

RELATIONS ENUMERABLE FROM POSITIVE INFORMATION

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ABSTRACT. We study countable structures from the viewpoint of enumeration reducibility. Since enumeration reducibility is based on only positive information, in this setting it is natural to consider structures given by their positive atomic diagram – the computable join of all relations of the structure. Fixing a structure \mathcal{A} , a natural class of relations in this setting are the relations R such that $R^{\hat{\mathcal{A}}}$ is enumeration reducible to the positive atomic diagram of $\hat{\mathcal{A}}$ for every $\hat{\mathcal{A}} \cong \mathcal{A}$ – the *relatively intrinsically positively enumerable (r.i.p.e.)* relations. We show that the r.i.p.e. relations are exactly the relations that are definable by Σ_1^p formulas, a subclass of the infinitary Σ_1^0 formulas. We then introduce a new natural notion of the jump of a structure and study its interaction with other notions of jumps. At last we show that positively enumerable functors, a notion studied by Csima, Rossegger, and Yu, are equivalent to a notion of interpretability using Σ_1^p formulas.

While in computable structure theory countable structures are classically studied up to Turing reducibility, researchers have successfully used enumeration reducibility to both contribute to the classical study and develop a beautiful theory on its own. One example of a contribution to classical theory is Soskov’s work on degree spectra and co-spectra [15] which allowed him to show that there is a degree spectrum of a structure such that the set of degrees whose ω -jump is in this spectrum is not the spectrum of a structure [14]. Another example is Kalimullin’s study of reducibilities

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between classes of structures [8] where he studied enumeration reducibility versions of Muchnik and Medvedev reducibility between classes of structures. This topic has also been studied by Knight, Miller, and Vanden Boom [9] and Calvert, Cummins, Knight, and Miller [2]. There appears to be a rich theory on these notions with interesting questions on the relationship between the enumeration reducibility and the classical versions. In this article we develop a novel approach to study relations that are enumeration reducible to every copy of a given structure.

More formally, say that given a countable relational structure \mathcal{A} , a relation R is *relatively intrinsically positively enumerable (r.i.p.e.)* in \mathcal{A} if for every copy \mathcal{B} of \mathcal{A} and every enumeration of the basic relations on \mathcal{B} , we can compute an enumeration of $R^{\mathcal{B}}$. We obtain a syntactic classification of the r.i.p.e. relations using the infinitary logic $L_{\omega_1\omega}$. Theorem 1.6 shows that the r.i.p.e. relations on a structure are precisely those that are definable by computable infinitary Σ_1^0 formulas in which neither negations nor implications occur.

This classification is motivated by a related classification of the r.i.c.e. relations. A relation R on a structure \mathcal{A} is *relatively intrinsically computably enumerable (r.i.c.e.)* if for every $\mathcal{B} \cong \mathcal{A}$, $R^{\mathcal{B}}$ is c.e. relative to the atomic diagram of \mathcal{B} . The following classification of r.i.c.e. relations is a special case of a theorem by Ash, Knight, Manasse and Slaman [1] and, independently, Chisholm [3]: A relation is r.i.c.e. in a structure \mathcal{A} if and only if it is definable in \mathcal{A} by a computable Σ_1^0 formula in the logic $L_{\omega_1\omega}$.

Since this result there has been much work on r.i.c.e. relations and related concepts. For a summary, see Fokina, Harizanov, and Melnikov [6]. One particularly interesting generalization of r.i.c.e. relations is due to Montalbán [12]. He extended the definition of r.i.c.e. relations from relations on ω^n to $\omega^{<\omega}$ and to sequences of relations, obtaining a classification similar to the one given in [1, 3]. This extension allows the development of a rich theory such as an intuitive definition of the jump of a structure, and an effective version of interpretability with a category theoretic analogue: A structure \mathcal{A} is *effectively interpretable* in a structure \mathcal{B} if and only if there is a computable functor from \mathcal{B} to \mathcal{A} as given in Harrison-Trainor, Melnikov,

Miller, and Montalbán [7]. For a complete development of this theory see Montalbán [11]. The main goal of this article is to develop a similar theory for r.i.p.e. relations.

Let \mathcal{A} be a countable structure in relational language L . We might assume that the universe of \mathcal{A} is ω , and in order to measure its computational complexity, identify it with the set $=^{\mathcal{A}} \oplus \neq^{\mathcal{A}} \oplus \bigoplus_{R_i \in L} R_i^{\mathcal{A}}$ which we call $P(\mathcal{A})$, the *positive diagram* of \mathcal{A} . It is Turing equivalent to the standard definition of the atomic diagram of a structure, which can be viewed as the set $=^{\mathcal{A}} \oplus \neq^{\mathcal{A}} \oplus \bigoplus_{R_i \in L} R_i^{\mathcal{A}} \oplus \overline{R_i^{\mathcal{A}}}$. Now, a relation $R \subseteq \omega^{<\omega}$ is r.i.c.e. if for every copy $\mathcal{B} \cong \mathcal{A}$, $R^{\mathcal{B}}$ is c.e. in $P(\mathcal{B})$. The r.i.p.e. relations are the natural analogue to the r.i.c.e. relations for enumeration reducibility. Recall that a set of natural numbers A is enumeration reducible to B , $A \leq_e B$ if there exists a c.e. set Ψ consisting of pairs $\langle D, x \rangle$ where D is a finite set under some fixed coding such that

$$A = \Psi^B = \{x : D \subseteq B \wedge \langle D, x \rangle \in \Psi\}.$$

Enumeration reducibility allows us to formally define the notion of a r.i.p.e. relation.

Definition 0.1. Let \mathcal{A} be a structure. A relation $R \subseteq A^{<\omega}$ is *relatively intrinsically positively enumerable* in \mathcal{A} , short r.i.p.e., if, for every copy $(\mathcal{B}, R^{\mathcal{B}})$, of (\mathcal{A}, R) , $R^{\mathcal{B}} \leq_e P(\mathcal{B})$. The relation is *uniformly relatively intrinsically positively enumerable* in \mathcal{A} , if the above enumeration reducibility is uniform in the copies of \mathcal{A} , that is, if there is a fixed enumeration operator Ψ such that $R^{\mathcal{B}} = \Psi^{P(\mathcal{B})}$ for every copy \mathcal{B} of \mathcal{A} .

The study of the computability theoretic properties of structures with respect to enumeration reducibility is an active research topic; see Soskova and Soskova [16] for a summary of results in this area.

0.1. Structure of the paper. In Section 1 we show that a relation is r.i.p.e. in a structure \mathcal{A} if and only if it is definable by a Σ_1^p formula in the language of \mathcal{A} , that is, a computable infinitary Σ_1^0 formula in which neither negations nor implications occur. We continue by defining a notion of reducibility between r.i.p.e. relations

and exhibiting a complete relation. Towards the end of Section 1 we study sets of natural numbers r.i.p.e. in a given structure and show that these are precisely those sets whose degrees lie in the structures co-spectrum, the set of enumeration degrees below every copy of the structure.

Section 2 is devoted to the study of the positive jump of a structure. We define the positive jump and study its degree theoretic properties. The main result of this section is that the enumeration degree spectrum of the positive jump of a structure is the set of jumps of enumeration degrees of the structure. The main tool to prove this result are r.i.p.e. generic enumerations which are studied at the beginning of the section. To finish the section we compare the enumeration degree spectra obtained by applying various operations on structures such as the positive jump, the classical jump and the totalization.

In Section 3 we define an effective version of interpretability using Σ_1^p formulas, positive interpretability. We show that a structure \mathcal{A} is positively interpretable in a structure \mathcal{B} if and only if there is a positive enumerable functor from the isomorphism class of \mathcal{B} to the isomorphism class of \mathcal{A} . Positive enumerable functors and related effectivizations of functors were studied by Csima, Rossegger and Yu [5]. Positive interpretability allows for a useful strengthening that preserves most properties of a structure, positive bi-interpretability. We show that two structures \mathcal{A} and \mathcal{B} are positive bi-interpretable if and only if their isomorphism classes are enumeration adjoint, that is there is an adjoint equivalence between $Iso(\mathcal{A})$ and $Iso(\mathcal{B})$ witnessed by enumeration operators.

1. FIRST RESULTS ON R.I.P.E. RELATIONS

In our proofs we will often build structures in stages by copying increasing pieces of finite substructures of a given structure \mathcal{A} . The following definitions will be useful for this.

Definition 1.1. Given an \mathcal{L} -structure \mathcal{A} and $\bar{a} \in A^{<\omega}$, let $P(\mathcal{A}) \upharpoonright \bar{a}$ denote the positive diagram of the substructure of \mathcal{A} with universe \bar{a} in the restriction of \mathcal{L} to the first $|\bar{a}|$ relation symbols.

Definition 1.2. Let \mathcal{A} be an \mathcal{L} -structure and $\bar{a} = \langle a_0, \dots, a_s \rangle \in A^s$. The set $P_{\mathcal{A}}(\bar{a})$ is the pullback of $P(\mathcal{A}) \upharpoonright \bar{a}$ along the index function of \bar{a} , i.e.,

$$\langle i, n_0, \dots, n_{r_i} \rangle \in P_{\mathcal{A}}(\bar{a}) \Leftrightarrow \langle i, a_{n_0}, \dots, a_{n_{r_i}} \rangle \in P(\mathcal{A}) \upharpoonright \bar{a}.$$

The main feature of Definition 1.2 is that if we approximate the positive diagram of a structure \mathcal{A} in stages by considering larger and larger tuples, i.e., $\lim_{s \in \omega} P(\mathcal{A}) \upharpoonright \bar{a}_s = P(\mathcal{A})$, then the limit of $P_{\mathcal{A}}(\bar{a}_s)$ gives a structure isomorphic to \mathcal{A} . We will use this fact to obtain generic copies of a given structure.

We denote by Ψ_e the e^{th} enumeration operator in a fixed computable enumeration of all enumeration operators and by $\Psi_{e,s}$ its stage s approximation. Without loss of generality we make the common assumption that $\Psi_{e,s}$ is finite and does not contain pairs $\langle n, D \rangle > s$. Notice that $\Psi_{e,s}$ itself is an enumeration operator. In our proofs we also frequently argue that a set A is enumeration reducible to a set B by using a characterization of enumeration reducibility due to Selman [13].

Theorem 1.3 (Selman [13]). For any $A, B \subseteq \omega$

$$A \leq_e B \quad \text{iff} \quad \forall X [B \text{ is c.e. in } X \Rightarrow A \text{ is c.e. in } X]$$

We refer the reader to Cooper [4] for a proof of this result and further background on enumeration reducibility and enumeration degrees.

Notice that by Definition 0.1 (\mathcal{A}, R) is technically not a first order structure as $R \subseteq \omega^{<\omega}$. We may however still think of it as a first order structure in the language expanded by relation symbols $(Q_i)_{i \in \omega}$, each Q_i of arity i , where $Q_i^{\mathcal{A}} = \{\bar{a} \in A^n : \bar{a} \in R^{\mathcal{A}}\}$. The positive diagrams $P(\mathcal{A}, R^{\mathcal{A}})$ and $P(\mathcal{A}, (Q_i^{\mathcal{A}})_{i \in \omega})$ are enumeration equivalent.

1.1. A syntactic characterization. The main purpose of this section is to show that being r.i.p.e. in a structure \mathcal{A} is equivalent to being definable by infinitary formulas in \mathcal{A} of the following type.

Definition 1.4. A *positive computable infinitary Σ_1^p formula* is a formula of the infinitary logic $L_{\omega_1\omega}$ of the form

$$\varphi(\bar{x}) \equiv \bigvee_{i \in I} \exists \bar{y}_i \varphi_i(\bar{x}, \bar{y}_i)$$

where each φ_i is a finite conjunct of atomic formulas, and the index set I is c.e..

Notice that the above definition includes all c.e. disjunctions of finitary Σ_1^0 formulas without negation and implication symbols, as every such formula is equivalent to a finite disjunction of conjunctions with each existential quantifier occurring in front of the conjunctions.

