Regular Convergence and Finite Element Methods for Eigenvalue Problems

Bo Gong* Jiguang Sun†

Abstract

Regular convergence, together with various other types of convergence, has been studied since the 1970s for the discrete approximations of linear operators. In this paper, we consider the eigenvalue approximation of compact operators whose spectral problem can be written as the eigenvalue problem of some holomophic Fredholm operator function. Focusing on the finite element methods (conforming, discontinuous Galerkin, etc.), we show that the regular convergence of discrete holomorphic operator functions follows from the approximation property of the finite element spaces and the compact convergence of the discrete operators in some suitable Sobolev space. The convergence for eigenvalues is then obtained using the discrete approximation theory for the eigenvalue problems of holomorphic Fredholm operator functions. The result can be used to show the convergence of various finite element methods for eigenvalue problems such as the Dirhcilet eigenvalue problem and the biharmonic eigenvalue problem.

1 Introduction

Eigenvalue problems of partial differential equations have many important applications in science and engineering, e.g., design of solar cells for clean energy, calculation of electronic structure in condensed matter, extraordinary optical transmission, nondestructive testing, photonic crystals, and biological sensing. Due to the flexibility in treating complex structures and rigorous theoretical justification, finite element methods have been widely used to compute eigenvalue problems [5, 19, 11, 2, 6, 23].

The study of the finite element methods for eigenvalue problems started in 1970s and has been an active research area since then. Many results obtained before 1990s can be found in the book chapter by Babuška and Osborn [2] (see also [6, 23] for some recent developments). The main functional analysis tool is the spectral perturbation theory for linear compact operators [17, 10]. Essentially, if the uniform convergence of the finite element solution operators to the continuous compact operator, the theory of Babuška and Osborn [2] can be employed to obtain the convergence for the eigenvalues

*Department of Mathematical Sciences, Beijing University of Technology, Beijing 100124, People?s Republic of China. *gongbo@bjut.edu.cn*

†Department of Mathematical Sciences, Michigan Technological University, Houghton, MI 49931, U.S.A. *jiguangs@mtu.edu*

and the associated eigenfunctions, i.e., all eigenpairs are approximated and there are no spurious modes.

While the uniform convergence of the discrete operators can be proved for some finite element methods such as the conforming finite element methods, it is impossible or challenging for many other finite element methods, e.g., the discontinuous Galerkin methods. Various methods has been proposed in literature to prove the convergence when the uniform convergence is not available [18, 1, 11]. In this paper, we consider the eigenvalue approximation of compact operators whose spectral problem and reformulate them as the eigenvalue problems of some holomophic Fredholm operator functions. The regular convergence of discrete holomorphic operator functions follows from the approximation property of the finite element spaces and the compact convergence of the discrete operators in some suitable Sobolev space. The convergence for exact eigenvalues and eigenfunctions is then obtained using the discrete approximation theory for the eigenvalue problems of holomorphic Fredholm operator functions in [15, 16]. This work extends the study in [14, 24] and provides an alter way to analyze the convergence of various finite element approximations for eigenvalue problems.

The rest of the paper is arranged as follows. In Section 2, we recall the discrete approximation scheme, different types of convergence for linear operators, and the abstract approximation theory for the eigenvalue problems of holomorphic Fredholm operator functions. Section 3 contains the study of regular convergence related to the finite element approximation operators for the partial differential equations in L^2 space. It turns out that the approximation property of the finite element space and the compact convergence of the discrete solution operator guarantee the regular convergence. Using the abstract approximation theory for holomorphic Fredholm operator functions, we show the convergence of various finite element methods for the Dirichlet eigenvalue problem in Section 4 and the biharmonic eigenvalue problem in Section 5. We end the paper with some conclusions and future work in Section 6.

2 Preliminaries

We introduce the discrete approximation scheme, different types of convergence, and the abstract approximation theory for the eigenvalue problems of holomorphic Fredholm operator functions. We refer the readers to [24, 21, 22, 10, 15, 16] for more details.

2.1 Discrete Approximation Scheme

Let X be a Banach space and $X_n (n \in \mathbb{N})$ be a sequence of approximation spaces for X. Let $P = \{p_n\}_{n\in\mathbb{N}}$ be a sequence of bounded linear operators $p_n : X \to X_n$ such that

$$\|p_n x\|_{X_n} \longrightarrow \|x\|_X \quad (n \in \mathbb{N}). \tag{2.1}$$

Definition 2.1. A sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in X_n$ is said to P-converge to $x \in X$ if

$$\|x_n - p_n x\|_{X_n} \longrightarrow 0 \quad (n \in \mathbb{N}).$$

We write it as $x_n \xrightarrow{P} x$ or simply $x_n \to x\,(n \in \mathbb{N})$.

If $\{x^{(n)}\}(n \in \mathbb{N})$ converges to x in X, it holds that $p_n x^{(n)} \xrightarrow{P} x$ (see Chp. 1 of [24]).

Definition 2.2. A sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in X_n$ is called (discrete) P-compact if for every $\mathbb{N}' \subset \mathbb{N}$ there exists $\mathbb{N}'' \subset \mathbb{N}'$ and an $x \in X$ such that $x_n \xrightarrow{P} x\,(n \in \mathbb{N}'')$.

Let Y be a Banach space. Denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators from X to Y. We denote by $\mathcal{N}(A) = \{x \in X : Ax = 0\}$ and $\mathcal{R}(A) = \{y \in Y : y = Ax, x \in X\}$ the null space and the range of the operator $A \in \mathcal{L}(X, Y)$, respectively.

Definition 2.3. An operator $A \in \mathcal{L}(X, Y)$ is called semi-Fredholm if $\mathcal{R}(A) \subset Y$ is closed and additionally $\mathcal{N}(A)$ has a finite dimension or $\mathcal{R}(A)$ has a finite codimension. If $\mathcal{R}(A)$ is closed and in addition dim$\mathcal{N}(A)$ and codim$\mathcal{R}(A)$ are both finite, A is called Fredholm. The index of a Fredholm operator is defined as

$$\text{ind}A = \text{dim}\mathcal{N}(A) - \text{codim}\mathcal{R}(A).$$

In the following, we define various notions of convergence for a linear operator [21, 24, 10]. They are point convergence, stable convergence, compact convergence and regular convergence. Let Y_n be a sequence of approximation spaces for Y and $Q = \{q_n\}_{n\in\mathbb{N}}$ be a sequence of linear operators $q_n : Y \to Y_n$ such that $\|q_n y\|_{Y_n} \longrightarrow \|y\|_Y\,(n \in \mathbb{N})$.

Definition 2.4. A sequence $\{A_n\}$ of linear operators $A_n \in \mathcal{L}(X_n, Y_n)$ converges (or PQ-converges, or converges discretely) to $A \in \mathcal{L}(X, Y)$ if the following relation holds for every P-converging sequence $\{x_n\}$:

$$x_n \xrightarrow{P} x\,(n \in \mathbb{N}) \Longrightarrow A_n x_n \xrightarrow{Q} Ax\,(n \in \mathbb{N}).$$

We write it as $A_n \longrightarrow A\,(n \in \mathbb{N})$ or $A_n \xrightarrow{PQ} A\,(n \in \mathbb{N})$.

Theorem 2.8 in [24]) claims the equivalence of $A_n \xrightarrow{PQ} A\,(n \in \mathbb{N})$ and

$$\|A_n\| \le const\,(n \in \mathbb{N}) \quad \text{and} \quad \|A_n p_n x - q_n Ax\| \to 0\,(n \in \mathbb{N})\ \forall x \in X.$$

We shall present a theorem for later use. It is a consequence of Theorem 2.8 and an exercise (Theorem 2-9) in [24].

Theorem 2.5. *If $p_n \in \mathcal{L}(E, E_n)$, $p_n E = E_n$ and $b = const.$ with*

$$\inf_{x\in E, p_n x = x_n} \|x\|_E \le b\|x_n\|_{E_n} \quad (n \in \mathbb{N}, \forall x_n \in E_n) \tag{2.2}$$

exists, then $\|A_n p_n x - q_n Ax\| \to 0\,(n \in \mathbb{N})\,\forall x \in X$ implies $\|A_n\| \le const\,(n \in \mathbb{N})$.

