

**FINITE FIELD MODELS OF POLYNOMIALS INTERPOLATING  
FOURIER COEFFICIENTS OF MODULAR FUNCTIONS FOR  
HECKE GROUPS**

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**Abstract**

Following work of Raleigh and Akiyama ([31], [1]), in our article [8] we considered (among other objects) families of weight zero meromorphic modular forms  $J_m$  for Hecke groups  $G(\lambda_m)$ . We conjectured that, for a certain uniformizing variable  $X_m$ , the  $J_m$  have Fourier expansions  $J_m = 1/X_m + \sum_{n=0}^{\infty} m^{-2n-2} A_n(m) X_m^n$ , where the  $A_n(x)$  are polynomials in  $\mathbb{Q}[x]$ . The present article is concerned with models  $\mathcal{A}_n[p](x)$  of the  $A_n(x)$ : polynomials representing self-maps of finite fields with characteristic  $p$ . The main content is a conjecture specifying  $\mathcal{A}_n[p](x)$  up to a multiplicative constant for certain families of  $n$  and  $p$ , based on numerical experiments.

## 1. Introduction

### 1.1. Why Hecke groups?

Here we describe an old puzzle to explain our interest in modular forms for Hecke groups and to advertise the puzzle itself, which we still have not solved, and indeed will not address in the present article.<sup>1</sup> C. L. Siegel ([37],[38]) established bounds on the least positive integer represented by a positive-definite even unimodular quadratic form in  $2h$  variables by first bounding the exponent of the first non-vanishing Fourier coefficient for a level one entire classical modular form  $T_h$  of weight  $h$  such that the constant term of  $T_h$  is non-vanishing. While working on an extension of Siegel's result on the non-vanishing of the  $T_h$  constant terms to level two modular forms, we came across the regularities described in equations (1) and (2) in numerical experiments. Let  $\Delta$  denote the weight twelve modular form for  $SL(2, \mathbb{Z})$  that generates Ramanujan's tau function, let  $j$  be the usual Hauptmodul normalized to have constant term 744, let  $d_b(n)$  be the sum of the digits in the base  $b$  expansion of  $n$ , and let  $C(f)$  stand for the constant term of the Fourier series of  $f$  in whatever uniformizing variable happens to be in question. Then (apparently)

$$\text{ord}_2(C(j^k)) = \text{ord}_2(C(1/\Delta^k)) = 3d_2(k) \quad (1)$$

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<sup>1</sup>We have a draft of a paper on this question in the folder "current draft" in our GitHub depository [5].

and

$$\text{ord}_3(C(j^k)) = \text{ord}_3(C(1/\Delta^k)) = d_3(k). \tag{2}$$

We listed many functions displaying analogous behavior in our 1998 article [9], together with a finer taxonomy based on congruences. (For example, if  $\text{ord}_3(n) = 1$ , then  $n$  may be congruent to 3 or 6 modulo 9.) Clearly, if only we had proofs of these statements and their analogues, we would know that the constant terms of  $1/\Delta^k, j^k$ , and their analogues are non-zero. In the level one case Siegel used different arguments to establish the non-vanishing of  $T_h$  constant terms, as we eventually did for their level two analogues ([9],[10].)

We wondered about the special role of the primes  $p = 2$  and  $3$  in equations (1) and (2): why these primes but not others? (This is the aforementioned puzzle.) We looked for patterns in the  $p$ -orders of constant terms of  $j$  and other modular forms for  $SL(2, \mathbb{Z})$  for  $p$  larger than three. Our search within  $SL(2, \mathbb{Z})$  came up empty, so we searched among the Hecke groups  $G(\lambda_n), n = 3, 4, \dots$  for the following reasons. The matrix group  $SL(2, \mathbb{Z})$  coincides with the Hecke group  $G(\lambda_3)$ , discussed below. It is isomorphic to the product of cyclic groups  $C_2 * C_3$ ; while in general  $G(\lambda_m) \cong C_2 * C_m$  for  $m = 3, 4, \dots$ . At first we hoped that primes  $p$  larger than three might manifest behavior analogous to that of three in equations (1) and (2) in Hecke groups  $G(\lambda_p)$ , but so far we have only found some more complicated patterns.

**1.2. Existence of the interpolating polynomials.**

For  $m = 3, 4, \dots$ , let  $\lambda_m = 2 \cos \pi/m$  and  $J_m$  be a certain meromorphic modular form built from a particular triangle function  $\phi_m$  for the Hecke group  $G(\lambda_m)$  with Fourier expansions  $J_m(\tau) = \sum_{n=-1}^{\infty} a_n(m)q_m^n$ , where  $q_m(\tau) = \exp 2\pi i\tau/\lambda_m$ . The groups  $G(\lambda_3)$  and  $SL(2, \mathbb{Z})$  coincide.<sup>2</sup> (There is more on triangle functions in the next section, and details of their construction are in [8].) For  $n = -1, 0, 1, 2$  and  $3$ , Raleigh [31] gave polynomials  $P_n(x)$  such that  $a_{-1}(m)^n q_m^{2n+2} a_n(m) = P_n(m)$  for  $m = 3, 4, \dots$ , and conjectured that similar relations hold for all positive integers  $n$ . This was proved by Akiyama [1].

In his 2021 article [8] the present writer suggested that such interpolating polynomials for higher weight Hecke-group modular forms should exist as well, and (acting on his own suggestion) tentatively identified some of them by Lagrange interpolation. In this section, we offer a more detailed existence argument. Beyond this introduction, the present article does not require such an argument because

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<sup>2</sup>For further details, the reader is referred to the books by Carathéodory [14], [15] and by Berndt and Knopp [4], the articles of Lehner and Raleigh [26], [31], to the dissertation of Leo [27], and to a summary, including pertinent references to that material, in the 2021 article [8]. Finally, the article by Hardy and Ramanujan on expansions of modular functions is reprinted in Ramanujan’s *Collected Papers* [20].

we will not construct models of interpolating polynomials other than (more or less) those of Raleigh and Akiyama. But a referee, whom we thank, asked us to comment on future directions of this investigation, and if the studies presented here are to be extended to higher-weight modular forms, the existence argument will be relevant. Our argument will apply to Hecke-group modular forms. The classical forms for  $SL(2, \mathbb{Z})$  lie within that class.

**1.2.1. Hecke’s theory of modular forms.**

Using the weight-raising properties of differentiation and the  $J_m$ , E. Hecke constructed certain families  $\mathcal{H}$  comprising modular forms of positive weight for each  $G(\lambda_m)$  sharing certain properties ([21], [4].) (The weight of  $g$  is not necessarily constant within such a family.) It seems apparent that Akiyama’s result can be extended: there should exist polynomials  $Q_{\mathcal{H},n}(x)$  interpolating the coefficient of  $X_m^n$  in the Fourier expansions of the members of Hecke families  $\mathcal{H}$ .

To make this precise, we review results of Hecke described in the book of Berndt and Knopp [4]. By Theorem 3.1 in that book, the region  $B(\lambda_m)$  defined below is a fundamental region for  $G(\lambda_m)$ .

