 3^3$. Then we have the minimum Hamming distance $d_H = 2$ by a magma progarm. Therefore, $\mathcal{C}$ is not an MDS symbol-pair code.

Similarly, when the generator ploynomial of $\mathcal{C}$ is one of

$$g(x) = (x-2)^3(x-3)$$

and

$$g(x) = (x-2)^2(x-3)^2,$$

$\mathcal{C}$ is still not an MDS symbol-pair code, since minimum Hamming distance is $d_H = 2$.

Now we present a new class of AMDS symbol-pair codes with the minimum symbol-pair distance 7.

Let $\mathcal{C}_1$ be the cyclic codes over $\mathbb{F}_p$. The generator ploynomial of $\mathcal{C}_1$ is

$$g_1(x) = (x-1)^4(x-\omega)\left(x-\omega^2\right),$$

where ω is a primitive l-th root of unity in $\mathbb{F}_p$.

Theorem 3.11. *$\mathcal{C}_1$ is an AMDS $(lp,\, 7)_p$ symbol-pair code, if l odd and $l \geqslant 3$.*

Proof. $\mathcal{C}_1$ is the cyclic code over $\mathbb{F}_p$ generated by

$$g_1(x) = (x-1)^4(x-\omega)\left(x-\omega^2\right).$$

By Lemma 2.6, one can derive that $\mathcal{C}_1$ is an $[lp,\ lp-6,\ 4]$ repeated-root cyclic codes code over $\mathbb{F}_p$. Lemma 2.3 yields that $d_p \geqslant 6$, since $\mathcal{C}_1$ is not an MDS cyclic code. To prove that $\mathcal{C}_1$ is an AMDS symbol-pair code with the minimum symbol-pair distance 7, it is sufficient to verify that there is no a codeword in $\mathcal{C}_1$ with the symbol-pair weight 6.

If there are codewords in $\mathcal{C}_1$ with Hamming weight 5 and symbol-pair weight 6, then its certain cyclic shift must have the form

$$(\star,\ \star,\ \star,\ \star,\ \star, 0_s)\,,$$

where each $\star$ denotes an element in $\mathbb{F}_p^*$ and 0_s is all-zero vector of length s. Without loss of generality, suppose that the constant term of $c(x)$ is 1. We denote that

$$c(x) = 1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4,$$

This leads to $\deg(c(x)) = 4 < 6 = \deg(g(x))$.

If $c \in \mathcal{C}$ has the symbol-pair weight 6 with the Hamming weight 4, then its certain cyclic shift must have the forms

$$(\star,\star,\star,0_{s_1},\star,0_{s_2})$$

or

$$(\star,\star,0_{s_1},\star,\star,0_{s_2})\,,$$

where each $\star$ denotes an element in $\mathbb{F}_p^*$ and 0_{s_1}, 0_{s_2} are all-zero vectors with lengths s_1 and s_2 respectively.

Case I. For the case of

$$(\star,\star,\star,0_{s_1},\star,0_{s_2})\,,$$

without loss of generality, we denote a codeword polynomial

$$c(x) = 1 + a_1x + a_2x^2 + a_3x^t\,.$$

It follows from $c^{(1)}(1) = c^{(2)}(1) = 0$ that

$$\begin{cases} a_1 + 2a_2 + ta_3 = 0, \\ 2a_2 + t(t-1)a_3 = 0. \end{cases}$$

By solving the system, we have $t(t-2)a_3 = a_1$. By $c^{(2)}(1) = 0$, we can conclude that

$$\begin{cases} t-2 \neq kp, k < l, \\ t-1 \neq kp, k < l, \\ t \neq kp, k < l. \end{cases}$$

This is a contradiction for $c^{(3)}(1) = 0$ and $a_3 \in \mathbb{F}_p^*$.

Case II. For the case of

$$(\star,\star,0_{s_1},\star,\star,0_{s_2})\,,$$

without loss of generality, we denote

$$c(x) = 1 + a_1x + a_2x^2 + a_3x^t + a_4x^{t+1}.$$

It follows from $c(1) = c^{(1)}(1) = 0$ that

$$\begin{cases} 1 + a_1 + a_2 + a_3 = 0, \\ a_1 + ta_2 + (t+1)a_3 = 0, \end{cases}$$

one can derive that

$$(t-1)a_2 + ta_3 - 1 = 0.$$

By $c^{(2)}(1) = 0$, we have

$$t(t-1)a_2 + t(t+1)a_3 = 0.$$

This leads to $t(a_3+1) = 0$. Therefore, we have $t = kp, 0 < k < l$ or $a_3 = -1$.

If $t = kp, 0 < k < l$, then

$$\begin{cases} 1 + a_1 + a_2 + a_3 = 0, \\ a_1 + a_3 = 0. \end{cases}$$

This indicates $a_1 = -a_3$ and $a_2 = -1$. Combined with $c(\omega) = c\left(\omega^2\right) = 0$, we have

$$\begin{cases} a_1\omega(\omega^t - 1) = 1 - \omega^t, \\ a_1\omega^2\left(\omega^{2t} - 1\right) = 1 - \omega^{2t}. \end{cases}$$

By solving the system,we have $\omega^{2t} = 1$, which contradicts l odd.

