Algebras over a symmetric fusion category and integrations

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Abstract

We study the symmetric monoidal 2-category of finite semisimple module categories over a symmetric fusion category. In particular, we study E_n-algebras in this 2-category and compute their E_n-centers for $n = 0, 1, 2$. We also compute the factorization homology of stratified surfaces with coefficients given by E_n-algebras in this 2-category for $n = 0, 1, 2$ satisfying certain anomaly-free conditions.

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1 Introduction

The mathematical theory of factorization homology is a powerful tool in the study of topological quantum field theories (TQFT). It was first developed by Lurie [L] under the name of 'topological chiral homology', which records its origin from Beilinson and Drinfeld's theory of chiral homology [BD, FG]. It was further developed by many people (see for example [CG, AF1, AFT1, AFT2, AFR, BBJ1, BBJ2]) and gained its current name from Francis [F].

Although the general theory of factorization homology has been well established, explicitly computing the factorization homology in any concrete examples turns out to be a non-trivial challenge. On a connected compact 1-dimensional manifold (or a 1-manifold), i.e. S^1, the factorization homology is just the usual Hochschild homology. On a compact 2-manifold, the computation is already highly nontrivial (see for example [BBJ1, BBJ2, AF2]). Motivated by the study of topological orders in condensed matter physics, Ai, Kong and Zheng carried out in [AKZ] the computation of perhaps the simplest (yet non-trivial) kind of factorization homology, i.e. integrating a unitary modular tensor category (UMTC) $\mathcal{A}$ (viewed as an E_2-algebra) over a compact 2-manifold Σ, denoted by $\int_\Sigma \mathcal{A}$. In physics, the category $\mathcal{A}$ is the category of anyons (or particle-like topological defects) in a 2d (spatial dimension) anomaly-free topological order (see [W] for a review). The result of this integration is a global observable defined on Σ. It turns out that this global observable is precisely the ground state degeneracy (GSD) of the 2d topological order on Σ. This fact remains to be true even if we introduce defects of codimension 1 and 2 as long as these defects are also anomaly-free. Mathematically, this amounts to computing the factorization homology on a disk-stratified 2-manifold with coefficient defined by assigning to each 2-cell a unitary modular tensor category, to each 1-cell a unitary fusion category (an E_1-algebra) and to each 0-cell an E_0-algebra, satisfying certain anomaly-free conditions (see [AKZ, Sec. 4]).

If the category $\mathcal{A}$ is not modular, i.e. the associated topological order is anomalous, the integral $\int_\Sigma \mathcal{A}$ gives a global observable beyond GSD. Mathematically, it is interesting to compute $\int_\Sigma \mathcal{A}$ for any braided monoidal category $\mathcal{A}$. In this work, we focus on a special situation that also has a clear physical meaning. It was shown in [LKW], a finite onsite symmetry of a 2d symmetry enriched topological (SET) order can be mathematically described by a symmetric fusion category $\mathcal{E}$, and the category of anyons in this SET order can be described by a UMTC over $\mathcal{E}$, which is roughly a unitary braided fusion category with Müger center given by $\mathcal{E}$ (see Def. 5.6 for a precise definition). This motivates us to compute the factorization homology on 2-manifolds but valued in the symmetric monoidal 2-category of finite semisimple module categories over $\mathcal{E}$, denoted by $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$. The symmetric tensor product in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ is defined by the relative tensor product $\boxtimes_{\mathcal{E}}$. We first study E_i-algebras in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ and their E_i-centers for $i = 0, 1, 2$. Then we derive the anomaly-free conditions for E_i-algebras in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ for $i = 0, 1, 2$. In the end, we compute the factorization homology on disk-stratified

2-manifolds with coefficients defined by assigning anomaly-free E_i-algebras in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ to each i-cells for $i = 0, 1, 2$. The main results of this work are Thm. 5.29, Thm. 5.30 and Thm. 5.32.

The layout of this paper is as follows. In Sec. 2, we introduce the tensor product $\boxtimes_{\mathcal{E}}$ and the symmetric monoidal 2-category $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$. In Sec. 3, we study E_i-algebras in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ and compute their E_i-centers for $i = 0, 1, 2$. In Sec. 4, we study the modules over a multifusion category over $\mathcal{E}$ and modules over a braided fusion category over $\mathcal{E}$. And we prove that two fusion categories over $\mathcal{E}$ are Morita equivalent in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ if and only if their E_1-centers are equivalent. In Sec. 5, we recall the theory of factorization homology and compute the factorization homology of stratified surfaces with coefficients given by E_i-algebras in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ for $i = 0, 1, 2$ satisfying certain anomaly-free conditions.

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2 The symmetric monoidal 2-category $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$

Notation 2.1. All categories considered in this paper are small categories. Let $\Bbbk$ be an algebraically closed field of characteristic zero. Let $\mathcal{E}$ be a symmetric fusion category over $\Bbbk$ with a braiding r. The category $\mathcal{V}\mathrm{ec}$ denotes the category of finite dimensional vector spaces over $\Bbbk$ and $\Bbbk$-linear maps.

Let $\mathcal{A}$ be a monoidal category. We denote $\mathcal{A}^{\mathrm{op}}$ the monoidal category which has the same tensor product of $\mathcal{A}$, but the morphism space is given by $\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(a, b) := \mathrm{Hom}_{\mathcal{A}}(b, a)$ for any objects $a, b \in \mathcal{A}$, and $\mathcal{A}^{\mathrm{rev}}$ the monoidal category which has the same underlying category $\mathcal{A}$ but equipped with the reversed tensor product $a \otimes^{\mathrm{rev}} b := b \otimes a$ for $a, b \in \mathcal{A}$. A monoidal category $\mathcal{A}$ is rigid if every object $a \in \mathcal{A}$ has a left dual a^L and a right dual a^R. The duality functors $\delta^L : a \mapsto a^L$ and $\delta^R : a \mapsto a^R$ induce monoidal equivalences $\mathcal{A}^{\mathrm{op}} \simeq \mathcal{A}^{\mathrm{rev}}$.

A braided monoidal category $\mathcal{A}$ is a monoidal category $\mathcal{A}$ equipped with a braiding $c_{a,b} : a \otimes b \to b \otimes a$ for any $a, b \in \mathcal{A}$. We denote $\overline{\mathcal{A}}$ the braided monoidal category which has the same monoidal category of $\mathcal{A}$ but equipped with the anti-braiding $\bar{c}_{a,b} = c^{-1}_{b,a}$.

A fusion subcategory of a fusion category we always mean a full tensor subcategory closed under taking of direct summands. Any fusion category $\mathcal{A}$ contains a trivial fusion subcategory $\mathcal{V}\mathrm{ec}$.

2.1 Module categories

Let $\mathrm{Cat}^{\mathrm{fs}}$ be the 2-category of finite semisimple $\Bbbk$-linear abelian categories, $\Bbbk$-linear functors, and natural transformations. The 2-category $\mathrm{Cat}^{\mathrm{fs}}$ equipped with Deligne's tensor product $\boxtimes$, the unit $\mathcal{V}\mathrm{ec}$ is a symmetric monoidal 2-category.

Let $\mathcal{C}, \mathcal{D}$ be multifusion categories. We define the 2-category $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}})$ as follows.

- Its objects are left $\mathcal{C}$-modules in $\mathrm{Cat}^{\mathrm{fs}}$. A left $\mathcal{C}$-module $\mathcal{M}$ in $\mathrm{Cat}^{\mathrm{fs}}$ is an object $\mathcal{M}$ in $\mathrm{Cat}^{\mathrm{fs}}$ equipped with a $\Bbbk$-bilinear functor $\odot : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$, a natural isomorphism $\lambda_{c,c',m} : (c \otimes c') \odot m \simeq c \odot (c' \odot m)$, and a unit isomorphism $l_m : \mathbb{1}_{\mathcal{C}} \odot m \simeq m$ for all $c, c' \in \mathcal{C}, m \in \mathcal{M}$ and the tensor unit $\mathbb{1}_{\mathcal{C}} \in \mathcal{C}$ satisfying some natural conditions.

- Its 1-morphisms are left $\mathcal{C}$-module functors. For left $\mathcal{C}$-modules $\mathcal{M}$, $\mathcal{N}$ in $\mathrm{Cat}^{\mathrm{fs}}$, a left $\mathcal{C}$-module functor from $\mathcal{M}$ to $\mathcal{N}$ is a pair (F, s^F), where $F : \mathcal{M} \to \mathcal{N}$ is a $\Bbbk$-linear functor and $s^F_{c,m} : F(c \odot m) \simeq c \odot F(m)$, $c \in \mathcal{C}$, $m \in \mathcal{M}$, is a natural isomorphism, satisfying some natural conditions.

- Its 2-morphisms are left $\mathcal{C}$-module natural transformations. A left $\mathcal{C}$-module natural transformation between two left $\mathcal{C}$-module functors $(F, s^F), (G, s^G) : \mathcal{M} \rightrightarrows \mathcal{N}$ is a natural transformation $\alpha : F \Rightarrow G$ such that the following diagram commutes for $c \in \mathcal{C}, m \in \mathcal{M}$:

$$\begin{tikzcd} F(c \odot m) \arrow[r, "s^F"] \arrow[d, "\alpha_{c\odot m}"'] & c \odot F(m) \arrow[d, "1\odot\alpha_m"] \\ G(c \odot m) \arrow[r, "s^G"] & c \odot G(m) \end{tikzcd} \tag{2.1}$$

Similarly, one can define the 2-category $\mathrm{RMod}_{\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}})$ of right $\mathcal{D}$-modules in $\mathrm{Cat}^{\mathrm{fs}}$ and the 2-category $\mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}})$ of $\mathcal{C}$-$\mathcal{D}$ bimodules in $\mathrm{Cat}^{\mathrm{fs}}$. We use $\mathrm{Fun}(\mathcal{M}, \mathcal{N})$ to denote the category of $\Bbbk$-linear functors from $\mathcal{M}$ to $\mathcal{N}$ and natural transformations. We use $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ (or $\mathrm{Fun}_{|\mathcal{C}}(\mathcal{M}, \mathcal{N})$) to denote the category of left (or right) $\mathcal{C}$-module functors from $\mathcal{M}$ to $\mathcal{N}$ and left (or right) $\mathcal{C}$-module natural transformations.

Remark 2.2. There is a bijective correspondence between $\Bbbk$-linear categories (or $\Bbbk$-linear functors) and $\mathcal{V}\mathrm{ec}$-modules (or $\mathcal{V}\mathrm{ec}$-module functors). For objects $\mathcal{C}, \mathcal{M}$ in $\mathrm{Cat}^{\mathrm{fs}}$, if $\odot : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ is a $\Bbbk$-bilinear functor, it is a balanced $\mathcal{V}\mathrm{ec}$-module functor. And a $\Bbbk$-bilinear functor $\odot : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ is equivalent to a $\Bbbk$-linear functor $\mathcal{C} \boxtimes \mathcal{M} \to \mathcal{M}$ by the universal functor $\boxtimes : \mathcal{C} \times \mathcal{M} \to \mathcal{C} \boxtimes \mathcal{M}$.

2.2 Tensor product

The following definitions are standard (see for example [ENO, Def. 3.1], [KZ, Def. 2.2.1]).

Definition 2.3. Let $\mathcal{M} \in \mathrm{RMod}_{\mathcal{E}}(\mathrm{Cat}^{\mathrm{fs}})$, $\mathcal{N} \in \mathrm{LMod}_{\mathcal{E}}(\mathrm{Cat}^{\mathrm{fs}})$ and $\mathcal{D} \in \mathrm{Cat}^{\mathrm{fs}}$. A *balanced $\mathcal{E}$-module functor* is a $\Bbbk$-bilinear functor $F : \mathcal{M} \times \mathcal{N} \to \mathcal{D}$ equipped with a natural isomorphism $b_{m,e,n} : F(m \odot e, n) \simeq F(m, e \odot n)$ for $m \in \mathcal{M}, n \in \mathcal{N}, e \in \mathcal{E}$, called the *balanced $\mathcal{E}$-module structure* on F, such that the diagram

$$\begin{tikzcd} F(m \odot (e_1 \otimes e_2), n) \arrow[rr, "b_{m,e_1\otimes e_2,n}"] \arrow[d, "\simeq"'] & & F(m, (e_1 \otimes e_2) \odot n) \arrow[d, "\simeq"] \\ F((m \odot e_1) \odot e_2, n) \arrow[r, "b_{m\odot e_1,e_2,n}"] & F(m \odot e_1, e_2 \odot n) \arrow[r, "b_{m,e_1,e_2\odot n}"] & F(m, e_1 \odot (e_2 \odot n)) \end{tikzcd} \tag{2.2}$$

commutes for $e_1, e_2 \in \mathcal{E}, m \in \mathcal{M}, n \in \mathcal{N}$.

A *balanced $\mathcal{E}$-module natural transformation* between two balanced $\mathcal{E}$-module functors $F, G : \mathcal{M} \times \mathcal{N} \rightrightarrows \mathcal{D}$ is a natural transformation $\alpha : F \Rightarrow G$ such that the diagram

$$\begin{tikzcd} F(m \odot e, n) \arrow[r, "b^F_{m,e,n}"] \arrow[d, "\alpha_{m\odot e,n}"'] & F(m, e \odot n) \arrow[d, "\alpha_{m,e\odot n}"] \\ G(m \odot e, n) \arrow[r, "b^G_{m,e,n}"'] & G(m, e \odot n) \end{tikzcd}$$

commutes for all $m \in \mathcal{M}, e \in \mathcal{E}, n \in \mathcal{N}$, where b^F and b^G are the balanced $\mathcal{E}$-module structures on F and G respectively. We use $\mathrm{Fun}^{\mathrm{bal}}_{\mathcal{E}}(\mathcal{M}, \mathcal{N}; \mathcal{D})$ to denote the category of balanced $\mathcal{E}$-module functors from $\mathcal{M} \times \mathcal{N}$ to $\mathcal{D}$, and balanced $\mathcal{E}$-module natural transformations.

Definition 2.4. Let $\mathcal{M} \in \mathrm{RMod}_{\mathcal{E}}(\mathrm{Cat}^{\mathrm{fs}})$ and $\mathcal{N} \in \mathrm{LMod}_{\mathcal{E}}(\mathrm{Cat}^{\mathrm{fs}})$. The *tensor product* of $\mathcal{M}$ and $\mathcal{N}$ over $\mathcal{E}$ is an object $\mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N}$ in $\mathrm{Cat}^{\mathrm{fs}}$, together with a balanced $\mathcal{E}$-module functor $\boxtimes_{\mathcal{E}} : \mathcal{M} \times \mathcal{N} \to \mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N}$, such that, for every object $\mathcal{D}$ in $\mathrm{Cat}^{\mathrm{fs}}$, composition with $\boxtimes_{\mathcal{E}}$ induces an equivalence of categories $\mathrm{Fun}(\mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N}, \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{bal}}_{\mathcal{E}}(\mathcal{M}, \mathcal{N}; \mathcal{D})$.

Remark 2.5. The tensor product of $\mathcal{M}$ and $\mathcal{N}$ over $\mathcal{E}$ is an object $\mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N}$ in $\mathrm{Cat}^{\mathrm{fs}}$ unique up to equivalence, together with a balanced $\mathcal{E}$-module functor $\boxtimes_{\mathcal{E}} : \mathcal{M} \times \mathcal{N} \to \mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N}$, such that for every object $\mathcal{D}$ in $\mathrm{Cat}^{\mathrm{fs}}$, for any $f \in \mathrm{Fun}^{\mathrm{bal}}_{\mathcal{E}}(\mathcal{M}, \mathcal{N}; \mathcal{D})$, there exists a pair $(\underline{f}, \eta)$ unique up to isomorphism, such that $f \simeq^{\eta} \underline{f} \circ \boxtimes_{\mathcal{E}}$, i.e.

$$\begin{tikzcd}
\mathcal{M} \times \mathcal{N} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[dr, "f"'] & \mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N} \arrow[d, "\exists ! \underline{f}"] \arrow[dl, phantom, "\eta \Nearrow"] \\
& \mathcal{D}
\end{tikzcd}$$

where $\underline{f}$ is a $\Bbbk$-linear functor in $\mathrm{Fun}(\mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N}, \mathcal{D})$, and $\eta : f \Rightarrow \underline{f} \circ \boxtimes_{\mathcal{E}}$ is a balanced $\mathcal{E}$-module natural transformation in $\mathrm{Fun}^{\mathrm{bal}}_{\mathcal{E}}(\mathcal{M}, \mathcal{N}; \mathcal{D})$. The notation $\simeq^{\eta}$ means that the natural isomorphism is induced by η. Given two objects f, g and a morphism $a : f \Rightarrow g$ in $\mathrm{Fun}^{\mathrm{bal}}_{\mathcal{E}}(\mathcal{M}, \mathcal{N}; \mathcal{D})$, there exist unique objects $\underline{f}, \underline{g} \in \mathrm{Fun}(\mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N}, \mathcal{D})$ such that $f \simeq^{\eta} \underline{f} \circ \boxtimes_{\mathcal{E}}$ and $g \simeq^{\xi} \underline{g} \circ \boxtimes_{\mathcal{E}}$. For any choice of $(a, \eta, \xi, \underline{f}, \underline{g})$, there exists a unique morphism $b : \underline{f} \Rightarrow \underline{g}$ in $\mathrm{Fun}(\mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N}, \mathcal{D})$ such that $\xi \circ a \circ \eta^{-1} = b * \mathrm{id}_{\boxtimes_{\mathcal{E}}}$.

2.3 The symmetric monoidal 2-category $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$

A left $\mathcal{E}$-module $\mathcal{M}$ in $\mathrm{Cat}^{\mathrm{fs}}$ is automatically a $\mathcal{E}$-bimodule category with the right $\mathcal{E}$-action defined as $m \odot e := e \odot m$, for $m \in \mathcal{M}, e \in \mathcal{E}$.

Definition 2.6. The 2-category $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ consists of the following data.

- Its objects are left $\mathcal{E}$-modules in $\mathrm{Cat}^{\mathrm{fs}}$.
- Its 1-morphisms are left $\mathcal{E}$-module functors.
- Its 2-morphisms are left $\mathcal{E}$-module natural transformations.
- The identity 1-morphism $1_{\mathcal{M}}$ for each object $\mathcal{M}$ is identity functor $1_{\mathcal{M}}$.
- The identity 2-morphism 1_F for each left $\mathcal{E}$-module functor $F : \mathcal{M} \to \mathcal{N}$ is the identity natural transformation 1_F.
- The vertical composition is the vertical composition of left $\mathcal{E}$-module natural transformations.
- Horizontal composition of 1-morphisms is the composition of left $\mathcal{E}$-module functors.
- Horizontal composition of 2-morphisms is the horizontal composition of left $\mathcal{E}$-module natural transformations.

It is routine to check the above data satisfy the axioms (i)-(vi) of [JY, Prop. 2.3.4]. We define a pseudo-functor $\boxtimes_{\mathcal{E}} : \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}} \times \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}} \to \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ in Sec. A.2. And the following theorem is proved in Sec. A.2 and Sec. A.3.

Theorem 2.7. The 2-category $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ is a symmetric monoidal 2-category.

3 Algebras and centers in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$

In this section, Sec. 3.1, Sec. 3.2 and Sec. 3.3 study E_0-algebras, E_1-algebras and E_2-algebras in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$, respectively. Sec. 3.4, Sec. 3.5 and Sec. 3.6 study E_0-centers, E_1-centers and E_2-centers in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$, respectively.

3.1 E_0-algebras

Definition 3.1. We define the 2-category $\mathrm{Alg}_{E_0}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ of E_0-algebras in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ as follows.

- Its objects are E_0-algebras in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$. An E_0-algebra in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ is a pair $(\mathcal{A}, A)$, where $\mathcal{A}$ is an object in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ and $A : \mathcal{E} \to \mathcal{A}$ is a 1-morphism in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$.
- For two E_0-algebras $(\mathcal{A}, A)$ and $(\mathcal{B}, B)$, a 1-morphism $F : (\mathcal{A}, A) \to (\mathcal{B}, B)$ in $\mathrm{Alg}_{E_0}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ is a 1-morphism $F : \mathcal{A} \to \mathcal{B}$ in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ and an invertible 2-morphism $F^0 : B \Rightarrow F \circ A$ in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$.
- For two 1-morphisms $F, G : (\mathcal{A}, A) \rightrightarrows (\mathcal{B}, B)$ in $\mathrm{Alg}_{E_0}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$, a 2-morphism $\alpha : F \Rightarrow G$ in $\mathrm{Alg}_{E_0}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ is a 2-morphism $\alpha : F \Rightarrow G$ in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ such that $(\alpha * 1_A) \circ F^0 = G^0$, i.e.

$$
\begin{tikzcd}
\mathcal{E} \arrow[r, "A"] \arrow[dr, bend right, "B"'] \arrow[dr, phantom, "F^0 \Nearrow"] & \mathcal{A} \arrow[d, bend right, "F"'] \arrow[d, bend left, "G"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \\
 & \mathcal{B}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{E} \arrow[r, "A"] \arrow[dr, "B"'] \arrow[dr, phantom, shift left=2, "G^0 \Nearrow"] & \mathcal{A} \arrow[d, "G"] \\
 & \mathcal{B}
\end{tikzcd}
\tag{3.1}
$$

3.2 E_1-algebras

Let $\mathcal{A}$ and $\mathcal{B}$ be two monoidal categories. A monoidal functor from $\mathcal{A}$ to $\mathcal{B}$ is a pair (F, J^F), where $F : \mathcal{A} \to \mathcal{B}$ is a functor and $J^F_{x,y} : F(x \otimes y) \simeq F(x) \otimes F(y)$, $x, y \in \mathcal{A}$, is a natural isomorphism such that $F(\mathbb{1}_{\mathcal{A}}) = \mathbb{1}_{\mathcal{B}}$ and a natural diagram commutes. A *monoidal natural transformation* between two monoidal functors $(F, J^F), (G, J^G) : \mathcal{A} \rightrightarrows \mathcal{B}$ is a natural transformation $\alpha : F \Rightarrow G$ such that the following diagram commutes for all $x, y \in \mathcal{A}$:

$$
\begin{tikzcd}
F(x \otimes y) \arrow[r, "J^F_{x,y}"] \arrow[d, "\alpha_{x \otimes y}"'] & F(x) \otimes F(y) \arrow[d, "\alpha_x, \alpha_y"] \\
G(x \otimes y) \arrow[r, "J^G_{x,y}"'] & G(x) \otimes G(y)
\end{tikzcd}
\tag{3.2}
$$

Given a monoidal category $\mathcal{M}$, the *Drinfeld center* of $\mathcal{M}$ is a braided monoidal category $Z(\mathcal{M})$. The objects of $Z(\mathcal{M})$ are pairs (x, z), where $x \in \mathcal{M}$ and $z_{x,m} : x \otimes m \simeq m \otimes x, m \in \mathcal{M}$ is a natural isomorphism such that the following diagram commutes for $m, m' \in \mathcal{M}$:

$$
\begin{tikzcd}
x \otimes m \otimes m' \arrow[rr, "z_{x,m \otimes m'}"] \arrow[dr, "z_{x,m}, 1"'] & & m \otimes m' \otimes x \\
 & m \otimes x \otimes m' \arrow[ur, "1, z_{x,m'}"'] &
\end{tikzcd}
$$

Recall the two equivalent definitions of a central functor in Def. A.1 and Def. A.2. The definitions of a fusion category over $\mathcal{E}$ and a braided fusion category over $\mathcal{E}$ are in [DNO].

Definition 3.2. The 2-category $\mathrm{Alg}_{E_1}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ consists of the following data.

- Its objects are multifusion categories over $\mathcal{E}$. *A multifusion category over* $\mathcal{E}$ is a multifusion category $\mathcal{A}$ equipped with a $\Bbbk$-linear central functor $T_{\mathcal{A}} : \mathcal{E} \to \mathcal{A}$. Equivalently, a multifusion category over $\mathcal{E}$ is a multifusion category $\mathcal{A}$ equipped with a $\Bbbk$-linear braided monoidal functor $T'_{\mathcal{A}} : \mathcal{E} \to Z(\mathcal{A})$.
- Its 1-morphisms are monoidal functors over $\mathcal{E}$. A *monoidal functor over* $\mathcal{E}$ between two multifusion categories $\mathcal{A}, \mathcal{B}$ over $\mathcal{E}$ is a $\Bbbk$-linear monoidal functor $(F, J) : \mathcal{A} \to \mathcal{B}$ equipped with a monoidal natural isomorphism $u_e : F(T_{\mathcal{A}}(e)) \to T_{\mathcal{B}}(e)$ in $\mathcal{B}$ for each $e \in \mathcal{E}$, called the structure of monoidal functor over $\mathcal{E}$ on F, such that the diagram

$$\begin{tikzcd} F(T_{\mathcal{A}}(e) \otimes x) \arrow[r, "J_{T_{\mathcal{A}}(e),x}"] \arrow[d, "F(z_{e,x})"'] & F(T_{\mathcal{A}}(e)) \otimes F(x) \arrow[r, "u_e{,}1"] & T_{\mathcal{B}}(e) \otimes F(x) \arrow[d, "\hat{z}_{e,F(x)}"] \\ F(x \otimes T_{\mathcal{A}}(e)) \arrow[r, "J_{x,T_{\mathcal{A}}(e)}"] & F(x) \otimes F(T_{\mathcal{A}}(e)) \arrow[r, "1{,}u_e"] & F(x) \otimes T_{\mathcal{B}}(e) \end{tikzcd} \tag{3.3}$$

  commutes for $e \in \mathcal{E}, x \in \mathcal{A}$. Here z and $\hat{z}$ are the central structures of the central functors $T_{\mathcal{A}} : \mathcal{E} \to \mathcal{A}$ and $T_{\mathcal{B}} : \mathcal{E} \to \mathcal{B}$ respectively.
- Its 2-morphisms are monoidal natural transformations over $\mathcal{E}$. A *monoidal natural transformation over* $\mathcal{E}$ between two monoidal functors $F, G : \mathcal{A} \rightrightarrows \mathcal{B}$ over $\mathcal{E}$ is a monoidal natural transformation $\alpha : F \Rightarrow G$ such that the following diagram commutes for $e \in \mathcal{E}$:

$$\begin{tikzcd} F(T_{\mathcal{A}}(e)) \arrow[rr, "\alpha_{T_{\mathcal{A}}(e)}"] \arrow[dr, "u_e"'] & & G(T_{\mathcal{A}}(e)) \arrow[dl, "v_e"] \\ & T_{\mathcal{B}}(e) & \end{tikzcd} \tag{3.4}$$

  where u and v are the structures of monoidal functors over $\mathcal{E}$ on F and G, respectively.

Remark 3.3. If $\mathcal{A}$ is a multifusion category over $\mathcal{E}$ such that $T'_{\mathcal{A}} : \mathcal{E} \to Z(\mathcal{A})$ is fully faithful, then $\mathcal{A}$ is a indecomposable. If $\mathcal{E} = \mathcal{V}\mathrm{ec}$, the functor $\mathcal{V}\mathrm{ec} \to Z(\mathcal{A})$ is fully faithful if and only if $\mathcal{A}$ is indecomposable. The condition "$\mathcal{E} \to Z(\mathcal{A}) \to \mathcal{A}$ is fully faithful" implies the condition "$\mathcal{E} \to Z(\mathcal{A})$ is fully faithful".

Lemma 3.4. Let $\mathcal{A}$ and $\mathcal{B}$ be two monoidal categories. Suppose that $T_{\mathcal{A}} : \mathcal{E} \to \mathcal{A}, T_{\mathcal{B}} : \mathcal{E} \to \mathcal{B}$ and $F : \mathcal{A} \to \mathcal{B}$ are monoidal functors, and $u : F \circ T_{\mathcal{A}} \Rightarrow T_{\mathcal{B}}$ is a monoidal natural isomorphism. Then $\mathcal{A}, \mathcal{B}$ are left $\mathcal{E}$-module categories, $T_{\mathcal{A}}, T_{\mathcal{B}}$ and F are left $\mathcal{E}$-module functors, and u is a left $\mathcal{E}$-module natural isomorphism.

Proof. The left $\mathcal{E}$-module structure on $\mathcal{A}$ is defined as $e \odot a := T_{\mathcal{A}}(e) \otimes a$ for all $e \in \mathcal{E}$ and $a \in \mathcal{A}$. The left $\mathcal{E}$-module structure on $T_{\mathcal{A}}$ is induced by the monoidal structure of $T_{\mathcal{A}}$. The left $\mathcal{E}$-module structure s^F on F is induced by $F(e \odot a) = F(T_{\mathcal{A}}(e) \otimes a) \to F(T_{\mathcal{A}}(e)) \otimes F(a) \xrightarrow{u_e,1} T_{\mathcal{B}}(e) \otimes F(a) = e \odot F(a)$. The left $\mathcal{E}$-module structure on $F \circ T_{\mathcal{A}}$ is induced by $F(T_{\mathcal{A}}(\tilde{e} \otimes e)) \to F(T_{\mathcal{A}}(\tilde{e}) \otimes T_{\mathcal{A}}(e)) \xrightarrow{s^F} T_{\mathcal{B}}(\tilde{e}) \otimes F(T_{\mathcal{A}}(e)) = \tilde{e} \odot F(T_{\mathcal{A}}(e))$ for $e, \tilde{e} \in \mathcal{E}$. The natural isomorphism u satisfy the diagram (2.1) by the diagram (3.2) of the monoidal natural isomorphism u. $\square$

Remark 3.5. A monoidal functor $F : \mathcal{A} \to \mathcal{B}$ over $\mathcal{E}$ is a left $\mathcal{E}$-module functor. If $\mathcal{A}$ is a multifusion category over $\mathcal{E}$ and $F : \mathcal{A} \to \mathcal{B}$ is an equivalence of multifusion categories, $\mathcal{B}$ is a multifusion category over $\mathcal{E}$. The central structure σ on the monoidal functor $F \circ T_{\mathcal{A}} : \mathcal{E} \to \mathcal{B}$ is induced by

$$\begin{tikzcd} F(T_{\mathcal{A}}(e)) \otimes b \arrow[r, "\simeq"] \arrow[d, dotted, "\sigma_{e,b}"'] & F(T_{\mathcal{A}}(e)) \otimes F(a) \arrow[r] \arrow[d, dotted, "\sigma_{e,F(a)}"] & F(T_{\mathcal{A}}(e) \otimes a) \arrow[d, "c_{e,a}"] \\ b \otimes F(T_{\mathcal{A}}(e)) & F(a) \otimes F(T_{\mathcal{A}}(e)) \arrow[l, "\simeq"] & F(a \otimes T_{\mathcal{A}}(e)) \arrow[l] \end{tikzcd}$$

for $e \in \mathcal{E}, b \in \mathcal{B}$, where c is the central structure of the functor $T_{\mathcal{A}} : \mathcal{E} \to \mathcal{A}$. Notice that for any object $b \in \mathcal{B}$, there is an object $a \in \mathcal{A}$ such that $b \simeq F(a)$ by the equivalence of F.

Example 3.6. If $\mathcal{C}$ is a multifusion category over $\mathcal{E}$, $\mathcal{C}^{\mathrm{rev}}$ is a multifusion category over $\mathcal{E}$ by the central functor $\mathcal{E} = \overline{\mathcal{E}} \xrightarrow{T_{\mathcal{C}}} \overline{Z(\mathcal{C})} \cong Z(\mathcal{C}^{\mathrm{rev}})$.

Example 3.7. Let $\mathcal{M}$ be a left $\mathcal{E}$-module in $\mathrm{Cat}^{\mathrm{fs}}$. $\mathrm{Fun}_{\mathcal{E}}(\mathcal{M},\mathcal{M})$ is a multifusion category by [EGNO, Cor. 9.3.3]. Moreover, $\mathrm{Fun}_{\mathcal{E}}(\mathcal{M},\mathcal{M})$ is a multifusion category over $\mathcal{E}$. We define a functor $T : \mathcal{E} \to \mathrm{Fun}_{\mathcal{E}}(\mathcal{M},\mathcal{M}), e \mapsto T^e := e \odot -$. The left $\mathcal{E}$-module structure on T^e is defined as $e \odot (\tilde{e} \odot m) \to (e \otimes \tilde{e}) \odot m \xrightarrow{r_{e,\tilde{e}},1} (\tilde{e} \otimes e) \odot m \to \tilde{e} \odot (e \odot m)$ for $\tilde{e} \in \mathcal{E}, m \in \mathcal{M}$. The monoidal structure J^T on T is induced by $T^{e\otimes e'} = (e \otimes e') \odot - \simeq e \odot (e' \odot -) = T^e \circ T^{e'}$ for $e, e' \in \mathcal{E}$. The central structure σ on T is induced by $T^e \circ G(m) = e \odot G(m) \simeq G(e \odot m) = G \circ T^e(m)$ for all $e \in \mathcal{E}, G \in \mathrm{Fun}_{\mathcal{E}}(\mathcal{M},\mathcal{M})$ and $m \in \mathcal{M}$.

Example 3.8. Let $\mathcal{C}$ and $\mathcal{D}$ be multifusion categories over $\mathcal{E}$. $\mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{D}$ is a multifusion category over $\mathcal{E}$. We define a monoidal functor $T_{\mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{D}} : \mathcal{E} \simeq \mathcal{E} \boxtimes_{\mathcal{E}} \mathcal{E} \xrightarrow{T_{\mathcal{C}} \boxtimes_{\mathcal{E}} T_{\mathcal{D}}} \mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{D}$ by $e \mapsto e \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{E}} \mapsto T_{\mathcal{C}}(e) \boxtimes_{\mathcal{E}} T_{\mathcal{D}}(\mathbb{1}_{\mathcal{E}}) = T_{\mathcal{C}}(e) \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{D}}$ for $e \in \mathcal{E}$. And the central structure σ on $T_{\mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{D}}$ is induced by

$$\begin{tikzcd}
T_{\mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{D}}(e) \otimes (c \boxtimes_{\mathcal{E}} d) \arrow[r, equal] \arrow[d, dotted, "\sigma_{e,c\boxtimes_{\mathcal{E}} d}"'] & (T_{\mathcal{C}}(e) \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{D}}) \otimes (c \boxtimes_{\mathcal{E}} d) \arrow[r, equal] & (T_{\mathcal{C}}(e) \otimes c) \boxtimes_{\mathcal{E}} (\mathbb{1}_{\mathcal{D}} \otimes d) \arrow[d, "z_{e,c} \boxtimes_{\mathcal{E}} \hat{z}_{\mathbb{1}_{\mathcal{D}},d}"] \\
(c \boxtimes_{\mathcal{E}} d) \otimes T_{\mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{D}}(e) \arrow[r, equal] & (c \boxtimes_{\mathcal{E}} d) \otimes (T_{\mathcal{C}}(e) \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{D}}) \arrow[r, equal] & (c \otimes T_{\mathcal{C}}(e)) \boxtimes_{\mathcal{E}} (d \otimes \mathbb{1}_{\mathcal{D}})
\end{tikzcd}$$

for $e \in \mathcal{E}$, $c \boxtimes_{\mathcal{E}} d \in \mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{D}$, where z and $\hat{z}$ are the central structures of the functors $T_{\mathcal{C}} : \mathcal{E} \to \mathcal{C}$ and $T_{\mathcal{D}} : \mathcal{E} \to \mathcal{D}$ respectively. Notice that $T_{\mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{D}}(e) \simeq 1_{\mathcal{C}} \boxtimes_{\mathcal{E}} T_{\mathcal{D}}(e)$.

An algebra A in a tensor category $\mathcal{A}$ is called *separable* if the multiplication morphism $m : A \otimes A \to A$ splits as a morphism of A-bimodules. Namely, there is an A-bimodule map $e : A \to A \otimes A$ such that $m \circ e = \mathrm{id}_A$.

Example 3.9. Let $\mathcal{C}$ be a multifusion category over $\mathcal{E}$ and A a separable algebra in $\mathcal{C}$. The category ${}_A\mathcal{C}_A$ of A-bimodules in $\mathcal{C}$ is a multifusion category by [DMNO, Prop. 2.7]. Moreover, ${}_A\mathcal{C}_A$ is a multifusion category over $\mathcal{E}$. We define a functor $I : \mathcal{E} \to {}_A\mathcal{C}_A$, $e \mapsto T_{\mathcal{C}}(e) \otimes A$. The left A-module structure on the right A-module $T_{\mathcal{C}}(e) \otimes A$ is defined as $A \otimes T_{\mathcal{C}}(e) \otimes A \xrightarrow{c^{-1}_{e,A},1} T_{\mathcal{C}}(e) \otimes A \otimes A \to T_{\mathcal{C}}(e) \otimes A$, where c is the central structure of the functor $T_{\mathcal{C}} : \mathcal{E} \to \mathcal{C}$. The monoidal structure on I is defined as $T_{\mathcal{C}}(e_1 \otimes e_2) \otimes A \simeq T_{\mathcal{C}}(e_1) \otimes T_{\mathcal{C}}(e_2) \otimes A \cong T_{\mathcal{C}}(e_1) \otimes A \otimes_A T_{\mathcal{C}}(e_2) \otimes A$ for $e_1, e_2 \in \mathcal{E}$. The central structure on I is induced by

$$I(e) \otimes_A x = T_{\mathcal{C}}(e) \otimes A \otimes_A x \xrightarrow{c_{e,A\otimes_A x}} A \otimes_A x \otimes T_{\mathcal{C}}(e) \cong x \otimes_A A \otimes T_{\mathcal{C}}(e) \xrightarrow{1,c^{-1}_{e,A}} x \otimes_A T_{\mathcal{C}}(e) \otimes A = x \otimes_A I(e)$$

for $e \in \mathcal{E}, x \in {}_A\mathcal{C}_A$.

3.3 E_2-algebras

Let $\mathcal{A}$ be a subcategory of a braided fusion category $\mathcal{C}$. The *centralizer of $\mathcal{A}$ in $\mathcal{C}$*, denoted by $\mathcal{A}'|_{\mathcal{C}}$, is defined by the full subcategory of objects $x \in \mathcal{C}$ such that $c_{a,x} \circ c_{x,a} = \mathrm{id}_{x\otimes a}$ for all $a \in \mathcal{A}$, where c is the braiding of $\mathcal{C}$. The *Müger center* of $\mathcal{C}$, denoted by $\mathcal{C}'$ or $\mathcal{C}'|_{\mathcal{C}}$, is the centralizer of $\mathcal{C}$ in $\mathcal{C}$. Let $\mathcal{B}$ be a fusion category over $\mathcal{E}$ such that $\mathcal{E} \to Z(\mathcal{B})$ is fully faithful. The centralizer of $\mathcal{E}$ in $Z(\mathcal{B})$ is denoted by $Z(\mathcal{B}, \mathcal{E})$ or $\mathcal{E}'|_{Z(\mathcal{B})}$.

Definition 3.10. The 2-category $\mathrm{Alg}_{E_2}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ consists of the following data.

- Its objects are braided fusion categories over $\mathcal{E}$. A *braided fusion category over* $\mathcal{E}$ is a braided fusion category $\mathcal{A}$ equipped with a $\Bbbk$-linear braided monoidal embedding $T_{\mathcal{A}} : \mathcal{E} \to \mathcal{A}'$. A braided fusion category $\mathcal{A}$ over $\mathcal{E}$ is non-degenerate if $T_{\mathcal{A}}$ is an equivalence.

- Its 1-morphisms are braided monoidal functors over $\mathcal{E}$. A *braided monoidal functor over* $\mathcal{E}$ between two braided fusion categories $\mathcal{A}, \mathcal{B}$ over $\mathcal{E}$ is a $\Bbbk$-linear braided monoidal functor $F : \mathcal{A} \to \mathcal{B}$ equipped with a monoidal natural isomorphism $u_e : F(T_{\mathcal{A}}(e)) \simeq T_{\mathcal{B}}(e)$ in $\mathcal{B}$ for all $e \in \mathcal{E}$.

- For two braided monoidal functors $F, G : \mathcal{A} \rightrightarrows \mathcal{B}$ over $\mathcal{E}$, a 2-morphism from F to G is a monoidal natural transformation $\alpha : F \Rightarrow G$ such that the diagram (3.4) commutes.

Remark 3.11. Let $\mathcal{A}$ be a braided fusion category over $\mathcal{E}$ and $\eta : \mathcal{A} \simeq \mathcal{B}$ is an equivalence of braided fusion categories. Then $\mathcal{B}$ is a braided fusion category over $\mathcal{E}$.

Example 3.12. If $\mathcal{D}$ is a braided fusion category over $\mathcal{E}$, $\overline{\mathcal{D}}$ is a braided fusion category over $\mathcal{E}$ by the braided monoidal embedding $\mathcal{E} = \overline{\mathcal{E}} \xrightarrow{T_{\mathcal{D}}} \overline{\mathcal{D}'} = \overline{\mathcal{D}}'$.

Example 3.13. Let $\mathcal{C}$ be a fusion category over $\mathcal{E}$ such that $\mathcal{E} \to Z(\mathcal{C})$ is fully faithful. $Z(\mathcal{C}, \mathcal{E})$ is a non-degenerate braided fusion category over $\mathcal{E}$. Next check that $Z(\mathcal{C}, \mathcal{E})' = \mathcal{E}$. On one hand, if $e \in \mathcal{E}$, we have $T_{\mathcal{C}}(e) \in Z(\mathcal{C}, \mathcal{E})'$. On the other hand, since $Z(\mathcal{C})' = \mathcal{V}\mathrm{ec} \subset \mathcal{E}$, we have $Z(\mathcal{C}, \mathcal{E})'|_{Z(\mathcal{C},\mathcal{E})} \subset Z(\mathcal{C}, \mathcal{E})'|_{Z(\mathcal{C})} = (\mathcal{E}'|_{Z(\mathcal{C})})'|_{Z(\mathcal{C})} = \mathcal{E}$. The central structure on $T_{Z(\mathcal{C},\mathcal{E})} : \mathcal{E} \to Z(\mathcal{C}, \mathcal{E})'$ is defined as $T_{\mathcal{C}}$.

If $\mathcal{C}$ is a non-degenerate braided fusion category over $\mathcal{E}$, there is a braided monoidal equivalence $Z(\mathcal{C}, \mathcal{E}) \simeq \mathcal{C} \boxtimes_{\mathcal{E}} \overline{\mathcal{C}}$ over $\mathcal{E}$ by [DNO, Cor. 4.4].

3.4 E_0-centers

A *contractible groupoid* is a non-empty category in which there is a unique morphism between any two objects. An object $\mathcal{X}$ in a monoidal 2-category B is called a *terminal object* if for each $\mathcal{Y} \in \mathsf{B}$, the hom category $\mathsf{B}(\mathcal{Y}, \mathcal{X})$ is a contractible groupoid. Here the hom category $\mathsf{B}(\mathcal{Y}, \mathcal{X})$ denotes the category of 1-morphisms from $\mathcal{Y}$ to $\mathcal{X}$ and 2-morphisms in B.

Definition 3.14. Let $\mathcal{A} = (\mathcal{A}, A) \in \mathrm{Alg}_{E_0}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. A *left unital $\mathcal{A}$-action on* $\mathcal{X} \in \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ is a 1-morphism $F : \mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{X} \to \mathcal{X}$ in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ together with an invertible 2-morphism α in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ as depicted in the following diagram:

$$\begin{tikzcd} & \mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{X} \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow\alpha"] & \\ \mathcal{E} \boxtimes_{\mathcal{E}} \mathcal{X} \arrow[ur, "A \boxtimes_{\mathcal{E}} 1_{\mathcal{X}}"] \arrow[rr] & & \mathcal{X} \end{tikzcd},$$

where the unlabeled arrow is given by the left $\mathcal{E}$-action on $\mathcal{X}$.

Definition 3.15. Let $\mathcal{X} \in \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$. The 2-category $\mathrm{Alg}_{E_0}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})_{\mathcal{X}}$ of left unital actions on $\mathcal{X}$ in $\mathrm{Alg}_{E_0}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ is defined as follows.

- The objects are left unital actions on $\mathcal{X}$.