Definition 1.5. A relation $R \subseteq \omega^{<\omega}$ is Σ_1^p -*definable with parameters* \bar{c} in a structure \mathcal{A} if there exists a uniformly computable sequence of Σ_1^p formulas $(\varphi_i(x_1, \dots, x_i, y_1, \dots, y_{|c|}))_{i \in \omega}$ such that for all $\bar{a} \in \omega^{<\omega}$

$$\bar{a} \in R \Leftrightarrow \mathcal{A} \models \varphi_{|\bar{a}|}(\bar{a}, \bar{c}).$$

Theorem 1.6. Let \mathcal{A} be a structure and $R \subseteq A^{<\omega}$ a relation on it. Then the following are equivalent:

- (i) R is relatively intrinsically positively enumerable in \mathcal{A} ,
- (ii) R is Σ_1^p -definable in \mathcal{A} with parameters.

Proof. Assuming (ii), there is a uniformly computable sequence $(\varphi_i(\bar{x}, \bar{z}))_{i \in \omega}$ of Σ_1^p formulas where each φ_i is of the form $\bigvee_{j \in \omega} \exists \bar{y}_{i,j} \psi_{i,j}(\bar{x}, \bar{y}_{i,j}, \bar{z})$ with the property that for every $\mathcal{B} \cong \mathcal{A}$ there is a tuple $\bar{c} \in \omega^{|\bar{z}|}$ such that for all $i \in \omega$ and $\bar{a} \in \omega^i$, $\mathcal{B} \models \varphi_i(\bar{a}, \bar{c})$ if and only if $\bar{a} \in R^{\mathcal{B}}$. Recall that each $\psi_{i,j}$ is a finite conjunction of atomic formulas, i.e., $\psi_{i,j} = \theta_1(\bar{x}, \bar{y}_{i,j}, \bar{z}) \wedge \dots \wedge \theta_n(\bar{x}, \bar{y}_{i,j}, \bar{z})$ for some $n \in \omega$. For $\theta(\bar{x})$ an atomic formula, let $\ulcorner \theta(\bar{a}) \urcorner$ be the function mapping $\theta(\bar{a})$ to its code in the positive diagram of a structure. For example, if $\theta(\bar{x}) = R_i(x_3, x_5)$, then $\ulcorner \theta(\bar{a}) \urcorner = \langle i+2, \langle a_3, a_5 \rangle \rangle$ for $\bar{a} \in \omega^{<\omega}$. Consider the set $X_{i,j}^{\bar{a}, \bar{b}, \bar{c}} = \{ \ulcorner \theta_k(\bar{a}, \bar{b}, \bar{c}) \urcorner : k < n \}$. Clearly, $X_{i,j}^{\bar{a}, \bar{b}, \bar{c}} \subseteq P(\mathcal{A})$ if and only if $\mathcal{A} \models \psi_{i,j}(\bar{a}, \bar{b}, \bar{c})$ for any L -structure \mathcal{A} . We define an enumeration operator Ψ by enumerating all pairs $\langle \bar{a}, X_{i,j}^{\bar{a}, \bar{b}, \bar{c}} \rangle$ into Ψ . We

have that

$$\begin{aligned}
\bar{a} \in \Psi^{P(\mathcal{B})} &\Leftrightarrow \exists \langle \bar{a}, X_{|\bar{a}|,j}^{\bar{a},\bar{b},\bar{c}} \rangle \in \Psi \wedge X_{|\bar{a}|,j}^{\bar{a},\bar{b},\bar{c}} \subseteq P(\mathcal{B}) \\
&\Leftrightarrow \mathcal{B} \models \exists \bar{y}_{|\bar{a}|,j} \psi_{|\bar{a}|,j}(\bar{a}, \bar{y}_{|\bar{a}|,j}, \bar{c}) \\
&\Leftrightarrow \mathcal{B} \models \varphi_{|\bar{a}|}(\bar{a}, \bar{c})
\end{aligned}$$

and thus R is r.i.p.e.

To show that (i) implies (ii) we build an enumeration $g : \omega \rightarrow \mathcal{A}$ by constructing a nested sequence of tuples $\{\bar{p}_s\}_{s \in \omega} \subseteq A^{<\omega}$ and letting $g = \bigcup_s \bar{p}_s$. We then define $\mathcal{B} = g^{-1}(\mathcal{A})$. Our goal is to produce a Σ_1^P definition of $R^{\mathcal{B}}$. To obtain it we try to force during the construction that $R^{\mathcal{B}} = g^{-1}(R) \neq \Psi_e^{P(\mathcal{B})}$. As R is r.i.p.e. this will fail for some e and we will use the failure to get the syntactic definition.

To construct \mathcal{B} we do the following. Let p_0 be the empty sequence. At stage $s + 1 = 2e + 1$ if the e -th element of A is not already in \bar{p}_s then we let $\bar{p}_{s+1} = \bar{p}_s \widehat{\ } e$. Otherwise let $\bar{p}_{s+1} = \bar{p}_s$. This guarantees that g is onto. At stage $s + 1 = 2e$ we try to force $\Psi_e^{P(\mathcal{B})} \not\subseteq g^{-1}(R)$ for which we need a tuple $\langle j_1, \dots, j_l \rangle \in \Psi_e^{P(\mathcal{B})}$ with $\langle g(j_1), \dots, g(j_l) \rangle \notin R$. To do this we ask if there is an extension \bar{q} of \bar{p}_s in the set

$$Q_e = \{\bar{q} \in A^{<\omega} : \exists l, j_1, \dots, j_l < |\bar{q}| [\langle j_1, \dots, j_l \rangle \in \Psi_e^{P_{\mathcal{A}}(\bar{q})} \text{ and } \langle q_{j_1}, \dots, q_{j_l} \rangle \notin R]\}$$

If there is one let $\bar{p}_{s+1} = \bar{q}$, otherwise let $\bar{p}_{s+1} = \bar{p}_s$. This concludes the construction.

If at any stage $s + 1 = 2e$ we succeed in defining $\bar{p}_{s+1} = \bar{q}$ for some $\bar{q} \in Q_e$ then we will have made $\Psi_e^{P(\mathcal{B})} \neq g^{-1}(R)$. But by our assumption this must fail for some $e \in \omega$. For this e , at stage $s + 1 = 2e$ we will not have been able to find an extension of \bar{p}_s in Q_e . We will use this to give a Σ_1^P definition of R with parameters \bar{p}_s .

Notice that if there is some $\bar{q} \supseteq \bar{p}_s$ and sub-tuple $\langle q_{j_1}, \dots, q_{j_l} \rangle$ such that $\langle j_1, \dots, j_l \rangle \in \Psi_e^{P_{\mathcal{A}}(\bar{q})}$ then we must have $\langle q_{j_1}, \dots, q_{j_l} \rangle \in R$ or else we will have that $\bar{q} \in Q_e$. We now show that R is equal to the set

$$\begin{aligned}
S = \{ \langle q_{j_1}, \dots, q_{j_l} \rangle \in A^{<\omega} : \text{for some } \bar{q} \in A^{<\omega} \text{ and } l, j_1, \dots, j_l < |\bar{q}| \text{ satisfying } \bar{q} \supseteq \bar{p}_s \\
\text{and } \langle j_1, \dots, j_l \rangle \in \Psi_e^{P_{\mathcal{A}}(\bar{q})} \}
\end{aligned}$$

By the previous paragraph $S \subseteq R$. If $\bar{a} \in R$ then there are indices $j_1, \dots, j_{|\bar{a}|}$ such that $\bar{a} = \langle g(j_1), \dots, g(j_{|\bar{a}|}) \rangle$ and so if we take a long enough initial segment of g it will witness the fact that $\bar{a} \in S$. Fix an enumeration $(\varphi_i^{at})_{i \in \omega}$ of all atomic formulas where without loss of generality the free variables of φ_i^{at} are a subset of $\{x_0, \dots, x_i\}$. The following is a Σ_1^p definition of S

$$\bigvee_{C \subseteq_{\text{fin}} \omega} \bigvee_{\langle j_1, \dots, j_{|\bar{a}|} \rangle \in W_e^C} \exists \bar{q} \supseteq \bar{p}_s [\langle q_{j_1}, \dots, q_{j_{|\bar{a}|}} \rangle = \bar{a} \wedge \bigwedge_{i \in C} [\varphi_i^{at} \frac{q_0}{x_0} \dots \frac{q_{|\bar{q}|}}{x_{|\bar{q}|}}]]$$

where the latter half of the formula is simply saying that $C \subseteq P_{\mathcal{A}}(\bar{q})$. \square

Corollary 1.7. Let \mathcal{A} be a structure and $R \subseteq A^{<\omega}$ be a relation on it. Then the following are equivalent:

- (i) R is uniformly relatively intrinsically positively enumerable in \mathcal{A} ,
- (ii) R is Σ_1^p -definable in \mathcal{A} without parameters.

Proof. For (i) \Rightarrow (ii) we mimic the proof of Theorem 1.6. Let Ψ_e be the fixed enumeration operator such that $R^{\mathcal{B}} = \Psi_e^{P(\mathcal{B})}$ and Q_e as above. Note that no \bar{q} can be in Q_e and so we mimic the construction of our set S with \bar{p}_s being the empty tuple.

For (ii) \Rightarrow (i) we again mimic the same direction in 1.6 excluding the parametrizing tuple \bar{c} to make the process uniform. \square

1.2. R.i.p.e. completeness. Similar to the study of computably enumerable sets we want to investigate notions of completeness for r.i.p.e. relations on a given structure. Before we obtain a natural example of a r.i.p.e. complete relation we have to define a suitable notion of reduction.

Definition 1.8. Given a structure \mathcal{A} and two relations $P, R \subseteq A^{<\omega}$, we say that P is *positively intrinsically one reducible* to R , and write $P \leq_{p1} R$, if for all $\mathcal{B} \cong \mathcal{A}$ $P(\mathcal{B}) \oplus P^{\mathcal{B}} \leq_1 P(\mathcal{B}) \oplus R^{\mathcal{B}}$.

Proposition 1.9. Positive intrinsic one reducibility (\leq_{p1}) is a reducibility.

Proof. Let \mathcal{A} be a structure with relations $P, Q, R \subseteq A^{<\omega}$. It is easy to see that \leq_{p1} is reflexive, since for any structure $\mathcal{B} \cong \mathcal{A}$ we have that $P(\mathcal{B}) \oplus P^{\mathcal{B}} \leq_1 P(\mathcal{B}) \oplus P^{\mathcal{B}}$, which means $P \leq_{p1} P$. To see that it is transitive assume that $P \leq_{p1} R$ and $R \leq_{p1} Q$ and let $\mathcal{B} \cong \mathcal{A}$. By assumption $P(\mathcal{B}) \oplus P^{\mathcal{B}} \leq_1 P(\mathcal{B}) \oplus R^{\mathcal{B}} \leq_1 P(\mathcal{B}) \oplus Q^{\mathcal{B}}$ and so $P \leq_{p1} Q$. \square

Definition 1.10. Fix a structure \mathcal{A} . A relation $R \subseteq A^{<\omega}$ is *r.i.p.e. complete* if R is r.i.p.e. and for every r.i.p.e. relation P on \mathcal{A} , $P \leq_{p1} R$.

The most natural way to obtain a complete set is to follow the construction of the Kleene set in taking the computable join of all r.i.p.e. sets. The result is a relation $R \subseteq \omega \times A^{<\omega}$ which can be seen as a uniform sequence of r.i.p.e. relations in the sense that there is a enumeration operator Ψ such that $\Psi^{\mathcal{A}^{[i]}} = R_i$. The issue is that this does not behave well under isomorphism. However, this can be easily overcome using the following coding. Given a relation $R \subseteq \omega \times A^{<\omega}$, we can identify it with a subset $R' \subseteq A^{<\omega}$ as follows. For any two elements $b, c \in A$ let

$$\overbrace{b \dots b}^{i \times} c \bar{a} \in R' \iff \langle i, \bar{a} \rangle \in R.$$

One can now easily see that R is a uniform sequence of r.i.p.e. relations if and only if the so obtained relation R' is uniformly r.i.p.e.

We can now give a natural candidate for a r.i.p.e. complete relation.