If $a_3 = -1$, by $c(1) = 0$, we can obtain that $a_1 = -a_2$. Combined with $c(\omega) = c\left(\omega^2\right) = 0$, we have

$$\begin{cases} a_1\omega\left(\omega^{t-1} - 1\right) = 1 - \omega^{t+1}, \\ a_1\omega^2\left(\omega^{2t-2} - 1\right) = 1 - \omega^{2t+2}. \end{cases}$$

Since ω is a primitive l-th root of unity, then

$$\omega\left(\omega^{t-1} + 1\right)\left(1 - \omega^{t+1}\right) = 1 - \omega^{2t+2}.$$

This implies that $\omega^t = 1$. Thus, $a_1 = -1, a_2 = 1$. By

$$t(t-1)a_2 + t(t+1)a_3 = 0,$$

we have $2t = kp$, which contradicts $\omega^t = 1$.

In order to prove that $\mathcal{C}_1$ is an AMDS symbol-pair code, we need to find a codeword with the symbol-pair weight 7. Since

$$c(x) = (x^p - 1)(x^{p-1} - 1) = x^{2p-1} - x^p - x^{p-1} + 1$$

is a codeword of $\mathcal{C}_1$ and $\omega_p(c(x)) = 7$, $\mathcal{C}_1$ is an AMDS $(lp,\ 7)_p$ symbol-pair code. □

3.2. MDS and AMDS Symbol-Pair Codes with length $3p$

In this subsection, we obtain all MDS symbol-pair codes of $d_p \leq 12$ and all AMDS symbol-pair codes of $d_p < 12$ from repeated-root cyclic codes with length $3p$. Furthermore, we discuss all minimum symbol-pair distance of the repeated-root cyclic codes with code length of $3p$, when the degree of generator ploynomials $\deg(g_{(r_1,r_2,r_3)}(x)) \leq 10$.

Let $\mathcal{C}_{(r_1,r_2,r_3)}$ be the repeated-root cyclic code over $\mathbb{F}_p$ and the generator ploynomial of $\mathcal{C}_{(r_1,r_2,r_3)}$ is

$$g_{(r_1,r_2,r_3)}(x) = (x-1)^{r_1}(x-\omega)^{r_2}(x-\omega^2)^{r_3}.$$

where ω is a primitive 3-th root of unity in $\mathbb{F}_p$ and $r_i \leq p-1, i = 1,2,3$.

Remark 3.12. *Let $C_{(r_1,r_2,r_3)} = \left\langle (x-1)^{r_1}(x-\omega)^{r_2}(x-\omega^2)^{r_3} \right\rangle$ have the same minimum symbol-pair distance, if the exponents of the three factors of the generator polynomial can be swapped.*

Proof. We first prove that such repeated-root cyclic codes

$$\widetilde{C} = \left\langle (x-\omega^i)^{r_1}(x-\omega^{i+1})^{r_2}(x-\omega^{i+2})^{r_3} \right\rangle$$

are the same codes for $i = 0, 1, 2$.

Without loss of generality, suppose that

$$\widetilde{g_1}(x) = (x-1)^{r_1}(x-\omega)^{r_2}(x-\omega^2)^{r_3},$$

$$\widetilde{g_2}(x) = (x-1)^{r_3}(x-\omega)^{r_1}(x-\omega^2)^{r_2}$$

and

$$\widetilde{g_3}(x) = (x-1)^{r_2}(x-\omega)^{r_3}(x-\omega^2)^{r_1}.$$

We denote that $\widetilde{g_1}(x)$, $\widetilde{g_2}(x)$ and $\widetilde{g_3}(x)$ represent the generator polynomials of $\widetilde{C_1}$, $\widetilde{C_2}$ and $\widetilde{C_3}$, respectively.

Let $y = \frac{x}{\omega^2}, z = \frac{x}{\omega}$, for the generator polynomial $\widetilde{g_1}(x)$ of $\widetilde{C_1}$, we can deduce the following results by deforming it.

$$\begin{aligned}
\widetilde{g_1}(x) &= (x-1)^{r_1}(x-\omega)^{r_2}(x-\omega^2)^{r_3} \\
&= \omega^{2(r_1+r_2+r_3)}\left(\frac{x}{\omega^2}-\frac{1}{\omega^2}\right)^{r_1}\left(\frac{x}{\omega^2}-\frac{\omega}{\omega^2}\right)^{r_2}\left(\frac{x}{\omega^2}-\frac{\omega^2}{\omega^2}\right)^{r_3} \\
&= \omega^{2(r_1+r_2+r_3)}(y-\omega)^{r_1}(y-\omega^2)^{r_2}(y-1)^{r_3} \\
&= \omega^{2(r_1+r_2+r_3)}\widetilde{g_2}(y) \\
&= \omega^{r_1+r_2+r_3}\left(\frac{x}{\omega}-\frac{1}{\omega}\right)^{r_1}\left(\frac{x}{\omega^2}-\frac{\omega}{\omega}\right)^{r_2}\left(\frac{x}{\omega}-\frac{\omega^2}{\omega}\right)^{r_3} \\
&= \omega^{r_1+r_2+r_3}(z-\omega^2)^{r_1}(z-1)^{r_2}(z-\omega)^{r_3} \\
&= \omega^{r_1+r_2+r_3}\widetilde{g_3}(z).
\end{aligned}$$

Thus, repeated-root cyclic codes $\widetilde{C}$ are the same codes for $i = 0, 1, 2$.