Proof. If the conditions of the above theorem are satisfied, by Theorem 2-9 in [24], one has the equivalence of $\|A_n p_n x - q_n Ax\| \to 0\,(n \in \mathbb{N})\,\forall x \in X$ and $A_n \xrightarrow{PQ} A\,(n \in \mathbb{N})$. By Theorem 2.8 in [24], $\|A_n\| \le const\,(n \in \mathbb{N})$. □

Definition 2.6. A sequence $\{A_n\}_{n\in\mathbb{N}}$ of linear operators $A_n \in \mathcal{L}(X_n, Y_n)$ converges stably to $A \in \mathcal{L}(X, Y)$ if the following two conditions are met:

1. $A_n \stackrel{PQ}{\longrightarrow} A \, (n \in \mathbb{N})$;
2. there is some $n_0 \in \mathbb{N}$ such that the inverse operators $A_n^{-1} \in \mathcal{L}(Y_n, X_n)$ exist for all $n \geq n_0$, where
$$\|A_n^{-1}\| \leq C \quad (n \geq n_0).$$

We write briefly $A_n \to A$ stably.

Definition 2.7. A sequence $\{A_n\}_{n\in\mathbb{N}}$ of linear operators $A_n \in \mathcal{L}(X_n, Y_n)$ converges compactly to $A \in \mathcal{L}(X, Y)$ if the following conditions are met:

1. $A_n \stackrel{PQ}{\longrightarrow} A \, (n \in \mathbb{N})$;
2. $\|x_n\|_{X_n} \leq C \, (n \in \mathbb{N}) \Longrightarrow \{A_n x_n\}$ Q-compact.

Definition 2.8. A sequence $\{A_n\}_{n\in\mathbb{N}}$ of linear operators $A_n \in \mathcal{L}(X_n, Y_n)$ converges regularly to $A \in \mathcal{L}(X, Y)$ if the following two conditions are satisfied:

1. $A_n \stackrel{PQ}{\longrightarrow} A \, (n \in \mathbb{N})$;
2. $\|x_n\|_{X_n} \leq C, \quad \{A_n x_n\}$ Q-compact $\Rightarrow \{x_n\}$ P-compact.

We write in short $A_n \to A$ regularly.

The following theorem from [24] (Theorem 2.55 therein) states a connection among the stable convergence, compact convergence, and regular convergence.

Theorem 2.9. *Let*
$$\begin{aligned} &B_n \to B \quad \textit{stably} \quad (B_n \in \mathcal{L}(X_n, Y_n), B \in \mathcal{L}(X, Y)), \\ &C_n \to C \quad \textit{compactly} \quad (C_n \in \mathcal{L}(X_n, Y_n), C \in \mathcal{L}(X, Y)), \end{aligned}$$
where $R(B) = F$. *Then*
$$A_n := B_n + C_n \longrightarrow B + C =: A \quad \textit{regularly}.$$

We also define the (discrete) uniform convergence for late use.

Definition 2.10. A sequence $\{A_n\}_{n\in\mathbb{N}}$ of linear operators $A_n \in \mathcal{L}(X_n, Y_n)$ converges uniformly to $A \in \mathcal{L}(X, Y)$ if $\|A_n p_n - q_n A\| \to 0$ as $n \to \infty$. We write in short $A_n \to A$ uniformly.

2.2 Holomorphic Fredholm Operator Functions

We now introduce the discrete approximation theory for eigenvalue problems of holomorphic Fredholm operator functions [15, 16]. Let X and Y be complex Banach spaces. Let $\Omega \subseteq \mathbb{C}$ be a compact simply connected region. Let $F : \Omega \to \mathcal{L}(X, Y)$ be a holomorphic operator function on Ω and, for each $\eta \in \Omega$, $F(\eta)$ is a Fredholm operator of index zero [13].

Definition 2.11. A complex number $\lambda \in \Omega$ is called an eigenvalue of F if there exists a nontrivial $x \in X$ such that $F(\lambda)x = 0$. The element x is called an eigenelement associated with λ.

The resolvent set $\rho(F)$ and the spectrum $\sigma(F)$ of $F(\cdot)$ are defined respectively as

$$\rho(F) = \{\eta \in \Omega : F(\eta)^{-1} \text{ exists and is bounded}\} \tag{2.3}$$

and

$$\sigma(F) = \Omega \setminus \rho(F). \tag{2.4}$$

If $\rho(F) \neq \emptyset$, since $F(\eta)$ is holomorphic, the spectrum $\sigma(F)$ has no cluster points in Ω and every $\lambda \in \sigma(F)$ is an eigenvalue for $F(\eta)$. Furthermore, the operator function $F^{-1}(\cdot)$ is meromorphic (see Section 2.3 of [15]). The dimension of $\mathcal{N}(F(\lambda))$ is called the geometric multiplicity of an eigenvalue λ.

Definition 2.12. An ordered sequence of elements $x_0, x_1, \ldots, x_k$ in X is called a Jordan chain of F at an eigenvalue λ if

$$F(\lambda)x_j + \frac{1}{1!}F^{(1)}(\lambda)x_{j-1} + \ldots + \frac{1}{j!}F^{(j)}(\lambda)x_0 = 0, \quad j = 0, 1, \ldots, k,$$

where $F^{(j)}$ denotes the jth derivative.

The length of any Jordan chain of an eigenvalue λ is finite. Denote by $m(F, \lambda, x_0)$ the length of a Jordan chain formed by an eigenelement x_0. The maximal length of all Jordan chains of λ is denoted by $\kappa(F, \lambda)$, called the ascent of λ. Elements of any Jordan chain of an eigenvalue λ are called generalized eigenelements of λ.

Definition 2.13. The closed linear hull of all generalized eigenelements of an eigenvalue λ, denoted by $G(\lambda)$, is called the generalized eigenspace of λ.

Let X_n, Y_n be Banach spaces, not necessarily subspaces of X, Y. Let $p_n \in \mathcal{L}(X, X_n)$ and $q_n \in \mathcal{L}(Y, Y_n)$ be such that

$$\lim_{n\to\infty} \|p_n x\|_{X_n} = \|x\|_X,\ x \in X \quad \text{and} \quad \lim_{n\to\infty} \|q_n y\|_{Y_n} = \|y\|_Y,\ y \in Y. \tag{2.5}$$

Denote by $\Phi_0(X, Y)$ the sets in $\mathcal{L}(X, Y)$ of all Fredholm operators of index zero. Consider a sequence of discrete operator functions

$$F_n : \Omega \to \Phi_0(X_n, Y_n), \quad n \in \mathbb{N}.$$

Assume that $\rho(F) \neq \emptyset$ and (b1) $F_n(\eta), \eta \in \Omega$, is a holomorphic Fredholm operator of index zero;

(b2) The sequence $\{F_n(\cdot)\}$ is equibounded on Ω, i.e., $\|F_n(\eta)\| \leq C, \eta \in \Omega$;

(b3) $\{F_n(\eta)\}_{n\in N}$ converges to $F(\eta)$ for every $\eta \in \Omega$ and $x \in X$, i.e.,

$$\|[F_n(\eta)p_n - p_n F(\eta)]x\|_{Y_n} \to 0;$$

(b4) $F_n(\eta)$ converges regularly to $F(\eta)$ for every $\eta \in \Omega$.

The following theorem states that all eigenvalues are approximated correctly (see [15, 16] or [3]).

Theorem 2.14. *Assume that (b1)-(b4) hold. For any $\lambda \in \sigma(F)$ there exists $n_0 \in \mathbb{N}$ and a sequence $\lambda_n \in \sigma(F_n)$, $n \geq n_0$, such that $\lambda_n \to \lambda$ as $n \to \infty$. For any sequence $\lambda_n \in \sigma(F_n)$ with this convergence property and the associate eigenelements $v_n^0 \in \mathcal{N}(F(\lambda_n)), \|v_n^0\|_{X_n} = 1$, one has that*

$$|\lambda_n - \lambda| \leq C\epsilon_n^{1/\kappa}, \tag{2.6}$$

$$\inf_{v\in\mathcal{N}(F(\lambda))} \|v_n^0 - p_n v\|_{X_n} \leq C\epsilon_n^{1/\kappa}, \tag{2.7}$$

where

$$\epsilon_n = \max_{|\eta-\lambda|\leq\delta} \max_{v\in G(\lambda)} \|F_n(\eta)p_n v - q_n F(\eta)v\|_{X_n}$$

for sufficiently small $\delta > 0$.