Definition 1.11. Let $\varphi_{i,j}^{\Sigma_1^p}$ be the i th formula with free variables x_1, \dots, x_j in a computable enumeration of all Σ_1^p formulas. The *positive Kleene predicate* relative to \mathcal{A} is $\vec{K}_p^{\mathcal{A}} = (K_i^{\mathcal{A}})_{i \in \omega}$, where

$$K_i^{\mathcal{A}} = \bigcup_{j \in \omega} \{\bar{a} \in A^j : \mathcal{A} \models \varphi_{i,j}^{\Sigma_1^p}(\bar{a})\}.$$

Notice that we defined the positive Kleene predicate as a sequence of relations instead of a single relation. It is slightly more convenient as we do not have to deal with coding. Another alternative definition would be to break down the sequence

even further and let the Kleene predicate be the sequence

$$(\{\bar{a} : \mathcal{A} \models \varphi_{i,j}^{\Sigma_1^p}(\bar{a})\})_{\langle i,j \rangle \in \omega}$$

so that $(\mathcal{A}, \vec{K}_p^{\mathcal{A}})$ is a first order structure. However, as all of these definitions are computationally equivalent these distinctions are irrelevant for our purpose.

Proposition 1.12. The positive Kleene predicate $\vec{K}_p^{\mathcal{A}}$ is uniformly r.i.p.e., and r.i.p.e. complete.

Proof. Since $\vec{K}_p^{\mathcal{A}}$ is defined in a Σ_1^p way without parameters we can use Corollary 1.7 to see that it is uniformly relatively intrinsically positively enumerable. Let R be any relation on \mathcal{A} of arity a_R . Notice that $R^{\mathcal{B}}$ is trivially Σ_1^p definable, and so there is a formula $\varphi_{i,a_R}^{\Sigma_1^p}$ such that $\bar{a} \in R^{\mathcal{B}} \Leftrightarrow \mathcal{B} \models \varphi_{i,a_R}^{\Sigma_1^p}(\bar{a})$. This shows that $P(\mathcal{B}) \oplus R^{\mathcal{B}} \leq_1 P(\mathcal{B}) \oplus \vec{K}_p^{\mathcal{B}}$. \square

1.3. R.i.p.e. sets of natural numbers. Our above discussion of sequences of r.i.p.e. relations allows us to code sets of natural numbers as r.i.p.e. relations.

Definition 1.13. A set $X \subseteq \omega$ is *r.i.p.e. in a structure \mathcal{A}* if the relation $R_X = X \times \emptyset$ is a r.i.p.e. relation, i.e., if the following relation is r.i.p.e.:

$$\{\overbrace{b \dots b}^{i \times} c : i \in X, b, c \in A\}$$

A natural question is which sets of natural numbers are r.i.p.e. in a given structure. One characterization can be derived directly from the definitions: The sets X such that $X \leq_e P(\mathcal{B})$ for all $\mathcal{B} \cong \mathcal{A}$. Another one can be given using co-spectra, a notion defined by Soskov [15]. Intuitively, the co-spectrum of a structure \mathcal{A} is the maximal ideal in the enumeration degrees such that every member of it is below every copy of \mathcal{A} . More formally.

Definition 1.14. The *co-spectrum* of a structure \mathcal{A} is

$$Co(\mathcal{A}) = \bigcap_{\mathcal{B} \cong \mathcal{A}} \{\mathbf{d} : \mathbf{d} \leq deg_e(P(\mathcal{B}))\}.$$

Let us point out that Soskov's definition of co-spectra appears to be different from ours. We will prove in Section 2 that the two definitions are equivalent. Given a tuple \bar{a} in some structure \mathcal{A} let $\Sigma_1^p\text{-tp}_{\mathcal{A}}(\bar{a})$ be the set of positive finitary Σ_1^0 formulas true of \mathcal{A} . The equivalence of Item 1 and Item 3 in Theorem 1.15 is the analogue to a well-known theorem of Knight [10] for total structures.

Theorem 1.15. The following are equivalent for every structure \mathcal{A} and $X \subseteq \omega$.

- (1) X is r.i.p.e. in \mathcal{A}
- (2) $\text{deg}_e(X) \in \text{Co}(\mathcal{A})$
- (3) X is enumeration reducible to $\Sigma_1^p\text{-tp}_{\mathcal{A}}(\bar{a})$ for some tuple $\bar{a} \in A^{<\omega}$.

Proof. If $\text{deg}_e(X) \in \text{Co}(\mathcal{A})$, then for all $\mathcal{B} \cong \mathcal{A}$, $X \leq_e P(\mathcal{B})$. Given an enumeration of X , enumerate $b^n c$ into R_X for all elements $b, c \in B$ whenever you see n enter X . The relation R_X clearly witnesses that X is r.i.p.e. in \mathcal{A} . This shows that Item 2 implies Item 1. On the other hand if X is r.i.p.e. in \mathcal{A} , then given any $B \cong \mathcal{A}$ and an enumeration of $R_X^{\mathcal{B}}$ build a set S by enumerating n into S whenever you see $b^n c$ enumerated into $R_X^{\mathcal{B}}$ for any two elements $b, c \in B$. Clearly $n \in S$ if and only if $n \in X$ and thus Item 1 implies Item 2.

To see that Item 3 implies Item 2 assume that X is enumeration reducible to the positive Σ_1 type of a tuple \bar{b} in any copy \mathcal{B} of \mathcal{A} . As the $\Sigma_1^p\text{-tp}_{\mathcal{B}}(\bar{b})$ is enumeration reducible to $P(\mathcal{B})$, by transitivity $X \leq_e P(\mathcal{B})$ for every $\mathcal{B} \cong \mathcal{A}$ and thus $\text{deg}_e(X) \in \text{Co}(\mathcal{A})$. At last, we show that Item 1 implies Item 3. Assume that X is r.i.p.e. in \mathcal{A} . Then there is a computable enumeration of Σ_1^p formulas ψ_n with parameters \bar{p} such that for some $\bar{a} \in A^{<\omega}$, $n \in X \Leftrightarrow \mathcal{A} \models \psi_n(\bar{a})$. Simultaneously enumerate $\Sigma_1^p\text{-tp}_{\mathcal{A}}(\bar{a})$ and the disjuncts in the formulas ψ_n . Whenever you see a disjunct of ψ_n that is also in $\Sigma_1^p\text{-tp}_{\mathcal{A}}(\bar{a})$ enumerate n . This gives an enumeration of X given an enumeration of $\Sigma_1^p\text{-tp}_{\mathcal{A}}(\bar{a})$ and thus $X \leq_e \Sigma_1^p\text{-tp}_{\mathcal{A}}(\bar{a})$ as required. \square

2. THE POSITIVE JUMP AND DEGREE SPECTRA

In this section we compare various definitions of the jump of a structure with respect to their enumeration degree spectra, a notion first studied by Soskov [15].

Before we introduce it, let us recall the definition of the jump of an enumeration degree.

Definition 2.1. Let $K_A = \{x \mid x \in \Psi_x^A\}$. The *enumeration jump* of a set A is $J_e(A) := A \oplus \overline{K_A}$. The jump of an e -degree $\mathbf{a} = \deg_e(A) = \{X : X \equiv_e A\}$ is $\mathbf{a}' = \deg_e(J_e(A))$.

Definition 2.2. The *positive jump* of a structure \mathcal{A} is the structure

$$PJ(\mathcal{A}) = (\mathcal{A}, \overline{K_p^{\mathcal{A}}}) = (\mathcal{A}, (\overline{K_i^{\mathcal{A}}})_{i \in \omega}).$$

We are interested in the degrees of enumerations of $P(\mathcal{A})$ and $P(PJ(\mathcal{A}))$. To be precise, let f be an enumeration of ω , that is, a surjective mapping $\omega \rightarrow \omega$, and for $X \subseteq \omega^{<\omega}$ let

$$f^{-1}(X) = \{\langle x_1, \dots \rangle : (f(x_1), \dots) \in X\}.$$

Then, given a structure \mathcal{A} let $f^{-1}(\mathcal{A}) = (\omega, f^{-1}(=), f^{-1}(\neq), f^{-1}(R_1^{\mathcal{A}}), f^{-1}(R_2^{\mathcal{A}}), \dots)$. This definition differs from the definition given in Soskov [15] where $f^{-1}(\mathcal{A})$ means what we will denote as $P(f^{-1}(\mathcal{A}))$, i.e.,

$$= \oplus \neq \oplus f^{-1}(=) \oplus f^{-1}(\neq) \oplus \bigoplus_{i \in \omega} f^{-1}(R_i^{\mathcal{A}}).$$

Using this notation we can see that for any structure \mathcal{A} , and any enumeration f of ω we have that $PJ(f^{-1}(\mathcal{A}))$ is the structure $(f^{-1}(\mathcal{A}), \overline{K_p^{f^{-1}(\mathcal{A})}})$. Thus

$$P(PJ(f^{-1}(\mathcal{A}))) = f^{-1}(=) \oplus f^{-1}(\neq) \oplus \bigoplus_{j \in \omega} \overline{K_j^{f^{-1}(\mathcal{A})}} \oplus \bigoplus_{i \in \omega} R_i^{f^{-1}(\mathcal{A})}.$$

If we instead apply the enumeration to $PJ(\mathcal{A})$ we will get the structure $f^{-1}(PJ(\mathcal{A}))$. Now, for every relation R on \mathcal{A} , $R^{f^{-1}(\mathcal{A})} = f^{-1}(R^{\mathcal{A}})$ and thus $\overline{K_j^{f^{-1}(\mathcal{A})}} = f^{-1}(\overline{K_j^{\mathcal{A}}})$. So $P(f^{-1}(PJ(\mathcal{A}))) = P(PJ(f^{-1}(\mathcal{A})))$.

2.1. R.i.p.e. generic presentations.

Definition 2.3. Let $A^* = \{\sigma \in A^{<\omega} : (\forall i \neq j < |\sigma|)[\sigma(i) \neq \sigma(j)]\}$. We say that $\gamma \in A^*$ *decides* an upwards closed subset $R \subseteq A$ if $\gamma \in R$ or $\sigma \notin R$ for all $\sigma \supseteq \gamma$. A 1–1 function $g : \omega \rightarrow \mathcal{A}$ is a *r.i.p.e.-generic enumeration*, if for every r.i.p.e.

subset $R \subseteq A^*$ there is an initial segment of g that decides R . \mathcal{B} is a *r.i.p.e.-generic presentation* of \mathcal{A} if it is the pull-back along a r.i.p.e.-generic enumeration of $P(\mathcal{A})$.

The following lemma is an analogue of the well-known result that there is a Δ_2^0 1-generic.

Lemma 2.4. Every structure \mathcal{A} has a r.i.p.e.-generic enumeration g such that $\text{Graph}(g) \leq_e P(PJ(\mathcal{A}))$. In particular, $P(g^{-1}(PJ(\mathcal{A}))) \leq_e P(PJ(\mathcal{A}))$.

Proof. Fix a $P(\mathcal{A})$ -computable enumeration $(S_i)_{i \in \omega}$ of all r.i.p.e. subsets of \mathcal{A} . We define our enumeration g to be the limit of an increasing sequence of strings $\{\bar{p}_s \in A^* : s \in \omega\}$. The strings \bar{p}_s are defined as follows:

- (1) $\bar{p}_{-1} = \langle \rangle$
- (2) To define \bar{p}_{s+1} , check if there is a $\bar{q} \in S_{s+1}$ such that $\bar{q} \supset \bar{p}_s$. If there is let \bar{p}_{s+1} be the least such tuple, otherwise define $\bar{p}_{s+1} = \bar{p}_s$.

Notice that $g = \bigcup_{s \in \omega} \bar{p}_s$ is onto as for every $n \in \omega$, $D_n = \{\bar{a} \in A^* : \exists j < |\bar{a}| \bar{a}(j) = n\}$ is a dense r.i.p.e. subset of A^* . Thus there is an e such that $S_e = D_n$ and \bar{p}_e forces into D_n . Thus, g is a 1–1 r.i.p.e.-generic enumeration of \mathcal{A} .

