### Averaging of random variables and fields.

### M.R.Formica, E.Ostrovsky, L.Sirota.

Università degli Studi di Napoli Parthenope, via Generale Parisi 13, Palazzo Pacanowsky, 80132, Napoli, Italy. e-mail: mara.formica@uniparthenope.it

Department of Mathematics and Statistics, Bar-Ilan University, 59200, Ramat Gan, Israel. e-mail: eugostrovsky@list.ru

Department of Mathematics and Statistics, Bar-Ilan University, 59200, Ramat Gan, Israel. e-mail: sirota3@bezeqint.net

#### Abstract

We will prove that by averaging of random variables (r.v.) and random fields (r.f.) its tails of distributions do not increase in comparison with the tails of source variables, essentially or almost exact, under very weak conditions.

Key words and phrases. Random variables (r.v.) and random fields (r.f.), spatial and conditional averaging, probability, Lebesgue - Riesz, Yudovich, Grand Lebesgue norm and spaces, Euclidean space, martingales, uniform integrability, tail of distribution, measurable functions, Young - Fenchel (Legendre) transform, Young's inequality, slowly varying function, natural function, normalized sums, filtration, generating function, upper and lover estimation, conditional expectation, rearrangement invariant (r.i.) space, examples, square variation.

### 1 Preliminary. Definitions. Notations.

Let  $(\Omega = \{\omega\}, \mathcal{B}, \mathbf{P})$  be non - trivial probability space with expectation  $\mathbf{E}$  and Variation Var and let  $(X = \{x\}, M, \mu)$  be another probability space:  $\mu(X) = 1$ . Let also  $\xi = \xi(x) = \xi(x, \omega), x \in X, \omega \in \Omega$  be numerical valued separable bi measurable random field.

There are at last two version to introduce the notion of an *averaging* in the probability theory.

A. Define

$$\xi[X] \stackrel{def}{=} \int_X \xi(x) \ \mu(dx) \ - \tag{1}$$

the spatial averaging.

**B.** Conditional averaging. Let F be some sigma - subalgebra of the source one  $\mathcal{B}$ . Define for arbitrary integrable numerical valued r.v.  $\eta$  the ordinary conditional expectation

$$\eta\{F\} \stackrel{def}{=} \mathbf{E}\eta/F.$$
 (2)

Our target in this preprint is to deduce the exact norm and tail of distribution estimations for both this averaging in the terms of tails and norms of the source datum.

We state as it is announced that by both the introduced averaging: of random variables (r.v.) and random fields (r.f.) its tails of distributions do not essentially exact increase in comparison with the tails of source variables, under very weak conditions.

Recall that the so - called *tail function*  $T_{\tau}(t)$  for the numerical valued r.v.  $\tau$  is defined as follows

$$T_{\tau}(t) \stackrel{\text{def}}{=} \mathbf{P}(|\tau| \ge t), \ t \ge 0.$$
(3)

Correspondingly, the tail function for the numerical valued random field  $\zeta = \zeta(x) = \zeta(x, \omega)$  is understood in the uniform sense

$$T_{\zeta}(t) \stackrel{def}{=} \sup_{x \in X} T_{\zeta(x)}(t), \ t \ge 0,$$

is of course the last sup is not trivial.

# 2 Averaging of random fields.

Proposition 2.1. Norm estimation.

Let  $(G, || \cdot || G)$  be arbitrary Banach functional space defined on the random variables (measurable functions)  $\zeta : \Omega \to R$ , such that

$$\forall x \in X \Rightarrow \xi(x) \in G.$$

Suppose also that the function  $\beta(x) := ||\xi(x)||G, x \in X$  is measurable. Then by virtue of integral version of triangle inequality

$$|| \xi[X] ||G = || \int_X \xi(x) \mu(dx) ||G \le$$

$$\int_{X} ||\xi(x)|| G \ \mu(dx) = \int_{X} \beta(x) \ \mu(dx), \tag{4}$$

if of course the last integral is finite.

