FINDING GLOBAL SOLUTIONS FOR A CLASS OF POSSIBLY NONCONVEX QCQP PROBLEMS THROUGH THE S-LEMMA

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ABSTRACT. In this paper we provide necessary and sufficient (KKT) conditions for global optimality for a new class of possibly nonconvex quadratically constrained quadratic programming (QCQP) problems, denoted by S-QCQP. The class consists of QCQP problems where the matrices of the quadratic components are formed by a scalar times the identity matrix. Our result relies on a generalized version of the S-Lemma, stated in the context of general QCQP problems. Moreover, we prove the exactness of the SDP and the SOCP relaxations for S-QCQP.

1. Introduction

The aim of this work is to provide a characterization for the global minima of a specific class of Quadratically Constrained Quadratic Programming (QCQP) problem. A generic (QCQP) problem is defined as in [24]:

$$\begin{aligned} & Minimize_{x \in \mathbb{R}^n} \ J(x) := x^T A_J x + 2b_J^T x + c_J \\ & s.t. \ f_k(x) := x^T A_k x + 2b_L^T x + c_k < 0, \ k = 1, ..., m \end{aligned} \tag{QCQP}$$

with $x \in \mathbb{R}^n$, $A_J, A_k \in S^n$, where S^n denotes the space of symmetric $n \times n$ matrices, $b_J, b_k \in \mathbb{R}^n$ and $c_J, c_k \in \mathbb{R}$ are given data for k = 1, ..., m.

We focus on (QCQP) problems which satisfy the following assumption:

Assumption 1. The matrices $A_J, A_k \ \forall \ k \in \{1, ..., m\}$ are of the form $A_J = a_J I, A_k = a_k I$ where I is the $n \times n$ identity matrix, and $a_J, a_k \in \mathbb{R}$.

When Assumption 1 holds, (QCQP) takes the form

$$\begin{aligned} & Minimize_{x \in \mathbb{R}^n} \quad J(x) := a_J x^T x + 2b_J^T x + c_J \\ & s.t. \quad f_k(x) := a_k x^T x + 2b_L^T x + c_k < 0, \quad k = 1, ..., m \end{aligned} \tag{S-QCQP}$$

Since in (S-QCQP) the matrices $A_J, A_k, \ k=1,...,m$ do not appear and we only have the scalars $a_J, a_k, \ k=1,...,m$, we propose to denote (S-QCQP) as scalar QCQP, i.e. S-QCQP.

Remark 1. The scalars $a_J, a_k, k = 1, ..., m$ can be also equal to zero for (S-QCQP). When $a_J = 0$ or $a_k = 0$ for some k, the corresponding J(x) or $f_k(x)$ are linear forms.

Key words and phrases. QCQP problems and KKT conditions and Jakubovich lemma and S-lemma and SDP relaxation and SOCP relaxation and S-QCQP.

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Consider the following KKT conditions:

(i)
$$\nabla (J + \sum_{k=1}^{m} \gamma_k f_k)(x^*) = 0$$

(ii) $\gamma_k f_k(x^*) = 0 \quad k \in \{1, ..., m\}$
(iii) $A_J + \sum_{k=1}^{m} \gamma_k A_k \succeq 0$

The main result of this work is to prove that (KKT) are necessary and sufficient optimality conditions for (S-QCQP), if m+1 < n, there exists x_0 such $f_k(x_0) < 0 \,\forall \, k \in \{1,...,m\}$ and if the following assumption holds.

Assumption 2. Assume that there exists a global minimum x^* of problem (QCQP) and there exists $\gamma \in \mathbb{R}^m_+ \setminus \mathbf{0}_m$ such that $A_J + \sum \gamma_k A_k \succeq 0$.

The assumption $\exists \gamma \in \mathbb{R}^m_+ \backslash \mathbf{0}_m$ such that $A_J + \sum \gamma_k A_k \succeq 0$, is a standard assumption in the literature related to QCQP, but it is non verified by nonconvex QP (quadratic problems with linear constraints only), since $A_J \not\succeq 0$. KKT conditions and the SDP relaxation for QP are studied for example in [13], [7].

The assumption that there exists a global minimum x^* of problem (S-QCQP), allows us to consider a heterogeneous set of cases, like the ones of following examples.

- ullet (S-QCQP) with nonconvex quadratic and linear constraints, when the Hessian of the objective function J is positive definite.
- (S-QCQP) where the objective function has a negative definite Hessian and convex quadratic and linear constraints.

The KKT conditions for (QCQP) with one convex quadratic constraint are studied for example in [22], [29]. In paper [15], the authors prove that (KKT) are necessary and sufficient for (QCQP) such that $\forall k \in \{J,1,...,m\}$ the matrices $H_k := \begin{pmatrix} A_k & b_k \\ b_k^T & c_k \end{pmatrix}$ are Z-matrices (which are matrices with non positive off diagonal elements), if there exists x_0 such that $f_k(x_0) < 0 \, \forall \, k \in \{1,...,m\}$.

In order to characterize the optimal value of Z-matrices QCQP, in [15] the authors propose a generalized version of the S-Lemma, in the sense described in Section 2-Preliminaries.

We apply the approach proposed in [15] for a specific subclass of (QCQP) to prove that the conditions (KKT) are necessary and sufficient optimality conditions for (S-QCQP), .

The main ingredients used in deriving the result of the present paper are a generalized version of the S-Lemma and the convexity of the set

$$\Omega_0 := \{(f_0(x), f_1(x), ..., f_m(x)) | x \in \mathbb{R}^n\} + int\mathbb{R}^{m+1}_+ \text{ with } f_0 := J(x) - J(x^*).$$

Then we prove the legitimacy of this approach in the context of (S-QCQP).

The paper [26] provides a comprehensive survey on the famous S-Lemma, while [16] explores the relations between the S-Lemma and the Lagrangian multipliers of (QCQP).

The S-Lemma was the first important results related to the more general S-procedure described in [11]. These important theorems are treated from an historical point view in [14]. Under data uncertainty, [4] applies a S-Lemma based approach similar to the one proposed by [15] and us. The paper [28] extend the results of [15] to Z-matrices (QCQP) infinite number of inequalities.

We will exploit (KKT) to prove that the convex relaxations (SDP) and (SOCP) of (S-QCQP) are exact. We say that a relaxation is exact when the objective values at a global minimum of the original problem and of the relaxation coincide.

Hence, in order to find the optimal value of (S-QCQP) problems, we can use solvers capable to deal with (SOCP) and (SDP) programs, such as the ones described in [9], [30],[17].

Even if the (SDP) is not exact for a (QCQP), it still provides a lower bound for the optimal value of the (QCQP), [2].

Many papers and books are dedicated to the convex relaxations of (QCQP) problem and, in particular, the semi-definite programming (SDP) relaxation and its modifications (see e.g. [31], [25], [21], [8]).

The algorithms proposed in literature often solves exactly the RLT-SDP relaxation of a QCQP, which improves the approximation of the solution given by the SDP relaxation, [2], [3]. The papers of [23] and [10] propose algorithms which provides a tighter lower bound to (QCQP) then the SDP relation. The algorithm proposed by [10], can deal also with a MIQCP (mixed-integer quadratically constrained program). The MIQCP can be defined by adding the following constraints to (QCQP): taking I:=1,...,n and $J\subset I,\ x_i\in\mathbb{N}\ \ \forall i\in J$ and $x_i\in\mathbb{R}^n\ \ \ \forall i\in I\backslash J$. Paper [12] offers an exhaustive survey on solvers which treat different types of QCQP and MIQCP, notwithstanding the elevated obsolescence rate of this kind of works.

In papers [6] and [19], the authors study the SDP relaxation in the framework of SD QCQP (see section 3.3), i.e. when all the matrices A_i , $i=\{0,...,m\}$ which appear in (QCQP) are SD (simultaneously diagonalizable), see section 6. In this framework, the SDP relaxation is equivalent to other convex relaxations as the Second Order Cone Programming (SOCP) relaxation, see [1], [17], [18], [20].

The organization of the paper is as follows. In Section 2, we provide the preliminaries concerning the notation, the S-Lemma, the Fermat rule and the convex separation theorem which are used in the sequel.

The main result of Section 3 is Theorem 7, which provides the global minima characterization for general (QCQP) problems in the form of (KKT) conditions.

In Section 4 we apply Theorem 7 to provide a characterization of the global solution for (S-QCQP). The main result is Theorem 8 which proves the convexity of the set Ω_0 . In Section 5 we discuss the exactness of the SDP and the SOCP relaxation for (S-QCQP), with the help of the equivalent convex relaxation provided by [6].

In Section 6, we make a comparison with the result provided in Section 4 and the results on global minima characterization which can be found in literature for (QCQP) when the number of constraints m=2. Finally, we solve the (KKT) condition for (S-QCQP) with m=2 and n>>m.