The set $\{\bar{p} : \exists \bar{q} \supset \bar{p}, \bar{q} \in S_e\}$ is Σ_1^p -definable in \mathcal{A} and so enumerable from $P(PJ(\mathcal{A}))$, which contains $P(\mathcal{A})$. The coset $\{\bar{p} : \forall \bar{q} \supset \bar{p}, \bar{q} \notin S_e\}$ is co-r.i.p.e. and so enumerable from $\overline{K_p^{\mathcal{A}}}$. Hence $P(PJ(\mathcal{A}))$ will be able to decide when a tuple \bar{p}_s belongs to the upward closure of S_e . Thus $\text{Graph}(g) \leq_e P(PJ(\mathcal{A}))$.

If we can enumerate the graph of g and also $PJ(\mathcal{A})$, then to enumerate $g^{-1}(PJ(\mathcal{A}))$, we wait for something of the form $(x, g(x)) \in \text{Graph}(g)$ to appear with $g(x) \in PJ(\mathcal{A})$ and enumerate x into the corresponding slice of $g^{-1}(PJ(\mathcal{A}))$. \square

R.i.p.e. generic presentations have many useful properties. One of them is that they are minimal in the sense that the only sets enumeration below a r.i.p.e. generic presentation are the r.i.p.e. sets.

Lemma 2.5. If \mathcal{B} is a r.i.p.e.-generic presentation of \mathcal{A} , then $X \subseteq \omega$ is r.i.p.e. in \mathcal{B} if and only if $X \leq_e P(\mathcal{B})$.

Proof. If X r.i.p.e. then it is enumerable from $P(\mathcal{B})$ by definition. Assume X is enumerable from $P(\mathcal{B})$. Then $X = \Psi_e^{P(\mathcal{B})}$ for some e . Recall the set Q_e from Theorem 1.6 which we know to be r.i.p.e. because we gave a Σ_1^p description of it.

$$Q_e = \{\bar{q} \in A^{<\omega} : \exists l, j_1, \dots, j_l < |\bar{q}| [\langle j_1, \dots, j_l \rangle \in \Psi_e^{P(\mathcal{B}(\bar{q}))} \text{ and } \langle q_{j_1}, \dots, q_{j_l} \rangle \notin X]\}.$$

Now because \mathcal{B} is r.i.p.e.-generic, there is some tuple $\langle 0, \dots, k-1 \rangle$ that decides Q_e . $\langle 0, \dots, k-1 \rangle \notin Q_e$ or we would contradict the fact that $X = \Psi_e^{P(\mathcal{B})}$ and so $\langle 0, \dots, k-1 \rangle$ forms the parameterizing tuple \bar{p}_s from the set S in Theorem 1.6. \square

Another useful property is that the degree of the positive jump of a r.i.p.e. generic agrees with the enumeration jump of the degree of their positive diagram.

Proposition 2.6. Let \mathcal{A} be a structure. For an arbitrary enumeration f of \mathcal{A} , let $\mathcal{B} = f^{-1}(\mathcal{A})$. Then $P(PJ(\mathcal{B})) \leq_e J_e(P(\mathcal{B}))$. Furthermore, if \mathcal{B} is a r.i.p.e.-generic presentation then $P(PJ(\mathcal{B})) \equiv_e J_e(P(\mathcal{B}))$.

Proof. Recall that $PJ(\mathcal{B}) = (\mathcal{B}, \overline{K_p^{\mathcal{B}}})$ and $J_e(P(\mathcal{B})) = P(\mathcal{B}) \oplus \overline{K}_{P(\mathcal{B})}$, so to show that $P(PJ(\mathcal{B})) \leq_e J_e(P(\mathcal{B}))$ it is sufficient to show that $\overline{K_p^{\mathcal{B}}} \leq_e \overline{K}_{P(\mathcal{B})}$. As $\overline{K_p^{\mathcal{B}}}$ is r.i.p.e., $\overline{K_p^{\mathcal{B}}} = \Psi_e^{P(\mathcal{B})}$ for some e . Thus $\overline{K_p^{\mathcal{B}}}$ appears as a slice of $\{\langle x, i \rangle : x \notin \Psi_i^{P(\mathcal{B})}\}$ and hence $\overline{K_p^{\mathcal{B}}} \leq_e \{\langle x, i \rangle : x \notin \Psi_i^{P(\mathcal{B})}\}$. The latter set is m -equivalent to $\overline{K}_{P(\mathcal{B})}$ and thus $\overline{K_p^{\mathcal{B}}} \leq_e \overline{K}_{P(\mathcal{B})}$.

It remains to show that $J_e(P(\mathcal{B})) \leq_e P(PJ(\mathcal{B}))$ if \mathcal{B} is r.i.p.e.-generic. For every e consider the set

$$R_e = \{\bar{q} \in B^* : e \in \Psi_e^{P(\mathcal{B}(\bar{q}))}\}.$$

Note that the R_e are closed upwards as subsets of B^* and r.i.p.e.. So since \mathcal{B} is r.i.p.e.-generic, for every e there is an initial segment of B^* that either is in R_e or such that no extension of it is in R_e . The set $Q_e = \{\bar{p} \in B^* : \forall(\bar{q} \supseteq \bar{p})(\bar{q} \notin R_e)\}$ of elements in B^* that are non-extendible in R_e is co-r.i.p.e. Also note that these sets are uniform in e , that is, given e we can compute the indices of R_e and Q_e as r.i.p.e., respectively, co-r.i.p.e. subsets of \mathcal{B} . Thus, given an enumeration of $P(PJ(\mathcal{B}))$ we can enumerate $P(\mathcal{B})$ and the sets Q_e and R_e . By genericity for all $e \in \omega$ there is an

initial segment of \bar{b} of B such that either $\bar{b} \in Q_e$ or $\bar{b} \in R_e$. So whenever we see such a \bar{b} in Q_e we enumerate e and thus obtain an enumeration $\bar{K}_{P(\mathcal{B})}$. \square

The above properties of r.i.p.e. generics are very useful to study how the enumeration degree spectra of structures and their positive jumps relate.

Definition 2.7 ([15]). Given a structure \mathcal{A} , define the set of enumerations of a structure $Enum(\mathcal{A}) = \{P(\mathcal{B}) : \mathcal{B} = f^{-1}(\mathcal{A}) \text{ for } f \text{ an enumeration of } \omega\}$. Further, let the *enumeration degree spectrum* of \mathcal{A} be the set

$$eSp(\mathcal{A}) = \{d_e(P(\mathcal{B})) : P(\mathcal{B}) \in Enum(\mathcal{A})\}$$

If \mathbf{a} is the least element of $eSp(\mathcal{A})$, then \mathbf{a} is called the *enumeration degree* of \mathcal{A} .

As mentioned after Definition 1.14, Soskov's definition of the co spectrum of a structure was slightly different [15]. He defined the co-spectrum of a structure \mathcal{A} as the set

$$\{\mathbf{d} : \forall(\mathbf{a} \in eSp(\mathcal{A})) \mathbf{d} \leq \mathbf{a}\}.$$

We now show that the two definitions are equivalent.

Proposition 2.8. For every structure \mathcal{A} , $Co(\mathcal{A}) = \{\mathbf{d} : \forall(\mathbf{a} \in eSp(\mathcal{A})) \mathbf{d} \leq \mathbf{a}\}$.

Proof. First note that $\{\mathbf{d} : \forall(\mathcal{C} \cong \mathcal{A}) \mathbf{d} \leq deg_e(P(\mathcal{C}))\} \supseteq \{\mathbf{d} : \forall(\mathbf{a} \in eSp(\mathcal{A})) \mathbf{d} \leq \mathbf{a}\}$ as every $P(\mathcal{C})$ is the pullback of \mathcal{A} along an injective enumeration.

On the other hand for any enumeration $f : \omega \rightarrow \omega$, $f^{-1}(\mathcal{A})/f^{-1}(=) \cong \mathcal{A}$. Consider the substructure \mathcal{B} of $f^{-1}(\mathcal{A})$ consisting of the least element in every $f^{-1}(=)$ equivalence class. Since $f^{-1}(\mathcal{A})/f^{-1}(=) \cong \mathcal{A}$, we get that $\mathcal{B} \cong \mathcal{A}$. As $P(f^{-1}(\mathcal{A}))$ contains both $f^{-1}(=)$ and $f^{-1}(\neq)$ we can compute an enumeration of $B \oplus \bar{B}$ from any enumeration of $P(f^{-1}(\mathcal{A}))$ and thus also the graph of its principal function p_B . Let $\mathcal{C} = p_B^{-1}(\mathcal{B})$. Then $\mathcal{C} \cong \mathcal{A}$, and $P(\mathcal{C}) \leq_e P(f^{-1}(\mathcal{A}))$. Thus,

$$\{\mathbf{d} : \forall(\mathbf{a} \in eSp(\mathcal{A})) \mathbf{d} \leq \mathbf{a}\} = \{\mathbf{d} : \forall(\mathcal{C} \cong \mathcal{A}) \mathbf{d} \leq deg_e(P(\mathcal{C}))\} = Co(\mathcal{A}).$$

\square

A set A is said to be *total* if $A \equiv_e A \oplus \overline{A}$. An enumeration degree is said to be total if it contains a total set, and a structure \mathcal{A} is total if $P(f^{-1}(\mathcal{A}))$ is a total set for every enumeration f .

Proposition 2.9. For every structure \mathcal{A} , $PJ(\mathcal{A})$ is a total structure.

Proof. First, note that we have the following equality for arbitrary relations R and enumerations f .

$$x \in f^{-1}(\overline{R}) \Leftrightarrow f(x) \in \overline{R} \Leftrightarrow f(x) \notin R \Leftrightarrow x \notin f^{-1}(R) \Leftrightarrow x \in \overline{f^{-1}(R)}$$

Recall that for every R_i in the language of \mathcal{A} , $R_i^{\mathcal{A}}$ is r.i.p.e. and that

$$P(f^{-1}(PJ(\mathcal{A}))) = f^{-1}(=) \oplus f^{-1}(\neq) \oplus f^{-1}(\overline{K_p^{\mathcal{A}}}) \oplus f^{-1}(R_1^{\mathcal{A}}) \oplus \dots$$

and we observe that all $R_i^{\mathcal{A}}$ are trivially r.i.p.e., uniformly in i , so in particular $\overline{f^{-1}(R_i^{\mathcal{A}})} \leq_e \overline{f^{-1}(K_p^{\mathcal{A}})} = f^{-1}(\overline{K_p^{\mathcal{A}}})$. Also, $f^{-1}(K_p^{\mathcal{A}}) \leq_e P(f^{-1}(\mathcal{A})) \leq_e P(f^{-1}(PJ(\mathcal{A})))$. \square

We will now see that the positive jump of a structure jumps, in the sense that the enumeration degree spectrum of the positive jump is indeed the set of jumps of the degrees in the enumeration degree spectrum of the structure. The following version of a Theorem by Soskova and Soskov [17] is essential to our proof.

Theorem 2.10 ([17, Theorem 1.2]). Let B be an arbitrary set of natural numbers. There exists a total set F such that

$$B \leq_e F \quad \text{and} \quad J_e(B) \equiv_e J_e(F).$$

Theorem 2.11. For any structure \mathcal{A} ,

$$eSp(PJ(\mathcal{A})) = \{\mathbf{a}' : \mathbf{a} \in eSp(\mathcal{A})\}$$

Proof. To show that $\{\mathbf{a}' : \mathbf{a} \in eSp(\mathcal{A})\} \subseteq eSp(PJ(\mathcal{A}))$, consider an arbitrary enumeration f of ω and let $\mathcal{B} = f^{-1}(\mathcal{A})$. Then from Proposition 2.6 we know that $P(PJ(\mathcal{B})) \leq_e J_e(P(\mathcal{B}))$. Note that $P(PJ(\mathcal{B})) = P(PJ(f^{-1}(\mathcal{A}))) = P(f^{-1}(PJ(\mathcal{A})))$, and $d_e(P(f^{-1}(PJ(\mathcal{A})))) \in eSp(PJ(\mathcal{A}))$. Since the enumeration jump is always total

and enumeration degree spectra are closed upwards with respect to total degrees, $d_e(J_e(P(\mathcal{B}))) \in eSp(PJ(\mathcal{B}))$.