3 Finite Element Approximations

To analyze finite element methods for eigenvalue problems of partial differential equations, we choose $X = Y = L^2(D)$ for the rest of the paper, where $D \subset \mathbb{R}^2$ is a Lipschitz polygonal domain. The result in this paper holds for higher dimension cases if the corresponding finite element methods approximation properties for the source problem are available. Denote by $\|\cdot\|$ the usual L^2-norm. Let T be the solution operator for the source problem associated to the eigenvalue problem. For example, the solution operator for the Poisson equation with homogeneous Dirichlet boundary condition is associated with the Dirichlet eigenvalue problem. In this section, we present some general results that can be used to prove the convergence of a large class of finite element methods for eigenvalue problems. It turns out that one only needs to verify the approximation property of the finite element spaces and the pointwise convergence of the solution operator for the source problem in L^2 norm.

Assume that $T \in \mathcal{L}(X, X)$ is compact and let $I : X \to X$ be the identity operator. The eigenvalue problem for T is to find $\lambda \neq 0$ and nontrivial $x \in X$ such that

$$Tx = \frac{1}{\lambda}x. \tag{3.1}$$

Define $F(\eta) : \Omega \to \mathcal{L}(X, X)$ such that

$$F(\eta) := T - \frac{1}{\eta}I, \quad \eta \in \Omega, \tag{3.2}$$

where Ω is a compact subset of $\mathbb{C}$ such that $0 \notin \Omega$. It is clear that $F(\cdot)$ is a holomorphic Fredholm operator function on Ω. The eigenvalue problem of $F(\cdot)$ is to find $\lambda \in \Omega$ and $x \in X$ such that

$$F(\lambda)x = 0. \tag{3.3}$$

Clearly the above eigenvalue problem is equivalent to the eigenvalue problem (3.1) for T.

Let $\mathcal{T}_n := \mathcal{T}_{h_n}$ be a regular triangular mesh for D with mesh size $h_n \to 0^+, n \to \infty$. Let $X_n \subset X$ be the associated finite element space endowed with the L^2-norm and $P = \{p_n\}, p_n : X \to X_n$, is the L^2-projection, i.e., for $f \in X, p_n f \in X_n$ is such that

$$(p_n f, x_n) = (f, x_n) \quad \text{for all } x_n \in X_n. \tag{3.4}$$

Let $I_n : X_n \to X_n$ be the identity operator. Assume that there exists a series of finite element approximation operators $T_n : X_n \to X_n$ for T. Define the discrete approximation operators

$$F_n(\eta) := T_n - \frac{1}{\eta} I_n, \quad \eta \in \Omega. \tag{3.5}$$

If $I_n \to I$ stably and $T_n \to T$ compactly, then using Theorem 2.9, it holds that

$$F_n(\eta) \to F(\eta) \quad \text{regularly for } \eta \in \Omega. \tag{3.6}$$

For the convergence analysis of the discrete eigenvalues of $F_n(\cdot)$ to those of $F(\cdot)$, we shall see that the following two conditions are sufficient.

(i) for $x \in X$, $\|p_n x - x\| \to 0$ as $n \to \infty$.

(ii) $T_n \to T$ compactly.

Condition (i) is the approximation property of the finite element spaces X_n and (ii) is the compact convergence of the finite element solution operators to the continuous solution operator for the source problem.

For the finite element spaces X_n consider in this paper, (i) holds. Then the stable convergence of I_n to I is implied by the discrete convergence of I_n to I, i.e., $I_n \to I$.

Lemma 3.1. *Let Condition (i) hold. Then $I_n \to I$ stably.*

Proof. Since (i) holds, $I_n \to I$. Furthermore, $I_n^{-1} : X_n \to X_n$ exists and is bounded, $I_n \to I$ stably. □

The following lemmas are on the sufficient conditions for the compact convergence.

Lemma 3.2. *Let Condition (i) hold and assume that $T_n \to T$ discretely. If $X_n \subset X'$ such that X' is compactly embedded in X, then $T_n \to T$ compactly.*

Proof. Let $\|x_n\| \leq C\,(n \in \mathbb{N})$. Then $\{T_n x_n\}\,(n \in \mathbb{N})$ is bounded in X'. Due to the compact embedding of X' into X, there exist a convergent subsequence of $\{T_n x_n\}\,(n \in \mathbb{N})$, denoted by $\{T_n x_n\}\,(n \in \mathbb{N}' \subset \mathbb{N})$, such that $T_n x_n \to y \in X, n \in \mathbb{N}'$. Hence $T_n \to T$ compactly. The proof is complete. □

Remark 3.3. For a finite element approximation of the source problem, one usually has a discrete solution operator $T_n' : X \to X_n$. In general, the discrete operator $T_n : X_n \to X_n$ is such that $T_n p_n = T_n'$.

Lemma 3.4. *Let Condition (i) hold and assume that $X_n \subset X, n \in \mathbb{N}$. Let $T : X \to X$ be a compact operator. If $T_n \to T$ uniformly, then $T_n \to T$ compactly.*

Proof. Let $\|x_n\| \le C\,(n \in \mathbb{N})$. Since T is compact, $\{Tx_n\}$ has a convergent subsequence $\{Tx_n\}_{n \in N'}, N' \subset \mathbb{N}$ such that $Tx_n \to y \in X$ as $N' \ni n \to \infty$. For $n \in N'$,

$$\begin{aligned}\|T_n x_n - p_n y\| &= \|T_n x_n - p_n T x_n + p_n T x_n - p_n y\| \\ &\le \|T_n p_n x_n - p_n T x_n\| + \|p_n T x_n - p_n y\| \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Hence $\{T_n x_n\}_{n \in \mathbb{N}}$ is discretely compact and $T_n \to T$ compactly. □

Theorem 3.5. *Let $T : X \to X$ be compact. Assume that $T_n \to T$ compactly and $I_n \to I$ stably. Then*

1. $\|F_n(\eta)\| \le C$ *for* $\eta \in \Omega$;
2. *For every* $\eta \in \Omega$ *and* $x \in X$, $\|[F_n(\eta)p_n - p_n F(\eta)]x\| \to 0$;
3. $F_n(\eta) \to F(\eta)$ *regularly for each* $\eta \in \Omega$.

Proof. 1. Since $X_n \subset X$ and p_n are the L^2-projections, we clearly have that $p_n \in \mathcal{L}(X, X_n)$ and $p_n X = X_n$. Furthermore, since p_n are orthogonal projections, (2.2) is satisfied by taking $x = x_n$ and $b = 1$. Since $T_n \to T$ discretely, T_n are uniformly bounded in n due to Theorem 2.5. It is clear that I_n are uniformly bounded. Then $\|F_n(\eta)\| \le C$ due to the fact that Ω is compact.

2. For a fixed $\eta \in \Omega$ and $x \in X$,

$$\begin{aligned}\|[F_n(\eta)p_n - p_n F(\eta)]x\| &= \left\|(T_n p_n - p_n T)x - \frac{1}{\eta}(I_n p_n - p_n I)x\right\| \\ &\le \|(T_n p_n - p_n T)x\| + \left\|\frac{1}{\eta}(I_n p_n - p_n I)x\right\| \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned}$$

3. Since $T_n \to T$ compactly and $I_n \to I$ stably, then $F_n(\eta) \to F(\eta)$ regularly for each $\eta \in \Omega$ due to Theorem 2.9.

□

Theorem 3.6. *Let $X = L^2(D)$ and $X_n \subset X$ be a sequence of finite element spaces. Let $P = \{p_n\}$ where $p_n : X \to X_n$ is the L^2-projection. Let $T : X \to X$ be a compact operator and $T_n : X_n \to X_n$ be a sequence of discrete operators. Assume that conditions (i) and (ii) hold. Then for an eigenvalue λ of T, there exist n_0 and a* *sequence of eigenvalues λ_n of T_n, $n > n_0$, such that $\lambda_n \to \lambda$ as $n \to \infty$. For any sequence $\lambda_n \in \sigma(T_n)$ with this convergence property and the associated eigenfunctions $x_n \in \mathcal{N}(F_n(\lambda_n))$, $\|x_n\| = 1$, it holds that*

$$|\lambda_n - \lambda| \leq C\epsilon_n^{1/\kappa}, \tag{3.7}$$

$$\inf_{x \in G(\lambda)} \|x_n - p_n x\| \leq C\epsilon_n^{1/\kappa}, \tag{3.8}$$

where $\epsilon_n = \|(T_n p_n - p_n T)\|$ and κ is the ascent of λ.