2. Preliminaries

Given a vector $x \in \mathbb{R}^n$, we say $x \geq 0$ when, $(\forall i \in \{1,...,m\})$, $x_i \geq 0$, $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x \geq 0\}$. $\mathbf{0}_n$ denotes the all-zero vector in \mathbb{R}^n . Given two vectors $x,y \in \mathbb{R}^n$, $\langle x,y \rangle := \sum_{i=1}^n x_i y_i$ denotes the inner product between x and y and $\langle x,y \rangle = x^T y = y^T x$. The corresponding norm is the Euclidean norm denoted by $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$.

Let S^n be the set of symmetric matrices in $\mathbb{R}^{n \times n}$. S^n_+ and S^n_{++} are the cones of symmetric matrices which are also positive semidefinite and positive definite, respectively. If a matrix A belongs to S^n_+ then we write $A \succeq 0$; if $A \in S^n_{++}$ then we write $A \succ 0$.

The general quadratic functional is of the form

$$f_k(x) := x^T A_k x + 2b_k^T x + c_k (2.1)$$

where $A_k \in S^n$, $b_k \in \mathbb{R}^n$ and $c_k \in \mathbb{R}$; k = 1, ..., m are the indices which refer to the constraints of (QCQP).

 $diag(A) \in \mathbb{R}^n$ is the vector of the elements in the main diagonal of $A \in S^n$. The trace inner product $\langle A,B \rangle$, between symmetric matrices A,B of dimension $n \times n$ is defined as $\langle A,B \rangle := Tr(B^TA) = \sum_{i=1}^n \sum_{j=1}^n a_{ij}b_{ij}$. Let $|\cdot|$ denote the cardinality of a set. C is an affine subspace if $C \neq \emptyset$ and $\forall \lambda \in \mathbb{R}$ and for all distinct $x,y \in C$

$$\lambda x + (1 - \lambda)y \in C$$

aff C denotes the affine hull of C, i.e. the smallest affine subspace of \mathbb{R}^n containing C. $B:=\{x\,|\,\|x\|\leq 1\}$ is the Euclidean unit ball in \mathbb{R}^n . $int\, C$ denotes the interior of C,

$$int C := \{ x \in C \mid (\exists \epsilon > 0) (x + \epsilon B) \subset C \}$$

 $ri\,C$ is the relative interior of C, i.e. :

$$ri C := \{ x \in \mathsf{aff} C \mid (\exists \, \epsilon > 0) \, (x + \epsilon B) \cap (\mathsf{aff} C) \subset C \} \tag{2.2}$$

An important theorem for the optimality conditions of (QCQP) is the S-lemma. We recall a generalized version of the S-Lemma that can be found in [26].

Theorem 1. (Yakubovich S-Lemma) Let $f,g:\mathbb{R}^n\to\mathbb{R}$ be quadratic functionals of the form (2.1) and suppose that there exists a point $x_0\in\mathbb{R}$ such that $g(x_0)<0$. The following statements (i) and (ii) are equivalent.

$$(\nexists x \in \mathbb{R}^n)$$
 s.t.
$$f(x) < 0$$
 (i) $g(x) \le 0$

$$\exists \gamma \ge 0 \text{ such that } f(x) + \gamma g(x) \ge 0 \ \forall x \in \mathbb{R}^n$$
 (ii)

Definition 1. Consider a collection of quadratic functionals $f_k : \mathbb{R}^n \to \mathbb{R}$ (k = 1, ..., m) of the form (2.1). A theorem of the alternative is called generalized version of the S-Lemma if it establishes under which assumptions on the functionals f_k only one between the following statements holds:

(1)
$$\exists x \in \mathbb{R}^n$$
 such that $f_k(x) < 0 \quad \forall k \in \{1, ..., m\}$

(2)
$$(\exists \gamma \in \mathbb{R}_+^m \backslash \mathbf{0}_m) \sum_{k=1}^m \gamma_k f_k(x) \ge 0 \ \forall x \in \mathbb{R}^n$$

Another important result is as follows.

Theorem 2. (Fermat necessary optimality conditions) Assume that $f: \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable on an open set $D \subset \mathbb{R}^n$. Then,

- if $x^* \in D$ is a local minimizer of f, then it must verify $\nabla f(x^*) = 0$.
- if f is twice continuously differentiable, then we also have $\nabla^2 f(x^*)$ positive semidefinite.

Moreover, in the convex case Theorem 2 can be rewritten in the following form.

Theorem 3. (Fermat necessary and sufficient optimality conditions) Assume that $f: \mathbb{R}^n \to \mathbb{R}$ is convex and continuously differentiable on an open set $D \subset \mathbb{R}^n$. Then, $x^* \in D$ is a global minimizer of f if and only if

$$\nabla f(x^*) = 0.$$

Let C_1 and C_2 be non-empty sets in \mathbb{R}^n .

Definition 2. ([27], section 11)

- A hyperplane H is said to separate C_1 and C_2 if C_1 is contained in one of the closed half spaces associated to H and C_2 is contained in the opposite closed half space.
- H is said to separate properly C₁ and C₂ if they are not both contained in H
 itself.

We are ready to state the convex separation theorem that will be crucial in the next section.

Theorem 4. ([27], Theorem 11.3)

Let C_1 and C_2 be non-empty convex sets in \mathbb{R}^n . In order that there exists a hyperplane that separates C_1 and C_2 properly, it is necessary and sufficient that $ri C_1$ and $ri C_2$ have no point in common.

For n-dimensional convex sets in \mathbb{R}^n , $aff C = \mathbb{R}^n$ and so, by (2.2), we have ri C = int C ([27], section 6).

Hence, we can rewrite Theorem 5 as follows.

Theorem 5. Let C_1 and C_2 be full dimensional non-empty convex sets in \mathbb{R}^n . In order that there exists a hyperplane that separates C_1 and C_2 properly, it is necessary and sufficient that $int \, C_1$ and $int \, C_2$ have no point in common.

3. Global Minima Characterization for General (QCQP)

In this section we characterize global minima of (QCQP) problem by (KKT) conditions derived with the help of a generalized form of the S-Lemma as defined in Definition 1.

Our approach is inspired by the one proposed in [15] to characterize the global minima of Z-matrices (QCQP), i.e. (QCQP) with the matrices

$$H_k := \begin{pmatrix} A_k & b_k \\ b_k^T & c_k \end{pmatrix} \quad k = 1, ..., m \text{ and } H_J := \begin{pmatrix} A_J & b_k \\ b_J^T & c_J \end{pmatrix}$$

$$\tag{3.1}$$

having all the off diagonal elements non positive.

In contrast to [15], we are not in the framework of Z-matrices (QCQP), but we consider the general case of (QCQP) problems.

In the sequel, we take into account the following additional assumption.

Assumption 3. Consider a collection of quadratic functionals $f_k(x) = x^T A_k x + b_k^T x + c_k \ k = 0, ..., m$. The set Ω_0 , with

$$\Omega_0 := \{ (f_0(x), f_1(x), ..., f_m(x)) \mid x \in \mathbb{R}^n \} + int \mathbb{R}_+^{m+1}$$
(3.2)

is convex.

We prove a generalized version of the S-Lemma, in the form of a theorem of the alternatives.

Theorem 6. If Assumption 3 holds, then exactly one of the following statements is valid:

(i)
$$\exists x \in \mathbb{R}^n$$
 such that $f_k(x) < 0 \ \forall k \in \{0, ..., m\}$

(ii)
$$(\exists \gamma \in \mathbb{R}^{m+1}_+ \setminus \boldsymbol{0}_{m+1}) \sum_{k=0}^m \gamma_k f_k(x) \ge 0 \ \forall x \in \mathbb{R}^n$$

Proof. The implication $[Not(ii)\Rightarrow(i)]$ is immediate (by contradiction). To show the implication $[Not(i)\Rightarrow(ii)]$, assume that (i) does not hold, i.e., the system

$$f_k(x) < 0 \quad \forall k \in \{0, ..., m\}$$
 (3.3)

has no solution. By the definition of Ω_0 in (3.2), the inconsistency of the system (3.3) implies that

$$\Omega_0 \cap (-\mathsf{int}\mathbb{R}^{m+1}_+) = \emptyset. \tag{3.4}$$

To see this, suppose by contrary, that there exists $y\in\Omega_0\cap(-\mathrm{int}\mathbb{R}^{m+1}_+)$. By the definition (3.2) of Ω_0 (for functions f_k , k=0,...,m), there exist $x_0\in\mathbb{R}^n$, $C_0,C_1\in\mathrm{int}\mathbb{R}^{m+1}_+$ such that

$$y = (f_0(x_0), ..., f_m(x_0)) + C_0 = -C_1 \in (-int\mathbb{R}^{m+1}_+),$$

i.e., $(f_0(x_0),...,f_m(x_0))=-C_0-C_1\in(-{\rm int}\mathbb{R}^{m+1}_+)$ contradictory to (3.3). This proves (3.4).