For instance, denote by  $L_p(\Omega, P) = L_p(\Omega)$ ,  $p \ge 1$  the classical Lebesgue -Riesz space consisting of all the numerical valued r.v.  $\nu$  having a finite norm

$$||\nu||_{\Omega,p} = ||\nu||L_p(P) \stackrel{def}{=} [\mathbf{E} |\nu|^p]^{1/p} = \left[\int_{\Omega} |\nu(\omega)|^p \mathbf{P}(d\omega)\right]^{1/p}$$

Then under formulated restrictions

$$||\xi[X]||L_p(\Omega) \le \int_X ||\xi(x)||L_p(\Omega) \cdot \mu(dx).$$
(5)

As other spaces may be considered, e.g. Marcinkiewicz, Lorentz ones. Let us bring an interest (in our opinion) example. The particular case of the classical Lorentz quasi - norm (and correspondent space) for the random variable  $\phi$  is defined as follows

$$||\phi||L^{q,\infty} \stackrel{def}{=} \sup_{t \ge 0} \left[ t^q T_{\phi}(t) \right]^{1/q}.$$

To be more precisely, there exists an actually norm which is equivalent to the described before quasi - norm [21].

We derive applying these spaces.

**Proposition 2.2.** Suppose for the introduced r.f  $\xi(x)$ 

$$\sup_{x \in X} T_{\xi(x)}(t) \le t^{-q}, \ t \ge 1, \ q = \text{const} > 1.$$

Then

$$\exists C = C(q) < \infty, \Rightarrow T_{\xi[X]}(t) \le C t^{-q}, t \ge 1.$$

Let us consider an important case for the r.v. having in general case an exponential decreasing tail of distribution.

### GRAND LEBESGUE SPACES.

Let  $b = \text{const}, 1 < b \leq \infty$ . Let also  $\psi = \psi(p), p \in (1, b)$  be certain numerical valued measurable strictly positive:  $\inf_{p \in [1,b)} \psi(p) > 0$  function, not necessary to be bounded. Denotation:  $\text{Dom}(\psi) \stackrel{\text{def}}{=} \{ p : \psi(p) < \infty \},\$ 

$$(1,b) := \sup\{\psi\}; \ \Psi[1,b) := \{ \psi: \ \sup\{\psi\} = (1,b) \},\$$

$$\Psi \stackrel{def}{=} \cup_{b>1} \Psi[1,b).$$

**Definition 2.1.**, see e.g. [15], [8], [10]. Let the function  $\psi(\cdot) \in \Psi[1, b)$ , which is named as *generating function* for introduced after space. The so - called *Grand Lebesgue Space*  $G\psi$  is defined as a set of all random variables (measurable functions)  $\tau$  having a finite norm

$$||\tau||G\psi \stackrel{def}{=} \sup_{p \in (1,b)} \left\{ \frac{||\tau||L_p(\Omega)}{\psi(p)} \right\}.$$
 (6)

The particular case of these spaces and under some additional restrictions on the generating function  $\psi = \psi(p)$  correspondent to the so - called *Yudovich spaces*, see [22], [23]. These spaces was applied at first in the theory of Partial Differential Equations (PDE), see [5], [6].

These spaces are complete Banach functional rearrangement invariant; they are investigated in many works, see e.g. [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19]. It is important for us in particular to note that there is exact of course up to finite multiplicative constant interrelations under certain natural conditions on the generating function between belonging the r.v.  $\tau$  to this space and it tail behavior. Indeed, assume for definiteness that  $\tau \in G\psi$  and moreover  $||\tau||G\psi = 1$ ; then

$$T_{\tau}(t) \le \exp\{-h^*(\ln t)\}, \ t \ge e,$$
(7)

where  $h(p) = h[\psi](p) := p \ln \psi(p)$  and  $h^*(\cdot)$  is famous Young - Fenchel (Legendre) transform of the function  $h(\cdot)$ :

$$h^*(u) \stackrel{def}{=} \sup_{p \in \text{Dom}(\psi)} (pu - h(p)).$$

Inversely, let the tail function  $T_{\tau}(t), t \ge 0$  be given. Introduce the following so - called *natural function* generated by  $\tau$ 

$$\psi_{\tau}(p) \stackrel{def}{=} \left[ p \int_0^\infty t^{p-1} T_{\tau}(t) dt \right]^{1/p} = ||\tau|| L_p(\Omega), \tag{8}$$

if it is finite for some value  $b \in (1, \infty]$ , following, it is finite at last for all the values  $p \in [1, b)$ .