Next, since ω and ω^2 are primitive 3-th root of unity in $\mathbb{F}_p$, we have $\widetilde{C_1}$ and $\widetilde{C_4}$ have the same minimum symbol-pair distance, where the generator polynomial of $\widetilde{C_4}$ is

$$\widetilde{g_4}(x) = (x-1)^{r_1}(x-\omega)^{r_3}(x-\omega^2)^{r_2}.$$

Therefore, all $\widetilde{C}$ have the same minimum symbol-pair distance, when the exponents of the three factors exchanged with each other. □

The above Remark 3.12 shows that the exponential positions of the three factors $x-1$, $x-\omega$ and $x-\omega^2$ of the generator polynomial of $C_{(r_1,r_2,r_3)}$ have the same minimum symbol-pair distance. Without loss of generality, suppose that

$$p-1 \geq r_1 \geq r_2 \geq r_3 \geq 0$$

in the next part of this subsection. Then we have the following theorem.

Theorem 3.13. *$\mathcal{C}_{(r_1,r_2,r_3)}$ is an MDS symbol-pair codes over $\mathbb{F}_p$, if one of the following two conditions is true*

1. $r_1 \leq 5$, $0 \leq r_2 - r_3 \leq 1$ *and* $r_1 = r_2 + r_3$,
2. $r_1 < 5$, $0 \leq r_2 - r_3 \leq 1$ *and* $r_1 = r_2 + r_3 + 1$.

Researchers in [9] and [14] given some proofs of Theorem 3.13, and the Theorem 3.1 in the previous paper also includes some proofs. Here we only need to prove that $\mathcal{C}_{(4,2,1)}$ is an MDS symbol-pair code.

Proposition 3.14. *$\mathcal{C}_{(4,2,1)}$ is an MDS $(3p,\,9)_p$ symbol-pair code.*

Proof. Since $\mathcal{C}_{(4,2,1)} = \left\langle (x-1)^4(x-\omega)^2(x-\omega^2)\right\rangle$, for any codeword $c \in \mathcal{C}_{(4,2,1)}$, we have

$$c(1) = c(\omega) = c\left(\omega^2\right) = c^{(1)}(1) = c^{(1)}(\omega) = c^{(2)}(1) = c^{(3)}(1) = 0.$$

By Lemma 2.6, $\mathcal{C}_{(4,2,1)}$ is a $[3p,\,3p-7,\,5]$ cyclic code over $\mathbb{F}_p$. Since $\mathcal{C}_{(4,2,1)}$ is a subcode of Lemma 2.4, we have $d_p \geqslant 8$.

To prove that $\mathcal{C}_{(4,2,1)}$ is an MDS $(3p,\,9)_p$ symbol-pair code, it suffices to verify that there does not exist codeword in $\mathcal{C}_{(4,2,1)}$ with symbol-pair weight 8. Then we have three cases to discuss.

Case I. If there are codewords with Hamming weight 5 and symbol-pair weight 8, then its certain cyclic shift must be one of the following forms

$$(\star,\star,0_{s_1},\star,\star,0_{s_2},\star,0_{s_3})$$

or

$$(\star,\star,\star,0_{s_1},\star,0_{s_2},\star,0_{s_3}),$$

where each $\star$ denotes an element in $\mathbb{F}_p^*$ and 0_{s_1}, 0_{s_2},0_{s_3} are all-zero vectors with lengths s_1, s_2 and s_3 respectively.

Subcase 1.1. For the case of

$$(\star,\star,0_{s_1},\star,\star,0_{s_2},\star,0_{s_3}),$$

without loss of generality, suppose that the constant term of $c(x)$ is 1. We denote that

$$c(x) = 1 + a_1x + a_2x^l + a_3x^{l+1} + a_4x^t.$$

When $t \equiv 0(\bmod\ 3)$ and $l \equiv 0(\bmod\ 3)$, it follows from $c(1) = c(\omega) = c\left(\omega^2\right) = 0$ that

$$\begin{cases} 1 + a_1 + a_2 + a_3 + a_4 = 0, \\ 1 + a_1\omega + a_2 + a_3\omega + a_4 = 0, \\ 1 + a_1\omega^2 + a_2 + a_3\omega^2 + a_4 = 0. \end{cases}$$

By solving the system, we have $a_1 = -a_3$. However, $c^{(1)}(1) = c^{(1)}(\omega) = 0$ induces that

$$\begin{cases} a_1 + ta_2 + (t+1)a_3 + la_4 = 0, \\ a_1 + ta_2\omega^2 + (t+1)a_3 + la_4\omega^2 = 0. \end{cases}$$

Together with $a_1 = -a_3$, one can immediately get can get $t(1-\omega^2)a_3 = 0$, which is impossible, since $t \equiv 0(\bmod\ 3)$ and the code length is $3p$.

When $l \equiv i(\bmod\ 3)$ and $t \equiv j(\bmod\ 3)$, $i, j = 0,1,2$, values in all i and j of **Subcase 1.1** are shown in the following Table 2.

Table 2: Summary of **Subcase 1.1**

i	j	Conditions	Results	Contradictory
0	0	$[\![1]\!],[\![2]\!]$	$t(1-\omega^2)a_3=0$	$t\leq 3p-2$
0	1	$[\![1]\!],[\![2]\!],[\![3]\!]$	$\omega^2-1=0$	$\omega^3=1$
0	2	$[\![1]\!],[\![3]\!]$	$a_1+a_3+a_4=0$, $a_1+a_3-a_4=0$	$a_4\in\mathbb{F}_p^*$
1	0	$[\![1]\!],[\![3]\!]$	$a_1+a_2+a_3=0$, $a_1+a_2-a_3=0$	$a_3\in\mathbb{F}_p^*$
1	1	$[\![1]\!],[\![3]\!]$	$3=0$	$p\neq 3$
1	2	$[\![1]\!],[\![3]\!]$	$3=0$	$p\neq 3$
2	0	$[\![1]\!],[\![3]\!]$	$a_1-a_2=0$, $a_1+a_2=0$	$a_1,a_2\in\mathbb{F}_p^*$.
2	1	$[\![1]\!],[\![3]\!]$	$a_1+a_2+a_4=0$, $a_1-a_2+a_4=0$	$a_2\in\mathbb{F}_p^*$
2	2	$[\![1]\!],[\![3]\!]$	$a_1+a_2+a_4=0$, $a_1-a_2-a_4=0$	$a_1\in\mathbb{F}_p^*$

