Moments and tail reciprocal connections for the random variables having generalized Gamma - Weibull distributions.

M.R.Formica, E.Ostrovsky, L.Sirota.

Università degli Studi di Napoli Parthenope, via Generale Parisi 13, Palazzo Pacanowsky, 80132, Napoli, Italy.
e-mail: mara.formica@uniparthenope.it

Department of Mathematics and Statistics, Bar-Ilan University, 59200, Ramat Gan, Israel.
e-mail: eugostrovsky@list.ru

Department of Mathematics and Statistics, Bar-Ilan University, 59200, Ramat Gan, Israel.
e-mail: sirota3@bezeqint.net

Abstract.

We establish the one - to one bilateral interrelations between an asymptotic behavior for the tail of distributions for random variables and its great moments evaluation.

Our results generalize the famous Richter's ones.

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1 Introduction. Notations. Statement of problem.

Let $(\Omega = \{\omega\}, \mathcal{B}, \mathbf{P})$ be certain non - trivial probability space with Expectation $\mathbf{E}$ and Variance Var.

Definition 1.1. We will say that the non - negative numerical valued random variable (r.v.) ξ has a generalized Gamma - Weibull distribution, write $\text{Law}(\xi) \in \mathbf{G} = \mathbf{G}(\Gamma, W; \ Q, \ \theta, \ \gamma, \ C, \ t_0)$, iff

$$T[\xi](t) \le t^\theta \, \exp(-C \, t^\gamma) \, Q(t), \ t \ge t_0 = \text{const} > 0. \tag{1}$$

Here and further $T[\xi](t) = \mathbf{P}(\xi \ge t)$ is the so - called *tail function* for the r.v. ξ; $\theta = \text{const} > -1$, γ, C, $t_0 = \text{const} > 0$, $Q = Q(t)$ is positive continuous function such that

$$\forall \alpha > 0 \ \Rightarrow \ \lim_{t \to \infty} Q(t)/t^\alpha = 0.$$

To be more precisely,

$$T[\xi](t) \le \min(1, \ t^\theta \, \exp(-C \, t^\gamma) \, Q(t) \,), \ t \ge 0. \tag{2}$$

The ordinary $p - th$, $p \ge 1$ moment for the (non - negative) random variable ξ will be denoted by $m_p = m_p[\xi]$:

$$m_p[\xi] \stackrel{def}{=} \mathbf{E} \, \xi^p, \ p \ge 1.$$

The classical Lebesgue - Riesz $||\xi||_p$ norm for the r.v. ξ is defined as ordinary

$$||\xi||_p \stackrel{def}{=} [\, m_p[\xi] \,]^{1/p} = [\, \mathbf{E}|\xi|^p \,]^{1/p}, \ p \ge 1.$$

As usually, $L_p = L_p(\Omega) := \{\xi : \ ||\xi||_p < \infty\}$.

Our aim in this report is to establish the bilateral exact connections between the tail behavior at infinity of the tail function for these variables and its moment function behavior, also at infinity.

These type estimates was applied in particular case in the theory of large deviations in the probability theory by W.Richter in articles [17], [18], [19]. Some interest applications of these estimates in the seismology may be found in the article of Gutenberg B., Richter C.F. [11].

Obtained in this report estimates complement and clarify ones (for these variables) in the works [2], [3], [4], [5], [7], [8], [9], [10], [12], [13], [14], [15], [16], chapters 1,2.

2 Main result. Direct estimation.

Suppose at first for instance for the non - negative r.v. ξ that

$$T[\xi](t) \le \min \left[\, t^\beta \, \exp(-t), \, 1 \, \right], \ t \ge t_0 = \text{const} > 0, \ \beta = \text{const} > -1. \tag{3}$$

We deduce using the *key relation*

$$\mathbf{E}|\xi|^p = p \int_0^\infty t^{p-1} \, T[\xi](t) \, dt : \tag{4}$$

$$\mathbf{E}|\xi|^p \le t_0^p + p \int_{t_0}^\infty t^{p+\beta-1} \, e^{-t} \, dt, \; \exists t_0 = \text{const} > 0, \; p \ge 1; \tag{5}$$

following

$$||\xi||_p \le \psi_\beta(p) \stackrel{def}{=} p^{1/p} \, \Gamma^{1/p}(p+\beta), \; p \ge 1. \tag{6}$$

The last relation may be rewritten as follows. Introduce the so - called Grand Lebesgue Space $G\psi_\beta$ consisting on all the random variables $\{\zeta\}$ having a finite norm

$$||\zeta||G\psi_\beta \stackrel{def}{=} \sup_{p \ge 1} \left\{ \frac{||\zeta||_p}{\psi_\beta(p)} \right\} < \infty; \tag{7}$$

then $||\xi||G\psi_\beta \le 1$.