Since Ω_0 and $-\text{int}\mathbb{R}^{m+1}_+$ are full dimensional in \mathbb{R}^{m+1} , non-empty and convex, by (3.4),

$$int \Omega_0 \cap (-int \mathbb{R}^{m+1}_+) = \emptyset,$$

we can apply Theorem 5. So there exists a hyperplane which separates Ω_0 and $-\mathrm{int}\mathbb{R}^{m+1}_+$ properly, i.e. $\gamma\in\mathbb{R}^m\backslash\mathbf{0}_m$ such that

$$\sum_{k=0}^{m} \gamma_k y_k \ge 0 \quad \forall \ y \in \Omega_0 \tag{3.5}$$

and $\sum_{k=0}^{m} \gamma_k y_k \leq 0 \quad \forall \ y \in (-int\mathbb{R}^{m+1}_+)$. This latter inequality shows that it must be $\gamma \in \mathbb{R}^{m+1}_+ \setminus \mathbf{0}_{m+1}$.

Consequently, by the definition (3.2) of Ω_0 , for k=0,...,m

$$y = (f_0(x), ..., f_m(x)) + C \in \Omega_0, \quad C \in \text{int}\mathbb{R}^{m+1}_{\perp}.$$

By this, and the formula (3.5), we get

$$\sum_{k=0}^{m} \gamma_k(f_k(x) + C_k) \ge 0 \quad \forall \ x \in \mathbb{R}^n, \quad C_k \in \text{int}\mathbb{R}_+.$$
 (3.6)

Consequently, it must be

$$\sum_{k=0}^{m} \gamma_k f_k(x) \ge 0 \quad \forall \ x \in \mathbb{R}^n.$$
 (3.7)

Otherwise, $\sum_{k=0}^m \gamma_k f_k(\bar{x}) < 0$ for some $\bar{x} \in \mathbb{R}^n$, and it would be possible to choose $C \in \operatorname{int}\mathbb{R}^{m+1}_+$ with components $C_k > 0$ small enough so as

$$\sum_{k=0}^{m} \gamma_k (f_k(\bar{x}) + C_k) < 0$$

which would contradict (3.5) since $(f_0(\bar{x}) + k_0, ..., f_m(\bar{x}) + k_m) \in \Omega_0$. Thus, (3.7) holds which proves (ii).

In the following, we exploit Theorem 6 to get necessary and sufficient optimality conditions for general (QCQP).

Theorem 7. Let Assumption 2 hold and x^* be a global minimizer of (QCQP). Define

$$f_0(x) := J(x) - J(x^*) = x^T A_J x + b_J^T x - (x^*)^T A_J x^* - 2b_J^T x^*$$

Let Assumption 3 hold, i.e. the set Ω_0 defined in (3.2) is convex.

Then the following Fritz-John conditions are necessary for optimality, i.e. there exists a vector $(\gamma_0,...,\gamma_m) \in \mathbb{R}^{m+1}_+ \backslash \mathbf{0}_{m+1}$ such that

(i)
$$\nabla(\gamma_0 J + \sum_{k=1}^m \gamma_k f_k)(x^*) = 0$$

(ii) $\gamma_k f_k(x^*) = 0 \quad k \in \{1, ..., m\}$
(iii) $\gamma_0 A_J + \sum_{k=1}^m \gamma_k A_k \succeq 0$ (3.8)

Moreover, if there exists a point $x_0 \in \mathbb{R}^n$ such that

$$f_k(x_0) < 0 \quad \forall k \in \{1, ..., m\}$$
 (3.9)

then there exists a vector $(\gamma_1,...,\gamma_m) \in \mathbb{R}^m_+ \backslash \mathbf{0}_m$ such that

(i)
$$\nabla (J + \sum_{k=1}^{m} \gamma_k f_k)(x^*) = 0$$

(ii) $\gamma_k f_k(x^*) = 0 \quad k \in \{1, ..., m\}$
(iii) $A_J + \sum_{k=1}^{m} \gamma_k A_k \succeq 0$

are necessary for optimality. Moreover, given a feasible x^* for problem (QCQP) satisfying (3.9), the conditions (KKT) are also sufficient for global optimality of x^* .

Proof. Let $f_0(x):=J(x)-J(x^*)$. Since x^* is a global minimizer of (QCQP), $f_0(x)\geq 0$ $\forall x$ feasible for (QCQP). Hence, the system $f_k(x)<0$ k=0,...,m has no solution. By Theorem 6, there exists $(\gamma_0,...,\gamma_m)\in\mathbb{R}_+^{m+1}\backslash\mathbf{0}_{m+1}$ such that

$$\tilde{L}(x) := \gamma_0 f_0(x) + \sum_{k=1}^m \gamma_k f_k(x) \ge 0 \quad \text{for all } x \in \mathbb{R}^n.$$
 (3.10)

In particular, for $x=x^*$, we have $\sum\limits_{k=1}^m \gamma_k f_k(x^*) \geq 0$. Since $\gamma_k f_k(x^*) \leq 0 \ \forall k \in \{1,...,m\}$, it must be $\gamma_k f_k(x^*) = 0 \ \forall k \in \{1,...,m\}$ which proves (ii) of (3.8). By (3.10), for all $x \in \mathbb{R}^n$

$$\gamma_0 J(x) + \sum_{k=1}^{m} \gamma_k f_k(x) \ge \gamma_0 J(x^*)$$
 (3.11)

Hence $L(x):=\gamma_0 J(x)+\sum\limits_{k=1}^m \gamma_k f_k(x)$ attains its minimum over \mathbb{R}^n at x^* . We can apply Theorem 2 for a twice continuously differentiable function L(x). The necessary optimality conditions $\nabla_x L(x^*)=0$ and $\nabla_x^2 L(x^*)\succeq 0$ are respectively equivalent to the conditions (i) and (iii) of (3.8), which finishes the proof of the first part.

Suppose now that (3.9) holds, i.e., there exists a point x_0 such that $f_k(x_0) < 0$ $\forall\, k=1,...,m.$ If it were $\gamma_0=0$, then by (3.10), it would be $\sum\limits_{k=1}^{m}\gamma_kf_k(x)\geq 0$ for all $x \in \mathbb{R}^n$, which would contradict (3.9). Hence $\gamma_0 > 0$ and the Fritz-John conditions becomes the KKT condition, i.e. (KKT) holds.

To complete the proof, we show that conditions (KKT) are also sufficient for optimality. Assume that there exists $x^* \in \mathbb{R}^n$ which is feasible to (QCQP) and $(\gamma_1, ..., \gamma_m) \in$ $\mathbb{R}^m_+ \backslash \mathbf{0}_m$ such that (KKT) holds. The Lagrangian for (QCQP) is:

$$L(x,\gamma) := J(x) + \sum_{k=1}^{m} \gamma_k f_k(x) = x^T A(\gamma) x + 2b(\gamma)^T x + c(\gamma)$$
 (3.12)

with
$$A(\gamma)=A_J+\sum\limits_{k=1}^m\gamma_kA_k$$
, $b(\gamma)=b_J+\sum\limits_{k=1}^m\gamma_kb_k$ and $c(\gamma)=c_J+\sum\limits_{k=1}^m\gamma_kc_k$. Notice that the Lagrangian $L(x,\gamma)$ is convex with respect to x , since $A(\gamma)=\sum\limits_{k=1}^m\gamma_kc_k$

 $A_J + \sum_{k=1}^m \gamma_k A_k \succeq 0$ by (KKT).

Hence x^* such that $\nabla_x L(x^*) = 0$ is the minimum of $L(x,\gamma)$ for γ fixed, by

By (KKT), $\gamma_k f_k(x^*) = 0$ for k = 1, ..., m. We have

$$J(x) + \sum_{k=1}^{m} \gamma_k f_k(x) \ge J(x^*) + \sum_{k=1}^{m} \gamma_k f_k(x^*) = J(x^*) \quad \forall x \in \mathbb{R}^n.$$
 (3.13)

For any x feasible for (QCQP), $f_k(x) \leq 0$ and hence

$$J(x) \ge J(x) + \sum_{k=1}^{m} \gamma_k f_k(x) \tag{3.14}$$

Combining (3.13) and (3.14), for any x feasible for (QCQP), we have

$$J(x) \ge J(x^*) \tag{3.15}$$

which proves that x^* is a global minimum for (QCQP).