* Conditions $[\![1]\!]$, $[\![2]\!]$ and $[\![3]\!]$ represent $c(1)=c(\omega)=c\left(\omega^2\right)=0$, $c^{(1)}(1)=c^{(1)}(\omega)=0$ and $c(1)+c(\omega)+c\left(\omega^2\right)=0$, respectively.

Subcase 1.2. For the subcase of

$$(\star,\star,\star,0_{s_1},\star,0_{s_2},\star,0_{s_3}),$$

without loss of generality, suppose that the constant term of $c(x)$ is 1. We denote that

$$c(x)=1+a_1x+a_2x^2+a_3x^l+a_4x^t.$$

Suppose that $l\equiv i(\bmod\ 3)$ and $t\equiv j(\bmod\ 3)$, $i,j=0,1,2$, similar to **Subcase 1.1**, we summarize all i and j of **Subcase 1.2** in the following Table 3.

Case II. If there are codewords with Hamming weight 6 and symbol-pair weight 8, then its certain cyclic shift must be one of the following forms

$$(\star,\star,\star,\star,\star,0_{s_1},\star,0_{s_2}),$$

$$(\star,\star,\star,\star,0_{s_1},\star,\star,0_{s_2})$$

or

$$(\star,\star,\star,0_{s_1},\star,\star,\star,0_{s_2}),$$

where each $\star$ denotes an element in $\mathbb{F}_p^*$ and 0_{s_1}, 0_{s_2} are all-zero vectors with lengths s_1 and s_2 respectively.

Table 3: Summary of **Subcase 1.2**

i	j	Conditions	Results	Contradictory
0	0	$[\![1]\!],[\![2]\!]$	$a_1-a_2=0$, $a_1+a_2=0$	$a_1,a_2\in\mathbb{F}_p^*$
0	1	$[\![1]\!],[\![2]\!]$	$a_1-a_2+a_4=0$, $a_1+a_2+a_4=0$	$a_2\in\mathbb{F}_p^*$
0	2	$[\![1]\!],[\![2]\!]$	$a_1+a_2+a_4=0$, $a_1-a_2-a_4=0$	$a_1\in\mathbb{F}_p^*$
1	1	$[\![1]\!],[\![2]\!]$	$3=0$	$p\neq 3$
1	2	$[\![1]\!],[\![2]\!]$	$3=0$	$p\neq 3$
2	2	$[\![1]\!],[\![2]\!]$	$3=0$	$p\neq 3$

* Conditions $[\![1]\!]$ and $[\![2]\!]$ represent $c(1)=c(\omega)=c(\omega^2)=0$ and $c(1)+c(\omega)+c(\omega^2)=0$, respectively.

Subcase 2.1. For the subcase of

$$(\star,\star,\star,\star,\star,0_{s_1},\star,0_{s_2}),$$

without loss of generality, suppose that the constant term of $c(x)$ is 1. We denote

$$c(x)=1+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^t.$$

When $t\equiv 0\pmod 3$, it can be derived from

$$\begin{cases} c(1)=c(\omega)=c(\omega^2)=0,\\ c(1)+c(\omega)+c(\omega^2)=0,\end{cases}$$

that

$$\begin{cases} a_1+a_2+a_4=0,\\ a_1-a_2+a_4=0,\end{cases}$$

which is impossible, since $a_2\in\mathbb{F}_p^*$ and p is an odd prime.

When $t\equiv 1\pmod 3$, with arguments similar to $t\equiv 0\pmod 3$, a contradiction can be obtained from

$$\begin{cases} c(1)=c(\omega)=c(\omega^2)=0,\\ c(1)+c(\omega)+c(\omega^2)=0.\end{cases}$$

When $t\equiv 2\pmod 3$, it follows from $c(1)=c(\omega)=c(\omega^2)=0$ that

$$\begin{cases} 1+a_1+a_2+a_3+a_4+a_5=0,\\ 1+a_1\omega+a_2\omega^2+a_3+a_4\omega+a_5\omega^2=0,\\ 1+a_1\omega^2+a_2\omega+a_3+a_4\omega^2+a_5\omega=0.\end{cases}$$

By solving the system, we have $1+a_3=0, a_1+a_4=0, a_2+a_5=0$. Then $c^{(1)}(1)=c^{(1)}(\omega)=0$ indicates

$$\begin{cases} a_1+2a_2+3a_3+4a_4+ta_5=0,\\ a_1+2a_2\omega+3a_3\omega^2+4a_4+ta_5\omega^2=0,\end{cases}$$

which means that $a_5 = \frac{3a_3\omega^2}{t-2} = \frac{3a_4\omega}{t-2}$ (since $t \equiv 2(\bmod\ 3)$, then p is not a divisor of $t-2$, otherwise $t-2 \geq 3p$).
By $c^{(2)}(1) = 0$, we have $t = 3+2\omega$. Together with $a_5 = \frac{3a_3\omega^2}{t-2} = \frac{3a_4\omega}{t-2}$ and $c^{(3)}(1) = 0$, one can derive that

$$6+24\omega+3t(t-1)\omega^2 = 0.$$

Then we have

$$2+8\omega+(3+2\omega)(2+2\omega)\omega^2 = 0.$$

Combining with $\omega^2 = -1-\omega$, we can obtain $3\omega^2 = 0$. This is impossible.