- Definition 1.** 1. Let  $\tau_{\lambda_m}$  be the intersection of the circle  $|\tau| = 1$  with the line  $\Re(\tau) = -\lambda_m/2$ .
2. Let  $B(\lambda_m) = \{\tau \in \mathbb{H} : \Re(\tau) < \lambda_m/2, |\tau| > 1\}$ .
3. Let  $g_m(\tau)$  be the unique function guaranteed to exist by the Riemann mapping function mapping  $B(\lambda_m)$  conformally and one-to-one onto the upper half plane such that  $g_m$  takes  $\tau_{\lambda_m}$  to zero,  $i$  to 1, and  $i\infty$  to itself. (Berndt and Knopp, pages 47–48.)
4. Let

$$f_{\lambda_m}(\tau) := \left\{ \frac{g'_m(\tau)^2}{g_m(\tau)(g_m(\tau) - 1)} \right\}^{1/(m-2)},$$

$$f_{i,m}(\tau) := \left\{ \frac{g'_m(\tau)^m}{g_m(\tau)^{m-1}(g_m(\tau) - 1)} \right\}^{1/(m-2)},$$

and

$$f_{\infty,m}(\tau) := \left\{ \frac{g'_m(\tau)^{2m}}{g_m(\tau)^{2m-2}(g_m(\tau) - 1)^m} \right\}^{1/(m-2)}.$$

By Theorem 5.5 in Berndt and Knopp [4], we know that the functions  $f_{\lambda_m}$ ,  $f_{i,m}$ , and  $f_{\infty,m}$  are modular for  $G(\lambda_m)$  with weights  $4/(m - 2)$ ,  $2m/(m - 2)$ , and  $4m/(m - 2)$ , respectively. (There is a subtlety about the multiplier in the functional equation for the modularity of  $f_{i,m}$  which we will pass over.)

Because of its uniqueness, we know that  $g_m = J_m$  from equation 2 in Raleigh’s article. Therefore, corresponding to the three  $f$ ’s, we have the following definitions.

**Definition 2.** 1. Let  $H_{\lambda,m}(\tau) :=$

$$\left\{ \frac{J'_m(\tau)^2}{J_m(\tau)(J_m(\tau) - 1)} \right\}^{1/(m-2)}.$$

2. Let  $H_{\lambda,4,m}(\tau) := H_{\lambda,m}(\tau)^{m-2}$ .

**Definition 3.** 1. Let  $H_{i,m}(\tau) :=$

$$\left\{ \frac{J'_m(\tau)^m}{J_m(\tau)^{m-1}(J_m(\tau) - 1)} \right\}^{1/(m-2)}.$$

2. Let  $H_{i,6,m}(\tau) :=$

$$\left\{ \frac{J'_m(\tau)^m}{J_m(\tau)^{m-1}(J_m(\tau) - 1)} \right\}^{3/m}.$$

**Definition 4.** 1. Let  $\Delta_{\infty,m}(\tau) :=$

$$\left\{ \frac{J'_m(\tau)^{2m}}{J_m(\tau)^{2m-2}(J_m(\tau) - 1)^m} \right\}^{1/(m-2)}.$$

2. Let  $\Delta_{\infty,12,m}(\tau) :=$

$$\left\{ \frac{J'_m(\tau)^{2m}}{J_m(\tau)^{2m-2}(J_m(\tau) - 1)^m} \right\}^{3/m}.$$

3. Let  $\Delta_m^\diamond(\tau) := H_{\lambda,m}(\tau)^3/J_m(\tau)$ .

4. Let  $\Delta_{12,m}^\diamond(\tau) := H_{\lambda,4,m}^3(\tau)/J_m(\tau)$ .

5. Let  $\Delta_m^\dagger(\tau) := H_{\lambda,4,m}(\tau)^3 - H_{i,6,m}(\tau)^2$ .

**Remark 1.** It is easy to see from the definitions (for example, in [36]) that in the classical case (subgroups of  $SL(2, \mathbb{Z})$ ), if  $f$  and  $g$  are modular for a particular group with weights  $\omega_f$  and  $\omega_g$ , and  $a$  is a rational number, then  $fg$  and  $f^a$  are modular for the same group, with weights  $\omega_f + \omega_g$  and  $a \cdot \omega_f$ , respectively. These statements hold in the case of the Hecke groups as well. Therefore it follows from Berndt and Knopp’s Theorem 5.5 that we have the following tables of weights:

$H_{\lambda,m}$	$H_{\lambda,4,m}$	$H_{i,m}$	$H_{i,6,m}$
$4/(m-2)$	4	$2m/(m-2)$	6

and

$\Delta_m^\diamond$	$\Delta_{12,m}^\diamond$	$\Delta_{\infty,m}$	$\Delta_{\infty,12,m}$	$\Delta_m^\dagger$
$12/(m-2)$	12	$4m/(m-2)$	12	12

**1.2.2. The existence argument.**

The main goal of the present article is to describe conjectures about finite field models of the  $J_m$  based on *SageMath* experiments [33]. Let <sup>3</sup>

$$J_m(\tau) = \sum_{n=-1}^{\infty} a_n(m)q_m(\tau)^n.$$

For integers  $m \geq 3$  and  $n = 0, 1, 2, 3$ ,  $m^{-2n-2}a_{-1}(m)^{-n}A_n(m) = a_n(m)$ , where  $A_n(x)$  is a polynomial with rational coefficients and degree  $2n + 2$ , such that the coefficient of  $x^n$  is zero when  $n$  is odd (Raleigh, [31]). As we said in the introduction, similar relations exist among the  $a_n$  for all positive  $n$ .

In section four of [8] we wrote the Fourier expansions of the  $J_m$  by replacing  $q_m$  with another variable  $X_m(\tau)$  in the expansions. (The number  $\tau$  is a generic element of the upper half-plane.) By Akiyama’s theorem, we have a series of the form  $\mathcal{J}(x, X_m) := \sum_{n=-1}^{\infty} \tilde{P}_n(x)X_m^n$  for polynomials  $\tilde{P}_n(x)$  in  $\mathbb{Q}[x]$  with the property that  $J_m = \mathcal{J}(m, X_m)$ . A reader willing to take that construction for granted can, for present purposes, regard the expansion of  $J_m$  in  $X_m$  (truncated to  $n$  terms) as the object defined in our *SageMath* code in the “dictionary” at the top of each notebook as  $\mathbf{J}(n, m)$ . Now, just to provide the reader with some context, we normalize the  $J_m$  themselves to obtain functions  $j_m$  such that  $j_3$  is (apparently) the usual  $j$  function.<sup>4</sup> We begin by defining an operator on infinite series in  $X_m$ . It has the effect when  $m = 3$  of recovering the Fourier series of a variety of standard modular forms.<sup>5</sup>

**Definition 5.** Let  $f = \sum_{n=a}^{\infty} k_n X_m^n$ , where  $k_n$  is a rational number for  $n = a, a + 1, \dots$ , and  $k_a \neq 0$ . Let  $g = \sum_{n=a}^{\infty} k_n (2^6 m^3 X_m)^n = \sum_{n=a}^{\infty} \tilde{k}_n X_m^n$  (say). Then

$$\bar{f} := g/\tilde{k}_a.$$

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<sup>3</sup>Relevant files in [7] are (1) the *SageMath* Jupyter notebook in which we generated Fourier expansions of the  $J_m$  and of the  $A_n(x)$ , namely “capital-J make data file1jun21.ipnyb”; (2) the notebook “capital-J polynomials make file.ipynb” in which we generated the data file “run14jun21no14.txt”, which is called in turn by many of our other notebooks; (3) a table for  $a_n(m)$  divided into several files: “run2jun21no11”, “run2jun21no12”, “run2jun21no13” and “run2jun21no14”; (4) a *Mathematica* notebook “conjecture 1.nb” documenting the table’s calculation; a table for the  $A_n(x)$  made in the same notebook under file name “run20apr21no5”; and (5) files generated for the same purposes in the SageMath Jupyter notebook “conjecture1no1.ipnyb” (which is also there.)

