HECKE'S THEOREM ON THE DIFFERENT FOR 3-MANIFOLDS

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ABSTRACT. Hecke has shown that the different of an extension of number fields is a square in the class group. We prove an analog for branched covers of closed 3-manifolds saying that the branch divisor is a square in the first homology group.

1. Introduction

Let E/F be an extension of number fields, let \mathcal{O}_E be the ring of integers of E, and let $\mathrm{Cl}(\mathcal{O}_E)$ be the class group of \mathcal{O}_E . One associates to the extension E/F the different $\mathcal{D}_{E/F}$, an ideal in \mathcal{O}_E , see [Ser79, Chapter 3]. Hecke has shown that as an element of $\mathrm{Cl}(\mathcal{O}_E)$, the different $\mathcal{D}_{E/F}$ is a square, namely there exists an ideal class $J \in \mathrm{Cl}(\mathcal{O}_E)$ such that $J^2 = \mathcal{D}_{E/F}$ in $\mathrm{Cl}(\mathcal{O}_E)$. Hecke's proof uses a reciprocity formula for Gauss sums, see [Arm67] and [Fro78] for a proof and a discussion of related results.

An analog of Hecke's theorem for finite separable extensions of fields of fractions of Dedekind domains fails in general, see [FST62]. However, there exists an analog in case E/F is a finite separable extension of function fields of curves over finite fields of odd characteristic, see [Arm67]. Another geometric analog of Hecke's theorem, based on similarities between the inverse of the different and the canonical bundle on a curve, is the theory of theta characteristics.

In this work we consider an analog of Hecke's theorem for 3-manifolds, as suggested by arithmetic topology. We refer to [Mor12] for the analogy between rings of integers and primes on the one hand, and 3-manifolds and knots on the other hand. The analog of $\operatorname{Spec}(\mathcal{O}_F)$ is a closed (not necessarily oriented) 3-manifold M. The map $\operatorname{Spec}(\mathcal{O}_E) \to \operatorname{Spec}(\mathcal{O}_F)$ is replaced by a cover $\pi \colon \widetilde{M} \to M$ branched over a link $L \subset M$, so \widetilde{M} is a closed 3-manifold and $\pi^{-1}(M \setminus L)$ is a covering space of $M \setminus L$. The inverse image of L under π is a link \widetilde{L} in \widetilde{M} .

For a prime ideal \mathfrak{p} of \mathcal{O}_E we denote by $e_{\mathfrak{p}}$ its ramification index, namely the largest positive integer e for which \mathfrak{p}^e contains $\mathfrak{p} \cap \mathcal{O}_F$. We view $\operatorname{Spec}(\mathcal{O}_E) \to \operatorname{Spec}(\mathcal{O}_F)$ as branched over the primes of \mathcal{O}_E that ramify, so \widetilde{L} is our analog for $\mathcal{R}_{E/F} = \{\mathfrak{p} \in \operatorname{Spec}(\mathcal{O}_E) : e_{\mathfrak{p}} > 1\}$. The analogy is perhaps closest in case $\operatorname{Spec}(\mathcal{O}_E) \to \operatorname{Spec}(\mathcal{O}_F)$ is tamely ramified, namely $e_{\mathfrak{p}}$ is coprime to $|\mathcal{O}_E/\mathfrak{p}|$ for every $\mathfrak{p} \in \operatorname{Spec}(\mathcal{O}_E)$. In this case the different of E/F is given by

$$\mathcal{D}_{E/F} = \prod_{\mathfrak{p} \in \mathcal{R}_{E/F}} \mathfrak{p}^{e_{\mathfrak{p}} - 1}.$$