- Let $((\mathcal{A}, A), F, \alpha_{\mathcal{A}})$ be a left unital $(\mathcal{A}, A)$-action on $\mathcal{X}$ and $((\mathcal{B}, B), G, \alpha_{\mathcal{B}})$ be a left unital $(\mathcal{B}, B)$-action on $\mathcal{X}$. A 1-morphism $(P, \rho) : ((\mathcal{A}, A), F, \alpha_{\mathcal{A}}) \to ((\mathcal{B}, B), G, \alpha_{\mathcal{B}})$ in $\mathrm{Alg}_{E_0}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})_{\mathcal{X}}$

is a 1-morphism $P : (\mathcal{A}, A) \to (\mathcal{B}, B)$ in $\mathrm{Alg}_{E_0}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}})$, equipped with an invertible 2-morphism $\rho : G \circ (P \boxtimes_{\mathcal{E}} 1_{\mathcal{X}}) \Rightarrow F$ in $\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}$, such that the following pasting diagram equality holds.

$$
\begin{tikzcd}
 & \mathcal{B} \boxtimes_{\mathcal{E}} \mathcal{X} \arrow[ddr, bend left, "G"] & \\
 & \mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{X} \arrow[u, "P\boxtimes_{\mathcal{E}} 1_{\mathcal{X}}"] \arrow[dr, "F"] \arrow[r, phantom, "\Downarrow\rho"] & \\
\mathcal{E} \boxtimes_{\mathcal{E}} \mathcal{X} \arrow[uur, bend left, "B\boxtimes_{\mathcal{E}} 1_{\mathcal{X}}"] \arrow[ur, "A\boxtimes_{\mathcal{E}} 1_{\mathcal{X}}"] \arrow[rr] \arrow[u, phantom, "P^0\boxtimes_{\mathcal{E}} 1 \Downarrow"] \arrow[rr, phantom, shift left=4, "\Downarrow\alpha_{\mathcal{A}}"] & & \mathcal{X}
\end{tikzcd}
=
\begin{tikzcd}
 & \mathcal{B} \boxtimes_{\mathcal{E}} \mathcal{X} \arrow[dr, "G"] & \\
\mathcal{E} \boxtimes_{\mathcal{E}} \mathcal{X} \arrow[ur, "B\boxtimes_{\mathcal{E}} 1_{\mathcal{X}}"] \arrow[rr] \arrow[rr, phantom, shift left=4, "\Downarrow\alpha_{\mathcal{B}}"] & & \mathcal{X}
\end{tikzcd}
$$

Here we choose the identity 2-morphism $\mathrm{id} : (P \boxtimes_{\mathcal{E}} 1_{\mathcal{X}}) \circ (A \boxtimes_{\mathcal{E}} 1_{\mathcal{X}}) \Rightarrow (P \circ A) \boxtimes_{\mathcal{E}} 1_{\mathcal{X}}$ for convenience.

- Given two 1-morphisms $(P, \rho), (Q, \sigma) : ((\mathcal{A}, A), F, \alpha_{\mathcal{A}}) \rightrightarrows ((\mathcal{B}, B), G, \alpha_{\mathcal{B}})$, a 2-morphism $\alpha : (P, \rho) \Rightarrow (Q, \sigma)$ in $\mathrm{Alg}_{E_0}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}})_{\mathcal{X}}$ is a 2-morphism $\alpha : P \Rightarrow Q$ in $\mathrm{Alg}_{E_0}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}})$ such that the following pasting diagram equality holds.

$$
\begin{tikzcd}
\mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{X} \arrow[rr, bend left, "P\boxtimes_{\mathcal{E}} 1_{\mathcal{X}}"] \arrow[rr, bend right, "Q\boxtimes_{\mathcal{E}} 1_{\mathcal{X}}"'] \arrow[rr, phantom, "\Downarrow\alpha\boxtimes_{\mathcal{E}} 1"] \arrow[dr, "F"'] & & \mathcal{B} \boxtimes_{\mathcal{E}} \mathcal{X} \arrow[dl, "G"] \\
 & \mathcal{X} \arrow[u, phantom, "\Downarrow\sigma"] &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{X} \arrow[rr, "P\boxtimes_{\mathcal{E}} 1_{\mathcal{X}}"] \arrow[dr, "F"'] & & \mathcal{B} \boxtimes_{\mathcal{E}} \mathcal{X} \arrow[dl, "G"] \\
 & \mathcal{X} \arrow[u, phantom, "\Downarrow\rho"] &
\end{tikzcd}
$$

An E_0-*center of the object* $\mathcal{X}$ *in* $\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}$ is a terminal object in $\mathrm{Alg}_{E_0}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}})_{\mathcal{X}}$.

Theorem 3.16. The E_0-center of a category $\mathcal{X} \in \mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}$ is given by the multifusion category $\mathrm{Fun}_{\mathcal{E}}(\mathcal{X}, \mathcal{X})$ over $\mathcal{E}$.

Proof. Suppose $(\mathcal{A}, A)$ is an E_0-algebra in $\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}$ and (F, u) as depicted in the following diagram

$$
\begin{tikzcd}
 & \mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{X} \arrow[dr, "F"] & \\
\mathcal{E} \boxtimes_{\mathcal{E}} \mathcal{X} \arrow[ur, "A\boxtimes_{\mathcal{E}} 1_{\mathcal{X}}"] \arrow[rr] \arrow[rr, phantom, shift left=4, "\Downarrow u"] & & \mathcal{X}
\end{tikzcd}
$$

is a unital $\mathcal{A}$-action on $\mathcal{X}$. In other words, $F : \mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{X} \to \mathcal{X}$ is a left $\mathcal{E}$-module functor and $u_{e,x} : F(A(e) \boxtimes_{\mathcal{E}} x) \to e \odot x, e \in \mathcal{E}, x \in \mathcal{X}$ is a natural isomorphism in $\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}$.

Recall that $(\mathrm{Fun}_{\mathcal{E}}(\mathcal{X}, \mathcal{X}), T)$ is an E_0-algebra in $\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}$ by Expl. 3.7.

$$
\begin{tikzcd}
 & \mathrm{Fun}_{\mathcal{E}}(\mathcal{X}, \mathcal{X}) \boxtimes_{\mathcal{E}} \mathcal{X} \arrow[dr, "G"] & \\
\mathcal{E} \boxtimes_{\mathcal{E}} \mathcal{X} \arrow[ur, "T\boxtimes_{\mathcal{E}} 1_{\mathcal{X}}"] \arrow[rr] \arrow[rr, phantom, shift left=4, "\Downarrow v"] & & \mathcal{X}
\end{tikzcd}
$$

Define a functor

$$G : \mathrm{Fun}_{\mathcal{E}}(\mathcal{X}, \mathcal{X}) \boxtimes_{\mathcal{E}} \mathcal{X} \to \mathcal{X}, \qquad f \boxtimes_{\mathcal{E}} x \mapsto f(x)$$

and a natural isomorphism

$$v_{e,x} = \mathrm{id}_{e \odot x} : G(T^e \boxtimes_{\mathcal{E}} x) = T^e(x) = e \odot x \to e \odot x, \quad e \in \mathcal{E}, x \in \mathcal{X}.$$

Then $((\mathrm{Fun}_{\mathcal{E}}(\mathcal{X}, \mathcal{X}), T), G, v)$ is a left unital $\mathrm{Fun}_{\mathcal{E}}(\mathcal{X}, \mathcal{X})$-action on $\mathcal{X}$.

We want to show that $\mathrm{Alg}_{E_0}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})_{\mathcal{X}}(\mathcal{A}, \mathrm{Fun}_{\mathcal{E}}(\mathcal{X},\mathcal{X}))$ is a contractible groupoid. First we want to show there exists a 1-morphism $(P,\rho) : \mathcal{A} \to \mathrm{Fun}_{\mathcal{E}}(\mathcal{X},\mathcal{X})$ in $\mathrm{Alg}_{E_0}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})_{\mathcal{X}}$. We define a functor $P : \mathcal{A} \to \mathrm{Fun}_{\mathcal{E}}(\mathcal{X},\mathcal{X})$ by $P(a) := F(a \boxtimes_{\mathcal{E}} -)$ for all $a \in \mathcal{A}$ and an invertible 2-morphism $P^0_e : T^e = e \odot - \Rightarrow P(A(e)) = F(A(e) \boxtimes_{\mathcal{E}} -)$ as u_e^{-1} for all $e \in \mathcal{E}$. The natural isomorphism ρ can be defined by

$$\rho_{a,x} = \mathrm{id}_{F(a\boxtimes_{\mathcal{E}} x)} : G(P(a) \boxtimes_{\mathcal{E}} x) = P(a)(x) = F(a \boxtimes_{\mathcal{E}} x) \to F(a \boxtimes_{\mathcal{E}} x)$$

for $a \in \mathcal{A}, x \in \mathcal{X}$. Then it suffices to show that the composition of morphisms

$$G(T^e \boxtimes_{\mathcal{E}} x) = e \odot x \xRightarrow{(P^0_e)_x = u^{-1}_{e,x}} F(A(e) \boxtimes_{\mathcal{E}} x) \xRightarrow{\rho_{A(e),x} = \mathrm{id}_{F(A(e)\boxtimes_{\mathcal{E}} x)}} F(A(e) \boxtimes_{\mathcal{E}} x) \xRightarrow{u_{e,x}} e \odot x$$

is equal to $v_{e,x} = \mathrm{id}_{e\odot x}$ by the definitions of P^0 and ρ.

Then we want to show that if there are two 1-morphisms $(Q_i,\sigma_i) : \mathcal{A} \to \mathrm{Fun}_{\mathcal{E}}(\mathcal{X},\mathcal{X})$ in $\mathrm{Alg}_{E_0}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})_{\mathcal{X}}$ for $i = 1,2$, there is a unique 2-morphism $\beta : (Q_1,\sigma_1) \Rightarrow (Q_2,\sigma_2)$ in $\mathrm{Alg}_{E_0}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})_{\mathcal{X}}$. The 2-morphism β in $\mathrm{Alg}_{E_0}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})_{\mathcal{X}}$ is a natural isomorphism $\beta : Q_1 \Rightarrow Q_2$ such that the equalities

$$\left(T \xRightarrow{Q^0_1} Q_1 \circ A \xRightarrow{\beta * 1_A} Q_2 \circ A\right) = \left(T \xRightarrow{Q^0_2} Q_2 \circ A\right) \tag{3.5}$$

and

$$\left(Q_1(a)(x) \xrightarrow{(\beta_a)_x} Q_2(a)(x) \xrightarrow{(\sigma_2)_{a,x}} F(a \boxtimes_{\mathcal{E}} x)\right) = \left(Q_1(a)(x) \xrightarrow{(\sigma_1)_{a,x}} F(a \boxtimes_{\mathcal{E}} x)\right) \tag{3.6}$$

hold for $a \in \mathcal{A}, x \in \mathcal{X}$. The second condition (3.6) implies that $(\beta_a)_x = (\sigma_2)^{-1}_{a,x} \circ (\sigma_1)_{a,x}$. This proves the uniqueness of β. For the existence of β, we want to show that β satisfy the first condition (3.5), i.e. β is a 2-morphism in $\mathrm{Alg}_{E_0}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. Since (Q_i,σ_i) are 1-morphisms in $\mathrm{Alg}_{E_0}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})_{\mathcal{X}}$, the composed morphism

$$e \odot x = T^e(x) \xrightarrow{(Q^0_i)_{e,x}} Q_i(A(e))(x) \xrightarrow{(\sigma_i)_{A(e),x}} F(A(e) \boxtimes_{\mathcal{E}} x) \xrightarrow{u_{e,x}} e \odot x$$

is equal to $v_{e,x} = \mathrm{id}_{e\odot x}$. It follows that the composition of morphisms

$$e \odot x \xrightarrow{(Q^0_1)_{e,x}} Q_1(A(e))(x) \xrightarrow{(\sigma_1)_{A(e),x}} F(A(e) \boxtimes_{\mathcal{E}} x) \xrightarrow{(\sigma_2)^{-1}_{A(e),x}} Q_2(A(e))(x) \xrightarrow{(Q^0_2)^{-1}_{e,x}} e \odot x$$

is equal to $\mathrm{id}_{e\odot x}$, i.e. $(Q^0_2)^{-1}_{e,x} \circ (\beta_{A(e)})_x \circ (Q^0_1)_{e,x} = \mathrm{id}_{e\odot x}$. This is precisely the first condition (3.5). Hence the natural transformation $\beta : Q_1 \Rightarrow Q_2$ defined by $(\beta_a)_x = (\sigma_2)^{-1}_{a,x} \circ (\sigma_1)_{a,x}$ is the unique 2-morphism $\beta : (Q_1,\sigma_1) \Rightarrow (Q_2,\sigma_2)$.

Finally, we also want to verify that the E_1-algebra structure on the E_1-center $\mathrm{Fun}_{\mathcal{E}}(\mathcal{X},\mathcal{X})$ coincides with the usual monoidal structure of $\mathrm{Fun}_{\mathcal{E}}(\mathcal{X},\mathcal{X})$ defined by the composition of functors. Recall that the E_1-algebra structure is induced by the iterated action

$$\mathrm{Fun}_{\mathcal{E}}(\mathcal{X},\mathcal{X}) \boxtimes_{\mathcal{E}} \mathrm{Fun}_{\mathcal{E}}(\mathcal{X},\mathcal{X}) \boxtimes_{\mathcal{E}} \mathcal{X} \xrightarrow{1\boxtimes_{\mathcal{E}} G} \mathrm{Fun}_{\mathcal{E}}(\mathcal{X},\mathcal{X}) \boxtimes_{\mathcal{E}} \mathcal{X} \xrightarrow{G} \mathcal{X}$$

By the construction given above, the induced tensor product $\mathrm{Fun}_{\mathcal{E}}(\mathcal{X},\mathcal{X}) \boxtimes_{\mathcal{E}} \mathrm{Fun}_{\mathcal{E}}(\mathcal{X},\mathcal{X}) \to \mathrm{Fun}_{\mathcal{E}}(\mathcal{X},\mathcal{X})$ is given by $f \boxtimes_{\mathcal{E}} g \mapsto G(f \boxtimes_{\mathcal{E}} G(g \boxtimes_{\mathcal{E}} -)) = f(g(-)) = f \circ g$. Hence, the E_1-algebra structure on $\mathrm{Fun}_{\mathcal{E}}(\mathcal{X},\mathcal{X})$ is the composition of functors, which is the usual monoidal structure on $\mathrm{Fun}_{\mathcal{E}}(\mathcal{X},\mathcal{X})$. □

3.5 E_1-centers

Definition 3.17. Let $\mathcal{X} \in \mathrm{Alg}_{E_1}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. The E_1-*center of* $\mathcal{X}$ *in* $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ is the E_0-center of $\mathcal{X}$ in $\mathrm{Alg}_{E_1}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$.

Theorem 3.18. Let $\mathcal{B}$ be a multifusion category over $\mathcal{E}$. Then the E_1-center of $\mathcal{B}$ in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ is the braided multifusion category $Z(\mathcal{B}, \mathcal{E})$ over $\mathcal{E}$.

Proof. Let $\mathcal{A}$ be a multifusion category over $\mathcal{E}$. A left unital $\mathcal{A}$-action on $\mathcal{B}$ in $\mathrm{Alg}_{E_1}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ is a monoidal functor $F : \mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{B} \to \mathcal{B}$ over $\mathcal{E}$ and a monoidal natural isomorphism u over $\mathcal{E}$ shown below:

$$\begin{tikzcd} & \mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B} \arrow[dr, "F"] & \\ \mathcal{E}\boxtimes_{\mathcal{E}}\mathcal{B} \arrow[ur, "T_{\mathcal{A}},1"] \arrow[rr, "\odot"'] \arrow[rr, phantom, shift left=4, "\Downarrow u"] & & \mathcal{B} \end{tikzcd}$$

More precisely, F is a functor equipped with natural isomorphisms $J^F : F(a_1 \boxtimes_{\mathcal{E}} b_1) \otimes F(a_2 \boxtimes_{\mathcal{E}} b_2) \xrightarrow{\simeq} F((a_1 \boxtimes_{\mathcal{E}} b_1) \otimes (a_2 \boxtimes_{\mathcal{E}} b_2))$, $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$, and $I^F : \mathbb{1}_{\mathcal{B}} \xrightarrow{\simeq} F(\mathbb{1}_{\mathcal{A}} \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{B}})$ satisfying certain commutative diagrams. The monoidal structure on the functor $\odot : \mathcal{E} \boxtimes_{\mathcal{E}} \mathcal{B} \to \mathcal{B}$, $e \boxtimes_{\mathcal{E}} b \mapsto e \odot b = T_{\mathcal{B}}(e) \otimes b$ is induced by $T_{\mathcal{B}}(e_1 \otimes e_2) \otimes (b_1 \otimes b_2) \simeq T_{\mathcal{B}}(e_1) \otimes T_{\mathcal{B}}(e_2) \otimes b_1 \otimes b_2 \xrightarrow{1, z_{e_2, b_1}, 1} T_{\mathcal{B}}(e_1) \otimes b_1 \otimes T_{\mathcal{B}}(e_2) \otimes b_2$ for $e_1, e_2 \in \mathcal{E}, b_1, b_2 \in \mathcal{B}$, where $(T_{\mathcal{B}}(e_2), z) \in Z(\mathcal{B})$. The structure of monoidal functor over $\mathcal{E}$ on $\odot$ is defined as $\odot(T_{\mathcal{E}\boxtimes_{\mathcal{E}}\mathcal{B}}(e)) = \odot(e \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{B}}) = e \odot \mathbb{1}_{\mathcal{B}} = T_{\mathcal{B}}(e) \otimes \mathbb{1}_{\mathcal{B}} \simeq T_{\mathcal{B}}(e)$. And u is a monoidal natural isomorphism $u_{e,b} : F(T_{\mathcal{A}}(e) \boxtimes_{\mathcal{E}} b) \xrightarrow{\simeq} e \odot b := T_{\mathcal{B}}(e) \otimes b$, $e \in \mathcal{E}, b \in \mathcal{B}$. Also one can show that $I^F = u^{-1}_{\mathbb{1}_{\mathcal{E}}, \mathbb{1}_{\mathcal{B}}}$. The structure of monoidal functor over $\mathcal{E}$ on F is $u_{e, \mathbb{1}_{\mathcal{B}}} : F(T_{\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}}(e)) = F(T_{\mathcal{A}}(e) \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{B}}) \simeq T_{\mathcal{B}}(e) \otimes \mathbb{1}_{\mathcal{B}} \simeq T_{\mathcal{B}}(e)$.

There is an obviously left unital $Z(\mathcal{B}, \mathcal{E})$-action on $\mathcal{B}$

$$\begin{tikzcd} & Z(\mathcal{B},\mathcal{E})\boxtimes_{\mathcal{E}}\mathcal{B} \arrow[dr, "G"] & \\ \mathcal{E}\boxtimes_{\mathcal{E}}\mathcal{B} \arrow[ur, "T_{\mathcal{B}},1"] \arrow[rr, "\odot"'] \arrow[rr, phantom, shift left=4, "\Downarrow v"] & & \mathcal{B} \end{tikzcd}$$

defined by $G : Z(\mathcal{B}, \mathcal{E}) \boxtimes_{\mathcal{E}} \mathcal{B} \xrightarrow{\mathbf{f},1} \mathcal{B} \boxtimes_{\mathcal{E}} \mathcal{B} \xrightarrow{\otimes} \mathcal{B}$ and $v_{e,b} := \mathrm{id}_{T_{\mathcal{B}}(e)\otimes b} : G(T_{\mathcal{B}}(e) \boxtimes_{\mathcal{E}} b) = T_{\mathcal{B}}(e) \otimes b \to e \odot b$, for $e \in \mathcal{E}, b \in \mathcal{B}$. The structure of monoidal functor over $\mathcal{E}$ on G is defined as $G(T_{\mathcal{B}}(e) \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{B}}) = T_{\mathcal{B}}(e) \otimes \mathbb{1}_{\mathcal{B}} \simeq T_{\mathcal{B}}(e)$.

First we want to show that $F(a \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{B}}) \in Z(\mathcal{B}, \mathcal{E})$ for $a \in \mathcal{A}$. Notice that $F(\mathbb{1}_{\mathcal{A}} \boxtimes_{\mathcal{E}} b) = F(T_{\mathcal{A}}(\mathbb{1}_{\mathcal{E}}) \boxtimes_{\mathcal{E}} b) \xrightarrow{u_{\mathbb{1}_{\mathcal{E}}, b}} \mathbb{1}_{\mathcal{E}} \odot b = b$. Since F is a monoidal functor over $\mathcal{E}$, it can be verified that the natural transformation γ (shown below)

$$\begin{tikzcd} F(a\boxtimes_{\mathcal{E}}\mathbb{1}_{\mathcal{B}})\otimes b \arrow[r, "1,u^{-1}_{\mathbb{1}_{\mathcal{E}},b}"] \arrow[d, dotted, "\gamma_{a,b}"'] & F(a\boxtimes_{\mathcal{E}}\mathbb{1}_{\mathcal{B}})\otimes F(\mathbb{1}_{\mathcal{A}}\boxtimes_{\mathcal{E}} b) \arrow[r, "J^F"] & F((a\otimes\mathbb{1}_{\mathcal{A}})\boxtimes_{\mathcal{E}}(\mathbb{1}_{\mathcal{B}}\otimes b)) \arrow[d, "\simeq,\simeq"] \\ b\otimes F(a\boxtimes_{\mathcal{E}}\mathbb{1}_{\mathcal{B}}) & F(\mathbb{1}_{\mathcal{A}}\boxtimes_{\mathcal{E}} b)\otimes F(a\boxtimes_{\mathcal{E}}\mathbb{1}_{\mathcal{B}}) \arrow[l, "u_{\mathbb{1}_{\mathcal{E}},b},1"] & F((\mathbb{1}_{\mathcal{A}}\otimes a)\boxtimes_{\mathcal{E}}(b\otimes\mathbb{1}_{\mathcal{B}})) \arrow[l, "J^F"] \end{tikzcd}$$

is a half-braiding on $F(a \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{B}}) \in \mathcal{B}$, for $a \in \mathcal{A}, b \in \mathcal{B}$. It is routine to check that the composition $T_{\mathcal{B}}(e) \otimes F(a \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{B}}) \to F(a \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{B}}) \otimes T_{\mathcal{B}}(e) \to T_{\mathcal{B}}(e) \otimes F(a \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{B}})$ equals to identity. Then $F(a \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{B}})$ belongs to $Z(\mathcal{B}, \mathcal{E})$.

We define a monoidal functor $P : \mathcal{A} \to Z(\mathcal{B}, \mathcal{E})$ by $P(a) := (F(a \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{B}}), \gamma_{a,-})$ with the monoidal structure induced by that of F:

$$J^P : \Big(P(a_1) \otimes P(a_2) = F(a_1 \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{B}}) \otimes F(a_2 \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{B}}) \xrightarrow{J^F} F((a_1 \otimes a_2) \boxtimes_{\mathcal{E}} (\mathbb{1}_{\mathcal{B}} \otimes \mathbb{1}_{\mathcal{B}})) = F((a_1 \otimes a_2) \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{B}}) = P(a_1 \otimes a_2)\Big)$$

$$I^P : \left(\mathbb{1}_{\mathcal{B}} \xrightarrow{I^F} F(\mathbb{1}_{\mathcal{A}} \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{B}}) = P(\mathbb{1}_{\mathcal{A}})\right)$$

The structure of monoidal functor over $\mathcal{E}$ on P is defined as $u_{e,\mathbb{1}_{\mathcal{B}}} : P(T_{\mathcal{A}}(e)) = F(T_{\mathcal{A}}(e) \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{B}}) \simeq T_{\mathcal{B}}(e) = T_{Z(\mathcal{B},\mathcal{E})}(e)$ for $e \in \mathcal{E}$.

Then we show that there exists a 1-morphism $(P, \rho) : \mathcal{A} \to Z(\mathcal{B}, \mathcal{E})$ in $\mathrm{Alg}_{E_0}(\mathrm{Alg}_{E_1}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}))_{\mathcal{B}}$. The invertible natural isomorphism $P^0 : T_{\mathcal{B}} \Rightarrow P \circ T_{\mathcal{A}}$ is defined by $T_{\mathcal{B}}(e) = e \odot \mathbb{1}_{\mathcal{B}} \xrightarrow{u^{-1}_{e,\mathbb{1}_{\mathcal{B}}}} F(T_{\mathcal{A}}(e) \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{B}}) = P(T_{\mathcal{A}}(e))$ for $e \in \mathcal{E}$. The monoidal natural isomorphism $\rho : G \circ (P \boxtimes_{\mathcal{E}} 1_{\mathcal{B}}) \Rightarrow F$ is defined by

$$\rho_{a,b} : F(a \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{B}}) \otimes b \xrightarrow{1, u^{-1}_{\mathbb{1}_{\mathcal{E}},b}} F(a \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{B}}) \otimes F(\mathbb{1}_{\mathcal{A}} \boxtimes_{\mathcal{E}} b) \xrightarrow{J^F} F((a \otimes \mathbb{1}_{\mathcal{A}}) \boxtimes_{\mathcal{E}} (\mathbb{1}_{\mathcal{B}} \otimes b)) = F(a \boxtimes_{\mathcal{E}} b)$$

for $a \in \mathcal{A}, b \in \mathcal{B}$. It is routine to check that the composition of 2-morphisms P^0, ρ and u is equal to the 2-morphism v.

Then we show that if there are two 1-morphisms $(Q_i, \sigma_i) : \mathcal{A} \to Z(\mathcal{B}, \mathcal{E})$ in $\mathrm{Alg}_{E_0}(\mathrm{Alg}_{E_1}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}))_{\mathcal{B}}$ for $i = 1, 2$, then there exists a unique 2-morphism $\beta : (Q_1, \sigma_1) \Rightarrow (Q_2, \sigma_2)$ in $\mathrm{Alg}_{E_0}(\mathrm{Alg}_{E_1}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}))_{\mathcal{B}}$. Such a β is a natural transformation $\beta : Q_1 \Rightarrow Q_2$ such that the equalities

$$\left(Q_1(a) \otimes b \xrightarrow{\beta_a, 1} Q_2(a) \otimes b \xrightarrow{(\sigma_2)_{a,b}} F(a \boxtimes_{\mathcal{E}} b)\right) = \left(Q_1(a) \otimes b \xrightarrow{(\sigma_1)_{a,b}} F(a \boxtimes_{\mathcal{E}} b)\right) \tag{3.7}$$

and

$$\left(T_{\mathcal{B}} \overset{Q_1^0}{\Longrightarrow} Q_1 \circ T_{\mathcal{A}} \overset{\beta * 1}{\Longrightarrow} Q_2 \circ T_{\mathcal{A}}\right) = \left(T_{\mathcal{B}} \overset{Q_2^0}{\Longrightarrow} Q_2 \circ T_{\mathcal{A}}\right) \tag{3.8}$$

hold for $a \in \mathcal{A}, b \in \mathcal{B}$. The first condition (3.7) implies that $\beta_a : Q_1(a) \to Q_2(a)$ is equal to the composition

$$Q_1(a) = Q_1(a) \otimes \mathbb{1}_{\mathcal{B}} \xrightarrow{(\sigma_1)_{a,\mathbb{1}_{\mathcal{B}}}} F(a \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{B}}) \overset{(\sigma_2)^{-1}_{a,\mathbb{1}_{\mathcal{B}}}}{\Longrightarrow} Q_2(a) \otimes \mathbb{1}_{\mathcal{B}} = Q_2(a)$$

This proves the uniqueness of β. It is routine to check that β_a is a morphism in $Z(\mathcal{B}, \mathcal{E})$ and β satisfy the second condition (3.8).

Finally, we also want to verify that the E_2-algebra structure on the E_1-center $Z(\mathcal{B}, \mathcal{E})$ coincides with the usual braiding structure on $Z(\mathcal{B}, \mathcal{E})$. The E_2-algebra structure is given by the monoidal functor $H : Z(\mathcal{B}, \mathcal{E}) \boxtimes_{\mathcal{E}} Z(\mathcal{B}, \mathcal{E}) \to Z(\mathcal{B}, \mathcal{E})$, which is induced by the iterated action

$$Z(\mathcal{B}, \mathcal{E}) \boxtimes_{\mathcal{E}} Z(\mathcal{B}, \mathcal{E}) \boxtimes_{\mathcal{E}} \mathcal{B} \xrightarrow{1,G} Z(\mathcal{B}, \mathcal{E}) \boxtimes_{\mathcal{E}} \mathcal{B} \xrightarrow{G} \mathcal{B}$$

with the monoidal structure given by

$$x_1 \otimes x_2 \otimes y_1 \otimes y_2 \otimes b_1 \otimes b_2 \xrightarrow{\gamma_{y_2,b_1}} x_1 \otimes x_2 \otimes y_1 \otimes b_1 \otimes y_2 \otimes b_2 \xrightarrow{\gamma_{x_2,y_1 \otimes b_1}} x_1 \otimes y_1 \otimes b_1 \otimes x_2 \otimes y_2 \otimes b_2$$

for $x_1 \boxtimes_{\mathcal{E}} y_1 \boxtimes_{\mathcal{E}} b_1, x_2 \boxtimes_{\mathcal{E}} y_2 \boxtimes_{\mathcal{E}} b_2$ in $Z(\mathcal{B}, \mathcal{E}) \boxtimes_{\mathcal{E}} Z(\mathcal{B}, \mathcal{E}) \boxtimes_{\mathcal{E}} \mathcal{B}$. Then by the construction given above, the induced functor $H : Z(\mathcal{B}, \mathcal{E}) \boxtimes_{\mathcal{E}} Z(\mathcal{B}, \mathcal{E}) \to Z(\mathcal{B}, \mathcal{E})$ maps $x \boxtimes_{\mathcal{E}} y$ to the object $G((1 \boxtimes_{\mathcal{E}} G)(x \boxtimes_{\mathcal{E}} y \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{B}})) = x \otimes y \otimes \mathbb{1}_{\mathcal{B}} = x \otimes y$ with the half-braiding

$$x \otimes y \otimes b \xrightarrow{\gamma_{y,b}} x \otimes b \otimes y \xrightarrow{\gamma_{x,b}} b \otimes x \otimes y$$

Thus the functor H coincides with the tensor product of $Z(\mathcal{B}, \mathcal{E})$. For $x_1 \boxtimes_{\mathcal{E}} y_1, x_2 \boxtimes_{\mathcal{E}} y_2 \in Z(\mathcal{B}, \mathcal{E}) \boxtimes_{\mathcal{E}} Z(\mathcal{B}, \mathcal{E})$, the monoidal structure of H is induced by

$$H((x_1 \boxtimes_{\mathcal{E}} y_1) \otimes (x_2 \boxtimes_{\mathcal{E}} y_2)) = x_1 \otimes x_2 \otimes y_1 \otimes y_2 \xrightarrow{\gamma_{x_2,y_1}} x_1 \otimes y_1 \otimes x_2 \otimes y_2 = H(x_1 \boxtimes_{\mathcal{E}} y_1) \otimes H(x_2 \boxtimes_{\mathcal{E}} y_2)$$

Equivalently, the braiding structure on $Z(\mathcal{B}, \mathcal{E})$ is given by $x \otimes y \xrightarrow{\gamma_{x,y}} y \otimes x$, which is the usual braiding structure on $Z(\mathcal{B}, \mathcal{E})$. □

3.6 E_2-centers

Definition 3.19. Let $\mathcal{X} \in \mathrm{Alg}_{E_2}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. The *$E_2$-center of $\mathcal{X}$ in* $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ is the E_0-center of $\mathcal{X}$ in $\mathrm{Alg}_{E_2}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$.

Theorem 3.20. Let $\mathcal{C}$ be a braided fusion category over $\mathcal{E}$. The E_2-center of $\mathcal{C}$ is the symmetric fusion category $\mathcal{C}'$ over $\mathcal{E}$.

Proof. Let $\mathcal{A}$ be a braided fusion category over $\mathcal{E}$. A left unital $\mathcal{A}$-action on $\mathcal{C}$ is a braided monoidal functor $F : \mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{C} \to \mathcal{C}$ over $\mathcal{E}$ and a monoidal natural isomorphism u over $\mathcal{E}$ shown below:

$$\begin{tikzcd} & \mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{C} \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow u"] & \\ \mathcal{E} \boxtimes_{\mathcal{E}} \mathcal{C} \arrow[ur, "T_{\mathcal{A}},1"] \arrow[rr, "\odot"'] & & \mathcal{C} \end{tikzcd}$$

More precisely, F is a monoidal functor over $\mathcal{E}$ (recall the proof of Thm. 3.18) such that the diagram

$$\begin{tikzcd} F(a_1 \boxtimes_{\mathcal{E}} x_1) \otimes F(a_2 \boxtimes_{\mathcal{E}} x_2) \arrow[r, "J^F"] \arrow[d, "c_{F(a_1 \boxtimes_{\mathcal{E}} x_1), F(a_2 \boxtimes_{\mathcal{E}} x_2)}"'] & F((a_1 \otimes a_2) \boxtimes_{\mathcal{E}} (x_1 \otimes x_2)) \arrow[d, "\tilde{c}_{a_1,a_2}, c_{x_1,x_2}"] \\ F(a_2 \boxtimes_{\mathcal{E}} x_2) \otimes F(a_1 \boxtimes_{\mathcal{E}} x_1) \arrow[r, "J^F"'] & F((a_2 \otimes a_1) \boxtimes_{\mathcal{E}} (x_2 \otimes x_1)) \end{tikzcd}$$

commutes for $a_1, a_2 \in \mathcal{A}, x_1, x_2 \in \mathcal{C}$, where $\tilde{c}$ and c are the half-braidings of $\mathcal{A}$ and $\mathcal{C}$ respectively. The braided structure on $\mathcal{E} \boxtimes_{\mathcal{E}} \mathcal{C}$ is defined as $T_{\mathcal{C}}(e_1 \otimes e_2) \otimes x_1 \otimes x_2 \xrightarrow{r_{e_1,e_2}, c_{x_1,x_2}} T_{\mathcal{C}}(e_2 \otimes e_1) \otimes x_2 \otimes x_1$, for $e_1 \boxtimes_{\mathcal{E}} x_1, e_2 \boxtimes_{\mathcal{E}} x_2 \in \mathcal{E} \boxtimes_{\mathcal{E}} \mathcal{C}$. Check that $\odot : \mathcal{E} \boxtimes_{\mathcal{E}} \mathcal{C} \to \mathcal{C}$ is a braided functor.

There is a left unital $\mathcal{C}'$-action on $\mathcal{C}$

$$\begin{tikzcd} & \mathcal{C}' \boxtimes_{\mathcal{E}} \mathcal{C} \arrow[dr, "G"] \arrow[d, phantom, "\Downarrow v"] & \\ \mathcal{E} \boxtimes_{\mathcal{E}} \mathcal{C} \arrow[ur, "T_{\mathcal{C}},1"] \arrow[rr] & & \mathcal{C} \end{tikzcd}$$

given by $G : \mathcal{C}' \boxtimes_{\mathcal{E}} \mathcal{C} \to \mathcal{C}, (z, x) \mapsto z \otimes x$ and $v_{e,x} := \mathrm{id}_{e \odot x} : G(T_{\mathcal{C}}(e) \boxtimes_{\mathcal{E}} x) = T_{\mathcal{C}}(e) \otimes x \to e \odot x$.

Next we want to show that there exists a 1-morphism $(P, \rho) : \mathcal{A} \to \mathcal{C}'$ in $\mathrm{Alg}_{E_0}(\mathrm{Alg}_{E_2}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}))_{\mathcal{C}}$. Since F is a braided monoidal functor over $\mathcal{E}$, the commutative diagram

$$\begin{tikzcd} F(a \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{C}}) \otimes x \arrow[r, "1,u^{-1}_{\mathbb{1}_{\mathcal{E}},x}"] \arrow[d, "c_{F(a \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{C}}),x}"'] & F(a \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{C}}) \otimes F(\mathbb{1}_{\mathcal{A}} \boxtimes_{\mathcal{E}} x) \arrow[r, "J^F"] \arrow[d, "c_{F(a \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{C}}), F(\mathbb{1}_{\mathcal{A}} \boxtimes_{\mathcal{E}} x)}"'] & F(a \boxtimes_{\mathcal{E}} x) \arrow[d] \arrow[dd, bend left, "1"] \\ x \otimes F(a \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{C}}) \arrow[r, "u^{-1}_{\mathbb{1}_{\mathcal{E}},x},1"] \arrow[d, "c_{x,F(a \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{C}})}"'] & F(\mathbb{1}_{\mathcal{A}} \boxtimes_{\mathcal{E}} x) \otimes F(a \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{C}}) \arrow[r, "J^F"] \arrow[d, "c_{F(\mathbb{1}_{\mathcal{A}} \boxtimes_{\mathcal{E}} x), F(a \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{C}})}"'] & F(a \boxtimes_{\mathcal{E}} x) \arrow[d] \\ F(a \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{C}}) \otimes x \arrow[r, "1,u^{-1}_{\mathbb{1}_{\mathcal{E}},x}"'] & F(a \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{C}}) \otimes F(\mathbb{1}_{\mathcal{A}} \boxtimes_{\mathcal{E}} x) \arrow[r, "J^F"] & F(a \boxtimes_{\mathcal{E}} x) \end{tikzcd}$$

implies that the equality $c_{x,F(a \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{C}})} \circ c_{F(a \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{C}}),x} = \mathrm{id}_{F(a \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{C}}) \otimes x}$ holds for $a \in \mathcal{A}$, $x \in \mathcal{C}$, i.e. $F(a \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{C}}) \in \mathcal{C}'$. Then we define the functor P by $P(a) := F(a \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{C}})$, and the monoidal structure of P is induced by that of F. The monoidal natural isomorphism $\rho : G \circ (P \boxtimes_{\mathcal{E}} 1_{\mathcal{C}}) \Rightarrow F$ is defined by

$$\rho_{e,x} : F(a \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{C}}) \otimes x \xrightarrow{1 \otimes u^{-1}_{\mathbb{1}_{\mathcal{E}},x}} F(a \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{C}}) \otimes F(\mathbb{1}_{\mathcal{A}} \boxtimes_{\mathcal{E}} x) \xrightarrow{J^F} F(a \boxtimes_{\mathcal{E}} x)$$

Then (P,ρ) is a 1-morphism in $\mathrm{Alg}_{E_0}(\mathrm{Alg}_{E_2}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}))_{\mathcal{C}}$.

It is routine to check that if there are two 1-morphisms $(Q_i,\sigma_i):\mathcal{A}\to\mathcal{C}'$, $i=1,2$, in $\mathrm{Alg}_{E_0}(\mathrm{Alg}_{E_2}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}))_{\mathcal{C}}$, there exists a unique 2-morphism $\beta:(Q_1,\sigma_1)\Rightarrow(Q_2,\sigma_2)$ in $\mathrm{Alg}_{E_0}(\mathrm{Alg}_{E_2}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}))_{\mathcal{C}}$. □

4 Representation theory and Morita theory in $\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}$

In this section, Sec. 4.1 and Sec. 4.2 study the modules over a multifusion category over $\mathcal{E}$ and bimodules in $\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}$. Sec. 4.3 and Sec. 4.4 prove that two fusion categories over $\mathcal{E}$ are Morita equivalent in $\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}$ if and only if their E_1-centers are equivalent. Sec. 4.5 studies the modules over a braided fusion category over $\mathcal{E}$.

4.1 Modules over a multifusion category over $\mathcal{E}$

Let $\mathcal{C}$ and $\mathcal{D}$ be multifusion categories over $\mathcal{E}$. We use z and $\hat{z}$ to denote the central structures of the central functors $T_{\mathcal{C}}:\mathcal{E}\to\mathcal{C}$ and $T_{\mathcal{D}}:\mathcal{E}\to\mathcal{D}$ respectively.

Definition 4.1. The 2-category $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}})$ consists of the following data.

- A class of objects in $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}})$. An object $\mathcal{M}\in\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}})$ is an object $\mathcal{M}\in\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}$ equipped with a monoidal functor $\phi:\mathcal{C}\to\mathrm{Fun}_{\mathcal{E}}(\mathcal{M},\mathcal{M})$ over $\mathcal{E}$.

  Equivalently, an object $\mathcal{M}\in\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}})$ is an object $\mathcal{M}$ both in $\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}$ and $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}})$ equipped with a monoidal natural isomorphism $u_e^{\mathcal{C}}:T_{\mathcal{C}}(e)\odot-\simeq e\odot-$ in $\mathrm{Fun}_{\mathcal{E}}(\mathcal{M},\mathcal{M})$ for each $e\in\mathcal{E}$, such that the functor $(c\odot-,s^{c\odot-})$ belongs to $\mathrm{Fun}_{\mathcal{E}}(\mathcal{M},\mathcal{M})$ for each $c\in\mathcal{C}$, and the diagram

$$\begin{tikzcd}
(T_{\mathcal{C}}(e)\otimes c)\odot- \arrow[r] \arrow[d, "z_{e,c},1"'] & T_{\mathcal{C}}(e)\odot(c\odot-) \arrow[r, "(u_e^{\mathcal{C}})_{c\odot-}"] & e\odot(c\odot-) \arrow[d, "s_{e,-}^{c\odot-}"] \\
(c\otimes T_{\mathcal{C}}(e))\odot- \arrow[r] & c\odot(T_{\mathcal{C}}(e)\odot-) \arrow[r, "1,u_e^{\mathcal{C}}"] & c\odot(e\odot-)
\end{tikzcd} \tag{4.1}$$

  commutes for $e\in\mathcal{E},c\in\mathcal{C},-\in\mathcal{M}$. We use a pair $(\mathcal{M},u^{\mathcal{C}})$ to denote an object $\mathcal{M}$ in $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}})$.

- For objects $(\mathcal{M},u^{\mathcal{C}}),(\mathcal{N},\bar{u}^{\mathcal{C}})$ in $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}})$, a 1-morphism $F:\mathcal{M}\to\mathcal{N}$ in $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}})$ is both a left $\mathcal{C}$-module functor $(F,s^F):\mathcal{M}\to\mathcal{N}$ and a left $\mathcal{E}$-module functor $(F,t^F):\mathcal{M}\to\mathcal{N}$ such that the following diagram commutes for $e\in\mathcal{E},m\in\mathcal{M}$:

$$\begin{tikzcd}
F(T_{\mathcal{C}}(e)\odot m) \arrow[r, "(u_e^{\mathcal{C}})_m"] \arrow[d, "s^F_{T_{\mathcal{C}}(e),m}"'] & F(e\odot m) \arrow[d, "t^F_{e,m}"] \\
T_{\mathcal{C}}(e)\odot F(m) \arrow[r, "(\bar{u}_e^{\mathcal{C}})_{F(m)}"'] & e\odot F(m)
\end{tikzcd} \tag{4.2}$$

- For 1-morphisms $F,G:\mathcal{M}\rightrightarrows\mathcal{N}$ in $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}})$, a 2-morphism from F to G is a left $\mathcal{C}$-module natural transformation from F to G. A left $\mathcal{C}$-module natural transformation is automatically a left $\mathcal{E}$-module natural transformation.

In the above definition, we take $\phi(c):=c\odot-$ for $c\in\mathcal{C}$. A left $\mathcal{D}^{\mathrm{rev}}$-module $\mathcal{M}$ is automatically a right $\mathcal{D}$-module, with the right $\mathcal{D}$-action defined by $m\odot d:=d\odot m$ for $m\in\mathcal{M},d\in\mathcal{D}$.

Definition 4.2. The 2-category $\mathrm{RMod}_{\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ consists of the following data.

- A class of objects in $\mathrm{RMod}_{\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. An object $\mathcal{M} \in \mathrm{RMod}_{\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ is an object $\mathcal{M} \in \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ equipped with a monoidal functor $\phi : \mathcal{D}^{\mathrm{rev}} \to \mathrm{Fun}_{\mathcal{E}}(\mathcal{M}, \mathcal{M})$ over $\mathcal{E}$.