As long as

$$\mathbf{E}|\tau|^{p} = p \int_{0}^{\infty} t^{p-1} T_{\tau}(t) dt = \psi_{\tau}^{p}(p), \ p \in [1, b),$$

we conclude that if the last *natural* for the r.v.  $\tau$  function  $\psi_{\tau}(p)$  is finite inside some non - trivial segment  $p \in [1, b), 1 < b \leq \infty$ , then

$$\tau \in G\psi_{\tau}; \quad ||\tau||G\psi_{\tau} = 1.$$

Further, let the estimate (7) be given. Suppose in addition that the generating function  $\psi = \psi(p), \ p \in \text{Dom}(\psi)$  is continuous and suppose in the case when  $b = \infty$ 

$$\lim_{p \to \infty} \frac{\psi(p)}{p} = 0.$$
(9)

Then the r.v.  $\tau$  belongs to the Grand Lebesgue Space  $G\psi$ :

$$||\tau||G\psi \le K[\psi] < \infty, \tag{10}$$

see e.g. [17].

These conditions on the generating function  $\psi(\cdot)$  are satisfied for example for the functions  $\psi_{m,L}(p)$  of the form

$$\psi_{m,L}(p) \stackrel{def}{=} p^{1/m} L(p), \ m = \text{const} > 1, \ \mathbf{b} = \infty,$$
(11)

where L = L(p) be some continuous strictly positive *slowly varying* at infinity function such that

$$\forall \theta > 0 \implies \sup_{p \ge 1} \left[ \frac{L(p^{\theta})}{L(p)} \right] = C(\theta) < \infty.$$
(12)

For instance,  $L(p) = [\ln(p+1)]^r$ ,  $r \in R$ .

We conclude that under formulated restrictions the r.v.  $\tau$  belongs to the space  $G\psi_{m,L}$ :

$$\sup_{p\geq 1} \left\{ \frac{||\tau||_{p,\Omega}}{\psi_{m,L}(p)} \right\} = C(m,L) < \infty$$
(13)

if and only if

$$T_{\tau}(u) \le \exp\left(-C_2(m,L) \ u^m/L(u)\right), \ u \ge e, \ \exists \ C_2(m,L) > 0.$$
 (14)

A very popular example of these spaces forms the so - called subgaussian space Sub = Sub( $\Omega$ ); it consists on the subgaussian random variables, for which  $\psi(p) = \psi_2(p) := \sqrt{p}$ :

$$||\tau||\operatorname{Sub} = ||\tau||G\psi_2 \stackrel{\text{def}}{=} \sup_{p\geq 1} \left[ \frac{||\tau||_{p,\Omega}}{\sqrt{p}} \right].$$
(15)

The r.v.  $\tau$  belongs to the subgaussian space  $Sub(\Omega)$  iff

$$\exists C > 0 \Rightarrow T_{\tau}(u) \le \exp(-Cu^2), \ u \ge 0.$$
(16)

**Remark 2.1.** As a rule, on the the r.v.  $\tau$  from the spaces  $G\psi_{m,L}$  is imposed the condition of *centering*:  $\mathbf{E}\tau = 0$ .

Another examples. Suppose that the r.v.  $\tau$  be such that

$$T_{\tau}(t) \leq T^{\beta,\gamma,L}(t), \ \beta > 1, \ \gamma > -1, \ L = L(t),$$

where

$$T^{\beta,\gamma,L}(t) \stackrel{def}{=} t^{-\beta} (\ln t)^{\gamma} L(\ln t), \ t \ge e$$

and  $L = L(t), t \ge e$  be as before slowly varying at infinity positive continuous function. It is known [17] that as  $p \in [1, \beta)$ 

$$\psi_{\tau}(p) = ||\tau||_{p} \le C_{1}(\beta, \gamma, L) \ (\beta - p)^{-(\gamma + 1)/\beta} \ L^{1/\beta}(1/(\beta - p)), \tag{17}$$

and conversely, if the relation (17) there holds, then

$$T_{\tau}(t) \leq C_7(\beta, \gamma, L) T^{\beta, \gamma+1, L}(t).$$

Herewith both this estimations are unimprovable.

**Theorem 2.1.** Denote for the random field  $\xi(x), x \in X$ 

$$Q(t) := \sup_{x \in X} T_{\xi(x)}(t), \ t \ge 0,$$
$$g_0(p) := \left[ p \int_0^\infty t^{p-1} Q(t) \ dt \right]^{1/p},$$

$$g(p) := p \ln g_0(p),$$

and suppose  $\exists b \in (1,\infty] \Rightarrow g(b) < \infty$ . We assert

$$T_{\xi[X]}(t) \le \exp(-g^*(\ln t)), \ t \ge e.$$
 (18)

**Proof.** We have

$$T_{\xi(x)}(t) \le Q(t), \ x \in X, \ t \ge 0.$$

Therefore the random variables  $\xi(x), x \in X$  belongs to the unit ball of the Grand Lebesgue Space  $G g_0$ :

$$\sup_{x \in X} ||\xi(x)|| G\kappa \le 1.$$

It remains to apply the proposition (7).

