#### A note about extension of functions

belonging to Sobolev - Grand Lebesgue Spaces.

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#### Abstract

We deduce an extension theorem for the so - called Sobolev - Grand Lebesgue Spaces defined on the suitable subsets of the whole finite - dimensional Euclidean space, and estimate the norms of correspondent extension operator, which may be choosed as linear.

Key words and phrases. Lebesgue - Riesz, Yudovich, ordinary Sobolev and Sobolev - Grand Lebesgue norm and spaces, Euclidean space, Lipschitz domain, ordinary and linear extension, linear bounded operator, norm, measurable functions, semi - space, generating function, estimation.

## 1 Introduction. Previous results.

Let B(G), where G is non - trivial subset of an Euclidean space  $R^d : G \subset R^d$ be a family of Banach spaces defined on the class of functions defined in turn on the support G; the norm in this space will be denoted ||f||B(G).

It will be presumed henceforth that all the considered domains G are closures of the non - empty open sets and are Lipschitzian. The case when  $G = R^d$  is trivial for us and may be excluded.

Put also for definiteness  $B = B(R^d)$ .

For instance, the classical Lebesgue - Riesz  $L_p(G)$  spaces equipped with the ordinary norm

$$||f||_p(G) = ||f||L_p(G) := \left[\int_G |f(x)|^p dx\right]^{1/p}, \ p \in [1,\infty), \ f: G \to R; \ x \in G,$$

as well as the famous Sobolev spaces  $W_p^m(G), m = 1, 2, ...$ :

$$||f||W_p^m(G) \stackrel{def}{=} \max_{\alpha, \ |\alpha| \le m} ||D^{\alpha}f||L_p(G), \tag{1}$$

 $||f||W_p^m := ||f||W_p^m(R^d);$  see e.g. [1], [24], [25], [26], [30], [31]. Here as ordinary

$$\alpha = \vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_d); \quad \alpha_j = 0, 1, 2, \dots$$

$$|\alpha| := \sum_{j=1}^{d} \alpha_j; \quad D^{\alpha} f := \frac{D^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_d^{\alpha_d}}$$

and all the derivatives are understood in the weak (Sobolev) sense. Of course,  $D^0 f = f$ .

Let the function  $f: G \to R$  belongs to some space B(G). By definition, the function  $\tilde{f}, R^d \to R$  is named as an *extension* of the function f (from the set G), iff

$$\forall x \in G \Rightarrow \hat{f}(x) = f(x)$$

and wherein  $||\tilde{f}||B < \infty$ .

If there exists a *linear bounded* operator  $L: \tilde{f} = Lf = L_G f, f \in B(G)$ , where again  $\tilde{f}$  is an extension for f, such that

$$K = K(G) = K(B,G) \stackrel{def}{=} \sup_{0 \neq f \in B(G)} \left\{ \frac{||Lf||B}{||f||B(G)} \right\} < \infty,$$

then the extension  $\tilde{f} = Lf = L_G f$  is named linear.

It is known, see e.g. [3], [4], [11], [31] that under imposed conditions on the Lipschitz domain G and for the Sobolev's spaces  $B(G) = W_p^m(G), m \ge 1, p \ge 1$  (the case m = 0 is trivial) there exist a linear extension operator. See also the works [18], [19], [25], [26], [29] etc.

We will ground in offered report that this linear operator there exists also for the so - called Sobolev - Grand Lebesgue Spaces.

## 2 Main result.

SOBOLEV - GRAND LEBESGUE SPACES.

Let  $(a,b) = \text{const}, 1 \leq a < b \leq \infty$ . Let also  $\psi = \psi(p), p \in (a,b)$  be certain numerical valued measurable strictly positive:  $\inf_{p \in (a,b)} \psi(p) > 0$  function, not necessary to be bounded. Denotation:

$$(a,b) := \operatorname{supp}(\psi); \ \Psi[\mathbf{a},\mathbf{b}] := \{ \ \psi : \ \operatorname{supp}(\psi) = (\mathbf{a},\mathbf{b}) \ \},$$
$$\Psi \stackrel{def}{=} \cup_{(a,b): \ 1 \le a \le b \le \infty} \Psi[a,b].$$

**Definition 2.1.**, see e.g. [28]. The Sobolev - Grand Lebesgue Space  $S[m, G, \psi]$  based on the set  $G, G \subset \mathbb{R}^d$  is defined as a set of all (measurable) functions having a finite norm

$$||f||S[m,G,\psi] \stackrel{def}{=} \sup_{p \in (a,b)} \left\{ \frac{||f||W_p^m(G)}{\psi(p)} \right\}.$$
(2)