  Equivalently, an object $\mathcal{M} \in \mathrm{RMod}_{\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ is an object $\mathcal{M}$ both in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ and $\mathrm{RMod}_{\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}})$ equipped with a monoidal natural isomorphism $u^{\mathcal{D}}_e : - \odot T_{\mathcal{D}}(e) \simeq e \odot -$ in $\mathrm{Fun}_{\mathcal{E}}(\mathcal{M}, \mathcal{M})$ for each $e \in \mathcal{E}$ such that the functor $(- \odot d, s^{-\odot d})$ belongs to $\mathrm{Fun}_{\mathcal{E}}(\mathcal{M}, \mathcal{M})$ for each $d \in \mathcal{D}$, and the diagram

$$\begin{tikzcd}
- \odot (d \otimes T_{\mathcal{D}}(e)) \arrow[r] \arrow[d, "{1, \check{z}^{-1}_{e,d}}"'] & (- \odot d) \odot T_{\mathcal{D}}(e) \arrow[r, "(u^{\mathcal{D}}_e)_{-\odot d}"] & e \odot (- \odot d) \arrow[d, "s^{-\odot d}_{e,-}"] \\
- \odot (T_{\mathcal{D}}(e) \otimes d) \arrow[r] & (- \odot T_{\mathcal{D}}(e)) \odot d \arrow[r, "{u^{\mathcal{D}}_e, 1}"'] & (e \odot -) \odot d
\end{tikzcd} \tag{4.3}$$

  commutes for $e \in \mathcal{E}, d \in \mathcal{D}, - \in \mathcal{M}$. We use a pair $(\mathcal{M}, u^{\mathcal{D}})$ to denote an object $\mathcal{M}$ in $\mathrm{RMod}_{\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$.

- For objects $(\mathcal{M}, u^{\mathcal{D}}), (\mathcal{N}, \bar{u}^{\mathcal{D}})$ in $\mathrm{RMod}_{\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$, a 1-morphism $F : \mathcal{M} \to \mathcal{N}$ in $\mathrm{RMod}_{\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ is both a right $\mathcal{D}$-module functor $(F, \tilde{s}^F) : \mathcal{M} \to \mathcal{N}$ and a left $\mathcal{E}$-module functor $(F, t^F) : \mathcal{M} \to \mathcal{N}$ such that the following diagram commutes for $e \in \mathcal{E}, m \in \mathcal{M}$:

$$\begin{tikzcd}
F(m \odot T_{\mathcal{D}}(e)) \arrow[r, "(u^{\mathcal{D}}_e)_m"] \arrow[d, "\tilde{s}^F_{m, T_{\mathcal{D}}(e)}"'] & F(e \odot m) \arrow[d, "t^F_{e,m}"] \\
F(m) \odot T_{\mathcal{D}}(e) \arrow[r, "(\bar{u}^{\mathcal{D}}_e)_{F(m)}"'] & e \odot F(m)
\end{tikzcd} \tag{4.4}$$

- For 1-morphisms $F, G : \mathcal{M} \rightrightarrows \mathcal{N}$ in $\mathrm{RMod}_{\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$, a 2-morphism from F to G is a right $\mathcal{D}$-module natural transformation from F to G.

Remark 4.3. Let $(\mathcal{M}, u^{\mathcal{D}})$ belongs to $\mathrm{RMod}_{\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. We explain the monoidal natural isomorphism $u_e : - \odot T_{\mathcal{D}}(e) \simeq e \odot -$ in $\mathrm{Fun}_{\mathcal{E}}(\mathcal{M}, \mathcal{M})$. The monoidal structure on $F : \mathcal{E} \to \mathrm{Fun}_{\mathcal{E}}(\mathcal{M}, \mathcal{M}), e \mapsto F^e := - \odot T_{\mathcal{D}}(e)$ is defined as $J_{e_1, e_2} : F^{e_1 \otimes e_2} = - \odot T_{\mathcal{D}}(e_1 \otimes e_2) \xrightarrow{r_{e_1,e_2}} - \odot T_{\mathcal{D}}(e_2 \otimes e_1) \to (- \odot T_{\mathcal{D}}(e_2)) \odot T_{\mathcal{D}}(e_1) = F^{e_1} \circ F^{e_2}$, for $e_1, e_2 \in \mathcal{E}$. The monoidal structure on $T : \mathcal{E} \to \mathrm{Fun}_{\mathcal{E}}(\mathcal{M}, \mathcal{M}), e \mapsto T^e := e \odot -$ is defined as $T^{e_1 \otimes e_2} = (e_1 \otimes e_2) \odot - \to e_1 \odot (e_2 \odot -) = T^{e_1} \circ T^{e_2}$, for $e_1, e_2 \in \mathcal{E}$. For each $e \in \mathcal{E}$, $u_e : - \odot T_{\mathcal{D}}(e) \to e \odot -$ is an isomorphism in $\mathrm{Fun}_{\mathcal{E}}(\mathcal{M}, \mathcal{M})$. That is, u_e is a left $\mathcal{E}$-module natural isomorphism. The monoidal natural isomorphism $u_e : - \odot T_{\mathcal{D}}(e) \to e \odot -$ satisfies the diagram

$$\begin{tikzcd}
- \odot T_{\mathcal{D}}(e_1 \otimes e_2) \arrow[r, "r_{e_1,e_2}"] \arrow[d, "u_{e_1 \otimes e_2}"'] & - \odot T_{\mathcal{D}}(e_2 \otimes e_1) \arrow[r] & (- \odot T_{\mathcal{D}}(e_2)) \odot T_{\mathcal{D}}(e_1) \arrow[d, "u_{e_1} * u_{e_2}"] \\
(e_1 \otimes e_2) \odot - \arrow[rr] & & e_1 \odot (e_2 \odot -)
\end{tikzcd}$$

where $u_{e_1} * u_{e_2}$ is defined as

$$\begin{tikzcd}
F^{e_1} \circ F^{e_2} = (- \odot T_{\mathcal{D}}(e_2)) \odot T_{\mathcal{D}}(e_1) \arrow[r, "{u_{e_2}, 1}"] \arrow[d, "(u_{e_1})_{-\odot T_{\mathcal{D}}(e_2)}"'] \arrow[dr, "u_{e_1} * u_{e_2}"] & (e_2 \odot -) \odot T_{\mathcal{D}}(e_1) \arrow[d, "(u_{e_1})_{e_2 \odot -}"] \\
e_1 \odot (- \odot T_{\mathcal{D}}(e_2)) \arrow[r, "{1, u_{e_2}}"'] & e_1 \odot (e_2 \odot -) = T^{e_1} \circ T^{e_2}
\end{tikzcd}$$

For any $d_1, d_2 \in \mathcal{D}$, the functors $(-\odot d_1, s^{-\odot d_1})$, $(-\odot d_2, s^{-\odot d_2})$ and $(-\odot (d_1 \otimes d_2), s^{-\odot(d_1\otimes d_2)})$ belong to $\mathrm{Fun}_{\mathcal{E}}(\mathcal{M},\mathcal{M})$. Consider the diagram:

$$\begin{tikzcd}
e\odot(m\odot(d_1\otimes d_2)) \arrow[rr, "s^{-\odot(d_1\otimes d_2)}_{e,m}"] \arrow[d, "{1,\lambda^{\mathcal{M}}_{m,d_1,d_2}}"'] & & (e\odot m)\odot(d_1\otimes d_2) \arrow[d, "\lambda^{\mathcal{M}}_{e\odot m,d_1,d_2}"] \\
e\odot((m\odot d_1)\odot d_2) \arrow[r, "s^{-\odot d_2}_{e,m\odot d_1}"'] & (e\odot(m\odot d_1))\odot d_2 \arrow[r, "{s^{-\odot d_1}_{e,m},1}"'] & ((e\odot m)\odot d_1)\odot d_2
\end{tikzcd} \tag{4.5}$$

where $\lambda^{\mathcal{M}}$ is the module associativity constraint of $\mathcal{M}$ in $\mathrm{RMod}_{\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}})$. Since the diagrams (4.3) and (A.1) commute and $m\odot - : \mathcal{D}\to\mathcal{M}$ is the functor for all $m\in\mathcal{M}$, the above diagram commutes. Then the natural isomorphism $-\odot(d_1\otimes d_2) \Rightarrow (-\odot d_1)\odot d_2$ is the left $\mathcal{E}$-module natural isomorphism.

Let $(\mathcal{M}, u^{\mathcal{C}})$ belong to $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. For any $c_1, c_2 \in \mathcal{C}$, the functors $(c_1\odot -, s^{c_1\odot -})$, $(c_2\odot -, s^{c_2\odot -})$ and $((c_1\otimes c_2)\odot -, s^{(c_1\otimes c_2)\odot -})$ belong to $\mathrm{Fun}_{\mathcal{E}}(\mathcal{M},\mathcal{M})$. Since the diagrams (4.1) and (A.1) commutes and $-\odot m : \mathcal{C}\to\mathcal{M}$ is the functor for all $m\in\mathcal{M}$, the natural isomorphism $(c_1\otimes c_2)\odot - \Rightarrow c_1\odot(c_2\odot -)$ is the left $\mathcal{E}$-module natural isomorphism.

Remark 4.4. Assume that $(\mathcal{M}, u^{\mathcal{D}})$ belongs to $\mathrm{RMod}_{\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. The right $\mathcal{E}$-module structure on $\mathcal{M}$ is defined as $m\bar{\odot}e := m\odot T_{\mathcal{D}}(e)$, $\forall m\in\mathcal{M}, e\in\mathcal{E}$. The module associativity constraint is defined as $\lambda^1_{m,e_1,e_2} : m\odot T_{\mathcal{D}}(e_1\otimes e_2) \to m\odot(T_{\mathcal{D}}(e_1)\otimes T_{\mathcal{D}}(e_2)) \to (m\odot T_{\mathcal{D}}(e_1))\odot T_{\mathcal{D}}(e_2)$, $\forall m\in\mathcal{M}, e_1, e_2\in\mathcal{E}$. Another right $\mathcal{E}$-module structure on $\mathcal{M}$ is defined as $m\odot e := e\odot m$, $\forall e\in\mathcal{E}, m\in\mathcal{M}$. The module associativity constraint is defined as $\lambda^2_{m,e_1,e_2} : m\odot(e_1\otimes e_2) = (e_1\otimes e_2)\odot m \xrightarrow{r_{e_1,e_2},1} (e_2\otimes e_1)\odot m \to e_2\odot(e_1\odot m) = (m\odot e_1)\odot e_2$, $\forall m\in\mathcal{M}, e_1, e_2\in\mathcal{E}$.

Check that the identity functor $\mathrm{id} : \mathcal{M}\to\mathcal{M}$ equipped with the natural isomorphism $s^{\mathrm{id}}_{m,e} : \mathrm{id}(m\bar{\odot}e) = m\odot T_{\mathcal{D}}(e) \xrightarrow{(u^{\mathcal{D}}_e)_m} e\odot m = \mathrm{id}(m)\odot e$ is a right $\mathcal{E}$-module functor by the monoidal natural isomorphism $u^{\mathcal{D}}_e : -\odot T_{\mathcal{D}}(e) \to e\odot -$.

Proposition 4.5. Let $(\mathcal{M}, u^{\mathcal{C}})$ belong to $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. The diagram

$$\begin{tikzcd}
\tilde{e}\odot(T_{\mathcal{C}}(e)\odot m) \arrow[r, "{1,(u^{\mathcal{C}}_e)_m}"] \arrow[d, "s^{T_{\mathcal{C}}(e)\odot -}_{\tilde{e},m}"'] & \tilde{e}\odot(e\odot m) \arrow[r] & (\tilde{e}\otimes e)\odot m \arrow[d, "{r_{\tilde{e},e},1}"] \\
T_{\mathcal{C}}(e)\odot(\tilde{e}\odot m) \arrow[r, "(u^{\mathcal{C}}_e)_{\tilde{e}\odot m}"'] & e\odot(\tilde{e}\odot m) \arrow[r] & (e\otimes\tilde{e})\odot m
\end{tikzcd} \tag{4.6}$$

commutes for $e, \tilde{e}\in\mathcal{E}$, $m\in\mathcal{M}$. Let $(\mathcal{M}, u^{\mathcal{D}})$ belong to $\mathrm{RMod}_{\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. The diagram

$$\begin{tikzcd}
(e\odot m)\odot T_{\mathcal{D}}(\tilde{e}) \arrow[r, "(u^{\mathcal{D}}_{\tilde{e}})_{e\odot m}"] \arrow[d, "s^{-\odot T_{\mathcal{D}}(\tilde{e})}_{e,m}"'] & \tilde{e}\odot(e\odot m) \arrow[r] & (\tilde{e}\otimes e)\odot m \arrow[d, "{r_{\tilde{e},e},1}"] \\
e\odot(m\odot T_{\mathcal{D}}(\tilde{e})) \arrow[r, "{1,(u^{\mathcal{D}}_{\tilde{e}})_m}"'] & e\odot(\tilde{e}\odot m) \arrow[r] & (e\otimes\tilde{e})\odot m
\end{tikzcd} \tag{4.7}$$

commutes for $e, \tilde{e}\in\mathcal{E}, m\in\mathcal{M}$. Here the functors $(T_{\mathcal{C}}(e)\odot -, s^{T_{\mathcal{C}}(e)\odot -})$ and $(-\odot T_{\mathcal{D}}(\tilde{e}), s^{-\odot T_{\mathcal{D}}(\tilde{e})})$ belong to $\mathrm{Fun}_{\mathcal{E}}(\mathcal{M},\mathcal{M})$.

Proof. Consider the diagram:

$$\begin{tikzcd}
\tilde{e}\odot(T_{\mathcal{C}}(e)\odot m) \arrow[r, "1{,}(u^{\mathcal{C}}_{e})_m"] \arrow[ddd, bend right=60, "s^{T_{\mathcal{C}}(e)\odot -}_{\tilde{e},m}"'] & \tilde{e}\odot(e\odot m) \arrow[r] & (\tilde{e}\otimes e)\odot m \arrow[ddd, bend left=60, "r_{\tilde{e},e},1"] \\
T_{\mathcal{C}}(\tilde{e})\odot(T_{\mathcal{C}}(e)\odot m) \arrow[u, "(u^{\mathcal{C}}_{\tilde{e}})_{T_{\mathcal{C}}(e)\odot m}"'] & (T_{\mathcal{C}}(\tilde{e})\otimes T_{\mathcal{C}}(e))\odot m \arrow[l] \arrow[d, "z_{\tilde{e},T_{\mathcal{C}}(e)}"'] & T_{\mathcal{C}}(\tilde{e}\otimes e)\odot m \arrow[l] \arrow[u, "u^{\mathcal{C}}_{\tilde{e}\otimes e}"] \arrow[d, "r_{\tilde{e},e},1"'] \\
T_{\mathcal{C}}(e)\odot(T_{\mathcal{C}}(\tilde{e})\odot m) \arrow[d, "1{,}(u^{\mathcal{C}}_{\tilde{e}})_m"] & (T_{\mathcal{C}}(e)\otimes T_{\mathcal{C}}(\tilde{e}))\odot m \arrow[l] & T_{\mathcal{C}}(e\otimes\tilde{e})\odot m \arrow[l] \arrow[d, "u^{\mathcal{C}}_{e\otimes\tilde{e}}"'] \\
T_{\mathcal{C}}(e)\odot(\tilde{e}\odot m) \arrow[r, "(u^{\mathcal{C}}_{e})_{\tilde{e}\odot m}"'] & e\odot(\tilde{e}\odot m) \arrow[r] & (e\otimes\tilde{e})\odot m
\end{tikzcd}$$

The top and bottom hexagon diagrams commute by the monoidal natural isomorphism $u^{\mathcal{C}}_e : T_{\mathcal{C}}(e)\odot - \simeq e\odot -$. The leftmost hexagon commutes by the diagram (4.1). The middle-right square commutes by the central functor $T_{\mathcal{C}} : \mathcal{E}\to\mathcal{C}$. The rightmost square commutes by the naturality of $u^{\mathcal{C}}_e$. Then the outward diagram commutes. One can check the diagram (4.7) commutes. □

For objects $\mathcal{M},\mathcal{N}$ in $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ (or $\mathrm{RMod}_{\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$), we use $\mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ (or $\mathrm{Fun}^{\mathcal{E}}_{|\mathcal{D}}(\mathcal{M},\mathcal{N})$) to denote the category of 1-morphisms $\mathcal{M}\to\mathcal{N}$, 2-morphisms in $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ (or $\mathrm{RMod}_{\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$).

Example 4.6. $\mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}}(\mathcal{M},\mathcal{M})$ is a multifusion category by [EGNO, Cor. 9.3.3]. Moreover, $\mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}}(\mathcal{M},\mathcal{M})$ is a multifusion category over $\mathcal{E}$. A functor $\hat{T} : \mathcal{E}\to\mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}}(\mathcal{M},\mathcal{M})$ is defined as $e\mapsto\hat{T}^e := T_{\mathcal{C}}(e)\odot -$. The left $\mathcal{C}$-module structure on $\hat{T}^e$ is defined as $s_{c,m} : T_{\mathcal{C}}(e)\odot(c\odot m)\to(T_{\mathcal{C}}(e)\otimes c)\odot m\xrightarrow{z_{e,c},1}(c\otimes T_{\mathcal{C}}(e))\odot m\to c\odot(T_{\mathcal{C}}(e)\odot m)$ for $c\in\mathcal{C}$, $m\in\mathcal{M}$. The left $\mathcal{E}$-module structure on $\hat{T}^e$ is defined as $T_{\mathcal{C}}(e)\odot(\tilde{e}\odot m)\xrightarrow{1,(u^{\mathcal{C}}_{\tilde{e}})^{-1}_m}T_{\mathcal{C}}(e)\odot(T_{\mathcal{C}}(\tilde{e})\odot m)\xrightarrow{s_{T_{\mathcal{C}}(\tilde{e}),m}}T_{\mathcal{C}}(\tilde{e})\odot(T_{\mathcal{C}}(e)\odot m)\xrightarrow{(u^{\mathcal{C}}_{\tilde{e}})_{T_{\mathcal{C}}(e)\odot m}}\tilde{e}\odot(T_{\mathcal{C}}(e)\odot m)$ for $\tilde{e}\in\mathcal{E}$, $m\in\mathcal{M}$. Then $\hat{T}^e$ belongs to $\mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}}(\mathcal{M},\mathcal{M})$.

The monoidal structure on $\hat{T}$ is induced by $T_{\mathcal{C}}(e_1\otimes e_2)\odot - \simeq (T_{\mathcal{C}}(e_1)\otimes T_{\mathcal{C}}(e_2))\odot - \simeq T_{\mathcal{C}}(e_1)\odot(T_{\mathcal{C}}(e_2)\odot -)$ for $e_1,e_2\in\mathcal{E}$. The central structure on $\hat{T}$ is a natural isomorphism $\sigma_{e,g} : \hat{T}^e\circ g(m) = T_{\mathcal{C}}(e)\odot g(m)\simeq g(T_{\mathcal{C}}(e)\odot m) = g\circ\hat{T}^e(m)$ for any $e\in\mathcal{E}$, $g\in\mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}}(\mathcal{M},\mathcal{M})$, $m\in\mathcal{M}$. The left (or right) $\mathcal{E}$-module structure on $\mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}}(\mathcal{M},\mathcal{M})$ is defined as $(e\odot f)(-) := T_{\mathcal{C}}(e)\odot f(-)$, (or $(f\odot e)(-) := f(T_{\mathcal{C}}(e)\odot -)$), for $e\in\mathcal{E}$, $f\in\mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}}(\mathcal{M},\mathcal{M})$ and $-\in\mathcal{M}$.

Proposition 4.7. Let $(\mathcal{M},u)$ and $(\mathcal{N},\bar{u})$ belong to $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. $f : \mathcal{M}\to\mathcal{N}$ is a 1-morphism in $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}})$. Then f belongs to $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$.

Proof. Notice that for a 1-morphism $f : \mathcal{M}\to\mathcal{N}$ in $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$, the left $\mathcal{C}$-action on f is compatible with the left $\mathcal{E}$-action on f. Assume $(f,s) : \mathcal{M}\to\mathcal{N}$ is a left $\mathcal{C}$-module functor. The left $\mathcal{E}$-module structure on f is given by $f(e\odot m)\xrightarrow{(u_e^{-1})_m}f(T_{\mathcal{C}}(e)\odot m)\xrightarrow{s_{T_{\mathcal{C}}(e),m}}T_{\mathcal{C}}(e)\odot f(m)\xrightarrow{(\bar{u}_e)_{f(m)}}e\odot f(m)$. □

Remark 4.8. The forgetful functor $\mathbf{f} : \mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}}(\mathcal{M},\mathcal{N})\to\mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{N})$, $(f,s,t)\mapsto(f,s)$ induces an equivalence in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$, where s and t are the left $\mathcal{C}$-module structure and the left $\mathcal{E}$-module structure on f respectively. Notice that t equals to the composition of u^{-1}, s and $\bar{u}$.

Let $(\mathcal{M},u)$ and $(\mathcal{M},\mathrm{id})$ belong to $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. Then the identity functor $\mathrm{id}_{\mathcal{M}} : (\mathcal{M},u)\to(\mathcal{M},\mathrm{id})$ induces an equivalence in $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$.

Example 4.9. Let A be a separable algebra in $\mathcal{C}$. We use $\mathcal{C}_A$ to denote the category of right A-modules in $\mathcal{C}$. By [DMNO, Prop. 2.7], the category $\mathcal{C}_A$ is a finite semisimple abelian category. $\mathcal{C}_A$ has a canonical left $\mathcal{C}$-module structure. The left $\mathcal{E}$-module structure on $\mathcal{C}_A$ is defined as $e\odot x := T_{\mathcal{C}}(e)\otimes x$ for any $e\in\mathcal{E}$, $x\in\mathcal{C}_A$. Then $(\mathcal{C}_A,\mathrm{id})$ belongs to $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$.

We use ${}_A\mathcal{C}_A$ to denote the category of A-bimodules in $\mathcal{C}$. By Prop. A.5, $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C}_A, \mathcal{C}_A)$ is equivalent to $({}_A\mathcal{C}_A)^{\mathrm{rev}}$ as multifusion categories over $\mathcal{E}$.

Proposition 4.10. Let $\mathcal{M} \in \mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. There is a separable algebra A in $\mathcal{C}$ such that $\mathcal{M} \simeq \mathcal{C}_A$ in $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$.

Proof. By [EGNO, Thm. 7.10.1], there is an equivalence $\eta : \mathcal{M} \simeq \mathcal{C}_A$ in $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}})$ for some separable algebra A in $\mathcal{C}$. By Prop. 4.7, η is an equivalence in $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. □

Definition 4.11. An object $\mathcal{M}$ in $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ is *faithful* if there exists $m \in \mathcal{M}$ such that $\mathbb{1}^i_{\mathcal{C}} \odot m \neq 0$ for every nonzero subobject $\mathbb{1}^i_{\mathcal{C}}$ of the unit object $\mathbb{1}_{\mathcal{C}}$.

Remark 4.12. Notice that $\mathbb{1}_{\mathcal{E}} \odot m \simeq T_{\mathcal{C}}(\mathbb{1}_{\mathcal{E}}) \odot m = \mathbb{1}_{\mathcal{C}} \odot m \neq 0$. If $\mathcal{C}$ is an indecomposable multifusion category over $\mathcal{E}$, any nonzero $\mathcal{M}$ in $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ is faithful.

Proposition 4.13. Suppose $\mathcal{M}$ is a faithful object in $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. There is an equivalence $\mathcal{C} \simeq \mathrm{Fun}^{\mathcal{E}}_{\mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}}(\mathcal{M},\mathcal{M})}(\mathcal{M}, \mathcal{M})$ of multifusion categories over $\mathcal{E}$.

Proof. By Prop. 4.10, there is a separable algebra A in $\mathcal{C}$ such that $\mathcal{M} \simeq \mathcal{C}_A$. By [EGNO, Thm. 7.12.11], the category $\mathrm{Fun}_{\mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{M})}(\mathcal{M}, \mathcal{M})$ is equivalent to the category of $A^R \otimes A$-bimodules in the category of A-bimodules. The latter category is equivalent to the category ${}_{A^R\otimes A}\mathcal{C}_{A^R\otimes A}$ of $A^R \otimes A$-bimodules. Then the functor $\Phi : \mathcal{C} \to {}_{A^R\otimes A}\mathcal{C}_{A^R\otimes A}$, $x \mapsto A^R \otimes x \otimes A$ is an equivalence by the faithfulness of $\mathcal{M}$. The monoidal structure on Φ is defined as

$$\Phi(x \otimes y) = A^R \otimes x \otimes y \otimes A \simeq A^R \otimes x \otimes A \otimes_{A^R\otimes A} A^R \otimes y \otimes A = \Phi(x) \otimes_{A^R\otimes A} \Phi(y)$$

for $x, y \in \mathcal{C}$, where the equivalence is due to $A \otimes_{A^R\otimes A} A^R \simeq \mathbb{1}_{\mathcal{C}}$. Recall the central structure on the monoidal functor $I : \mathcal{E} \to {}_{A^R\otimes A}\mathcal{C}_{A^R\otimes A}$ in Expl. 3.9. The structure of monoidal functor over $\mathcal{E}$ on Φ is induced by $\Phi(T_{\mathcal{C}}(e)) = A^R \otimes T_{\mathcal{C}}(e) \otimes A \xrightarrow{z^{-1}_{e,A^R},1} T_{\mathcal{C}}(e) \otimes A^R \otimes A = I(e)$ for $e \in \mathcal{E}$.

By Rem. 4.8 and Prop. A.5, we have the equivalences $\mathrm{Fun}^{\mathcal{E}}_{\mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}}(\mathcal{M},\mathcal{M})}(\mathcal{M}, \mathcal{M}) \simeq \mathrm{Fun}_{\mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{M})}(\mathcal{M}, \mathcal{M}) \simeq {}_{A^R\otimes A}\mathcal{C}_{A^R\otimes A} \simeq \mathcal{C}$ of multifusion categories over $\mathcal{E}$. □

4.2 Bimodules in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$

Let $\mathcal{C}$ and $\mathcal{D}$ be multifusion categories over $\mathcal{E}$. We use z and $\hat{z}$ to denote the central structures of the central functors $T_{\mathcal{C}} : \mathcal{E} \to \mathcal{C}$ and $T_{\mathcal{D}} : \mathcal{E} \to \mathcal{D}$ respectively.

Definition 4.14. The 2-category $\mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ consists of the following data.

- A class of objects in $\mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. An object $\mathcal{M} \in \mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ is an object $\mathcal{M}$ both in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ and $\mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}})$ equipped with monoidal natural isomorphisms $u^{\mathcal{C}}_e : T_{\mathcal{C}}(e) \odot - \simeq e \odot -$ and $u^{\mathcal{D}}_e : - \odot T_{\mathcal{D}}(e) \simeq e \odot -$ in $\mathrm{Fun}_{\mathcal{E}}(\mathcal{M}, \mathcal{M})$ for each $e \in \mathcal{E}$ such that the functor $(c \odot - \odot d, s^{c\odot-\odot d})$ belongs to $\mathrm{Fun}_{\mathcal{E}}(\mathcal{M}, \mathcal{M})$ for each $c \in \mathcal{C}$, $d \in \mathcal{D}$, and the diagrams

$$\begin{tikzcd}
(T_{\mathcal{C}}(e) \otimes c) \odot - \odot d \arrow[r] \arrow[d, "{z_{e,c},1,1}"'] & T_{\mathcal{C}}(e) \odot (c \odot - \odot d) \arrow[r, "(u^{\mathcal{C}}_e)_{c\odot-\odot d}"] & e \odot (c \odot - \odot d) \arrow[d, "s^{c\odot-\odot d}"] \\
(c \otimes T_{\mathcal{C}}(e)) \odot - \odot d \arrow[r] & c \odot (T_{\mathcal{C}}(e) \odot -) \odot d \arrow[r, "{1,u^{\mathcal{C}}_e,1}"] & c \odot (e \odot -) \odot d
\end{tikzcd} \tag{4.8}$$

$$\begin{tikzcd}
c \odot - \odot (d \otimes T_{\mathcal{D}}(e)) \arrow[r] \arrow[d, "{1,1,\hat{z}^{-1}_{e,d}}"'] & (c \odot - \odot d) \odot T_{\mathcal{D}}(e) \arrow[r, "(u^{\mathcal{D}}_e)_{c\odot-\odot d}"] & e \odot (c \odot - \odot d) \arrow[d, "s^{c\odot-\odot d}"] \\
c \odot - \odot (T_{\mathcal{D}}(e) \otimes d) \arrow[r] & c \odot (- \odot T_{\mathcal{D}}(e)) \odot d \arrow[r, "{1,u^{\mathcal{D}}_e,1}"] & c \odot (e \odot -) \odot d
\end{tikzcd} \tag{4.9}$$

commute for all $e \in \mathcal{E}, c \in \mathcal{C}, d \in \mathcal{D}$. We use a triple $(\mathcal{M}, u^{\mathcal{C}}, u^{\mathcal{D}})$ to denote an object $\mathcal{M}$ in $\mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$.

- For objects $(\mathcal{M}, u^{\mathcal{C}}, u^{\mathcal{D}})$, $(\mathcal{N}, \bar{u}^{\mathcal{C}}, \bar{u}^{\mathcal{D}})$ in $\mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$, a 1-morphism $F : \mathcal{M} \to \mathcal{N}$ in $\mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ is a 1-morphism $F : \mathcal{M} \to \mathcal{N}$ both in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ and $\mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}})$ such that the diagrams (4.2) and (4.4) commute.
- For 1-morphisms $F, G : \mathcal{M} \rightrightarrows \mathcal{N}$ in $\mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$, a 2-morphism from F to G is a $\mathcal{C}$-$\mathcal{D}$ bimodule natural transformation from F to G.

For objects $\mathcal{M}, \mathcal{N}$ in $\mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$, we use $\mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}|\mathcal{D}}(\mathcal{M}, \mathcal{N})$ to denote the category of 1-morphisms $\mathcal{M} \to \mathcal{N}$, 2-morphisms in $\mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$.

Let $(\mathcal{M}, u^{\mathcal{C}}, u^{\mathcal{D}}), (\mathcal{N}, \bar{u}^{\mathcal{C}}, \bar{u}^{\mathcal{D}})$ belong to $\mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. A monoidal natural isomorphism $v^{\mathcal{M}}$ is defined as $v^{\mathcal{M}}_e : T_{\mathcal{C}}(e) \odot - \xRightarrow{u^{\mathcal{C}}_e} e \odot - \xRightarrow{(u^{\mathcal{D}}_e)^{-1}} - \odot T_{\mathcal{D}}(e)$ for $e \in \mathcal{E}$, $- \in \mathcal{M}$. Similarly, a monoidal natural isomorphism $v^{\mathcal{N}}$ is defined as $v^{\mathcal{N}}_e := (\bar{u}^{\mathcal{D}}_e)^{-1} \circ \bar{u}^{\mathcal{C}}_e$. A 1-morphism $F : \mathcal{M} \to \mathcal{N}$ in $\mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ satisfies the following diagram for $e \in \mathcal{E}, m \in \mathcal{M}$:

$$\begin{tikzcd}
F(T_{\mathcal{C}}(e) \odot m) \arrow[r, "(v^{\mathcal{M}}_e)_m"] \arrow[d] & F(m \odot T_{\mathcal{D}}(e)) \arrow[d] \\
T_{\mathcal{C}}(e) \odot F(m) \arrow[r, "(v^{\mathcal{N}}_e)_{F(m)}"'] & F(m) \odot T_{\mathcal{D}}(e)
\end{tikzcd} \tag{4.10}$$

Remark 4.15. Plugging $c = \mathbb{1}_{\mathcal{C}}$ into the diagram (4.8) and $d = \mathbb{1}_{\mathcal{D}}$ into the diagram (4.9), the diagrams

$$\begin{tikzcd}
T_{\mathcal{C}}(e) \odot (m \odot d) \arrow[r] \arrow[d, "(u^{\mathcal{C}}_e)_{m\odot d}"'] & (T_{\mathcal{C}}(e) \odot m) \odot d \arrow[d, "(u^{\mathcal{C}}_e)_m{,}1"] \\
e \odot (m \odot d) \arrow[r, "s^{-\odot d}"'] & (e \odot m) \odot d
\end{tikzcd}
\qquad
\begin{tikzcd}
(c \odot m) \odot T_{\mathcal{D}}(e) \arrow[r] \arrow[d, "(u^{\mathcal{D}}_e)_{c\odot m}"'] & c \odot (m \odot T_{\mathcal{D}}(e)) \arrow[d, "1{,}(u^{\mathcal{D}}_e)_m"] \\
e \odot (c \odot m) \arrow[r, "s^{c\odot -}"'] & c \odot (e \odot m)
\end{tikzcd} \tag{4.11}$$

commute for $m \in \mathcal{M}$. Since the diagrams (4.11) and (4.6) commute, the diagram

$$\begin{tikzcd}
(T_{\mathcal{C}}(e) \odot m) \odot T_{\mathcal{D}}(\tilde{e}) \arrow[r, "(u^{\mathcal{C}}_e)_m{,}1"] \arrow[d] & (e \odot m) \odot T_{\mathcal{D}}(\tilde{e}) \arrow[r, "(u^{\mathcal{D}}_{\tilde{e}})_{e\odot m}"] & \tilde{e} \odot (e \odot m) \arrow[r] & (\tilde{e} \otimes e) \odot m \arrow[d, "r_{e,\tilde{e}}{,}1"] \\
T_{\mathcal{C}}(e) \odot (m \odot T_{\mathcal{D}}(\tilde{e})) \arrow[r, "1{,}(u^{\mathcal{D}}_{\tilde{e}})_m"'] & T_{\mathcal{C}}(e) \odot (\tilde{e} \odot m) \arrow[r, "(u^{\mathcal{C}}_e)_{\tilde{e}\odot m}"'] & e \odot (\tilde{e} \odot m) \arrow[r] & (e \otimes \tilde{e}) \odot m
\end{tikzcd} \tag{4.12}$$

commutes for $e, \tilde{e} \in \mathcal{E}$, $m \in \mathcal{M}$. Since the diagrams (4.11), (4.1) and (4.3) commute, the diagrams

$$\begin{tikzcd}
T_{\mathcal{C}}(e) \odot (m \odot d) \arrow[r, "(v^{\mathcal{M}}_e)_{m\odot d}"] \arrow[d] & (m \odot d) \odot T_{\mathcal{D}}(e) \arrow[r] & m \odot (d \otimes T_{\mathcal{D}}(e)) \arrow[d, "1{,}\hat{z}^{-1}_{e,d}"] \\
(T_{\mathcal{C}}(e) \odot m) \odot d \arrow[r, "(v^{\mathcal{M}}_e)_m{,}1"'] & (m \odot T_{\mathcal{D}}(e)) \odot d \arrow[r] & m \odot (T_{\mathcal{D}}(e) \otimes d)
\end{tikzcd}$$

$$\begin{tikzcd}
(T_{\mathcal{C}}(e) \otimes c) \odot m \arrow[r] \arrow[d, "z_{e,c}{,}1"'] & T_{\mathcal{C}}(e) \odot (c \odot m) \arrow[r, "(v^{\mathcal{M}}_e)_{c\odot m}"] & (c \odot m) \odot T_{\mathcal{D}}(e) \arrow[d] \\
(c \otimes T_{\mathcal{C}}(e)) \odot m \arrow[r] & c \odot (T_{\mathcal{C}}(e) \odot m) \arrow[r, "1{,}(v^{\mathcal{M}}_e)_m"'] & c \odot (m \odot T_{\mathcal{D}}(e))
\end{tikzcd}$$

commute for $e \in \mathcal{E}, d \in \mathcal{D}, c \in \mathcal{C}, m \in \mathcal{M}$.

Proposition 4.16. Let $\mathcal{A}, \mathcal{B}$ be multifusion categories over $\mathcal{E}$. There is an equivalence of 2-categories

$$\mathrm{LMod}_{\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}^{\mathrm{rev}}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}) \simeq \mathrm{BMod}_{\mathcal{A}|\mathcal{B}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$$

Proof. An object $\mathcal{M} \in \mathrm{LMod}_{\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}^{\mathrm{rev}}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ is an object $\mathcal{M} \in \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ equipped with a monoidal functor $\phi : \mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{B}^{\mathrm{rev}} \to \mathrm{Fun}_{\mathcal{E}}(\mathcal{M}, \mathcal{M})$ over $\mathcal{E}$. Given an object $\mathcal{M}$ in $\mathrm{LMod}_{\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}^{\mathrm{rev}}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$, we want to define an object $(\mathcal{M}, u^{\mathcal{A}}, u^{\mathcal{B}})$ in $\mathrm{BMod}_{\mathcal{A}|\mathcal{B}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. The left $\mathcal{A}$-action on $\mathcal{M}$ is defined as $a \odot m := \phi^{a\boxtimes_{\mathcal{E}}\mathbb{1}_{\mathcal{B}}}(m)$ for $a \in \mathcal{A}$, $m \in \mathcal{M}$, and the unit $\mathbb{1}_{\mathcal{B}} \in \mathcal{B}^{\mathrm{rev}}$. And the right $\mathcal{B}$-action on $\mathcal{M}$ is defined as $m \odot b := \phi^{\mathbb{1}_{\mathcal{A}}\boxtimes_{\mathcal{E}} b}(m)$ for $b \in \mathcal{B}$, $m \in \mathcal{M}$, and the unit $\mathbb{1}_{\mathcal{A}} \in \mathcal{A}$. By Expl. 3.8, we have $T_{\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}^{\mathrm{rev}}}(e) = T_{\mathcal{A}}(e) \boxtimes_{\mathcal{E}} \mathbb{1}_{\mathcal{B}}$ and $T_{\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}^{\mathrm{rev}}}(e) \simeq \mathbb{1}_{\mathcal{A}} \boxtimes_{\mathcal{E}} T_{\mathcal{B}}(e)$. Recall the central structure on $T : \mathcal{E} \to \mathrm{Fun}_{\mathcal{E}}(\mathcal{M}, \mathcal{M})$ in Expl.3.7. The structure of monoidal functor over $\mathcal{E}$ on ϕ gives the monoidal natural isomorphisms $u^{\mathcal{A}}$ and $u^{\mathcal{B}}$ and the commutativity of diagrams (4.8) and (4.9).

Given objects $\mathcal{M}, \mathcal{N}$ and a 1-morphism $f : \mathcal{M} \to \mathcal{N}$ in $\mathrm{LMod}_{\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}^{\mathrm{rev}}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$, f satisfy the diagrams (4.2) and (4.4). For two 1-morphisms $f, g : \mathcal{M} \rightrightarrows \mathcal{N}$ in $\mathrm{LMod}_{\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}^{\mathrm{rev}}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$, a 2-morphism $\alpha : f \Rightarrow g$ in $\mathrm{LMod}_{\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}^{\mathrm{rev}}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ is a left $\mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{B}^{\mathrm{rev}}$-module natural transformation. If $\mathcal{B}^{\mathrm{rev}} = \mathcal{E}$, α is a left $\mathcal{A}$-module natural transformation. If $\mathcal{A} = \mathcal{E}$, α is a right $\mathcal{B}$-module natural transformation.

Conversely, given an object $(\mathcal{M}, u^{\mathcal{A}}, u^{\mathcal{B}})$ in $\mathrm{BMod}_{\mathcal{A}|\mathcal{B}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$, we want to define a monoidal functor $\phi : \mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{B}^{\mathrm{rev}} \to \mathrm{Fun}_{\mathcal{E}}(\mathcal{M}, \mathcal{M})$ over $\mathcal{E}$. For $a \boxtimes_{\mathcal{E}} b \in \mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{B}^{\mathrm{rev}}$, we define $\phi^{a\boxtimes_{\mathcal{E}} b} := (a \boxtimes_{\mathcal{E}} b) \odot - := a \odot - \odot b$ for $- \in \mathcal{M}$. For $a_1 \boxtimes_{\mathcal{E}} b_1$, $a_2 \boxtimes_{\mathcal{E}} b_2 \in \mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{B}^{\mathrm{rev}}$, the monoidal structure on ϕ is defined as $\phi^{(a_1\boxtimes_{\mathcal{E}} b_1)\otimes(a_2\boxtimes_{\mathcal{E}} b_2)} = \phi^{(a_1\otimes a_2)\boxtimes_{\mathcal{E}}(b_1\otimes^{\mathrm{rev}} b_2)} = (a_1 \otimes a_2) \odot - \odot (b_2 \otimes b_1) \simeq a_1 \odot (a_2 \odot - \odot b_2) \odot b_1 = \phi^{a_1\boxtimes_{\mathcal{E}} b_1} \circ \phi^{a_2\boxtimes_{\mathcal{E}} b_2}$. The structure of monoidal functor over $\mathcal{E}$ on ϕ is defined as $\phi^{T_{\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}^{\mathrm{rev}}}(e)} = \phi^{T_{\mathcal{A}}(e)\boxtimes_{\mathcal{E}}\mathbb{1}_{\mathcal{B}}} = T_{\mathcal{A}}(e) \odot - \odot \mathbb{1}_{\mathcal{B}} \xrightarrow{u^{\mathcal{A}},1} e \odot - \odot \mathbb{1}_{\mathcal{B}} \simeq e \odot - = T^e$ for $e \in \mathcal{E}$.

Given an object $(\mathcal{N}, \bar{u}^{\mathcal{A}}, \bar{u}^{\mathcal{B}})$ and a 1-morphism $f : \mathcal{M} \to \mathcal{N}$ in $\mathrm{BMod}_{\mathcal{A}|\mathcal{B}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$, we want to define a 1-morphism f in $\mathrm{LMod}_{\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}^{\mathrm{rev}}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. The left $\mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{B}^{\mathrm{rev}}$-module structure on f is defined as $f((a \boxtimes_{\mathcal{E}} b) \odot m) = f(a \odot m \odot b) \xrightarrow{s^f} a \odot f(m \odot b) \xrightarrow{1,t^f} a \odot f(m) \odot b = (a \boxtimes_{\mathcal{E}} b) \odot f(m)$ for $a \boxtimes_{\mathcal{E}} b \in \mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{B}^{\mathrm{rev}}$, $m \in \mathcal{M}$, where s^f and t^f are the left $\mathcal{A}$-module structure and the right $\mathcal{B}$-module structure on f respectively. It is routine to check that f satisfy the diagram (4.2).

For 1-morphisms $f, g : \mathcal{M} \rightrightarrows \mathcal{N}$ in $\mathrm{BMod}_{\mathcal{A}|\mathcal{B}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$, a 2-morphism $\alpha : f \Rightarrow g$ in $\mathrm{BMod}_{\mathcal{A}|\mathcal{B}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ is an $\mathcal{A}$-$\mathcal{B}$ bimodule natural transformation. It is routine to check that $\alpha : f \Rightarrow g$ is a left $\mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{B}^{\mathrm{rev}}$-module natural transformation. □

Example 4.17. Let $\mathcal{C}$ be a multifusion category over $\mathcal{E}$. The left $\mathcal{E}$-module structure on $\mathcal{C}$ is defined as $e \odot c := T_{\mathcal{C}}(e) \otimes c$ for $e \in \mathcal{E}, c \in \mathcal{C}$. For $c \in \mathcal{C}$, the functor $(c \otimes -, s^{c\otimes -}) : \mathcal{C} \to \mathcal{C}$ belongs to $\mathrm{Fun}_{\mathcal{E}}(\mathcal{C}, \mathcal{C})$, where the natural isomorphism $s^{c\otimes -}_{e,-} : c \otimes (e \odot -) = c \otimes T_{\mathcal{C}}(e) \otimes - \xrightarrow{z^{-1}_{e,c},1} T_{\mathcal{C}}(e) \otimes c \otimes - = e \odot (c \otimes -)$. Then $(\mathcal{C}, \mathrm{id}_e : T_{\mathcal{C}}(e) \otimes - = e \odot -)$ belongs to $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$.

For $c \in \mathcal{C}$, the functor $(- \otimes c, s^{-\otimes c}) : \mathcal{C} \to \mathcal{C}$ belongs to $\mathrm{Fun}_{\mathcal{E}}(\mathcal{C}, \mathcal{C})$, where the natural isomorphism $s^{-\otimes c}_{e,-} : (e \odot -) \otimes c = (T_{\mathcal{C}}(e) \otimes -) \otimes c \xrightarrow{\simeq} T_{\mathcal{C}}(e) \otimes (- \otimes c) = e \odot (- \otimes c)$. The category $\mathcal{C}$ equipped with the monoidal natural isomorphism $u_e : - \otimes T_{\mathcal{C}}(e) \xrightarrow{z^{-1}_{e,-}} T_{\mathcal{C}}(e) \otimes - = e \odot -$ belongs to $\mathrm{RMod}_{\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$.