The theory of these spaces is explained in many works, see e.g. [2], [3], [4], [5], [7], [8], [9], [10], [12], [13], [14], [15], [16]. We will use further some methods offered in these articles.

Let us return to the relation (3), assuming that it is given.

We have by virtue of the *key relation* (4)

$$\mathbf{E}\xi^p \le p \int_0^\infty t^{\beta+p-1} \, e^{-t} \, dt = p \, \Gamma(p+\beta), \tag{8}$$

where as ordinary $\Gamma(\cdot)$ denotes an ordinary Euler's Gamma function.

Inversely, let the inequality (8) be given. We intent to evaluate the tail function for the r.v. ξ.

One can use the classical Markov - Tchebychev's inequality for all the values of the parameters p, t such that $p, t \ge \beta + 1$

$$T[\xi](t) \le t^{-p} \, p \, \Gamma(p+\beta).$$

One can deduce choosing the value $p := t$, which is asymptotically as $t \to \infty$ optimal:

$$T[\xi](t) \le t^{-t} \, t \, \Gamma(t+\beta) \le t^{-t} \, \Gamma(t+\beta+1). \tag{9}$$

We obtain applying the famous Stirling's formula for the values $p \in (\beta+1, \infty)$ and denoting

$$c_2 := e^{1/12} \, \sqrt{2\pi} :$$

$$T[\xi](t) \le c_2 \ (t+\beta)^{\beta+1/2} \ e^{-t}. \tag{10}$$

Remark 2.1. Note that there is a "gap" of a size $\sim t^{1/2}$ as $t \to \infty$ between the tail estimate (3) and moment one (8) But the estimate (10) is asymptotically non - improvable; for instance, for the r.v. η having the following tail behavior

$$T[\eta](t) = C_0 \ t^{\beta} \ \exp(-t), \ t > t_1 = \mathrm{const} > 0.$$

Let's move on to the rigorous considerations. Suppose that our non - negative r.v. ξ is such that there exists a constant $\beta > -1$ and a positive numerical valued function $L = L(s), \ s > 0$ such that

$$\mathbf{E}\xi^p \le (p+\beta)^{p+\beta} \ e^{-(p+\beta)} \ L(p+\beta). \tag{11}$$

We apply once more the Markov - Tchebychev's inequality

$$T[\xi](t) \le \exp[\ -p \ln t + (p+\beta)\ln(p+\beta) - (p+\beta) + \ln L(p+\beta) \ ].$$

One can choose in the last inequality the value $p = t - \beta, \ t \ge \beta + 1$.

To summarize:

Proposition 2.1. We deduce under formulated conditions, namely, positivity of the r.v. ξ and, especially, under the condition (11):

$$T[\xi](t) \le t^{\beta} \ e^{-t} \ L(t), \ t \ge \beta + 1. \tag{12}$$

Remark 2.2. It is no hard to deduce the *lower bound* for the moment function $m_p[\xi]$ from the *lower tail* estimate. Namely, assume now that for the non - negative random variable ξ there holds the following tail estimate

$$T[\xi](t) \ge \ \min \left[\ t^{\beta} \ \exp(-t), \ 1 \ \right], \ t \ge t_0 > 0, \ \beta = \mathrm{const} > -1. \tag{13}$$

We deduce using again the *key relation* (4) and the saddle - point method that for all the greatest values $p \ge p_0 = \mathrm{const} > 1$

$$\mathbf{E}|\xi|^p \ge C(t_0, \beta) \ p \ \Gamma(p+\beta), \ \exists C(t_0) \in (0, \infty). \tag{14}$$

3 Main result. Inverse estimation.

Inversely, let the estimate (12) be given. We want to evaluate for this r.v. ξ its the moments function $m[\xi]_p = \mathbf{E}\xi^p$ for all the sufficiently greatest values p, of course, under additional restrictions on the function $L = L(t)$.