Subcase 2.2. For the subcase of

$$(\star,\star,\star,\star,0_{s_1},\star,\star,0_{s_2}),$$

without loss of generality, suppose that the constant term of $c(x)$ is 1. We denote that

$$c(x) = 1+a_1x+a_2x^2+a_3x^3+a_4x^t+a_5x^{t+1}.$$

When $t \equiv 0(\bmod\ 3)$ and $t \equiv 2(\bmod\ 3)$, with arguments similar to the previous $t \equiv 0(\bmod\ 3)$ of **Subcase 2.1**, a contradiction can be derived from $c(1) = c(\omega) = c\left(\omega^2\right) = 0$ again.

When $t \equiv 1(\bmod\ 3)$, with arguments similar to $t \equiv 2(\bmod\ 3)$ of **Subcase 2.1**, $1+a_3 = 0, a_1+a_4 = 0$, and $a_2+a_5 = 0$ can be obtained from $c(1) = c(\omega) = c\left(\omega^2\right) = 0$.
Then $a_5 = a_4\omega = \frac{3a_3\omega^2}{t-1}$ can be derived from $c^{(1)}(1) = c^{(1)}(\omega) = 0$. $c^{(2)}(1) = 0$ means $t = -\omega$. Finally, combined with $c^{(3)}(1) = 0$, we have $\omega^2-\omega = 0$, a contradiction again.

Subcase 2.3. For the subcase of

$$(\star,\star,\star,0_{s_1},\star,\star,\star,0_{s_2}),$$

without loss of generality, suppose that the constant term of $c(x)$ is 1. We denote that

$$c(x) = 1+a_1x+a_2x^2+a_3x^t+a_4x^{t+1}+a_5x^{t+2}.$$

When $t \equiv 0(\bmod\ 3)$, it follows from $c(1) = c(\omega) = c\left(\omega^2\right) = 0$ that

$$\begin{cases} 1+a_1+a_2+a_3+a_4+a_5 = 0, \\ 1+a_1\omega+a_2\omega^2+a_3+a_4\omega+a_5\omega^2 = 0, \\ 1+a_1\omega^2+a_2\omega+a_3+a_4\omega^2+a_5\omega = 0. \end{cases}$$

By solving the system, we have $1+a_3 = 0, a_1+a_4 = 0, a_2+a_5 = 0$. Then $c^{(1)}(1) = c^{(1)}(\omega) = 0$ indicates

$$\begin{cases} a_1+2a_2+ta_3+(t+1)a_4+(t+2)a_5 = 0, \\ a_1+2a_2\omega+ta_3\omega^2+(t+1)a_4+(t+2)a_5\omega = 0, \end{cases}$$

which means that $a_5 = a_4\omega = a_3\omega^2$. Then $c^{(2)}(1) = 0$ implies that

$$2a_2+t(t-1)a_3+t(t+1)a_4+(t+1)(t+2)a_5 = 0,$$

which implies $t(3\omega^2+\omega-1 = 0)$. Since $t \equiv 0(\bmod 3), \omega = -\omega^2-1$ and the code length $3p$, we have $2(\omega^2-1) = 0$, a contradiction.

When $t \equiv 1 \pmod{3}$, with arguments similar to the $t \equiv 0 \pmod{3}$ of **Subcase 2.3**, by

$$\begin{cases} c(1) = c(\omega) = c\left(\omega^2\right) = 0, \\ c^{(1)}(1) = c^{(1)}(\omega) = 0, \end{cases}$$

we have $a_4 = a_3\omega = \frac{a_5\omega^2(t+2)}{t-1}$. Together with $c^{(2)}(1) = 0$, $\omega^2 - \omega = 0$ can be derived, which is impossible.

When $t \equiv 2 \pmod{3}$, similarly, we can derive $a_5 = a_4\omega = \frac{a_3\omega^2(t-2)}{t+1}$ from

$$\begin{cases} c(1) = c(\omega) = c\left(\omega^2\right) = 0, \\ c^{(1)}(1) = c^{(1)}(\omega) = 0. \end{cases}$$

Then, together with $c^{(2)}(1) = 0$, we have $t = p+1$. It follows from $c^{(1)}(1) = c^{(1)}(\omega) = 0$ that

$$\begin{cases} (t-1)a_3 + (t+1)a_4 + (t+1)a_5 = 0, \\ (t-1)a_3\omega + (t+1)a_4\omega^2 + (t+1)a_5 = 0. \end{cases}$$

Combined with $t = p+1$, we have $\omega^2 = 1$, which contradicts that ω is primitive 3-th root of unity in $\mathbb{F}_p$.

Case III. If there are codewords in $\mathcal{C}$ with Hamming weight 7 and symbol-pair weight 8, then its certain cyclic shift must have the form

$$(\star, \star, \star, \star, \star, \star, \star, 0_s),$$

where each $\star$ denotes an element in $\mathbb{F}_p^*$ and 0_s is all-zero vector of length s. Without loss of generality, suppose that the constant term of $c(x)$ is 1. We denote

$$c(x) = 1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6.$$

This leads to $\deg(c(x)) = 6 < 7 = \deg(g(x))$.