For $c, \tilde{c} \in \mathcal{C}$, the functor $c \otimes - \otimes \tilde{c} : \mathcal{C} \to \mathcal{C}$ equipped with the natural isomorphism

$$s^{c\otimes -\otimes \tilde{c}}_{e,-} : c \otimes (e \odot -) \otimes \tilde{c} = c \otimes T_{\mathcal{C}}(e) \otimes - \otimes \tilde{c} \xrightarrow{z^{-1}_{e,c},1,1} T_{\mathcal{C}}(e) \otimes c \otimes - \otimes \tilde{c} = e \odot (c \otimes - \otimes \tilde{c})$$

beongs to $\mathrm{Fun}_{\mathcal{E}}(\mathcal{M}, \mathcal{M})$. Then $(\mathcal{C}, \mathrm{id}_e, u_e)$ belongs to $\mathrm{BMod}_{\mathcal{C}|\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$.

Theorem 4.18. Let $\mathcal{C}$ be a multifusion category over $\mathcal{E}$ such that $\mathcal{E} \to Z(\mathcal{C})$ is fully faithful. There is an equivalence of multifusion categories over $\mathcal{E}$:

$$\mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}|\mathcal{C}}(\mathcal{C}, \mathcal{C}) \simeq Z(\mathcal{C}, \mathcal{E})$$

Proof. Let us recall the proof of a monoidal equivalence $\mathrm{Fun}_{\mathcal{C}\boxtimes\mathcal{C}^{\mathrm{rev}}}(\mathcal{C}, \mathcal{C}) \simeq Z(\mathcal{C})$ in [EGNO, Prop. 7.13.8]. Let F belong to $\mathrm{Fun}_{\mathcal{C}\boxtimes\mathcal{C}^{\mathrm{rev}}}(\mathcal{C}, \mathcal{C})$. Since F is a right $\mathcal{C}$-module functor, we have $F = d \otimes -$ for some $d \in \mathcal{C}$. Since F is a left $\mathcal{C}$-module functor, we have a natural isomorphism

$$d \otimes (x \otimes y) = F(x \otimes y) \xrightarrow{s_{x,y}} x \otimes F(y) = x \otimes (d \otimes y) \quad x, y \in \mathcal{C}$$

Taking $y = \mathbb{1}_{\mathcal{C}}$, we obtain a natural isomorphism $\gamma_d = s_{-,\mathbb{1}_{\mathcal{C}}} : d \otimes - \xrightarrow{\simeq} - \otimes d$. The compatibility conditions of γ_d correspond to the axioms of module functors. Then (d, γ_d) belongs to $Z(\mathcal{C})$. And the composition of $\mathcal{C}$-bimodule functors of $\mathcal{C}$ corresponds to the tensor product of objects of $Z(\mathcal{C})$.

Moreover, F belongs to $\mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}|\mathcal{C}}(\mathcal{C}, \mathcal{C})$. Taking $m = \mathbb{1}_{\mathcal{C}}, F = d \otimes -$ in the diagram (4.10), the following square commutes:

$$\begin{tikzcd}
d \otimes (T_{\mathcal{C}}(e) \otimes \mathbb{1}_{\mathcal{C}}) \arrow[rr, "1{,}z_{e,\mathbb{1}_{\mathcal{C}}}"] \arrow[d, "\gamma_d{,}1"'] & & d \otimes (\mathbb{1}_{\mathcal{C}} \otimes T_{\mathcal{C}}(e)) \arrow[d] \\
T_{\mathcal{C}}(e) \otimes (d \otimes \mathbb{1}_{\mathcal{C}}) \arrow[rr, "z_{e,d\otimes\mathbb{1}_{\mathcal{C}}}"] \arrow[dr, "z_{e,d}{,}1"'] & & (d \otimes \mathbb{1}_{\mathcal{C}}) \otimes T_{\mathcal{C}}(e) \\
& d \otimes T_{\mathcal{C}}(e) \otimes \mathbb{1}_{\mathcal{C}} \arrow[ur, "1{,}z_{e,\mathbb{1}_{\mathcal{C}}}"'] &
\end{tikzcd}$$

The triangle commutes by the diagram (A.1). Then we obtain $z_{e,d} \circ \gamma_d = \mathrm{id}_{d\otimes T_{\mathcal{C}}(e)}$, i.e. $(d, \gamma_d) \in Z(\mathcal{C}, \mathcal{E})$. It is routine to check that the functor $\mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}|\mathcal{C}}(\mathcal{C}, \mathcal{C}) \to Z(\mathcal{C}, \mathcal{E})$ is a monoidal functor over $\mathcal{E}$. □

Example 4.19. Let A, B be separable algebras in a multifusion category $\mathcal{C}$ over $\mathcal{E}$. We use ${}_A\mathcal{C}_B$ to denote the category of A-B bimodules in $\mathcal{C}$. The left $\mathcal{E}$-module structure on ${}_A\mathcal{C}_B$ is defined as $e \odot x := T_{\mathcal{C}}(e) \otimes x$ for $e \in \mathcal{E}, x \in {}_A\mathcal{C}_B$. We use q_x and p_x to denote the left A-action and right B-action on x respectively. The right B-action on $T_{\mathcal{C}}(e) \otimes x$ is induced by $T_{\mathcal{C}}(e) \otimes x \otimes B \xrightarrow{1,p_x} T_{\mathcal{C}}(e) \otimes x$. The left A-action on $T_{\mathcal{C}}(e) \otimes x$ is induced by $A \otimes T_{\mathcal{C}}(e) \otimes x \xrightarrow{z^{-1}_{e,A},1} T_{\mathcal{C}}(e) \otimes A \otimes x \xrightarrow{1,q_x} T_{\mathcal{C}}(e) \otimes x$. The module associativity constraint is given by $\lambda_{e_1,e_2,x} : (e_1 \otimes e_2) \odot x = T_{\mathcal{C}}(e_1 \otimes e_2) \otimes x \to T_{\mathcal{C}}(e_1) \otimes T_{\mathcal{C}}(e_2) \otimes x = e_1 \odot (e_2 \odot x)$, for $e_1, e_2 \in \mathcal{E}, x \in {}_A\mathcal{C}_B$. The unit isomorphism is given by $l_x : \mathbb{1}_{\mathcal{E}} \odot x = T_{\mathcal{C}}(\mathbb{1}_{\mathcal{E}}) \otimes x = \mathbb{1}_{\mathcal{C}} \otimes x \to x$. Check that $\lambda_{e_1,e_2,x}$ and l_x belong to ${}_A\mathcal{C}_B$.

The right $\mathcal{E}$-action on ${}_A\mathcal{C}_B$ is defined as $x \odot e := x \otimes T_{\mathcal{C}}(e), e \in \mathcal{E}, x \in {}_A\mathcal{C}_B$. The left A-action on $x \otimes T_{\mathcal{C}}(e)$ is defined as $A \otimes x \otimes T_{\mathcal{C}}(e) \xrightarrow{q_x,1} x \otimes T_{\mathcal{C}}(e)$. The right B-action on $x \otimes T_{\mathcal{C}}(e)$ is defined as $x \otimes T_{\mathcal{C}}(e) \otimes B \xrightarrow{1,z_{e,B}} x \otimes B \otimes T_{\mathcal{C}}(e) \xrightarrow{p_x,1} x \otimes T_{\mathcal{C}}(e)$. The module associativity constraint is defined as $\lambda_{x,e_1,e_2} : x \odot (e_1 \otimes e_2) = x \otimes T_{\mathcal{C}}(e_1 \otimes e_2) \to x \otimes T_{\mathcal{C}}(e_1) \otimes T_{\mathcal{C}}(e_2) = (x \odot e_1) \odot e_2$, for $x \in {}_A\mathcal{C}_B$, $e_1, e_2 \in \mathcal{E}$. The unit isomorphism is defined as $r_x : x \odot \mathbb{1}_{\mathcal{E}} = x \otimes T_{\mathcal{C}}(\mathbb{1}_{\mathcal{E}}) = x \otimes \mathbb{1}_{\mathcal{C}} \to x$. Check that λ_{x,e_1,e_2} and r_x belong to ${}_A\mathcal{C}_B$. Check that ${}_A\mathcal{C}_B$ equipped with the monoidal natural isomorphism $v_e : T_{\mathcal{C}}(e) \otimes x \xrightarrow{z_{e,x}} x \otimes T_{\mathcal{C}}(e)$ belongs to $\mathrm{BMod}_{\mathcal{E}|\mathcal{E}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$.

Also one can check that ${}_A\mathcal{C}$ belongs to $\mathrm{BMod}_{\mathcal{E}|\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ and $\mathcal{C}_B$ belongs to $\mathrm{BMod}_{\mathcal{C}|\mathcal{E}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$.

Example 4.20. Let $\mathcal{M}$ belongs to $\mathrm{RMod}_{\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. Then $\mathcal{M}$ belongs to $\mathrm{BMod}_{\mathcal{E}|\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. The $\mathcal{E}$-$\mathcal{D}$ bimodule structure on $\mathcal{M}$ is defined as $(e \odot m) \odot d \xrightarrow{(s^{-\odot d}_{e,m})^{-1}} e \odot (m \odot d)$ for any $e \in \mathcal{E}, m \in \mathcal{M}$. Since $(- \odot d, s^{-\odot d})$ belongs to $\mathrm{Fun}_{\mathcal{E}}(\mathcal{M}, \mathcal{M})$ and the diagram (4.5) commutes, M is an $\mathcal{E}$-$\mathcal{D}$ bimodule category.

The functor $e \odot - \odot d : \mathcal{M} \to \mathcal{M}$ equipped with the natural isomorphism $s^{e\odot-\odot d}_{\tilde{e},-} : \tilde{e} \odot ((e \odot -) \odot d) \xrightarrow{s^{-\odot d}_{\tilde{e},e\odot-}} (\tilde{e} \odot (e \odot -)) \odot d \xrightarrow{s^{e\odot-}_{\tilde{e},-},1} (e \odot (\tilde{e} \odot -)) \odot d$ is a left $\mathcal{E}$-module functor, where $s^{e\odot-}_{\tilde{e},-} : \tilde{e} \odot (e \odot -) \simeq (\tilde{e} \otimes e) \odot - \xrightarrow{r_{\tilde{e},e}} (e \otimes \tilde{e}) \odot - \simeq e \odot (\tilde{e} \odot -)$ for $\tilde{e} \in \mathcal{E}$. The object $\mathcal{M}$ both in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ and $\mathrm{BMod}_{\mathcal{E}|\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}})$ equipped with the monoidal natural isomorphisms $u^{\mathcal{E}}_e = \mathrm{id} : e \odot - = e \odot -$ and $u^{\mathcal{D}}_e : - \odot T_{\mathcal{D}}(e) \simeq e \odot -$ belongs to $\mathrm{BMod}_{\mathcal{E}|\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. The monoidal natural isomorphism $u^{\mathcal{D}}_e$ satisfies the diagram (4.9) by the diagrams (4.3) and (4.7).

Example 4.21. Let $\mathcal{C}, \mathcal{D}$ be multifusion categories over $\mathcal{E}$ and $(\mathcal{M}, u^{\mathcal{C}}, u^{\mathcal{D}}) \in \mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. The $\mathcal{D}$-$\mathcal{C}$ bimodule structure on the category $\mathcal{M}^{L|\mathrm{op}|L}$ is defined as $d \odot^L m \odot^L c := c^L \odot m \odot d^L$ for $d \in \mathcal{D}, c \in \mathcal{C}, m \in \mathcal{M}$. Then $(\mathcal{M}^{L|\mathrm{op}|L}, \tilde{u}^{\mathcal{D}}, \tilde{u}^{\mathcal{C}})$ belongs to $\mathrm{BMod}_{\mathcal{D}|\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. The left $\mathcal{E}$-module structure on $\mathcal{M}^{L|\mathrm{op}|L}$ is defined as $e \odot^L m := e^L \odot m$ for $e \in \mathcal{E}$, $m \in \mathcal{M}$. The monoidal natural isomorphism $\tilde{u}^{\mathcal{D}}$ is defined as $T_{\mathcal{D}}(e) \odot^L m = m \odot T_{\mathcal{D}}(e)^L \simeq m \odot T_{\mathcal{D}}(e^L) \xrightarrow{u^{\mathcal{D}}_{e^L}} e^L \odot m$. The monoidal natural isomorphism $\tilde{u}^{\mathcal{C}}$ is defined as $m \odot^L T_{\mathcal{C}}(e) = T_{\mathcal{C}}(e)^L \odot m \simeq T_{\mathcal{C}}(e^L) \odot m \xrightarrow{u^{\mathcal{C}}_{e^L}} e^L \odot m$.

Example 4.22. Let $\mathcal{C}, \mathcal{D}, \mathcal{P}$ be multifusion categories over $\mathcal{E}$, and $(\mathcal{M}, u^{\mathcal{C}}, u^{\mathcal{D}}) \in \mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$, $(\mathcal{N}, \bar{u}^{\mathcal{C}}, \bar{u}^{\mathcal{P}}) \in \mathrm{BMod}_{\mathcal{C}|\mathcal{P}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. Then $(\mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}}(\mathcal{M}, \mathcal{N}), \tilde{u}^{\mathcal{D}}, \tilde{u}^{\mathcal{P}})$ belongs to $\mathrm{BMod}_{\mathcal{D}|\mathcal{P}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. The left $\mathcal{E}$-module structure on $\mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ is defined as $(e \odot f)(-) := T_{\mathcal{C}}(e) \odot f(-)$, for $e \in \mathcal{E}, f \in \mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$. The $\mathcal{D}$-$\mathcal{P}$ bimodule structure on $\mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ is defined as $(d \odot f \odot p)(-) := f(- \odot d) \odot p$ for any $d \in \mathcal{D}, p \in \mathcal{P}$. Let $v^{\mathcal{M}}_e := (u^{\mathcal{D}}_e)^{-1} \circ u^{\mathcal{C}}_e$ and $v^{\mathcal{N}}_e := (\bar{u}^{\mathcal{P}}_e)^{-1} \circ \bar{u}^{\mathcal{C}}_e$. The monoidal natural isomorphism $\tilde{u}^{\mathcal{D}}$ is defined as $(T_{\mathcal{D}}(e) \odot f)(-) = f(- \odot T_{\mathcal{D}}(e)) \xrightarrow{(v^{\mathcal{M}}_e)^{-1}} f(T_{\mathcal{C}}(e) \odot -) \xrightarrow{s^f} T_{\mathcal{C}}(e) \odot f(-) = (e \odot f)(-)$. The monoidal natural isomorphism $\tilde{u}^{\mathcal{P}}$ is defined as $(f \odot T_{\mathcal{P}}(e))(-) = f(-) \odot T_{\mathcal{P}}(e) \xrightarrow{(v^{\mathcal{N}}_e)^{-1}} T_{\mathcal{C}}(e) \odot f(-) = (e \odot f)(-)$.

4.3 Invertible bimodules in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$

Definition 4.23. Let $\mathcal{C}$ be a multifusion category over $\mathcal{E}$, and $(\mathcal{M}, u^{\mathcal{M}}) \in \mathrm{RMod}_{\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$, $(\mathcal{N}, u^{\mathcal{N}}) \in \mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ and $\mathcal{D} \in \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$. A *balanced $\mathcal{C}$-module functor $F : \mathcal{M} \times \mathcal{N} \to \mathcal{D}$ in* $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ consists of the following data.

- $F : \mathcal{M} \times \mathcal{N} \to \mathcal{D}$ is an $\mathcal{E}$-bilinear bifunctor. That is, for each $n \in \mathcal{N}$, $(F(-, n), s^{F1}) : \mathcal{M} \to \mathcal{D}$ is a left $\mathcal{E}$-module functor, where
$$s^{F1}_{e,m} : F(e \odot m, n) \simeq e \odot F(m, n), \quad \forall e \in \mathcal{E}, m \in \mathcal{M}$$
is a natural isomorphism. For each $g : n \to n'$ in $\mathcal{N}$, $F(-, g) : F(-, n) \Rightarrow F(-, n')$ is a left $\mathcal{E}$-module natural transformation. And for each $m \in \mathcal{M}$, $(F(m, -), s^{F2}) : \mathcal{N} \to \mathcal{D}$ is a left $\mathcal{E}$-module functor, where
$$s^{F2}_{e,n} : F(m, e \odot n) \simeq e \odot F(m, n), \quad \forall e \in \mathcal{E}, n \in \mathcal{N}$$
is a natural isomorphism. For each $f : c \to c'$ in $\mathcal{C}$, $F(f, -) : F(c, -) \Rightarrow F(c', -)$ is a left $\mathcal{E}$-module natural transformation.
- $F : \mathcal{M} \times \mathcal{N} \to \mathcal{D}$ is a balanced $\mathcal{E}$-module functor (recall Def. 2.3), where the balanced $\mathcal{E}$-module structure on F is defined as
$$\hat{b}_{m,e,n} : F(m \odot e, n) = F(e \odot m, n) \xrightarrow{s^{F1}_{e,m}} e \odot F(m, n) \xrightarrow{(s^{F2}_{e,n})^{-1}} F(m, e \odot n).$$

- $F : \mathcal{M}\times\mathcal{N} \to \mathcal{D}$ is a balanced $\mathcal{C}$-module functor (recall Def. 2.3), where $b_{m,c,n} : F(m\odot c, n) \simeq F(m, c\odot n)$, $\forall m \in \mathcal{M}, c \in \mathcal{C}, n \in \mathcal{N}$, is the balanced $\mathcal{C}$-module structure on F. And $b_{m,c,n}$ is a left $\mathcal{E}$-module natural isomorphism. That is, the following diagram commutes

$$\begin{tikzcd}
F(e\odot(m\odot c), n) \arrow[r, "s^{F1}_{e,m\odot c}"] \arrow[d, "s^{-\odot c}_{e,m}{,}1"'] & e\odot F(m\odot c, n) \arrow[dd, "1{,}b_{m,c,n}"] \\
F((e\odot m)\odot c, n) \arrow[d, "b_{e\odot m,c,n}"'] & \\
F(e\odot m, c\odot n) \arrow[r, "s^{F1}_{e,m}"'] & e\odot F(m, c\odot n)
\end{tikzcd} \tag{4.13}$$

where the functor $(-\odot c, s^{-\odot c}) \in \mathrm{Fun}_{\mathcal{E}}(\mathcal{M},\mathcal{M})$, $\forall c \in \mathcal{C}$.

such that the followng diagram commutes

$$\begin{tikzcd}
F(m\odot T_{\mathcal{C}}(e), n) \arrow[rr, "b_{m,T_{\mathcal{C}}(e),n}"] \arrow[d, "(u^{\mathcal{M}}_e)_m{,}1"'] & & F(m, T_{\mathcal{C}}(e)\odot n) \arrow[d, "1{,}(u^{\mathcal{N}}_e)_n"] \\
F(e\odot m, n) \arrow[r, "s^{F1}_{e,m}"'] \arrow[rr, bend left, "\hat{b}_{m,e,n}"] & e\odot F(m,n) \arrow[r, "(s^{F2}_{e,n})^{-1}"'] & F(m, e\odot n)
\end{tikzcd} \tag{4.14}$$

We use $\mathrm{Fun}^{\mathrm{ball}|\mathcal{E}}_{\mathcal{C}}(\mathcal{M},\mathcal{N};\mathcal{D})$ to denote the category of balanced $\mathcal{C}$-module functors in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$, and natural transformations both in $\mathrm{Fun}^{\mathrm{bal}}_{\mathcal{C}}(\mathcal{M},\mathcal{N};\mathcal{D})$ and $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$.

The *tensor product of* $\mathcal{M}$ *and* $\mathcal{N}$ *over* $\mathcal{C}$ is an object $\mathcal{M}\boxtimes_{\mathcal{C}}\mathcal{N}$ in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$, together with a balanced $\mathcal{C}$-module functor $\boxtimes_{\mathcal{C}} : \mathcal{M}\times\mathcal{N} \to \mathcal{M}\boxtimes_{\mathcal{C}}\mathcal{N}$ in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$, such that, for every object $\mathcal{D}$ in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$, composition with $\boxtimes_{\mathcal{C}}$ induces an equivalence $\mathrm{Fun}_{\mathcal{E}}(\mathcal{M}\boxtimes_{\mathcal{C}}\mathcal{N},\mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{ball}|\mathcal{E}}_{\mathcal{C}}(\mathcal{M},\mathcal{N};\mathcal{D})$.

Proposition 4.24. For $e_1, e_2 \in \mathcal{E}$, $m \in \mathcal{M}, n \in \mathcal{N}$, the following diagram commutes

$$\begin{tikzcd}
F(e_1\odot m, e_2\odot n) \arrow[r, "s^{F1}_{e_1,m}"] \arrow[d, "s^{F2}_{e_2,n}"'] & e_1\odot F(m, e_2\odot n) \arrow[r, "1{,}s^{F2}_{e_2,n}"] & e_1\odot e_2\odot F(m,n) \arrow[dl, "r_{e_1,e_2}{,}1"] \\
e_2\odot F(e_1\odot m, n) \arrow[r, "1{,}s^{F1}_{e_1,m}"'] & e_2\odot e_1\odot F(m,n) &
\end{tikzcd}$$

Proof. Since $F : \mathcal{M}\times\mathcal{N} \to \mathcal{D}$ is a balanced $\mathcal{E}$-module functor, the following outward diagram commutes.

$$\begin{tikzcd}
F((e_1\otimes e_2)\odot m, n) \arrow[r, "s^{F1}_{e_1\otimes e_2,m}"] \arrow[d, "r_{e_1,e_2}{,}1"'] & (e_1\otimes e_2)\odot F(m,n) \arrow[r, "(s^{F2}_{e_1\otimes e_2,n})^{-1}"] \arrow[d, "r_{e_1,e_2}"] & F(m,(e_1\otimes e_2)\odot n) \\
F(e_2\odot e_1\odot m, n) \arrow[r, "s^{F1}_{e_2\otimes e_1,m}"] \arrow[d, "s^{F1}_{e_2,e_1\odot m}"'] & e_2\odot e_1\odot F(m,n) & \\
e_2\odot F(e_1\odot m, n) \arrow[r, "(s^{F2}_{e_2,n})^{-1}"'] \arrow[ur, "1{,}s^{F1}_{e_1,m}"'] & F(e_1\odot m, e_2\odot n) \arrow[r, "s^{F1}_{e_1,m}"'] & e_1\odot F(m, e_2\odot n) \arrow[uu, "(s^{F2}_{e_1,e_2\odot n})^{-1}"'] \arrow[uul, bend right, "1{,}s^{F2}_{e_2,n}"']
\end{tikzcd}$$

The two triangles commute since $(F(-,n), s^{F1}) : \mathcal{M} \to \mathcal{D}$ and $(F(m,-), s^{F2}) : \mathcal{N} \to \mathcal{D}$ are left $\mathcal{E}$-module functors. The square commutes by the naturality of s^{F1}. Then the pentagon commutes. $\square$

Proposition 4.25. For $e \in \mathcal{E}, m \in \mathcal{M}, c \in \mathcal{C}, n \in \mathcal{N}$, the diagram

$$\begin{tikzcd}
F((e \odot m) \odot c, n) \arrow[r, "b_{e\odot m,c,n}"] \arrow[d, "(s^{-\odot c}_{e,m})^{-1}"'] & F(e \odot m, c \odot n) \arrow[r, "\hat{b}_{m,e,c\odot n}"] & F(m, e \odot (c \odot n)) \arrow[d, "s^{c\odot -}_{e,n}"] \\
F(e \odot (m \odot c), n) \arrow[r, "\hat{b}_{m\odot c,e,n}"'] & F(m \odot c, e \odot n) \arrow[r, "b_{m,c,e\odot n}"'] & F(m, c \odot (e \odot n))
\end{tikzcd} \tag{4.15}$$

commutes, where the functors $(-\odot c, s^{-\odot c}) \in \mathrm{Fun}_{\mathcal{E}}(\mathcal{M}, \mathcal{M})$ and $(c\odot -, s^{c\odot -}) \in \mathrm{Fun}_{\mathcal{E}}(\mathcal{N}, \mathcal{N})$, $\forall c \in \mathcal{C}$.

Proof. Consider the following diagram:

$$\begin{tikzcd}
F(m \odot (T_{\mathcal{C}}(e) \otimes c), n) \arrow[r, "z_{e,c},1"] \arrow[d, "(u^{\mathcal{M}}_e)_m,1"] \arrow[ddddd, bend right=60, "b_{m,T_{\mathcal{C}}(e)\otimes c,n}"'] & F(m \odot (c \otimes T_{\mathcal{C}}(e)), n) \arrow[d, "(u^{\mathcal{M}}_e)_{m\odot c},1"'] \arrow[ddddd, bend left=60, "b_{m,c\otimes T_{\mathcal{C}}(e),n}"] \\
F((e \odot m) \odot c, n) \arrow[r, "(s^{-\odot c}_{e,m})^{-1}"'] \arrow[d, "b_{e\odot m,c,n}"] & F(e \odot (m \odot c), n) \arrow[d, "\hat{b}_{m\odot c,e,n}"'] \\
F(e \odot m, c \odot n) \arrow[d, "\hat{b}_{m,e,c\odot n}"] & F(m \odot c, e \odot n) \arrow[d, "b_{m,c,e\odot n}"'] \\
F(m, e \odot (c \odot n)) \arrow[r, "s^{c\odot -}_{e,n}"] & F(m, c \odot (e \odot n)) \\
F(m, (T_{\mathcal{C}}(e) \otimes c) \odot n)) \arrow[u, "1,(u^{\mathcal{N}}_e)_{c\odot n}"'] \arrow[r, "1,z_{e,c}"'] & F(m, (c \otimes T_{\mathcal{C}}(e)) \odot n)) \arrow[u, "1,(u^{\mathcal{N}}_e)_n"]
\end{tikzcd} \tag{4.16}$$

Here z is the central structure of the central functor $T_{\mathcal{C}} : \mathcal{E} \to \mathcal{C}$. The middle-top and middle-down squares commute by the diagrams (4.1) and (4.3). The leftmost diagram commutes by the diagram

$$\begin{tikzcd}
F(m \odot (T_{\mathcal{C}}(e) \otimes c), n) \arrow[rr, "b_{m,T_{\mathcal{C}}(e)\otimes c,n}"] \arrow[d] & & F(m, (T_{\mathcal{C}}(e) \otimes c) \odot n) \arrow[d] \\
F((m \odot T_{\mathcal{C}}(e)) \odot c, n) \arrow[r, "b_{m\odot T_{\mathcal{C}}(e),c,n}"] \arrow[d, "(u^{\mathcal{M}}_e)_m,1"'] & F(m \odot T_{\mathcal{C}}(e), c \odot n) \arrow[r, "b_{m,T_{\mathcal{C}}(e),c\odot n}"] \arrow[d, "(u^{\mathcal{M}}_e)_m,1"] & F(m, T_{\mathcal{C}}(e) \odot (c \odot n)) \arrow[d, "1,(u^{\mathcal{N}}_e)_{c\odot n}"] \\
F((e \odot m) \odot c, n) \arrow[r, "b_{e\odot m,c,n}"'] & F(e \odot m, c \odot n) \arrow[r, "\hat{b}_{m,e,c\odot n}"'] & F(m, e \odot (c \odot n))
\end{tikzcd}$$

The top pentagon commutes by the balanced $\mathcal{C}$-module functor $F : \mathcal{M} \times \mathcal{N} \to \mathcal{D}$. The left-down square commutes by the naturality of the balanced $\mathcal{C}$-module structure b on F. The right-down square commutes by the diagram (4.14). One can check that the rightmost diagram of (4.16) commutes. Then the middle hexagon of (4.16) commutes. □

Corollary 4.26. By the commutativities of the diagrams (4.13) and (4.15), the following diagram commutes

$$\begin{tikzcd}
F(m \odot c, e \odot n) \arrow[r, "s^{F2}_{e,n}"] \arrow[d, "b_{m,c,e\odot n}"'] & e \odot F(m \odot c, n) \arrow[dd, "1,b_{m,c,n}"] \\
F(m, c \odot (e \odot n)) \arrow[d, "1,(s^{c\odot -}_{e,n})^{-1}"'] & \\
F(m, e \odot (c \odot n)) \arrow[r, "s^{F2}_{e,c\odot n}"'] & e \odot F(m, c \odot n)
\end{tikzcd}$$

Example 4.27. Let $\mathcal{C}, \mathcal{D}, \mathcal{P}$ be multifusion categories over $\mathcal{E}$, $(\mathcal{M}, u^{\mathcal{C}}, u^{\mathcal{D}}) \in \mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ and $(\mathcal{N}, \bar{u}^{\mathcal{D}}, \bar{u}^{\mathcal{P}}) \in \mathrm{BMod}_{\mathcal{D}|\mathcal{P}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. Then $(\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}, \tilde{u}^{\mathcal{C}}, \tilde{u}^{\mathcal{P}})$ belongs to $\mathrm{BMod}_{\mathcal{C}|\mathcal{P}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. The left $\mathcal{E}$-module structure on $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$ is defined as $e \odot (m \boxtimes_{\mathcal{D}} n) := (e \odot m) \boxtimes_{\mathcal{D}} n$, for $e \in \mathcal{E}$, $m \boxtimes_{\mathcal{D}} n \in \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$. The $\mathcal{C}$-$\mathcal{P}$ bimodule structure on $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$ is defined as $c \odot (m \boxtimes_{\mathcal{D}} n) \odot p := (c \odot m) \boxtimes_{\mathcal{D}} (n \odot p)$, for $c \in \mathcal{C}, p \in \mathcal{P}$. The monoidal natural isomorphism $\tilde{u}^{\mathcal{C}}$ is induced by $T_{\mathcal{C}}(e) \odot (m \boxtimes_{\mathcal{D}} n) = (T_{\mathcal{C}}(e) \odot m) \boxtimes_{\mathcal{D}} n \xrightarrow{(u^{\mathcal{C}}_e)_m, 1} (e \odot m) \boxtimes_{\mathcal{D}} n = e \odot (m \boxtimes_{\mathcal{D}} n)$. The monoidal isomorphism $\tilde{u}^{\mathcal{P}}$ is induced by $(m \boxtimes_{\mathcal{D}} n) \odot T_{\mathcal{P}}(e) = m \boxtimes_{\mathcal{D}} (n \odot T_{\mathcal{P}}(e)) \xrightarrow{1, (\bar{u}^{\mathcal{P}}_e)_n} m \boxtimes_{\mathcal{D}} (e \odot n) \xrightarrow{1, (\bar{u}^{\mathcal{D}}_e)^{-1}_n} m \boxtimes_{\mathcal{D}} (T_{\mathcal{D}}(e) \odot n) \xrightarrow{b^{-1}_{m, T_{\mathcal{D}}(e), n}} (m \odot T_{\mathcal{D}}(e)) \boxtimes_{\mathcal{D}} n \xrightarrow{(u^{\mathcal{D}}_e)_m, 1} (e \odot m) \boxtimes_{\mathcal{D}} n = e \odot (m \boxtimes_{\mathcal{D}} n)$, where b is the balanced $\mathcal{D}$-module structure on $\boxtimes_{\mathcal{D}} : \mathcal{M} \times \mathcal{N} \to \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$.

Let $\mathcal{C}$ be a multifusion category over $\mathcal{E}$ and $\mathcal{M} \in \mathrm{LMod}_{\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. Then $\mathcal{M}$ is *enriched* in $\mathcal{C}$. That is, there exists an object $[x, y]_{\mathcal{C}} \in \mathcal{C}$ and a natural isomorphism $\mathrm{Hom}_{\mathcal{M}}(c \odot x, y) \simeq \mathrm{Hom}_{\mathcal{C}}(c, [x, y]_{\mathcal{C}})$ for $c \in \mathcal{C}$, $x, y \in \mathcal{M}$. The category $\mathcal{C}_A$ is enriched in $\mathcal{C}$ and we have $[x, y]_{\mathcal{C}} = (x \otimes_A y^R)^L$ for $x, y \in \mathcal{C}_A$ by [EGNO, Expl. 7.9.8]. By Prop. A.4, the diagram

$$\begin{tikzcd}
T_{\mathcal{C}}(e) \otimes x \otimes_A y^R \arrow[r, "c_{e, x \otimes_A y^R}"] \arrow[d, "c_{e,x}, 1"'] & x \otimes_A y^R \otimes T_{\mathcal{C}}(e) \\
x \otimes T_{\mathcal{C}}(e) \otimes_A y^R \arrow[r] & x \otimes_A T_{\mathcal{C}}(e) \otimes y^R \arrow[u, "1, c_{e, y^R}"']
\end{tikzcd}$$

commutes for $e \in \mathcal{E}, x, y \in \mathcal{C}_A$, where c is the central structure of the central functor $T_{\mathcal{C}} : \mathcal{E} \to \mathcal{C}$.

Let $\mathcal{C}$ be a multifusion category over $\mathcal{E}$ and A, B be separable algebras in $\mathcal{C}$. By Prop. A.6, we have the following statements.

- There is an equivalence ${}_A\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{C}_B \xrightarrow{\simeq} {}_A\mathcal{C}_B$, $x \boxtimes_{\mathcal{C}} y \mapsto x \otimes y$ in $\mathrm{BMod}_{\mathcal{E}|\mathcal{E}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$.
- There is an equivalence $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C}_A, \mathcal{C}_B) \xrightarrow{\simeq} {}_A\mathcal{C}_B$, $f \mapsto f(A)$ in $\mathrm{BMod}_{\mathcal{E}|\mathcal{E}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$, whose inverse is defined as $x \mapsto - \otimes_A x$.

Proposition 4.28. Let $\mathcal{C}, \mathcal{B}, \mathcal{D}$ be multifusion categories over $\mathcal{E}$ and $\mathcal{M} \in \mathrm{BMod}_{\mathcal{C}|\mathcal{B}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ and $\mathcal{N} \in \mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. The functor $\Phi : \mathcal{M}^{L|\mathrm{op}|L} \boxtimes_{\mathcal{C}} \mathcal{N} \to \mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$, $m \boxtimes_{\mathcal{C}} n \mapsto [-, m]^R_{\mathcal{C}} \odot n$, is an equivalence of $\mathcal{B}$-$\mathcal{D}$-bimodules in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$.

Proof. There are equivalences of categories $\mathcal{M}^{L|\mathrm{op}|L} \boxtimes_{\mathcal{C}} \mathcal{N} \simeq \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N}) \simeq \mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ by [KZ, Cor. 2.2.5] and Rem. 4.8. The $\mathcal{B}$-$\mathcal{D}$ bimodule structure on Φ is induced by

$$(b \odot^L m) \boxtimes_{\mathcal{C}} (n \odot d) = (m \odot b^L) \boxtimes_{\mathcal{C}} (n \odot d) \mapsto [-, m \odot b^L]^R_{\mathcal{C}} \odot (n \odot d) \simeq ([- \odot b, m]^R_{\mathcal{C}} \odot n) \odot d = b \odot ([-, m]^R_{\mathcal{C}} \odot n) \odot d$$

for $m \in \mathcal{M}, n \in \mathcal{N}, b \in \mathcal{B}, d \in \mathcal{D}$, where the equivalence is due to the canonical isomorphisms $\mathrm{Hom}_{\mathcal{C}}(c, [-, m \odot b^L]_{\mathcal{C}}) \simeq \mathrm{Hom}_{\mathcal{M}}(c \odot -, m \odot b^L) \simeq \mathrm{Hom}_{\mathcal{M}}(c \odot - \odot b, m) \simeq \mathrm{Hom}_{\mathcal{C}}(c, [- \odot b, m]_{\mathcal{C}})$ for $c \in \mathcal{C}$. The left $\mathcal{E}$-module structure on Φ is induced by the left $\mathcal{B}$-module structure on Φ. Recall Expl. 4.21, 4.27 and 4.22. It is routine to check that Φ satisfy the diagram (4.10). □

Definition 4.29. Let $\mathcal{C}, \mathcal{D}$ be multifusion categories over $\mathcal{E}$ and $\mathcal{M} \in \mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. $\mathcal{M}$ is right dualizable, if there exists an $\mathcal{N} \in \mathrm{BMod}_{\mathcal{D}|\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ equipped with bimodule functors $u : \mathcal{D} \to \mathcal{N} \boxtimes_{\mathcal{C}} \mathcal{M}$ and $v : \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N} \to \mathcal{C}$ in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ such that the composed bimodule functors

$$\mathcal{M} \simeq \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{D} \xrightarrow{1_{\mathcal{M}} \boxtimes_{\mathcal{D}} u} \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N} \boxtimes_{\mathcal{C}} \mathcal{M} \xrightarrow{v \boxtimes_{\mathcal{C}} 1_{\mathcal{M}}} \mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{M} \simeq \mathcal{M}$$

$$\mathcal{N} \simeq \mathcal{D} \boxtimes_{\mathcal{D}} \mathcal{N} \xrightarrow{u \boxtimes_{\mathcal{D}} 1_{\mathcal{N}}} \mathcal{N} \boxtimes_{\mathcal{C}} \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N} \xrightarrow{1_{\mathcal{N}} \boxtimes_{\mathcal{C}} v} \mathcal{N} \boxtimes_{\mathcal{C}} \mathcal{C} \simeq \mathcal{N}$$

in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ are isomorphic to the identity functor. In this case, the $\mathcal{D}$-$\mathcal{C}$ bimodule $\mathcal{N}$ in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ is left dualizable.

Proposition 4.30. The right dual of $\mathcal{M}$ in $\mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ is given by a $\mathcal{D}$-$\mathcal{C}$ bimodule $\mathcal{M}^{L|\mathrm{op}|L}$ in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ equipped with two maps u and v defined as follows:

$$
\begin{aligned}
u :& \mathcal{D} \to \mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{M}) \simeq \mathcal{M}^{L|\mathrm{op}|L} \boxtimes_{\mathcal{C}} \mathcal{M}, \qquad d \mapsto - \odot d,\\
v :& \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{M}^{L|\mathrm{op}|L} \to \mathcal{C}, \qquad x \boxtimes_{\mathcal{D}} y \mapsto [x,y]^R_{\mathcal{C}}
\end{aligned} \tag{4.17}
$$

Proof. By [AKZ, Thm. 4.6], the object $\mathcal{M}^{L|\mathrm{op}|L}$ in $\mathrm{BMod}_{\mathcal{D}|\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}})$, equipped with the maps u and v, are the right dual of $\mathcal{M}$ in $\mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}})$. It is routine to check that u is a $\mathcal{D}$-bimodule functor in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ and v is a $\mathcal{C}$-bimodule functor in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$. □

Definition 4.31. Let $\mathcal{C}, \mathcal{D}$ be multifusion categories over $\mathcal{E}$. An $\mathcal{M} \in \mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ is *invertible* if there is an equivalence $\mathcal{D}^{\mathrm{rev}} \simeq \mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}}(\mathcal{M},\mathcal{M})$ of multifusion categories over $\mathcal{E}$. If such an invertible $\mathcal{M}$ exists, $\mathcal{C}$ and $\mathcal{D}$ are said to be *Morita equivalent in* $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$.

Proposition 4.32. Let $\mathcal{M}$ belong to $\mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. The following conditions are equivalent.

(i) $\mathcal{M}$ is invertible,

(ii) The functor $\mathcal{D}^{\mathrm{rev}} \to \mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}}(\mathcal{M},\mathcal{M}), d \mapsto - \odot d$ is an equivalence of multifusion categories over $\mathcal{E}$,

(iii) The functor $\mathcal{C} \to \mathrm{Fun}^{\mathcal{E}}_{|\mathcal{D}}(\mathcal{M},\mathcal{M}), c \mapsto c \odot -$ is an equivalence of multifusion categories over $\mathcal{E}$.

Proof. We obtain (i) $\Leftrightarrow$ (ii) by the Def. 4.31. Since $\mathrm{Fun}^{\mathcal{E}}_{\mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}}(\mathcal{M},\mathcal{M})}(\mathcal{M},\mathcal{M})$ and $\mathcal{C}$ are equivalent as multifusion categories over $\mathcal{E}$ by Prop.4.13, we obtain (ii) $\Leftrightarrow$ (iii). □

4.4 Characterization of Morita equivalence in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$

Convention 4.33. Throughout this subsection, we consider multifusion categories $\mathcal{C}$ over $\mathcal{E}$ with the property that $\mathcal{E} \to Z(\mathcal{C})$ is fully faithful.

Let $\mathcal{C}$ and $\mathcal{D}$ be multifusion categories over $\mathcal{E}$. We use β and γ to denote the central structures of the central functors $T_{\mathcal{C}} : \mathcal{E} \to \mathcal{C}$ and $T_{\mathcal{D}} : \mathcal{E} \to \mathcal{D}$ respectively.

Theorem 4.34. Let $\mathcal{M}$ be invertible in $\mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. The left action of $Z(\mathcal{C},\mathcal{E})$ and the right action of $Z(\mathcal{D},\mathcal{E})$ on $\mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}|\mathcal{D}}(\mathcal{M},\mathcal{M})$ induce an equivalence of multifusion categories over $\mathcal{E}$

$$Z(\mathcal{C},\mathcal{E}) \xrightarrow{L} \mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}|\mathcal{D}}(\mathcal{M},\mathcal{M}) \xleftarrow{R} Z(\mathcal{D},\mathcal{E})$$

Moreover, $Z(\mathcal{C},\mathcal{E})$ and $Z(\mathcal{D},\mathcal{E})$ are equivalent as braided multifusion categories over $\mathcal{E}$.

Proof. Since $\mathcal{M}$ is invertible, the functor $\mathcal{C} \to \mathrm{Fun}_{\mathcal{D}^{\mathrm{rev}}}(\mathcal{M},\mathcal{M}), z \mapsto z \odot -$ is a monoidal equivalence over $\mathcal{E}$. Then the induced monoidal equivalence $L : Z(\mathcal{C},\mathcal{E}) \xrightarrow{(z,\beta_{z,-}) \mapsto (z\odot -, \beta_{z,-})} \mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}|\mathcal{D}}(\mathcal{M},\mathcal{M})$ is constructed as follows.

- An object $z \in Z(\mathcal{C},\mathcal{E})$ is an object $z \in \mathcal{C}$, equipped with a half-braiding $\beta_{z,c} : z \otimes c \to c \otimes z$ for all $c \in \mathcal{C}$, such that the composition $z \otimes T_{\mathcal{C}}(e) \xrightarrow{\beta_{z,T_{\mathcal{C}}(e)}} T_{\mathcal{C}}(e) \otimes z \xrightarrow{\beta_{T_{\mathcal{C}}(e),z}} z \otimes T_{\mathcal{C}}(e), e \in \mathcal{E}$, equals to identity.
- An object $z \odot -$ in $\mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}|\mathcal{D}}(\mathcal{M},\mathcal{M})$ is an object $z \odot -$ in $\mathrm{Fun}^{\mathcal{E}}_{\mathcal{D}^{\mathrm{rev}}}(\mathcal{M},\mathcal{M})$ for $z \in \mathcal{C}$, equipped with a natural isomorphism $z \odot c \odot - \xrightarrow{\beta_{z,c}} c \odot z \odot -$ for $c \in \mathcal{C}, - \in \mathcal{M}$. The left $\mathcal{E}$-module structure on $z \odot -$ is induced by Prop. 4.7. Notice that $z \odot -$ satisfies the diagram (4.10) by the last diagram in Rem. 4.15 and the equality $\beta_{T_{\mathcal{C}}(e),z} = \beta^{-1}_{z,T_{\mathcal{C}}(e)}$.

It is routine to check that L is a monoidal functor over $\mathcal{E}$. By the same reason, the functor $R : Z(\mathcal{D}, \mathcal{E}) \simeq Z(\mathcal{D}, \mathcal{E})^{\mathrm{rev}} \xrightarrow{\simeq} \mathrm{Fun}^{\mathcal{E}}_{\mathcal{C}|\mathcal{D}}(\mathcal{M}, \mathcal{M})$ is defined by $(a, \gamma_{a,-}) \mapsto (- \odot a, \gamma_{a,-})$, where the second $\gamma_{a,-}$ is a natural isomorphism $- \odot a \odot d \xrightarrow{\gamma_{a,d}} - \odot d \odot a$ for $d \in \mathcal{D}$. Thus $Z(\mathcal{C}, \mathcal{E}) \simeq Z(\mathcal{D}, \mathcal{E})$.