We have denoting $M(t) := \ln L(t), \ t \ge e; \ z := p + \beta - 1$ using again the *key relation* (4) that

$$\mathbf{E}\xi^p \le p \int_0^\infty t^{p+\beta-1} \ e^{-t} \ L(t) \ dt =$$

$$p \int_0^\infty \exp(z \ \ln t - t + M(t)) \ dt =$$

$$pz \int_0^\infty \exp(z \ln z + z \ln v - zv + M(zv)) \ dv.$$

Let us suppose now that the function $M(\cdot)$ is slowly varying at infinity:

$$\forall v > 0 \ \Rightarrow \ \lim_{z\to\infty} \left\{ \ \frac{M(zv)}{M(z)} \ \right\} = 1. \tag{15}$$

Proposition 3.1. We deduce under this (and previous) conditions:

$$\mathbf{E}\xi^p \le C \ p \ z \ e^{z \ln z} e^{M(z)} \int_0^\infty v^z \ e^{-zv} \ dv =$$

$$C \ p \ \Gamma(p + \beta) \ L(p + \beta - 1).$$

Remark 3.1. Ii is sufficient to suppose instead the condition (15) the following one

$$\sup_{z\ge e} \sup_{v\ge 1} \left\{ \ \frac{M(zv)}{M(z)} \ \right\} < \infty. \tag{16}$$

4 Generalizations.

It is no hard to generalize obtained result on the case when

$$T[\xi](t) \le t^\theta \ \exp\left(\ -t^\gamma \ \right) \ Q(t), \ \xi \ge 0, \ t \ge t_0 = \mathrm{const} > 0, \tag{17}$$

where $Q = Q(t), \ t > 0$ is some positive a.e. continuous function,

$$\theta = \mathrm{const} \ge 0, \ \gamma = \mathrm{const} > 0.$$

Indeed, this case may be reduced to the investigated above by means of the changing random variable $\eta = \xi^\gamma$, so that

$$T[\eta](t) = \mathbf{P}(\xi > t^{1/\gamma}) \le t^{\theta/\gamma} \, e^{-t} \, Q\left(t^{1/\gamma}\right).$$

Proposition 4.1.

Thus, if the function $S(t) = \exp Q(t^{1/\gamma})$ is positive continuous and slowly varying at infinity, then as $q \in (e, \infty)$

$$\mathbf{E}\, \xi^q \le \frac{q}{\gamma}\, \Gamma\left(\frac{q+\theta}{\gamma}\right)\, Q\left(\frac{q+\theta}{\gamma}\right). \tag{18}$$

The inverse proposition is also true in the following sense.

Proposition 4.2.

Let the estimate (18) be given under at the same restrictions. Then it follows once more from the Tchebychev - Markov inequality

$$T[\xi](t^{1/\gamma}) \le t \cdot t^{-t} \cdot \Gamma(t + \theta/\gamma) \cdot Q(t + \theta/\gamma), \; t \ge 1, \tag{19}$$

and by virtue of the Stirling's approximation

$$T[\xi](t^{1/\gamma}) \le c_2 \cdot t^{1/2} \cdot e^{-t} \cdot (t + \theta/\gamma)^{\theta/\gamma} \cdot Q(t + \theta/\gamma), \; t \ge 1. \tag{20}$$

5   Tauberian theorems.

Tauberian theorems are called as ordinary propositions connecting behavior of certain function, for instance, tail one, with some corresponding behavior its transform, for example, power series, Laplace, Dunkle, Fourier etc. transforms, [20]. We intent to ground in this section the *fine* interrelations between tail behavior $T[\xi](t)$ for the random variable ξ as $t \to \infty$ and asymptotic behavior its moment function $m_p[\xi]$ also as $p \to \infty$.

Analogous problem for the r.v. satisfying the Kramer's condition is investigated in [16], chapter 1, section 1.4., pp. 33 - 35. Some applications in the reliability theory see in [1].