As a result, $\mathcal{C}_{(4,2,1)}$ is an MDS $(3p,\ 9)_p$ symbol-pair code. □

Based on Reference [9], [12], [14] , Remark 3.12, Proposition 3.14 to Proposition 3.16 and Corollary 3.17 in this paper, all known MDS symbol-pair codes with $n-k \leq 10$ from repeated-root cyclic codes $\mathcal{C}_{(r_1,r_2,r_3)}$, which are listed in the following Table 4.

Next, we will explain that there is no MDS symbol pair code except in Table 4, when the degree of generator polynomials $g_{(r_1,r_2,r_3)}(x)$ does not exceed 10.

Similarly, all AMDS symbol-pair codes with $d_p < 12$ can be deduced from the following propositions. We list all AMDS symbol-pair codes in the following Table 5.

In what follows, let's determine the minimum symbol-pair distance for $\mathcal{C}_{(r_1,r_2,r_3)}$ in the previous paper by using the following propositions.

Proposition 3.15.

1. *The minimum distance of* $\mathcal{C}_{(0,r_2,0)}$ *is 4, when* $2 \leq r_2 \leq p-1$.
2. *The minimum distance of* $\mathcal{C}_{(2,1,0)}$ *is 5.*
3. *The minimum distance of* $\mathcal{C}_{(r_1,r_2,0)}$ *is 6, when* $r_1 + r_2 \geq 4, r_2 \geq 1$.
4. *The minimum distance of* $\mathcal{C}_{(2,r_2,r_3)}$ *is 6, when* $2 \leq r_2 + r_3 \leq 4$.

Table 4: All MDS symbol-pair codes with length $3p$ for $d_p \leq 12$

r_1	r_2	r_3	$(n-k,d_p)_p$	Reference or Proposition
0	2	0	$(2,4)_p$	Trivially($r_1 = 0$)
2	1	0	$(3,5)_p$	Reference [9]
2	1	1	$(4,6)_p$	Reference [9]
3	1	0	$(4,6)_p$	Reference [12]
3	1	1	$(5,7)_p$	Reference [9]
3	2	1	$(6,8)_p$	Reference [9]
4	2	2	$(8,10)_p$	Reference [14]
5	3	2	$(10,12)_p$	Reference [14]
2	2	0	$(4,6)_p$	Proposition 3.4
4	2	1	$(7,9)_p$	Proposition 3.14

* These MDS symbol-pair codes are constructed by repeated-root cyclic codes.

Proof. For $\mathcal{C}_{(0,r_2,0)} = \langle (x-\omega)^{r_2} \rangle$, Lemma 2.6 shows $d_H(\mathcal{C}_{(0,r_2,0)}) = 2$, then one can obtain $d_p(\mathcal{C}_{(0,r_2,0)}) = 4$ by Lemma 2.3.

Corollary 3.7 proves the minimum distance of $\mathcal{C}_{(2,1,0)}$ is $d_p = 5$.

Since $\mathcal{C}_{(r_1,r_2,0)}$ is a subcode of Theorem 3.7 and Lemma 2.6 means $d_H(\mathcal{C}_{(r_1,r_2,0)}) = 3$. Then we have $d_p(\mathcal{C}_{(r_1,r_2,0)}) = 6$.

Since $\mathcal{C}_{(2,r_2,r_3)}$ is a subcode of $\mathcal{C}_{(2,1,1)}$ and $\mathcal{C}_{(2,1,1)}$ is an MDS symbol-pair code with the mimnmum symbol-pair distance 6, when $2 \leq r_1 + r_2 \leq 4$ and $r_1, r_2 \in \mathbb{F}_p^*$. Then we can deduce $d_p(\mathcal{C}_{(2,r_2,r_3)}) = 6$. □

We can obtain all MDS and AMDS symbol-pair codes with length $3p$ with a minimum symbol-pair distance of 4 to 6 using Proposition 3.15. Then, we check for MDS and AMDS symbol-pair codes with a symbol pair distance of 7 to 12.

Proposition 3.16.

1. *The minimum distance of* $\mathcal{C}_{(r_1,1,1)}$ *is 7, when* $3 \leq r_1 \leq p-1$.
2. *The minimum distance of* $\mathcal{C}_{(3,r_2,r_3)}$ *is 8, when* $3 \leq r_2 + r_3 \leq 6, r_3 \geq 1$.
3. *The minimum distance of* $\mathcal{C}_{(r_1,r_2,1)}$ *is 9, when* $2 \leq r_2 \leq r_1 \leq p-1, r_1 \geq 4$.
4. *The minimum distance of* $\mathcal{C}_{(r_1,r_2,r_3)}$ *is 10, when* r_1*,* r_2 *and* r_3 *meet any of the following two conditions*
   - $4 \leq r_1 \leq p-1$ *and* $r_2 = r_3 = 2$*,*
   - $r_1 = 4$ *and* $4 \leq r_2 + r_3 \leq 8, r_3 \geq 1$.

Proof. In reference [9], $\mathcal{C}_{(2,1,1)}$, $\mathcal{C}_{(3,1,1)}$ and $\mathcal{C}_{(3,2,1)}$ are MDS symbol-pair codes with symbol-pair distances of 6, 7 and 8, respectively. Reference [14] proved $\mathcal{C}_{(4,2,2)}$ and $\mathcal{C}_{(5,3,2)}$ are MDS symbol-pair codes with symbol-pair distances of 10 and 12, respectively.