To show $eSp(PJ(\mathcal{A})) \subseteq \{\mathbf{a}' : \mathbf{a} \in eSp(\mathcal{A})\}$, let us look at $P(f^{-1}(PJ(\mathcal{A})))$ for some enumeration f of ω , and again write $\mathcal{B} = f^{-1}(\mathcal{A})$ so that $P(f^{-1}(PJ(\mathcal{A}))) = P(PJ(\mathcal{B}))$. Since $PJ(\mathcal{A})$ is a total structure, we know that $P(PJ(\mathcal{B}))$ is total. By Lemma 2.4 we can use $P(PJ(\mathcal{B}))$ to enumerate a r.i.p.e.-generic enumeration g of \mathcal{B} such that $P(g^{-1}(PJ(\mathcal{B}))) \leq_e P(PJ(\mathcal{B}))$. Then, letting $\mathcal{C} = g^{-1}(\mathcal{B})$ and using the latter part of Proposition 2.6 we know that $J_e(P(\mathcal{C})) \equiv_e P(PJ(\mathcal{C}))$ which makes $J_e(P(\mathcal{C})) \leq_e P(PJ(\mathcal{B}))$. Using Theorem 2.10 there is a total set F such that $P(\mathcal{C}) \leq_e F$ and $J_e(F) \equiv_e J_e(P(\mathcal{C})) \leq_e P(PJ(\mathcal{B}))$. As F and $P(PJ(\mathcal{B}))$ are total, we have that $F' \leq_T P(PJ(\mathcal{B}))$.

We can now apply the relativized jump inversion theorem for the Turing degrees to get a set $Z \geq_T F$ such that $Z' \equiv_T P(PJ(\mathcal{B}))$. For this Z , we have that $\iota(deg(Z')) = d_e(J_e(\chi_Z)) = d_e(P(PJ(\mathcal{B}))) \in eSp(PJ(\mathcal{A}))$ and $d_e(P(\mathcal{B})) \leq d_e(F) \leq d_e(\chi_Z)$. Since χ_Z is total and enumeration degree spectra are upwards closed in the total degrees, this means $d_e(\chi_Z) \in eSp(\mathcal{B}) \subseteq eSp(\mathcal{A})$. So in particular, $d_e(J_e(\chi_Z)) = d_e(P(PJ(\mathcal{B}))) \in \{\mathbf{a}' : \mathbf{a} \in eSp(\mathcal{A})\}$. \square

We now compare the enumeration degree spectrum of the positive jump of a structure \mathcal{A} with the spectrum of the original structure and the spectrum of its traditional jump. The latter is the spectrum of the structure obtained by adding the sequence of relations

$$\vec{K}^{\mathcal{A}} = \left(\bigcup_{j \in \omega} \{\bar{a} \in \omega^j : \mathcal{A} \models \varphi_{i,j}^{\Sigma_1^c}(\bar{a})\} \right)_{i \in \omega}$$

where $\varphi_{i,j}^{\Sigma_1^c}(\bar{x})$ is the i^{th} formula with j free variables in a fixed computable enumeration of all computable infinitary Σ_1^0 formulas. Let $J(\mathcal{A}) = (\mathcal{A}, \vec{K}^{\mathcal{A}})$ denote the traditional jump of \mathcal{A} . This notion of a jump was defined by Montalbán [12] and is similar to several other notions of jumps of structures that arose independently such as Soskova's notion using Marker extensions [17] or Stukachev's version using Σ -definability [18].

We will also consider the structure \mathcal{A}^+ , the totalization of \mathcal{A} given by

$$\mathcal{A}^+ = (A, (R_i^A, \overline{R_i^A})_{i \in \omega}).$$

We will not compare the enumeration degree spectra directly but instead the sets of enumerations. This gives more insight than comparing the degree spectra as we can make use of the following analogues to Muchnik and Medvedev reducibility for enumeration degrees. Given $\mathfrak{A}, \mathfrak{B} \subseteq P(\omega)$ we say that $\mathfrak{A} \leq_{we} \mathfrak{B}$, \mathfrak{A} is *weakly enumeration reducible* to \mathfrak{B} , if for every $B \in \mathfrak{B}$ there is $A \in \mathfrak{A}$ such that $A \leq_e B$. We say that $\mathfrak{A} \leq_{se} \mathfrak{B}$, \mathfrak{A} is *strongly enumeration reducible* to \mathfrak{B} , if there is an enumeration operator Ψ such that for every $B \in \mathfrak{B}$, $\Psi^B \in \mathfrak{A}$.

It is not hard to see that given an enumeration of \mathcal{A}^+ one can enumerate an enumeration of $J(\mathcal{A})$ and the converse holds trivially. We thus have the following.

Proposition 2.12. For every structure \mathcal{A} , $deg_e(J(\mathcal{A})) = deg_e(\mathcal{A}^+)$. In particular $Enum(J(\mathcal{A})) \equiv_{se} Enum(\mathcal{A}^+)$.

Proof. By replacing every subformula of the form $\neg R_i(x_1, \dots, x_m)$ by the formula $\overline{R_i}(x_1, \dots, x_m)$ we get a Σ_1^p formula in the language of \mathcal{A}^+ . Similarly, given a Σ_1^p formula in the language of \mathcal{A}^+ we can obtain a Σ_1^c formula in the language of \mathcal{A} by substituting subformulas of the form $\overline{R_i}(x_1, \dots, x_m)$ with $\neg R_i(x_1, \dots, x_m)$. Indeed we get a computable bijection between the Σ_1^p formulas in the language of \mathcal{A}^+ and the Σ_1^c formulas of \mathcal{A} . Thus $J(\mathcal{A}) \equiv_e (\mathcal{A}^+, \vec{K}_p^{\mathcal{A}^+})$, but $\mathcal{A}^+ \equiv_e (\mathcal{A}^+, \vec{K}_p^{\mathcal{A}^+})$ given the first equivalence. All of these equivalence are witnessed by fixed enumeration operators and thus $Enum(J(\mathcal{A})) \equiv_{se} Enum(\mathcal{A}^+)$. \square

Proposition 2.12 is not very surprising, as adding a r.i.c.e. set such as $\vec{K}^{\mathcal{A}}$ to the totalization of \mathcal{A} won't change its enumeration degree. We will thus consider a slightly different notion of jump by adding the coset of $\vec{K}^{\mathcal{A}}$:

$$T(\mathcal{A}) = (\mathcal{A}, \overline{\vec{K}^{\mathcal{A}}})$$

Clearly $T(\mathcal{A}) \equiv_T J(\mathcal{A})$; however, the two sets are not necessarily enumeration equivalent as the following two propositions show.

Proposition 2.13. Let \mathcal{A} be a structure. For every presentation $\hat{\mathcal{A}}$ of \mathcal{A} , $P(\hat{\mathcal{A}}) \leq_e D(\hat{\mathcal{A}}) = P(\hat{\mathcal{A}}^+) \leq_e PJ(\hat{\mathcal{A}}) \leq_e T(\hat{\mathcal{A}})$. In particular

$$Enum(\mathcal{A}) \leq_{se} Enum(\mathcal{A}^+) \leq_{se} Enum(PJ(\mathcal{A})) \leq_{se} Enum(T(\mathcal{A})).$$

Proof. Straightforward from the definitions. \square

Proposition 2.14. There is a structure \mathcal{A} such that

$$Enum(\mathcal{A}) \not\leq_{we} Enum(\mathcal{A}^+) \not\leq_{we} Enum(PJ(\mathcal{A})) \not\leq_{we} Enum(T(\mathcal{A})).$$

Proof. Sacks showed that there is an incomplete c.e. set X of high Turing degree. Thus, X has enumeration degree $deg_e(X) = \mathbf{0}_e$ and $deg_T(X') = \mathbf{0}''$. Let \mathcal{A} be the graph coding X as follows. \mathcal{A} contains a single element a with a loop, i.e. $aE^{\mathcal{A}}a$ and one circle of length $n + 1$ for every $n \in \omega$. Let y be the least element in \mathcal{A} that is part of the cycle of length $n + 1$. If $n \in X$, then connect a to y , i.e., $aE^{\mathcal{A}}y$. This finishes the construction.

As X is c.e., \mathcal{A} has enumeration degree $\mathbf{0}_e$. However, $\mathbf{0}_e \notin eSp(\mathcal{A}^+)$, as \mathcal{A}^+ has enumeration degree $deg_e(X \oplus \overline{X}) > \mathbf{0}_e$. So, in particular $Enum(\mathcal{A}) \not\leq_{we} Enum(\mathcal{A}^+)$. By Theorem 2.11 the enumeration degree of $PJ(\mathcal{A})$ is $\mathbf{0}'_e$ and $\mathbf{0}'_e \ni \emptyset' \oplus \overline{\emptyset'} >_e X \oplus \overline{X}$, so $Enum(\mathcal{A}^+) \not\leq_{we} Enum(PJ(\mathcal{A}))$. For the last inequality notice that both $T(\mathcal{A})$ and $PJ(\mathcal{A})$ are total structures and that by the analogue of Theorem 2.11 for the traditional jumps of structures we have that the enumeration degree of $T(\mathcal{A})$ is $deg_e(\emptyset'' \oplus \overline{\emptyset''})$. As mentioned above the enumeration degree of $PJ(\mathcal{A})$ is $\mathbf{0}'_e$, $\emptyset' \oplus \overline{\emptyset'} \in \mathbf{0}'_e$, and $\emptyset' \oplus \overline{\emptyset'} <_e \emptyset'' \oplus \overline{\emptyset''}$. So, in particular, $Enum(PJ(\mathcal{A})) \not\leq_{we} Enum(T(\mathcal{A}))$. \square

For total structures the traditional notion of the jump and the positive jump coincide.

Proposition 2.15. Let \mathcal{A} be a structure, then $Enum(PJ(\mathcal{A}^+)) \equiv_{se} Enum(T(\mathcal{A}))$ and $eSp(PJ(\mathcal{A}^+)) = eSp(T(\mathcal{A}))$.

Proof. This proof is similar to the proof of Proposition 2.12. Mutatis mutandis. \square

3. FUNCTORS

When comparing structures with respect to their enumerations it is natural to want to use only positive information. Csima, Rossegger, and Yu [5] introduced the notion of a positive enumerable functor which uses only the positive diagrams of structures. Reductions based on this notion preserve desired properties such as enumeration degree spectra of structures.

Recall that a functor F from a class of structures \mathfrak{C} to a class \mathfrak{D} maps structures in \mathfrak{C} to structures in \mathfrak{D} and isomorphisms $f : \mathcal{A} \rightarrow \mathcal{B}$ to isomorphisms $F(f) : F(\mathcal{A}) \rightarrow F(\mathcal{B})$ with the property that $F(id_{\mathcal{A}}) = id_{F(\mathcal{A})}$ and $F(f \circ g) = F(f) \circ F(g)$ for all isomorphisms f and g and structures $\mathcal{A} \in \mathfrak{C}$.

Definition 3.1. A functor $F : \mathfrak{C} \rightarrow \mathfrak{D}$ is *positive enumerable* if there is a pair (Ψ, Ψ_*) where Ψ and Ψ_* are enumeration operators such that for all $\mathcal{A}, \mathcal{B} \in \mathfrak{C}$

- (1) $\Psi^{P(\mathcal{A})} = P(F(\mathcal{A}))$,
- (2) for all $f \in Hom(\mathcal{A}, \mathcal{B})$, $\Psi_*^{P(\mathcal{A}) \oplus Graph(f) \oplus P(\mathcal{B})} = Graph(F(f))$.

For ease of notation, when a graph of a function occurs in an oracle, we will simply write the name of the function to represent it.

An alternative and purely syntactical method of comparing classes of structures is through the model-theoretic notion of interpretability. Since we are restricting ourselves to positive information, we introduce a new notion of interpretability that uses Σ_1^p formulas.