The prime ideals in $\mathcal{R}_{E/F}$ are analogous to the components of the link \widetilde{L} . For each component \widetilde{K} of this link, let the ramification index $e_{\widetilde{K}}$ be the number of times the image under π of a small loop around \widetilde{K} wraps around $\pi(\widetilde{K})$. An analog of $\text{Cl}(\mathcal{O}_E)$ is $H_1(\widetilde{M}, \mathbb{Z})$, and a homology class is a square if and only if its image in

$$H_1(\widetilde{M}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong H_1(\widetilde{M}, \mathbb{Z}/2\mathbb{Z})$$

vanishes. Our analogy of $\mathcal{D}_{E/F}$, or rather of its class in $\text{Cl}(\mathcal{O}_E)/\text{Cl}(\mathcal{O}_E)^2$, is the branch divisor

$$\mathcal{D}_{\pi} = \sum_{\widetilde{K} \text{ a component of } \widetilde{L}} (e_{\widetilde{K}} - 1)[\widetilde{K}] \in H_1(\widetilde{M}, \mathbb{Z}/2\mathbb{Z})$$

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of π . Since we are working with $\mathbb{Z}/2\mathbb{Z}$ -coefficients, it is not necessary to fix an orientation of \widetilde{K} , nor is the sign of $e_{\widetilde{K}}$ significant.

Theorem 1.1. Let \widetilde{M} and M be closed 3-manifolds, and let $\pi \colon \widetilde{M} \to M$ be a cover branched over a link in M. Then the branch divisor \mathcal{D}_{π} represents the trivial class in $H_1(\widetilde{M}, \mathbb{Z}/2\mathbb{Z})$.

2. A CENTRAL EXTENSION OF THE HYPEROCTAHEDRAL GROUP

Let n be a positive integer, and let S_n be the symmetric group. Recall the hyperoctahedral group

$$B_n = (\mathbb{Z}/2\mathbb{Z})^n \times S_n$$

where S_n acts on $(\mathbb{Z}/2\mathbb{Z})^n$ by permuting the coordinates.

Let H_n be the group consisting of pairs $(a,b) \in (\mathbb{Z}/2\mathbb{Z})^n \times \mathbb{Z}/2\mathbb{Z}$ with group law

$$(a_1, b_1)(a_2, b_2) = (a_1 + a_2, b_1 + b_2 + \sum_{1 \le i < j \le n} a_{1,i} a_{2,j}).$$

A straightforward computation shows that this law is associative, and that the inverse of (a, b) is

$$(a,b+\sum_{1\leq i< j\leq n}a_ia_j).$$

Projection onto the first factor exhibits H_n as a central extension of $(\mathbb{Z}/2\mathbb{Z})^n$ by $\mathbb{Z}/2\mathbb{Z}$.

For $1 \le i \le n$ we donte by e_i the *i*th unit vector in $(\mathbb{Z}/2\mathbb{Z})^n$, set $x_i = (e_i, 0) \in H_n$, and $\epsilon = (0, 1) \in H_n$. We denote the unit element $(0, 0) \in H_n$ by 1. We can check that

(2.1)
$$x_i^2 = \epsilon^2 = 1, \quad 1 \le i \le n,$$

that

$$(2.2) x_i x_j = \epsilon x_j x_i, \quad 1 \le i, j \le n, \ i \ne j,$$

and that

$$\epsilon x_i = x_i \epsilon, \quad 1 \le i \le n.$$

Furthermore, the relations in Eq. (2.1), Eq. (2.2), and Eq. (2.3) among the generators $x_1, \ldots, x_n, \epsilon$ define the group H_n since using these relations every word in $x_1, \ldots, x_n, \epsilon$ can be brought to the form $x_{i_1} \cdots x_{i_k} \epsilon^{\delta}$ with $1 \le i_1 < i_2 < \cdots < i_k \le n$ and $\delta \in \{0, 1\}$.

We therefore have an action of S_n on H_n by automorphisms via

$$\sigma(x_i) = x_{\sigma(i)}, \ \sigma(\epsilon) = \epsilon, \quad \sigma \in S_n, \ 1 \le i \le n.$$

Let $G_n = H_n \times S_n$ be the semidirect product defined using this action. Since $\epsilon \in H_n$ is central and S_n -invariant, it lies in the center of G_n , so

$$G_n/\langle \epsilon \rangle = G_n/\{1, \epsilon\} \cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n = B_n.$$

We see that G_n is a central extension of B_n by $\mathbb{Z}/2\mathbb{Z}$. We denote by β_n the class in $H^2(B_n, \mathbb{Z}/2\mathbb{Z})$ corresponding to this extension.