4. Global minima characterization for (S-QCQP)

In the present section we use the results of Section 3 to provide (KKT) characterization of global minima for (S-QCQP). The main result of this section is Theorem

Let x^* be the global minimum of (S-QCQP). As in (7), we use the notation

$$f_0(x) := J(x) - J(x^*) = a_J ||x||^2 + b_J^T x - a_J ||x^*||^2 - 2b_J^T x^*$$

In the case of (S-QCQP), the set Ω_0 , defined in (3.2), take the form

$$\Omega_0 := \{ (f_0(x), f_1(x), ..., f_m(x)) | x \in \mathbb{R}^n \} + int \mathbb{R}^{m+1}_+$$

where

$$f_k(x) := a_k ||x||^2 + 2b_k^T x + c_k \quad k = 0, ..., m$$

with $a_k \in \mathbb{R}$.

Theorem 8. Consider problem (S-QCQP) with m+1 < n. The set Ω_0 defined in (3.2) *is convex.*

Proof. In order to show that $\Omega_0 \subset \mathbb{R}^{m+1}$ is convex, take any $v := (v_0, ..., v_m), w := (w_0, ..., w_m) \in \Omega_0$ and $\lambda \in (0, 1)$. There exist $x_v, x_w \in \mathbb{R}^n$ such that

$$f_k(x_v) < v_k \text{ and } f_k(x_w) < w_k \ \forall k \in \{0, ..., m\}.$$
 (4.1)

Consider the convex combination $\lambda v + (1 - \lambda)w$. Let $(\lambda v + (1 - \lambda)w)_k$ be the k-th component of $\lambda v + (1 - \lambda)w$. By (4.1), we have

$$\lambda f_k(x_v) + (1 - \lambda)f_k(x_w) < (\lambda v + (1 - \lambda)w)_k \quad \forall k \in \{0, ..., m\}.$$
 (4.2)

In order to prove that Ω_0 is convex, we show that the convex combination $\lambda v + (1-\lambda)w$ belongs to Ω_0 , i.e. there exists \tilde{x} such that $\forall\,k\in\{0,...,m\}$

$$f_k(\tilde{x}) < (\lambda v + (1 - \lambda)w)_k. \tag{4.3}$$

Formulas (4.2) and (4.3) together imply that Ω_0 is convex if there exists $\tilde{x} \in \mathbb{R}^n$ such that $\forall k \in \{0,...,m\}$

$$f_k(\tilde{x}) \le \lambda f_k(x_v) + (1 - \lambda) f_k(x_w). \tag{4.4}$$

Note that if $x_v=x_w$, Then (4.4) trivially holds for $\tilde{x}=x_v=x_w$. From now on, we assume that x_v and x_w are distinct vectors, $x_v\neq x_w$. Let us take \tilde{x} such that

$$\tilde{x} \in \mathcal{S}^n := \{ x \in \mathbb{R}^n \mid ||x||^2 = \lambda ||x_v||^2 + (1 - \lambda) ||x_w||^2 \}. \tag{4.5}$$

In general, the set \mathcal{S}^n is a sphere centered at zero with radius $\lambda \|x_v\|^2 + (1-\lambda)\|x_w\|^2$. In particular case, when both $x_v = x_w = 0$ the set \mathcal{S}^n reduces to $\{0\}$, but this is impossible, since we assumed that x_v and x_w are distinct vectors.

Clearly, \tilde{x} satisfies (4.3) if $\forall k \in \{0,...,m\}$:

$$f_k(\tilde{x}) \le \lambda f_k(x_v) + (1 - \lambda) f_k(x_w) < \lambda v_k + (1 - \lambda) w_k.$$

$$a_k \|\tilde{x}\|^2 + 2b_k^T \tilde{x} + c_k \le a_k (\lambda \|x_v\|^2 + (1 - \lambda) \|x_w\|^2) + 2b_k^T (\lambda x_v + (1 - \lambda) x_w) + c_k.$$
(4.6)

To ensure (4.6), we show that we can choose \tilde{x} satisfying (4.5) such that

$$b_k^T \tilde{x} \le b_k^T (\lambda x_v + (1 - \lambda) x_w),$$

$$b_k^T (\tilde{x} - (\lambda x_v + (1 - \lambda) x_w)) < 0.$$
(4.7)

Define $y:=\tilde{x}-(\lambda x_v+(1-\lambda)x_w)\in\mathbb{R}^n$. Observe that $y=y(\lambda,\tilde{x})$, where $\lambda\in(0,1)$ and $\tilde{x}\in\mathcal{S}^n$. Clearly, the solution set of the system of inequalities

$$b_k^T y \le 0 \ k \in \{0, ..., m\}$$

includes all the solutions of the system of homogeneous equations

$$b_k^T y = 0 \quad k \in \{0, ..., m\}$$

$$\begin{bmatrix} b_1^0 & \cdots & b_n^0 \\ \vdots & \ddots & \vdots \\ b_1^m & \cdots & b_n^m \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$(4.8)$$

In the sequel we will look for $y = \tilde{x} - (\lambda x_v + (1 - \lambda)x_w) \in \mathbb{R}^n$ among the solutions of the system of equations (4.8). Let

$$B = \begin{bmatrix} b_1^0 & \cdots & b_n^0 \\ \vdots & \ddots & \vdots \\ b_1^m & \cdots & b_n^m \end{bmatrix}$$

By assumption, $rank(B) \le m+1 < n$. The rank-nullity theorem states that the solutions of system (4.8) form a vector space of dimension p = n - rank(B).

Let $u_1,...,u_p$ be a basis for the vector space of solutions to system (4.8). Given some scalars $\alpha_i \in \mathbb{R}$, $i \in \{1,...,p\}$ a solution y of (4.8) can be written as $y = \sum_{i=1}^p \alpha_i u_i$.

To complete the proof, we need to show that there exist some scalars α_i $i \in \{1,...,p\}$ such that

$$\tilde{x} = \sum_{i=1}^{p} \alpha_i u_i + \lambda x_v + (1 - \lambda) x_w \in \mathcal{S}^n, \tag{4.9}$$

where S^n is defined as in (4.5). In fact, by choosing \tilde{x} as in (4.9), $y = \sum_{i=1}^p \alpha_i u_i$ is a solution of (4.8) and $\tilde{x} \in S^n$ satisfies (4.7). We have to prove that one can choose α_i , i = 1, ..., p such that

$$\tilde{x} = \sum_{i=1}^{p} \alpha_i u_i + \lambda x_v + (1 - \lambda) x_w \in \mathcal{S}^n$$

$$\| \sum_{i=1}^{p} \alpha_i u_i + \lambda x_v + (1 - \lambda) x_w \|^2 = \lambda \|x_v\|^2 + (1 - \lambda) \|x_w\|^2$$
(4.10)

Below we show that we can only consider y of the form $y=\alpha_*u_*$, where an index * is chosen arbitrarily from $\{1,...,p\}$. Indeed, given an index $*\in\{1,...,p\}$, set $\alpha_i=0$ for $i\in\{1,...,p\}\setminus\{*\}$ and $\alpha_*\neq 0$. With this choice of $\alpha_i,\ i=1,...,p$ we can rewrite (4.10) as

$$\tilde{x} = \alpha_* u_* + \lambda x_v + (1 - \lambda) x_w$$

and consequently, we need to find $\alpha_* \in \mathbb{R}$ such that

$$\|\alpha_* u_* + \lambda x_v + (1 - \lambda) x_w\|^2 =$$

$$= \alpha_*^2 \|u_*\|^2 + \|\lambda x_v + (1 - \lambda) x_w\|^2 + 2\alpha_* \langle u_*, \lambda x_v + (1 - \lambda) x_w \rangle =$$

$$= \lambda \|x_v\|^2 + (1 - \lambda) \|x_w\|^2$$
(4.11)

By Corollary 2.14 of [5],

$$\|(\lambda x_v + (1 - \lambda)x_w)\|^2 = \lambda \|x_v\|^2 + (1 - \lambda)\|x_w\|^2 - \lambda(1 - \lambda)\|x_v - x_w\|^2.$$
 (4.12)

Hence, (4.11) becomes

$$\alpha_*^2 \|u_*\|^2 + 2\alpha_* \langle u_*, \lambda x_v + (1 - \lambda) x_w \rangle - \lambda (1 - \lambda) \|x_v - x_w\|^2 = 0$$
 (4.13)

Note that by (4.13) it must be $\alpha_* \neq 0$ since x_v and x_w are distinct vectors. There exists α_* such that (4.13) holds (i.e. $\tilde{x} \in \mathcal{S}^n$), if and only if

$$\Delta := (\langle u_*, (\lambda x_v + (1 - \lambda)x_w) \rangle)^2 + \lambda (1 - \lambda) \|u_*\|^2 \|x_v - x_w\|^2 \ge 0,$$

which holds for every $x_v, x_w \in \mathbb{R}^n$.

By the definition of y and (4.9), $y(\tilde{x})=a_*u_*$ belongs to the vector space of the solutions of (4.8). Hence, (4.7) holds by choosing \tilde{x} as in (4.9), with $\alpha_i=0$ for $i\in\{1,...,p\}\setminus\{*\}$ and $\alpha_*\neq 0$ satisfying (4.13).

