A note about extension of functions

belonging to Sobolev - Grand Lebesgue Spaces.

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Abstract

We deduce an extension theorem for the so - called Sobolev - Grand Lebesgue Spaces defined on the suitable subsets of the whole finite - dimensional Euclidean space, and estimate the norms of correspondent extension operator, which may be choosed as linear.

Key words and phrases. Lebesgue - Riesz, Yudovich, ordinary Sobolev and Sobolev - Grand Lebesgue norm and spaces, Euclidean space, Lipschitz domain, ordinary and linear extension, linear bounded operator, norm, measurable functions, semi - space, generating function, estimation.

1 Introduction. Previous results.

Let $B(G)$, where G is non-trivial subset of an Euclidean space $R^d: G \subset R^d$ be a family of Banach spaces defined on the class of functions defined in turn on the support G ; the norm in this space will be denoted $||f||B(G)$.

It will be presumed henceforth that all the considered domains G are closures of the non-empty open sets and are Lipschitzian. The case when $G = R^d$ is trivial for us and may be excluded.

Put also for definiteness $B = B(R^d)$.

For instance, the classical Lebesgue - Riesz $L_p(G)$ spaces equipped with the ordinary norm

$$
||f||_p(G) = ||f||L_p(G) := \left[\int_G |f(x)|^p \ dx \right]^{1/p}, \ p \in [1, \infty), \ f: G \to R; \ x \in G,
$$

as well as the famous Sobolev spaces $W_p^m(G)$, $m = 1, 2, \ldots$:

$$
||f||W_p^m(G) \stackrel{def}{=} \max_{\alpha, |\alpha| \le m} ||D^{\alpha}f||L_p(G), \tag{1}
$$

 $||f||W_p^m := ||f||W_p^m(R^d);$ see e.g. [1], [24], [25], [26], [30], [31]. Here as ordinary

$$
\alpha = \vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_d); \quad \alpha_j = 0, 1, 2, \dots;
$$

$$
|\alpha| := \sum_{j=1}^d \alpha_j; \quad D^{\alpha} f := \frac{D^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_d^{\alpha_d}}
$$

and all the derivatives are understood in the weak (Sobolev) sense. Of course, $D^0 f = f$.

Let the function $f: G \to R$ belongs to some space $B(G)$. By definition, the function $\tilde{f}, R^d \to R$ is named as an extension of the function f (from the set G), iff

$$
\forall x \in G \Rightarrow \tilde{f}(x) = f(x)
$$

and wherein $||\tilde{f}||B < \infty$.

If there exists a *linear bounded* operator $L: \tilde{f} = Lf = L_Gf, f \in B(G)$, where again \tilde{f} is an extension for f, such that

$$
K = K(G) = K(B, G) \stackrel{def}{=} \sup_{0 \neq f \in B(G)} \left\{ \frac{||Lf||B}{||f||B(G)} \right\} < \infty,
$$

then the extension $\tilde{f} = Lf = L_Gf$ is named linear.

It is known, see e.g. [3], [4], [11], [31] that under imposed conditions on the Lipschitz domain G and for the Sobolev's spaces $B(G) = W_p^m(G)$, $m \ge 1$, $p \ge 1$ (the case $m = 0$ is trivial) there exist a linear extension operator. See also the works [18], [19], [25], [26], [29] etc.

We will ground in offered report that this linear operator there exists also for the so - called Sobolev - Grand Lebesgue Spaces.

2 Main result.

Sobolev - Grand Lebesgue Spaces.

Let $(a, b) = \text{const}, 1 \le a < b \le \infty$. Let also $\psi = \psi(p), p \in (a, b)$ be certain numerical valued measurable strictly positive: $\inf_{p\in (a,b)} \psi(p) > 0$ function, not necessary to be bounded. Denotation:

$$
(a, b) := \text{supp}(\psi); \ \Psi[a, b] := \{ \ \psi : \ \text{supp}(\psi) = (a, b) \ \},
$$

$$
\Psi \stackrel{\text{def}}{=} \bigcup_{(a, b): 1 \le a < b \le \infty} \Psi[a, b].
$$