<sup>4</sup>Except to repeat a conjecture from our 2021 article, we do not study the  $j_m$  in this one. But the reader will notice in conjecture 1 below that that interpolating polynomials behave a little more nicely for  $j_m$  than for  $J_m$ .

<sup>5</sup>The substitution involved appears in [27]; see our article [8] for a fuller acknowledgement of our debt to Leo.

**Definition 6.** With  $J_m(\tau) = 1/X_m + \sum_{n=0}^{\infty} a_n(m)X_m^n$ , we set<sup>6</sup>

$$j_m(\tau) := \overline{J}_m = 1/X_m + \sum_{n \geq 0} c_m(n)X_m^n \text{ (say).}$$

The Fourier expansion of  $j_3$  is <sup>7</sup>

$$j_3(\tau) = 1/X_3(\tau) + 744 + 196884X_3(\tau) + 21493760X_3(\tau)^2 + \dots,$$

which matches the standard expansion  $j(\tau) =$

$$1/\exp(2\pi i\tau) + 744 + 196884 \exp(2\pi i \cdot \tau) + 21493760 \exp(2\pi i \cdot 2 \cdot \tau) + \dots$$

We make the following

**Definition 7.** Let  $\mathcal{F} = \{f_3, \dots, f_m, \dots\}$  where  $f_m$  is modular for  $G(\lambda_m)$ . Then we write the Fourier expansion of  $f_m^k$  in powers of  $X_m$  as

$$f_m(\tau)^k = \sum_n A_{\mathcal{F},k,m}(n)X_m^n.$$

**Proposition 1.** Let  $\mathcal{K} = \{J_3, J_4, \dots\}$  and  $\overline{\mathcal{K}} = \{j_3, j_4, \dots\}$ . Then there exist polynomials  $Q_{\mathcal{K},k,n}(x)$  and  $Q_{\overline{\mathcal{K}},k,n}(x)$  in  $\mathbb{Q}[x]$  such that

$$J_m(\tau)^k = \sum_{n=-k}^{\infty} Q_{\mathcal{K},k,n}(m)X_m(\tau)^n$$

and

$$j_m(\tau)^k = \sum_{n=-k}^{\infty} Q_{\overline{\mathcal{K}},k,n}(m)X_m(\tau)^n.$$

In other words,  $A_{\mathcal{K},k,m}(n) = Q_{\mathcal{K},k,n}(m)$  and  $A_{\overline{\mathcal{K}},k,m}(n) = Q_{\overline{\mathcal{K}},k,n}(m)$  for  $k = 1, 2, \dots, m = 3, 4, \dots$ , and  $n = -k, 1 - k, \dots$

For  $k$  equal to one, the first claim is just Akiyama’s theorem and the claim for  $k$  not equal to one is then obvious. The second statement follows immediately.

**Proposition 2.** With  $k$  as in proposition 1, let

$$\mathcal{H} = \{H_{\lambda,m}\}, \{H_{\lambda,4,m}\}, \{H_{i,m}\}, \{H_{i,6,m}\},$$

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<sup>6</sup>Some code for  $j_m$  Fourier expansions appearing in *SageMath* notebooks cited below was generated in [7], notebook “j from scratch.ipynb”, which employs a “dictionary” (the definitions at the top of the notebook) distinct from the corresponding dictionaries in the notebooks where it is reproduced.

<sup>7</sup>See equation (23) of Serre’s book [36], section 3, and the *SageMath* notebook “jpower constant term NewmanShanks 26oct22.ipynb” in our repository [5].

$$\{\Delta_m^\circ\}, \{\Delta_{12,m}^\circ\}, \{\Delta_{\infty,m}\}, \text{ or } \{\Delta_m^\dagger\},$$

permitting  $m$  to range over the integers greater than two. Then there exist polynomials  $Q_{\mathcal{H},k,n}(x)$  in  $\mathbb{Q}[x]$  such that the elements  $f_3, f_4, \dots$  of  $\mathcal{H}_k$  have Fourier expansions

$$f_m(\tau) = \sum_n Q_{\mathcal{H},k,n}(m) X_m(\tau)^n.$$

For  $k$  equal to one, we justify this as follows. After substituting  $\mathcal{J}(x, X_m)$  (the series defined in the paragraph succeeding remark 1 above) for  $J_m$  in the various clauses of definitions 2 - 4, the right sides become rational functions of fractional powers of various series in powers of  $X_m$  with coefficients in  $\mathbb{Q}[x]$ , which by purely formal operations should be expressible as other series in powers of  $X_m$  with coefficients in  $\mathbb{Q}[x]$ , from which we recover Fourier expansions of each of the defined functions by setting  $x$  equal to  $m$ . The statement for  $k$  other than one follows easily.

**1.3. Finite field models.**

**1.3.1. How do finite field models help?**

We want to understand as much as we can about the  $A_n(x)$ . The most obvious ways to analyze polynomials are through their coefficients and their roots. We have not found patterns among the coefficients in these polynomials. In experiments that we have not published, on the other hand, we did see that in some situations the roots of relevant polynomials appear to be confined to the real axis, but we had no proofs.<sup>8</sup> We usually could not do better than approximations. The roots of polynomials in a finite field, on the other hand, can be enumerated exactly, not just in theory but (when the field is small enough) in practice, simply by doing a brute-force search. We found that (by clearing denominators of a prime  $p$ ) we arrived at models of the  $A_n(x)$  in fields of characteristic  $p$  that displayed regularities determined by the residue class of  $n$  modulo  $p$ . These regularities suggest, to the author, at least, that the  $A_n$  in his 2021 article were correctly identified. They are also evidence that the Fourier coefficient of the original  $J_m$  at  $X^n$  is governed somehow by  $n$  modulo  $p$  for each prime  $p$  and all  $n$ . This vague claim is the moral of our story.<sup>9</sup>

**1.3.2. A sketch of the models.**

If  $P(x)$  is in  $\mathbb{Q}[x]$ , we write  $\mathcal{K}_p(P(x))$  for a certain polynomial in  $\mathbb{Q}[x]$  that agrees with  $P(x)$  up to a multiplicative constant. The definition of the map  $\mathcal{K}_p$  guarantees

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<sup>8</sup>In some circumstances this is a significant property. For example, the proposition that all of the roots of Jensen polynomials are real is equivalent to the Riemann hypothesis ([30], [19], [18], [17].)