- **Remark 2.** (1) Observe that the assumption m+1 < n is important for the validity of the presented proof of Theorem 8. Otherwise, if rank B = m+1 = n, the only solution of the system (4.8) is y = 0 which implies that it must be $\tilde{x} = \lambda x_v + (1-\lambda)x_w \in \mathcal{S}^n$, i.e. $\lambda x_v \perp (1-\lambda)x_w$ which can hardly be satisfied.
 - (2) Under the assumption of Theorem 8, there could exist a component $i \in \{0,...,m\}$ such that $f_i(x) := \|x\|^2$ and a component $j \in \{0,...,m\}$ such that $f_j(x) := -\|x\|^2$.

In this case, \tilde{x} must satisfies

$$\begin{cases} \|\tilde{x}\|^2 \le \lambda \|x_v\|^2 + (1-\lambda)\|x_w\|^2 \\ -\|\tilde{x}\|^2 \le -(\lambda \|x_v\|^2 + (1-\lambda)\|x_w\|^2) \end{cases}$$

The above proves that, in order to complete the proof of Theorem 8, we cannot choose \tilde{x} such that

$$\|\tilde{x}\|^2 \neq \lambda \|x_v\|^2 + (1-\lambda)\|x_w\|^2$$

This motivates our approach of looking for suitable $ilde{x}$ from among elements of \mathcal{S}^n

(3) Let us note that in the special case where $x_v, x_w \neq 0$ and $u_* = \pm (\lambda x_v + (1 - \lambda)x_w)$ for some $* \in \{1, ..., p\}$, the formula (4.10) reduces to

$$\tilde{x} = (\alpha_* \pm 1)((\lambda x_v + (1 - \lambda)x_w)$$

and $\tilde{x} \in \mathcal{S}^n$ iff

$$|\alpha_* \pm 1| = \frac{\lambda ||x_v||^2 + (1 - \lambda)||x_w||^2}{||\lambda x_v + (1 - \lambda)x_w||}$$

Theorem 8 allows us to prove that (KKT) conditions are necessary and sufficient optimality conditions for (S-QCQP) with m+1 < n under some standard assumptions, as stated in the following theorem.

Theorem 9. Consider (S-QCQP) such that m+1 < n. Let Assumption 2 holds and x^* be a global minimizer of (S-QCQP). Define the matrices H_0, H_k k=1,...,m and Ω_0 as in Theorem 7. Then the Fritz-John conditions (3.8) are necessary for optimality. Moreover, if there exists a point $x_0 \in \mathbb{R}^n$ such that

$$f_k(x_0) < 0 \quad \forall k \in \{1, ..., m\}$$

then (KKT) are necessary and sufficient for global optimality of x^* , which is feasible for (S-QCQP).

Proof. By Theorem 8, the set Ω_0 defined as in (3.2) is convex, i.e. Assumption 3 holds for the quadratic functionals which appear in (S-QCQP). We can apply Theorem 7 to complete the proof.

Remark 3. The condition (iii) of (KKT), i.e. $A_J + \sum\limits_{k=1}^m \gamma_k A_k \succeq 0$ can be rewritten

as $(a_J + \sum\limits_{k=1}^m \gamma_k a_k)I \succeq 0$ when Assumption 1 holds, i.e. for (S-QCQP). Since all

the eigenvalues of $(a_J + \sum\limits_{k=1}^m \gamma_k a_k)I$ are equal to $a_J + \sum\limits_{k=1}^m \gamma_k a_k$, (iii) takes the form

$$a_J + \sum_{k=1}^m \gamma_k a_k \ge 0.$$

5. SDP and SOCP relaxations exactness for (S-QCQP)

Problem (QCQP) can be rewritten as

$$Minimize_{x \in \mathbb{R}^n} Tr(A_J, X) + 2b_J^T x$$

 $s.t. \ Tr(A_k, X) + 2b_k^T x + c_k \le 0 \ k = 1, ..., m$ (5.1)
 $X = xx^T$

It is possible to relax the constraint $X = xx^T$ with $X - xx^T \succeq 0$, which is equivalent to the semidefinite constraint $\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0$. We obtain the semidefinite or Shor relaxation for (QCQP), [31]:

$$\begin{aligned} &Minimize_{x \in \mathbb{R}^n} Tr(A_J, X) + 2b_J^T x\\ &s.t. \ Tr(A_k, X) + 2b_k^T x + c_k \leq 0 \ k = 1, ..., m\\ &\binom{1}{x} \quad x^T \\ & X \end{pmatrix} \succeq 0 \end{aligned} \tag{SDP}$$

Consider a diagonal (QCQP), which means that the matrices A_k k = J, 1, ..., m are diagonal. Let $\alpha_k \in \mathbb{R}^n$ be the vector of all the diagonal entries of A_k , $\forall k$. Then it is possible to rewrite a diagonal (QCQP) as follow:

$$\begin{aligned} & Minimize_{x,y \in \mathbb{R}^n} & \alpha_J^T y + 2b_J^T x \\ & s.t. & \alpha_k^T y + 2b_k^T x + c_j \leq 0 & (\forall \, k \in \{1,...,m\}) \\ & s.t. & x_i^2 - y_i = 0, \ \forall \, i \end{aligned} \tag{5.2}$$

If we relax the constraints with $x_i^2 - y_i \le 0$ ($\forall i$), we obtain the following convex relaxation as in [6]:

$$\begin{aligned} &Minimize_{x,y\in\mathbb{R}^n} \quad \alpha_J^T y + 2b_J^T x\\ &s.t. \quad \alpha_k^T y + 2b_k^T x + c_k \leq 0 \quad (\forall\, k\in\{1,...,m\})\\ &s.t. \quad x_i^2 - y_i \leq 0, \quad \forall\, i \end{aligned} \tag{SDP2}$$

For diagonal QCQP, (SDP2) and (SDP) are equivalent, as stated in the next theorem:

Theorem 10. Consider a diagonal (QCQP). The SDP relaxation, (SDP), can be rewritten as the relaxation (SDP2).

Proof. Let (X,x) be feasible for (SDP), then $X-xx^T\succeq 0$. The principal minors of order 1 of $X-xx^T$ are $X_{ii}-x_ix_i$ i=1,...,n, i.e. the elements on the main diagonal of the symmetric matrix $X-xx^T$. If $X-xx^T\succeq 0$ then a necessary condition for $X-xx^T$ to be positive semidefinite is $X_{ii}\geq x_ix_i=x_i^2$ i=1,...,n (Sylvester criterion). Since $A_k\in S^n,\ k=J,1,...,m$, are diagonal then $diag(A_k)^Tdiag(X)=Tr(A_kX)\ \forall k$. So, only the elements in the main diagonal of X are relevant for the objective function and the constraints of (SDP). We can substitute the constraint $\begin{pmatrix} X & x \\ x^T & 1 \end{pmatrix}\succeq 0$ with $X_{ii}>x_i^2$ $\forall i\in\{1,...,n\}$.

$$\begin{split} X_{ii} & \geq x_i^2 \quad \forall \, i \in \{1,...,n\}. \\ \text{Taking } y & \in \mathbb{R}^n \text{ such that } y_j = X_{jj} \quad \forall \, j \in \{1,...,n\}, \text{ the proof is complete.} \end{split}$$

Since (SDP2) is a convex problem, under the Slater condition, the KKT conditions are necessary and sufficient for global optimality. Consider the (SDP2) relaxation for (S-QCQP). We have that every component of α_k , k=J,1,...,m, is equal to a

constant a_k . For $\nu_i \geq 0$ i=1,...,n and $\gamma_k \geq 0$ k=1,...,m, The KKT conditions for the (SDP2) relaxation for (S-QCQP) are:

$$a_J + \sum_{k=1}^{m} \gamma_k a_k - \nu_i = 0 \quad \forall i \in \{1, ..., n\}$$
 (5.3)

$$2b_{Ji} + 2\sum_{k=1}^{m} \gamma_k b_{ki} + 2\nu_i x_i = 0 \quad \forall i \in \{1, ..., n\}$$
(5.4)

$$\gamma_k(a_k \sum_{i=1}^n y_i + 2\sum_{i=1}^n b_{ki} x_i + c_k) = 0 \quad \forall k \in \{1, ..., m\}$$
 (5.5)

$$\nu_i(x_i^2 - y_i) = 0 \quad \forall i \tag{5.6}$$

$$f_k(y,x) := a_k \sum_{i=1}^n y_i + 2\sum_{i=1}^n b_{ki} x_i + c_k \le 0 \quad \forall k \in \{1, ..., m\}$$
 (5.7)

$$x_i^2 - y_i \le 0 \quad \forall i \in \{1, ..., n\}$$
 (5.8)