**Remark 2.2.** A more general version: assume that there exists a finite a.e. and non - negative measurable numerical valued function  $\theta = \theta(x), x \in X$  such that

$$T_{\xi(x)}(t) \le \exp\left\{ -h_{\psi}^*[\ln(t/\theta(x))] \right\}, \ t \ge e,$$

then

$$||\xi(x)||G\psi \le C_4[\psi] \ \theta(x), \ C_4[\psi] < \infty,$$

therefore

$$||\xi[X]||G\psi \le C_4[\psi] \int_X \theta(x) \ \mu(dx),$$

with correspondent tail estimate

$$T_{\xi[X]} \le \exp\left[-h_{\psi}^*(\ln t/C_5)\right], \ t \ge eC_5,$$
 (19)

where

$$C_5 = C_4[\psi] \int_X \theta(x) \ \mu(dx),$$

if of course the last integral is finite.

Example 2.1. Suppose

$$\sup_{x \in X} T_{\xi(x)}(u) \le \exp\left(-u^m / L(u)\right), \ u \ge e,$$
(20)

where as before m = const > 1, L = L(p) is some continuous strictly positive slowly varying at infinity function such that

$$\forall \theta > 0 \Rightarrow \sup_{p \ge 1} \left[ \frac{L(p^{\theta})}{L(p)} \right] = C(\theta) < \infty.$$
 (21)

We conclude that under formulated above restrictions that  $\exists C_5(m,L) > 0 \Rightarrow$ 

$$T_{\xi[X]}(u) \le \exp\{ -C_5(m,L) \ u^m/L(u) \}, \ u \ge e.$$
 (22)

Obviously, the last estimate (22) is non - improvable, of course. up to multiplicative constant  $C_5(m, L)$ .

Example 2.2. Suppose now in the introduced notations and restrictions

$$\sup_{x \in X} T_{\xi(x)}(u) \le T^{\beta,\gamma,L}(t), \ t \ge 1,$$
(23)

We derive arguing similarly to the previous example

$$T_{\xi[X]}(u) \le T^{\beta,\gamma+1,L}(C_6(\beta,\gamma,L) \ t), \ t \ge e, \ C_6 > 0,$$
(24)

and this estimate is also non - improvable still in the case when the set X consists in a single point [18].

**Example 2.3.** Let us show that the proposition of theorem 2.1 is not true if we replace the averaging onto the classical *normalized* sums of the centered random variables. Let  $\{\zeta_i\}, i = 1, 2, \ldots; \zeta = \zeta_1$  be a sequence if independent symmetrical distributed r.v. such that

$$\forall y \ge 0 \Rightarrow T_{\zeta}(y) = \exp(-y^q), \ q = \text{const} > 2.$$

Denote

$$S_n := n^{-1/2} \sum_{i=1}^n \zeta_i; \ \sigma^2 = \sigma^2(q) \stackrel{def}{=} \operatorname{Var}(\zeta) \in (0, \infty).$$

We derive by virtue of CLT for all the positive values y, say for  $y \ge 1$ 

$$\sup_{n} \mathbf{P}(S_n > y) \ge \lim_{n \to \infty} \mathbf{P}(S_n > y) \ge \exp\left(-C_0 y^2\right),$$

 $C_0 = C_0(q) \in (0, \infty)$ ; so that the tails of distribution of the r.v.  $S_n$  are "much heavier" as ones for the source variable  $\zeta$ .

## 3 Conditional averaging.

Let as in the first section (2)  $\nu \stackrel{def}{=} \eta\{F\} \stackrel{def}{=} \mathbf{E}\eta/F$ , where F is some sub - sigma field of the source sigma algebra **B**.