<sup>9</sup>The congruences for the coefficients  $c(n)$  of the  $j$  function discussed in the articles [2], [24], and [25], and also in in Serre’s book [36] (chapter VII, section 3) are clearly related to this claim.

that  $\text{ord}_p$  is non-negative on all coefficients of  $\mathcal{K}_p(P(x))$ —a property that allows us to define its models in finite fields of characteristic  $p$ . We write  $\mathbb{A}(n, p)$  for the smallest splitting field (in a weak sense) of  $A_n(x)$  over  $\mathbb{F}_p$ . We also write  $\mathcal{A}_n[p](x)$  for  $\mathcal{K}_p(A_n(x))$  considered as a polynomial self-map of  $\mathbb{A}(n, p)$ . These are the models of our title. We study the  $\mathcal{A}_n[p](x)$  with numerical experiments and make conjectures describing their roots completely for primes  $p$  less than or equal to seven.

To recover the  $A_n(m)$  from the  $\mathcal{A}_n[p](x)$ , our experiments suggest that it would be most useful to have in hand  $\mathcal{A}_n[p](x)$  for all primes  $p$  less than or equal to the smallest prime greater than  $n$ . But the feasibility of computing with  $p$  declines with the size of  $\mathbb{A}(n, p)$ , which becomes large for  $n$  in certain residue classes modulo  $p$ . Consequently, our conjectures only address primes less than or equal to seven. If our conjectures are valid and the behavior of the  $\mathbb{A}(n, p)$  is similar for all larger primes as well, then the Fourier coefficients  $A_n(m)$  of the  $J_m = 1/X_m + \sum_{n=0}^{\infty} A_n(m)X_m^n$  behave in a uniform way that is independent of the Hecke group  $G(\lambda_m)$  and depends only on congruences satisfied by the indices  $n$ .

Our identification of their roots for  $p$  less than or equal to seven only specifies the  $\mathcal{A}_n[p](x)$  themselves up to a multiplicative constant, and we have not been able to understand how these constants vary with  $n$ , even for fixed small values of  $p$ . On the other hand, the original polynomials  $A_n(x)$  behaved much better in this regard in our experiments. We were able to write them as the product of a product of monic polynomials and an explicitly known rational number. (See conjectures 1 and 2 below.) This seems to lessen the urgency of solving the problem of understanding more fully the multiplicative constants associated with the  $\mathcal{A}_n[p](x)$ .

## 2. Triangle functions.

The material in the present section was sketched more fully in [8], which also includes citations to an even more thorough exposition of much of it in the second volume of Carathéodory [15]. Let  $\mathbb{Z}, \mathbb{Q}, \mathbb{C}$  and  $\mathbb{H}$  denote, respectively, the set of rational integers, the set of rational numbers, the set of complex numbers, and the set of complex numbers with positive imaginary parts. (We will reserve the letter  $\tau$  for elements of the upper half-plane, and  $z$  for generic complex numbers.) We write  $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$ , and we equip  $\mathbb{H}^*$  with the Poincaré metric. Figures  $T$  made by three geodesics of  $\mathbb{H}^*$  are called hyperbolic or circular-arc triangles. Let  $\lambda_m = 2 \cos \pi/m$ . For  $m = 3, 4, \dots$ , we define the Hecke group  $G(\lambda_m)$  as the discrete group generated by the maps  $z \rightarrow -1/z$  and  $z \rightarrow z + \lambda_m$ . The full modular group  $SL(2, \mathbb{Z})$  is identical to  $G(\lambda_3)$ .

For our purposes, Schwarz triangles  $T$  are hyperbolic triangles in  $\mathbb{H}^*$  with certain restrictions on the angles at the vertices. From a Euclidean point of view, their sides are vertical rays, segments of vertical rays, semicircles orthogonal to the real axis and meeting it at points  $(r, 0)$  with  $r$  rational, or arcs of such semicircles. We choose  $\lambda, \mu$  and  $\nu$ , all non-negative, such that  $\lambda + \mu + \nu < 1$ ; then the angles of  $T$  are  $\lambda\pi, \mu\pi$ , and  $\nu\pi$ . By reflecting  $T$  across one of its edges, we get another Schwarz triangle. The reflection between two triangles in  $\mathbb{H}^*$  is effected by a Möbius transformation, so the orbit of  $T$  under repeated reflections is associated to a collection of Möbius transformations. The group generated by these transformations is a triangle group.<sup>10</sup> By the Riemann Mapping Theorem there is a conformal, onto map  $\phi : T \mapsto \mathbb{H}^*$  called a triangle function.

Hecke groups are triangle groups  $H$  that act properly discontinuously on  $\mathbb{H}$  [21]. This means that for compact  $K \subset \mathbb{H}$ , the set  $\{\mu \in H \text{ s.t. } K \cap \mu(K) \neq \emptyset\}$  is finite. Recall that  $G(\lambda_m)$  is the Hecke group generated by the maps  $z \mapsto -1/z$  and  $z \mapsto z + \lambda_m$ . Hecke established that  $G(\lambda_m)$  has the structure of a free product of cyclic groups  $C_2 * C_m$ , generalizing the relation ([36], [13])  $SL(2, \mathbb{Z}) = C_2 * C_3$ .

Let  $\rho = -\exp(-\pi i/m) = -\cos(\pi/m) + i \sin(\pi/m)$ , and let  $T_m \subset \mathbb{H}^*$  denote the hyperbolic triangle with vertices  $\rho, i$ , and  $i\infty$ . The corresponding angles are  $\pi/m, \pi/2$  and  $0$  respectively. Let  $\phi_{\lambda_m}$  be a triangle function for  $T_m$ . The function  $\phi_{\lambda_m}$  has a pole at  $i\infty$  and period  $\lambda_m$ . For  $P, Q \in \mathbb{H}^*$ , let us write  $P \equiv_H Q$  when  $\mu \in H$  and  $Q = \mu(P)$ . Then  $\phi_{\lambda_m}$  extends to a function  $J_m : \mathbb{H}^* \rightarrow \mathbb{H}^*$  by declaring that  $J_m(P) = J_m(Q)$  if and only if  $P \equiv_H Q$ .  $J_m$  is a modular function (a meromorphic modular form of weight zero) for  $G(\lambda_m)$ .

Schwarz ([34] or [35]), Lehner [26] and Raleigh [31] studied Schwarz triangle functions, which map hyperbolic triangles  $T$  in the extended upper half  $z$ -plane onto the extended upper half  $w$ -plane. For certain  $T = T_m$ , a triangle function  $\phi_{\lambda_m} : T \rightarrow \mathbb{H}^*$  extends to a map  $J_m : \mathbb{H}^* \rightarrow \mathbb{H}^*$  invariant under modular transformations from  $G(\lambda_m)$ .

### 3. Polynomial interpolation of Fourier coefficients of modular functions for Hecke groups.

We reproduce parts of conjecture 1 and conjecture 2 from our earlier article [8].