Proof. Let $\Omega \subset \mathbb{C}$ be compact such that $\lambda \in \Omega$ and $\lambda \notin \partial\Omega$. Since T is compact, for $F(\cdot)$ defined in (3.2), $\rho(F) \neq \emptyset$.

We check the conditions (b1)-(b4). Since $0 \notin \Omega$, $F(\eta), \eta \in \Omega$, is holomorphic Fredholm operator of index zero and thus (b1) holds. Since Ω is a compact set, $\frac{1}{\eta} I_n$ is equibounded. The operators T_n is uniformly bounded since T is bounded and (ii) is satisfied (see Theorem (2.55) in [24]). Thus $\{F_n(\cdot)\}$ is equibounded, i.e., (b2) holds. For $\eta \in \Omega$ and $x \in X$, (b3) holds since

$$\begin{aligned}
\|[F_n(\eta)p_n - p_n F(\eta)]x\| &= \left\| \left[\left(T_n - \frac{1}{\eta} I_n \right) p_n - p_n \left(T - \frac{1}{\eta} I \right) \right] x \right\| \\
&= \left\| \left[(T_n p_n - p_n T) - \left(\frac{1}{\eta} I_n p_n - \frac{1}{\eta} p_n I \right) \right] x \right\| \\
&\leq \|(T_n p_n - p_n T)x\| \\
&\to 0
\end{aligned}$$

due to conditions (i) and (ii) and the fact that $(I_n p_n - p_n I)x = 0$. The regular convergence of $F_n(\eta)$ to F follows Theorem 3.5 and thus (b4) holds.

Finally, we have that

$$\max_{|\eta - \lambda| \leq \delta} \max_{x \in G(\lambda)} \|F_n(\eta)p_n x - p_n F(\eta)x\| = \max_{x \in G(\lambda)} \|(T_n p_n - p_n T)x\| \leq \|T_n p_n - p_n T\|.$$

The proof is complete by applying Theorem 2.14. □

Remark 3.7. In view of Remark 3.3, if $X_n \subset X$, one has that

$$\|(T_n' - p_n T)x\| \leq \|(T_n' - T)x\| + \|T_n x - p_n T_n x\|. \tag{3.9}$$

Hence the consistency error ϵ_n is bounded by the sum of the error of the finite element solution and the approximation error of the discrete space.

The above theorem (c.f. Theorem 2.14) claims that all eigenvalues and eigenfunctions are approximated correctly. Note that condition (b3) (point-wise convergence) is not sufficient to rule out spurious discrete eigenvalues. We refer the readers to [7] for some mixed finite element methods that produce spurious eigenvalues.

We end this section with the following lemma which can be useful to study the convergence for the eigenvalue problem in a space different than $L^2(D)$.

Lemma 3.8. *Let X and Y be Banach spaces such that $Y \subset X$. Assume that the embedding of Y into X is compact. Let $T : X \to Y \subset X$ be a bounded linear operator. Then the restriction of T on Y, $T|_Y : Y \to Y$, is compact.*

Proof. Let $\{x_n\}, n \in \mathbb{N}$ be a bounded sequence in Y. Due to the compact embedding of Y into X, there exists a convergent subsequence $\{x_{n'}\}, n' \in N' \subset \mathbb{N}$, in X. Let $x = \lim_{n' \to \infty} x_{n'}$ and $y = Tx \in Y$. Clearly, $\lim_{n' \to \infty} Tx_{n'} = y$, i.e., $\{Tx_{n'}\}$ converges to $y \in Y$. The proof is complete. □

4 Dirichlet Eigenvalue Problem

In this section, we analyze the convergence of several finite element methods for the Dirichlet eigenvalue problem. Let $D \subset \mathbb{R}^2$ be a bounded Lipschitz polygonal domain. The Dirichlet eigenvalue problem is to find $\lambda \in \mathbb{R}$ and $u \neq 0$ such that

$$-\Delta u = \lambda u \qquad \text{in } D, \tag{4.1a}$$

$$u = 0 \qquad \text{on } \partial D. \tag{4.1b}$$

The associated source problem is, given f, to find u such that

$$-\Delta u = f \qquad \text{in } D, \tag{4.2a}$$

$$u = 0 \qquad \text{on } \partial D. \tag{4.2b}$$

For $f \in L^2(D)$, the weak formulation of (4.2) is to find $u \in H_0^1(D)$ such that

$$a(u, v) = (f, v) \quad \text{for all } \ v \in H_0^1(D), \tag{4.3}$$

where

$$a(u, v) = \int_D \nabla u \cdot \nabla \bar{v}\, dx, \quad (f, v) = \int_D f \bar{v}\, dx.$$

The variational formulation for the eigenvalue problem is to find $\lambda \in \mathbb{R}$ and $u \in H_0^1(D)$ such that

$$a(u, v) = \lambda (u, v) \qquad \text{for all } \ v \in H_0^1(D), \tag{4.4}$$

There exists a unique solution $u \in H_0^1(D)$ to (4.3). Furthermore, $u \in H^{1+\alpha}(D)$, where $1/2 < \alpha \leq 1$ ($\alpha = 1$ if D is convex) is the elliptic regularity index (see, e.g., Sec. 3.2 in [23]). Due to the wellposedness of (4.3) and the compact imbedding of $H_0^1(D)$ into $L^2(D)$, there exists a compact solution operator to (4.3)

$$T : L^2(D) \to L^2(D) \quad \text{such that} \quad Tf = u. \tag{4.5}$$

Assuming that $\lambda \neq 0$, the Dirichlet eigenvalue problem is equivalent to the operator eigenvalue problem of finding $\lambda \in \mathbb{R}$ and $u \in L^2(D)$ such that

$$T(\lambda u) = u. \tag{4.6}$$

Define a nonlinear operator function $F : \Omega \to \mathcal{L}(L^2(D), L^2(D))$ by

$$F(\eta) := T - \frac{1}{\eta} I, \quad \eta \in \Omega. \tag{4.7}$$

Clearly, λ is a Dirichlet eigenvalue if and only if λ is an eigenvalue of $F(\cdot)$.

Lemma 4.1. *Let $\Omega \subset \mathbb{C}\backslash\{0\}$ be a compact set. Then $F(\cdot) : \Omega \to \mathcal{L}(L^2(D), L^2(D))$ is a holomorphic Fredholm operator function of index zero.*

Proof. It is clear that $F(\cdot)$ is holomorphic in Ω. Since T is compact and I is the identity operator, $F(\lambda)$ is a Fredholm operator of index zero. □

Let X_n be a finite element space associated with $\mathcal{T}_{h_n}$ endowed with the L^2-norm $\|\cdot\|$. Let p_n be the L^2-projection from $L^2(D)$ to X_n such that $\|x_n - x\| \to 0$ as $n \to \infty$ for $x \in X$, where $x_n = p_n x$. For example, if X_n is the linear Lagrange element space, it holds that

$$\|p_n x - x\| \leq C h_n^r \|x\|_{H^r(D)}, \quad 0 \leq r \leq 2. \tag{4.8}$$

Assume that there exists a finite element solution operator $T_n : X_n \to X_n$ such that $T_n f_n = u_n$. In general, one has the convergence of the finite element method for the source problem, i.e.,

$$\lim_{n\to\infty} \|T_n p_n f - p_n T f\| = 0 \quad \text{for all } f \in X. \tag{4.9}$$

Next we investigate several finite element methods for the Dirichlet eigenvalue problem using the results in Section 3.

4.1 Conforming Finite Element Method

Let X_n be the Lagrange element space equipped with the L^2-norm $\|\cdot\|_{X_n} = \|\cdot\|$. One has that $X_n \subset H_0^1(D) \subset X$. The discrete formulation for the source problem (4.3) is as follows. For $f \in L^2(D)$, find $u_n \in X_n$ such that,

$$a(u_n, v_n) = (f_n, v_n) \quad \text{for all } v_n \in X_n, \tag{4.10}$$

where $a(u_n, v_n) := (\nabla u_n, \nabla v_n)$ and $f_n = p_n f$.