Let $\sigma, \tau \in B_n$ be two elements that commute, let $\widetilde{\sigma}, \widetilde{\tau}$ be lifts to G_n , and define

$$\phi(\sigma,\tau) = [\widetilde{\sigma},\widetilde{\tau}] = \widetilde{\sigma}\widetilde{\tau}\widetilde{\sigma}^{-1}\widetilde{\tau}^{-1} \in \langle \epsilon \rangle \cong \mathbb{Z}/2\mathbb{Z}.$$

Since G_n is a central extension of B_n , the above is indeed independent of the choice of lifts. As every element in $\mathbb{Z}/2\mathbb{Z}$ is its own inverse, we see that

(2.4)
$$\phi(\sigma,\tau) = [\widetilde{\sigma},\widetilde{\tau}] = [\widetilde{\tau},\widetilde{\sigma}]^{-1} = [\widetilde{\tau},\widetilde{\sigma}] = \phi(\tau,\sigma).$$

We denote by

$$C_{B_n}(\sigma) = \{ \tau \in B_n : \sigma\tau = \tau\sigma \}$$

the centralizer of σ in B_n .

Proposition 2.1. For every $\sigma \in B_n$ the map that sends $\tau \in C_{B_n}(\sigma)$ to $\phi(\sigma, \tau)$ is a homomorphism.

Proof. For every $\tau_1, \tau_2 \in C_{B_n}(\sigma)$ we have

$$\phi(\sigma, \tau_1 \tau_2) = \widetilde{\sigma} \widetilde{\tau}_1 \widetilde{\tau}_2 \widetilde{\sigma}^{-1} \widetilde{\tau}_2^{-1} \widetilde{\tau}_1^{-1}, \quad \phi(\sigma, \tau_1) \phi(\sigma, \tau_2) = \widetilde{\sigma} \widetilde{\tau}_1 \widetilde{\sigma}^{-1} \widetilde{\tau}_1^{-1} [\widetilde{\sigma}, \widetilde{\tau}_2]$$

so after cancelling $\widetilde{\sigma}\widetilde{\tau}_1$, it remains to check that

$$\widetilde{\tau}_2\widetilde{\sigma}^{-1}\widetilde{\tau}_2^{-1}\widetilde{\tau}_1^{-1} = \widetilde{\sigma}^{-1}\widetilde{\tau}_1^{-1}[\widetilde{\sigma},\widetilde{\tau}_2].$$

After multiplying by $\widetilde{\sigma}$ from the left, we just need to check that $[\widetilde{\sigma}, \widetilde{\tau_2}]$ commutes with $\widetilde{\tau_1}$. This is indeed the case because $[\widetilde{\sigma}, \widetilde{\tau_2}]$ lies in the central subgroup $\{1, \epsilon\}$ of G_n .

Corollary 2.2. For every $\tau \in B_n$ the map that sends $\sigma \in C_{B_n}(\tau)$ to $\phi(\sigma, \tau)$ is a homomorphism.

Proof. For $\sigma_1, \sigma_2 \in C_{B_n}(\tau)$ we get from Eq. (2.4) and Proposition 2.1 that

$$\phi(\sigma_1\sigma_2,\tau) = \phi(\tau,\sigma_1\sigma_2) = \phi(\tau,\sigma_1)\phi(\tau,\sigma_2) = \phi(\sigma_1,\tau)\phi(\sigma_2,\tau).$$

as required.

Proposition 2.3. For a k-cycle $\sigma = (i_1 \dots i_k) \in S_n \leq B_n$, and

$$\tau = e_{i_1} + \dots + e_{i_k} \in (\mathbb{Z}/2\mathbb{Z})^n \le B_n$$

we have $\phi(\sigma,\tau) = \epsilon^{k-1}$. For every $\alpha \in S_n \leq B_n$ with $\alpha(i_1) = i_1, \ldots, \alpha(i_k) = i_k$ we have $\phi(\alpha,\tau) = 1$.