Table 5: All AMDS symbol-pair codes with length $3p$ for $d_p < 12$

r_1	r_2	r_3	$(n-k,d_p)_p$	Reference or Proposition
4	3	2	$(9,10)_p$	Reference [14]
0	3	0	$(3,4)_p$	Proposition 3.15
2	2	1	$(5,6)_p$	Proposition 3.15
3	2	0	$(5,6)_p$	Proposition 3.15
3	2	2	$(7,8)_p$	Proposition 3.16
3	3	1	$(7,8)_p$	Proposition 3.16
4	1	0	$(5,6)_p$	Proposition 3.15
4	1	1	$(6,7)_p$	Proposition 3.16
4	3	1	$(8,9)_p$	Proposition 3.16
5	2	1	$(8,9)_p$	Proposition 3.16
5	2	2	$(9,10)_p$	Proposition 3.16

* These AMDS symbol-pair codes are constructed by repeated-root cyclic codes.

1. Since $\mathcal{C}_{(r_1,1,1)}$ is a subcode of $\mathcal{C}_{(3,1,1)}$, we have $d_p(\mathcal{C}_{(r_1,1,1)}) \geq 7$. Then the minimum symbol-pair distance $d_p(\mathcal{C}_{(r_1,1,1)}) = 7$ can be obtained by $\omega_p(c(x)) = 7$, where
$$c(x) = 1 - x - x^p + x^{2p+1}$$
is a codeword of $\mathcal{C}_{(r_1,1,1)}$.
2. Reference [9] proved that $\mathcal{C}_{(3,2,1)}$ is an MDS $(3p,8)_p$ symbol-pair code. Since $3 \leq r_2+r_3 \leq 6$, we have $\mathcal{C}_{(3,r_2,r_3)}$ is subcode of $\mathcal{C}_{(3,2,1)}$ and $d_p(\mathcal{C}_{(3,r_2,r_3)}) \geq 8$.
By Lemma 2.6, we can deduce that the minimum Hamming distance of $\mathcal{C}_{(3,r_2,r_3)}$ is 4, which implies $d_p(\mathcal{C}_{(3,r_2,r_3)}) \leq 8$.
Therefore, the minimum distance of $\mathcal{C}_{(3,r_2,r_3)}$ is $d_p = 8$, when $3 \leq r_2+r_3 \leq 6, r_3 \geq 1$.
3. With arguments similar as the proof of case, since $\mathcal{C}_{(4,2,1)}$ is an MDS symbol-pair code with the minimum symbol-pair distance 9, we can deduce that the minimum symbol-pair distance of $\mathcal{C}_{(r_1,r_2,1)}$ is $d_p(\mathcal{C}_{(r_1,r_2,1)}) \geq 9$. Next, we prove that there are symbol-pair codewords with symbol-pair weight $d_p = 9$ in $\mathcal{C}_{(r_1,r_2,1)}$.
For the codeword
$$c(x) = (x-1)^p(x-\omega)^p(x-\omega^2) = x^{2p+1} - \omega^2 x^{2p} + \omega^2 x^{p+1} - \omega x^p + \omega x - 1,$$
it is easy to verify that $c(x)$ is a codeword polynomial of $\mathcal{C}_{(r_1,r_2,1)}$ with the symbol-pair weight 9. Thus, we have $d_p(\mathcal{C}_{(r_1,r_2,1)}) = 9$.
4. For $4 \leq r_1 \leq p-1$ and $r_2 = r_3 = 2$, with arguments similar as the proof of case, since $\mathcal{C}_{(4,2,2)}$ is an MDS symbol-pair code with $d_p = 10$ and $\mathcal{C}_{(r_1,2,2)}$ is a subcode of $\mathcal{C}_{(4,2,2)}$, we have $d_p(\mathcal{C}_{(r_1,2,2)}) \geq 10$. Note that the codeword polynomial
$$c(x) = 1 - x^2 + 2x^{p+1} + x^{p+2} - x^{2p} - 2x^{2p+1}$$

is a codeword of $\mathcal{C}_{(r_1,2,2)}$ and $\omega_p(c(x)) = 10$. Therefore, we derive the minimum symbol-pair distance of $\mathcal{C}_{(r_1,2,2)}$ is 10.
For $r_1 = 4$ and $4 \le r_2 + r_3 \le 8$, since $\mathcal{C}_{(4,r_2,r_3)}$ is a subcode of $\mathcal{C}_{(4,2,2)}$, we have $d_p(\mathcal{C}_{(4,r_2,r_3)}) \ge 10$. However, Lemma 2.6 shows that $d_H(\mathcal{C}_{(4,r_2,r_3)}) = 5$, which implies that $d_p(\mathcal{C}_{(4,r_2,r_3)}) \le 10$. Thus, we can deduce $d_p(\mathcal{C}_{(4,r_2,r_3)}) = 10$.

□

According to Proposition 3.16, we can obtain all MDS and AMDS symbol-pair codes with length $3p$ with a minimum symbol-pair distance of 7 to 10. We can also deduce that $\mathcal{C}_{(r_1,r_2,r_3)}$ is an MDS symbol-pair code with $d_p = 12$, iff $(r_1, r_2, r_3) = (5,3,2)$. Furthermore, there dose not exist codeword with a minimum symbol pair distance of 11, when the degree of the generator polynomials $\deg(g_{(r_1,r_2,r_3)}(x))$ are 9 and 10. The following corollary can be drawn.

Corollary 3.17. *The repeated-root cyclic code $\mathcal{C}_{(r_1,r_2,r_3)}$ must not be the MDS and the AMDS symbol-pair code with a minimum symbol-pair distance 11.*

Based on the above Proposition 3.14 to Proposition 3.16, Corollary 3.17 and Remark 3.12, we complete the proof of Theorem 3.13. In addition, we also explain that the MDS symbol-pair codes in Table 4 and the AMDS symbol-pair codes in Table 5 are all cases that meet the requirements.