Assume that

$$T_{\eta}(t) \leq R(t), t \geq 0;$$

where R = R(t) is non-negative bounded:  $R(t) \leq 1$  non-increasing measurable function such that  $R(\infty) = 0$ . Introduce the  $\Psi$  function

$$\delta(p) = \delta[R](p) \stackrel{def}{=} \left[ p \int_0^\infty t^{p-1} R(t) dt \right]^{1/p},$$

and suppose  $\delta(\cdot) \in \Psi$ ; then

$$\mathbf{E}|\eta|^p \le \delta^p(p).$$

Since the function  $g(y) = |y|^p, p \ge 1$  is convex, one can apply the Jensen's inequality for the conditional expectations

$$\mathbf{E}|\nu|^p \leq \mathbf{E}|\eta|^p \leq \delta^p(p), \ p \in [1,b);$$

following  $\nu \in G\delta$  and we obtain the inequality

$$T_{\nu}(t) \le \exp(-v^*(\ln t)), \ t \ge e,$$
 (25)

where

$$v(p) = p \ln \delta(p), \ p \in \text{Dom}(\delta).$$

To summarize.

Proposition 3.1. We obtain under formulated notations and restrictions

$$||\nu||G\delta \le 1,\tag{26}$$

and hence the estimation (25) there holds.

The examples may be considered alike ones in the previous section, under at the same restrictions.

Example 3.1. If

$$T_{\eta}(u) \le \exp\{-u^m/L(u)\}, \ u \ge e,$$
 (27)

then

$$T_{\nu}(u) \le \exp\{-C_7(m,L) \ u^m/L(u)\}, \ u \ge e, \ C_7 > 0.$$
 (28)

Example 3.2. Suppose

$$T_{\eta}(u) \le T^{\beta,\gamma,L}(t), \ t \ge 1, \ \beta > 1, \ \gamma > -1.$$
 (29)

We derive

$$T_{\nu}(u) \le T^{\beta,\gamma+1,L}(C_8(\beta,\gamma,L) \ t), \ t \ge e, \ C_8 > 0,$$
 (30)

and this estimate is also essentially non - improvable, see e.g. [18].

**Example 3.3.** Let  $\Omega = (0, 1)$  equipped with the classical Lebesgue measure. Define the r.v.  $\eta = \omega^{-\alpha}, \ \omega \in (0, 1); \ \alpha = \text{const} \in (0, 1)$  and put

$$F := \{ \emptyset, (0, 1/2], (1/2, 1), (0, 1) \}.$$

We have

$$\mathbf{E}|\eta|^p = \frac{1}{1 - \alpha p}, \ 1 \le p < 1/\alpha$$

and  $\mathbf{E}|\eta|^p = \infty, \ p \ge 1/\alpha$ . The r.v.  $\nu = \mathbf{E}\eta/F$  has a form

$$\nu(\omega) = 2^{\alpha}/(1-\alpha), \ \omega \in (0, 1/2]$$

and

$$\nu(\omega) = (2 - 2^{\alpha})/(1 - \alpha), \ \omega \in (1/2, 1).$$

Thus, the r.v.  $\nu$  is *bounded*, despite that the source r.v.  $\eta$  has not all the moments  $\mathbf{E}[\eta]^p$ ,  $p \ ge1/\alpha$ .

## 4 Applications to the martingale theory.

#### A. UNIFORM INTEGRABLE MARTINGALES.

Suppose that there is certain *filtration* on the source probability sigma - field  $\mathcal{B}$ , i.e. an increasing sequence of sigma - subfields  $\{F_n\}$  of canonical one  $\mathcal{B}$ , and let  $(\kappa_n, F_n)$ ,  $n = 1, 2, \ldots$  be an *uniform integrable* martingale:

$$\mathbf{E}\kappa_m/F_k = \kappa_k, \ 1 \le k \le m$$

We conclude by virtue of theorem J.Doob there exists with probability one a limit

$$\kappa \stackrel{def}{=} \lim_{n \to \infty} \kappa_n. \tag{31}$$

Assume that

$$T_{\kappa}(t) \le U(t), \ t \ge 0,$$

where U = U(t) is some non - negative bounded  $U(t) \leq 1$  non - increasing function for which  $\lim U(t) = 0, t \to \infty$ . Introduce as before the following  $\Psi$ -function

$$\rho(p) \stackrel{def}{=} \left[ p \int_0^\infty t^{p-1} U(t) dt \right]^{1/p},$$

and we suppose  $\rho \in \Psi(b)$  for some value  $b \in (1, \infty]$ .