A. Direct assertion.

Proposition 5.1. Suppose that for the non - negative r.v. ξ

$$\lim_{t\to\infty}\left[\frac{|\ln T_\xi(t)|}{t}\right]=1. \qquad (21)$$

Then

$$\lim_{p\to\infty}\left\{\frac{||\xi||_p}{p/e}\right\}=1. \qquad (22)$$

Proof. Let $\delta\in(0,1)$ be an arbitrary fixed number. There exists a value $t_0=t_0(\delta)\in(0,\infty)$ such that

$$t\geq t_0 \Rightarrow T[\xi](t)\leq\exp(-t(1-\delta)).$$

We estimate for the values $p>p_0=\text{const}>1$:

$$\mathbf{E}\xi^p = p\int_0^\infty t^{p-1}\, T[\xi](t)\, dt \leq p\int_0^{t_0} t^{p-1}\, dt+$$

$$p\int_{t_0}^\infty t^{p-1}\, \exp(-t(1-\delta))\, dt \leq t_0^p + p\int_0^\infty t^{p-1}\, \exp(\, -t(1-\delta)\,)\, dt =$$

$$t_0^p + \frac{\Gamma(p+1)}{(1-\delta)^p}.$$

It follows again from the Stirling's formula that

$$\overline{\lim}_{p\to\infty}\frac{||\xi||_p}{p/e}\leq 1,$$

and quite analogously

$$\underline{\lim}_{p\to\infty}\frac{||\xi||_p}{p/e}\geq 1.$$

Thus, the proposition 5.1 is proved.

B. Inverse proposition.

Proposition 5.2. Suppose that for the non - negative r.v. ξ

$$\lim_{p\to\infty}\left\{\frac{||\xi||_p}{p/e}\right\}=1. \qquad (23)$$

Then

$$\lim_{t\to\infty}\left[\frac{|\ln T_\xi(t)|}{t}\right]=1. \qquad (24)$$

Proof. *Upper estimate.* Let again $\delta\in(0,1)$ be an arbitrary fixed number. There exists a value $p_0=p_0(\delta)>1$ such that

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$$\forall p \geq p_0 \ \Rightarrow \ \frac{||\xi||_p}{p/e} \leq 1 + \delta,$$

following

$$\forall p \geq p_0 \ \Rightarrow \ ||\xi||_p \leq (1 + \delta) \cdot (p/e).$$

We apply once again the famous Tchebychev - Markov's inequality for all the sufficiently great values p

$$T[\xi](t) \leq \frac{(1 + \delta)^p \ p^p}{e^p \ t^p} = \exp\left(-p \ln t + p \ln(1 + \delta) + p \ln p - p \right).$$

One can choose as above $p := t, \ t \geq e$:

$$\ln T[\xi](t) \leq -t + t \ln(1 + \delta).$$

Therefore,

$$\overline{\lim}_{t \to \infty} \left[\frac{|\ln T_\xi(t)|}{t} \right] \leq 1. \tag{25}$$

Let us deduce now the *lover estimate.* Given:

$$(1 - \delta)^p \ \frac{p^p}{e^p} \leq \mathbf{E}\xi^p \leq (1 + \delta)^p \ \frac{p^p}{e^p}, \ p \geq p_0 = p_0(\delta) > e.$$

Put $\xi = e^\tau$; recall that $\xi > 0$. Then we have for the value $\lambda \geq e$

$$(1 - \delta)^\lambda \ \exp(\lambda \ \ln \lambda - \lambda) \leq \mathbf{E}e^{\lambda \ \tau} \leq (1 + \delta)^\lambda \ \exp(\lambda \ \ln \lambda - \lambda).$$

The announced lower estimate

$$\underline{\lim}_{t \to \infty} \left[\frac{|\ln T_\xi(t)|}{t} \right] \geq 1 \tag{26}$$

is grounded, up to changing variables, in particular in the monograph [16], chapter 1, section 1.4; see also [1].

6   Concluding remarks.

It is interest in our opinion to deduce some multidimensional version on these result.

References

[1] **Bagdasarov D,R., Ostrovsky E.I.** *Bilateral estimation of the reliability function.* In: Diagnosis and forecasting of reliability for nuclear power devices. Proseedings of the kathedra ASU OIATE, Obninsk, 1995, **10,** 37 - 39.