Proposion 3.16 shows that the condition does not satisfy Theorem 3.13, when $r \ge 5$ and $r_1 = r_2 + r_3 + 1$. Next, we use an example to illustrate that the conditions of Theorem 3.13 are also no longer applicable, when $r > 5$ and $r_1 = r_2 + r_3$.

Example 3.18. *Let $\mathcal{C}$ and be a repeated-root cyclic code over $\mathbb{F}_7$ and the generator ploynomial of $\mathcal{C}$ is*

$$g(x) = (x-1)^6(x-2)^3(x-4)^3,$$

where $\omega = 2$ is a 3-th primitive element in $\mathbb{F}_7$ and $2^2 = 4$.

Then we have the minimum Hamming distance $d_H = 7$ by a magma progarm. Reference [14] shows the symbol-pair distance $d_p \ge 12$. The magma program also shows that both vectors

$$\boldsymbol{a} = [0\,0\,1\,0\,0\,0\,0\,0\,0\,6\,6\,3\,3\,3\,4\,4\,4\,5\,1\,1\,1]$$

and

$$\boldsymbol{b} = [0\,0\,0\,0\,0\,0\,1\,0\,0\,1\,1\,1\,5\,4\,4\,4\,3\,3\,3\,6\,6]$$

are in $\mathcal{C}$. Let $\boldsymbol{c} = \boldsymbol{a}+\boldsymbol{b}$, we have

$$\boldsymbol{c} = [0\,0\,1\,0\,0\,0\,1\,0\,0\,0\,0\,4\,1\,0\,1\,1\,0\,1\,4\,0\,0],$$

which is also in $\mathcal{C}$. We can easily deduce $\omega_p(\boldsymbol{c}) = 13$. Therefore, $\mathcal{C}$ is not an MDS symbol-pair code.

4. Conclusion

In this paper, employing repeated-root cyclic codes, some new classes of MDS and AMDS symbol-pair codes over $\mathbb{F}_p$ with lengths lp and $3p$ are provided. We give some more general generator polynomials about MDS $(lp,5)_p$ and $(lp,6)_p$ symbol-pair codes. We also present a class of AMDS $(lp,7)_p$ symbol-pair codes. For length $3p$, we provide all MDS symbol-pair codes with $d_p \le 12$ and also provide all AMDS symbol-pair codes with $d_p < 12$.

References

[1] Y. Cassuto and M. Blaum, Codes for symbol-pair read channels, in Proc. IEEE Int. Symp. Inf. Theory (ISIT2010) 988-992.

[2] Y. Cassuto and M. Blaum, Codes for symbol-pair read channels, IEEE Trans. Inf. Theory 57(12)(2011) 8011–8020.

[3] Y. Cassuto and S. Litsyn, Symbol-pair codes: Algebraic constructions and asymptotic bounds, in Proc. IEEE Int. Symp. Inf. Theory, (ISIT2011) 2348–2352.

[4] Y. M. Chee, H. M. Kiah, and C. Wang, "Maximum distance separable symbol-pair codes, in Proc. IEEE Int. Symp. Inf. Theory (ISIT2012) 2886-2890.

[5] Y. M. Chee, L. Ji, H. M. Kiah, C. Wang, and J. Yin, Maximum distance separable codes for symbol-pair read channels, IEEE Trans. Inf. Theory 59(11)(2013) 7259-7267.

[6] X. Kai, S. Zhu, and P. Li, A construction of new MDS symbol-pair codes, IEEE Trans. Inf. Theory 61(11)(2015) 5828-5834.

[7] X. Kai, S. Zhu, Y. Zhao, H. Luo, and Z. Chen, New MDS symbol-pair codes from repeated-root codes, IEEE Commun. Letters 22(3)(2018) 462-465.

[8] S. Li and G. Ge, Constructions of maximum distance separable symbol-pair codes using cyclic and constacyclic codes, Des. Codes Cryptogr 84(3)(2017) 359-372.

[9] B. Chen, L. Lin, and H. Liu, Constacyclic symbol-pair codes: lower bounds and optimal constructions, IEEE Trans. Inf. Theory 63(12)(2017) 7661-7666.

[10] H. Q. Dinh, B. T. Nguyen, A. K. Singh, and S. Sriboonchitta, On the symbol-pair distance of repeated-root constacyclic codes of prime power lengths, IEEE Trans. Inf. Theory 64(4)(2018) 2417–2430.

[11] H. Q. Dinh, B. T. Nguyen and S. Sriboonchitta , MDS Symbol-Pair Cyclic Codes of Length $2p^s$ over $\mathbb{F}_{p^m}$, IEEE Transactions on Information Theory 66(1)(2020) 240-262.

[12] W. Zhao, Research on algebraic theory of several types of error-correcting codes, Thesis (Ph.D.), South China University of Technology, 2019.

[13] W. Zhao, S. Yang, K. W. Shum and X. Tang, Repeated-Root Constacyclic Codes With Pair-Metric, IEEE Commun. Letters 25(8)(2021), 2502-2506.

[14] J. Ma and J. Luo, MDS symbol-pair codes from repeated-root cyclic codes, Des. Codes Cryptogr 90(1)(2022) 121-137.

[15] G. Castagnoli, J. L. Massey, P. A. Schoeller, and N. von Seemann, On repeated-root cyclic codes, IEEE Trans. Inf. Theory 37(2)(1991) 337-342.