We get on the basis of proposition 3.1 using the fact that  $\kappa_n = \mathbf{E}\kappa/F_n$ 

### Proposition 4.1.

$$||\kappa_n||G\rho \le 1 \tag{32}$$

with correspondent tail estimation (7).

Further, define the following finite a.e. random variable

$$\hat{\kappa} := \sup_n \kappa_n$$

and a new  $\Psi$  – function

$$\hat{\rho}(p) := p/(p-1) \cdot \rho(p), \ p \in (1,b).$$

Proposition 4.2.

$$||\hat{\kappa}||G\hat{\rho} \le 1,\tag{33}$$

with correspondent tail estimation.

**Proof** is quite alike to the previous one; one can apply the famous Doob's inequality

$$\mathbf{E}|\hat{\kappa}|^{p} \leq \left[ \begin{array}{c} p\\ p-1 \end{array} \right]^{p} \cdot \rho^{p}(p) = \left[ \begin{array}{c} \hat{\rho}(p) \end{array} \right]^{p}.$$

**Remark 4.1.** The proposition 4.2 allows a simple (particular) inversion. Assume namely that the estimation (33) is satisfied. Then the r.v.  $\{\kappa_n\}$  are uniform integrable and, following, by virtue of theorem J.Doob there exists a limit a.e.

 $\mathbf{P}(\lim_{n \to \infty} \kappa_n = \kappa) = 1. \tag{34}$ 

It follows from (34) and (33) that in turn

$$\sup_{n} ||\kappa_n||G\hat{\rho} < \infty.$$

#### **B.** GENERAL CASE: A NON - UNIFORM INTEGRABLE MARTINGALES.

Let again  $(\kappa_n, F_n)$  be a martingale; we do not suppose in this subsection that it is uniform integrable. Then the sequence  $\kappa_n$  must be normed; and as it is noted in [20] the natural norming sequence may be choosed as a its square variation

$$[\kappa]_n^2 \stackrel{def}{=} \kappa_1^2 + \sum_{i=2}^n (\kappa_i - \kappa_{i-1})^2, \ n \ge 2.$$
(35)

But the square variation sequence is *random*; therefore we offer its *expectation* as a capacity of norming sequence

$$\sigma_n^2 \stackrel{def}{=} \mathbf{E}[\kappa]_n^2, \ n \ge 2; \quad \sigma_1^2 := \mathbf{E}\kappa_1^2.$$
(36)

It will be presumed further that  $\sigma_n \in (0, \infty), n \ge 1$ .

For instance, if our martingale may be represented as a sum of the centered independent r.v.

$$\kappa_n = \sum_{j=1}^n \epsilon_j, \quad F_n = \sigma\{ \epsilon_j, \ 1 \le j \le n \},$$

such that  $\operatorname{Var}(\epsilon_j) < \infty$ , then

$$\sigma_n^2 = \sum_{j=1}^n \operatorname{Var}(\epsilon_j).$$

Further, suppose as above that

$$\sup_{n} T_{\kappa_n/\sigma_n}(t) \le W(t), \ t \ge 1,$$

where  $W(\cdot)$  is non - negative bounded  $W \leq 1$  non - increasing function such that  $W(\infty) = 0$ . Define

$$\upsilon(p) := \left[ p \int_0^\infty t^{p-1} W(t) \ dt \right]^{1/p}, \quad \hat{\upsilon}(p) := p\upsilon(p).$$

#### Proposition 4.3.

One can apply the famous Burkholder's inequality [2]

$$||\max_{k=1,2,...,n} \kappa_k||_p \le p \ v(p) = \hat{v}(p),$$

or equally

$$\left|\left|\max_{k=1,2,\dots,n}\kappa_k\right|\right|G\hat{\upsilon}\leq 1$$

with correspondent tail estimate (7).

## 5 Concluding remarks.

It is interest in our opinion to generalize obtained results on the variables, and martingales, taking values in Banach spaces.

Acknowledgement. The first author has been partially supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and by Università degli Studi di Napoli Parthenope through the project "sostegno alla Ricerca individuale".

## References

 V.V. Buldygin V.V., D.I.Mushtary, E.I.Ostrovsky, M.I.Pushalsky. New Trends in Probability Theory and Statistics. Mokslas, (1992), V.1, p. 78 - 92; Amsterdam, Utrecht, New York, Tokyo.