The necessary and sufficient optimality conditions for (S-QCQP) are (KKT). They can be rewritten, for $\gamma_k \ge 0$ k=1,...,m, as follow:

$$a_J + \sum_{k=1}^m \gamma_k a_k \ge 0 \tag{5.9}$$

$$2b_J + 2\sum_{k=1}^{m} \gamma_k b_k + 2[a_J + \sum_{k=1}^{m} \gamma_k a_k]x = 0$$
(5.10)

$$\gamma_k(a_k \sum_{i=1}^n x_i^2 + 2\sum_{i=1}^n b_{ki} x_i + c_k) = 0 \quad \forall k \in \{1, ..., m\}$$
 (5.11)

$$f_k(x) := a_k \sum_{i=1}^n x_i^2 + 2 \sum_{i=1}^n b_{ki} x_i + c_k \le 0 \quad \forall k \in \{1, ..., m\}$$
 (5.12)

Theorem 11. Consider (S-QCQP) such that m+1 < n. Let Assumption 2 holds and x^* be a global minimizer of (S-QCQP). Assume that there exists x_0 such that $f_k(x_0) < 0 \ \forall \ k\{1,...,m\}$. Then (SDP2) is exact and (x^*,y^*) , with $y_i^* = (x^*)^2 \ \forall \ i \in \{1,...,n\}$ is a global optimum for (SDP2).

Proof. By Theorem 7, the conditions (5.9)-(5.12) are necessary and sufficient for the optimality of (S-QCQP). Hence x^* fulfills all the conditions (5.9)-(5.12). Now consider the conditions (5.3)-(5.8). By setting $x_i^2 = y_i \,\forall\, i$, the system composed by (5.3)-(5.8) can be rewritten as (5.9)-(5.12). Hence, (x^*,y^*) , with $y_i^* = (x^*)^2 \,\forall\, i$ fulfill also the KKT conditions (5.3)-(5.8) for (SDP2). Since the KKT conditions are necessary and sufficient for the optimality of (SDP2), (x^*,y^*) is an optimal solution of (SDP2). Notice that in x^* the objective function of (S-QCQP) yields the same value of the objective function of (SDP2) in (x^*,y^*) . So (SDP2) is an exact convex relaxation. \Box

Remark 4. By Theorem 10, for a diagonal (QCQP), if (SDP2) is exact, then (SDP) is also exact. An analogous result was obtained in [32].

The next result can be seen as an adaptation of Theorem 3.5 from [17] for (S-QCQP).

Theorem 12. Problem (SDP2) can be rewritten as the following Second Order Cone Program (SOCP)

$$Minimize_{\mathbf{x} \in \mathbb{R}^{3n}} \sum_{i=1}^{n} (a_J, a_J, b_{Ji})^T \mathbf{x}_i$$

$$\sum_{i=1}^{n} (a_k, a_k, b_{ki})^T \mathbf{x}_i + c_k \le 0 \quad \forall k \in 1, ..., m$$

$$\mathbf{x}_i \in \mathcal{L}_i$$
(SOCP)

where $\mathbf{x}_i = \left[\frac{y_i+1}{2}, \frac{y_i-1}{2}, x_i\right]$ and the second order cones \mathcal{L}_i are defined as

$$\mathcal{L}_i := \{(\frac{y_i+1}{2}, \frac{y_i-1}{2}, x_i) | \ \ x_i, y_i \in \mathbb{R} \ \text{and} \ \frac{y_i+1}{2} \geq \|[\frac{y_i-1}{2}, x_i]\|\}$$

Proof. The constraints $y_i \geq x_i^2$ $i \in \{1,...,n\}$ can be rewritten as

$$\frac{y_i + 1}{2} \ge \|[x_i, \frac{y_i - 1}{2}]\|$$

In fact, $y_i \geq x_i^2$ implies $y_i \geq 0$. Moreover, $\frac{y_i^2}{4} - \frac{y_i}{2} + \frac{1}{4} \geq 0 \ \forall y_i \in \mathbb{R}$ and $x_i^2 \geq 0$. Hence,

$$\frac{y_i+1}{2} = \sqrt{y_i + \frac{y_i^2}{4} - \frac{y_i}{2} + \frac{1}{4}} \ge \sqrt{x_i^2 + \frac{y_i^2}{4} - \frac{y_i}{2} + \frac{1}{4}} = \|[x_i, \frac{y_i-1}{2}]\| \Leftrightarrow y_i \ge x_i^2$$

which proves that the constraints $y_i \geq x_i^2$ are equivalent to $(\frac{y_i+1}{2}, \frac{y_i-1}{2}, x_i) \in \mathcal{L}_i \ \forall i \in \{1, ..., n\}.$

Define the vector $\mathbf{x} \in \mathbb{R}^{3n}$ such that we can group every three components into the vector

$$\mathbf{x}_i := \left[\frac{y_i + 1}{2}, \frac{y_i - 1}{2}, x_i\right] \ i \in \{1, ..., n\}$$

Notice that, for $k \in \{J, 1, ..., m\}$, we have:

$$a_k \sum_{i=1}^n y_i + 2\sum_{i=1}^n b_{ki} x_i = a_k \sum_{i=1}^n \frac{y_i + 1}{2} + a_k \sum_{i=1}^n \frac{y_i - 1}{2} + 2\sum_{i=1}^n b_{ki} x_i = \sum_{i=1}^n (a_k, a_k, b_{ki})^T \mathbf{x}_i$$

Remark 5. Consider problem (S-QCQP) and its convex relaxation (SDP2). Assume that (\bar{x}, \bar{y}) is an optimal solution of (SDP2) and there exist $\bar{\nu}_i, \bar{\gamma}_k$ k=1,...,m; i=1,...,n not all null, such that the KKT conditions (5.3)-(5.8) are satisfied in $(\bar{x}, \bar{y}, \bar{\nu}, \bar{\gamma})$.

Notice that $\bar{\nu}_i$ are all equal to $a_J + \sum\limits_{k=1}^m \gamma_k a_k$, and so we will consider just a single $\bar{\nu} \in \mathbb{R}_+$. If $\bar{\nu} > 0$, then $\bar{y}_i = \bar{x}_i^2 \ \forall i \in \{1,...,n\}$ and hence \bar{x} is also feasible and optimal for (S-QCQP) and (SDP2) is exact.

6. Solving the KKT system for nonconvex S-QCQP

In this section, we consider a S-QCQP problem of the form

$$Minimize_{x \in \mathbb{R}^n} \ J(x) := 1/2||x - z||^2 \ s.t. \ x \in A$$
 (P1)

where

$$A := \{x \in \mathbb{R}^n \mid f_k(x) := a_k ||x||^2 + b_k^T x + c_k \le 0, \ k = 1, ..., m\} \text{ and } m + 1 < n\}$$

Note that in order to provide a characterization of the global minima of (S-QCQP), in Theorem 7 we assumed that a global solution $x^* \in A$ of (S-QCQP) exists. This assumption is not restrictive for (P1) due to the strong convexity of the objective J. Hence, the problem (P1) is solvable.

Assume that there exists $x_0 \in A$ such that $f_k(x_0) < 0$ for k=1,...,m. By Theorem 8, the set Ω_0 is convex.

By Theorem 7, an element x^* feasible for (P1) is a global minimizer of problem (P1) if and only if conditions (KKT) hold. Taking into account that

$$\nabla f_k(x) = 2a_k x + b_k, \quad k = 1, ..., m$$

(KKT) conditions for (P1) take the form: there exists a vector $(\gamma_1,...,\gamma_m) \in \mathbb{R}_+^m \setminus \mathbf{0}_m$ such that

$$x^* + \sum_{k=1}^m \gamma_k (2a_k x^* + b_k) = z \quad \text{stationarity}$$

$$\gamma_k f_k(x^*) = 0 \quad k = 1, ..., m \quad \text{complementarity}$$

$$1 + 2 \sum_{k=1}^m \gamma_k a_k \ge 0, \quad x^* \in A \quad \text{dual and primal feasibility}$$
 (KKT-P1)

KKT conditions (KKT-P1) can be equivalently rewritten as

$$(1.) \ x^*(1 + \sum_{k=1}^m \gamma_k 2a_k) = z - \sum_{k=1}^m \gamma_k b_k \text{ stationarity}$$

$$(2.) \ \gamma_k[a_k \|x^*\|^2 + b_k^T x^* + c_k] = 0 \quad k = 1, ..., m \text{ complementarity}$$

$$(3.) \ 1 + 2 \sum_{k=1}^m \gamma_k a_k \ge 0, \ x^* \in A \text{ feasibility.}$$

$$(6.1)$$

In the case $z=\mathbf{0}_n$, (KKT-P1) reduces to finding $(\gamma_1,...,\gamma_m)\in\mathbb{R}_+^m\backslash\mathbf{0}_m$

$$(1.) \ \ \underbrace{1 + 2\sum_{k=1}^{m} \gamma_k a_k}_{=m} \ge 0, \ \ \text{dual feasibility}$$

- (2.) $f_k(x^*) = a_k ||x^*||^2 + b_k^T x^* + c_k \le 0$ k = 1, ..., m primal feasibility (KKT-P10)
- (3.) $\gamma_k f_k(x^*) = 0$ k = 1, ..., m complementarity.