Definition 3.2. A structure $\mathcal{A} = (A, P_0^{\mathcal{A}}, \dots)$ is *positively interpretable* in a structure \mathcal{B} if there exists a Σ_1^p definable sequence of relations $(Dom_{\mathcal{A}}^{\mathcal{B}}, \overline{Dom_{\mathcal{A}}^{\mathcal{B}}}, \sim, \not\sim, R_0, \dots)$ in the language of \mathcal{B} such that

- (1) $Dom_{\mathcal{A}}^{\mathcal{B}} \subseteq B^{<\omega}$, $\overline{Dom_{\mathcal{A}}^{\mathcal{B}}} = B^{<\omega} \setminus Dom_{\mathcal{A}}^{\mathcal{B}}$,
- (2) \sim is an equivalence relation on $Dom_{\mathcal{A}}^{\mathcal{B}}$ and $\not\sim$ its corelation,
- (3) $R_i \subseteq (B^{<\omega})^{a_{P_i}}$ ¹ is closed under \sim on $Dom_{\mathcal{A}}^{\mathcal{B}}$,

¹ a_{P_i} is the arity of P_i

and there exists a function $f_{\mathcal{B}}^{\mathcal{A}} : Dom_{\mathcal{A}}^{\mathcal{B}} \rightarrow \mathcal{A}$, the *interpretation of \mathcal{A} in \mathcal{B}* , which induces an isomorphism:

$$(Dom_{\mathcal{A}}^{\mathcal{B}}, R_0, \dots) / \sim \cong (A, P_0^{\mathcal{A}}, \dots)$$

We seek to provide enumeration analogues to the results proven in [7], starting with the following.

Theorem 3.3. There is a positive enumerable functor $F : Iso(\mathcal{B}) \rightarrow Iso(\mathcal{A})$ if and only if \mathcal{A} is positively interpretable in \mathcal{B} .

Proof. Suppose that \mathcal{A} is positively interpretable in \mathcal{B} using $(Dom_{\mathcal{A}}^{\mathcal{B}}, \overline{Dom_{\mathcal{A}}^{\mathcal{B}}}, \sim, \not\sim, R_0^{\mathcal{B}}, \dots)$. We want to construct a functor $F : Iso(\mathcal{B}) \rightarrow Iso(\mathcal{A})$, so let $\tilde{\mathcal{B}} \in Iso(\mathcal{B})$. Since the relations are all Σ_1^p definable they are uniformly r.i.p.e. and so uniformly enumerable from $P(\tilde{\mathcal{B}})$ by Corollary 1.7. Fix an enumeration of $\omega^{<\omega}$. Using the fact that both $Dom_{\mathcal{A}}^{\tilde{\mathcal{B}}}, \sim$, and their complements are uniformly enumerable from $P(\tilde{\mathcal{B}})$ we can uniformly enumerate a map $\tilde{\tau} : \omega \rightarrow Dom_{\mathcal{A}}^{\tilde{\mathcal{B}}} / \sim$ sending elements of ω to \sim equivalence classes in $Dom_{\mathcal{A}}^{\tilde{\mathcal{B}}}$. We define $F(\tilde{\mathcal{B}})$ to be the pullback along $\tilde{\tau}$ of the structure $(Dom_{\mathcal{A}}^{\tilde{\mathcal{B}}} / \sim, \overline{Dom_{\mathcal{A}}^{\tilde{\mathcal{B}}} / \sim}, R_0^{\tilde{\mathcal{B}}} / \sim, \dots)$. Since our relations are all uniformly enumerable from $P(\tilde{\mathcal{B}})$, we have that $P(F(\tilde{\mathcal{B}}))$ is enumerable from $P(\tilde{\mathcal{B}})$.

Now to define our functor on isomorphisms, if we have $f : \tilde{\mathcal{B}} \rightarrow \hat{\mathcal{B}}$ then letting $\tilde{\tau} : \omega \rightarrow Dom_{\mathcal{A}}^{\tilde{\mathcal{B}}} / \sim$ and $\hat{\tau} : \omega \rightarrow Dom_{\mathcal{A}}^{\hat{\mathcal{B}}} / \sim$ be defined as above, and extending f to $\tilde{\mathcal{B}}^{<\omega}$ in the natural way, we let

$$F(f) = \hat{\tau}^{-1} \circ f \circ \tilde{\tau} : \tilde{\mathcal{A}} \rightarrow \hat{\mathcal{A}}$$

Since $\tilde{\tau}$ and $\hat{\tau}$ are uniformly enumerable from $\tilde{\mathcal{B}}$ and $\hat{\mathcal{B}}$ respectively, we have that this set $Graph(F(f))$ is uniformly enumerable from $P(\tilde{\mathcal{B}}) \oplus Graph(f) \oplus P(\hat{\mathcal{B}})$.

Now suppose that there is a positive enumerable functor $F = (\Psi, \Psi_*)$ from $Iso(\mathcal{B})$ to $Iso(\mathcal{A})$. We want to produce the Σ_1^p sequence of relations providing the positive interpretation of \mathcal{B} in \mathcal{A} .

In what follows, we will often write $P(\bar{b})$ instead of $P_{\mathcal{B}}(\bar{b})$ when it is clear from context which structure \mathcal{B} is meant.

We can view any finite disjoint tuple \bar{b} as a map $i \mapsto b_i$ for $i < |\bar{b}|$. Note that viewing \bar{b} as such a map, if f is any permutation of ω extending \bar{b} , then $P_{\mathcal{B}}(\bar{b}) \subseteq P(\mathcal{B}_f)$ where $\mathcal{B}_f = f^{-1}(\mathcal{B})$.

Let $Dom_{\mathcal{A}}^{\mathcal{B}}$ be the set of pairs $(\bar{b}, i) \in \mathcal{B}^{<\omega} \times \omega$ such that

$$(i, i) \in \Psi_{*, |\bar{b}|}^{P(\bar{b}) \oplus \lambda \upharpoonright |\bar{b}| \oplus P(\bar{b})}$$

where λ is the identity function. Both $Dom_{\mathcal{A}}^{\mathcal{B}}$ and its corelation $\overline{Dom_{\mathcal{A}}^{\mathcal{B}}}$ are clearly uniformly r.i.p.e.

For $(\bar{b}, i), (\bar{c}, j) \in Dom_{\mathcal{A}}^{\mathcal{B}}$ we let $(\bar{b}, i) \sim (\bar{c}, j)$ exactly if there exists a finite tuple \bar{d} which does not mention elements from \bar{b} or \bar{c} , such that if \bar{b}' lists the elements that occur in \bar{b} but not \bar{c} and \bar{c}' lists the elements in \bar{c} but not in \bar{b} , and if $\sigma = (\bar{c}\bar{b}'\bar{d})^{-1} \circ \bar{b}\bar{c}'\bar{d}$ then

$$(i, j) \in \Psi_{*, |\bar{b}\bar{c}\bar{d}|}^{P(\bar{b}\bar{c}'\bar{d}) \oplus \sigma \oplus P(\bar{c}\bar{b}'\bar{d})} \quad \text{and} \quad (j, i) \in \Psi_{*, |\bar{b}\bar{c}\bar{d}|}^{P(\bar{c}\bar{b}'\bar{d}) \oplus \sigma^{-1} \oplus P(\bar{b}\bar{c}'\bar{d})}.$$

It is easy to see that \sim is uniformly r.i.p.e.. Rather than showing immediately that the complement of \sim is uniformly r.i.p.e., we define a clearly uniformly r.i.p.e. relation $\not\sim$, and then show that it is indeed the complement of \sim .

For $(\bar{b}, i), (\bar{c}, j) \in Dom_{\mathcal{A}}^{\mathcal{B}}$ we say $(\bar{b}, i) \not\sim (\bar{c}, j)$ if there exist $k \neq j, l \neq i$, and a finite tuple \bar{d} which does not mention elements from \bar{b} or \bar{c} , such that if \bar{b}' lists the elements that occur in \bar{b} but not \bar{c} and \bar{c}' lists the elements in \bar{c} but not in \bar{b} , and if $\sigma = (\bar{c}\bar{b}'\bar{d})^{-1} \circ \bar{b}\bar{c}'\bar{d}$ then

$$(i, k) \in \Psi_{*, |\bar{b}\bar{c}\bar{d}|}^{P(\bar{b}\bar{c}'\bar{d}) \oplus \sigma \oplus P(\bar{c}\bar{b}'\bar{d})} \quad \text{or} \quad (j, l) \in \Psi_{*, |\bar{b}\bar{c}\bar{d}|}^{P(\bar{c}\bar{b}'\bar{d}) \oplus \sigma^{-1} \oplus P(\bar{b}\bar{c}'\bar{d})}.$$

Claim 3.3.1. $\not\sim$ is the complement of \sim

Proof. We want to show that, for any tuples $(\bar{b}, i), (\bar{c}, j) \in Dom_{\mathcal{A}}^{\mathcal{B}}$, exactly one of $(\bar{b}, i) \sim (\bar{c}, j)$, $(\bar{b}, i) \not\sim (\bar{c}, j)$ hold. Let \bar{b}' list the elements in \bar{b} but not \bar{c} , and let \bar{c}' list the elements in \bar{c} but not \bar{b} . Let $\sigma = (\bar{c}\bar{b}'\bar{d})^{-1} \circ \bar{b}\bar{c}'\bar{d}$. Let $f, g : \omega \rightarrow \mathcal{A}$ be bijections extending $\bar{b}\bar{c}', \bar{c}\bar{b}'$ respectively which agree on all inputs $i > |\bar{b}\bar{c}'| = |\bar{c}\bar{b}'|$. We can then pull back f and g to get structures $\mathcal{B}_f, \mathcal{B}_g$. Then $h = g^{-1} \circ f$ is an isomorphism

extending σ which is constant on all $i > |\sigma|$. Hence

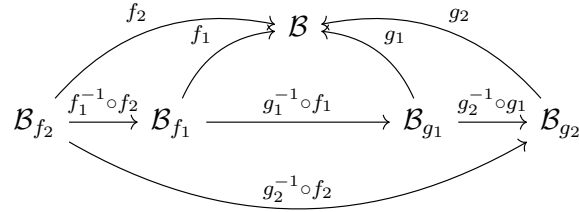
$$(i, h(i)) \in \Psi_*^{P(\mathcal{B}_f) \oplus h \oplus P(\mathcal{B}_g)} \quad \text{and} \quad (j, h(j)) \in \Psi_*^{P(\mathcal{B}_f) \oplus h \oplus P(\mathcal{B}_g)}$$

If $h(i) = j$ and $h(j) = i$, then taking a long enough initial segment of h witnesses $(\bar{b}, i) \sim (\bar{c}, j)$. If however $h(i) \neq j$ or $h(j) \neq i$, then a long enough initial segment of h witnesses $(\bar{b}, i) \not\sim (\bar{c}, j)$.

We now assume towards a contradiction that both $(\bar{b}, i) \sim (\bar{c}, j)$, and $(\bar{b}, i) \not\sim (\bar{c}, j)$ and that their equivalence is witnessed by \bar{d}_1, σ and their inequivalence is witnessed by k, l, \bar{d}_2 and τ . Without loss of generality assume that $(i, k) \in \Psi_*^{P(\bar{b}\bar{c}'\bar{d}_2) \oplus \tau \oplus P(\bar{c}\bar{b}'\bar{d}_2)}$. We also have $(i, j) \in \Psi_*^{P(\bar{b}\bar{c}'\bar{d}_1) \oplus \sigma \oplus P(\bar{c}\bar{b}'\bar{d}_1)}$. Let

$$f_1 \supset \bar{b}\bar{c}'\bar{d}_1 \quad g_1 \supset \bar{c}\bar{b}'\bar{d}_1 \quad f_2 \supset \bar{b}\bar{c}'\bar{d}_2 \quad g_2 \supset \bar{c}\bar{b}'\bar{d}_2$$

Then we have isomorphisms as shown below.