[2] **V.V. Buldygin V.V., D.I.Mushtary, E.I.Ostrovsky, M.I.Pushalsky.** *New Trends in Probability Theory and Statistics.* Mokslas, (1992), V.1, p. 78 - 92; Amsterdam, Utrecht, New York, Tokyo.

[3] **Capone C, Formica M.R, Giova R.** *Grand Lebesgue spaces with respect to measurable functions.* Nonlinear Analysis 2013; 85: 125 - 131.

[4] **Capone C, and Fiorenza A.** *On small Lebesgue spaces.* Journal of function spaces and applications. 2005; 3; 73 - 89.

[5] **S.V.Ermakov, and E. I. Ostrovsky.** *Continuity Conditions, Exponential Estimates, and the Central Limit Theorem for Random Fields.* Moscow, VINITY, (1986), (in Russian).

[6] **M.V.Fedoryuk.** *The saddle-point method.* Moscow, Science, (1977) (In Russian).

[7] **Fiorenza A., and Karadzhov G.E.** *Grand and small Lebesgue spaces and their analogs.* Consiglio Nationale Delle Ricerche, Instituto per le Applicazioni del Calcoto Mauro Picone, Sezione di Napoli, Rapporto tecnicon. 272/03, (2005).

[8] **A. Fiorenza, M. R. Formica** and **A. Gogatishvili.** *On grand and small Lebesgue and Sobolev spaces and some applications to PDE's. Differ. Equ. Appl.* **10** (2018), no. 1, 21–46.

[9] **A. Fiorenza, M. R. Formica, A. Gogatishvili, T. Kopaliani** and **J. M. Rakotoson.** *Characterization of interpolation between grand, small or classical Lebesgue spaces.* Preprint arXiv:1709.05892, Nonlinear Anal., to appear.

[10] **A. Fiorenza, M. R. Formica** and **J. M. Rakotoson.** *Pointwise estimates for $G\Gamma$-functions and applications.* Differential Integral Equations **30** (2017), no. 11-12, 809–824.

[11] **Gutenberg, B., Richter, C. F.** (1956). *Magnitude and Energy of Earthquakes. Annali di Geofisica,* **9:** 1 - 15, (PDF). Archived from the original (PDF) on 2017-08-09. Retrieved 2015-05-13.

[12] **Yu.V. Kozachenko and E.I. Ostrovsky.** *Banach spaces of random variables of subgaussian type.* Theory Probab. Math. Stat., Kiev, (1985), v. 43, 42 - 56, (in Russian).

[13] **Kozachenko Yu.V., Ostrovsky E., Sirota L.** *Relations between exponential tails, moments and moment generating functions for random variables and vectors.* arXiv:1701.01901v1 [math.FA] 8 Jan 2017

[14] **Kozachenko Yu.V., Ostrovsky E., Sirota L.** *Equivalence between tails, Grand Lebesgue Spaces and Orlicz norms for random variables without Kramer's condition.* Bulletin of KSU, Kiev, 2018, **4,** pp. 20 - 29.

[15] **E.Liflyand, E.Ostrovsky, L.Sirota.** *Structural Properties of Bilateral Grand Lebesgue Spaces.* Turk. J. Math.; **34,** (2010), 207 - 219.

[16] **E.I. Ostrovsky.** *Exponential Estimations for Random Fields.* Moscow - Obninsk, OINPE, (1999), in Russian.

[17] **W.Richter**. *A local limit theorem for large deviations.* Dokl. Akad. Nauk SSSR, Vol. **115**, (1957), pp. 53 - 56. (In Russian.)

[18] **W.Richter**. *Local limit theorems for large deviations.* Teor. Veroyatnost. i Primenen., Vol. 2, (1957), pp. 214 - 229.

[19] **W.Richter**. *Multidimensional local limit theorems for large deviations.* Teor. Veroyatnost. i Primenen., Vol. 3, (1958), pp. 107 - 114.

[20] **Tauber, Alfred.** (1897). *Ein Satz aus der Theorie der unendlichen Reihen.* [A theorem about infinite series]. Monatshefte fur Mathematik und Physik, (in German). **8**, 273 - 277. doi:10.1007/BF01696278. JFM 28.0221.02. S2CID 120692627.