(4.)
$$\sum_{k=1}^{m} \gamma_k b_k = -x^* w \text{ stationarity}$$

Notice that if $z \neq \mathbf{0}_n$, we can always apply conditions (KKT-P10) after changing variable, i.e. taking $\bar{x} := x - z$. Since conditions (KKT-P10) are easier to handle, in the sequel we will consider always $z = \mathbf{0}_n$.

Consider a nonconvex and unbounded feasible set A for problem (P1). We can describe the feasible set A by using the terminology introduced by [33]. Consider a concave constraints $f_i(x)$, for some $i \in \{1,...,m\}$ (we have $a_i < 0$). We say that $-f_i(x)$ induces a hollow in \mathbb{R}^n . Notice that in [33], it is assumed that the hollows induced by different constraints are non intersecting, while we do not have this assumption.

In the literature, it is not common to consider a possibly nonconvex and unbounded feasible set as described above, if the number of constraints is greater than 2. Hence, in this section, we will compare our result with the optimality condition found in [15] for (QCQP) with two constraints and in [6] and [20] for simultaneously diagonalizable (SD) (QCQP) with m=2. ((QCQP) is SD if there exists a real matrix S such that the matrices S^TA_kS , k=J,1...,m are all diagonal). Then, we solve system (6.1) for (P1) with m=2.

Theorem 4.4. by [15] is the following.

Theorem 13. (Theorem 4.4. by [15]) Consider a (QCQP) such that Assumption 2 holds. Assume that $m=2, n\geq 3$ and $\exists \gamma_1, \gamma_2 \in \mathbb{R}$ such that $\gamma_1 H_1 + \gamma_2 H_2 \succ 0$, with H defined as in (3.1). Then the Fritz-John conditions (3.8) are necessary for global optimality. Moreover, if $\exists x_0 \in \mathbb{R}^n$ such that $f_1(x_0) < 0$ and $f_2(x_0) < 0$, the conditions (KKT) are necessary and sufficient for global optimality for a point x^* feasible for (QCQP).

Remark 6. For (S-QCQP) with m=2, Theorem 13 and Theorem 9 provide the same result under different assumptions. Hence, when m=2, we can replace the assumption n>m+1=3 in Theorem 9 with $n\geq 3$ and $\exists \, \gamma_1,\gamma_2\in \mathbb{R}$ such that $\gamma_1H_1+\gamma_2H_2\succ 0$, i.e.

$$\exists \ \gamma_1, \gamma_2 \in \mathbb{R} \ \text{s.t.} \ (\gamma_1 a_1 + \gamma_2 a_2) > 0, \ \ (\gamma_1 c_1 + \gamma_2 c_2)(\gamma_1 a_1 + \gamma_2 a_2) - \|\gamma_1 b_1 + \gamma_2 b_2\|^2 > 0.$$

Consider an SD (QCQP) with m=2. In [6], it is shown that if there exists a KKT point $(\bar{x},\bar{y},\bar{\gamma_1},\bar{\gamma_2},\bar{\nu_i}\ i=1,...,n)$ such that only one between $\bar{\gamma_1},\bar{\gamma_2}$ is strictly greater than 0, then (SDP2) is exact. Also, it is possible to recover the global solution x^* of the SD (QCQP) with m=2 with simple calculations. Paper [20] takes the result of [6] as starting point and provides the conditions such that (SDP2) is exact for SD (QCQP) with m=2 and $\bar{\gamma_1}>0,\bar{\gamma_2}>0$. However, the following example from [20] shows that the number n of dimensions of the space can still be an issue.

Example 14. (Example 3.1 from [20]) Consider the following problem (S-QCQP) problem.

$$Minimize_{x \in \mathbb{R}} - x^2 + x \quad s.t. \quad x^2 \le 1 \quad -x \le 0$$
 (6.2)

Note that (SDP2) is not exact.

6.1. Solving (KKT) conditions for (P1) with two constraints. In the present subsection we solve the problem of finding a global solution x^* to (P1) with the help (KKT-P10), by direct inspection of all possible configurations $\gamma_i \neq 0$, $i \in I$, $\gamma_i = 0$, $i \in \{1,2\} \setminus I$, where $I \subseteq \{1,2\}$ is a subset of $\{1,2\}$. We are interested in the case such that n >> 3.

The number of all possible configuration is 4, with

$$I \in \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$$

Assume $\mathbf{0}_n$ is not feasible. Then we can exclude the case where $I=\emptyset$.

6.1.1. I is a single element set. We start with the case where I is a single element set. Without loss of generality, set $I = \{1\}$ and $\gamma_1 > 0, \gamma_2 = 0$.

Conditions (1.)-(4.) of (KKT-P10) take the form

(1.)
$$w := 1 + 2\gamma_1 a_1 \ge 0$$
, dual feasibility

(2.)
$$f_2(x^*) = a_2 ||x^*||^2 + b_2^T x^* + c_2 \le 0$$
 primal feasibility (6.3)

(3.)
$$f_1(x^*) = a_1 ||x^*||^2 + b_1^T x^* + c_1 = 0$$
 complementarity

(4.) $\gamma_1 b_1 = -x^* w$ stationarity

Theorem 15. Let $\gamma_1 > 0, \gamma_2 = 0$. Let w = 0, i.e. $\gamma_1 = -\frac{1}{2a_1} > 0$. If $b_1 = 0$ and there exists a solution x^* of the system

$$\begin{cases} ||x^*||^2 = -\frac{c_1}{a_1} \\ b_2^T x^* \le a_2 \frac{c_1}{a_1} - c_2 \end{cases}$$
 (6.4)

Then x^* is a global solution of (P1).

Proof. Consider the case w=0. By (1.) of (6.3), if $a_1 \geq 0$, then w=0 is impossible. Hence, the case w=0 could appear only if $a_1<0$. Moreover,

$$||x^*||^2 = -\frac{c_1}{a_1}.$$

We must have $b_1 = 0$, not to generate contradiction in (4.).

The (KKT) system (6.3) takes the form

(1.)
$$w:=1+2\gamma_1a_1=0 \Rightarrow \gamma_1=-\frac{1}{2a_1}>0,$$
 dual feasibility

$$(2.) \ b_2^T x^* \le a_2 \frac{c_1}{a_1} - c_2 \text{ primal feasibility}$$
 (6.5)

$$(3.) \ \|x^*\|^2 = -\frac{c_1}{a_1} \ \text{complementarity}$$

The next remark is about the feasibility of system (6.4).

Remark 7. Consider the linear constraint $b_2^T x^* \le a_2 \frac{c_1}{a_1} - c_2$. By assumption, $x = \mathbf{0}_n$ does not belong to the feasible set. Hence, if (6.4) is feasible, then the hyperplane

$$b_2^T x = a_2 \frac{c_1}{a_1} - c_2,$$

must intersect or be tangent to the n-dimensional sphere $||x||^2 = -\frac{c_1}{a_1}$.

We can find the least square solution \bar{x} of $b_2^T x^* \leq a_2 \frac{c_1}{a_1} - c_2$. If

$$\|\bar{x}\|^2 \le \|x^*\|^2 = -\frac{c_i}{a_i}$$

then there exists a solution x^* which satisfies (6.4).

Let us now consider the case w>0. The case w>0 is simpler than the case w=0considered in Theorem 15. In fact, by (4.)

$$(x^*)^T x^* = \frac{\gamma_i^2}{w^2} b_1^T b_1,$$

and by the complementarity condition (3.),

$$f_1(x^*) = a_1 \frac{\gamma_1^2}{w^2} b_1^T b_1 - \frac{1}{w} \gamma_1 b_1^T b_1 + c_1 = 0$$
 (6.6)

$$\Delta = \frac{(b_1^T b_1)^2}{w^2} - 4a_1 c_1 \frac{b_1^T b_1}{w^2} = \frac{b_1^T b_1}{w^2} [b_1^T b_1 - 4a_1 c_1]$$

If $[b_1^Tb_1 - 4a_1c_1] \ge 0$, then (6.6), has two (or a double) roots: γ_1^1 and γ_1^2 . If at least one of them is positive and it is such that w > 0, then it satisfies the dual feasibility conditions (1.) and

$$x^* = \frac{-1}{w} \gamma_1 b_1$$

is a candidate for a solution to (P1) provided it satisfies the primal feasibility conditions (2.) of (6.3).