Proof. We take $\widetilde{\sigma} = (i_1 \dots i_k), \ \widetilde{\tau} = x_{i_1} \dots x_{i_k}$ and get that

$$\phi(\sigma,\tau) = \widetilde{\sigma}\widetilde{\tau}\widetilde{\sigma}^{-1} \cdot \widetilde{\tau}^{-1} = \sigma(x_{i_1} \cdots x_{i_k}) \cdot (x_{i_1} \cdots x_{i_k})^{-1} = x_{\sigma(i_1)} \cdots x_{\sigma(i_k)} \cdot (x_{i_1} \cdots x_{i_k})^{-1}$$
$$= x_{i_2} \cdots x_{i_k} x_{i_1} \cdot (x_{i_1} \cdots x_{i_k})^{-1} = \epsilon^{k-1} x_{i_1} \cdots x_{i_k} \cdot (x_{i_1} \cdots x_{i_k})^{-1} = \epsilon^{k-1}.$$

Taking $\widetilde{\alpha} = \alpha$ we see that

$$\phi(\alpha,\tau) = \widetilde{\alpha}\widetilde{\tau}\widetilde{\alpha}^{-1} \cdot \widetilde{\tau}^{-1} = \alpha(x_{i_1} \cdots x_{i_k}) \cdot (x_{i_1} \cdots x_{i_k})^{-1} = x_{\alpha(i_1)} \cdots x_{\alpha(i_k)} \cdot (x_{i_1} \cdots x_{i_k})^{-1} = 1.$$
 as claimed. \Box

Corollary 2.4. Let $\sigma \in S_n \leq B_n$ whose disjoint cycles are

$$C_1 = (i_{1,1} \dots i_{1,d_1}), \dots, C_j = (i_{j,1} \dots i_{j,d_j}), \quad \sum_{r=1}^{j} d_r = n,$$

and let $\tau \in C_{B_n}(\sigma)$. Then there exists a (unique) choice of $\tau' \in C_{S_n}(\sigma)$ and $\lambda_1, \ldots, \lambda_j \in \mathbb{Z}/2\mathbb{Z}$ such that

(2.5)
$$\tau = \tau' v, \quad v = \sum_{r=1}^{j} \lambda_r (e_{i_{r,1}} + \dots + e_{i_{r,d_r}})$$

and

$$\phi(\sigma,\tau) = \epsilon^{\sum_{r=1}^{j} \lambda_r (d_r - 1)}.$$

Proof. The ability to express τ as in Eq. (2.5) is immediate from the definition of the group law in B_n . From Proposition 2.1, Corollary 2.2, and Proposition 2.3 we therefore get that

$$\phi(\sigma,\tau) = \phi\left(\sigma,\tau' \cdot \sum_{r=1}^{j} \lambda_r (e_{i_{r,1}} + \dots + e_{i_{r,d_r}})\right) = \phi(\sigma,\tau') \cdot \prod_{r=1}^{j} \phi(\sigma,e_{i_{r,1}} + \dots + e_{i_{r,d_r}})^{\lambda_r}$$
$$= [\sigma,\tau'] \cdot \prod_{r=1}^{j} \prod_{s=1}^{j} \phi(C_s,e_{i_{r,1}} + \dots + e_{i_{r,d_r}})^{\lambda_r} = 1 \cdot \prod_{r=1}^{j} \epsilon^{\lambda_r(d_r-1)} = \epsilon^{\sum_{r=1}^{j} \lambda_r(d_r-1)}$$

as required. \Box

We keep the notation of Corollary 2.4 and denote by O_1, \ldots, O_z the orbits of the action by conjugation of the subgroup of S_n generated by τ' on $\{C_1, \ldots, C_j\}$. For $1 \leq y \leq z$ we let $I_y \subseteq \{1, \ldots, n\}$ be the set of all indices that appear in one of the cycles in O_y , and define the permutation $\tau'_y \in S_n$ by

$$\tau_y'(i) = \begin{cases} \tau'(i) & i \in I_y \\ i & i \notin I_y. \end{cases}$$

We have a disjoint union

$$\bigcup_{y=1}^{z} I_y = \{1, \dots, n\}$$

hence $\tau' = \tau'_1 \cdots \tau'_z$ and the permutations τ'_1, \dots, τ'_z commute. We put

$$\tau_y = \tau'_y v_y, \quad v_y = \sum_{\substack{1 \le r \le j \\ C_r \in O_y}} \lambda_r (e_{i_{r,1}} + \dots + e_{i_{r,d_r}})$$

and get that

$$\tau = \tau_1' v_1 \cdots \tau_z' v_z$$

where the factors $\tau'_1 v_1, \ldots, \tau'_z v_z$ commute.