The discrete problem (4.10) has a unique solution u_n such that $\|u_n\| \leq C\|f_n\|$. Let u and u_n be the solutions of (4.3) and (4.10), respectively. The classical finite element error analysis gives that (see, e.g., [9])

$$\|u - u_n\| \leq C h_n^{2\alpha} \|f\|. \tag{4.11}$$

Let $T_n : X_n \to X_n$, $T_n f_n = u_n$ be the finite element solution operator of (4.10). Since

$$\|u_n - p_n u\| \leq \|u_n - u\| + \|u - p_n u\|,$$

using (4.11), one obtains that

$$\lim_{n\to\infty} \|T_n p_n f - p_n T f\| = 0 \quad \text{for all } f \in X. \tag{4.12}$$

Define a discrete operator function $F_n : \Omega \to \mathcal{L}(X_n, X_n)$

$$F_n(\eta) := T_n - \frac{1}{\eta} I_n. \tag{4.13}$$

Lemma 4.2. *$F_n \to F$ regularly.*

Proof. Let $X' = H_0^1(D)$. Since $T_n : X_n \to X', n \in \mathbb{N}$ are uniformly bounded, due to the compact embedding of X' into X and Lemma 3.2, $T_n \to T$ compactly. Lemma 2.9 implies that $F_n \to F$ regularly. The proof is complete. □

The convergence for the Lagrange finite element method of finding λ_n and u_n such that

$$a(u_n, v_n) = \lambda_n (u_n, v_n) \quad \text{for all } v_n \in X_n,$$

follows from (4.8), (4.11), and Theorem 3.6.

Theorem 4.3. *Let $\lambda \in \sigma(F)$. There exists $n_0 \in \mathbb{N}$ and a sequence $\lambda_n \in \sigma(F_n)$, $n \geq n_0$, such that $\lambda_n \to \lambda$ as $n \to \infty$. For any sequence $\lambda_n \in \sigma(F_n)$ with this convergence property and the associated eigenfunction u_n, $\|u_n\| = 1$, one has that*

$$|\lambda_n - \lambda| \leq C h_n^{2\alpha} \quad \text{and} \quad \|u_n - u\| \leq C h_n^{2\alpha}, \tag{4.14}$$

where u is some eigenfunction associated to λ with $\|u\| = 1$.

Proof. Let $\Omega \subset \mathbb{C}$ be a simply connected set such that $0 \notin \Omega$ and $\lambda \in \Omega$. It is clear that $\rho(F) \cap \Omega \neq \emptyset$. For $X = Y = L^2(D)$, $X_n = Y_n$ and $p_n = q_n$ being the L^2-projection from X onto X_n, condition (i) in Theorem 3.6 holds. Due to (4.12), (ii) of Theorem 3.6 holds. Since T is self-adjoint, each eigenvalue has ascent $\kappa = 1$. From (4.11), it holds that $\epsilon_n \leq C h_n^{2\alpha}$. Hence (4.22) holds due to Theorem 3.6. □

Remark 4.4. For convex domains, one has that $\alpha = 1$ and obtains second order of convergence for the eigenvalues and eigenfunctions using the linear Lagrange element.

Remark 4.5. We refer the readers to [27] which uses the holomorphic operator approach but a different proof of regular convergence.

4.2 Interior Penalty Discontinuous Galerkin Methods

We consider the interior penalty discontinuous Galerkin methods for the Dirichlet eigenvalue problem. Following the notations in [1], let h_K be the diameter of the element $K \in \mathcal{T}_n$. Denote by $\mathcal{F}_n^I$ and $\mathcal{F}_n^B$ be the sets of the interior edges and boundary edges of $\mathcal{T}_n$, respectively. Let $\mathcal{F}_n := \mathcal{F}_n^I \cup \mathcal{F}_n^B$. Let $\boldsymbol{w}$ and v be piecewise smooth vector-valued and scalar-valued functions. Let $F \in \mathcal{F}_n^I$ be an interior face share by two elements K^+ and K^- with outward norm $\boldsymbol{\nu}^\pm$. Denote by $\boldsymbol{w}^\pm$ and $v^\pm$ on $\partial K^\pm$ taken from within $K^\pm$, respectively. The jumps across F are defined by

$$[\![\boldsymbol{w}]\!] = \boldsymbol{w}^+ \cdot \boldsymbol{\nu}^+ + \boldsymbol{w}^- \cdot \boldsymbol{\nu}^-, \quad [\![v]\!] = v^+ \boldsymbol{\nu}^+ + v^- \boldsymbol{\nu}^-$$

and the averages are defined by

$$\{\!\{\boldsymbol{w}\}\!\} = \frac{1}{2}\left(\boldsymbol{w}^+ + \boldsymbol{w}^-\right), \quad \{\!\{v\}\!\} = \frac{1}{2}\left(v^+ + v^-\right).$$

For $F \in \mathcal{F}_n^B$, one simply defines

$$[\![\boldsymbol{w}]\!] = \boldsymbol{w} \cdot \boldsymbol{\nu}, \quad [\![v]\!] = v\boldsymbol{\nu}, \quad \{\!\{\boldsymbol{w}\}\!\} = \boldsymbol{w}, \quad \{\!\{v\}\!\} = v.$$

Define the discontinuous Galerkin space X_n by

$$X_n := \{v \in L^2(D) : v|_K \in P^\ell(K), K \in \mathcal{T}_n\}, \tag{4.15}$$

where $P^\ell(K)$ is the space of polynomials of degree at most $\ell \geq 1$ on K.

Let p_n be the L^2-projection of $f \in L^2(D)$ to X_n such that

$$(p_n f, v_n) = (f, v_n) \quad \text{for all } v_n \in X_n. \tag{4.16}$$

Then condition (i) in Theorem 3.6 holds.

We consider the symmetric interior penalty (SIP) DG methods in primal form for the Poisson equation. Find $u_n \in X_n$ such that

$$a_n(u_n, v_n) = (f_n, v_n) \quad \text{for all } v_n \in X_n \tag{4.17}$$

where $a_n : X_n \times X_n \to \mathbb{C}$ is defined by

$$\begin{aligned} a_n(u_n, v_n) \quad &= \quad (\nabla_n u_n, \nabla_n v_n) - \int_{\mathcal{F}_n} \{\!\{\nabla_n u_n\}\!\} \cdot [\![\bar{v}_n]\!] ds \\ &\quad - \int_{\mathcal{F}_n} \{\!\{\nabla_n \bar{v}_n\}\!\} \cdot [\![u_n]\!] ds - s_n(u_n, v_n) \quad \text{for } u_n, v_n \in X_n. \end{aligned}$$

The interior penalty family are defined by choosing the the stabilization form $s_n(\cdot, \cdot)$ as

$$s_n(u_n, v_n) = \int_{\mathcal{F}_h} \gamma h_n^{-1} [\![u_n]\!][\![\bar{v}_n]\!] ds,$$

for $\gamma > 0$ independent of the mesh size.

The finite element space X_n can be endowed with a second norm $\|\cdot\|_n$:

$$\|v_n\|_n = \|\nabla_n v_n\| + \|h^{-1/2}[\![v_n]\!]\|^2_{\mathcal{F}_n}, \tag{4.18}$$

where ∇_n is the element-wise gradient operator. For $f \in H_0^1(\Omega)$ or $f \in X_n$, the Poincaré inequality holds (Property 1 in [1])

$$\|f\| \leq C\|f\|_n, \tag{4.19}$$

where C is a constant depends on D but not on the mesh.