Additionally, observe that if $a_1 = 0$, then w = 1 and (6.6) takes the form

$$c_1 = \gamma_1 b_1^T b_1, \quad \gamma_1 = \frac{b_1^T b_1}{c_1}.$$
 (6.7)

Notice that by (6.7) $c_1 > 0$. Otherwise, there is no positive γ_1 which satisfies (6.6).

6.1.2. I is not a single element set. Consider the following theorem

Theorem 16. Assume that $\mathbf{0}_n$ is not feasible and there does not exist a KKT point such that $I = \{1\}$ or $I = \{2\}$. Then there exists a (KKT) point such that $I = \{1, 2\}$.

Proof. Since $\mathbf{0}_n$ is not feasible, there does not exist a KKT point such that $I=\emptyset$. By assumption there does not exist a KKT point such that $I=\{1\}$ or $I=\{2\}$. On the other side, an optimal point of (P1) must exist since the Hessian of the objective function is positive definite and the KKT conditions (KKT-P10) are necessary and sufficient for optimality by Theorem 9. Hence there exists a KKT point such that $I=\{1,2\}$.

Conditions (1.)-(4.) of (KKT-P10) take the following form. There exist $\gamma_1 \geq 0, \gamma_2 \geq 0$ not both null such that:

- (1.) $w := 1 + 2\gamma_1 a_1 + 2\gamma_2 a_2 \ge 0$, dual feasibility
- (2.) $f_i(x^*) = a_i ||x^*||^2 + b_i^T x^* + c_i = 0$ i + 1, 2 complementarity (6.8)
- (3.) $\gamma_1 b_1 + \gamma_2 b_2 = -x^* w$ stationarity

Assumption 4. Assume that the vectors b_1, b_2 of the active constraints are linearly independent.

Observe that, whenever n is very big, the Assumption 4 is not very restrictive and it is likely to be satisfied automatically.

Theorem 17. Consider problem (P1) satisfying Assumption 4 such that $\mathbf{0}_m$ is not feasible for (P1). Assume that $a_1 \neq 0$. Let $I = \{1,2\}$ and n > 3. Consider the following system of one quadratic polynomial and one linear equations with respect to the variables γ_1, γ_2 .

$$(1.) \begin{cases} \frac{1}{a_1 w} b_1^T (\gamma_1 b_1 + \gamma_2 b_2) - \frac{c_1}{a_1} = \frac{1}{a_2 w} b_2^T (\gamma_1 b_1 + \gamma_2 b_2) - \frac{c_2}{a_2} & \text{if } a_2 \neq 0 \\ \frac{1}{w} b_2^T (\gamma_1 b_1 + \gamma_2 b_2) + c_2 = 0 & \text{if } a_2 = 0 \end{cases}$$

$$(2.) \sum_{i \in I} \gamma_i^2 [\|b_i\|^2 - 2\frac{a_i}{a_1} b_1^T b_i + 4\frac{c_1}{a_1} a_i^2] + \sum_{i \in I} \gamma_i [4a_i \frac{c_1}{a_1} - \frac{b_1^T b_i}{a_1}] -$$

$$-2\gamma_1 \gamma_2 (\frac{a_2}{a_1} \|b_1\|^2 + 4a_1 a_2) + \frac{c_1}{a_1} = 0$$

$$(6.9)$$

If there exist $\gamma_1, \gamma_2 > 0$, not both zero, which solve (6.9), and

$$w = 1 + 2\gamma_1 a_1 + 2\gamma_2 a_2 > 0 \tag{6.10}$$

then

$$x^* = -\frac{1}{w}(\gamma_1 b_1 + \gamma_2 b_2)$$

is a global solution of (P1).

Proof. By the fact that 0 is not feasible for (P1), Assumption 4 implies that the condition (1.) of (6.8) for any index set I become w>0. Indeed, if it were w=0, then 0-vector would be represented as a linear nonzero combination of vectors b_1,b_2 by (3.)of (6.8), which would imply that b_1 and b_2 are linearly dependent, contrary to Assumption 4.

By the stationarity condition (3.) of (6.8),

$$x^* = -\frac{1}{w}(\gamma_1 b_1 + \gamma_2 b_2), \quad (x^*)^T x^* = \frac{1}{w^2}(\gamma_1 b_1 + \gamma_2 b_2)^T (\gamma_1 b_1 + \gamma_2 b_2)$$
 (6.11)

Focus on the complementarity, (2.) of (6.8). Since $a_1 \neq 0$ by assumption, we have

$$f_1(x^*) = a_1 ||x^*||^2 + b_1^T x^* + c_1 = 0$$

$$||x^*||^2 = -\frac{b_1^T x^*}{a_1} - \frac{c_1}{a_1}$$

$$||x^*||^2 = \frac{1}{a_1 w} b_1^T (\gamma_1 b_1 + \gamma_2 b_2) - \frac{c_1}{a_1}$$
(6.12)

Plugging (6.12) into the complementarity condition (2.), we have

$$(1.) ||x^*||^2 = \frac{1}{a_1 w} b_1^T (\gamma_1 b_1 + \gamma_2 b_2) - \frac{c_1}{a_1}$$

$$(2.) \begin{cases} \frac{1}{a_1 w} b_1^T (\gamma_1 b_1 + \gamma_2 b_2) - \frac{c_1}{a_1} = \frac{1}{a_2 w} b_2^T (\gamma_1 b_1 + \gamma_2 b_2) - \frac{c_2}{a_2} \text{ if } a_2 \neq 0 \\ \frac{1}{w} b_2^T (\gamma_1 b_1 + \gamma_2 b_2) + c_2 = 0 \text{ if } a_2 = 0 \end{cases}$$

$$(6.13)$$

Equation (1.) of system (6.13), can be rewritten as

$$\begin{aligned} &\|x^*\|^2 = \frac{1}{a_1 w} b_1^T (\gamma_1 b_1 + \gamma_2 b_2) - \frac{c_1}{a_1} \\ &\frac{1}{w^2} (\gamma_1 b_1 + \gamma_2 b_2)^T (\gamma_1 b_1 + \gamma_2 b_2) = \frac{1}{a_1 w} b_1^T (\gamma_1 b_1 + \gamma_2 b_2) - \frac{c_1}{a_1} \\ &\sum_{i \in I} \gamma_i^2 \|b_i\|^2 + 2\gamma_1 \gamma_2 b_1^T b_2 = \frac{w}{a_1} b_1^T (\gamma_1 b_1 + \gamma_2 b_2) - w^2 \frac{c_1}{a_1} \\ &\dots = \frac{1}{a_1} b_1^T (\gamma_1 b_1 + \gamma_2 b_2) + 2\gamma_1 b_1^T (\gamma_1 b_1 + \gamma_2 b_2) + 2\frac{a_2}{a_1} \gamma_2 b_1^T (\gamma_1 b_1 + \gamma_2 b_2) - w^2 \frac{c_1}{a_1} \\ &\sum_{i \in I} \gamma_i^2 [\|b_i\|^2 - 2\frac{a_i}{a_1} b_1^T b_i] - 2\gamma_1 \gamma_2 \frac{a_2}{a_1} \|b_1\|^2 = \frac{1}{a_1} b_1^T [\sum_{i \in I} \gamma_i b_i] - w^2 \frac{c_1}{a_1} \end{aligned}$$

Moreover,

$$w^{2} = (1 + 2\gamma_{1}a_{1} + 2\gamma_{2}a_{2})^{2} = 1 + 4\sum_{i \in I} \gamma_{i}^{2} a_{i}^{2} + 8\gamma_{1}\gamma_{2}a_{1}a_{2} + 4\sum_{i \in I} a_{i}\gamma_{i}$$

The polynomial quadratic equation becomes

$$\sum_{i \in I} \gamma_i^2 [\|b_i\|^2 - 2\frac{a_i}{a_1} b_1^T b_i + 4\frac{c_1}{a_1} a_i^2] + \sum_{i \in I} \gamma_i [4a_i \frac{c_1}{a_1} - \frac{b_1^T b_i}{a_1}] - 2\gamma_1 \gamma_2 (\frac{a_2}{a_1} \|b_1\|^2 + 4a_1 a_2) + \frac{c_1}{a_1} = 0$$
(6.14)

7. Conclusions

We provide necessary and sufficient (KKT) conditions for global optimality for a new class of possibly nonconvex quadratically constrained quadratic programming (QCQP) problems, denoted by (S-QCQP). Our result relies on a generalized version of the S-Lemma, Theorem 6, stated in the context of general (QCQP) problems. Hence, we believe that this paper can also be the groundwork for providing conditions for global optimality for other classes of (QCQP). In section 5, we prove the exactness of the SDP and the SOCP relaxations for (S-QCQP). In section 6, we solve the (KKT) condition for S-QCQP when the number of constraints m=2 and n>>m.

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