Suppose $R^{-1} \circ L : Z(\mathcal{C}, \mathcal{E}) \to Z(\mathcal{D}, \mathcal{E})$ carries z, z' to d, d', respectively. The diagram

$$\begin{tikzcd}
z \odot (z' \odot x) \arrow[r, "\simeq"] \arrow[d, "\beta_{z,z'}, 1_x"'] & (z' \odot x) \odot d \arrow[r, "\simeq"] \arrow[d, "\simeq"] & (x \odot d') \odot d \arrow[d, "1_x, \gamma_{d',d}"] \\
z' \odot (z \odot x) \arrow[r, "\simeq"'] & z' \odot (x \odot d) \arrow[r, "\simeq"'] & (x \odot d) \odot d'
\end{tikzcd}$$

commutes for $x \in \mathcal{M}$. Since the isomorphism $z \odot - \simeq - \odot d$ is a left $\mathcal{C}$-module natural isomorphism, the left square commutes. Since the isomorphism $z' \odot - \simeq - \odot d'$ is a right $\mathcal{D}$-module natural isomorphism, the right square commutes. Then the commutativity of the outer square implies that the equivalence $R^{-1} \circ L$ preserves braidings. The equivalence $R^{-1} \circ L$, equipped with the monoidal natural isomorphism $L(T_{\mathcal{C}}(e)) = T_{\mathcal{C}}(e) \odot - \xrightarrow{v_e^{\mathcal{M}}} - \odot T_{\mathcal{D}}(e) = R(T_{\mathcal{D}}(e))$, is the braided equivalence over $\mathcal{E}$. □

Lemma 4.35. Let $\mathcal{C}$ be a fusion category over $\mathcal{E}$ such that the central functor $T_{\mathcal{C}} : \mathcal{E} \to \mathcal{C}$ is fully faithful. Let $\mathbf{f} : Z(\mathcal{C}, \mathcal{E}) \to \mathcal{C}$ and $I_{\mathcal{C}} : \mathcal{C} \to Z(\mathcal{C}, \mathcal{E})$ denote the forgetful functor and its right adjoint.

(1) There is a natural isomorphism $I_{\mathcal{C}}(x) \cong [\mathbb{1}_{\mathcal{C}}, x]_{Z(\mathcal{C},\mathcal{E})}$ for all $x \in \mathcal{C}$.

(2) The object $A := I_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}})$ is a connected étale algebra in $Z(\mathcal{C}, \mathcal{E})$; moreover for any $x \in \mathcal{C}$, the object $I_{\mathcal{C}}(x)$ has a natural structure of a right A-module.

(3) The functor $I_{\mathcal{C}}$ induces an equivalence of fusion categories $\mathcal{C} \simeq Z(\mathcal{C}, \mathcal{E})_A$ over $\mathcal{E}$. Notice that $Z(\mathcal{C}, \mathcal{E})_A$ is the category of right A-modules in $Z(\mathcal{C}, \mathcal{E})$.

Proof. For any $z \in Z(\mathcal{C}, \mathcal{E}), x \in \mathcal{C}$, we have the equivalences $\mathrm{Hom}_{Z(\mathcal{C},\mathcal{E})}(z, I_{\mathcal{C}}(x)) \simeq \mathrm{Hom}_{\mathcal{C}}(z, x) \simeq \mathrm{Hom}_{Z(\mathcal{C},\mathcal{E})}(z, [\mathbb{1}_{\mathcal{C}}, x]_{Z(\mathcal{C},\mathcal{E})})$. By Yoneda lemma, we obtain $I_{\mathcal{C}}(x) \cong [\mathbb{1}_{\mathcal{C}}, x]_{Z(\mathcal{C},\mathcal{E})}$.

Since $T_{\mathcal{C}} : \mathcal{E} \to \mathcal{C}$ is fully faithful, the forgetful functor $\mathbf{f} : Z(\mathcal{C}, \mathcal{E}) \to \mathcal{C}$ is surjective by [DNO, Lem. 3.12]. By [DMNO, Lem. 3.5], the object A is a connected étale algebra and there is a monoidal equivalence $\mathcal{C} \simeq Z(\mathcal{C}, \mathcal{E})_A$. More explicitly, for any object $x \in \mathcal{C}$, the object $I_{\mathcal{C}}(x) = [\mathbb{1}_{\mathcal{C}}, x]_{Z(\mathcal{C},\mathcal{E})}$ is a right A-module and the monoidal functor

$$I_{\mathcal{C}} = [\mathbb{1}_{\mathcal{C}}, -]_{Z(\mathcal{C},\mathcal{E})} : \mathcal{C} \to Z(\mathcal{C}, \mathcal{E})_A$$

is a monoidal equivalence. The left A-module structure on $I_{\mathcal{C}}(x)$ is given by $A \otimes I_{\mathcal{C}}(x) \xrightarrow{\beta_{A, I_{\mathcal{C}}(x)}} I_{\mathcal{C}}(x) \otimes A \to I_{\mathcal{C}}(x)$. One can check that for $x = \mathbf{f}(z) \in \mathcal{C}$ with $z \in Z(\mathcal{C}, \mathcal{E})$, one have $I_{\mathcal{C}}(x) \cong z \otimes A$ (as A-modules). The monoidal structure on $I_{\mathcal{C}}$ is induced by

$$\mu_{x,y} : I_{\mathcal{C}}(x \otimes y) = [\mathbb{1}_{\mathcal{C}}, \mathbf{f}(z) \otimes y]_{Z(\mathcal{C},\mathcal{E})} \simeq z \otimes [\mathbb{1}_{\mathcal{C}}, y]_{Z(\mathcal{C},\mathcal{E})} = z \otimes A \otimes_A I_{\mathcal{C}}(y) = I_{\mathcal{C}}(x) \otimes_A I_{\mathcal{C}}(y)$$

for $x, y \in \mathcal{C}$. Since $\mathbf{f}$ is surjective, $\mu_{x,y}$ is always an isomorphism. $Z(\mathcal{C}, \mathcal{E})_A$ can be identified with a subcategory of the fusion category ${}_AZ(\mathcal{C}, \mathcal{E})_A$. Recall the central structure on the functor $\mathcal{E} \to {}_AZ(\mathcal{C}, \mathcal{E})_A$ by Ex. 3.9. The structure of monoidal functor over $\mathcal{E}$ on $I_{\mathcal{C}}$ is induced by $I_{\mathcal{C}}(T_{\mathcal{C}}(e)) = [\mathbb{1}_{\mathcal{C}}, T_{\mathcal{C}}(e)]_{Z(\mathcal{C},\mathcal{E})} \simeq T_{\mathcal{C}}(e) \otimes A$. □

Lemma 4.36. Let $\mathcal{C}$ and $\mathcal{D}$ be fusion categories over $\mathcal{E}$ such that the central functors $T_{\mathcal{C}} : \mathcal{E} \to \mathcal{C}$ and $T_{\mathcal{D}} : \mathcal{E} \to \mathcal{D}$ are fully faithful. Suppose that $Z(\mathcal{C}, \mathcal{E})$ is equivalent to $Z(\mathcal{D}, \mathcal{E})$ as braided fusion categories over $\mathcal{E}$. We have $\mathrm{FPdim}(\mathcal{C}) = \mathrm{FPdim}(\mathcal{D})$ and $\mathrm{FPdim}(I_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}})) = \mathrm{FPdim}(I_{\mathcal{D}}(\mathbb{1}_{\mathcal{D}})) = \frac{\mathrm{FPdim}(\mathcal{C})}{\mathrm{FPdim}(\mathcal{E})}$, where FPdim is the Frobenius-Perron dimension.

Proof. $Z(\mathcal{C},\mathcal{E})$ is a subcategory of $Z(\mathcal{C})$. By [DGNO, Thm. 3.14], we obtain the equation

$$\mathrm{FPdim}(Z(\mathcal{C},\mathcal{E}))\mathrm{FPdim}(Z(\mathcal{C},\mathcal{E})') = \mathrm{FPdim}(Z(\mathcal{C}))\mathrm{FPdim}(Z(\mathcal{C},\mathcal{E})\cap Z(\mathcal{C})')$$

Since the equations $Z(\mathcal{C},\mathcal{E})' = \mathcal{E}$, $Z(\mathcal{C})' = \mathcal{V}\mathrm{ec}$ and $\mathrm{FPdim}(Z(\mathcal{C})) = \mathrm{FPdim}(\mathcal{C})^2$ (recall [EGNO, Thm. 7.16.6]) hold, we get the equation

$$\mathrm{FPdim}(Z(\mathcal{C},\mathcal{E})) = \frac{\mathrm{FPdim}(\mathcal{C})^2}{\mathrm{FPdim}(\mathcal{E})} \tag{4.18}$$

Since $Z(\mathcal{C},\mathcal{E}) \simeq Z(\mathcal{D},\mathcal{E})$ and the numbers $\mathrm{FPdim}(\mathcal{C})$ and $\mathrm{FPdim}(\mathcal{D})$ are positive, $\mathrm{FPdim}(\mathcal{C}) = \mathrm{FPdim}(\mathcal{D})$.

Since $\mathbf{f} : Z(\mathcal{C},\mathcal{E}) \to \mathcal{C}$ is surjective, we get the equation

$$\mathrm{FPdim}(I_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}})) = \frac{\mathrm{FPdim}(Z(\mathcal{C},\mathcal{E}))}{\mathrm{FPdim}(\mathcal{C})} = \frac{\mathrm{FPdim}(\mathcal{C})}{\mathrm{FPdim}(\mathcal{E})}$$

by [EGNO, Lem. 6.2.4] and the equation (4.18). Then we have $\mathrm{FPdim}(I_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}})) = \mathrm{FPdim}(I_{\mathcal{D}}(\mathbb{1}_{\mathcal{D}}))$. □

Lemma 4.37. Suppose that $f : Z(\mathcal{C}) \xrightarrow{\simeq} Z(\mathcal{D})$ is an equivalence of braided multifusion categories and $u_e : f(T_{\mathcal{C}}(e)) \simeq T_{\mathcal{D}}(e)$ is a monoidal natural isomorphism in $Z(\mathcal{D})$ for all $e \in \mathcal{E}$. Then f induces an equivalence $Z(\mathcal{C},\mathcal{E}) \simeq Z(\mathcal{D},\mathcal{E})$ of braided multifusion categories over $\mathcal{E}$.

Proof. Suppose that $f : Z(\mathcal{C}) \to Z(\mathcal{D})$ maps $(x, \beta_{x,-})$ to $(f(x), \gamma_{f(x),-})$. If the object $(x, \beta_{x,-})$ belongs to $Z(\mathcal{C},\mathcal{E})$, the object $(f(x), \gamma_{f(x),-})$ belongs to $Z(\mathcal{D},\mathcal{E})$ by the commutativity of the following diagram.

$$\begin{tikzcd}
f(x\otimes T_{\mathcal{C}}(e)) \arrow[r] \arrow[d, "\beta_{x,T_{\mathcal{C}}(e)}"'] & f(x)\otimes f(T_{\mathcal{C}}(e)) \arrow[r, "1{,}u_e"] \arrow[d, "\gamma_{f(x),f(T_{\mathcal{C}}(e))}"] & f(x)\otimes T_{\mathcal{D}}(e) \arrow[d, "\gamma_{f(x),T_{\mathcal{D}}(e)}"] \\
f(T_{\mathcal{C}}(e)\otimes x) \arrow[r] \arrow[d, "\beta_{T_{\mathcal{C}}(e),x}"'] & f(T_{\mathcal{C}}(e))\otimes f(x) \arrow[r, "u_e{,}1"] \arrow[d, "\gamma_{f(T_{\mathcal{C}}(e)),f(x)}"] & T_{\mathcal{D}}(e)\otimes f(x) \arrow[d, "\gamma_{T_{\mathcal{D}}(e),f(x)}"] \\
f(x\otimes T_{\mathcal{C}}(e)) \arrow[r] & f(x)\otimes f(T_{\mathcal{C}}(e)) \arrow[r, "1{,}u_e"] & f(x)\otimes T_{\mathcal{D}}(e)
\end{tikzcd}$$

Since f is the braided functor, the left two squares commute. The right-upper square commutes by the naturality of $\gamma_{f(x),-}$. The right-down square commutes by reason that u_e is a natural isomorphism in $Z(\mathcal{D})$. Since the equation $\beta_{T_{\mathcal{C}}(e),x} \circ \beta_{x,T_{\mathcal{C}}(e)} = \mathrm{id}$ holds, we obtain the equation $\gamma_{T_{\mathcal{D}}(e),f(x)} \circ \gamma_{f(x),T_{\mathcal{D}}(e)} = \mathrm{id}$. Then f induces an equivalence $Z(\mathcal{C},\mathcal{E}) \simeq Z(\mathcal{D},\mathcal{E})$. □

Example 4.38. Let $\mathcal{C}$ be a fusion category over $\mathcal{E}$ and A a separable algebra in $\mathcal{C}$. By [EGNO, Rem. 7.16.3], there is a monoidal equivalence $\Phi : Z(\mathcal{C}) \to Z({}_A\mathcal{C}_A)$, $(z, \beta_{z,-}) \mapsto (z\otimes A, \beta_{z\otimes A})$, where $\beta_{z\otimes A}$ is induced by

$$z\otimes A\otimes_A x \cong z\otimes x \xrightarrow{\beta_{z,x}} x\otimes z \cong x\otimes_A A\otimes z \xrightarrow{1,\beta_{z,A}^{-1}} x\otimes_A z\otimes A, \quad \forall x \in {}_A\mathcal{C}_A$$

Φ induces the monoidal equivalence $Z(\mathcal{C},\mathcal{E}) = \mathcal{E}'|_{Z(\mathcal{C})} \simeq \mathcal{E}'|_{Z({}_A\mathcal{C}_A)} = Z({}_A\mathcal{C}_A,\mathcal{E})$. Recall the central structure on the functor $I : \mathcal{E} \to {}_A\mathcal{C}_A$ in Ex. 3.9. We obtain $\Phi(T_{\mathcal{C}}(e)) = T_{\mathcal{C}}(e)\otimes A = I(e)$. Then $Z(\mathcal{C},\mathcal{E}) \simeq Z({}_A\mathcal{C}_A,\mathcal{E})$ is the monoidal equivalence over $\mathcal{E}$.

Let $\mathcal{C}_A$ be an indecomposable left $\mathcal{C}$-module in $\mathrm{Cat}^{\mathrm{fs}}$. By [EGNO, Prop. 8.5.3], $\Phi : Z(\mathcal{C}) \simeq Z({}_A\mathcal{C}_A)$ is the equivalence of braided fusion categories. By Lem.4.37, $\Phi : Z(\mathcal{C},\mathcal{E}) \simeq Z({}_A\mathcal{C}_A,\mathcal{E})$ is the equivalence of braided fusion categories over $\mathcal{E}$.

Lemma 4.39. Let $\mathcal{C}$ be a fusion category over $\mathcal{E}$ and $\mathcal{M}$ an indecomposable left $\mathcal{C}$-module in $\mathrm{Cat}^{\mathrm{fs}}$. Then $\mathrm{FPdim}(\mathcal{C}) = \mathrm{FPdim}(\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M}))$.

Proof. Since $\mathcal{M}$ is a left $\mathcal{C}$-module in $\mathrm{Cat}^{\mathrm{fs}}$, there is a separable algebra A in $\mathcal{C}$ such that $\mathcal{M} \simeq \mathcal{C}_A$. Recall the equivalences $Z(\mathcal{C}, \mathcal{E}) \simeq Z({}_A\mathcal{C}_A, \mathcal{E})$ in Expl. 4.38 and ${}_A\mathcal{C}_A \simeq \mathrm{Fun}_{\mathcal{C}}(\mathcal{C}_A, \mathcal{C}_A)^{\mathrm{rev}}$ in Prop. A.5. Then we get the equations

$$\frac{\mathrm{FPdim}(\mathcal{C})^2}{\mathrm{FPdim}(\mathcal{E})} = \mathrm{FPdim}(Z(\mathcal{C}, \mathcal{E})) = \mathrm{FPdim}(Z({}_A\mathcal{C}_A, \mathcal{E})) = \frac{\mathrm{FPdim}({}_A\mathcal{C}_A)^2}{\mathrm{FPdim}(\mathcal{E})} = \frac{\mathrm{FPdim}(\mathrm{Fun}_{\mathcal{C}}(\mathcal{C}_A, \mathcal{C}_A))^2}{\mathrm{FPdim}(\mathcal{E})}$$

The first and third equations are due to the equation (4.18). Since the Frobenius-Perron dimensions are positive, the result follows. ◻

Thm. 8.12.3 of [EGNO] says that two finite tensor categories $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent if and only if $Z(\mathcal{C})$ and $Z(\mathcal{D})$ are equivalent as braided tensor categories. The statement and the proof idea of Thm. 4.40 comes from which of Thm. 8.12.3 in [EGNO].

Theorem 4.40. Let $\mathcal{C}$ and $\mathcal{D}$ be fusion categories over $\mathcal{E}$ such that the central functors $T_{\mathcal{C}} : \mathcal{E} \to \mathcal{C}$ and $T_{\mathcal{D}} : \mathcal{E} \to \mathcal{D}$ are fully faithful. $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ if and only if $Z(\mathcal{C}, \mathcal{E})$ and $Z(\mathcal{D}, \mathcal{E})$ are equivalent as braided fusion categories over $\mathcal{E}$.

Proof. The "only if" direction is proved in Thm. 4.34.

Let $\mathcal{C}, \mathcal{D}$ be fusion categories over $\mathcal{E}$ such that there is an equivalence $a : Z(\mathcal{C}, \mathcal{E}) \xrightarrow{\simeq} Z(\mathcal{D}, \mathcal{E})$ as braided fusion categories over $\mathcal{E}$. Since $I_{\mathcal{D}}(\mathbb{1}_{\mathcal{D}})$ is a connected étale algebra in $Z(\mathcal{D}, \mathcal{E})$, $L := a^{-1}(I_{\mathcal{D}}(\mathbb{1}_{\mathcal{D}}))$ is a connected étale algebra in $Z(\mathcal{C}, \mathcal{E})$. By Lem. 4.35, there is an equivalence

$$\mathcal{D} \simeq Z(\mathcal{C}, \mathcal{E})_L$$

of fusion categories over $\mathcal{E}$.

By [DMNO, Prop. 2.7], the category $\mathcal{C}_L$ of L-modules in $\mathcal{C}$ is semisimple. Note that the algebra L is indecomposable in $Z(\mathcal{C}, \mathcal{E})$ but L might be decomposable as an algebra in $\mathcal{C}$, i.e. the category ${}_L\mathcal{C}_L$ is a multifusion category. It has a decomposition

$${}_L\mathcal{C}_L = \bigoplus_{i,j\in J} \left({}_L\mathcal{C}_L\right)_{ij}$$

where J is a finite set and each $\left({}_L\mathcal{C}_L\right)_{ii}$ is a fusion category. Let $L = \bigoplus_{i\in J} L_i$ be the decomposition of L such that ${}_{L_i}\mathcal{C}_{L_i} \simeq \left({}_L\mathcal{C}_L\right)_{ii}$. Here $L_i, i \in J$, are indecomposable algebras in $\mathcal{C}$ such that the multiplication of L is zero on $L_i \otimes L_j, i \neq j$.

Next we want to show that there is an equivalence $Z(\mathcal{C}, \mathcal{E})_L \simeq {}_{L_i}\mathcal{C}_{L_i}$ of fusion categories over $\mathcal{E}$. Consider the following commutative diagram of monoidal functors over $\mathcal{E}$:

$$\begin{tikzcd}
Z(\mathcal{C}, \mathcal{E}) \arrow[rrr, "z\mapsto z\otimes L_i"] \arrow[d, "z\mapsto z\otimes L"'] & & & Z({}_{L_i}\mathcal{C}_{L_i}, \mathcal{E}) \arrow[d, "\mathbf{f}"] \\
Z(\mathcal{C}, \mathcal{E})_L \subset {}_LZ(\mathcal{C}, \mathcal{E})_L \arrow[r, "\mathbf{f}"'] & {}_L\mathcal{C}_L \arrow[rr, "\pi_i"'] & & {}_{L_i}\mathcal{C}_{L_i}
\end{tikzcd}$$

π_i is projection and $\pi_i(x \otimes L) = x \otimes L_i$. The top arrow is the equivalence by Expl. 4.38. Next we calculate the Frobenius-Perron dimensions of the categories $Z(\mathcal{C}, \mathcal{E})_L$ and ${}_{L_i}\mathcal{C}_{L_i}$:

$$\mathrm{FPdim}(Z(\mathcal{C}, \mathcal{E})_L) = \frac{\mathrm{FPdim}(Z(\mathcal{C}, \mathcal{E}))}{\mathrm{FPdim}(L)} = \mathrm{FPdim}(\mathcal{C}) = \mathrm{FPdim}({}_{L_i}\mathcal{C}_{L_i})$$

The first equation is due to [DMNO, Lem. 3.11]. The second equation is due to $\mathrm{FPdim}(L) = \mathrm{FPdim}(I_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}})) = \mathrm{FPdim}(Z(\mathcal{C},\mathcal{E}))/\mathrm{FPdim}(\mathcal{C})$ by Lem. 4.36. The third equation is due to Lem. 4.39. Since $\pi_i \circ \mathbf{f}$ is also surjective, $\pi_i \circ \mathbf{f}$ is an equivalence. Then we have monoidal equivalences over $\mathcal{E}$: $\mathcal{D} \simeq Z(\mathcal{C},\mathcal{E})_L \simeq {}_{L_i}\mathcal{C}_{L_i} \simeq \mathrm{Fun}_{\mathcal{C}}(\mathcal{C}_{L_i},\mathcal{C}_{L_i})^{\mathrm{rev}}$. □

4.5 Modules over a braided fusion category over $\mathcal{E}$

Let $\mathcal{C}$ and $\mathcal{D}$ be braided fusion categories over $\mathcal{E}$. In this subsection, fusion categories $\mathcal{M}$ over $\mathcal{E}$ with the property that $\mathcal{E} \to Z(\mathcal{M})$ is fully faithful.

Definition 4.41. The 2-category $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Alg}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}))$ consists of the following data.

- A class of objects in $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Alg}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}))$. An object $\mathcal{M} \in \mathrm{LMod}_{\mathcal{C}}(\mathrm{Alg}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}))$ is a fusion category $\mathcal{M}$ over $\mathcal{E}$ equipped with a braided monoidal functor $\phi_{\mathcal{M}} : \overline{\mathcal{C}} \to Z(\mathcal{M},\mathcal{E})$ over $\mathcal{E}$.

- For objects $\mathcal{M},\mathcal{N}$ in $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Alg}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}))$, a 1-morphism $F : \mathcal{M} \to \mathcal{N}$ in $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Alg}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}))$ is a monoidal functor $F : \mathcal{M} \to \mathcal{N}$ equipped with a monoidal isomorphism $u^{\mathcal{M}\mathcal{N}} : F \circ \phi_{\mathcal{M}} \Rightarrow \phi_{\mathcal{N}}$ such that the diagram
$$\begin{tikzcd} F(\phi_{\mathcal{M}}(c)\otimes m) \arrow[r] \arrow[d, "\beta^{\mathcal{M}}_{c,m}"'] & F(\phi_{\mathcal{M}}(c))\otimes F(m) \arrow[r, "u^{\mathcal{M}\mathcal{N}}_c{,}1"] & \phi_{\mathcal{N}}(c)\otimes F(m) \arrow[d, "\beta^{\mathcal{N}}_{c,F(m)}"] \\ F(m\otimes\phi_{\mathcal{M}}(c)) \arrow[r] & F(m)\otimes F(\phi_{\mathcal{M}}(c)) \arrow[r, "1{,}u^{\mathcal{M}\mathcal{N}}_c"'] & F(m)\otimes\phi_{\mathcal{N}}(c) \end{tikzcd} \tag{4.19}$$
commutes for $c \in \overline{\mathcal{C}}, m \in \mathcal{M}$, where $(\phi_{\mathcal{M}}(c),\beta^{\mathcal{M}}) \in Z(\mathcal{M},\mathcal{E})$ and $(\phi_{\mathcal{N}}(c),\beta^{\mathcal{N}}) \in Z(\mathcal{N},\mathcal{E})$.

- For 1-morphisms $F,G : \mathcal{M} \rightrightarrows \mathcal{N}$ in $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Alg}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}))$, a 2-morphism $\alpha : F \Rightarrow G$ in $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Alg}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}))$ is a monoidal isomorphism α such that the diagram
$$\begin{tikzcd} F(\phi_{\mathcal{M}}(c)) \arrow[rr, "\alpha_{\phi_{\mathcal{M}}(c)}"] \arrow[dr, "u^{\mathcal{M}\mathcal{N}}_c"'] & & G(\phi_{\mathcal{M}}(c)) \arrow[dl, "\tilde{u}^{\mathcal{M}\mathcal{N}}_c"] \\ & \phi_{\mathcal{N}}(c) & \end{tikzcd}$$
commutes for $c \in \overline{\mathcal{C}}$, where $u^{\mathcal{M}\mathcal{N}}$ and $\tilde{u}^{\mathcal{M}\mathcal{N}}$ are the monoidal isomorphisms on F and G respectively.

Remark 4.42. If $F : \mathcal{M} \to \mathcal{N}$ is a 1-morphism in $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Alg}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}))$, F is a left $\overline{\mathcal{C}}$-module functor and a monoidal functor over $\mathcal{E}$. By Lem. 3.4, the left $\overline{\mathcal{C}}$-module structure s^F on F is defined as $F(c\odot m) = F(\phi_{\mathcal{M}}(c)\otimes m) \to F(\phi_{\mathcal{M}}(c))\otimes F(m) \xrightarrow{u^{\mathcal{M}\mathcal{N}}_c,1} \phi_{\mathcal{N}}(c)\otimes F(m) = c\odot F(m)$ for all $c \in \overline{\mathcal{C}}, m \in \mathcal{M}$. Let $u^{\mathcal{C}\mathcal{M}} : \phi_{\mathcal{M}} \circ T_{\mathcal{C}} \Rightarrow T_{\mathcal{M}}$ and $u^{\mathcal{C}\mathcal{N}} : \phi_{\mathcal{N}} \circ T_{\mathcal{C}} \Rightarrow T_{\mathcal{N}}$ be the structures of monoidal functors over $\mathcal{E}$ on $\phi_{\mathcal{M}}$ and $\phi_{\mathcal{N}}$ respectively. The structure of monoidal functor over $\mathcal{E}$ on F is induced by the composition $v : F \circ T_{\mathcal{M}} \xRightarrow{1,(u^{\mathcal{C}\mathcal{M}})^{-1}} F \circ \phi_{\mathcal{M}} \circ T_{\mathcal{C}} \xRightarrow{u^{\mathcal{M}\mathcal{N}},1} \phi_{\mathcal{N}} \circ T_{\mathcal{C}} \xRightarrow{u^{\mathcal{C}\mathcal{N}}} T_{\mathcal{N}}$.

The 2-category $\mathrm{RMod}_{\mathcal{D}}(\mathrm{Alg}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}))$ consists of the following data.

- An object $\mathcal{M} \in \mathrm{RMod}_{\mathcal{D}}(\mathrm{Alg}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}))$ is a fusion category $\mathcal{M}$ over $\mathcal{E}$ equipped with a braided monoidal functor $\phi_{\mathcal{M}} : \mathcal{D} \to Z(\mathcal{M},\mathcal{E})$ over $\mathcal{E}$.

- 1-morphisms and 2-morphisms are similar with which in the Def. 4.41.

And the 2-category $\mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Alg}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}))$ consists of the following data.

- An object $\mathcal{M} \in \mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Alg}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}))$ is a fusion category $\mathcal{M}$ over $\mathcal{E}$ equipped with a braided monoidal functor $\phi_{\mathcal{M}} : \overline{\mathcal{C}} \boxtimes_{\mathcal{E}} \mathcal{D} \to Z(\mathcal{M}, \mathcal{E})$ over $\mathcal{E}$. An object $\mathcal{M} \in \mathrm{BMod}_{\mathcal{C}|\mathcal{D}}(\mathrm{Alg}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}))$ is closed if $\phi_{\mathcal{M}}$ is an equivalence.
- 1-morphisms and 2-morphisms are similar with which in the Def. 4.41.

5 Factorization homology

In this section, Sec. 5.1 recalls the definitions of unitary categories, unitary fusion categories and unitary modular tensor categories over $\mathcal{E}$ (see [LKW, Def. 3.15, 3.16, 3.21]). Sec. 5.2 recalls the theory of factorization homology. Sec. 5.3 and Sec. 5.4 compute the factorization homology of stratified surfaces with coefficients given by $\mathrm{UMTC}_{/\mathcal{E}}$'s.

5.1 Unitary categories

Definition 5.1. A *$*$-category* $\mathcal{C}$ is a $\mathbb{C}$-linear category equipped with a functor $* : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ which acts as the identity map on objects and is anti-linear and involutive on morphisms. More explicitly, for any objects $x, y \in \mathcal{C}$, there is a map $* : \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{C}}(y, x)$, such that

$$(g \circ f)^* = f^* \circ g^*, \quad (\lambda f)^* = \bar{\lambda} f^*, \quad (f^*)^* = f$$

for $f : u \to v, g : v \to w, h : x \to y, \lambda \in \mathbb{C}^\times$. Here $\mathbb{C}$ denotes the field of complex numbers.

A *$*$-functor* between two $*$-categories $\mathcal{C}$ and $\mathcal{D}$ is a $\mathbb{C}$-linear functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(f^*) = F(f)^*$ for all $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$. A $*$-category is called *unitary* if it is finite and the $*$-operation is positive, i.e. $f \circ f^* = 0$ implies $f = 0$.

Definition 5.2. A *unitary fusion category* $\mathcal{C}$ is both a fusion category and a unitary category such that $*$ is compatible with the monoidal structures, i.e.

$$(g \otimes h)^* = g^* \otimes h^*, \quad \forall g : v \to w, h : x \to y$$

$$\alpha^*_{x,y,z} = \alpha^{-1}_{x,y,z}, \quad \gamma^*_x = \gamma^{-1}_x, \quad \rho^*_x = \rho^{-1}_x$$

for $x, y, z, v, w \in \mathcal{C}$, where α, γ, ρ are the associativity, the left unit and the right unit constraints respectively. A unitary braided fusion category is a unitary fusion category $\mathcal{C}$ with a braiding c such that $c^*_{x,y} = c^{-1}_{x,y}$ for any $x, y \in \mathcal{C}$.

A *monoidal $*$-functor* between unitary fusion categories is a monoidal functor $(F, J) : \mathcal{C} \to \mathcal{D}$, such that F is a $*$-functor and $J^*_{x,y} = J^{-1}_{x,y}$ for $x, y \in \mathcal{C}$. A *braided $*$-functor* between unitary braided fusion categories is both a monoidal $*$-functor and a braided functor.

Remark 5.3. Let $\mathcal{C}$ be a unitary fusion category. $\mathcal{C}$ admits a canonical spherical structure. The unitary center $Z^*(\mathcal{C})$ is defined as the fusion subcategory of the Drinfeld center $Z(\mathcal{C})$, where $(x, c_{x,-}) \in Z^*(\mathcal{C})$ if $c^*_{x,-} = c^{-1}_{x,-}$. $Z^*(\mathcal{C})$ is a unitary braided fusion category and $Z^*(\mathcal{C})$ is braided equivalent to $Z(\mathcal{C})$ by [GHR, Prop. 5.24].

Definition 5.4. A *unitary $\mathcal{E}$-module category* $\mathcal{C}$ is an object $\mathcal{C}$ in $\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}$ such that $\mathcal{C}$ is a unitary category, and the $*$ is compatible with the $\mathcal{E}$-module structure, i.e.

$$(i \odot j)^* = i^* \odot j^*, \qquad \lambda^*_{e,\tilde{e},x} = \lambda^{-1}_{e,\tilde{e},x}, \qquad l^*_x = l^{-1}_x$$

for $i : e \to \tilde{e} \in \mathcal{E}, j : x \to y \in \mathcal{C}$, where λ and l are the module associativity and the unit constraints respectively. Notice that symmetric fusion categories are all unitary.

Let $\mathcal{C}, \mathcal{D}$ be unitary $\mathcal{E}$-module categories. An *$\mathcal{E}$-module $*$-functor* is an $\mathcal{E}$-module functor $(F, s) : \mathcal{C} \to \mathcal{D}$ such that F is a $*$-functor and $s^*_{e,x} = s^{-1}_{e,x}$ for $e \in \mathcal{E}, x \in \mathcal{C}$.

Remark 5.5. Let $\mathcal{C}$ be an indecomposable unitary $\mathcal{E}$-module category. Then the full subcategory $\mathrm{Fun}^*_{\mathcal{E}}(\mathcal{C},\mathcal{C}) \subset \mathrm{Fun}_{\mathcal{E}}(\mathcal{C},\mathcal{C})$ of $\mathcal{E}$-module $*$-functors is a unitary fusion category. And the embedding $\mathrm{Fun}^*_{\mathcal{E}}(\mathcal{C},\mathcal{C}) \to \mathrm{Fun}_{\mathcal{E}}(\mathcal{C},\mathcal{C})$ is the monoidal equivalence by [GHR, Thm, 5.3].

Definition 5.6. A *unitary fusion category over* $\mathcal{E}$ is a unitary fusion category $\mathcal{A}$ equipped with a braided $*$-functor $T'_{\mathcal{A}} : \mathcal{E} \to Z(\mathcal{A})$ such that the central functor $\mathcal{E} \to \mathcal{A}$ is fully faithful. A *unitary braided fusion category over* $\mathcal{E}$ is a unitary braided fusion category $\mathcal{C}$ equipped with a braided $*$-embedding $T_{\mathcal{C}} : \mathcal{E} \to \mathcal{C}'$. A *unitary modular tensor category over* $\mathcal{E}$ *(or* $\mathrm{UMTC}_{/\mathcal{E}}$*)* is a unitary braided fusion category $\mathcal{C}$ over $\mathcal{E}$ such that $\mathcal{C}' \simeq \mathcal{E}$.

Let $\mathcal{C}$ be a unitary fusion category.

Definition 5.7. Let $(A, m : A \otimes A \to A, \eta : \mathbb{1}_{\mathcal{C}} \to A)$ be an algebra in $\mathcal{C}$. A *$*$-Frobenius algebra in* $\mathcal{C}$ is an algebra A in $\mathcal{C}$ such that the comultiplication $m^* : A \to A \otimes A$ is an A-bimodule map. Let A be a $*$-Frobenius algebra in $\mathcal{C}$ and $\mathcal{M}$ a unitary left $\mathcal{C}$-module category. A *left $*$-A-module in* $\mathcal{M}$ is a left A-module $(M, q : A \otimes M \to M)$ such that $q^* : M \to A \otimes M$ is a left A-module map.

Remark 5.8. A $*$-Frobenius algebra in $\mathcal{C}$ is separable. The full subcategory ${}_A\mathcal{M}^* \subset {}_A\mathcal{M}$ of left $*$-A-modules in $\mathcal{M}$ is a unitary category. The embedding ${}_A\mathcal{M}^* \to {}_A\mathcal{M}$ is an equivalence. Similarly, one can define $\mathcal{M}^*_A$ and ${}_A\mathcal{M}^*_A$.

If the object $(x^L, \mathrm{ev}_x : x^L \otimes x \to \mathbb{1}_{\mathcal{C}}, \mathrm{coev}_x : \mathbb{1}_{\mathcal{C}} \to x \otimes x^L)$ is a left dual of x in $\mathcal{C}$, then $(x^L, \mathrm{coev}^*_x : x \otimes x^L \to \mathbb{1}_{\mathcal{C}}, \mathrm{ev}^*_x : \mathbb{1}_{\mathcal{C}} \to x^L \otimes x)$ is the right dual of x in $\mathcal{C}$. Here we choose the duality maps ev_x and coev_x are *normalized*. That is, the induced composition

$$\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, x \otimes -) \xrightarrow{\mathrm{ev}_x} \mathrm{Hom}_{\mathcal{C}}(x^L, -) \xrightarrow{\mathrm{ev}^*_x} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, - \otimes x)$$

is an isometry. Then the normalized left dual x^L is unique up to canonical unitary isomorphism. Let (A, m, η) be a $*$-Frobenius algebra in $\mathcal{C}$. The object $(A, \eta^* \circ m : A \otimes A \to \mathbb{1}_{\mathcal{C}}, m^* \circ \eta : \mathbb{1}_{\mathcal{C}} \to A \otimes A)$ is the left (or right) dual of A in $\mathcal{C}$.

Definition 5.9. A $*$-Frobenius algebra A in $\mathcal{C}$ is *symmetric* if the two morphisms $\Phi_1 = \Phi_2$ in $\mathrm{Hom}_{\mathcal{C}}(A, A^L)$, where

$$\Phi_1 := [(\eta^* \circ m) \otimes \mathrm{id}_{A^L}] \circ (\mathrm{id}_A \otimes \mathrm{coev}_A) \quad \text{and} \quad \Phi_2 := [\mathrm{id}_{A^L} \otimes (\eta^* \circ m)] \circ (\mathrm{ev}^*_A \otimes \mathrm{id}_A)$$

The following proposition comes from Hao Zheng's lessons.

Proposition 5.10. Let $\mathcal{M}$ be a unitary left $\mathcal{C}$-module category. Then there exists a symmetric $*$-Frobenius algebra A such that $\mathcal{M} \simeq \mathcal{C}^*_A$ as unitary left $\mathcal{C}$-module categories.

5.2 Factorization homology for stratified surfaces

The theory of factorization homology (of stratified spaces) is in [AF1, AFT2, AF2].

Definition 5.11. Let $\mathrm{Mfld}^{\mathrm{or}}_n$ be the topological category whose objects are oriented n-manifolds without boundary. For any two oriented n-manifolds M and N, the morphism space $\mathrm{Hom}_{\mathrm{Mfld}^{\mathrm{or}}_n}(M, N)$ is the space of all orientation-preserving embeddings $e : M \to N$, endowed with the compact-open topology. We define $\mathcal{M}\mathrm{fld}^{\mathrm{or}}_n$ to be the symmetric monoidal ∞-category associated to the topological category $\mathrm{Mfld}^{\mathrm{or}}_n$. The symmetric monoidal structure is given by disjoint union.

Definition 5.12. The symmetric monoidal ∞-category $\mathcal{D}\mathrm{isk}^{\mathrm{or}}_n$ is the full subcategory of $\mathcal{M}\mathrm{fld}^{\mathrm{or}}_n$ whose objects are disjoint union of finitely many n-dimensional Euclidean spaces $\coprod_I \mathbb{R}^n$ equipped with the standard orientation.

Definition 5.13. Let $\mathcal{V}$ be a symmetric monoidal ∞-category. An *n-disk algebra* in $\mathcal{V}$ is a symmetric monoidal functor $A : \mathcal{D}\mathrm{isk}_n^{\mathrm{or}} \to \mathcal{V}$.

Let $\mathcal{V}_{\mathrm{uty}}$ be the symmetric monoidal (2,1)-category of unitary categories. The tensor product of $\mathcal{V}_{\mathrm{uty}}$ is Deligne tensor product $\boxtimes$. Expl. 3.5 of [AKZ] gives examples of 0-, 1-, 2-disk algebras in $\mathcal{V}_{\mathrm{uty}}$. A unitary braided fusion category gives a 2-disk algebra in $\mathcal{V}_{\mathrm{uty}}$. A 1-disk algebra in $\mathcal{V}_{\mathrm{uty}}$ is a unitary monoidal category. A 0-disk algebra in $\mathcal{V}_{\mathrm{uty}}$ is a pair $(\mathcal{P}, p)$, where $\mathcal{P}$ is a unitary category and $p \in \mathcal{P}$ is a distinguished object. We guess that the n-disk algebra in $\mathcal{V}_{\mathrm{uty}}$ equipped with the compatible $\mathcal{E}$-module structure, is the n-disk algebra both in $\mathcal{V}_{\mathrm{uty}}$ and $\mathrm{Cat}_{\mathcal{E}}^{\mathrm{fs}}$, for $n = 0, 1, 2$.

Assumption 5.14. Let $\mathcal{V}_{\mathrm{uty}}^{\mathcal{E}}$ be the symmetric monoidal (2,1)-category of unitary $\mathcal{E}$-module categories. We assume that a unitary braided fusion category over $\mathcal{E}$ gives a 2-disk algebra in $\mathcal{V}_{\mathrm{uty}}^{\mathcal{E}}$, a unitary fusion category over $\mathcal{E}$ gives a 1-disk algebra in $\mathcal{V}_{\mathrm{uty}}^{\mathcal{E}}$, and a unitary $\mathcal{E}$-module category equipped with a distinguished object gives a 0-disk algebra in $\mathcal{V}_{\mathrm{uty}}^{\mathcal{E}}$.

Definition 5.15. An *(oriented) stratified surface* is a pair $(\Sigma, \Sigma \xrightarrow{\pi} \{0, 1, 2\})$ where Σ is an oriented surface and π is a map. The subspace $\Sigma_i := \pi^{-1}(i)$ is called the *i-stratum* and its connected components are called *i-cells*. These data are required to satisfy the following properties.

(1) Σ_0 and $\Sigma_0 \cup \Sigma_1$ are closed subspaces of Σ.

(2) For each point $x \in \Sigma_1$, there exists an open neighborhood U of x such that $(U, U \cap \Sigma_1, U \cap \Sigma_0) \cong (\mathbb{R}^2, \mathbb{R}^1, \emptyset)$.

(3) For each point $x \in \Sigma_0$, there exists an open neighborhood V of x and a finite subset $I \subset S^1$, such that $(V, V \cap \Sigma_1, V \cap \Sigma_0) \cong (\mathbb{R}^2, C(I)\backslash\{\text{cone point}\}, \{\text{cone point}\})$, where $C(I)$ is the open cone of I defined by $C(I) = I \times [0, 1)/I \times \{0\}$.

(4) Each 1-cell is oriented, and each 0-cell is equipped with the standard orientation.

There are three important types of stratified 2-disks shown in [AKZ, Expl. 3.14].

Definition 5.16. We define $\mathrm{Mfld}^{\mathrm{str}}$ to be the topological category whose objects are stratified surfaces and morphism space between two stratified surfaces M and N are embeddings $e : M \to N$ that preserve the stratifications, and the orientations on 1-, 2-cells. We define $\mathcal{M}\mathrm{fld}^{\mathrm{str}}$ to be the symmetric monoidal ∞-category associated to the topological category $\mathrm{Mfld}^{\mathrm{str}}$. The symmetric monoidal structure is given by disjoint union.

Definition 5.17. Let M be a stratified surface. We define $\mathcal{D}\mathrm{isk}_M^{\mathrm{str}}$ to be the full subcategory of $\mathcal{M}\mathrm{fld}^{\mathrm{str}}$ consisting of those disjoint unions of stratified 2-disks that admit at least one morphism into M.

Definition 5.18. Let $\mathcal{V}$ be a symmetric monoidal ∞-category. A *coefficient* on a stratified surface M is a symmetric monoidal functor $A : \mathcal{D}\mathrm{isk}_M^{\mathrm{str}} \to \mathcal{V}$.

A coefficient A provides a map from each i-cell of M to an i-disk algebra in $\mathcal{V}$.

Definition 5.19. Let $\mathcal{V}$ be a symmetric monoidal ∞-category, M a stratified surface, and $A : \mathcal{D}\mathrm{isk}_M^{\mathrm{str}} \to \mathcal{V}$ a coefficient. The *factorization homology* of M with coefficient in A is an object of $\mathcal{V}$ defined as follows:

$$\int_M A := \mathrm{Colim}\Big((\mathcal{D}\mathrm{isk}_M^{\mathrm{str}})_{/M} \xrightarrow{i} \mathcal{D}\mathrm{isk}_M^{\mathrm{str}} \xrightarrow{A} \mathcal{V}\Big)$$

where $(\mathcal{D}\mathrm{isk}_M^{\mathrm{str}})_{/M}$ is the over category of stratified 2-disks embedded in M. And the notation $\mathrm{Colim}\Big((\mathcal{D}\mathrm{isk}_M^{\mathrm{str}})_{/M} \xrightarrow{A \circ i} \mathcal{V}\Big)$ denotes the colimit of the functor $A \circ i$.

Definition 5.20. A *collar-gluing* for an oriented n-manifold M is a continuous map $f : M \to [-1,1]$ to the closed interval such that restriction of f to the preimage of $(-1,1)$ is a manifold bundle. We denote a collar-gluing $f : M \to [-1,1]$ by the open cover $M_- \cup_{M_0\times\mathbb{R}} M_+ \simeq M$, where $M_- = f^{-1}([-1,1))$, $M_+ = f^{-1}((-1,1])$ and $M_0 = f^{-1}(0)$.