3. Proof of Theorem 1.1

It suffices to show, for each $\alpha \in H^1(\widetilde{M}, \mathbb{Z}/2\mathbb{Z})$, that the pairing of the branch divisor \mathcal{D}_{π} with α vanishes, namely

$$\sum_{\widetilde{K} \text{ a component of } \widetilde{L}} (e_{\widetilde{K}}-1) \langle [\widetilde{K}], \alpha \rangle = 0$$

or equivalently

$$\sum_{K \text{ a component of } L} \sum_{\widetilde{K} \text{ a component of } \pi^{-1}(K)} (e_{\widetilde{K}} - 1) \langle [\widetilde{K}], \alpha \rangle = 0.$$

Associated to α is a degree two covering space $N \to \widetilde{M}$. Let n be the degree of $\pi \colon \widetilde{M} \to M$ which is locally constant away from L, thus constant. Away from L, we get that N is a degree 2 covering space of a degree n covering space, hence has monodromy group contained in the wreath product

$$S_2 \wr S_n = (\mathbb{Z}/2\mathbb{Z}) \wr S_n = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n = B_n.$$

We thus have a map $H^2(B_n, \mathbb{Z}/2\mathbb{Z}) \to H^2(M \setminus L, \mathbb{Z}/2\mathbb{Z})$, and we denote by $\gamma \in H^2(M \setminus L, \mathbb{Z}/2\mathbb{Z})$ the image of β_n .

Consider a tubular neighborhood Q of L and let $S = \partial Q$ be its boundary, a union of tori. Each such torus T corresponds to a unique component K of L - the boundary of a tubular neighborhood of K is T. Since S bounds a 3-manifold in $M \setminus L$, i.e. the complement of the tubular neighborhood Q, our cohomology class γ integrates to 0 on S. It follows that

$$\sum_{T \text{ a component of } S} \int_{T} \gamma = 0.$$

It is therefore sufficient to prove that

(3.1)
$$\int_{T} \gamma = \sum_{\widetilde{K} \text{ a component of } \pi^{-1}(K)} (e_{\widetilde{K}} - 1) \langle [\widetilde{K}], \alpha \rangle.$$

Since T is a torus, a covering of T with monodromy B_n , i.e. a homomorphism from $\pi_1(T)$ to B_n , is given by a pair of elements $m, \ell \in B_n$ that commute, where m represents a meridian and ℓ represents a longitude. From the standard cell decomposition of the torus, we can see that

$$\int_T \gamma = \phi(m, \ell).$$

Since the $\mathbb{Z}/2\mathbb{Z}$ -covering $N \to \widetilde{M}$ is unbranched over every component \widetilde{K} of $\pi^{-1}(K)$, the monodromy of the meridian m does not swap the two components of the covering, and therefore m is (up to conjugation) contained in $S_n \leq B_n$.

We shall use here the notation of Corollary 2.4 and the paragraph following it for $\sigma = m$ and $\tau = \ell$, in particular we write $\ell = \ell'v$ as in Eq. (2.5). The components of $\pi^{-1}(K)$ are naturally in bijection with the orbits of the action by conjugation of the subgroup of S_n generated by ℓ' on the set of disjoint cycles $\{C_1, \ldots, C_j\}$ of m. We denote by $O_{\widetilde{K}}$ the orbit corresponding to a component \widetilde{K} of $\pi^{-1}(K)$. As in Eq. (2.6), we can write

$$\ell = \prod_{\widetilde{K}} \ell_{\widetilde{K}}, \quad \ell_{\widetilde{K}} = \ell'_{\widetilde{K}} v_{\widetilde{K}}, \quad v_{\widetilde{K}} = \sum_{\substack{1 \le r \le j \\ C_r \in O_{\widetilde{K}}}} \lambda_r (e_{i_{r,1}} + \dots + e_{i_{r,d_r}}).$$