Definition 2.1., see e.g. [28]. The Sobolev - Grand Lebesgue Space $S[m, G, \psi]$ based on the set $G, G \subset \mathbb{R}^d$ is defined as a set of all (measurable) functions having a finite norm

$$
||f||S[m, G, \psi] \stackrel{def}{=} \sup_{p \in (a, b)} \left\{ \frac{||f||W_p^m(G)}{\psi(p)} \right\}.
$$
 (2)

The particular case $m = 0$, i.e. when

$$
||f||G\psi = ||f||G\psi(a,b) \stackrel{def}{=} ||f|| |S[0, G, \psi] = \sup_{p \in (a,b)} \left\{ \frac{||f||L_p(G)}{\psi(p)} \right\} \tag{3}
$$

and under some additional restrictions on the *generating function* $\psi = \psi(p)$ correspondent to the so-called Yudovich spaces, see [32], [33]. These spaces was applied at first in the theory of Partial Differential Equations (PDE), see [7], [8].

A general case of these spaces when $m = 0$ are named as the classical Grand Lebesgue Spaces (GLS) $G\psi$, $\psi \in \Psi$. These spaces are investigated in many works, see e.g. [9], [10], [12], [13], [14], [15], [16], [17], [20], [21], [22], [23], [27]. The general case of Sobolev - Grand Lebesgue Spaces appears at first perhaps in the article [28], where was investigated the modulus of continuity of the functions belonging to these spaces.

Theorem 2.1. Assume that all the formulated before restrictions are satisfied, indeed: that the Lipschitz domain G is closure of the non - empty open subset of whole Euclidean space R^d . We propose that for arbitrary Sobolev - Grand Lebesgue Space $S[m, G, \psi]$ there exists a linear bounded extension operator $L = L_G$.

Proof. One can suppose $d \geq 2$ and that $G = \vec{x} = \{x_i\}$ ${x_1, x_2, \ldots, x_{d-1}, x_d}$, where $\vec{x} = (\tilde{x}, x_d); \tilde{x} = \{x_1, x_2, \ldots, x_{d-1}\}$; so that $\vec{x} \in G \Leftrightarrow x_d \geq 0$; on the other words, upper semi - space; see e.g. [3], [4], [11]. Let us define the following extension operator $Lf(x) := f(x), x \in G, f(\cdot) \in S[m, G, \psi];$

$$
Lf(x) := \sum_{k=1}^{d+1} c_k f(\tilde{x}, -kx_d), \ x_d < 0.
$$

The coefficients ${c_k}$ may be uniquely determined from the following system of linear equations

$$
\sum_{k=1}^{m+1} (-k)^{l} c_{k} = 1; l = 0, 1, ..., d.
$$

Suppose that $f \in S[m, G, \psi]$; one can assume without loss of generality $||f||S[m, G, \psi] = 1$. Then for all the values $p \in (a, b)$

$$
||f||W_p^m \le \psi(p), \ p \in (a, b),
$$

therefore

$$
\forall p \in (a, b), \quad \forall \alpha : |\alpha| \le m \Rightarrow ||D^{\alpha}f||_p(G) \le \psi(p).
$$

Introduce the functions

$$
g_k(\tilde{x}, y) := f(\tilde{x}, -ky), \ y \le 0;
$$

then

$$
D^{\alpha}g_k = (-k)^{\alpha_d} f^{(\alpha)}(\tilde{x}, -ky),
$$

$$
||D^{\alpha}g_k||_p(G) = k^{\alpha_d - 1/p} ||D^{\alpha}f||_p(G) \le k^{\alpha_d} ||D^{\alpha}f||_p(G),
$$

therefore

$$
\forall \alpha : |\alpha| \le m \Rightarrow \sum_{k=1}^{m+1} |c_k| k^m \cdot ||D^{\alpha} f||_p(G) \le
$$

$$
\sum_{k=1}^{m+1} |c_k| k^m \cdot \psi(p), \ p \in (a, b).
$$

Following, by virtue of triangle inequality for Lebesgue - Riesz spaces

$$
||L[f]||W_p^m(R^d \setminus G) \le C(d, m) \psi(p), C(d, m) < \infty,
$$

$$
||Lf||S[m, R^d, \psi] \le 1 + C(d, m) < \infty,
$$

Q.E.D.

3 Concluding remarks.

It is interest in our opinion to compute the exact value of extension constant for Sobolev - Grand Lebesgue Spaces, as well as to generalize the extension theorem on the anisotropic spaces.

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