Theorem 5.21. ([AF1] Lem. 3.18). Suppose $\mathcal{V}$ is presentable and the tensor product $\otimes : \mathcal{V}\times\mathcal{V} \to \mathcal{V}$ preserves small colimits for both variables. Then the factorization homology satisfies $\otimes$-excision property. That is, for any collar-gluing $M_- \cup_{M_0\times\mathbb{R}} M_+ \simeq M$, there is a canonical equivalence:

$$\int_M A \simeq \int_{M_-} A \bigotimes_{\int_{M_0\times\mathbb{R}} A} \int_{M_+} A$$

Remark 5.22. If U is contractible, there is an equivalence $\int_U A \simeq A$ in $\mathcal{V}$.

Generalization of the $\otimes$-excision property is the *pushforward property.* Let M be an oriented m-manifold, N an oriented n-manifold, possibly with boundary, and A an m-disk algebra in a $\otimes$-presentable ∞-category $\mathcal{V}$. Let $f : M \to N$ be a continuous map which fibers over the interior and the boundary of N. There is a pushforward functor f_* sends an m-disk algebra A on M to the n-disk algebra f_*A on N. Given an embedding $e : U \to N$ where $U = \mathbb{R}^n$ or $\mathbb{R}^{n-1}\times[0,1)$, an n-disk algebra f_*A is defined as $(f_*A)(U) := \int_{f^{-1}(e(U))} A$. Then there is a canonical equivalence in $\mathcal{V}$

$$\int_N f_*A \simeq \int_M A \tag{5.1}$$

5.3 Preparation

Lemma 5.23. Let $\mathcal{C}$ be a multifusion category over $\mathcal{E}$ such that $\mathcal{E} \to Z(\mathcal{C})$ is fully faithful. Then the functor $\mathcal{C} \boxtimes_{Z(\mathcal{C},\mathcal{E})} \mathcal{C}^{\mathrm{rev}} \to \mathrm{Fun}_{\mathcal{E}}(\mathcal{C},\mathcal{C})$ given by $a \boxtimes_{Z(\mathcal{C},\mathcal{E})} b \mapsto a \otimes - \otimes b$ is an equivalence of multifusion categories over $\mathcal{E}$.

Proof. $\mathcal{C}^{\mathrm{rev}}$ and $\mathcal{C}$ are the same as categories. The composed equivalence (as categories):

$$\mathcal{C} \boxtimes_{Z(\mathcal{C},\mathcal{E})} \mathcal{C} \xrightarrow{\mathrm{id}\boxtimes_{Z(\mathcal{C},\mathcal{E})}\delta^L} \mathcal{C} \boxtimes_{Z(\mathcal{C},\mathcal{E})} \mathcal{C}^{\mathrm{op}} \xrightarrow{v} \mathcal{C}^{\mathrm{rev}} \boxtimes_{\mathcal{E}} \mathcal{C}$$

carries $a \boxtimes_{Z(\mathcal{C},\mathcal{E})} b \mapsto a \boxtimes_{Z(\mathcal{C},\mathcal{E})} b^L \mapsto [a, b^L]^R_{\mathcal{C}^{\mathrm{rev}}\boxtimes_{\mathcal{E}}\mathcal{C}}$, where v is induced by Thm. 4.18 and Eq. (4.17). Notice that the object $\mathcal{C}$ in $\mathrm{LMod}_{\mathcal{C}^{\mathrm{rev}}\boxtimes_{\mathcal{E}}\mathcal{C}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ is faithful. The composed equivalence

$$\mathcal{C}^{\mathrm{rev}} \boxtimes_{\mathcal{E}} \mathcal{C} \xrightarrow{\delta^R\boxtimes_{\mathcal{E}}\mathrm{id}} \mathcal{C}^{\mathrm{op}} \boxtimes_{\mathcal{E}} \mathcal{C} \to \mathrm{Fun}_{\mathcal{E}}(\mathcal{C},\mathcal{C})$$

$$c \boxtimes_{\mathcal{E}} d \mapsto c^R \boxtimes_{\mathcal{E}} d \mapsto [-, c^R]^R_{\mathcal{E}} \odot d$$

maps $[a, b^L]^R_{\mathcal{C}^{\mathrm{rev}}\boxtimes_{\mathcal{E}}\mathcal{C}}$ to a functor $f \in \mathrm{Fun}_{\mathcal{E}}(\mathcal{C},\mathcal{C})$. Note that $\mathrm{Hom}_{\mathcal{C}}([x,c^R]^R_{\mathcal{E}}\odot d, y) \simeq \mathrm{Hom}_{\mathcal{E}}([x,c^R]^R_{\mathcal{E}}, [d,y]_{\mathcal{E}}) \simeq \mathrm{Hom}_{\mathcal{E}}(\mathbb{1}_{\mathcal{E}}, [d,y]_{\mathcal{E}} \otimes [x,c^R]_{\mathcal{E}}) \simeq \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}\boxtimes_{\mathcal{E}}\mathcal{C}}(c^R \boxtimes_{\mathcal{E}} d, x \boxtimes_{\mathcal{E}} y) \simeq \mathrm{Hom}_{\mathcal{C}^{\mathrm{rev}}\boxtimes_{\mathcal{E}}\mathcal{C}}(c \boxtimes_{\mathcal{E}} d, x^L \boxtimes_{\mathcal{E}} y)$, which implies

$$\mathrm{Hom}_{\mathcal{C}}(f(x), y) \simeq \mathrm{Hom}_{\mathcal{C}^{\mathrm{rev}}\boxtimes_{\mathcal{E}}\mathcal{C}}([a,b^L]^R_{\mathcal{C}^{\mathrm{rev}}\boxtimes_{\mathcal{E}}\mathcal{C}}, x^L \boxtimes_{\mathcal{E}} y) \simeq \mathrm{Hom}_{\mathcal{C}}(a \otimes x \otimes b, y)$$

i.e. $f \simeq a \otimes - \otimes b$. Here the second equivalence above holds by the equivalence $(x^L \boxtimes_{\mathcal{E}} y) \otimes [a, b^L]_{\mathcal{C}^{\mathrm{rev}}\boxtimes_{\mathcal{E}}\mathcal{C}} \simeq [a, (x^L \boxtimes_{\mathcal{E}} y) \odot b^L]_{\mathcal{C}^{\mathrm{rev}}\boxtimes_{\mathcal{E}}\mathcal{C}} = [a, y \otimes b^L \otimes x^L]_{\mathcal{C}^{\mathrm{rev}}\boxtimes_{\mathcal{E}}\mathcal{C}}$.

Then the functor $\Phi : \mathcal{C} \boxtimes_{Z(\mathcal{C},\mathcal{E})} \mathcal{C}^{\mathrm{rev}} \to \mathrm{Fun}_{\mathcal{E}}(\mathcal{C},\mathcal{C}), a \boxtimes_{Z(\mathcal{C},\mathcal{E})} b \mapsto a \otimes - \otimes b$ is a monoidal equivalence. Recall the central structures of the functors $T_{\mathcal{C}\boxtimes_{Z(\mathcal{C},\mathcal{E})}\mathcal{C}^{\mathrm{rev}}} : \mathcal{E} \to \mathcal{C} \boxtimes_{Z(\mathcal{C},\mathcal{E})} \mathcal{C}^{\mathrm{rev}}$ and $T : \mathcal{E} \to \mathrm{Fun}_{\mathcal{E}}(\mathcal{C},\mathcal{C})$ in Expl. 3.8 and Expl. 3.7 respectively. The structure of monoidal functor over $\mathcal{E}$ on Φ is induced by $\Phi \circ T_{\mathcal{C}\boxtimes_{Z(\mathcal{C},\mathcal{E})}\mathcal{C}^{\mathrm{rev}}}(e) = T_{\mathcal{C}}(e) \otimes - \otimes \mathbb{1}_{\mathcal{C}} \simeq T_{\mathcal{C}}(e) \otimes - = T^e$. □

Lemma 5.24. Let $\mathcal{C}$ be a multifusion category over $\mathcal{E}$ such that $\mathcal{E} \to Z(\mathcal{C})$ is fully faithful and $\mathcal{X}$ a left $\mathcal{C}$-module. There is an equivalence in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$

$$\mathcal{C} \boxtimes_{Z(\mathcal{C},\mathcal{E})} \mathrm{Fun}_{\mathcal{C}}(\mathcal{X},\mathcal{X}) \simeq \mathrm{Fun}_{\mathcal{E}}(\mathcal{X},\mathcal{X})$$

Proof. Corollary 3.6.18 of [Su] says that there is an equivalence

$$\mathrm{Fun}_{\mathcal{C}}(\mathcal{X},\mathcal{X}) \simeq \mathrm{Fun}_{\mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{C}^{\mathrm{rev}}}(\mathcal{C}, \mathrm{Fun}_{\mathcal{E}}(\mathcal{X},\mathcal{X}))$$

We have equivalences $\mathcal{C}\boxtimes_{Z(\mathcal{C},\mathcal{E})}\mathcal{C}^{\mathrm{rev}}\boxtimes_{\mathcal{C}^{\mathrm{rev}}}\mathcal{C}^{\mathrm{op}}\boxtimes_{\mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{C}^{\mathrm{rev}}}\mathrm{Fun}_{\mathcal{E}}(\mathcal{X},\mathcal{X}) \simeq \mathrm{Fun}_{\mathcal{E}}(\mathcal{C},\mathcal{C})\boxtimes_{\mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{C}^{\mathrm{rev}}}\mathrm{Fun}_{\mathcal{E}}(\mathcal{X},\mathcal{X}) \simeq \mathcal{C}^{\mathrm{op}} \boxtimes_{\mathcal{E}} \mathcal{C} \boxtimes_{\mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{C}^{\mathrm{rev}}} \mathrm{Fun}_{\mathcal{E}}(\mathcal{X},\mathcal{X}) \simeq \mathrm{Fun}_{\mathcal{E}}(\mathcal{X},\mathcal{X})$. The first equivalence holds by the Lem. 5.23. □

Lemma 5.25. Let $\mathcal{C}$ be a semisimple finite left $\mathcal{E}$-module. There is an equivalence $\mathcal{C}^{\mathrm{op}} \boxtimes_{\mathrm{Fun}_{\mathcal{E}}(\mathcal{C},\mathcal{C})} \mathcal{C} \simeq \mathcal{E}$ in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$.

Proof. The left $\mathrm{Fun}_{\mathcal{E}}(\mathcal{C},\mathcal{C})$-action on $\mathcal{C}$ is defined as $f \odot x := f(x)$ for $f \in \mathrm{Fun}_{\mathcal{E}}(\mathcal{C},\mathcal{C}), x \in \mathcal{C}$. The composed equivalence

$$\mathcal{C}^{\mathrm{op}} \boxtimes_{\mathrm{Fun}_{\mathcal{E}}(\mathcal{C},\mathcal{C})} \mathcal{C} \simeq \mathcal{C} \boxtimes_{\mathrm{Fun}_{\mathcal{E}}(\mathcal{C},\mathcal{C})^{\mathrm{rev}}} \mathcal{C}^{\mathrm{op}} \simeq \mathcal{E}$$

carries $a \boxtimes_{\mathrm{Fun}_{\mathcal{E}}(\mathcal{C},\mathcal{C})} b \mapsto b \boxtimes_{\mathrm{Fun}_{\mathcal{E}}(\mathcal{C},\mathcal{C})^{\mathrm{rev}}} a \mapsto [b,a]^R_{\mathcal{E}}$, where the second equivalence is due to Thm. 4.13 and Eq. (4.17). □

5.4 Computation of factorization homology

Modules over a fusion category over $\mathcal{E}$ and modules over a braided fusion category over $\mathcal{E}$ can be generalized to the unitary case automatically. Let $\mathcal{C}$ be a unitary fusion category over $\mathcal{E}$. A closed object in $\mathrm{LMod}_{\mathcal{C}}(\mathcal{V}^{\mathcal{E}}_{\mathrm{uty}})$ is an object $\mathcal{M} \in \mathcal{V}^{\mathcal{E}}_{\mathrm{uty}}$ equipped with a monoidal equivalence $(\psi, u) : \mathcal{C} \to \mathrm{Fun}_{\mathcal{E}}(\mathcal{M},\mathcal{M})$ over $\mathcal{E}$ such that ψ is a monoidal $*$-functor and $u_e^* = u_e^{-1}$ for $e \in \mathcal{E}$. Let $\mathcal{A}$ and $\mathcal{B}$ be unitary braided fusion categories over $\mathcal{E}$. A closed object in $\mathrm{BMod}_{\mathcal{A}|\mathcal{B}}(\mathrm{Alg}(\mathcal{V}^{\mathcal{E}}_{\mathrm{uty}}))$ is a unitary fusion category $\mathcal{M}$ over $\mathcal{E}$ equipped with a braided monoidal equivalence $(\phi, u) : \overline{\mathcal{A}} \boxtimes_{\mathcal{E}} \mathcal{B} \to Z(\mathcal{M},\mathcal{E})$ over $\mathcal{E}$ such that ϕ is a braided $*$-functor and $u_e^* = u_e^{-1}$ for $e \in \mathcal{E}$.

Definition 5.26. A coefficient system $A : \mathcal{D}\mathrm{isk}^{\mathrm{str}}_M \to \mathcal{V}^{\mathcal{E}}_{\mathrm{uty}}$ on a stratified surface M is called *anomaly-free in* $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ if the following conditions are satisfied:

- The target label for a 2-cell is given by a $\mathrm{UMTC}_{/\mathcal{E}}$.
- The target label for a 1-cell between two adjacent 2-cells labeled by $\mathcal{A}$(left) and $\mathcal{B}$(right) is given by a closed object in $\mathrm{BMod}_{\mathcal{A}|\mathcal{B}}(\mathrm{Alg}(\mathcal{V}^{\mathcal{E}}_{\mathrm{uty}}))$.
- The target label for a 0-cell as the one depicted in Figure 1 is given by a 0-disk algebra $(\mathcal{P}, p)$ in $\mathcal{V}^{\mathcal{E}}_{\mathrm{uty}}$, where the unitary $\mathcal{E}$-module category $\mathcal{P}$ is equipped with the structure of a closed left $\int_{M\setminus\{0\}} A$-module, i.e.

$$\int_{M\setminus\{0\}} A \simeq \mathrm{Fun}_{\mathcal{E}}(\mathcal{P},\mathcal{P})$$

Example 5.27. A stratified 2-disk M is shown in Fig. 1. An anomaly-free coefficient system A on M in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ is determined by its target labels shown in Fig. 1

- The target labels for 2-cells: $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$ are $\mathrm{UMTC}_{/\mathcal{E}}$'s.

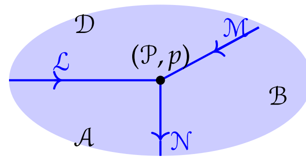

Figure 1: The figure depicts a stratified 2-disk with an anomaly-free coefficient system A in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ determined by its target labels.

- The target labels for 1-cells: $\mathcal{L}$ is a closed object in $\mathrm{BMod}_{\mathcal{A}|\mathcal{D}}(\mathrm{Alg}(\mathcal{V}^{\mathcal{E}}_{\mathrm{uty}}))$, $\mathcal{M}$ a closed object in $\mathrm{BMod}_{\mathcal{D}|\mathcal{B}}(\mathrm{Alg}(\mathcal{V}^{\mathcal{E}}_{\mathrm{uty}}))$ and $\mathcal{N}$ is a closed object in $\mathrm{BMod}_{\mathcal{A}|\mathcal{B}}(\mathrm{Alg}(\mathcal{V}^{\mathcal{E}}_{\mathrm{uty}}))$.
- The target labels for 0-cells: $(\mathcal{P}, p)$ is a closed left module over $\mathcal{L} \boxtimes_{\mathcal{A}^{\mathrm{rev}} \boxtimes_{\mathcal{E}} \mathcal{D}} (\mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}^{\mathrm{rev}})$.

The data of the coefficient system $A : \mathcal{D}\mathrm{isk}^{\mathrm{str}}_{M} \to \mathcal{V}^{\mathcal{E}}_{\mathrm{uty}}$ shown in Fig. 1 are denoted as

$$A = (\mathcal{A}, \mathcal{B}, \mathcal{D}; \mathcal{L}, \mathcal{M}, \mathcal{N}; (\mathcal{P}, p))$$

Example 5.28. Let $\mathcal{C}$ be a $\mathrm{UMTC}_{/\mathcal{E}}$. Consider an open disk $\mathring{\mathbb{D}}$ with two 0-cells p_1, p_2. And a coefficient system assigns $\mathcal{C}$ to the unique 2-cell and assigns $(\mathcal{C}, x_1), (\mathcal{C}, x_2)$ to the 0-cells p_1, p_2, respectively. By the $\otimes$-excision property, we have

$$\int_{(\mathring{\mathbb{D}};\emptyset;p_1 \sqcup p_2)} (\mathcal{C}; \emptyset; (\mathcal{C}, x_1), (\mathcal{C}, x_2)) \simeq \big(\mathcal{C}; \emptyset; (\mathcal{C}, x_1) \otimes_{\mathcal{C}} (\mathcal{C}, x_2)\big) \simeq \big(\mathcal{C}; \emptyset; (\mathcal{C}, x_1 \otimes x_2)\big)$$

Notice the equivalence $\mathcal{C} \otimes_{\mathcal{C}} \mathcal{C} \simeq \mathcal{C}$ is defined as $x \otimes_{\mathcal{C}} y \mapsto x \otimes y$, whose inverse is defined as $m \mapsto \mathbb{1}_{\mathcal{C}} \otimes_{\mathcal{C}} m$ for $x, y, m \in \mathcal{C}$.

Consider an open disk $\mathring{\mathbb{D}}$ with finitely many 0-cells $p_1, \ldots, p_n$. And a coefficient system assigns $\mathcal{C}$ to the unique 2-cell and assigns $(\mathcal{C}, x_1), \ldots, (\mathcal{C}, x_n)$ to the 0-cells $p_1, \ldots, p_n$, respectively. We have

$$\int_{(\mathring{\mathbb{D}};\emptyset;p_1,\ldots,p_n)} (\mathcal{C}; \emptyset; (\mathcal{C}, x_1), \ldots, (\mathcal{C}, x_n)) \simeq \big(\mathcal{C}; \emptyset; (\mathcal{C}, x_1 \otimes \cdots \otimes x_n)\big)$$

Theorem 5.29. Let $\mathcal{C}$ be a $\mathrm{UMTC}_{/\mathcal{E}}$ and $x_1, \ldots, x_n \in \mathcal{C}$. Consider the stratified sphere S^2 without 1-stratum but with finitely many 0-cells $p_1, \ldots, p_n$. Suppose a coefficient system assigns $\mathcal{C}$ to the unique 2-cell and assigns $(\mathcal{C}, x_1), \ldots, (\mathcal{C}, x_n)$ to the 0-cells $p_1, \ldots, p_n$, respectively. We have

$$\int_{(S^2;\emptyset;p_1,\ldots,p_n)} (\mathcal{C}; \emptyset; (\mathcal{C}, x_1), \ldots, (\mathcal{C}, x_n)) \simeq \big(\mathcal{E}, [\mathbb{1}_{\mathcal{C}}, x_1 \otimes \cdots \otimes x_n]_{\mathcal{E}}\big) \tag{5.2}$$

Proof. If we map the open stratified disk $(\mathring{\mathbb{D}}; \emptyset; p_1, \ldots, p_n)$ to the open stratified disk $(\mathring{\mathbb{D}}; \emptyset; p)$ and map the points $p_1, \ldots, p_n$ to the point p. We have the following equivalence by Expl. 5.28

$$\int_{(\mathring{\mathbb{D}};\emptyset;p_1,\ldots,p_n)} (\mathcal{C}; \emptyset; (\mathcal{C}, x_1), \ldots, (\mathcal{C}, x_n)) \simeq \int_{(\mathring{\mathbb{D}};\emptyset;p)} (\mathcal{C}; \emptyset; (\mathcal{C}, x_1 \otimes \cdots \otimes x_n))$$

On the stratified sphere $(S^2; \emptyset; p)$, we add an oriented 1-cell $S^1 \setminus p$ from p to p, labelled by the 1-disk algebra $\mathcal{C}$ obtained by forgetting its 2-disk algebra structure. We project the stratified sphere $(S^2; S^1 \setminus p; p)$ directly to a closed stratified 2-disk $(\mathbb{D}; S^1 \setminus p; p)$ as shown in Fig. 2 (a). Notice that this projection preserves the stratification. Applying the pushforward property (5.1) and the $\otimes$-excision property, we reduce the problem to the computation of the factorization homology of the stratified 2-disk.

$$\int_{(S^2;\emptyset;p_1,\ldots,p_n)} (\mathcal{C}; \emptyset; (\mathcal{C}, x_1), \ldots, (\mathcal{C}, x_n)) \simeq \int_{(\mathbb{D};S^1\setminus p;p)} \big(\mathcal{C} \boxtimes_{\mathcal{E}} \overline{\mathcal{C}}; \mathcal{C}; (\mathcal{C}, x_1 \otimes \cdots \otimes x_n)\big)$$

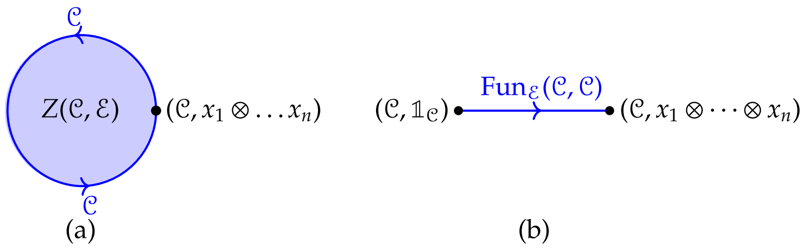

Figure 2: The figure depicts the two steps in computing the factorization homology of a sphere with the coefficient system given by a $\mathrm{UMTC}_{/\mathcal{E}}$.

Notice that $\mathcal{C} \boxtimes_\mathcal{E} \overline{\mathcal{C}} \simeq Z(\mathcal{C},\mathcal{E})$. Next we project the stratified 2-disk vertically onto the closed interval $[-1,1]$ as shown in Fig. 2 (b). Notice that $\mathcal{C} \boxtimes_{Z(\mathcal{C},\mathcal{E})} \mathcal{C}^{\mathrm{rev}} \simeq \mathrm{Fun}_\mathcal{E}(\mathcal{C},\mathcal{C})$. The final result is expressed as a tensor product:

$$\int_{(S^2;\emptyset;p_1,\dots,p_n)} (\mathcal{C};\emptyset;(\mathcal{C},x_1),\dots,(\mathcal{C},x_n)) \simeq \Big(\mathcal{C} \boxtimes_{\mathrm{Fun}_\mathcal{E}(\mathcal{C},\mathcal{C})} \mathcal{C}, \mathbb{1}_\mathcal{C} \boxtimes_{\mathrm{Fun}_\mathcal{E}(\mathcal{C},\mathcal{C})} (x_1 \otimes \cdots \otimes x_n)\Big)$$

By Lem. 5.25 and Lem. A.9, the composed equivalence

$$\mathcal{C} \boxtimes_{\mathrm{Fun}_\mathcal{E}(\mathcal{C},\mathcal{C})} \mathcal{C} \simeq \mathcal{C}^{\mathrm{op}} \boxtimes_{\mathrm{Fun}_\mathcal{E}(\mathcal{C},\mathcal{C})} \mathcal{C} \simeq \mathcal{C} \boxtimes_{\mathrm{Fun}_\mathcal{E}(\mathcal{C},\mathcal{C})^{\mathrm{rev}}} \mathcal{C}^{\mathrm{op}} \simeq \mathcal{E}$$

carries $x \boxtimes_{\mathrm{Fun}_\mathcal{E}(\mathcal{C},\mathcal{C})} y \mapsto x^R \boxtimes_{\mathrm{Fun}_\mathcal{E}(\mathcal{C},\mathcal{C})} y \mapsto y \boxtimes_{\mathrm{Fun}_\mathcal{E}(\mathcal{C},\mathcal{C})^{\mathrm{rev}}} x^R \mapsto [y, x^R]^R_\mathcal{E} \cong [x^R, y]_\mathcal{E}$. Taking $x = \mathbb{1}_\mathcal{C}$ and $y = x_1 \otimes \cdots \otimes x_n$ in the above composed equivalence, we obtain Eq. (5.2). □

Theorem 5.30. Let $\mathcal{C}$ be a $\mathrm{UMTC}_{/\mathcal{E}}$ and $x_1,\dots,x_n \in \mathcal{C}$. Let Σ_g be a closed stratified surface of genus g without 1-stratum but with finitely many 0-cells $p_1,\dots,p_n$. Suppose a coefficient system assigns $\mathcal{C}$ to the unique 2-cell and assigns $(\mathcal{C},x_1),\dots,(\mathcal{C},x_n)$ to the 0-cells $p_1,\dots,p_n$, respectively. We have

$$\int_{(\Sigma_g;\emptyset;p_1,\dots,p_n)} (\mathcal{C};\emptyset;(\mathcal{C},x_1),\dots,(\mathcal{C},x_n)) \simeq \Big(\mathcal{E}, [\mathbb{1}_\mathcal{C}, x_1 \otimes \cdots \otimes x_n \otimes (\eta^{-1}(A) \otimes_{T(A)} \eta^{-1}(A))^{\otimes g}]_\mathcal{E}\Big) \tag{5.3}$$

where A is a symmetric $*$-Frobenius algebra in $\mathcal{E}$ such that there exists an equivalence $\eta : \mathcal{C} \simeq \mathcal{E}_A$ in $\mathcal{V}^\mathcal{E}_{\mathrm{uty}}$ and $T : \mathcal{E} \to \mathcal{C}'$ is the braided embedding.

Proof. Since $\mathcal{C}$ is a unitary $\mathcal{E}$-module category, there exists a symmetric $*$-Frobenius algebra A in $\mathcal{C}$ such that $\mathcal{C} \simeq^\eta \mathcal{E}_A$ in $\mathcal{V}^\mathcal{E}_{\mathrm{uty}}$. Notice that Eq. (5.3) holds for genus $g = 0$ by Thm. 5.29. Now we assume $g > 0$. The proof of Thm. 5.29 implies that $\int_{S^1\times\mathbb{R}} \mathcal{C} \simeq \mathrm{Fun}_\mathcal{E}(\mathcal{C},\mathcal{C})$. By Prop.A.6, Lem. A.7 and Lem. A.8, the composed equivalence of categories

$$\mathrm{Fun}_\mathcal{E}(\mathcal{E}_A,\mathcal{E}_A) \simeq {}_A\mathcal{E}_A \simeq {}_A\mathcal{E} \boxtimes_\mathcal{E} \mathcal{E}_A \simeq \mathcal{E}_A \boxtimes_\mathcal{E} \mathcal{E}_A$$

carries $\mathrm{id} \mapsto A \mapsto \bar{p} \boxtimes_\mathcal{E} \bar{q} \mapsto \bar{p} \boxtimes_\mathcal{E} \bar{q}$, where $\bar{p} \boxtimes_\mathcal{E} \bar{q} := \mathrm{colim}\Big((A \otimes A) \boxtimes_\mathcal{E} A \rightrightarrows A \boxtimes_\mathcal{E} A\Big)$. Then the equivalence $\mathrm{Fun}_\mathcal{E}(\mathcal{C},\mathcal{C}) \simeq \mathcal{C} \boxtimes_\mathcal{E} \mathcal{C}$ carries $\mathrm{id}_\mathcal{C} \mapsto p \boxtimes_\mathcal{E} q := \mathrm{Colim}\Big(\eta^{-1}(A \otimes A) \boxtimes_\mathcal{E} \eta^{-1}(A) \rightrightarrows \eta^{-1}(A) \boxtimes_\mathcal{E} \eta^{-1}(A)\Big)$.

Therefore, we have $\int_{S^1\times\mathbb{R}} \mathcal{C} \simeq \Big(\mathcal{C} \boxtimes_\mathcal{E} \mathcal{C}, p \boxtimes_\mathcal{E} q\Big)$. As a consequence, when we compute the factorization homology, we can replace a cylinder $S^1 \times \mathbb{R}$ by two open 2-disks with two 0-cells as shown on the Fig. 3, both of which are labelled by $\Big(\mathcal{C} \boxtimes_\mathcal{E} \mathcal{C}, p \boxtimes_\mathcal{E} q\Big)$, or labelled by $(\mathcal{C}, p)$ and $(\mathcal{C}, q)$.

(a) (b)

Figure 3: Figure (a) shows a stratified cylinder with a coefficient system $(\mathcal{C};\mathcal{M};\emptyset)$, where $\mathcal{C}$ is a UMTC$_{/\mathcal{E}}$ and $\mathcal{M}$ is closed in $\mathrm{BMod}_{\mathcal{C}|\mathcal{C}}(\mathrm{Alg}(\mathcal{V}^{\mathcal{E}}_{\mathrm{uty}}))$. Figure (b) shows a disjoint union of two open disks with 2-cells labeled by $\mathcal{C}$, 1-cells labeled by $\mathcal{M}$ and $\mathcal{M}^{\mathrm{rev}}$, and 0-cells labeled by $\mathcal{X}$ and $\mathcal{X}^{\mathrm{op}}$.

In this way, the genus is reduced by one. By induction, we obtain the equation

$$\begin{aligned}
&\int_{(\Sigma_g;\emptyset;p_1,\dots,p_n)} (\mathcal{C};\emptyset;(\mathcal{C},x_1),\dots,(\mathcal{C},x_n)) \simeq \int_{(\Sigma_{g-1};\emptyset;p_1,\dots,p_n,p_{n+1},p_{n+2})} \Big(\mathcal{C};\emptyset;(\mathcal{C},x_1),\dots,(\mathcal{C},x_n),(\mathcal{C},p),(\mathcal{C},q)\Big)\\
&\simeq \int_{(\Sigma_0;\emptyset;p_1,\dots,p_n,\dots,p_{n+2g-1},p_{n+2g})} \Big(\mathcal{C};\emptyset;(\mathcal{C},x_1),\dots,(\mathcal{C},x_n),(\mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{C},p\boxtimes_{\mathcal{E}} q)^g\Big)\\
&\simeq \Big(\mathcal{E},[\mathbb{1}_{\mathcal{C}},x_1\otimes\cdots\otimes x_n\otimes(p\otimes q)^{\otimes g}]_{\mathcal{E}}\Big)
\end{aligned}$$

where the notation $(\mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{C},p\boxtimes_{\mathcal{E}} q)^g$ denotes g copies of $(\mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{C},p\boxtimes_{\mathcal{E}} q)$ and

$$\begin{aligned}
p\otimes q &\simeq \mathrm{Colim}\Big(\eta^{-1}(A\otimes A)\otimes\eta^{-1}(A) \rightrightarrows \eta^{-1}(A)\otimes\eta^{-1}(A)\Big)\\
&\simeq \mathrm{Colim}\Big(\eta^{-1}(A)\otimes T(A)\otimes\eta^{-1}(A) \rightrightarrows \eta^{-1}(A)\otimes\eta^{-1}(A)\Big)\\
&\simeq \eta^{-1}(A)\otimes_{T(A)}\eta^{-1}(A)
\end{aligned}$$

Since factorization homology and $p\boxtimes_{\mathcal{E}} q$ are both defined by colimits, we exchange the order of two colimits in the first equivalence. The second equivalence is induced by the composed equivalence $\eta^{-1}(A\otimes A)\simeq A\odot\eta^{-1}(A) = T(A)\otimes\eta^{-1}(A)\simeq\eta^{-1}(A)\otimes T(A)$. Since $T(A)$ is an algebra in $\mathcal{C}$, we obtain the last equivalence. □

Example 5.31. The unitary category $\mathbb{H}$ denotes the category of finite dimensional Hilbert spaces. Let $\mathcal{E}=\mathbb{H}$ and $\mathcal{C}=$ UMTC. We want to choose an algebra $A\in\mathbb{H}$ such that $\mathcal{C}\simeq^{\eta}\mathbb{H}_A$. Suppose that $\eta^{-1}(A)\cong A$ and $T(A)\cong A$. Then $\eta^{-1}(A)\otimes_{T(A)}\eta^{-1}(A)\cong A\otimes_A A\cong A$ and

$$\int_{(\Sigma_g;\emptyset;p_1,\dots,p_n)} (\mathcal{C};\emptyset;(\mathcal{C},x_1),\dots,(\mathcal{C},x_n)) \simeq \Big(\mathbb{H},\mathrm{Hom}(\mathbb{1}_{\mathcal{C}},x_1\otimes\cdots\otimes x_n\otimes A^{\otimes g})\Big)$$

The set $O(\mathcal{C})$ denotes the set of isomorphism classes of simple objects in $\mathcal{C}$. If $\eta^{-1}(A) = \oplus_{i\in O(\mathcal{C})} i^R\otimes i = T(A)$, the distinguished object is $\mathrm{Hom}(\mathbb{1}_{\mathcal{C}},x_1\otimes\cdots\otimes x_n\otimes(\oplus_i i^R\otimes i)^{\otimes g})$. If $A=\oplus_{i\in O(\mathcal{C})}\mathbb{C}$ and $\eta^{-1}(A)=\oplus_{i\in O(\mathcal{C})} i = T(A)$, the distinguished object is $\mathrm{Hom}(\mathbb{1}_{\mathcal{C}},x_1\otimes\cdots\otimes(\oplus_{i\in O(\mathcal{C})} i)^{\otimes g})$.

Theorem 5.32. Let $(S^1\times\mathbb{R};\mathbb{R})$ be the stratified cylinder shown in Fig. 3. in which the target label $\mathcal{C}$ is a UMTC$_{/\mathcal{E}}$ and the target label $\mathcal{M}$ is closed in $\mathrm{BMod}_{\mathcal{C}|\mathcal{C}}(\mathrm{Alg}(\mathcal{V}^{\mathcal{E}}_{\mathrm{uty}}))$. We have

$$\int_{(S^1\times\mathbb{R};\mathbb{R})} (\mathcal{C};\mathcal{M};\emptyset) \simeq \mathrm{Fun}_{\mathcal{E}}(\mathcal{X},\mathcal{X})$$

where $\mathcal{X}$ is the unique (up to equivalence) left $\mathcal{C}$-module in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ such that $\mathcal{M}\simeq\mathrm{Fun}_{\mathcal{C}}(\mathcal{X},\mathcal{X})$.

Proof. By the equivalences $Z(\mathcal{M}^{\mathrm{rev}}, \mathcal{E}) \simeq \mathcal{C} \boxtimes_{\mathcal{E}} \overline{\mathcal{C}} \simeq Z(\mathcal{C}, \mathcal{E})$, there exists a $\mathcal{C}$-module $\mathcal{X}$ such that $\mathcal{M} \simeq \mathrm{Fun}_{\mathcal{C}}(\mathcal{X}, \mathcal{X})$ by Thm. 4.40. Therefore, we have $\int_{(S^1\times\mathbb{R};\mathbb{R})}(\mathcal{C}; \mathcal{M}; \emptyset) \simeq \mathcal{C} \boxtimes_{Z(\mathcal{C},\mathcal{E})} \mathcal{M} \simeq \mathcal{C} \boxtimes_{Z(\mathcal{C},\mathcal{E})} \mathrm{Fun}_{\mathcal{C}}(\mathcal{X}, \mathcal{X}) \simeq \mathrm{Fun}_{\mathcal{E}}(\mathcal{X}, \mathcal{X})$, which maps $\mathbb{1}_{\mathcal{C}} \boxtimes_{Z(\mathcal{C},\mathcal{E})} \mathbb{1}_{\mathcal{M}}$ to $\mathrm{id}_{\mathcal{X}}$. The last equivalence is due to Thm. 5.24. □

Conjecture 5.33. Given any closed stratified surface Σ and an anomaly-free coefficient system A in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ on Σ, we have $\int_\Sigma A \simeq (\mathcal{E}, u_\Sigma)$, where u_Σ is an object in $\mathcal{E}$.

A Appendix

A.1 Central functors and other results

Let $\mathcal{D}$ be a braided monoidal category with the braiding c and $\mathcal{M}$ a monoidal category.

Definition A.1. A *central structure* of a monoidal functor $F : \mathcal{D} \to \mathcal{M}$ is a braided monoidal functor $F' : \mathcal{D} \to Z(\mathcal{M})$ such that $F = \mathbf{f} \circ F'$, where $\mathbf{f} : Z(\mathcal{M}) \to \mathcal{M}$ is the forgetful functor.

A *central functor* is a monoidal functor equipped with a central structure. For any monoidal functor $F : \mathcal{D} \to \mathcal{M}$, the central structure of F given in Def. A.1 is equivalent to the central structure of F given in the Def. A.2.

Definition A.2. A *central structure* of a monoidal functor $F : \mathcal{D} \to \mathcal{M}$ is a natural isomorphism $\sigma_{d,m} : F(d) \otimes m \to m \otimes F(d)$, $d \in \mathcal{D}$, $m \in \mathcal{M}$ which is natural in both variables such that the diagrams

$$\begin{tikzcd} F(d)\otimes m\otimes m' \arrow[rr, "\sigma_{d,m\otimes m'}"] \arrow[dr, "\sigma_{d,m},1"'] & & m\otimes m'\otimes F(d) \\ & m\otimes F(d)\otimes m' \arrow[ur, "1,\sigma_{d,m'}"'] & \end{tikzcd} \tag{A.1}$$

$$\begin{tikzcd} F(d)\otimes F(d')\otimes m \arrow[r, "1,\sigma_{d',m}"] \arrow[d, "J_{d,d'},1"'] & F(d)\otimes m\otimes F(d') \arrow[r, "\sigma_{d,m},1"] & m\otimes F(d)\otimes F(d') \arrow[d, "1,J_{d,d'}"] \\ F(d\otimes d')\otimes m \arrow[rr, "\sigma_{d\otimes d',m}"] & & m\otimes F(d\otimes d') \end{tikzcd} \tag{A.2}$$

$$\begin{tikzcd} F(d)\otimes F(d') \arrow[r, "J_{d,d'}"] \arrow[d, "\sigma_{d,F(d')}"'] & F(d\otimes d') \arrow[d, "F(c_{d,d'})"] \\ F(d')\otimes F(d) \arrow[r, "J_{d',d}"] & F(d'\otimes d) \end{tikzcd} \tag{A.3}$$

commute for any $d, d' \in \mathcal{D}$ and $m, m' \in \mathcal{M}$, where J is the monoidal structure of F.

Proposition A.3. Suppose $F : \mathcal{D} \to \mathcal{M}$ is a central functor. For any $d \in \mathcal{D}, m \in \mathcal{M}$, the following two diagrams commute

$$\begin{tikzcd} F(d)\otimes \mathbb{1}_{\mathcal{M}} \arrow[rr, "\sigma_{d,\mathbb{1}_{\mathcal{M}}}"] \arrow[dr, "r_{F(d)}"'] & & \mathbb{1}_{\mathcal{M}}\otimes F(d) \arrow[dl, "l_{F(d)}"] \\ & F(d) & \end{tikzcd} \qquad \begin{tikzcd} F(\mathbb{1}_{\mathcal{D}})\otimes m \arrow[rr, "\sigma_{\mathbb{1}_{\mathcal{D}},m}"] \arrow[dr, "l_m"'] & & m\otimes F(\mathbb{1}_{\mathcal{D}}) \arrow[dl, "r_m"] \\ & m & \end{tikzcd} \tag{A.4}$$

Here $l_m : F(\mathbb{1}_{\mathcal{D}}) \otimes m = \mathbb{1}_{\mathcal{M}} \otimes m \to m$ and $r_m : m \otimes F(\mathbb{1}_{\mathcal{D}}) = m \otimes \mathbb{1}_{\mathcal{M}} \to m$, $m \in \mathcal{M}$ are the unit isomorphisms of the monoidal category $\mathcal{M}$.

Proof. Consider the diagram:

$$\begin{tikzcd}
 & F(d)\otimes(\mathbb{1}_\mathcal{M}\otimes\mathbb{1}_\mathcal{M}) \arrow[r, "\sigma_{d,\mathbb{1}_\mathcal{M}\otimes\mathbb{1}_\mathcal{M}}"] \arrow[d, "1{,}l_{\mathbb{1}_\mathcal{M}}"] & (\mathbb{1}_\mathcal{M}\otimes\mathbb{1}_\mathcal{M})\otimes F(d) \arrow[d, "l_{\mathbb{1}_\mathcal{M}}{,}1"] \arrow[dr] & \\
(F(d)\otimes\mathbb{1}_\mathcal{M})\otimes\mathbb{1}_\mathcal{M} \arrow[ur] \arrow[r, "r_{F(d)}{,}1"] \arrow[dr, "\sigma_{d,\mathbb{1}_\mathcal{M}}{,}1"'] & F(d)\otimes\mathbb{1}_\mathcal{M} \arrow[r, "\sigma_{d,\mathbb{1}_\mathcal{M}}"] & \mathbb{1}_\mathcal{M}\otimes F(d) & \mathbb{1}_\mathcal{M}\otimes(\mathbb{1}_\mathcal{M}\otimes F(d)) \arrow[l, "l_{\mathbb{1}_\mathcal{M}\otimes F(d)}"'] \\
 & (\mathbb{1}_\mathcal{M}\otimes F(d))\otimes\mathbb{1}_\mathcal{M} \arrow[u, "l_{F(d)}{,}1"'] \arrow[r] & \mathbb{1}_\mathcal{M}\otimes(F(d)\otimes\mathbb{1}_\mathcal{M}) \arrow[ul, "l_{F(d)\otimes\mathbb{1}_\mathcal{M}}"'] \arrow[ur, "1{,}\sigma_{d,\mathbb{1}_\mathcal{M}}"'] &
\end{tikzcd}$$

The outward hexagon commutes by the diagram (A.1). The left-upper, right-upper and middle-bottom triangles commute by the monoidal category $\mathcal{M}$. The middle-up square commutes by the naturality of the central structure $\sigma_{d,m} : F(d)\otimes m \to m\otimes F(d)$, $\forall d \in \mathcal{D}, m \in \mathcal{M}$. The right-down square commutes by the naturality of the unit isomorphism $l_m : \mathbb{1}_\mathcal{M}\otimes m \simeq m$, $m \in \mathcal{M}$. Then the left-down triangle commutes. Since $-\otimes\mathbb{1}_\mathcal{M} \simeq \mathrm{id}_\mathcal{M}$ is the natural isomorphism, the left triangle of (A.4) commutes.

Consider the diagram:

$$\begin{tikzcd}
 & m\otimes F(\mathbb{1}_\mathcal{D}\otimes\mathbb{1}_\mathcal{D}) \arrow[dl, "1{,}J"'] \arrow[d, "1{,}r_{\mathbb{1}_\mathcal{D}}"] & \\
m\otimes F(\mathbb{1}_\mathcal{D})\otimes F(\mathbb{1}_\mathcal{D}) \arrow[r, shift left, "1{,}r_{F(\mathbb{1}_\mathcal{D})}"] \arrow[r, shift right, "r_{m\otimes F(\mathbb{1}_\mathcal{D})}"'] & m\otimes F(\mathbb{1}_\mathcal{D}) & F(\mathbb{1}_\mathcal{D}\otimes\mathbb{1}_\mathcal{D})\otimes m \arrow[ul, "\sigma_{\mathbb{1}_\mathcal{D}\otimes\mathbb{1}_\mathcal{D},m}"'] \arrow[dl, "r_{\mathbb{1}_\mathcal{D}}{,}1"] \arrow[dd, "J{,}1"] \\
 & F(\mathbb{1}_\mathcal{D})\otimes m \arrow[u, "\sigma_{\mathbb{1}_\mathcal{D},m}"] & \\
F(\mathbb{1}_\mathcal{D})\otimes m\otimes F(\mathbb{1}_\mathcal{D}) \arrow[uu, "\sigma_{\mathbb{1}_\mathcal{D},m}{,}1"] \arrow[ur, bend left, "r_{F(\mathbb{1}_\mathcal{D})\otimes m}"] \arrow[ur, "1{,}r_m"'] & & F(\mathbb{1}_\mathcal{D})\otimes F(\mathbb{1}_\mathcal{D})\otimes m \arrow[ul, bend right, "r_{F(\mathbb{1}_\mathcal{D})}{,}1"'] \arrow[ul, "1{,}l_m"] \arrow[ll, "1{,}\sigma_{\mathbb{1}_\mathcal{D},m}"]
\end{tikzcd}$$

The outward diagram commutes by the diagram (A.2). The right-upper square commutes by the naturality of the central structure $\sigma_{d,m} : F(d)\otimes m \to m\otimes F(d)$, $d \in D, m \in \mathcal{M}$. The left square commutes by the naturality of the unit isomorphism $r_m : m\otimes\mathbb{1}_\mathcal{M} \simeq m$, $m \in \mathcal{M}$. The left-upper and right-down triangles commute by the monoidal functor F. Three parallel arrows equal by the triangle diagrams of the monoidal category $\mathcal{M}$. Then the bottom triangle commutes. Since $F(\mathbb{1}_\mathcal{D})\otimes - = \mathbb{1}_\mathcal{M}\otimes - \simeq \mathrm{id}_\mathcal{M}$ is the natural isomorphism, the right triangle of (A.4) commutes. □

Let A be a separable algebra in a multifusion category $\mathcal{C}$ over $\mathcal{E}$. We use ${}_A\mathcal{C}$ (or $\mathcal{C}_A$, ${}_A\mathcal{C}_A$) to denote the category of left A-modules (or right A-modules, A-bimodules) in $\mathcal{C}$.