Since F is a functor,

$$F(g_2^{-1} \circ f_2) = F(g_2^{-1} \circ g_1) \circ F(g_1^{-1} \circ f_1) \circ F(f_1^{-1} \circ f_2).$$

Since $(\bar{b}, i) \in \text{Dom}_{\mathcal{A}}^{\mathcal{B}}$ we know that $(i, i) \in \Psi_*^{P(\bar{b}) \oplus \lambda \uparrow |\bar{b}| \oplus P(\bar{b})}$. Notice that $f_1^{-1} \circ f_2 \supset \lambda \uparrow |\bar{b}\bar{c}'|$ and $P(\mathcal{B}_{f_1}) \supset P(\bar{b}), P(\mathcal{B}_{f_2}) \supset P(\bar{b})$. Thus

$$(i, i) \in \Psi_*^{P(\mathcal{B}_{f_2}) \oplus (f_1^{-1} \circ f_2) \oplus P(\mathcal{B}_{f_1})} = \text{Graph}(F(f_1^{-1} \circ f_2)) \Rightarrow F(f_1^{-1} \circ f_2)(i) = i.$$

Similarly, since $(\bar{c}, j) \in \text{Dom}_{\mathcal{A}}^{\mathcal{B}}$,

$$(j, j) \in \Psi_*^{P(\bar{c}) \oplus \lambda \uparrow |\bar{c}| \oplus P(\bar{c})} \Rightarrow (j, j) \in \Psi_*^{P(\mathcal{B}_{g_1}) \oplus (g_2^{-1} \circ g_1) \oplus P(\mathcal{B}_{g_2})} = \text{Graph}(F(g_2^{-1} \circ g_1)).$$

Following from our choices for f_1, g_1, f_2, g_2 we have that $g_1^{-1} \circ f_1 \supset \sigma$ and $g_2^{-1} \circ f_2 \supset \tau$, so

$$(i, j) \in \Psi_*^{P(\bar{b}\bar{c}'\bar{d}_1) \oplus \sigma \oplus P(\bar{c}\bar{b}'\bar{d}_1)} \Rightarrow (i, j) \in \Psi_*^{P(\mathcal{B}_{f_1}) \oplus (g_1^{-1} \circ f_1) \oplus P(\mathcal{B}_{g_1})} = \text{Graph}(F(g_1^{-1} \circ f_1))$$

$$(i, k) \in \Psi_*^{P(\bar{b}\bar{c}'\bar{d}_2) \oplus \tau \oplus P(\bar{c}\bar{b}'\bar{d}_2)} \Rightarrow (i, k) \in \Psi_*^{P(\mathcal{B}_{f_2}) \oplus (g_2^{-1} \circ f_2) \oplus P(\mathcal{B}_{g_2})} = \text{Graph}(F(g_2^{-1} \circ f_2)).$$

The first three equations tell us that

$$F(g_2^{-1} \circ g_1) \circ F(g_1^{-1} \circ f_1) \circ F(f_1^{-1} \circ f_2)(i) = F(g_2^{-1} \circ g_1) \circ F(g_1^{-1} \circ f_1)(i) = F(g_2^{-1} \circ g_1)(j) = j$$

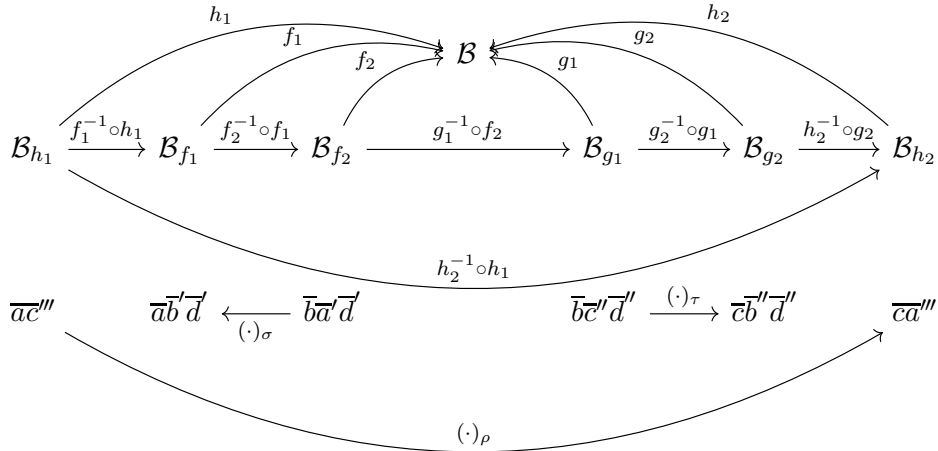
whereas the fourth equation tells us that $F(g_2^{-1} \circ f_2)(i) = k$. This contradicts our earlier statement of F being a functor, and so only one of $\sim, \not\sim$ can hold at once. \square

Claim 3.3.2. The relation \sim is an equivalence relation on $\text{Dom}_{\mathcal{A}}^{\mathcal{B}}$.

Proof. Let $(\bar{a}, i), (\bar{b}, j), (\bar{c}, k) \in \text{Dom}_{\mathcal{A}}^{\mathcal{B}}$. It is reflexive, since $(\bar{a}, i) \in \text{Dom}_{\mathcal{A}}^{\mathcal{B}}$ means that $(i, i) \in \Psi_*^{P(\bar{a}) \oplus \lambda \upharpoonright |\bar{a}| \oplus P(\bar{a})}$, and so the equivalence is witnessed by the empty tuple and $\lambda \upharpoonright |\bar{a}|$. If $(\bar{a}, i) \sim (\bar{b}, j)$ via \bar{d}, σ , then $(\bar{b}, j) \sim (\bar{a}, i)$ via \bar{d}, σ^{-1} . Now assume that $(\bar{a}, i) \sim (\bar{b}, j)$ and it is witnessed by $\bar{a}', \bar{b}'\bar{d}', \sigma$ and $(\bar{b}, j) \sim (\bar{c}, k)$ via $\bar{b}'', \bar{c}'', \bar{d}'', \tau$. Let \bar{a}''' and \bar{c}''' list the elements of $\bar{a} \setminus \bar{c}$ and $\bar{c} \setminus \bar{a}$ respectively. Choose bijections as follows

$$\begin{aligned} f_1 &\supset \bar{a}\bar{b}'\bar{d}' & g_1 &\supset \bar{b}\bar{c}''\bar{d}'' & h_1 &\supset \bar{a}\bar{c}''' \\ f_2 &\supset \bar{b}\bar{a}'\bar{d}' & g_2 &\supset \bar{c}\bar{b}''\bar{d}'' & h_2 &\supset \bar{c}\bar{a}''' \end{aligned}$$

such that h_1 and h_2 agree outside an initial segment of length $|\bar{a}| + |\bar{c}'''|$.



Since F is a functor we have

$$F(h_2^{-1} \circ h_1) = F(h_2^{-1} \circ g_2) \circ F(g_2^{-1} \circ g_1) \circ F(g_1^{-1} \circ f_2) \circ F(f_2^{-1} \circ f_1) \circ F(f_1^{-1} \circ h_1).$$

Since $(\bar{a}, i) \in \text{Dom}_{\mathcal{A}}^{\mathcal{B}}$, and $f_1^{-1} \circ h_1 \supset \lambda \upharpoonright |\bar{a}|$ we can show as we did in Claim 3.3.1 that $F(f_1^{-1} \circ h_1)(i) = i$. Similarly, $F(g_1^{-1} \circ f_2)(j) = j$, $F(h_2^{-1} \circ g_2)(k) = k$. By assumption $F(f_2^{-1} \circ f_1)(i) = j$ and $F(g_2^{-1} \circ g_1)(j) = k$. Thus

$$(i, k) \in \text{Graph}(F(h_2^{-1} \circ h_1)) = \Psi_*^{P(\mathcal{B}_{h_1}) \oplus (h_2^{-1} \circ h_1) \oplus P(\mathcal{B}_{h_2})}$$

Similarly one can show that $(k, i) \in \Psi_*^{P(\mathcal{B}_{h_2}) \oplus (h_1^{-1} \circ h_2) \oplus P(\mathcal{B}_{h_1})}$. Since h_1 and h_2 agree outside of the initial segment of length $|\bar{a}| + |\bar{c}'''|$ if we take a long enough \bar{d}''' and let $\rho \subset h_2^{-1} \circ h_1$ be the permutation sending $\bar{a}\bar{c}'''\bar{d}'''$ to $\bar{c}\bar{a}'''\bar{d}'''$ we witness that $(\bar{a}, i) \sim (\bar{c}, k)$. \square

Claim 3.3.3. For all $i \in \omega$, there is some $n \in \omega$ such that for $\bar{c} = \mathcal{B} \upharpoonright n$, we have $(\bar{c}, i) \in \text{Dom}_{\mathcal{A}}^{\mathcal{B}}$.

Proof. Let λ be the identity function. Since functors map the identity to the identity, $(i, i) \in \text{Graph}(\lambda) = \Psi_*^{P(\mathcal{B}) \oplus \lambda \oplus P(\mathcal{B})}$, so by the use principle there is a sufficiently long initial segment of \mathcal{B} will have $(\bar{c}, i) \in \text{Dom}_{\mathcal{A}}^{\mathcal{B}}$. \square

Claim 3.3.4. For $(\bar{b}, i) \in \text{Dom}_{\mathcal{A}}^{\mathcal{B}}$, there is an initial segment $\bar{c} = \mathcal{B} \upharpoonright n$ of \mathcal{B} and $j \in \omega$ such that $(\bar{b}, i) \sim (\bar{c}, j)$.

Proof. Let m be greater than the maximum number in the tuple \bar{b} , and let \bar{c}' list the numbers less than or equal to m not occurring in \bar{b} . Let $\bar{c} = \mathcal{B} \upharpoonright m$, and let $f \supset \bar{c}^{-1} \circ \bar{b}\bar{c}'$ be defined by $f(n) = n$ for all $n \geq m$. Then $(i, j) \in \text{Graph}(F(f)) = \Psi_*^{P(\mathcal{B}_f) \oplus f \oplus P(\mathcal{B})}$ for some j . So by the use principle, there exists \bar{d} such that $(i, j) \in \Psi_*^{P(\bar{b}\bar{c}'\bar{d}) \oplus \sigma \oplus P(\bar{c}\bar{d})}$ where $\sigma = (\bar{c}\bar{d})^{-1} \circ \bar{b}\bar{c}'\bar{d}$, witnessing that $(\bar{b}, i) \sim (\bar{c}, j)$. \square

Claim 3.3.5. If $(\bar{b}, i), (\bar{c}, j) \in \text{Dom}_{\mathcal{A}}^{\mathcal{B}}$ and $\bar{b} \subseteq \bar{c}$ then $(\bar{b}, i) \sim (\bar{c}, j)$ iff $i = j$.

Proof. To see this we note that since $(\bar{b}, i) \in \text{Dom}_{\mathcal{A}}^{\mathcal{B}}$, we have $(i, i) \in \Psi_*^{P(\bar{b}) \oplus \lambda \upharpoonright |\bar{b}| \oplus P(\bar{b})}$. So since $\bar{b} \subseteq \bar{c}$, by the use principle $(i, i) \in \Psi_*^{P(\bar{c}) \oplus \lambda \upharpoonright |\bar{c}| \oplus P(\bar{c})}$, so $(\bar{b}, i) \sim (\bar{c}, i)$. Now let \bar{d}, σ witness that $(\bar{b}, i) \sim (\bar{c}, j)$. Then $\sigma \supseteq \lambda \upharpoonright |\bar{b}|$ and the oracle $P(\bar{b}\bar{c}'\bar{d}) \oplus \sigma \oplus P(\bar{c}\bar{d})$ extends $P(\bar{b}) \oplus \lambda \upharpoonright |\bar{b}| \oplus P(\bar{b})$. So by the use principle $(i, i) \in \Psi_*^{P(\bar{b}\bar{c}'\bar{d}) \oplus \sigma \oplus P(\bar{c}\bar{d})}$. As $\Psi_*^{P(\bar{b}\bar{c}'\bar{d}) \oplus \sigma \oplus P(\bar{c}\bar{d})}$ must extend to the graph of a function, this shows that $j = i$. \square

We now define relations R_i on $Dom_{\mathcal{A}}^{\mathcal{B}}$. For each relation P_i of arity $p(i)$ we let $(\bar{b}_1, k_1), \dots, (\bar{b}_{p(i)}, k_{p(i)})$ be in R_i if there is an initial segment $\bar{c} = \mathcal{B} \upharpoonright n$ of \mathcal{B} and $j_1, \dots, j_{p(i)} \in \omega$ such that $(\bar{b}_l, k_l) \sim (\bar{c}, j_l)$ for all l and $P_i(j_1, \dots, j_{p(i)}) \in \Psi^{P(\bar{c})}$. Note that by Claim 3.3.5, it does not matter which initial segment is chosen.