For γ large enough, it is well-known that there exists a unique solution u_n to (4.17). Denote the discrete solution operator by $T_n : X_n \to X_n$. According to Property 2 in [1], for the discrete solution u_n to (4.17) and the exact solution u to (4.3), it holds that

$$\|u - u_n\|_n \leq Ch^\alpha \|f\| \quad \text{for } f \in L^2(D). \tag{4.20}$$

Using the approximation property of p_n and the Poincaré inequality (4.19), for $f \in L^2(D)$, one has that

$$\begin{aligned} \|T_n p_n f - p_n T f\| \quad &= \quad \|u_n - p_n u\| \\ &\leq \quad \|u_n - u\| + \|u - p_n u\| \\ &\leq \quad \|u_n - u\|_n + \|u - p_n u\| \\ &\leq \quad Ch^\alpha \|f\| + Ch^\alpha \|u\|_{H^1(D)} \\ &\leq \quad Ch^\alpha \|f\|. \end{aligned} \tag{4.21}$$

Similarly, we define the operator function $F_n : \Omega \to \mathcal{L}(X_n, X_n)$ such that

$$F_n(\eta) := T_n - \frac{1}{\eta} I_n,$$

where X_n is the discontinuous Galerkin space defined in (4.15), T_n is the discrete solution operator to (4.17), and I_n is the identity operator from X_n to X_n.

Theorem 4.6. *Let $\lambda \in \sigma(F)$. There exists $n_0 \in \mathbb{N}$ and a sequence $\lambda_n \in \sigma(F_n)$, $n \geq n_0$, such that $\lambda_n \to \lambda$ as $n \to \infty$. For any sequence $\lambda_n \in \sigma(F_n)$ with this convergence property and the associated eigenfunction u_n, $\|u_n\| = 1$, one has that*

$$|\lambda_n - \lambda| \leq C h_n^{\alpha} \quad \text{and} \quad \|u_n - u\| \leq C h_n^{\alpha}, \tag{4.22}$$

where u is some eigenfunction associated to λ with $\|u\| = 1$.

Proof. From (4.21), $T_n \to T$ uniformly. Thus $F_n(\eta)$ converges to $F(\eta)$ regularly due to Lemma 3.4. Then (4.22) follows Theorem 3.6. □

We refer the readers to [25] which uses the holomorphic operator approach but a different discrete norm for X_n and a different proof.

Remark 4.7. Note that the convergence is not optimal since (4.21) is obtained using the Poincaré inequality and the convergence of the DG solution in $\|\cdot\|_{X_n}$ norm. If a higher order of convergence of u_n in L^2 norm is available, the convergence orders in Theorem (4.6) can be improved.

4.3 Nonconforming Crouzeix-Raviart Method

We consider the nonconforming piecewise linear finite element space of Crouzeix-Raviart [4]:

$$\begin{aligned} X_n := \{v : v|_K \in \mathcal{P}_1 &\text{ is continuous at the midpoints of the edges of } K \\ &\text{and } v = 0 \text{ at the midpoints on } \partial D\}, \end{aligned} \tag{4.23}$$

where $\mathcal{P}_1$ denotes the space of polynomials of degree less or equal to 1.

Define the bilinear form on X_n

$$a_n(u_n, v_n) := \sum_{K \in \mathcal{T}_n} \int_K \nabla u_n \cdot \nabla v_n dK, \quad u_n, v_n \in X_n.$$

The discrete problem is to find $u_n \in X_n$ such that

$$a_n(u_n, v_n) = (f_n, v_n) \quad \text{for all } v_n \in X_n, \tag{4.24}$$

where $f_n = p_n f$. There exists a unique solution u_n to (4.24). Denote the discrete solution operator to be $T_n : X_n \to X_n$ such that

$$u_n = T_n f_n.$$

Using the error estimate in L^2-norm (Theorem 1.5 of Chp. III in [4]) and the property of the L^2-projection, one has that

$$\|p_n T f - T_n p_n f\| \leq C h^{2\alpha} \|f\|, \tag{4.25}$$

which implies $T_n \to T$ uniformly.

Define the discrete operator function $F_n : \Omega \to \mathcal{L}(X_n, X_n)$ such that

$$F_n(\eta) := T_n - \frac{1}{\eta} I_n,$$

where X_n is the non-conforming Crouzeix-Raviart finite element space defined in (4.23), T_n is the discrete solution operator to (4.24), and I_n is the identity operator from X_n to X_n. Using Theorem 3.6, we obtain the convergence of the non-conforming Crouzeix-Raviart method. It is proof is similar to the previous ones and thus omitted.

Theorem 4.8. *Let $\lambda \in \sigma(F)$. There exists $n_0 \in \mathbb{N}$ and a sequence $\lambda_n \in \sigma(F_n)$, $n \geq n_0$, such that $\lambda_n \to \lambda$ as $n \to \infty$. For any sequence $\lambda_n \in \sigma(F_n)$ with this convergence property and the associated eigenfunction u_n, $\|u_n\| = 1$, one has that*

$$|\lambda_n - \lambda| \leq C h_n^{2\alpha} \quad \textit{and} \quad \|u_n - u\| \leq C h_n^{2\alpha}, \tag{4.26}$$

where u is some eigenfunction associated to λ with $\|u\| = 1$.

Remark 4.9. The Crouzeix-Raviart element is non-conforming in the sense that it is not a subspace of $H_0^1(D)$, which is the solution space for (4.3). Note that we use $L^2(D)$ other than $H^1(D)$. Similar comment applies to the Morley element method for the biharmonic eigenvalue problem in the next section.

5 The Biharmonic Eigenvalue Problem

We now consider the biharmonic eigenvalue problem. Let D denote a bounded Lipschitz polygonal domain in $\mathbb{R}^2$ with boundary ∂D. Let ν denote the unit outward normal to ∂D. The biharmonic eigenvalue problem with clamped plate boundary condition is to find $\lambda \in \mathbb{R}$ and $u \neq 0$ such that

$$\Delta^2 u = \lambda u \qquad \text{in } D, \tag{5.1a}$$

$$u = \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial D. \tag{5.1b}$$

The associated source problem is as follows. Given a function f, find a function u such that

$$\Delta^2 u = f \qquad \text{in } D, \tag{5.2a}$$

$$u = \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial D. \tag{5.2b}$$

Define

$$H_0^2(D) := \left\{ v \in H^2(D) : v = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial D \right\},$$

and a sesquilinear form $a : H_0^2(D) \times H_0^2(D)$ such that

$$a(u,v) := (\triangle u, \Delta v). \tag{5.3}$$

The weak formulation for (5.2) is, for $f \in L^2(D)$, to find $u \in H_0^2(D)$ such that

$$a(u,v) = (f,v) \quad \text{for all } v \in H_0^2(D). \tag{5.4}$$

The weak formulation for (5.1) is to find $\lambda \in \mathbb{R}$ and $u \in H_0^2(D), u \neq 0$ such that

$$a(u,v) = \lambda(u,v) \quad \text{for all } v \in H_0^2(D). \tag{5.5}$$

There exists a unique solution u to (5.5) belonging to $H^{2+\alpha}(D)$ for some $\alpha \in (1/2, 1]$ such that

$$\|u\|_{H^{2+\alpha}(D)} \leq C\|f\|, \tag{5.6}$$

where the constant C depending only on D. When D is convex, $\alpha = 1$. The parameter α is referred as the index of elliptic regularity for the biharmonic equation.

Consequently, there exists a solution operator $T : L^2(D) \to L^2(D)$ such that, given $f \in L^2(D)$,

$$a(Tf,v) = (f,v) \quad \text{for all } v \in H_0^2(D). \tag{5.7}$$

It is obvious that T is self-adjoint due to the symmetry of $a(\cdot,\cdot)$ and compact due to the compact imbedding of $H_0^2(D)$ into $L^2(D)$. Similar to the case of the Dirichlet eigenvalue problem, we define a holomorphic Fredholm operator function $F : \Omega \to \mathcal{L}(L^2(D), L^2(D))$ such that

$$F(\eta) := T - \frac{1}{\eta} I, \quad \eta \in \Omega. \tag{5.8}$$

5.1 Argyris Element Method

We consider the Argyris element (see, e.g., Section 4.2 of [23]), which is H^2-conforming for triangular meshes $\mathcal{T}_{h_n}$. Denote the associated finite element space by X_n.