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<sup>10</sup>In the terminology of Isant and Grau [3], page 45, these are Fuchsian groups of signature  $(0; 1/\lambda, 1/\mu, 1/\nu)$ . We have not as yet examined the possibility of extending the experiments of our article [8] (and of the present article) to a broader class of Fuchsian groups. Movasati [28] provides relevant code for such studies, based on results of the article by Doran, Gannon and himself [16].

**Conjecture 1.** Let the Fourier expansion of  $j_m(\tau)$  be

$$j_m = 1/X_m + \sum_{n \geq 0} c_m(n)X_m^n.$$

1. For each integer  $n$  greater than  $-2$ , there exists a polynomial  $C_n(x) \in \mathbb{Q}[x]$  that satisfies the relation  $c_m(n) = C_n(m)$  for  $m = 3, 4, \dots$  <sup>11</sup>
2. Let  $\{\phi_{-1}, \phi_0, \dots\} = \{1, 24, \dots\}$  be the McKay-Thompson series of class 4A [40]. For some degree  $2n$ , irreducible, monic polynomial  $\gamma_n(x)$  in  $\mathbb{Q}[x]$  we have <sup>12</sup>:

$$C_n(x) = \phi_n \cdot (x - 2)(x + 2)x^{n+1}\gamma_n(x).$$

3. The function  $j_3$  is identical to the modular function on  $SL(2, \mathbb{Z})$  usually denoted  $j$ .<sup>13</sup>

**Conjecture 2.** Let <sup>14</sup> the Fourier expansion of  $J_m(\tau)$  be

$$J_m = \sum_{n=-1}^{\infty} a_m(n)X_m^n.$$

1. <sup>15</sup>
  - (a) There exist polynomials  $A_n(x)$  such that  $A_{-1}(x) \equiv 1$ ,  $A_0(x) = 3x^2 + 4$ ,  $A_1(x) = 69x^4 - 8x^2 - 48$ , and  $A_n(m) = m^{2n+2}a_m(n)$  for  $m = 3, 4, \dots$
  - (b) Let  $C_n(x)$  be as in Conjecture 1. Then

$$A_n(x) = 2^{-6n-6}x^{-n-1}C_n(x).$$

2. Let  $\pi_n$  be the set of prime numbers dividing the denominator of at least one non-zero coefficient of  $A_n$ . Then the following statements are true <sup>16</sup>
  - (a)  $\pi_2 = \{3\}$ .
  - (b) If  $\pi_n$  is ordered by size, it contains no gaps. That is, if  $p$  and  $p'$  are consecutive elements of  $\pi_n$  with  $p = p_k$  and  $p' = p_j$ , then  $j = k + 1$ .
  - (c) If  $n$  is an odd prime, then

$$\pi_n = \{2, \dots, k, \dots, p\}_{k \text{ prime}}$$

where  $p$  is the greatest prime less than  $n$ .

<sup>11</sup>[7], notebook “conjecture 1.nb”.

<sup>12</sup>[7], notebook “conjecture 1 clause 2.ipynb”.

<sup>13</sup>[7], notebook “conjecture 1 clause 3.nb”.

<sup>14</sup>Relevant documents in [7] are notebooks “conjecture 2.nb”, “conjecture2no1.ipynb”, “capital-J make data file1jun21.ipynb” and associated data files.

<sup>15</sup>For clause 1, see [7], notebooks “conjecture 2.nb”, “conjecture 2 clause 1b.ipynb”, and “conjecture 2 clause 1b no2.ipynb”.

<sup>16</sup>[7], notebook “conjecture 2 clause 2 w code 14jun21.ipynb”.

(d) If  $n$  is composite and  $n + 1$  is prime, then

$$\pi_n = \{2, \dots, k, \dots, n + 1\}_{k \text{ prime}}.$$

(e) If  $n$  and  $n + 1$  are both composite, then

$$\pi_n = \{2, \dots, k, \dots, p\}_{k \text{ prime}}$$

where  $p$  is the greatest prime less than  $n$ .

**Remark 2.** Comparing the notation of clause 1(a) of conjecture 2 with that of proposition 1, we see that  $A_n(m) = m^{2n+2}Q_{\mathcal{K},1,n}(m) = m^{2n+2}A_{\mathcal{K},1,m}(n)$ .

#### 4. Finite-field models of interpolating polynomials.

Interpreting a polynomial  $f(x)$  as a self-map on a finite field  $\mathbb{F}_{p^k}$  is not possible if the prime  $p$  divides the denominator of one of the coefficients of  $f(x)$ . It appears that all primes less than  $n$  occur in the denominators of coefficients of the interpolating polynomials  $A_n(x)$  and that, as a result, we cannot make such an interpretation when  $p$  is less than  $n$ . To get around this difficulty, we clear denominators to obtain polynomials, either with integer coefficients, or at least with denominators not divisible by a specified prime number, before applying maps from  $\mathbb{Q}[x]$  to  $\mathbb{F}_{p^k}[x]$  defined below to arrive at finite field models of the interpolating polynomials.<sup>17</sup>

**Definition 8.** 1. If  $F$  is a finite field of characteristic  $p$ , let  $0_F$  and  $1_F$  denote the additive and multiplicative identities of  $F$ , respectively. Furthermore if  $n$  is a positive integer, let  $n_F$  denote the sum in  $F$  of  $n$  copies of  $1_F$ , and let  $(-n)_F$  be the additive inverse in  $F$  of  $n_F$ . If  $n_1$  and  $n_2$  are positive integers such that  $\text{ord}_p(n_2) \geq 0$  let  $(n_1/n_2)_F$  be the quotient in  $F$  of  $(n_1)_F$  and  $(n_2)_F$ . We refer to the map  $x \mapsto x_F$  as coercion.

2. (a) Let  $\mathbb{A}(n, p)$  be the smallest characteristic  $p$  field in which  $A_n(x)$  splits into factors of degree zero or one.
- (b) Let  $\mathbb{A}(n, p)^*$  be the (cyclic) multiplicative group  $\mathbb{A}(n, p) \setminus \{0_F\}$ .
3. (a) Let  $s_A(n, p)$  denote the vector space dimension  $[\mathbb{A}(n, p) : \mathbb{F}_p]$ .
- (b) Integers  $n_1$  and  $n_2$  are  $(A, p)$ -equivalent if  $n_1 \equiv n_2$  modulo  $p$  and  $s_A(n_1, p) = s_A(n_2, p)$ .

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<sup>17</sup>In practice, these maps are *SageMath* coercions. Coercion is a concept from type theory. For a recent discussion, see Buzzard's preprint [11]. For *SageMath*'s own explanations of its coercion routines, see the documents [42] and [22]. *Caveat:* In the article, we define the maps mathematically; coercion in *SageMath*, on the other hand, is defined by code. Our conjectures are based on the validity (which is left to the judgement of the reader) of the correspondence between these definitions.