The particular case m = 0, i.e. when

$$||f||G\psi = ||f||G\psi(a,b) \stackrel{def}{=} ||f|||S[0,G,\psi] = \sup_{p \in (a,b)} \left\{ \frac{||f||L_p(G)}{\psi(p)} \right\}$$
(3)

and under some additional restrictions on the generating function  $\psi = \psi(p)$  correspondent to the so - called Yudovich spaces, see [32], [33]. These spaces was applied at first in the theory of Partial Differential Equations (PDE), see [7], [8].

A general case of these spaces when m = 0 are named as the classical Grand Lebesgue Spaces (GLS)  $G\psi$ ,  $\psi \in \Psi$ . These spaces are investigated in many works, see e.g. [9], [10], [12], [13], [14], [15], [16], [17], [20], [21], [22], [23], [27]. The general case of Sobolev - Grand Lebesgue Spaces appears at first perhaps in the article [28], where was investigated the modulus of continuity of the functions belonging to these spaces.

**Theorem 2.1.** Assume that all the formulated before restrictions are satisfied, indeed: that the Lipschitz domain G is closure of the non - empty open subset of whole Euclidean space  $\mathbb{R}^d$ . We propose that for arbitrary Sobolev - Grand Lebesgue Space  $S[m, G, \psi]$  there exists a *linear bounded* extension operator  $L = L_G$ .

**Proof.** One can suppose  $d \ge 2$  and that  $G = \vec{x} = \{x_j\} = \{x_1, x_2, \ldots, x_{d-1}, x_d\}$ , where  $\vec{x} = (\tilde{x}, x_d)$ ;  $\tilde{x} = \{x_1, x_2, \ldots, x_{d-1}\}$ ; so that  $\vec{x} \in G \Leftrightarrow x_d \ge 0$ ; on the other words, upper semi - space; see e.g. [3], [4], [11]. Let us define the following extension operator  $Lf(x) := f(x), x \in G, f(\cdot) \in S[m, G, \psi]$ ;

$$Lf(x) := \sum_{k=1}^{d+1} c_k f(\tilde{x}, -kx_d), \ x_d < 0.$$

The coefficients  $\{c_k\}$  may be uniquely determined from the following system of linear equations

$$\sum_{k=1}^{m+1} (-k)^l \ c_k = 1; \ l = 0, 1, \dots, d.$$

Suppose that  $f \in S[m, G, \psi]$ ; one can assume without loss of generality  $||f||S[m, G, \psi] = 1$ . Then for all the values  $p \in (a, b)$ 

$$||f||W_p^m \le \psi(p), \ p \in (a,b),$$

therefore

$$\forall p \in (a, b), \quad \forall \alpha : |\alpha| \le m \implies ||D^{\alpha}f||_p(G) \le \psi(p).$$

Introduce the functions

$$g_k(\tilde{x}, y) := f(\tilde{x}, -ky), \ y \le 0;$$

then

$$D^{\alpha}g_{k} = (-k)^{\alpha_{d}} f^{(\alpha)}(\tilde{x}, -ky),$$
$$||D^{\alpha}g_{k}||_{p}(G) = k^{\alpha_{d}-1/p}||D^{\alpha}f||_{p}(G) \le k^{\alpha_{d}}||D^{\alpha}f||_{p}(G),$$

therefore

$$\begin{aligned} \forall \alpha : |\alpha| &\leq m \; \Rightarrow \sum_{k=1}^{m+1} |c_k| \; k^m \cdot ||D^{\alpha}f||_p(G) \leq \\ & \sum_{k=1}^{m+1} |c_k| \; k^m \cdot \psi(p), \; p \in (a,b). \end{aligned}$$

Following, by virtue of triangle inequality for Lebesgue - Riesz spaces

$$\begin{split} ||L[f]||W_p^m(R^d \setminus G) &\leq C(d,m) \ \psi(p), \ C(d,m) < \infty, \\ ||Lf||S[m,R^d,\psi] &\leq 1 + C(d,m) < \infty, \end{split}$$

Q.E.D.

# 3 Concluding remarks.

It is interest in our opinion to compute the exact value of extension constant for Sobolev - Grand Lebesgue Spaces, as well as to generalize the extension theorem on the anisotropic spaces.

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