- Burkholder D.L. Distribution function inequalities for Martingales. Annals Probab., 1973, V.1, 19 - 42.
- [3] Capone C, Formica M.R, Giova R. Grand Lebesgue spaces with respect to measurable functions. Nonlinear Analysis 2013; 85: 125 - 131.
- [4] Capone C, and Fiorenza A. On small Lebesgue spaces. Journal of function spaces and applications. 2005; 3; 73 89.
- [5] Qionglei Chen, Changxing Miao and Xiaoxin Zheng. The two dimensional Euler equation in Yudovich and bmo - type spaces. arXiv:1311.0934v4 [math.AP] 10 Jan 2019
- [6] Gianluca Crippa and Giorgio Stefani. An elementary proof of existence and uniqueness for the Euler flow in localized Yudovich spaces. arXiv:2110.15648v1 [math.AP] 29 Oct 2021
- [7] S. V. Ermakov, E. I. Ostrovsky. Central limit theorem for weakly dependent Banach space valued random variables. Theory Probab. Appl., 30, 2, (1986), 391 - 394.
- [8] S.V.Ermakov, and E. I. Ostrovsky. Continuity Conditions, Exponential Estimates, and the Central Limit Theorem for Random Fields. Moscow, VINITY, (1986), (in Russian).
- [9] Fiorenza A., and Karadzhov G.E. Grand and small Lebesgue spaces and their analogs. Consiglio Nationale Delle Ricerche, Instituto per le Applicazioni del Calcoto Mauro Picone, Sezione di Napoli, Rapporto tecnicon. 272/03, (2005).
- [10] A. Fiorenza, M. R. Formica and A. Gogatishvili. On grand and small Lebesgue and Sobolev spaces and some applications to PDE's. Differ. Equ. Appl. 10 (2018), no. 1, 21–46.
- [11] A. Fiorenza, M. R. Formica, A. Gogatishvili, T. Kopaliani and J. M. Rakotoson. Characterization of interpolation between grand, small or classical Lebesgue spaces. Preprint arXiv:1709.05892, Nonlinear Anal., to appear.
- [12] A. Fiorenza, M. R. Formica and J. M. Rakotoson. Pointwise estimates for GΓ-functions and applications. Differential Integral Equations 30 (2017), no. 11-12, 809–824.
- [13] M. R. Formica and R. Giova. Boyd indices in generalized grand Lebesgue spaces and applications. Mediterr. J. Math. 12 (2015), no. 3, 987–995.
- [14] M.R.Formica, E.Ostrovsky, L.Sirota Central Limit Theorem in Lebesgue-Riesz spaces for weakly dependent random sequences. arXiv:1912.00338v2 [math.PR] 3 Dec 2019
- [15] Yu.V. Kozachenko and E.I. Ostrovsky. Banach spaces of random variables of subgaussian type. Theory Probab. Math. Stat., Kiev, (1985), v. 43, 42 - 56, (in Russian).
- [16] Kozachenko Yu.V., Ostrovsky E., Sirota L. Relations between exponential tails, moments and moment generating functions for random variables and vectors. arXiv:1701.01901v1 [math.FA] 8 Jan 2017
- [17] Kozachenko Yu.V., Ostrovsky E., Sirota L. Equivalence between tails, Grand Lebesgue Spaces and Orlicz norms for random variables without Kramer's condition. Bulletin of KSU, Kiev, 2018, 4, pp. 20 - 29.
- [18] E.Liflyand, E.Ostrovsky, L.Sirota. Structural Properties of Bilateral Grand Lebesgue Spaces. Turk. J. Math.; 34, (2010), 207 - 219.
- [19] E.I. Ostrovsky. Exponential Estimations for Random Fields. Moscow Obninsk, OINPE, (1999), in Russian.
- [20] Pekshir G., Shiryaev A.N. Khintchine's inequalities and martingale extension of scope of their actions. Uspekhi Mathem. Nauk, 1995, V. 50, Issue 5 (305), 3 - 63, (in Russian).

- [21] E.M.Stein. Interpolation of linear operators. Trans. Amer. Math. Soc., 83; 482 492, 1956.
- [22] V. I. Yudovich. Nonstationary flow of an ideal incompressible liquid. Zh. Vych. Mat., 3, (1963), 1032 - 1066.
- [23] V. I. Yudovich. Uniqueness theorem for the basic nonstationary problem in the dynamics of an ideal incompressible fluid. Mathematical Research Letters, 2. (1995), 27 - 38.