We denote the number of cycles in $O_{\widetilde{K}}$ by $t_{\widetilde{K}}$, note that each such cycle is of length $e_{\widetilde{K}}$, and set

$$d_{\widetilde{K}}=\#\{1\leq r\leq j: C_r\in O_{\widetilde{K}},\ \lambda_r=1\}.$$

It follows from Corollary 2.4 that $\phi(m, \ell_{\widetilde{K}}) \equiv (e_{\widetilde{K}} - 1)d_{\widetilde{K}} \mod 2$, so from Corollary 2.2 we get that

$$\phi(m,\ell) = \sum_{\widetilde{K} \text{ a component of } \pi^{-1}(K)} \phi(m,\ell_{\widetilde{K}}) \equiv \sum_{\widetilde{K} \text{ a component of } \pi^{-1}(K)} (e_{\widetilde{K}} - 1) d_{\widetilde{K}} \mod 2.$$

It is therefore enough to show that $d_{\widetilde{K}} \equiv \langle [\widetilde{K}], \alpha \rangle \mod 2$.

Let C be a longitude curve in a tubular neighborhood of \widetilde{K} . Then $[C] = [\widetilde{K}]$ as homology classes in $H_1(\widetilde{M}, \mathbb{Z}/2\mathbb{Z})$, so it suffices to show that $d_{\widetilde{K}} \equiv \langle [C], \alpha \rangle \mod 2$. The projection of [C] to T is $a[m] + t_{\widetilde{K}}[\ell]$ for some $a \in \mathbb{Z}$. Thus, the action of C on the covering space $N \to M$ is given by $m^a \ell^{t_{\widetilde{K}}}$. We have

$$m^a \ell^{t_{\widetilde{K}}} = m^a (\ell' v)^{t_{\widetilde{K}}} = m^a \ell'^{t_{\widetilde{K}}} \cdot (v + \ell'(v) + \dots + \ell'^{t_{\widetilde{K}}-1}(v)).$$

The pairing $\langle [C], \alpha \rangle$ is nonzero if and only if the monodromy along C of the covering $N \to \widetilde{M}$ is nontrivial, which happens if and only if the action of $m^a \ell^t \tilde{\kappa}$ sends one branch of this covering to the other, and that occurs if and only if the kth entry of $v + \ell'(v) + \cdots + \ell'^t \tilde{\kappa}^{-1}(v)$ is nonzero for some (equivalently, every) index $1 \le k \le n$ that belongs to one of the cycles in $O_{\widetilde{K}}$. It is therefore sufficient to show that

$$d_{\widetilde{K}} \equiv (v + \ell'(v) + \dots + \ell'^{t_{\widetilde{K}}-1}(v))_k \mod 2.$$

We have

$$(v + \ell'(v) + \dots + \ell'^{t_{\tilde{K}}-1}(v))_k = v_k + \ell'(v)_k + \dots + \ell'^{t_{\tilde{K}}-1}(v)_k = v_k + v_{\ell'^{-1}(k)} + \dots + v_{\ell'^{-t_{\tilde{K}}+1}(k)}.$$

By the orbit-stabilizer theorem, each of the $t_{\widetilde{K}}$ cycles in $O_{\widetilde{K}}$ contains exactly one of the $t_{\widetilde{K}}$ elements $k, \ell'^{-1}(k), \ldots, \ell'^{-t_{\widetilde{K}}+1}(k)$. Thus, from Eq. (2.5) we get that

$$v_k + v_{\ell'^{-1}(k)} + \dots + v_{\ell'^{-t}\widetilde{K}^{+1}(k)} = \sum_{\substack{1 \le r \le j \\ C_r \in O_{\widetilde{k'}}}} \lambda_r \equiv d_{\widetilde{K}} \mod 2,$$

as desired.

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