The discrete problem for the source problem (5.2) can be stated as follows. For $f \in L^2(D)$, find $u_n \in X_n \subset H_0^2(D)$ such that

$$a(u_n, v_n) = (p_n f, v_n) \quad \text{for all } v_n \in X_n. \tag{5.9}$$

There exists a unique solution u_n to (5.9) such that

$$\|u - u_n\|_{H^2(D)} \leq C h_n^\alpha \|f\|. \tag{5.10}$$

The discrete formulation for the eigenvalue problem (5.1) is to find $\lambda_n \in \mathbb{R}$ and $u_n \in X_n$, $u_n \neq 0$ such that

$$a(u_n, v_n) = \lambda_n (u_n, v_n) \quad \text{for all } v_n \in X_n. \tag{5.11}$$

Using a duality argument (Chp. II of [4]) and (5.10), one has that

$$\|u - u_n\| \leq C h_n^{2\alpha} \|f\|. \tag{5.12}$$

Consequently, the discrete solution operator $T_n : X_n \to X_n$ is such that

$$\|T_n p_n f - p_n T f\| \leq C h_n^{2\alpha} \|f\| \quad \text{for } f \in L^2(D). \tag{5.13}$$

Since $X_n \subset H_0^2(D), n \in \mathbb{N}$ and the embedding of $H_0^2(D)$ into X is compact, one has that $T_n \to T$ compactly due to Lemma 3.2.

Define a discrete operator function $F_n : \Omega \to \mathcal{L}(X_n, X_n)$

$$F_n(\eta) := T_n - \frac{1}{\eta} I_n. \tag{5.14}$$

Then $F_n \to F$ regularly. Using a similar argument as for the Dirichlet eigenvalue problem, we obtain the following convergence theorem of the Argyris element method for the biharmonic eigenvalue problem.

Theorem 5.1. *Let $\lambda \in \sigma(F)$. There exists $n_0 \in \mathbb{N}$ and a sequence $\lambda_n \in \sigma(F_n)$, $n \geq n_0$, such that $\lambda_n \to \lambda$ as $n \to \infty$. For any sequence $\lambda_n \in \sigma(F_n)$ with this convergence property and the associated eigenfunction u_n, $\|u_n\| = 1$, one has that*

$$|\lambda_n - \lambda| \leq C h_n^{2\alpha} \quad \text{and} \quad \|u_n - u\| \leq C h_n^{2\alpha}, \tag{5.15}$$

where u is some eigenfunction associated to λ with $\|u\| = 1$.

5.2 C^0 interior penalty discontinuous Galerkin method

We consider the C^0 interior penalty discontinuous Galerkin method (C^0 IPG) for the biharmonic equation [8] (see also [26] using the holomorphic operator approach but a different proof). Let $X_n \subset H^1(\Omega)$ be the Lagrange finite element space of order $k \geq 2$ associated with $\mathcal{T}_{h_n}$. Let $\mathcal{E}_{h_n}$ be the set of the edges in $\mathcal{T}_{h_n}$. For edges $e \in \mathcal{E}_{h_n}$ that are the common edge of two adjacent triangles $K_\pm \in \mathcal{T}_{h_n}$ and for $v \in X_n$, we define the jump of the flux to be

$$[\![\partial v / \partial n_e]\!] = \frac{\partial v_{K_+}}{\partial n_e}\Big|_e - \frac{\partial v_{K_-}}{\partial n_e}\Big|_e,$$

where n_e is the unit normal pointing from K_- to K_+. Let

$$\frac{\partial^2 v}{\partial n_e^2} = n_e \cdot (\triangle v) n_e$$

and define the average normal-normal derivative to be

$$\left\{\!\!\left\{ \frac{\partial^2 v}{\partial n_e^2} \right\}\!\!\right\} = \frac{1}{2}\left(\frac{\partial^2 v_{K_+}}{\partial n_e^2} + \frac{\partial^2 v_{K_-}}{\partial n_e^2} \right).$$

For $e \in \partial D$, we take n_e to be the unit outward normal and define

$$[\![\partial v/\partial n_e]\!] = -\frac{\partial v}{\partial n_e} \quad \text{and} \quad \left\{\!\!\left\{ \frac{\partial^2 v}{\partial n_e^2} \right\}\!\!\right\} = \frac{\partial^2 v}{\partial n_e^2}.$$

Given $f_n \in X_n$, $f_n = p_n f$, the corresponding C^0 IPG method for the source problem is to find $u_h \in X_n$ such that

$$a_n(u_n, v_n) = (f_n, v_n) \quad \text{for all } v \in X_n, \tag{5.16}$$

where

$$\begin{aligned} a_n(w,v) = &\sum_{K\in\mathcal{T}_{h_n}} \int_K D^2 w : D^2 v \,\mathrm{d}x \\ &+ \sum_{e\in\mathcal{E}_{h_n}} \int_e \left\{\!\!\left\{ \frac{\partial^2 w}{\partial n_e^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial v}{\partial n_e} \right]\!\!\right] + \left\{\!\!\left\{ \frac{\partial^2 v}{\partial n_e^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial w}{\partial n_e} \right]\!\!\right] \mathrm{d}s \\ &+ \sigma \sum_{e\in\mathcal{E}_{h_n}} \frac{1}{|e|} \int_e \left[\!\!\left[\frac{\partial w}{\partial n_e} \right]\!\!\right] \left[\!\!\left[\frac{\partial v}{\partial n_e} \right]\!\!\right] \mathrm{d}s, \end{aligned} \tag{5.17}$$

where $D^2 w : D^2 v = \sum_{i,j=1}^2 w_{x_i x_j} v_{x_i x_j}$ is the Frobenius inner product of the Hessian matrices of w and v, and $\sigma > 0$ is a (sufficiently large) penalty parameter.

There exists discrete solution operators $T_n : X_n \to X_n$ to (5.16) such that $T_n \to T$.

The C^0 IPG method for the biharmonic eigenvalue problem is to find $\lambda_n \in \mathbb{R}$ and $u_n \neq 0$ such that

$$a_n(u_n, v_n) = \lambda_n (u_n, v_n) \quad \text{for all } v_n \in X_n. \tag{5.18}$$

Consequently, define

$$F_n(\eta) = T_n - \frac{1}{\eta} I_n.$$

Let $X' = H_0^1(D)$. Since $T_n : X_n \to X', n \in \mathbb{N}$ are uniformly bounded, due to the compact embedding of X' into X and Lemma 3.2, $T_n \to T$ compactly. Lemma 2.9 implies that $F_n \to F$ regularly. Then the following convergence theorem holds for the C^0 IPG method.

Theorem 5.2. *Let $\lambda \in \sigma(F)$. There exists $n_0 \in \mathbb{N}$ and a sequence $\lambda_n \in \sigma(F_n)$, $n \geq n_0$, such that $\lambda_n \to \lambda$ as $n \to \infty$. For any sequence $\lambda_n \in \sigma(F_n)$ with this convergence property and the associated eigenfunction u_n, $\|u_n\| = 1$, one has that*

$$|\lambda_n - \lambda| \leq C h_n^{2\alpha} \quad \textit{and} \quad \|u_n - u\| \leq C h_n^{2\alpha}, \tag{5.19}$$

where u is some eigenfunction associated to λ with $\|u\| = 1$.

Proof. Due to Lemma 1 in [8], there exists a unique discrete solution u_n to (5.18), such that

$$\|u - u_n\| \leq Ch_n^{2\alpha}\|f\| \quad \text{for } f \in L^2(D). \tag{5.20}$$

One has that

$$\|T_n p_n f - p_n T f\| \leq Ch_n^{2\alpha}\|f\| \quad \text{for } f \in L^2(D). \tag{5.21}$$

□

5.3 Morley Element Method

We consider the Morley element space X_n. Let p_n be the L^2-projection from X onto X_n. For $f \in X$, the Morley finite element method for the biharmonic equation is to find $u_n \in X_n$ such that

$$a_n(u_n, v_n) = (p_n f, v_n) \quad \text{for all } v_n \in X_n, \tag{5.22}$$

where

$$a_n(u_n, v_n) := \sum_{K \in \mathcal{T}_n} \int_K D^2 u_n : D^2 v_n dK, \quad u_n, v_n \in X_n.$$

There exists a unique solution $u_n \in X_n$ such that (see, e.g., Section 4.4.2 of [23])

$$\|u - u_n\|_{2,h_n} \leq Ch_n^{\alpha}\|f\|, \tag{5.23}$$

where the mesh dependent norm $\|\cdot\|_{2,h_n}$ is defined as

$$\|u\|_{2,h_n} = \sum_{K \in \mathcal{T}_n} (u, u)_{H^2(K)},$$

Let $T_n : X_n \to X_n$ be the discrete solution operator to (5.22). Since $\|u\| \leq \|u\|_{2,h_n}$, due to (5.23), it holds that

$$\|u - u_n\| \leq Ch_n^{\alpha}\|f\|.$$

As a consequence, one has that

$$\|T_n p_n f - p_n T f\| \leq Ch_n^{\alpha}\|f\| \quad \text{for } f \in X. \tag{5.24}$$

Letting

$$F_n(\eta) = T_n - \frac{1}{\eta} I_n, \quad \eta \in \Omega,$$

we obtain the following convergence theorem for the Morley element method.