Proposition A.4. Let $\mathcal{C}$ be a multifusion category over $\mathcal{E}$ and A a separable algebra in $\mathcal{C}$. Then the diagram

$$\begin{tikzcd}
T_\mathcal{C}(e)\otimes x\otimes_A y^R \arrow[r, "c_{e,x\otimes_A y^R}"] \arrow[d, "c_{e,x}{,}1"'] & x\otimes_A y^R\otimes T_\mathcal{C}(e) \\
x\otimes T_\mathcal{C}(e)\otimes_A y^R \arrow[r, "h"'] & x\otimes_A T_\mathcal{C}(e)\otimes y^R \arrow[u, "1{,}c_{e,y^R}"']
\end{tikzcd}$$

commutes for $e \in \mathcal{E}$, $x, y \in \mathcal{C}_A$, where c is the central structure of the central functor $T_\mathcal{C} : \mathcal{E} \to \mathcal{C}$.

Proof. The functor $y \mapsto y^R$ defines an equivalence of right $\mathcal{C}$-modules $(\mathcal{C}_A)^{\mathrm{op}|L} \simeq {}_A\mathcal{C}$. For $x \in \mathcal{C}_A$, we use p_x to denote the right A-action on x. For $y^R \in {}_A\mathcal{C}$, we use q_{y^R} to denote the left A-action on y^R. Obviously, $T_\mathcal{C}(e)\otimes x$ belongs to $\mathcal{C}_A$ and $y^R\otimes T_\mathcal{C}(e)$ belongs to ${}_A\mathcal{C}$. The right A-action on

$x \otimes T_{\mathcal{C}}(e)$ is induced by $x \otimes T_{\mathcal{C}}(e) \otimes A \xrightarrow{1,c_{e,A}} x \otimes A \otimes T_{\mathcal{C}}(e) \xrightarrow{p_x,1} x \otimes T_{\mathcal{C}}(e)$. The left A-action on $T_{\mathcal{C}}(e) \otimes y^R$ is induced by $A \otimes T_{\mathcal{C}}(e) \otimes y^R \xrightarrow{c_{e,A}^{-1},1} T_{\mathcal{C}}(e) \otimes A \otimes y^R \xrightarrow{1,q_{y^R}} T_{\mathcal{C}}(e) \otimes y^R$. It is routine to check that $c_{e,x}$ is a morphism in $\mathcal{C}_A$ and c_{e,y^R} is a morphism in ${}_A\mathcal{C}$.

The morphism $c_{e,x\otimes_A y^R}$ is induced by

$$\begin{tikzcd}
T_{\mathcal{C}}(e) \otimes x \otimes A \otimes y^R \arrow[r, shift left, "{1,p_x,1}"] \arrow[r, shift right, "{1,1,q_{y^R}}"'] \arrow[d, "c_{e,x\otimes A\otimes y^R}"'] & T_{\mathcal{C}}(e) \otimes x \otimes y^R \arrow[r] \arrow[d, "c_{e,x\otimes y^R}"] & T_{\mathcal{C}}(e) \otimes x \otimes_A y^R \arrow[d, "c_{e,x\otimes_A y^R}"] \\
x \otimes A \otimes y^R \otimes T_{\mathcal{C}}(e) \arrow[r, shift left, "{p_x,1,1}"] \arrow[r, shift right, "{1,q_{y^R},1}"'] & x \otimes y^R \otimes T_{\mathcal{C}}(e) \arrow[r] & x \otimes_A y^R \otimes T_{\mathcal{C}}(e)
\end{tikzcd}$$

The composition $(1_x \otimes_A c_{e,y^R}) \circ h \circ (c_{e,x} \otimes_A 1_{y^R})$ is induced by

$$\begin{tikzcd}
T_{\mathcal{C}}(e) \otimes x \otimes A \otimes y^R \arrow[r, shift left, "{1,p_x,1}"] \arrow[r, shift right, "{1,1,q_{y^R}}"'] \arrow[d, "{c_{e,x},1,1}"'] \arrow[ddd, bend right=60, "c_{e,x\otimes A\otimes y^R}"'] & T_{\mathcal{C}}(e) \otimes x \otimes y^R \arrow[r] \arrow[d, "{c_{e,x},1}"] & T_{\mathcal{C}}(e) \otimes x \otimes_A y^R \arrow[d, "{c_{e,x},1}"] \\
x \otimes T_{\mathcal{C}}(e) \otimes A \otimes y^R \arrow[r, shift left, "p_{x\otimes T_{\mathcal{C}}(e)}"] \arrow[r, shift right, "{1,1,q_{y^R}}"'] \arrow[d, "{1,c_{e,A},1}"'] & x \otimes T_{\mathcal{C}}(e) \otimes y^R \arrow[r] \arrow[d, "1"] & x \otimes T_{\mathcal{C}}(e) \otimes_A y^R \arrow[d, "h"] \\
x \otimes A \otimes T_{\mathcal{C}}(e) \otimes y^R \arrow[r, shift left, "{p_x,1,1}"] \arrow[r, shift right, "{1,q_{T_{\mathcal{C}}(e)\otimes y^R}}"'] \arrow[d, "{1,1,c_{e,y^R}}"'] & x \otimes T_{\mathcal{C}}(e) \otimes y^R \arrow[r] \arrow[d, "{1,c_{e,y^R}}"] & x \otimes_A T_{\mathcal{C}}(e) \otimes y^R \arrow[d, "{1,c_{e,y^R}}"] \\
x \otimes A \otimes y^R \otimes T_{\mathcal{C}}(e) \arrow[r, shift left, "{p_x,1,1}"] \arrow[r, shift right, "{1,q_{y^R},1}"'] & x \otimes y^R \otimes T_{\mathcal{C}}(e) \arrow[r] & x \otimes_A y^R \otimes T_{\mathcal{C}}(e)
\end{tikzcd}$$

Since $c_{e,x\otimes y^R} = (1_x \otimes c_{e,y^R}) \circ (c_{e,x} \otimes 1_{y^R})$, the composition $(1_x \otimes_A c_{e,y^R}) \circ h \circ (c_{e,x} \otimes_A 1_{y^R})$ equals to $c_{e,x\otimes_A y^R}$ by the universal property of coequalizers. □

Proposition A.5. Let $\mathcal{C}$ be a multifusion category over $\mathcal{E}$ and A a separable algebra in $\mathcal{C}$. There is an equivalence $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C}_A, \mathcal{C}_A) \simeq ({}_A\mathcal{C}_A)^{\mathrm{rev}}$ of multifusion categories over $\mathcal{E}$.

Proof. By [EGNO, Prop. 7.11.1], the functor $\Phi : ({}_A\mathcal{C}_A)^{\mathrm{rev}} \to \mathrm{Fun}_{\mathcal{C}}(\mathcal{C}_A, \mathcal{C}_A)$ is defined as $x \mapsto - \otimes_A x$ and the inverse of Φ is defined as $f \mapsto f(A)$. The monoidal structure on Φ is defined as

$$\Phi(x \otimes_A^{\mathrm{rev}} y) = - \otimes_A (y \otimes_A x) \simeq (- \otimes_A y) \otimes_A x = \Phi(x) \circ \Phi(y)$$

for $x, y \in ({}_A\mathcal{C}_A)^{\mathrm{rev}}$. Recall the central structures on the functors $I : \mathcal{E} \to ({}_A\mathcal{C}_A)^{\mathrm{rev}}$ and $\hat{T} : \mathcal{E} \to \mathrm{Fun}_{\mathcal{C}}(\mathcal{C}_A, \mathcal{C}_A)$ in Expl. 3.9 and Expl. 4.6 respectively. The structure of monoidal functor over $\mathcal{E}$ on Φ is induced by

$$\Phi(I(e)) = \Phi(T_{\mathcal{C}}(e) \otimes^{\mathrm{rev}} A) = - \otimes_A (A \otimes T_{\mathcal{C}}(e)) \cong - \otimes T_{\mathcal{C}}(e) \xrightarrow{c_{e,-}^{-1}} T_{\mathcal{C}}(e) \otimes - = \hat{T}^e$$

for $e \in \mathcal{E}$, where c is the central structure of the functor $T_{\mathcal{C}} : \mathcal{E} \to \mathcal{C}$. Next we want to check that Φ is a monoidal functor over $\mathcal{E}$. Consider the diagram for $e \in \mathcal{E}, x \in ({}_A\mathcal{C}_A)^{\mathrm{rev}}$:

$$\begin{tikzcd}
\Phi(I(e) \otimes_A^{\mathrm{rev}} x) \arrow[r] \arrow[d, "\Phi(\sigma_{e,x})"'] & \Phi(I(e)) \circ \Phi(x) \arrow[r] & \hat{T}^e \circ \Phi(x) \arrow[d, "\bar{\sigma}_{e,\Phi(x)}"] \\
\Phi(x \otimes_A^{\mathrm{rev}} I(e)) \arrow[r] & \Phi(x) \circ \Phi(I(e)) \arrow[r] & \Phi(x) \circ \hat{T}^e
\end{tikzcd}$$

The central structure $\sigma_{e,x}$ is induced by $x\otimes_A A\otimes T_\mathcal{C}(e) \xrightarrow{c^{-1}_{e,x\otimes_A A}} T_\mathcal{C}(e)\otimes x\otimes_A A \cong T_\mathcal{C}(e)\otimes A\otimes_A x \xrightarrow{c_{e,A},1} A\otimes T_\mathcal{C}(e)\otimes_A x$. The central structure $\bar{\sigma}_{e,\Phi(x)}$ is induced by $T_\mathcal{C}(e)\otimes(-\otimes_A x) \simeq (T_\mathcal{C}(e)\otimes -)\otimes_A x$. The commutativity of the above diagram is due to the commutativity of the following diagram

$$\begin{tikzcd}
-\otimes_A T_\mathcal{C}(e)\otimes x\otimes_A A \arrow[r, "{1,c_{e,x\otimes_A A}}"] \arrow[d, "\cong"'] & -\otimes_A x\otimes_A A\otimes T_\mathcal{C}(e) \arrow[r, "{c^{-1}_{e,-\otimes_A x\otimes_A A}}"] \arrow[d, dotted, "\Phi(\sigma_{e,x})"'] & T_\mathcal{C}(e)\otimes -\otimes_A x\otimes_A A \arrow[d, "\cong"] \\
-\otimes_A T_\mathcal{C}(e)\otimes A\otimes_A x \arrow[r, "{1,c_{e,A},1}"'] & -\otimes_A A\otimes T_\mathcal{C}(e)\otimes_A x \arrow[r, "{c^{-1}_{e,-\otimes_A A},1}"'] & T_\mathcal{C}(e)\otimes -\otimes_A A\otimes_A x
\end{tikzcd}$$

The upper horizontal composition $c^{-1}_{e,-\otimes_A x}\circ(1\otimes_A c_{e,x})$ is induced by

$$\begin{tikzcd}
-\otimes A\otimes T_\mathcal{C}(e)\otimes x \arrow[r, shift left, "{p_-,1,1}"] \arrow[r, shift right, "{1,q_{T_\mathcal{C}(e)\otimes x}}"'] \arrow[d, "{1,1,c_{e,x}}"'] \arrow[dd, bend right=60, "{c^{-1}_{e,-\otimes A},1}"'] & -\otimes T_\mathcal{C}(e)\otimes x \arrow[r] \arrow[d, "{1,c_{e,x}}"] & -\otimes_A T_\mathcal{C}(e)\otimes x \arrow[d, "{1,c_{e,x}}"] \arrow[dd, dotted, bend right, "{c^{-1}_{e,-},1}"'] \\
-\otimes A\otimes x\otimes T_\mathcal{C}(e) \arrow[r, shift left, "{p_-,1,1}"] \arrow[r, shift right, "{1,q_x,1}"'] \arrow[d, "c^{-1}_{e,-\otimes A\otimes x}"'] & -\otimes x\otimes T_\mathcal{C}(e) \arrow[r] \arrow[d, "c^{-1}_{e,-\otimes x}"] & -\otimes_A x\otimes T_\mathcal{C}(e) \arrow[d, "c^{-1}_{e,-\otimes_A x}"] \\
T_\mathcal{C}(e)\otimes -\otimes A\otimes x \arrow[r, shift left, "{1,p_-,1}"] \arrow[r, shift right, "{1,1,q_x}"'] & T_\mathcal{C}(e)\otimes -\otimes x \arrow[r] & T_\mathcal{C}(e)\otimes -\otimes_A x
\end{tikzcd}$$

Here $(-,p_-)$ and (A,m) belong to $\mathcal{C}_A$ and (x,p_x,q_x) belong to ${}_A\mathcal{C}_A$. $q_{T_\mathcal{C}(e)\otimes x}$ is defined as $A\otimes T_\mathcal{C}(e)\otimes x \xrightarrow{c^{-1}_{e,A},1} T_\mathcal{C}(e)\otimes A\otimes x \xrightarrow{1,q_x} T_\mathcal{C}(e)\otimes x$. The lower horizontal composition $c^{-1}_{e,-\otimes_A A}\circ(1\otimes_A c_{e,A})$ is induced by

$$\begin{tikzcd}
-\otimes A\otimes T_\mathcal{C}(e)\otimes A \arrow[r, shift left, "{p_-,1,1}"] \arrow[r, shift right, "{1,q_{T_\mathcal{C}(e)\otimes A}}"'] \arrow[d, "{1,1,c_{e,A}}"'] \arrow[dd, bend right=60, "{c^{-1}_{e,-\otimes A},1}"'] & -\otimes T_\mathcal{C}(e)\otimes A \arrow[r] \arrow[d, "{1,c_{e,A}}"] & -\otimes_A T_\mathcal{C}(e)\otimes A \arrow[d, "{1,c_{e,A}}"] \arrow[dd, dotted, bend right, "{c^{-1}_{e,-},1}"'] \\
-\otimes A\otimes A\otimes T_\mathcal{C}(e) \arrow[r, shift left, "{p_-,1,1}"] \arrow[r, shift right, "{1,m,1}"'] \arrow[d, "c^{-1}_{e,-\otimes A\otimes A}"'] & -\otimes A\otimes T_\mathcal{C}(e) \arrow[r] \arrow[d, "c^{-1}_{e,-\otimes A}"] & -\otimes_A A\otimes T_\mathcal{C}(e) \arrow[d, "c^{-1}_{e,-\otimes_A A}"] \\
T_\mathcal{C}(e)\otimes -\otimes A\otimes A \arrow[r, shift left, "{1,p_-,1}"] \arrow[r, shift right, "{1,1,m}"'] & T_\mathcal{C}(e)\otimes -\otimes A \arrow[r] & T_\mathcal{C}(e)\otimes -\otimes_A A
\end{tikzcd}$$

Since $x\otimes_A A\cong x\cong A\otimes_A x$, the compositions $c^{-1}_{e,-\otimes_A x\otimes_A A}\circ(1\otimes_A c_{e,x\otimes_A A})$ and $(c^{-1}_{e,-\otimes_A A}\otimes_A 1_x)\circ(1\otimes_A c_{e,A}\otimes_A 1_x)$ are equal by the universal property of cokernels. □

Proposition A.6. Let $\mathcal{C}$ be a multifusion category over $\mathcal{E}$ and A, B be separable algebras in $\mathcal{C}$.

(1) There is an equivalence ${}_A\mathcal{C}\boxtimes_\mathcal{C}\mathcal{C}_B \xrightarrow{\simeq} {}_A\mathcal{C}_B$, $x\boxtimes_\mathcal{C} y\mapsto x\otimes y$ in $\mathrm{BMod}_{\mathcal{E}|\mathcal{E}}(\mathrm{Cat}^{\mathrm{fs}}_\mathcal{E})$.

(2) There is an equivalence $\mathrm{Fun}_\mathcal{C}(\mathcal{C}_A,\mathcal{C}_B) \xrightarrow{\simeq} {}_A\mathcal{C}_B$, $f\mapsto f(A)$ in $\mathrm{BMod}_{\mathcal{E}|\mathcal{E}}(\mathrm{Cat}^{\mathrm{fs}}_\mathcal{E})$, whose inverse is defined as $x\mapsto -\otimes_A x$.

Proof. (1) The functor $\Phi: {}_A\mathcal{C}\boxtimes_\mathcal{C}\mathcal{C}_B\to {}_A\mathcal{C}_B$, $x\boxtimes_\mathcal{C} y\mapsto x\otimes y$ is an equivalence by [KZ, Thm. 2.2.3]. Recall the $\mathcal{E}$-$\mathcal{E}$ bimodule structure on ${}_A\mathcal{C}_B$ and ${}_A\mathcal{C}\boxtimes_\mathcal{C}\mathcal{C}_B$ by Expl. 4.19 and Expl. 4.27 respectively. The left $\mathcal{E}$-module structure on Φ is defined as

$$\Phi(e\odot(x\boxtimes_\mathcal{C} y)) = \Phi((T_\mathcal{C}(e)\otimes x)\boxtimes_\mathcal{C} y) = (T_\mathcal{C}(e)\otimes x)\otimes y \to T_\mathcal{C}(e)\otimes(x\otimes y) = e\odot\Phi(x\boxtimes_\mathcal{C} y)$$

for $e \in \mathcal{E}$, $x \boxtimes_{\mathcal{C}} y \in {}_A\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{C}_B$. The right $\mathcal{E}$-module structure on Φ is defined as

$$\Phi((x \boxtimes_{\mathcal{C}} y) \odot e) = \Phi(x \boxtimes_{\mathcal{C}} (y \otimes T_{\mathcal{C}}(e))) = x \otimes (y \otimes T_{\mathcal{C}}(e)) \to (x \otimes y) \otimes T_{\mathcal{C}}(e) = \Phi(x \boxtimes_{\mathcal{C}} y) \odot e$$

Check that Φ satisfies the diagram (4.10).

$$\begin{tikzcd}
\Phi((T_{\mathcal{C}}(e) \otimes x) \boxtimes_{\mathcal{C}} y) \arrow[r, "c_{e,x},1"] \arrow[d] & \Phi((x \otimes T_{\mathcal{C}}(e)) \boxtimes_{\mathcal{C}} y) \arrow[r, "b_{x,T_{\mathcal{C}}(e),y}"] & \Phi(x \boxtimes_{\mathcal{C}} (T_{\mathcal{C}}(e) \otimes y)) \arrow[r, "1,c_{e,y}"] & \Phi(x \boxtimes_{\mathcal{C}} (y \otimes T_{\mathcal{C}}(e))) \arrow[d] \\
T_{\mathcal{C}}(e) \otimes \Phi(x \boxtimes_{\mathcal{C}} y) \arrow[rrr, "c_{e,x\otimes y}"'] & & & \Phi(x \boxtimes_{\mathcal{C}} y) \otimes T_{\mathcal{C}}(e)
\end{tikzcd}$$

Here c is the central structure of the central functor $T_{\mathcal{C}} : \mathcal{E} \to \mathcal{C}$. The above diagram commutes by the diagram (A.1).

(2) Since $\mathcal{C}_A$ and $\mathcal{C}_B$ belongs to $\mathrm{BMod}_{\mathcal{C}|\mathcal{E}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$, the category $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C}_A, \mathcal{C}_B)$ belongs to $\mathrm{BMod}_{\mathcal{E}|\mathcal{E}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. The $\mathcal{E}$-$\mathcal{E}$ bimodule structure on $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C}_A, \mathcal{C}_B)$ in Expl. 4.22 is defined as

$$(e \odot f \odot \tilde{e})(-) := f(- \otimes T_{\mathcal{C}}(e)) \otimes T_{\mathcal{C}}(\tilde{e})$$

for $e, \tilde{e} \in \mathcal{E}$, $f \in \mathrm{Fun}_{\mathcal{C}}(\mathcal{C}_A, \mathcal{C}_B)$ and $- \in \mathcal{C}_A$.

The functor $\Psi : {}_A\mathcal{C}_B \to \mathrm{Fun}_{\mathcal{C}}(\mathcal{C}_A, \mathcal{C}_B)$, $x \mapsto \Psi^x := -\otimes_A x$ is an equivalence by [KZ, Cor. 2.2.6]. The left $\mathcal{E}$-module structure on Ψ is defined as

$$\Psi^{e\odot x} = - \otimes_A (T_{\mathcal{C}}(e) \otimes x) \cong (- \otimes T_{\mathcal{C}}(e)) \otimes_A x = \Psi^x(- \otimes T_{\mathcal{C}}(e)) = e \odot \Psi^x$$

The right $\mathcal{E}$-module structure on Ψ^x is defined as

$$\Psi^{x\odot e} = - \otimes_A (x \otimes T_{\mathcal{C}}(e)) \cong (- \otimes_A x) \otimes T_{\mathcal{C}}(e) = \Psi^x \odot e$$

Recall the monoidal natural isomorphism $(v_e)_{\Psi^x} : e \odot \Psi^x \Rightarrow \Psi^x \odot e$ in Expl. 4.22:

$$(e \odot \Psi^x)(-) = \Psi^x(-\otimes T_{\mathcal{C}}(e)) \xrightarrow{c^{-1}_{e,-}} \Psi^x(T_{\mathcal{C}}(e)\otimes -) \xrightarrow{s^{\Psi^x}} T_{\mathcal{C}}(e)\otimes\Psi^x(-) \xrightarrow{c_{e,\Psi^x(-)}} \Psi^x(-)\otimes T_{\mathcal{C}}(e) = (\Psi^x \odot e)(-)$$

Check Ψ satisfies the diagram (4.10).

$$\begin{tikzcd}
\Psi^{e\odot x} = - \otimes_A (T_{\mathcal{C}}(e) \otimes x) \arrow[r] \arrow[dd, "1,c_{e,x}"'] & (- \otimes T_{\mathcal{C}}(e)) \otimes_A x = \Psi^x(- \otimes T_{\mathcal{C}}(e)) \arrow[d, "c^{-1}_{e,-},1"] \\
& T_{\mathcal{C}}(e) \otimes - \otimes_A x \arrow[d, "c_{e,-\otimes_A x}"] \\
\Psi^{x\odot e} = - \otimes_A (x \otimes T_{\mathcal{C}}(e)) \arrow[r] & (- \otimes_A x) \otimes T_{\mathcal{C}}(e) = \Psi^x(-) \otimes T_{\mathcal{C}}(e)
\end{tikzcd}$$

The above diagram commutes by Prop. A.4. □

Lemma A.7. Let M and N be separable algebras in $\mathcal{E}$. The functor $\Phi : {}_M\mathcal{E} \boxtimes_{\mathcal{E}} \mathcal{E}_N \to {}_M\mathcal{E}_N$, $x \boxtimes_{\mathcal{E}} y \mapsto x \otimes y$ is an equivalence of categories. The inverse of Φ is defined as $z \mapsto \mathrm{Colim}\Big((M \otimes M) \boxtimes_{\mathcal{E}} z \rightrightarrows M \boxtimes_{\mathcal{E}} z\Big)$ for any $z \in {}_M\mathcal{E}_N$.

Proof. The inverse of Φ is denoted by Ψ.

$$\begin{aligned}
\Psi \circ \Phi(x \boxtimes_{\mathcal{E}} y) &= \Psi(x \otimes y) = \mathrm{Colim}\Big(\boxtimes_{\mathcal{E}} (M \otimes M, x \otimes y) \rightrightarrows \boxtimes_{\mathcal{E}}(M, x \otimes y)\Big) \\
&\simeq \mathrm{Colim}\Big(\boxtimes_{\mathcal{E}} (M \otimes M \otimes x, y) \rightrightarrows \boxtimes_{\mathcal{E}}(M \otimes x, y)\Big) \simeq \boxtimes_{\mathcal{E}}(M \otimes_M x, y) \simeq x \boxtimes_{\mathcal{E}} y
\end{aligned}$$

The first equivalence is due to the balanced $\mathcal{E}$-module functor $\boxtimes_{\mathcal{E}}$. The second equivalence holds because the functor $\boxtimes_{\mathcal{E}}$ preserves colimits.

$$\begin{aligned}\Phi \circ \Psi(z) = \Phi\Big(\mathrm{Colim}\Big((M \otimes M) \boxtimes_{\mathcal{E}} z \rightrightarrows M \boxtimes_{\mathcal{E}} z\Big)\Big) &\simeq \mathrm{Colim}\Big(\Phi((M \otimes M) \boxtimes_{\mathcal{E}} z) \rightrightarrows \Phi(M \boxtimes_{\mathcal{E}} z)\Big) \\ &\simeq \mathrm{Colim}\Big((M \otimes M) \otimes z \rightrightarrows M \otimes z\Big) \simeq M \otimes_M z \simeq z\end{aligned}$$

The first equivalence holds because Φ preserves colimits. □

Lemma A.8. Let A be a separable algebra in $\mathcal{E}$. There is an equivalence ${}_A\mathcal{E} \simeq \mathcal{E}_A$ of right $\mathcal{E}$-module categories.

Proof. We define a functor $F : {}_A\mathcal{E} \to \mathcal{E}_A$, $(x, q_x : A \otimes x \to x) \mapsto (x, p_x : x \otimes A \xrightarrow{r_{x,A}} A \otimes x \xrightarrow{q_x} x)$ and a functor $G : \mathcal{E}_A \to {}_A\mathcal{E}$, $(y, p_y : y \otimes A \to y) \mapsto (y, q_y : A \otimes y \xrightarrow{r_{A,y}} y \otimes A \xrightarrow{p_y} y)$, where r is the braiding of $\mathcal{E}$. Since $r_{x,y} \circ r_{y,x} = \mathrm{id}_{x \otimes y}$ for all $x, y \in \mathcal{E}$, then $F \circ G = \mathrm{id}$ and $G \circ F = \mathrm{id}$.

The right $\mathcal{E}$-action on $\mathcal{E}_A$ is defined as $(y, p_y) \otimes e = (y \otimes e, p_{y \otimes e} : y \otimes e \otimes A \xrightarrow{1, r_{e,A}} y \otimes A \otimes e \xrightarrow{p_y, 1} y \otimes e)$. We have $F((x, q_x) \otimes e) = F(x \otimes e, q_{x \otimes e} : A \otimes x \otimes e \xrightarrow{q_x, 1} x \otimes e) = (x \otimes e, p_{x \otimes e} : x \otimes e \otimes A \xrightarrow{r_{x \otimes e, A}} A \otimes x \otimes e \xrightarrow{q_x, 1} x \otimes e)$ and $F(x, q_x) \otimes e = (x, p_x) \otimes e = (x \otimes e, p_{x \otimes e} : x \otimes e \otimes A \xrightarrow{1, r_{e,A}} x \otimes A \otimes e \xrightarrow{r_{x,A}, 1} A \otimes x \otimes e \xrightarrow{q_x, 1} x \otimes e)$. Then the right $\mathcal{E}$-module structure on F is the identity natural isomorphism $F((x, q_x) \otimes e) = F(x, q_x) \otimes e$. □

Lemma A.9. Let $\mathcal{C}$ and $\mathcal{M}$ be pivotal fusion categories and $\mathcal{M}$ a left $\mathcal{C}$-module in $\mathrm{Cat}^{\mathrm{fs}}$. There are isomorphisms $[x, y]^R_{\mathcal{C}} \cong [y, x]_{\mathcal{C}} \cong [x, y]^L_{\mathcal{C}}$ for $x, y \in \mathcal{M}$.

Proof. Since $\mathcal{M}$ is a pivotal fusion category, there is a one-to-one correspondence between traces on $\mathcal{M}$ and natural isomorphisms

$$\eta^{\mathcal{M}}_{x,y} : \mathrm{Hom}_{\mathcal{M}}(x, y) \to \mathrm{Hom}_{\mathcal{M}}(y, x)^*$$

for $x, y \in \mathcal{M}$ by [S, Prop. 4.1]. Here both $\mathrm{Hom}_{\mathcal{M}}(-,-)$ and $\mathrm{Hom}(-,-)^*$ are functors from $\mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{V}\mathrm{ec}$. For $c \in \mathcal{C}$, we have composed natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(c, [x, y]_{\mathcal{C}}) \simeq \mathrm{Hom}_{\mathcal{M}}(c \odot x, y) &\xrightarrow{\eta^{\mathcal{M}}} \mathrm{Hom}_{\mathcal{M}}(y, c \odot x)^* \simeq \mathrm{Hom}_{\mathcal{M}}(c^L \odot y, x)^* \\ &\simeq \mathrm{Hom}_{\mathcal{C}}(c^L, [y, x]_{\mathcal{C}})^* \simeq \mathrm{Hom}_{\mathcal{C}}([y, x]^R_{\mathcal{C}}, c)^* \xrightarrow{(\eta^{\mathcal{C}})^{-1}} \mathrm{Hom}_{\mathcal{C}}(c, [y, x]^R_{\mathcal{C}}),\end{aligned}$$

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(c, [x, y]_{\mathcal{C}}) \simeq \mathrm{Hom}_{\mathcal{M}}(c \odot x, y) \simeq \mathrm{Hom}_{\mathcal{M}}(x, c^R \odot y) &\xrightarrow{\eta^{\mathcal{M}}} \mathrm{Hom}_{\mathcal{M}}(c^R \odot y, x)^* \\ \simeq \mathrm{Hom}_{\mathcal{C}}(c^R, [y, x]_{\mathcal{C}})^* &\xrightarrow{(\eta^{\mathcal{C}})^{-1}} \mathrm{Hom}_{\mathcal{C}}([y, x]_{\mathcal{C}}, c^R) \simeq \mathrm{Hom}_{\mathcal{C}}(c, [y, x]^L_{\mathcal{C}})\end{aligned}$$

By Yoneda lemma, we obtain $[x, y]^R_{\mathcal{C}} \cong [y, x]_{\mathcal{C}} \cong [x, y]^L_{\mathcal{C}}$. □

A.2 The monoidal 2-category $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$

For objects $\mathcal{M}, \mathcal{N}$ in a 2-category B, the hom category $\mathsf{B}(\mathcal{M}, \mathcal{N})$ denotes the category of 1-morphisms from $\mathcal{M}$ to $\mathcal{N}$ in B and 2-morphisms in B. For 1-morphisms $f, g \in \mathsf{B}(\mathcal{M}, \mathcal{N})$, the set $\mathsf{B}(\mathcal{M}, \mathcal{N})(f, g)$ denotes the set of all 2-morphisms in B with domain f and codomain g.

Definition A.10. The product 2-category $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}} \times \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ is the 2-category defined by the following data:

- The objects are pairs $(\mathcal{A}, \mathcal{B})$ for $\mathcal{A}, \mathcal{B} \in \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$.
- For objects $(\mathcal{A}, \mathcal{B}), (\mathcal{C}, \mathcal{D}) \in \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}} \times \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$, a 1-morphism from $(\mathcal{A}, \mathcal{B})$ to $(\mathcal{C}, \mathcal{D})$ is a pair (f, g) where $f : \mathcal{A} \to \mathcal{C}$ and $g : \mathcal{B} \to \mathcal{D}$ are 1-morphisms in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$.
- The identity 1-morphism of an object $(\mathcal{A}, \mathcal{B})$ is $1_{(\mathcal{A},\mathcal{B})} := (1_{\mathcal{A}}, 1_{\mathcal{B}})$.
- For 1-morphisms $(f, g), (p, q) \in (\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}} \times \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})((\mathcal{A}, \mathcal{B}), (\mathcal{C}, \mathcal{D}))$, a 2-morphism from (f, g) to (p, q) is a pair (α, β) where $\alpha : f \Rightarrow p$ and $\beta : g \Rightarrow q$ are 2-morphisms in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$.
- For 1-morphisms $(f, g), (p, q), (m, n) \in (\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}} \times \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})((\mathcal{A}, \mathcal{B}), (\mathcal{C}, \mathcal{D}))$, and 2-morphisms $(\alpha, \beta) \in (\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}} \times \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})((\mathcal{A}, \mathcal{B}), (\mathcal{C}, \mathcal{D}))((f, g), (p, q))$, and $(\gamma, \delta) \in (\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}} \times \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})((\mathcal{A}, \mathcal{B}), (\mathcal{C}, \mathcal{D}))((p, q), (m, n))$, the vertical composition is $(\gamma, \delta) \circ (\alpha, \beta) := (\gamma \circ \alpha, \delta \circ \beta)$.
- For 1-morphisms $(f, g) \in (\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}} \times \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})((\mathcal{A}, \mathcal{B}), (\mathcal{C}, \mathcal{D})), (p, q) \in (\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}} \times \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})((\mathcal{C}, \mathcal{D}), (\mathcal{M}, \mathcal{N}))$, the horizontal composition of 1-morphisms is $(p, q) \circ (f, g) := (p \circ f, q \circ g)$.
- For 1-morphisms $(f, g), (f', g') \in (\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}} \times \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})((\mathcal{A}, \mathcal{B}), (\mathcal{C}, \mathcal{D}))$, and $(p, q), (p', q') \in (\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}} \times \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})((\mathcal{C}, \mathcal{D}), (\mathcal{M}, \mathcal{N}))$, and 2-morphisms $(\alpha, \beta) \in (\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}} \times \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})((\mathcal{A}, \mathcal{B}), (\mathcal{C}, \mathcal{D}))((f, g), (f', g'))$, and $(\gamma, \delta) \in (\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}} \times \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})((\mathcal{C}, \mathcal{D}), (\mathcal{M}, \mathcal{N}))((p, q), (p', q'))$, the horizontal composition of 2-morphisms is $(\gamma, \delta) * (\alpha, \beta) := (\gamma * \alpha, \delta * \beta)$.

It is routine to check that the above data satisfy the axioms (i)-(vi) of [JY, Prop. 2.3.4].

Next, we define a pseudo-functor $\boxtimes_{\mathcal{E}} : \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}} \times \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}} \to \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ as follows.

- For each object $(\mathcal{A}, \mathcal{B}) \in \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}} \times \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$, an object $\mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{B}$ in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ exists (unique up to equivalence).
- For a 1-morphism $(f, g) \in (\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}} \times \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})((\mathcal{A}, \mathcal{B}), (\mathcal{C}, \mathcal{D}))$, a 1-morphism $f \boxtimes_{\mathcal{E}} g : \mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{B} \to \mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{D}$ in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ is induced by the universal property of the tensor product $\boxtimes_{\mathcal{E}}$:

$$\begin{tikzcd}
\mathcal{A} \times \mathcal{B} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, "f{,}g"'] \arrow[dr, phantom, "t_{fg} \Nearrow"] & \mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{B} \arrow[d, "\exists! f \boxtimes_{\mathcal{E}} g"] \\
\mathcal{C} \times \mathcal{D} \arrow[r, "\boxtimes_{\mathcal{E}}"'] & \mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{D}
\end{tikzcd}$$

Notice that for all $x \in \mathcal{A}, e \in \mathcal{E}, y \in \mathcal{B}$, the balanced $\mathcal{E}$-module structure on the functor $\boxtimes_{\mathcal{E}} \circ (f \times g)$ is induced by

$$f(x \odot e) \boxtimes_{\mathcal{E}} g(y) \xrightarrow{(s^r_f)^{-1} \boxtimes_{\mathcal{E}} 1} (f(x) \odot e) \boxtimes_{\mathcal{E}} g(y) \xrightarrow{b^{\mathcal{C}\mathcal{D}}_{f(x),e,g(y)}} f(x) \boxtimes_{\mathcal{E}} (e \odot g(y)) \xrightarrow{1 \boxtimes_{\mathcal{E}} s^l_g} f(x) \boxtimes_{\mathcal{E}} g(e \odot y)$$

where $(g, s^l_g) : \mathcal{B} \to \mathcal{D}$ is the left $\mathcal{E}$-module functor, $(f, s^r_f) : \mathcal{A} \to \mathcal{C}$ is the right $\mathcal{E}$-module functor, and the natural isomorphism $b^{\mathcal{C}\mathcal{D}}$ is the balanced $\mathcal{E}$-module structure on the functor $\boxtimes_{\mathcal{E}} : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{D}$.

For a 2-morphism $(\alpha, \beta) : (f, g) \Rightarrow (p, q)$ in $(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}} \times \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})((\mathcal{A}, \mathcal{B}), (\mathcal{C}, \mathcal{D}))$, a 2-morphism $\alpha \boxtimes_{\mathcal{E}} \beta : f \boxtimes_{\mathcal{E}} g \Rightarrow p \boxtimes_{\mathcal{E}} q$ in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ is defined by the universal property of $\boxtimes_{\mathcal{E}}$:

$$\begin{tikzcd}
\mathcal{A} \times \mathcal{B} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, "f{,}g"'] \arrow[dr, phantom, "\overset{t_{fg}}{\Rightarrow}"] & \mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{B} \arrow[d, bend right=60, "f \boxtimes_{\mathcal{E}} g"'] \arrow[d, bend left=60, "p \boxtimes_{\mathcal{E}} q"] \arrow[d, phantom, "\overset{\exists! \alpha \boxtimes_{\mathcal{E}} \beta}{\Rightarrow}"] \\
\mathcal{C} \times \mathcal{D} \arrow[r, "\boxtimes_{\mathcal{E}}"'] & \mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{D}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{A} \times \mathcal{B} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, bend right=60, "f{,}g"'] \arrow[d, bend left=60, "p{,}q"] \arrow[d, phantom, "\overset{\alpha{,}\beta}{\Rightarrow}"] \arrow[dr, phantom, "\overset{t_{pq}}{\Rightarrow}"] & \mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{B} \arrow[d, "p \boxtimes_{\mathcal{E}} q"] \\
\mathcal{C} \times \mathcal{D} \arrow[r, "\boxtimes_{\mathcal{E}}"'] & \mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{D}
\end{tikzcd}$$

It is routine to check that $\boxtimes_{\mathcal{E}} : (\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}} \times \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})((\mathcal{A},\mathcal{B}),(\mathcal{C},\mathcal{D})) \to \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}(\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}, \mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{D})$ is a local functor. That is, for 2-morphisms $(\alpha,\beta) : (f,g) \Rightarrow (p,q)$ and $(\delta,\tau) : (p,q) \Rightarrow (m,n)$ in $(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}} \times \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})((\mathcal{A},\mathcal{B}),(\mathcal{C},\mathcal{D}))$, The equations $(\delta\circ\alpha)\boxtimes_{\mathcal{E}}(\tau\circ\beta) = (\delta\boxtimes_{\mathcal{E}}\tau)\circ(\alpha\boxtimes_{\mathcal{E}}\beta)$ and $1_f\boxtimes_{\mathcal{E}} 1_g = 1_{f\boxtimes_{\mathcal{E}} g}$ hold.

- For all 1-morphisms $f\boxtimes_{\mathcal{E}} g : \mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B} \to \mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{D}$, $p\boxtimes_{\mathcal{E}} q : \mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{D} \to \mathcal{M}\boxtimes_{\mathcal{E}}\mathcal{N}$ in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$, the lax functoriality constraint $(p\boxtimes_{\mathcal{E}} q)\circ(f\boxtimes_{\mathcal{E}} g) \simeq^{t^{pq}_{fg}} (p\circ f)\boxtimes_{\mathcal{E}}(q\circ g)$ is defined by the universal property of $\boxtimes_{\mathcal{E}}$:

$$\begin{tikzcd} & \mathcal{A}\times\mathcal{B} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[dl, "{f,g}"'] \arrow[d, phantom, "\overset{t_{fg}}{\Rightarrow}"] & \mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B} \arrow[dl, "\exists! f\boxtimes_{\mathcal{E}} g"'] \arrow[dd, "\exists!(p\circ f)\boxtimes_{\mathcal{E}}(q\circ g)"] \arrow[dd, phantom, "\overset{\exists! t^{pq}_{fg}}{\Rightarrow}"', shift right=4] \\ \mathcal{C}\times\mathcal{D} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[dr, "{p,q}"'] & \mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{D} \arrow[dr, "\exists! p\boxtimes_{\mathcal{E}} q"] \arrow[d, phantom, "\overset{t_{pq}}{\Rightarrow}"] & \\ & \mathcal{M}\times\mathcal{N} \arrow[r, "\boxtimes_{\mathcal{E}}"'] & \mathcal{M}\boxtimes_{\mathcal{E}}\mathcal{N} \end{tikzcd} = \begin{tikzcd} & \mathcal{A}\times\mathcal{B} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[dl, "{f,g}"'] \arrow[dd, "{p\circ f, q\circ g}"'] \arrow[ddr, phantom, "\overset{t_{pf,qg}}{\Rightarrow}"] & \mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B} \arrow[dd, "\exists!(p\circ f)\boxtimes_{\mathcal{E}}(q\circ g)"] \\ \mathcal{C}\times\mathcal{D} \arrow[dr, "{p,q}"'] & & \\ & \mathcal{M}\times\mathcal{N} \arrow[r, "\boxtimes_{\mathcal{E}}"'] & \mathcal{M}\boxtimes_{\mathcal{E}}\mathcal{N} \end{tikzcd}$$

where the identity 2-morphism is always abbreviated.

- For 1-morphisms $1_{\mathcal{A}}\boxtimes_{\mathcal{E}} 1_{\mathcal{B}} : \mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B} \to \mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}$ in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$, the lax unity constraint $1_{\mathcal{A}}\boxtimes_{\mathcal{E}} 1_{\mathcal{B}} \simeq^{t_{\mathcal{A}\mathcal{B}}} 1_{\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}}$ is defined by the universal property of $\boxtimes_{\mathcal{E}}$:

$$\begin{tikzcd} \mathcal{A}\times\mathcal{B} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, bend right=50, "{1_{\mathcal{A}},1_{\mathcal{B}}}"'] \arrow[dr, phantom, "\overset{t_{1_{\mathcal{A}},1_{\mathcal{B}}}}{\Rightarrow}"] & \mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B} \arrow[d, bend right=50, "1_{\mathcal{A}}\boxtimes_{\mathcal{E}}1_{\mathcal{B}}"'] \arrow[d, bend left=50, "1_{\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}}"] \arrow[d, phantom, "\overset{\exists! t_{\mathcal{A}\mathcal{B}}}{\Rightarrow}"] \\ \mathcal{A}\times\mathcal{B} \arrow[r, "\boxtimes_{\mathcal{E}}"'] & \mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B} \end{tikzcd} = \begin{tikzcd} \mathcal{A}\times\mathcal{B} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, bend right=50, "{1_{\mathcal{A}},1_{\mathcal{B}}}"'] \arrow[d, bend left=50, "1_{\mathcal{A}\times\mathcal{B}}"] \arrow[dr, phantom, "\overset{\mathrm{id}}{\Rightarrow}"] & \mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B} \arrow[d, "1_{\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}}"] \\ \mathcal{A}\times\mathcal{B} \arrow[r, "\boxtimes_{\mathcal{E}}"'] & \mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B} \end{tikzcd}$$

where we choose the identity 2-morphism $\mathrm{id} : \boxtimes_{\mathcal{E}}\circ 1_{\mathcal{A}\times\mathcal{B}} \Rightarrow 1_{\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}}\circ\boxtimes_{\mathcal{E}}$ for convenience.

It is routine to check that the above data satisfy the lax associativity, the lax left and right unity of [JY, (4.1.3),(4.1.4)].