4. (a) If  $P(x) = \sum_{n \in I} (N_n/D_n)x^n \in \mathbb{Q}[x]$  where  $N_n$  and  $D_n$  are relatively prime integers,  $d(P(x)) := \text{lcm}\{D_n\}_{n \in I}$ .  
 (b) With  $\mu_p(P(x)) = p^{\text{ord}_p(d(P(x)))}$ ,

$$K(P(x)) := d(P(x)) \times P(x)$$

and

$$\mathcal{K}_p(P(x)) := \mu_p(P(x)) \times P(x).$$

5. Here we define finite field models of the  $A_n(x)$ .

- (a) Let  $\alpha_j, j = 0, 1, 2, \dots, u$ , be rational integers, and let  $K(A_n(x)) = \sum_{j=0}^u \alpha_j x^j$ .  
 Then

$$A_n[p](x) := \sum_{j=0}^u (\alpha_j)_F x^j.$$

where  $F = \mathbb{A}(n, p)$ .

- (b) Let  $\alpha_j^*, j = 0, 1, 2, \dots, u$ , be rational numbers and let  $\mathcal{K}_p(A_n(x)) = \sum_{j=0}^u \alpha_j^* x^j$ .  
 By construction,  $\text{ord}_p(\alpha_j^*) \geq 0$ . Then

$$\mathcal{A}_n[p](x) := \sum_{j=0}^u (\alpha_j^*)_F x^j.$$

where  $F = \mathbb{A}(n, p)$ .

6. (a) Let  $\text{mod}(n, p)$  be the element  $\rho$  of  $\{0, 1, \dots, p-1\}$  such that  $n \equiv \rho \pmod{p}$ .  
 (b) We set  $\delta(n, p) := (n - \text{mod}(n, p))/p$ .  
 (c) We set  $a(n, p) :=$  the leading coefficient not divisible by  $p$  in  $K(A_n(x))$ .  
 (d) We set  $a^*(n, p) :=$  the leading coefficient not divisible by  $p$  in  $\mathcal{K}_p(A_n(x))$ .  
 (e) We set  $r(n, p) := \text{mod}(a(n, p), p)$ .  
 (f) We set  $r^*(n, p) := \text{mod}(a^*(n, p), p)$ .  
 (g) We set  $\alpha(n, p) := r(n, p)_F$  where  $F = \mathbb{A}(n, p)$ .  
 (h) We set  $\alpha^*(n, p) := r^*(n, p)_F$  where  $F = \mathbb{A}(n, p)$ .  
 (i) We write the Frobenius map as  $\phi_p : F \rightarrow F$  with  $\phi_p(s) := s^p$ .  
 (j) For  $s$  in  $F$ , the  $\phi_p$ -orbit of  $s$  is written  $\mathcal{O}_p$ .

- Remark 3.** 1. The models  $\mathcal{A}_n[p](x)$  and  $A_n[p](x)$  agree up to multiplicative constants. Therefore, the assertions summarized in tables 1 - 3 below are valid for both of them. We require the  $A_n[p](x)$  in order to make the argument in remark 5 concerning Frobenius orbits of the roots. But there is a loss of information in  $A_n[p](x)$  from the clearing of denominators by the  $K$ -operator which can only be reflected in the relevant multiplicative constant. To be more precise,  $K$  takes distinct polynomials in  $\mathbb{Q}[x]$  to a single element of  $\mathbb{Z}[x]$ , and so it would seem to be difficult to constrain  $A_n(x)$  using  $K(A_n(x))$  or the corresponding models  $A_n[p](x)$ . On the other hand, let the set of prime numbers  $\pi_n$  be as in conjecture 2, clause 2 and let  $\nu_n$  be the largest element of  $\pi_n$ . Then the obstacles to deriving conditions on  $A_n(x)$  from information about the images  $\{\mathcal{K}_p(A_n(x))\}_{p \leq \nu_n}$  and the corresponding models  $\{\mathcal{A}_n[p](x)\}_{p \leq \nu_n}$  would seem to be less formidable.
2. We have not considered the question whether or not the  $\mathbb{A}(n, p)$  are splitting fields in the full sense of, say, Definition A.5.7 in the book [29]. (The extra condition is this: for a splitting field  $S$  of a polynomial  $P(x)$ , the roots of  $P(x)$  generate  $S$  over the base field.)
3. Typically when  $n_1$  and  $n_2$  are  $(A, p)$ -equivalent, the polynomials  $\mathcal{A}_{n_1}[p](x)$  and  $\mathcal{A}_{n_2}[p](x)$  are distinct (tables 1-3.) Of course, the set of self-maps of a finite set is also finite, so the infinite family  $\{\mathcal{A}_n[p](x)\}_n$  represents only a finite set of self-maps on the  $\mathbb{A}(n, p)$ , and there is a natural equivalence relation on  $\{\mathcal{A}_n[p](x)\}_n$  induced by this fact; we will study it below.

The conjectures below are based on numerical experiments documented in our repository [6] in the range of  $n$ -values that were accessible to our tests, namely,  $-1 \leq n \leq 200$ .

**Conjecture 3.** <sup>18</sup> Let  $p_1$  and  $p_2$  be prime numbers such that  $p_1 < p_2$ .<sup>19</sup>

1. <sup>20</sup>
  - (a) If  $A_n[p](x)$  is factored over  $\mathbb{A}(n, p)$  as a field element  $\gamma$  times a product of monic polynomials, then  $\gamma = \alpha(n, p)$ ; by construction,  $\gamma$  belongs to  $\{1_{\mathbb{A}(n,p)}, 2_{\mathbb{A}(n,p)}, \dots, (p-1)_{\mathbb{A}(n,p)}\}$ .
  - (b) If  $\mathcal{A}_n[p](x)$  is factored over  $\mathbb{A}(n, p)$  as a field element  $\gamma$  times a product of monic polynomials, then  $\gamma = \alpha^*(n, p)$  and  $\gamma$  belongs to  $\{1_{\mathbb{A}(n,p)}, 2_{\mathbb{A}(n,p)}, \dots, (p-1)_{\mathbb{A}(n,p)}\}$ .

<sup>18</sup>[6], notebooks with titles “conjecture 4 ...” (*sic.*) for various primes  $p$ ; these notebooks serve several purposes, hence the file names. In the file names, “res” stands for “residue”, *i.e.*,  $\text{mod}(n, p)$ , and “sd” stands for “splitting degree”, *i.e.*,  $s_A(n, p)$ .

<sup>19</sup>[6], notebook ‘conjecture 3 for  $A_n$ .ipynb’.

<sup>20</sup>[6], Jupyter notebook “numerical term on extension fields 17may22.ipynb”

2. If  $p$  is a prime greater than 3,  $p < n$  and  $p \leq 17$ , then  $s_A(n, p)$  is determined by  $\text{mod}(n, p)$ ; moreover,  $s_A(n, 2) \equiv 1$  identically. Here are tables for  $p = 3, 5$  and 7.<sup>21</sup>

$\text{mod}(n, 3)$	0	1	2
$s_A(n, 3)$	1	1, 2, 3	1, 2, 3, 4

$\text{mod}(n, 5)$	0	1	2	3	4
$s_A(n, 5)$	1, 2	4	4	6	1

$\text{mod}(n, 7)$	0	1	2	3	4	5	6
$s_A(n, 7)$	4	4	2	2	6	5	1

3. If  $p$  is a prime number greater than three and  $n \equiv p - 1 \pmod{p}$ , then  $s_A(n, p) = 1$ .
4. (a) Either  $s_A(0, p) = 1$  or  $s_A(0, p) = 2$ .
- (b) If  $p$  is a prime number greater than three and  $s_A(0, p) = 1$ , then  $p \equiv 1 \pmod{3}$ .
- (c) If  $p$  is a prime number greater than three and  $s_A(0, p) = 2$ , then  $p \equiv 2 \pmod{3}$ .