Theorem 5.3. *Let $\lambda \in \sigma(F)$. There exists $n_0 \in \mathbb{N}$ and a sequence $\lambda_n \in \sigma(F_n)$, $n \geq n_0$, such that $\lambda_n \to \lambda$ as $n \to \infty$. For any sequence $\lambda_n \in \sigma(F_n)$ with this convergence property and the associated eigenfunction u_n, $\|u_n\| = 1$, one has that*

$$|\lambda_n - \lambda| \leq Ch_n^{\alpha} \quad \textit{and} \quad \|u_n - u\| \leq Ch_n^{\alpha}, \tag{5.25}$$

where u is some eigenfunction associated to λ with $\|u\| = 1$.

Remark 5.4. The convergence order can be improved if a sharper error estimate in L^2-norm is available, see, e.g., [12].

6 Conclusions and Future Work

In this paper, we present a general approach to prove the convergence of various finite element methods for eigenvalue problems, including the conforming methods, discontinuous Galerkin methods, and non-conforming methods. Using the abstract approximation theory for eigenvalue problems of holomorphic Fredholm operator functions, one needs to verify the property of the L^2-projection and the compact convergence of the discrete solution operators in L^2-norm. The result has the potential to prove the convergence of many other finite elements methods for eigenvalue problems such as the mixed finite element methods, virtual element methods, and weak Galerkin methods.

We use $L^2(D)$ and the L^2-projection for the finite element spaces. Thus the convergence of the eigenfunctions is also in L^2-norm. However, this framework also works if one chooses other spaces and projections, for example, $H^1(D)$ and H^1-projection to the finite element spaces for the Dirichlet eigenvalue problem. Then one can obtain the convergence of the eigenfunctions in H^1-norm.

As seen above, the convergence order obtained using Theorem 3.6 for certain method is not optimal. However, if the optimal convergence order for the source problem is available in L^2-norm, the optimal convergence order for the eigenfunctions follows immediately. There are many interesting projects related to the proposed approach, e.g., 1) to prove the convergence of other methods, e.g., the virtual element methods; 2) to prove the convergence of eigenvectors in other norms, e.g., the maximum norm; 3) to treat other eigenvalue problems, e.g., the Steklov eigenvalue problem; 4) to treat nonlinear eigenvalue problems, e.g., band structure calculation for photonic crystals of dispersive media and the nonlinear plate vibrations [27, 28, 20].

References

[1] P. F. Antonietti, A. Buffa, and I. Perugia, *Discontinuous Galerkin approximation of the Laplace eigenproblem.* Comput. Methods Appl. Mech. Engrg. 195 (2006), no. 25-28, 3483-3503.

[2] I. Babuška and J. Osborn, Eigenvalue problems. Handbook of numerical analysis, Vol. II, 641-787, Handb. Numer. Anal., II, North-Holland, Amsterdam, 1991.

[3] W.-J. Beyn, Y. Latushkin, and J. Rottmann-Matthes, *Finding eigenvalues of holomorphic Fredholm operator pencils using boundary value problems and contour integrals.* Integr. Equ. Oper. Theory 78 (2014), 155-211.

[4] D. Braess, *Finite elements. Theory, fast solvers, and applications in solid mechanics.* 2nd ed. Cambridge University Press, Cambridge, 2001.

[5] J. H. Bramble and J. E. Osborn, *Rate of convergence estimates for non-selfadjoint eigenvalue approximations*, Math. Comp. **27**(1973), 525-549.

[6] D. Boffi, *Finite element approximation of eigenvalue problems*, Acta Numer. **19**(2010), 1-120.

[7] D. Boffi, F. Brezzi, and L. Gastaldi, *On the problem of spurious eigenvalues in the approximation of linear elliptic problems in mixed form.* Math. Comp. 69 (2000), no. 229, 121-140.

[8] S.C. Brenner, P. B. Monk and J. Sun, *C^0 interior penalty Galerkin method for biharmonic eigenvalue problems.* Lect. Notes Comput. Sci. Eng., 106, 3-15, Springer, Cham, 2015

[9] S.C. Brenner and L.R. Scott, The mathematical theory of finite elements methods, 3rd Edition, Texts in Applied Mathematics, Springer, 2008.

[10] F. Chatelin, Spectral approximation of linear operators. Computer Science and Applied Mathematics. Academic Press, New York, 1983.

[11] J. Descloux, N. Nassif, J. Rappaz, *On spectral approximation Part 1. The problem of convergence*, RAIRO Anal. Numér. **12**(1978), 97-112.

[12] D. Gallistl, *Morley finite element method for the eigenvalues of the biharmonic operator.* IMA J. Numer. Anal. 35 (2015), no. 4, 1779-1811.

[13] I. Gohberg and J. Leiterer, Holomorphic operator functions of one variable and applications. Methods from complex analysis in several variables. Operator Theory: Advances and Applications, 192. Birkhäuser Verlag, Basel, 2009.

[14] R.D. Grigorieff and H. Jeggle, *Approximation von Eigenwertproblemen bei nichlinearer Parameterabhängigkeit.* Manuscripta Math. 10 (1973), 245-271.

[15] O. Karma, *Approximation in eigenvalue problems for holomorphic Fredholm operator functions. I.* Numer. Funct. Anal. Optim. **17** (1996), no. 3-4, 365–387.

[16] O. Karma, *Approximation in eigenvalue problems for holomorphic Fredholm operator functions. II. (Convergence rate).* Numer. Funct. Anal. Optim. **17** (1996), no. 3-4, 389–408.

[17] T. Kato, Perturbation theory for linear operators. Berlin-Heidelberg-New York, Springer, 1966.

[18] B. Mercier, J. Osborn, J. Rappaz, and P.A. Raviart, *Eigenvalue approximation by mixed and hybrid methods.* Math. Comp. 36 (1981), no. 154, 427-453.

[19] J.E. Osborn, *Spectral approximation for compact operators.* Math. Comput. **29**(1975), 712-725.

[20] X. Pang, J. Sun, and Z. Zhang, *FE-holomorphic operator function method for nonlinear plate vibrations with elastically added masses.* J. Comput. Appl. Math. 410 (2022), Paper No. 114156.

[21] F. Stummel, *Diskrete Konvergenz linearer Operatoren. I.* Math. Ann. 190 (1970/71), 45-92.

[22] F. Stummel, *Diskrete Konvergenz linearer Operatoren. II.* Math. Z. 120 (1971), 231-264.

[23] J. Sun and A. Zhou, Finite element methods for eigenvalue problems. CRC Press, Taylor & Francis Group, Boca Raton, 2016.

[24] G. Vainikko, Funktionalanalysis der Diskretisierungsmethoden. Mit Englischen und Russischen Zusammenfassungen. Teubner-Texte zur Mathematik. B. G. Teubner Verlag, Leipzig, 1976.

[25] Y. Xi and X. Ji, *A holomorphic operator function approach for the Laplace eigenvalue problem using discontinuous Galerkin method.* CSIAM Trans. Appl. Math. 2 (2021), no. 4, 776-792.

[26] Y. Xi and X. Ji, *A new method using C0IPG for the biharmonic eigenvalue problem.* J. Sci. Comput. 90 (2022), no. 3, Paper No. 81.

[27] W. Xiao, B. Gong, J. Sun, and Z. Zhang, *A new finite element approach for the Dirichlet eigenvalue problem.* Appl. Math. Lett. 105 (2020), 106295.

[28] W. Xiao, B. Gong, J. Sun, and Z. Zhang, *Finite element calculation of photonic band structures for frequency dependent materials.* J. Sci. Comput. 87 (2021), no. 1, Paper No. 27.