Remark A.11. The left (or right) $\mathcal{E}$-module structure on $\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}$ is induced by

$$\begin{tikzcd} \mathcal{E}\times\mathcal{A}\times\mathcal{B} \arrow[r, "{1,\boxtimes_{\mathcal{E}}}"] \arrow[d, "{\odot,1}"'] \arrow[dr, phantom, "t^{l}_{\mathcal{A}\mathcal{B}}\Nearrow"] & \mathcal{E}\times\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B} \arrow[d, "\exists!\odot"] \\ \mathcal{A}\times\mathcal{B} \arrow[r, "\boxtimes_{\mathcal{E}}"'] & \mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}\times\mathcal{B}\times\mathcal{E} \arrow[r, "{\boxtimes_{\mathcal{E}},1}"] \arrow[d, "{1,\odot}"'] \arrow[dr, phantom, "t^{r}_{\mathcal{A}\mathcal{B}}\Nearrow"] & \mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}\times\mathcal{E} \arrow[d, "\exists!\odot"] \\ \mathcal{A}\times\mathcal{B} \arrow[r, "\boxtimes_{\mathcal{E}}"'] & \mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B} \end{tikzcd}$$

The n-fold product 2-category $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}\times\cdots\times\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ is written as $(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})^n$ such that $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ has a set of objects. The 2-category $\mathrm{Cat}^{\mathrm{ps}}((\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})^n, \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$ contains pseudofunctors $(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})^n \to \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ as objects, strong transformations between such pseudofunctors as 1-morphisms, and modifications between such strong transformations as 2-morphisms.

Lemma A.12. We claim that $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ is a monoidal 2-category.

Proof. A monoidal 2-category $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ consists of the following data.

i The 2-category $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ is equipped with the pseudo-functor $\boxtimes_{\mathcal{E}} : \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}\times\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}} \to \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$ and the tensor unit $\mathcal{E}$.

ii The associator is a strong transformation $\alpha : \boxtimes_{\mathcal{E}} \circ (\boxtimes_{\mathcal{E}} \times \mathrm{id}) \Rightarrow \boxtimes_{\mathcal{E}} \circ (\mathrm{id} \times \boxtimes_{\mathcal{E}})$ in the 2-cateogry $\mathrm{Cat}^{\mathrm{ps}}((\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})^3, \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. For each $(\mathcal{A}, \mathcal{B}, \mathcal{C}) \in (\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})^3$, α contains an invertible 1-morphism $\alpha_{\mathcal{A},\mathcal{B},\mathcal{C}} : (\mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{B}) \boxtimes_{\mathcal{E}} \mathcal{C} \to \mathcal{A} \boxtimes_{\mathcal{E}} (\mathcal{B} \boxtimes_{\mathcal{E}} \mathcal{C})$ induced by

$$\begin{tikzcd}
\mathcal{A}\times\mathcal{B}\times\mathcal{C} \arrow[r, "\boxtimes_{\mathcal{E}}\circ(\boxtimes_{\mathcal{E}}\times\mathrm{id})"] \arrow[dr, "\boxtimes_{\mathcal{E}}\circ(\mathrm{id}\times\boxtimes_{\mathcal{E}})"'] & (\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B})\boxtimes_{\mathcal{E}}\mathcal{C} \arrow[d, "\exists!\alpha_{\mathcal{A},\mathcal{B},\mathcal{C}}"] \arrow[dl, phantom, "d^{\alpha}_{\mathcal{A}\mathcal{B}\mathcal{C}}\Nearrow"] \\
 & \mathcal{A}\boxtimes_{\mathcal{E}}(\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{C})
\end{tikzcd}$$

For each 1-morphism $(f_1, f_2, f_3) : (\mathcal{A}, \mathcal{B}, \mathcal{C}) \to (\mathcal{A}', \mathcal{B}', \mathcal{C}')$ in $(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})^3$, α contains an invertible 2-morphism $\alpha_{f_1,f_2,f_3} : (f_1 \boxtimes_{\mathcal{E}} (f_2 \boxtimes_{\mathcal{E}} f_3)) \circ \alpha_{\mathcal{A},\mathcal{B},\mathcal{C}} \Rightarrow \alpha_{\mathcal{A}',\mathcal{B}',\mathcal{C}'} \circ ((f_1 \boxtimes_{\mathcal{E}} f_2) \boxtimes_{\mathcal{E}} f_3)$ induced by

$$\begin{tikzcd}
\mathcal{A}\times\mathcal{B}\times\mathcal{C} \arrow[r, "{\boxtimes_{\mathcal{E}},1}"] \arrow[dd, "{f_1,f_2,f_3}"'] \arrow[dr, "{1,\boxtimes_{\mathcal{E}}}"'] & (\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B})\times\mathcal{C} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, phantom, "d^{\alpha}_{\mathcal{A}\mathcal{B}\mathcal{C}}\Nearrow"] & (\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B})\boxtimes_{\mathcal{E}}\mathcal{C} \arrow[d, "\alpha_{\mathcal{A},\mathcal{B},\mathcal{C}}"] \arrow[dr, "(f_1\boxtimes_{\mathcal{E}}f_2)\boxtimes_{\mathcal{E}}f_3"] & \\
 & \mathcal{A}\times(\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{C}) \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, "{f_1,f_2\boxtimes_{\mathcal{E}}f_3}"'] \arrow[dl, phantom, "{1_{f_1},t_{f_2f_3}}\Nearrow"] \arrow[dr, phantom, "t_{f_1,f_2f_3}\Nearrow"] & \mathcal{A}\boxtimes_{\mathcal{E}}(\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{C}) \arrow[r, phantom, "\overset{\exists!\alpha_{f_1f_2f_3}}{\Rrightarrow}"] \arrow[d, "f_1\boxtimes_{\mathcal{E}}(f_2\boxtimes_{\mathcal{E}}f_3)"] & (\mathcal{A}'\boxtimes_{\mathcal{E}}\mathcal{B}')\boxtimes_{\mathcal{E}}\mathcal{C}' \arrow[dl, "\alpha_{\mathcal{A}',\mathcal{B}',\mathcal{C}'}"] \\
\mathcal{A}'\times\mathcal{B}'\times\mathcal{C}' \arrow[r, "{1,\boxtimes_{\mathcal{E}}}"'] & \mathcal{A}'\times(\mathcal{B}'\boxtimes_{\mathcal{E}}\mathcal{C}') \arrow[r, "\boxtimes_{\mathcal{E}}"'] & \mathcal{A}'\boxtimes_{\mathcal{E}}(\mathcal{B}'\boxtimes_{\mathcal{E}}\mathcal{C}') &
\end{tikzcd}$$

$$\|$$

$$\begin{tikzcd}
\mathcal{A}\times\mathcal{B}\times\mathcal{C} \arrow[r, "{\boxtimes_{\mathcal{E}},1}"] \arrow[dd, "{f_1,f_2,f_3}"'] & (\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B})\times\mathcal{C} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, "{(f_1\boxtimes_{\mathcal{E}}f_2),f_3}"] \arrow[dl, phantom, "\overset{t_{f_1f_2,1_{f_3}}}{\Rrightarrow}"] \arrow[dr, phantom, "\overset{t_{f_1f_2,f_3}}{\Rrightarrow}"] & (\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B})\boxtimes_{\mathcal{E}}\mathcal{C} \arrow[dr, "(f_1\boxtimes_{\mathcal{E}}f_2)\boxtimes_{\mathcal{E}}f_3"] & \\
 & (\mathcal{A}'\boxtimes_{\mathcal{E}}\mathcal{B}')\times\mathcal{C}' \arrow[rr, "\boxtimes_{\mathcal{E}}"] \arrow[drr, phantom, "\overset{d^{\alpha}_{\mathcal{A}'\mathcal{B}'\mathcal{C}'}}{\Rrightarrow}"] & & (\mathcal{A}'\boxtimes_{\mathcal{E}}\mathcal{B}')\boxtimes_{\mathcal{E}}\mathcal{C}' \arrow[dl, "\alpha_{\mathcal{A}',\mathcal{B}',\mathcal{C}'}"] \\
\mathcal{A}'\times\mathcal{B}'\times\mathcal{C}' \arrow[r, "{1,\boxtimes_{\mathcal{E}}}"'] \arrow[ur, "{\boxtimes_{\mathcal{E}},1}"] & \mathcal{A}'\times(\mathcal{B}'\boxtimes_{\mathcal{E}}\mathcal{C}') \arrow[r, "\boxtimes_{\mathcal{E}}"'] & \mathcal{A}'\boxtimes_{\mathcal{E}}(\mathcal{B}'\boxtimes_{\mathcal{E}}\mathcal{C}') &
\end{tikzcd}$$

iii The left unitor and right unitor are strong transformations $l : \mathcal{E} \boxtimes_{\mathcal{E}} - \Rightarrow -$ and $r : - \boxtimes_{\mathcal{E}} \mathcal{E} \Rightarrow -$ in $\mathrm{Cat}^{\mathrm{ps}}(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}, \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. For each $\mathcal{A} \in \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$, l and r contain invertible 1-morphisms $l_{\mathcal{A}} : \mathcal{E} \boxtimes_{\mathcal{E}} \mathcal{A} \to \mathcal{A}$ and $r_{\mathcal{A}} : \mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{E} \to \mathcal{A}$ respectively.

$$\begin{tikzcd}
\mathcal{E}\times\mathcal{A} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[dr, "\odot"'] & \mathcal{E}\boxtimes_{\mathcal{E}}\mathcal{A} \arrow[d, "\exists!l_{\mathcal{A}}"] \arrow[dl, phantom, "d^{l}_{\mathcal{A}}\Nearrow"] \\
 & \mathcal{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A}\times\mathcal{E} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[dr, "\odot"'] & \mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{E} \arrow[d, "\exists!r_{\mathcal{A}}"] \arrow[dl, phantom, "d^{r}_{\mathcal{A}}\Nearrow"] \\
 & \mathcal{A}
\end{tikzcd}$$

For each 1-morphism $f : \mathcal{A} \to \mathcal{B}$ in $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$, l and r contain invertible 2-morphisms $\beta^l_f : f \circ l_{\mathcal{A}} \Rightarrow l_{\mathcal{B}} \circ (1_{\mathcal{E}} \boxtimes_{\mathcal{E}} f)$ and $\beta^r_f : f \circ r_{\mathcal{A}} \Rightarrow r_{\mathcal{B}} \circ (f \boxtimes_{\mathcal{E}} 1_{\mathcal{E}})$ respectively.

$$\begin{tikzcd}
\mathcal{E}\times\mathcal{A} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[dd, "{1_{\mathcal{E}},f}"'] \arrow[dr, "\odot"'] & \mathcal{E}\boxtimes_{\mathcal{E}}\mathcal{A} \arrow[d, "l_{\mathcal{A}}"] \arrow[dl, phantom, "d^{l}_{\mathcal{A}}\Nearrow"] \arrow[dr, "1_{\mathcal{E}}\boxtimes_{\mathcal{E}}f"] & \\
 & \mathcal{A} \arrow[d, "f"] \arrow[r, phantom, "\overset{\exists!\beta^l_f}{\Rrightarrow}"] \arrow[dl, phantom, "\overset{s^l_f}{\Rrightarrow}"] & \mathcal{E}\boxtimes_{\mathcal{E}}\mathcal{B} \arrow[dl, "l_{\mathcal{B}}"] \\
\mathcal{E}\times\mathcal{B} \arrow[r, "\odot"'] & \mathcal{B} &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{E}\times\mathcal{A} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[dd, "{1_{\mathcal{E}},f}"'] \arrow[dr, phantom, "\overset{t_{1_{\mathcal{E}}f}}{\Rrightarrow}"] & \mathcal{E}\boxtimes_{\mathcal{E}}\mathcal{A} \arrow[dr, "1_{\mathcal{E}}\boxtimes_{\mathcal{E}}f"] & \\
 & \overset{d^l_{\mathcal{B}}}{\Rrightarrow} & \mathcal{E}\boxtimes_{\mathcal{E}}\mathcal{B} \arrow[dl, "l_{\mathcal{B}}"] \\
\mathcal{E}\times\mathcal{B} \arrow[r, "\odot"'] \arrow[urr, bend left, "\boxtimes_{\mathcal{E}}"] & \mathcal{B} &
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathcal{A}\times\mathcal{E} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[dr, "\odot"'] \arrow[dd, "{f,1_{\mathcal{E}}}"'] & \mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{E} \arrow[d, "r_{\mathcal{A}}"] \arrow[dr, "f\boxtimes_{\mathcal{E}}1_{\mathcal{E}}"] \arrow[dl, phantom, "d^r_{\mathcal{A}}\Uparrow"] & \\
& \mathcal{A} \arrow[d, "f"] \arrow[r, phantom, "\overset{\exists!\beta^r_f}{\Rightarrow}"] & \mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{E} \arrow[dl, "r_{\mathcal{B}}"] \\
\mathcal{B}\times\mathcal{E} \arrow[r, "\odot"'] \arrow[ur, phantom, "\overset{s^r_f}{\Rightarrow}"] & \mathcal{B} &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{A}\times\mathcal{E} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[dd, "{f,1_{\mathcal{E}}}"'] & \mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{E} \arrow[dr, "f\boxtimes_{\mathcal{E}}1_{\mathcal{E}}"] \arrow[d, phantom, "\overset{t_{f1_{\mathcal{E}}}}{\Rightarrow}"] & \\
& {} & \mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{E} \arrow[dl, "r_{\mathcal{B}}"] \arrow[dl, phantom, bend left, "\overset{d^r_{\mathcal{B}}}{\Rightarrow}"] \\
\mathcal{B}\times\mathcal{E} \arrow[r, "\odot"'] \arrow[urr, bend left, "\boxtimes_{\mathcal{E}}"] & \mathcal{B} &
\end{tikzcd}
$$

where $(f, s^l_f) : \mathcal{A} \to \mathcal{B}$ is a left $\mathcal{E}$-module functor and $(f, s^r_f) : \mathcal{A} \to \mathcal{B}$ is a right $\mathcal{E}$-module functor.

iv The pentagonator is a modification π in $\mathrm{Cat}^{\mathrm{ps}}((\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})^4, \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. For each $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$, π consists of an invertible 2-morphism $\pi_{\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}} : (1_{\mathcal{A}} \boxtimes_{\mathcal{E}} \alpha_{\mathcal{B},\mathcal{C},\mathcal{D}}) \circ \alpha_{\mathcal{A},\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{C},\mathcal{D}} \circ (\alpha_{\mathcal{A},\mathcal{B},\mathcal{C}} \boxtimes_{\mathcal{E}} 1_{\mathcal{D}}) \Rightarrow \alpha_{\mathcal{A},\mathcal{B},\mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{D}} \circ \alpha_{\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B},\mathcal{C},\mathcal{D}}$ induced by (where, for example, $\mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{B}$ is abbreviated to $\mathcal{A}\mathcal{B}$):

$$
\begin{tikzcd}
\mathcal{A}\times\mathcal{B}\times\mathcal{C}\times\mathcal{D} \arrow[r, "{\boxtimes_{\mathcal{E}},1,1}"] \arrow[dr, "{1,\boxtimes_{\mathcal{E}},1}"'] \arrow[ddd, "{1,1,\boxtimes_{\mathcal{E}}}"'] & \mathcal{A}\mathcal{B}\times\mathcal{C}\times\mathcal{D} \arrow[r, "{\boxtimes_{\mathcal{E}},1}"] \arrow[d, phantom, "\overset{d^{\alpha}_{\mathcal{A}\mathcal{B}\mathcal{C}},1_{1_{\mathcal{D}}}}{\Rightarrow}"] & (\mathcal{A}\mathcal{B})\mathcal{C}\times\mathcal{D} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, "{\alpha_{\mathcal{A},\mathcal{B},\mathcal{C}},1_{\mathcal{D}}}"'] \arrow[dr, phantom, "t_{\alpha_{\mathcal{A},\mathcal{B},\mathcal{C}},1_{\mathcal{D}}}\Uparrow"] & ((\mathcal{A}\mathcal{B})\mathcal{C})\mathcal{D} \arrow[d, "\alpha_{\mathcal{A},\mathcal{B},\mathcal{C}}\boxtimes_{\mathcal{E}}1_{\mathcal{D}}"] \arrow[dr, bend left, "\alpha_{\mathcal{A}\mathcal{B},\mathcal{C},\mathcal{D}}"] & \\
& \mathcal{A}\times\mathcal{B}\mathcal{C}\times\mathcal{D} \arrow[r, "{\boxtimes_{\mathcal{E}},1}"'] \arrow[d, "{1,\boxtimes_{\mathcal{E}}}"'] \arrow[ddl, phantom, "\overset{1_{1_{\mathcal{A}}},d^{\alpha}_{\mathcal{B}\mathcal{C}\mathcal{D}}}{\Rightarrow}"] & \mathcal{A}(\mathcal{B}\mathcal{C})\times\mathcal{D} \arrow[r, "\boxtimes_{\mathcal{E}}"'] \arrow[d, phantom, "d^{\alpha}_{\mathcal{A},\mathcal{B}\mathcal{C},\mathcal{D}}\Uparrow"] & (\mathcal{A}(\mathcal{B}\mathcal{C}))\mathcal{D} \arrow[d, "\alpha_{\mathcal{A},\mathcal{B}\mathcal{C},\mathcal{D}}"] \arrow[r, phantom, "\overset{\exists!\pi_{\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}}}{\Rightarrow}"] & (\mathcal{A}\mathcal{B})(\mathcal{C}\mathcal{D}) \arrow[ddl, "\alpha_{\mathcal{A},\mathcal{B},\mathcal{C}\mathcal{D}}"] \\
& \mathcal{A}\times(\mathcal{B}\mathcal{C})\mathcal{D} \arrow[rr, "\boxtimes_{\mathcal{E}}"] \arrow[d, "{1_{\mathcal{A}},\alpha_{\mathcal{B},\mathcal{C},\mathcal{D}}}"'] & {} \arrow[d, phantom, "t_{1_{\mathcal{A}},\alpha_{\mathcal{B},\mathcal{C},\mathcal{D}}}\Uparrow"] & \mathcal{A}((\mathcal{B}\mathcal{C})\mathcal{D}) \arrow[d, "1_{\mathcal{A}}\boxtimes_{\mathcal{E}}\alpha_{\mathcal{B},\mathcal{C},\mathcal{D}}"'] & \\
\mathcal{A}\times\mathcal{B}\times\mathcal{C}\mathcal{D} \arrow[r, "{1,\boxtimes_{\mathcal{E}}}"] & \mathcal{A}\times\mathcal{B}(\mathcal{C}\mathcal{D}) \arrow[rr, "\boxtimes_{\mathcal{E}}"] & {} & \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D})) &
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}
\mathcal{A}\times\mathcal{B}\times\mathcal{C}\times\mathcal{D} \arrow[r, "{\boxtimes_{\mathcal{E}},1,1}"] \arrow[d, "{1,1,\boxtimes_{\mathcal{E}}}"'] & \mathcal{A}\mathcal{B}\times\mathcal{C}\times\mathcal{D} \arrow[r, "{\boxtimes_{\mathcal{E}},1}"] \arrow[d, "{1,\boxtimes_{\mathcal{E}}}"'] & (\mathcal{A}\mathcal{B})\mathcal{C}\times\mathcal{D} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, phantom, "d^{\alpha}_{\mathcal{A}\mathcal{B},\mathcal{C},\mathcal{D}}\Uparrow"] & ((\mathcal{A}\mathcal{B})\mathcal{C})\mathcal{D} \arrow[d, "\alpha_{\mathcal{A}\mathcal{B},\mathcal{C},\mathcal{D}}"] \\
\mathcal{A}\times\mathcal{B}\times\mathcal{C}\mathcal{D} \arrow[r, "{\boxtimes_{\mathcal{E}},1}"'] \arrow[dr, "{1,\boxtimes_{\mathcal{E}}}"'] & \mathcal{A}\mathcal{B}\times\mathcal{C}\mathcal{D} \arrow[rr, "\boxtimes_{\mathcal{E}}"] & {} \arrow[d, phantom, "d^{\alpha}_{\mathcal{A},\mathcal{B},\mathcal{C}\mathcal{D}}\Uparrow"] & (\mathcal{A}\mathcal{B})(\mathcal{C}\mathcal{D}) \arrow[d, "\alpha_{\mathcal{A},\mathcal{B},\mathcal{C}\mathcal{D}}"] \\
& \mathcal{A}\times\mathcal{B}(\mathcal{C}\mathcal{D}) \arrow[rr, "\boxtimes_{\mathcal{E}}"'] & {} & \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))
\end{tikzcd}
$$

v The middle 2-unitor μ is a modification in $\mathrm{Cat}^{\mathrm{ps}}((\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})^2, \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. For each $(\mathcal{B}, \mathcal{A}) \in (\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})^2$, μ consists of an invertible 2-morphism $\mu_{\mathcal{B},\mathcal{A}} : (1_{\mathcal{B}} \boxtimes_{\mathcal{E}} l_{\mathcal{A}}) \circ \alpha_{\mathcal{B},\mathcal{E},\mathcal{A}} \Rightarrow 1_{\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A}} \circ (r_{\mathcal{B}} \boxtimes_{\mathcal{E}} 1_{\mathcal{A}})$ induced by

$$
\begin{tikzcd}
\mathcal{B}\times\mathcal{E}\times\mathcal{A} \arrow[r, "{\boxtimes_{\mathcal{E}},1}"] \arrow[d, "{1,\boxtimes_{\mathcal{E}}}"'] \arrow[dd, bend right=60, "{1_{\mathcal{B}},\odot}"'] & \mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{E}\times\mathcal{A} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, phantom, "\overset{d^{\alpha}_{\mathcal{B}\mathcal{E}\mathcal{A}}}{\Rightarrow}"] & (\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{E})\boxtimes_{\mathcal{E}}\mathcal{A} \arrow[dl, "\alpha_{\mathcal{B},\mathcal{E},\mathcal{A}}"'] \arrow[d, "r_{\mathcal{B}}\boxtimes_{\mathcal{E}}1_{\mathcal{A}}"] \\
\mathcal{B}\times\mathcal{E}\boxtimes_{\mathcal{E}}\mathcal{A} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, "{1,l_{\mathcal{A}}}"] \arrow[d, phantom, shift right=8ex, "\overset{1_{1_{\mathcal{B}}},d^l_{\mathcal{A}}}{\Rightarrow}"] \arrow[dr, phantom, "\overset{t_{1_{\mathcal{B}},l_{\mathcal{A}}}}{\Rightarrow}"] & \mathcal{B}\boxtimes_{\mathcal{E}}(\mathcal{E}\boxtimes_{\mathcal{E}}\mathcal{A}) \arrow[dr, "1_{\mathcal{B}}\boxtimes_{\mathcal{E}}l_{\mathcal{A}}"'] \arrow[r, phantom, "\overset{\exists!\mu_{\mathcal{B},\mathcal{A}}}{\Rightarrow}"] & \mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A} \arrow[d, "1_{\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A}}"] \\
\mathcal{B}\times\mathcal{A} \arrow[rr, "\boxtimes_{\mathcal{E}}"'] & & \mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A}
\end{tikzcd}
$$

$$\|$$

$$\begin{tikzcd}
\mathcal{B}\times\mathcal{E}\times\mathcal{A} \arrow[rr, "\boxtimes_{\mathcal{E}},1"] \arrow[ddd, "1_{\mathcal{B}},\odot"'] \arrow[drr, "\odot{,}1_{\mathcal{A}}"'] & \arrow[d, phantom, "\overset{d^r_{\mathcal{B}},1_{1_{\mathcal{A}}}}{\Rightarrow}"] & \mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{E}\times\mathcal{A} \arrow[rr, "\boxtimes_{\mathcal{E}}"] \arrow[d, "r_{\mathcal{B}},1_{\mathcal{A}}"] & \arrow[d, phantom, "\overset{t_{r_{\mathcal{B}},1_{\mathcal{A}}}}{\Rightarrow}"] & (\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{E})\boxtimes_{\mathcal{E}}\mathcal{A} \arrow[d, "r_{\mathcal{B}}\boxtimes_{\mathcal{E}}1_{\mathcal{A}}"] \\
 & & \mathcal{B}\times\mathcal{A} \arrow[rr, "\boxtimes_{\mathcal{E}}"'] \arrow[ddrr, "\boxtimes_{\mathcal{E}}"'] & & \mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A} \arrow[dd, "1_{\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A}}"] \\
 & \arrow[r, phantom, "\overset{b^{\mathcal{B}\mathcal{A}}}{\Rightarrow}"] & & \arrow[r, phantom, "\overset{\mathrm{id}}{\Rightarrow}"] & \\
\mathcal{B}\times\mathcal{A} \arrow[rrrr, "\boxtimes_{\mathcal{E}}"'] & & & & \mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A}
\end{tikzcd}$$

where $b^{\mathcal{B}\mathcal{A}}$ is the balanced $\mathcal{E}$-module structure on the functor $\boxtimes_{\mathcal{E}}:\mathcal{B}\times\mathcal{A}\to\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A}$.

vi The left 2-unitor λ is a modification in $\mathrm{Cat}^{\mathrm{ps}}((\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})^2,\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. For each $(\mathcal{B},\mathcal{A})\in(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})^2$, λ consists of an invertible 2-morphism $\lambda_{\mathcal{B}\mathcal{A}}: l_{\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A}}\circ\alpha_{\mathcal{E},\mathcal{B},\mathcal{A}}\Rightarrow l_{\mathcal{B}}\boxtimes_{\mathcal{E}}1_{\mathcal{A}}$ induced by

$$\begin{tikzcd}
\mathcal{E}\times\mathcal{B}\times\mathcal{A} \arrow[r, "\boxtimes_{\mathcal{E}},1"] \arrow[dr, "1{,}\boxtimes_{\mathcal{E}}"'] \arrow[ddr, bend right, "\odot{,}1"'] & \mathcal{E}\boxtimes_{\mathcal{E}}\mathcal{B}\times\mathcal{A} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, phantom, "d^{\alpha}_{\mathcal{E}\mathcal{B}\mathcal{A}}\Nearrow"] & (\mathcal{E}\boxtimes_{\mathcal{E}}\mathcal{B})\boxtimes_{\mathcal{E}}\mathcal{A} \arrow[d, "\alpha_{\mathcal{E},\mathcal{B},\mathcal{A}}"] \arrow[dd, bend left=60, "l_{\mathcal{B}}\boxtimes_{\mathcal{E}}1_{\mathcal{A}}"] \\
 & \mathcal{E}\times\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[dr, "\odot"] \arrow[d, phantom, "\overset{t^l_{\mathcal{B},\mathcal{A}}}{\Rightarrow}"] & \mathcal{E}\boxtimes_{\mathcal{E}}(\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A}) \arrow[d, "l_{\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A}}"] \arrow[d, phantom, "\overset{d^l_{\mathcal{B},\mathcal{A}}}{\Rightarrow}"', shift right=6] \arrow[d, phantom, "\overset{\exists!\lambda_{\mathcal{B},\mathcal{A}}}{\Rightarrow}", shift left=8] \\
 & \mathcal{B}\times\mathcal{A} \arrow[r, "\boxtimes_{\mathcal{E}}"'] & \mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{E}\times\mathcal{B}\times\mathcal{A} \arrow[r, "\boxtimes_{\mathcal{E}},1"] \arrow[dr, bend right, "\odot{,}1"'] \arrow[r, phantom, "d^l_{\mathcal{B}},1_{1_{\mathcal{A}}}\Nearrow"', shift right=20] & \mathcal{E}\boxtimes_{\mathcal{E}}\mathcal{B}\times\mathcal{A} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, "l_{\mathcal{B}},1_{\mathcal{A}}"] & (\mathcal{E}\boxtimes_{\mathcal{E}}\mathcal{B})\boxtimes_{\mathcal{E}}\mathcal{A} \arrow[d, "l_{\mathcal{B}}\boxtimes_{\mathcal{E}}1_{\mathcal{A}}"] \\
 & \mathcal{B}\times\mathcal{A} \arrow[r, "\boxtimes_{\mathcal{E}}"'] \arrow[ur, phantom, "\overset{t_{l_{\mathcal{B}},1_{\mathcal{A}}}}{\Rightarrow}"] & \mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A}
\end{tikzcd}$$

vii The right 2-unitor ρ is a modification in $\mathrm{Cat}^{\mathrm{ps}}((\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})^2,\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})$. For each $(\mathcal{B},\mathcal{A})$ in $(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}})^2$, ρ consists of an invertible 2-morphism $\rho_{\mathcal{B},\mathcal{A}}:(1_{\mathcal{B}}\boxtimes_{\mathcal{E}}r_{\mathcal{A}})\circ\alpha_{\mathcal{B},\mathcal{A},\mathcal{E}}\Rightarrow r_{\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A}}$ induced by

$$\begin{tikzcd}
\mathcal{B}\times\mathcal{A}\times\mathcal{E} \arrow[r, "\boxtimes_{\mathcal{E}},1"] \arrow[dr, "1{,}\boxtimes_{\mathcal{E}}"'] \arrow[ddr, bend right, "1{,}\odot"'] & \mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A}\times\mathcal{E} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, phantom, "d^{\alpha}_{\mathcal{B}\mathcal{A}\mathcal{E}}\Nearrow"] & (\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A})\boxtimes_{\mathcal{E}}\mathcal{E} \arrow[d, "\alpha_{\mathcal{B},\mathcal{A},\mathcal{E}}"] \arrow[dd, bend left=60, "r_{\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A}}"] \\
 & \mathcal{B}\times\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{E} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, "1{,}r_{\mathcal{A}}"] \arrow[d, phantom, "\overset{1_{1_{\mathcal{B}}},d^r_{\mathcal{A}}}{\Rightarrow}"', shift right=10] \arrow[r, phantom, "\overset{t_{1_{\mathcal{B}},r_{\mathcal{A}}}}{\Rightarrow}"', shift right=16] & \mathcal{B}\boxtimes_{\mathcal{E}}(\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{E}) \arrow[d, "1_{\mathcal{B}}\boxtimes_{\mathcal{E}}r_{\mathcal{A}}"] \arrow[d, phantom, "\overset{\exists!\rho_{\mathcal{B},\mathcal{A}}}{\Rightarrow}", shift left=8] \\
 & \mathcal{B}\times\mathcal{A} \arrow[r, "\boxtimes_{\mathcal{E}}"'] & \mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{B}\times\mathcal{A}\times\mathcal{E} \arrow[r, "\boxtimes_{\mathcal{E}},1"] \arrow[d, "1{,}\odot"'] & \mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A}\times\mathcal{E} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[dr, "\odot"'] \arrow[d, phantom, "\overset{t^r_{\mathcal{B},\mathcal{A}}}{\Rightarrow}"] & (\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A})\boxtimes_{\mathcal{E}}\mathcal{E} \arrow[d, "r_{\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A}}"] \arrow[d, phantom, "\overset{d^r_{\mathcal{B},\mathcal{A}}}{\Rightarrow}"', shift right=10] \\
\mathcal{B}\times\mathcal{A} \arrow[rr, "\boxtimes_{\mathcal{E}}"'] & & \mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A}
\end{tikzcd}$$

It is routine to check that α, l, r satisfy the lax unity and the lax naturality of [JY, Def. 4.2.1], and π, μ, λ, ρ satisfy the modification axiom of [JY, Def. 4.4.1]. It is routine to check that the above data satisfy the non-abelian 4-cocycle condition, the left normalization and the right normalization of [JY, (11.2.14), (11.2.16), (11.2.17)]. □

A.3 The symmetric monoidal 2-category $\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$

Let $(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}},\boxtimes_{\mathcal{E}},\mathcal{E},\alpha,l,r,\pi,\mu,\lambda,\rho)$ be the monoidal 2-category. For objects $\mathcal{A},\mathcal{B}\in\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$, the braiding τ consists of an invertible 1-morphism $\tau_{\mathcal{A},\mathcal{B}}:\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}\to\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A}$ defined as

$$\begin{tikzcd}
\mathcal{A}\times\mathcal{B} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, "s_{\mathcal{A},\mathcal{B}}"'] \arrow[dr, phantom, "d^{\tau}_{\mathcal{A},\mathcal{B}}\Nearrow"] & \mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B} \arrow[d, "\exists!\tau_{\mathcal{A}\mathcal{B}}"] \\
\mathcal{B}\times\mathcal{A} \arrow[r, "\boxtimes_{\mathcal{E}}"'] & \mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A}
\end{tikzcd}$$

where s switches the two objects. For objects $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$, the left hexagonator $R_{-|--}$ and the right hexagonator $R_{--|-}$ consist of invertible 2-morphisms $R_{\mathcal{A}|\mathcal{B},\mathcal{C}}$ and $R_{\mathcal{A},\mathcal{B}|\mathcal{C}}$ respectively.

$$
\begin{tikzcd}
\mathcal{A}\times\mathcal{B}\times\mathcal{C} \arrow[r, "\boxtimes_{\mathcal{E}},1"] \arrow[dr, "1,\boxtimes_{\mathcal{E}}"] \arrow[dd, "s_{\mathcal{A},\mathcal{B}\times\mathcal{C}}"'] & \mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}\times\mathcal{C} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, phantom, "d^{\alpha}_{\mathcal{A},\mathcal{B},\mathcal{C}}\Nearrow"] & (\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B})\boxtimes_{\mathcal{E}}\mathcal{C} \arrow[d, "\alpha_{\mathcal{A},\mathcal{B},\mathcal{C}}"'] \arrow[dr, "\tau_{\mathcal{A},\mathcal{B}},1"] & \\
 & \mathcal{A}\times\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{C} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, "\tau_{\mathcal{A},\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{C}}"'] \arrow[r, phantom, shift right=6, "d^{\tau}_{\mathcal{A},\mathcal{B}\mathcal{C}}\Nearrow"] & \mathcal{A}\boxtimes_{\mathcal{E}}(\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{C}) \arrow[d, "\tau_{\mathcal{A},\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{C}}"'] \arrow[r, phantom, "\overset{\exists ! R_{\mathcal{A}|\mathcal{B},\mathcal{C}}}{\Longrightarrow}"] & (\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A})\boxtimes_{\mathcal{E}}\mathcal{C} \arrow[d, "\alpha_{\mathcal{B},\mathcal{A},\mathcal{C}}"] \\
\mathcal{B}\times\mathcal{C}\times\mathcal{A} \arrow[r, "\boxtimes_{\mathcal{E}},1"] \arrow[dr, "1,\boxtimes_{\mathcal{E}}"'] & \mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{C}\times\mathcal{A} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, phantom, "d^{\alpha}_{\mathcal{B},\mathcal{C},\mathcal{A}}\Nearrow"] & (\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{C})\boxtimes_{\mathcal{E}}\mathcal{A} \arrow[d, "\alpha_{\mathcal{B},\mathcal{C},\mathcal{A}}"'] & \mathcal{B}\boxtimes_{\mathcal{E}}(\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{C}) \arrow[dl, "1,\tau_{\mathcal{A},\mathcal{C}}"] \\
 & \mathcal{B}\times\mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{A} \arrow[r, "\boxtimes_{\mathcal{E}}"'] & \mathcal{B}\boxtimes_{\mathcal{E}}(\mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{A}) &
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}
\mathcal{A}\times\mathcal{B}\times\mathcal{C} \arrow[r, "\boxtimes_{\mathcal{E}},1"] \arrow[d, "s_{\mathcal{A},\mathcal{B}},1"'] \arrow[dd, bend right=60, "s_{\mathcal{A},\mathcal{B}\times\mathcal{C}}"'] \arrow[dr, phantom, "d^{\tau}_{\mathcal{A}\mathcal{B}},1\Nearrow"] & \mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}\times\mathcal{C} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, "\tau_{\mathcal{A}\mathcal{B}},1"] \arrow[dr, phantom, "t_{\tau,1}\Nearrow"] & (\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B})\boxtimes_{\mathcal{E}}\mathcal{C} \arrow[d, "\tau_{\mathcal{A},\mathcal{B}},1"] \\
\mathcal{B}\times\mathcal{A}\times\mathcal{C} \arrow[r, "\boxtimes_{\mathcal{E}},1"] \arrow[d, "1,s_{\mathcal{A},\mathcal{C}}"'] \arrow[dr, "1,\boxtimes_{\mathcal{E}}"] & \mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A}\times\mathcal{C} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[dr, phantom, "d^{\alpha}_{\mathcal{B},\mathcal{A},\mathcal{C}}\Nearrow"] & (\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A})\boxtimes_{\mathcal{E}}\mathcal{C} \arrow[d, "\alpha_{\mathcal{B},\mathcal{A},\mathcal{C}}"] \\
\mathcal{B}\times\mathcal{C}\times\mathcal{A} \arrow[r, phantom, "\overset{1,d^{\tau}_{\mathcal{A},\mathcal{C}}}{\Longrightarrow}"] \arrow[dr, "1,\boxtimes_{\mathcal{E}}"'] & \mathcal{B}\times\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{C} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, "1,\tau_{\mathcal{A},\mathcal{C}}"] \arrow[dr, phantom, "t_{1,\tau}\Nearrow"] & \mathcal{B}\boxtimes_{\mathcal{E}}(\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{C}) \arrow[d, "1,\tau_{\mathcal{A},\mathcal{C}}"] \\
 & \mathcal{B}\times\mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{A} \arrow[r, "\boxtimes_{\mathcal{E}}"'] & \mathcal{B}\boxtimes_{\mathcal{E}}(\mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{A})
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{A}\times\mathcal{B}\times\mathcal{C} \arrow[r, "1,\boxtimes_{\mathcal{E}}"] \arrow[dr, "\boxtimes_{\mathcal{E}},1"'] \arrow[dd, "s_{\mathcal{A}\times\mathcal{B},\mathcal{C}}"'] & \mathcal{A}\times\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{C} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, phantom, "d^{\alpha^{-1}}_{\mathcal{A},\mathcal{B},\mathcal{C}}\Nearrow"] & \mathcal{A}\boxtimes_{\mathcal{E}}(\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{C}) \arrow[d, "\alpha^{-1}_{\mathcal{A},\mathcal{B},\mathcal{C}}"] \arrow[dr, "1,\tau_{\mathcal{B},\mathcal{C}}"] & \\
 & \mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}\times\mathcal{C} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, "s_{\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B},\mathcal{C}}"'] \arrow[r, phantom, shift right=6, "d^{\tau}_{\mathcal{A}\mathcal{B},\mathcal{C}}\Nearrow"] & (\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B})\boxtimes_{\mathcal{E}}\mathcal{C} \arrow[d, "\tau_{\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B},\mathcal{C}}"'] \arrow[r, phantom, "\overset{\exists ! R_{\mathcal{A},\mathcal{B}|\mathcal{C}}}{\Longrightarrow}"] & \mathcal{A}\boxtimes_{\mathcal{E}}(\mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{B}) \arrow[d, "\alpha^{-1}_{\mathcal{A},\mathcal{C},\mathcal{B}}"] \\
\mathcal{C}\times\mathcal{A}\times\mathcal{B} \arrow[r, "1,\boxtimes_{\mathcal{E}}"] \arrow[dr, "\boxtimes_{\mathcal{E}},1"'] & \mathcal{C}\times\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, phantom, "d^{\alpha^{-1}}_{\mathcal{C},\mathcal{A},\mathcal{B}}\Nearrow"] & \mathcal{C}\boxtimes_{\mathcal{E}}(\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}) \arrow[d, "\alpha^{-1}_{\mathcal{C},\mathcal{A},\mathcal{B}}"] & (\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{C})\boxtimes_{\mathcal{E}}\mathcal{B} \arrow[dl, "\tau_{\mathcal{A},\mathcal{C}},1"] \\
 & \mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{A}\times\mathcal{B} \arrow[r, "\boxtimes_{\mathcal{E}}"'] & (\mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{A})\boxtimes_{\mathcal{E}}\mathcal{B} &
\end{tikzcd}
$$

$$\|$$

$$\begin{tikzcd}
\mathcal{A}\times\mathcal{B}\times\mathcal{C} \arrow[r, "1{,}\boxtimes_{\mathcal{E}}"] \arrow[d, "1{,}s_{\mathcal{B},\mathcal{C}}"'] \arrow[dd, bend right=60, "s_{\mathcal{A}\times\mathcal{B},\mathcal{C}}"'] \arrow[dr, phantom, "1{,}d^{\tau}_{\mathcal{B},\mathcal{C}}\Nearrow"] & \mathcal{A}\times\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{C} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, "1{,}\tau_{\mathcal{B},\mathcal{C}}"] \arrow[dr, phantom, "t_{1,\tau}\Nearrow"] & \mathcal{A}\boxtimes_{\mathcal{E}}(\mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{C}) \arrow[d, "1{,}\tau_{\mathcal{B},\mathcal{C}}"] \\
\mathcal{A}\times\mathcal{C}\times\mathcal{B} \arrow[r, "1{,}\boxtimes_{\mathcal{E}}"] \arrow[d, "s_{\mathcal{A},\mathcal{C}}{,}1"'] \arrow[dr, "\boxtimes_{\mathcal{E}}{,}1"] & \mathcal{A}\times\mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{B} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[dr, phantom, "d^{\alpha^{-1}}_{\mathcal{A},\mathcal{C},\mathcal{B}}\Nearrow"] & \mathcal{A}\boxtimes_{\mathcal{E}}(\mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{B}) \arrow[d, "\alpha^{-1}_{\mathcal{A},\mathcal{C},\mathcal{B}}"] \\
\mathcal{C}\times\mathcal{A}\times\mathcal{B} \arrow[r, phantom, "\overset{d^{\tau}_{\mathcal{A},\mathcal{C}}{,}1}{\Rightarrow}"] \arrow[dr, "\boxtimes_{\mathcal{E}}{,}1"'] & \mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{C}\times\mathcal{B} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, "\tau_{\mathcal{A},\mathcal{C}}{,}1"] \arrow[dr, phantom, "t_{\tau,1}\Nearrow"] & (\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{C})\boxtimes_{\mathcal{E}}\mathcal{B} \arrow[d, "\tau_{\mathcal{A},\mathcal{C}}{,}1"] \\
 & \mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{A}\times\mathcal{B} \arrow[r, "\boxtimes_{\mathcal{E}}"'] & (\mathcal{C}\boxtimes_{\mathcal{E}}\mathcal{A})\boxtimes_{\mathcal{E}}\mathcal{B}
\end{tikzcd}$$

For objects $\mathcal{A},\mathcal{B}\in\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}}$, the syllepsis ν consists of an invertible 2-morphism $\nu_{\mathcal{A},\mathcal{B}}$ defined as

$$\begin{tikzcd}
\mathcal{A}\times\mathcal{B} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, "s_{\mathcal{A},\mathcal{B}}"'] \arrow[dr, phantom, "d^{\tau}_{\mathcal{A},\mathcal{B}}\Nearrow"] & \mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B} \arrow[d, "\tau_{\mathcal{A},\mathcal{B}}"] \arrow[dd, bend left=60, "1_{\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}}"] \\
\mathcal{B}\times\mathcal{A} \arrow[r, "\boxtimes_{\mathcal{E}}"'] \arrow[d, "s_{\mathcal{B},\mathcal{A}}"'] \arrow[dr, phantom, "d^{\tau}_{\mathcal{B},\mathcal{A}}\Nearrow"] & \mathcal{B}\boxtimes_{\mathcal{E}}\mathcal{A} \arrow[d, "\tau_{\mathcal{B},\mathcal{A}}"] \arrow[r, phantom, "\overset{\exists!\nu_{\mathcal{A},\mathcal{B}}}{\Rightarrow}"] & {} \\
\mathcal{A}\times\mathcal{B} \arrow[r, "\boxtimes_{\mathcal{E}}"'] & \mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{A}\times\mathcal{B} \arrow[r, "\boxtimes_{\mathcal{E}}"] \arrow[d, "s_{\mathcal{A},\mathcal{B}}"'] \arrow[dd, bend left=60, "1_{\mathcal{A}\times\mathcal{B}}"] & \mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B} \arrow[dd, "1_{\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}}"] \\
\mathcal{B}\times\mathcal{A} \arrow[d, "s_{\mathcal{B},\mathcal{A}}"'] & \\
\mathcal{A}\times\mathcal{B} \arrow[r, "\boxtimes_{\mathcal{E}}"'] & \mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}
\end{tikzcd}$$

where we choose the identity 2-morphism $\mathrm{id}:\boxtimes_{\mathcal{E}}\circ 1_{\mathcal{A}\times\mathcal{B}}\Rightarrow 1_{\mathcal{A}\boxtimes_{\mathcal{E}}\mathcal{B}}\circ\boxtimes_{\mathcal{E}}$ for convenience. It is routine to check that $(\mathrm{Cat}^{\mathrm{fs}}_{\mathcal{E}},\tau,R_{-|--},R_{--|-},\nu)$ is a symmetric monoidal 2-category [JY, Def. 12.1.6, 12.1.15, 12.1.19].

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