**Remark 4.** 1. We have not found statements like clause 4 about  $s_A(n, p)$  for values of  $n$  other than zero.

2. We found clause 4 after reading Sloane’s A002476 [39] and related pages.

**Remark 5.** By construction, the coefficients of  $K(A_n(x))$  are rational integers; hence the coefficients of  $A_n[p](x)$  lie in the prime sub-field of  $\mathbb{A}(n, p)$ . Let  $f(x)$  be a polynomial self-map of  $\mathbb{A}(n, p)$  with coefficients in its prime sub-field. If  $t$  is a generator of  $\mathbb{A}(n, p)^*$ , then  $t^{kp}$  is a root of  $f(x)$  whenever  $t^k$  is such a root. (Proof: the Frobenius map  $\phi_p : x \mapsto x^p$  is an automorphism of  $\mathbb{A}(n, p)$  and fixes its prime sub-field.<sup>22</sup> So applying  $\phi_p$  to the equation  $f(x) = 0$  gives  $f(\phi_p(x)) = 0$ .) Consequently, the roots of  $A_n[p](x)$  and  $\mathcal{A}_n[p]$  are (identical) collections of complete  $\phi_p$  orbits. (In particular, each root of the form  $n_{\mathbb{A}(n, p)}$  is a  $\phi_p$  fixed point.)

<sup>21</sup>[6], notebook “conjecture 3 clause 2.ipynb”

<sup>22</sup>For example, see Chapter VII, section 5 of Lang’s book [23].

**Conjecture 4.**<sup>23</sup> We state conjectures on  $\mathcal{A}_n[p](x)$  for given  $(A, p)$  equivalence class data and  $p = 2, 3, 5$  and  $7$ . Let  $t(n, p) = t$  (say) be a generator of  $\mathbb{A}(n, p)^*$ . Excepting members  $n_F$  of the prime sub-field, we display the non-zero roots of  $\mathcal{A}_n[p](x)$  as  $\phi_p$ -orbits  $\mathcal{O}_p(t^k)$  for some  $k$ .<sup>24</sup> We do not make a general conjecture regarding the sizes of the Frobenius orbits, but for  $p = 2, 3$ , or  $5$ , let  $r$  be a non-zero root of  $\mathcal{A}_n[p](x)$ . If  $r$  belongs to the prime sub-field of  $\mathbb{A}(n, p)$  so that, for some rational integer  $n, r = n_F$ , then  $r$  is a fixed point of  $\phi_p$ :  $\#\mathcal{O}_p(r) = 1$ . Otherwise,  $\#\mathcal{O}_p(r) = s_A(n, p)$ .

The situation for  $p = 7$  is more complicated. If we write  $r = t^k$ , denote the base  $p$  expansion of  $k$  as  $X_p(k)$ , padded with zeros, necessary, so that its length is equal to  $s_A(n, p)$ , then for any prime  $p$  the length of  $\mathcal{O}_p(r)$  is the same as the length of the orbit under cyclic permutations of  $X_p(k)$ . For the smaller  $p$ ,  $X_p(k)$  exhibits no internal symmetries, but  $X_7(k)$  does. For example, if  $n \equiv 4 \pmod{7}$  and  $s_A(n, 7) = 6$ , we find that  $\mathcal{A}_n[7]$  has roots  $r_1 = t^k$  for  $k = 29412$  and  $r_2 = t^k$  for  $k = 88236$ . Here  $X_7(29412) = (5, 1, 5, 1, 5, 1)$  and  $X_7(88236) = (1, 5, 1, 5, 1, 5)$ . These expansions comprise a complete orbit under cyclic permutation. Consequently,  $r_1$  and  $r_2$  comprise a complete orbit under  $\phi_7$ . On the other hand, by a similar analysis,  $s_A(n, p) = 1$  implies that all roots of  $\mathcal{A}_n[p]$  are fixed points of  $\phi_p$ , and  $s_A(n, p) = 2$  implies that all roots  $r$  of  $\mathcal{A}_n[p]$  are either fixed by  $\phi_p$  or satisfy  $\#\mathcal{O}_p(r) = 2$ . Roots  $r = t^k$  fixed by  $\phi_p$  must belong to the prime sub-field (for example, the notes of Cañez [12], pp. 27-28).

We have dropped constant factors from our tables. For example,  $3_F(x - 1_F)$  would be listed simply as  $x - 1_F$ .<sup>25</sup> We use the  $\delta(n, p)$  notation defined above and write  $\delta(n, p) = \delta$ ; the arguments will be clear. The letter “u” means unrestricted. When we wish to abbreviate, “F” means  $\mathbb{A}(n, p)$ .

1.  $\mathcal{A}_n[2] = x^{2n+2}$ .

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<sup>23</sup>This conjecture is based on numerical data available in our GitHub repository [6] in files with filenames beginning “conjecture 4 p = (\*), res=(\*\*), sd=(\*\*\*)” for various integers (\*), etc. When these are loaded, they show titles beginning “conjecture 4 clause 4”. The reader should disregard the phrase “clause 4”.

<sup>24</sup>A reader who consults our *Sagemath* notebooks in the repository [6] will find that we kept track of the action of Frobenius by computing the (appropriately padded) base  $p$  expansions of the discrete logarithms  $k$ ; when  $\phi_p(t^k) = t^j$ , the expansion of  $j$  is the image of the expansion of  $k$  under a cyclic permutation. For example, in the case  $p = 3, n \equiv 1 \pmod{3}, s_A(n, 3) = 2$  (row 4 of table 2), our *SageMath* code outputs a root  $t + 2_F$  with discrete logarithm base 3 expansion  $(1, 2)$ . This tells us that  $t + 2 = t^7$  in  $\mathbb{A}(n, 3)$  because  $(1, 2)$  is the base 3 expansion of 7, and that the length of  $\mathcal{O}_3(t + 2)$  is 2, because the length of the orbit of  $(1, 2)$  under cyclic permutation is 2.

<sup>25</sup>What we think might be true about the constant factors is stated in conjecture 3. We cannot specify them *ab initio*.

2. Table for  $p = 3$ :

$\text{mod}(n, 3)$	$s_A(n, 3)$	$\mathcal{A}_n[3](x)$
0	1	$x^2(x - 1_F)^{2\delta}(x - 2_F)^{2\delta}$
1	1	$x^2(x - 1_F)^{2\delta}(x - 2_F)^{2\delta}$
1	1	$x^2(x - 1_F)^{2\delta-2}(x - 2_F)^{2\delta-2}$
1	2	$x^8(x - 1_F)^{2\delta-6}(x - 2_F)^{2\delta-6}$ $\times \prod_{s \in \mathcal{O}_3(t)} (x - s)^2$ $\times \prod_{s \in \mathcal{O}_3(t^2)} (x - s)^2$ $\times \prod_{s \in \mathcal{O}_3(t^7)} (x - s)^2$
2	1	$(x - 1_F)^{2\delta+1}(x - 2_F)^{2\delta+1}$
2	1	$x^6(x - 1_F)^{2\delta-1}(x - 2_F)^{2\delta-1}$

TABLE 1:  $\mathcal{A}_n[3]$  UP TO CONSTANT FACTORS

(In the ambiguous cases, we have not deciphered the conditions that choose between the polynomials listed in table 1 for  $\mathcal{A}_n[3]$ .)

3. Table for  $p = 5$ :

$\text{mod}(n, 5)$	$s_A(n, 5)$	$\mathcal{A}_n[5](x)$
0	1	$(x - 1_F)^{2\delta+2}(x - 4_F)^{2\delta+2} \times$ $(x - 2_F)^{2\delta}(x - 3_F)^{2\delta}$
1	4	$(x - 1_F)^{2\delta}(x - 4_F)^{2\delta} \times$ $(x - 2_F)^{2\delta}(x - 3_F)^{2\delta} \times$ $\prod_{s \in \mathcal{O}_5(t^{91})}(x - s)$
2	4	$(x - 1_F)^{2\delta}(x - 4_F)^{2\delta} \times$ $(x - 2_F)^{2\delta+1}(x - 3_F)^{2\delta+1} \times$ $\prod_{s \in \mathcal{O}_5(t^{169})}(x - s)$
3	6	$(x - 1_F)^{2\delta}(x - 4_F)^{2\delta} \times$ $(x - 2_F)^{2\delta+1}(x - 3_F)^{2\delta+1} \times$ $\prod_{s \in \mathcal{O}_5(t^{2961})}(x - s)$
4	1	$(x - 1_F)^{2\delta+2}(x - 2_F)^{2\delta+2} \times$ $(x - 3_F)^{2\delta+2}(x - 4_F)^{2\delta+2}$

TABLE 2:  $\mathcal{A}_n[5]$  UP TO CONSTANT FACTORS.

4. Table for  $p = 7$ :

$\text{mod}(n, 7)$	$s_A(n, 7)$	$\mathcal{A}_n[7](x)$
0	4	$(x - 1_F)^{2\delta-2}(x - 6_F)^{2\delta-2} \times$ $(x - 2_F)^{2\delta}(x - 5_F)^{2\delta} \times$ $(x - 3_F)^{2\delta+1}(x - 4_F)^{2\delta+1} \times$ $\prod_{s \in \mathcal{O}_7(t^{173})} (x - s)$ $\prod_{s \in \mathcal{O}_7(t^{260})} (x - s) \times$
1	4	$(x - 1_F)^{2\delta}(x - 2_F)^{2\delta} \times$ $(x - 3_F)^{2\delta}(x - 3_F)^{2\delta} \times$ $(x - 5_F)^{2\delta}(x - 6_F)^{2\delta} \times$ $\prod_{s \in \mathcal{O}_7(t^{75})} (x - s)$
2	2	$(x - 1_F)^{2\delta}(x - 3_F)^{2\delta} \times$ $(x - 4_F)^{2\delta}(x - 6_F)^{2\delta} \times$ $(x - 2_F)^{2\delta+1}(x - 5_F)^{2\delta+1} \times$ $\prod_{s \in \mathcal{O}_7(t^4)} (x - s)^2$

PART ONE OF TABLE 3:  $\mathcal{A}_n[7]$  UP TO CONSTANT FACTORS.

This is a continuation of the preceding table:

$\text{mod}(n, 7)$	$s_A(n, 7)$	$\mathcal{A}_n[7](x)$
3	2	$(x - 1_F)^{2\delta}(x - 3_F)^{2\delta} \times$ $(x - 4_F)^{2\delta}(x - 6_F)^{2\delta} \times$ $(x - 2_F)^{2\delta+1}(x - 5_F)^{2\delta+1} \times$ $\prod_{s \in \mathcal{O}_7(t^7)}(x - s) \times$ $\prod_{s \in \mathcal{O}_7(t^{12})}(x - s) \times$ $\prod_{s \in \mathcal{O}_7(t^{25})}(x - s)$
4	6	$(x - 1_F)^{2\delta}(x - 3_F)^{2\delta} \times$ $(x - 4_F)^{2\delta}(x - 6_F)^{2\delta} \times$ $(x - 2_F)^{2\delta+1}(x - 5_F)^{2\delta+1} \times$ $\prod_{s \in \mathcal{O}_7(t^{29412})}(x - s) \times$ $\prod_{s \in \mathcal{O}_7(t^{41280})}(x - s) \times$ $\prod_{s \in \mathcal{O}_7(t^{81528})}(x - s)$

PART TWO OF TABLE 3:  $\mathcal{A}_n[7]$  UP TO CONSTANT FACTORS.

This is part three of the preceding table:

$\text{mod}(n, 7)$	$s_A(n, 7)$	$\mathcal{A}_n[7](x)$
5	5	$(x - 1_F)^{2\delta}(x - 3_F)^{2\delta} \times$ $(x - 4_F)^{2\delta}(x - 6_F)^{2\delta} \times$ $(x - 2_F)^{2\delta+1}(x - 5_F)^{2\delta+1} \times$ $\prod_{s \in \mathcal{O}_7(t^{1513})} (x - s) \times$ $\prod_{s \in \mathcal{O}_7(t^{11020})} (x - s)$
6	1	$(x - 1_F)^{2\delta+2}(x - 2_F)^{2\delta+2} \times$ $(x - 3_F)^{2\delta+2}(x - 4_F)^{2\delta+2} \times$ $(x - 5_F)^{2\delta+2}(x - 6_F)^{2\delta+2}$

PART THREE OF TABLE 3:  $\mathcal{A}_n[7]$  UP TO CONSTANT FACTORS.

In the case of the prime  $p = 7$ , there was only one example for which  $n \equiv 0 \pmod{7}$  and  $s_A(n, 7) = 1$  in the range of our observations. Namely, with  $F = \mathbb{A}(0, 7)$ ,  $\mathcal{A}_0[7] = 3_F(x - 1_F)(x - 6_F)$ . For the other cases, we have table 3 above. By remarks and footnotes above, it is only necessary to state the value of  $\#\mathcal{O}_7$  on a root of  $\mathcal{A}_n[7]$  when  $s_A(n, 7)$  is larger than 2. For roots of  $\mathcal{A}_n[7]$  where  $n \equiv 4 \pmod{7}$ ,  $\#\mathcal{O}_7(t^{29412}) = 2$ ,  $\#\mathcal{O}_7(t^{41290}) = 3$ , and  $\#\mathcal{O}_7(t^{81528}) = 3$ . For roots of  $\mathcal{A}_n[7]$  where  $n \equiv 5 \pmod{7}$ ,  $\#\mathcal{O}_7(t^{1513}) = 5$  and  $\#\mathcal{O}_7(t^{11020}) = 5$ .

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