We are now in position to define the isomorphism

$$\mathfrak{F} : (\mathcal{A}, P_0, P_1 \dots) \rightarrow (Dom_{\mathcal{A}}^{\mathcal{B}}/\sim, R_0/\sim, R_1/\sim, \dots)$$

by $\mathfrak{F}(i) = (\bar{c}, i)$ where $\bar{c} = \mathcal{B} \upharpoonright n$ for the least n such that $(\bar{c}, i) \in Dom_{\mathcal{A}}^{\mathcal{B}}$. Note that such (\bar{c}, i) exists by Claim 3.3.3. It then follows from Claim 3.3.4 and Claim 3.3.5 that \mathfrak{F} is a bijection. The bijection respects the relations by definition, and so \mathfrak{F} is an isomorphism. \square

In the above theorem we not only show the existence of an interpretation given a functor, but provide a method for transforming a functor F into an interpretation. Using the other direction of the proof we can turn this interpretation back into a new functor. We shall call this new induced functor I^F . We would like I^F to agree with our original functor F in some fashion, and so we introduce the following definitions.

Definition 3.4. A functor $F : \mathfrak{C} \rightarrow \mathfrak{D}$ is *enumeration isomorphic* to a functor $G : \mathfrak{C} \rightarrow \mathfrak{D}$ if there is an enumeration operator Λ such that for any $\mathcal{A} \in \mathfrak{C}$, $\Lambda^{P(\mathcal{A})} : F(\mathcal{A}) \rightarrow G(\mathcal{A})$ is an isomorphism. Moreover, for any morphism $h \in Hom(\mathcal{A}, \mathcal{B})$ in \mathfrak{C} when viewing $\Lambda^{P(\mathcal{A})}, \Lambda^{P(\mathcal{B})}$ as isomorphisms, $\Lambda^{P(\mathcal{B})} \circ F(h) = G(h) \circ \Lambda^{P(\mathcal{A})}$. That is, the diagram below commutes.

$$\begin{array}{ccc} F(\mathcal{A}) & \xrightarrow{\Lambda^{P(\mathcal{A})}} & G(\mathcal{A}) \\ F(h) \downarrow & & \downarrow G(h) \\ F(\mathcal{B}) & \xrightarrow{\Lambda^{P(\mathcal{B})}} & G(\mathcal{B}) \end{array}$$

Proposition 3.5. Let $F : Iso(\mathcal{B}) \rightarrow Iso(\mathcal{A})$ be positive enumerable and $I^F : Iso(\mathcal{B}) \rightarrow Iso(\mathcal{A})$ be the functor obtained from Theorem 3.3. Then F and I^F are enumeration isomorphic.

Proof. Given a presentation $\mathcal{B} \in Iso(\mathcal{B})$ let $\mathfrak{F}^{\mathcal{B}} : \mathcal{B} \rightarrow Dom_{\mathcal{A}}^{\mathcal{B}}$ be the map obtained in Theorem 3.3 and let $\tau^{\mathcal{B}} : \omega \rightarrow Dom_{\mathcal{A}}^{\mathcal{B}} / \sim$ be the bijection obtained in the other direction of the same proof. $\tau^{\mathcal{B}}$ gives rise to an isomorphism $\mathcal{I}^F(\mathcal{B}) \rightarrow Dom_{\mathcal{A}}^{\mathcal{B}}$. We know that both can be enumerated uniformly from a given presentation of \mathcal{B} and so $(\tau^{\mathcal{B}})^{-1} \circ \mathfrak{F}^{\mathcal{B}}$ is also uniformly enumerable from \mathcal{B} . Thus there is some enumeration operator Λ such that for any presentation \mathcal{B}

$$\Lambda^{P(\mathcal{B})} = (\tau^{\mathcal{B}})^{-1} \circ \mathfrak{F}^{\mathcal{B}} : F(\mathcal{B}) \rightarrow \mathcal{I}^F(\mathcal{B})$$

To show Λ is an enumeration isomorphism we want to show the following diagram commutes for all $\tilde{\mathcal{B}}, \hat{\mathcal{B}} \in Iso(\mathcal{B})$ and all morphisms $h : \tilde{\mathcal{B}} \rightarrow \hat{\mathcal{B}}$. We extend h to a map $\tilde{\mathcal{B}}^{<\omega} \rightarrow \hat{\mathcal{B}}^{<\omega}$ and then restrict to $Dom_{\mathcal{A}}^{\tilde{\mathcal{B}}} \rightarrow Dom_{\mathcal{A}}^{\hat{\mathcal{B}}}$.

$$\begin{array}{ccccc}
 & & \Lambda^{P(\tilde{\mathcal{B}})} & & \\
 & \nearrow & & \searrow & \\
 F(\tilde{\mathcal{B}}) & \xrightarrow{\mathfrak{F}^{\tilde{\mathcal{B}}}} & Dom_{\mathcal{A}}^{\tilde{\mathcal{B}}} & \xleftarrow{\tau^{\tilde{\mathcal{B}}}} & \mathcal{I}^F(\tilde{\mathcal{B}}) \\
 F(h) \downarrow & & h \downarrow & & \downarrow \mathcal{I}^F(h) \\
 F(\hat{\mathcal{B}}) & \xrightarrow{\mathfrak{F}^{\hat{\mathcal{B}}}} & Dom_{\mathcal{A}}^{\hat{\mathcal{B}}} & \xleftarrow{\tau^{\hat{\mathcal{B}}}} & \mathcal{I}^F(\hat{\mathcal{B}}) \\
 & \searrow & & \nearrow & \\
 & & \Lambda^{P(\hat{\mathcal{B}})} & &
 \end{array}$$

The right hand square commutes since $I^F(h)$ is defined to be $\tau^{\hat{\mathcal{B}}} \circ h \circ (\tau^{\tilde{\mathcal{B}}})^{-1}$. To see that the left square commutes take $i \in F(\tilde{\mathcal{B}})$. Then $F(h)(i) = j$ for some $j \in F(\hat{\mathcal{B}})$ and $\mathfrak{F}^{\tilde{\mathcal{B}}}(i) = (\bar{a}, i)$, $\mathfrak{F}^{\hat{\mathcal{B}}}(j) = (\bar{b}, j)$ where \bar{a} and \bar{b} are initial segments of ω . We want to show that $h(\bar{a}, i) = (h(\bar{a}), i) \sim^{\hat{\mathcal{B}}} (\bar{b}, j)$.

Since $(i, j) \in \Psi_*^{P(\tilde{\mathcal{B}}) \oplus h \oplus P(\hat{\mathcal{B}})}$ we can get $(i, j) \in \Psi_{*, |\bar{a}| |\bar{b}|}^{P_{\tilde{\mathcal{B}}}(\bar{a}) \oplus h \upharpoonright |\bar{a}| \oplus P_{\hat{\mathcal{B}}}(\bar{b})}$ by extending \bar{a} and \bar{b} . Note that $P_{\hat{\mathcal{B}}}(\bar{a}) = P_{\hat{\mathcal{B}}}(h(\bar{a}))$ and assume without loss of generality that \bar{b} contains both \bar{a} and $h(\bar{a})$. Since \bar{b} is an initial segment, the map associated to it is the identity. So the map $\sigma = \bar{b}^{-1} \circ h(\bar{a}) \bar{b}'$ is an initial segment of $h \upharpoonright |\bar{a}|$. Hence

$$(i, j) \in \Psi_{*, |\bar{a}\bar{b}|}^{P_{\hat{\mathcal{B}}}(h(\bar{a})) \oplus \sigma \oplus P_{\hat{\mathcal{B}}}(\bar{b})} \quad \text{and} \quad (j, i) \in \Psi_{*, |\bar{a}\bar{b}|}^{P_{\hat{\mathcal{B}}}(\bar{b}) \oplus (\sigma)^{-1} \oplus P_{\hat{\mathcal{B}}}(h(\bar{a}))}$$

□

Clearly if we have a functor $F : Iso(\mathcal{A}) \rightarrow Iso(\mathcal{B})$, then every enumeration of \mathcal{A} computes an enumeration of \mathcal{B} . In order to preserve enumeration degree spectra of structures we need the relationship between the two isomorphism classes to be even stronger. In [5] positive enumerable bi-transformability was introduced and it was shown that two positive enumerable bi-transformable structures have the same enumeration degree spectra. The next definition is the same as positive enumerable bi-transformability. We chose to rename it, as we learned that the notion is not new, but rather an effectivization of the highly influential notion of an adjoint equivalence of categories in category theory.

Definition 3.6. An *enumeration adjoint equivalence* of categories \mathfrak{C} and \mathfrak{D} consists of a tuple $(F, G, \Lambda_{\mathfrak{C}}, \Lambda_{\mathfrak{D}})$ where $F : \mathfrak{C} \rightarrow \mathfrak{D}$ and $G : \mathfrak{D} \rightarrow \mathfrak{C}$ are positive enumerable functors, $\Lambda_{\mathfrak{C}}$ and $\Lambda_{\mathfrak{D}}$ witness enumeration isomorphisms between the compositions of $G \circ F$ and $Id_{\mathfrak{C}}$, respectively $F \circ G$ and $Id_{\mathfrak{D}}$, and the two isomorphisms are mapped to each other. I.e.,

$$F(\Lambda_{\mathfrak{C}}^{P(\mathcal{A})}) = \Lambda_{\mathfrak{D}}^{P(F(\mathcal{A}))} \quad \text{and} \quad G(\Lambda_{\mathfrak{D}}^{P(\mathcal{B})}) = \Lambda_{\mathfrak{C}}^{P(G(\mathcal{B}))}$$

for all $\mathcal{A} \in \mathfrak{C}$ and $\mathcal{B} \in \mathfrak{D}$. If there is an enumeration adjoint equivalence between $Iso(\mathcal{A})$ and $Iso(\mathcal{B})$ then we say that \mathcal{A} and \mathcal{B} are *enumeration adjoint*.

We will show that the following notion based on positive interpretability is equivalent to enumeration adjointness.

Definition 3.7. Two structures \mathcal{A} and \mathcal{B} are *positively bi-interpretable* if there are effective interpretations of one in the other such that the compositions

$$f_{\mathcal{B}}^{\mathcal{A}} \circ \hat{f}_{\mathcal{A}}^{\mathcal{B}} : Dom_{\mathcal{B}}^{Dom_{\mathcal{A}}^{\mathcal{B}}} \rightarrow \mathcal{B} \quad \text{and} \quad f_{\mathcal{A}}^{\mathcal{B}} \circ \hat{f}_{\mathcal{B}}^{\mathcal{A}} : Dom_{\mathcal{A}}^{Dom_{\mathcal{B}}^{\mathcal{A}}} \rightarrow \mathcal{A}$$

are uniformly ripe in \mathcal{B} and \mathcal{A} respectively. (Here the function $\hat{f}_{\mathcal{A}}^{\mathcal{B}} : (Dom_{\mathcal{A}}^{\mathcal{B}})^{<\omega} \rightarrow \mathcal{A}^{<\omega}$ is the canonical extension of $f_{\mathcal{A}}^{\mathcal{B}} : Dom_{\mathcal{A}}^{\mathcal{B}} \rightarrow \mathcal{A}$ mapping $Dom_{\mathcal{B}}^{Dom_{\mathcal{A}}^{\mathcal{B}}}$ to $Dom_{\mathcal{B}}^{\mathcal{A}}$)

Theorem 3.8. \mathcal{A} and \mathcal{B} are positively bi-interpretable if and only if they are enumeration adjoint.

Since the proof of Theorem 3.3 works in the enumeration setting, this proof will go through exactly as in [7] *mutatis mutandis*.

4. CONCLUSION

When we restrict ourselves to only the positive information of a structure, the notion of a r.i.p.e. relation is a natural analogue to r.i.c.e. relations. Whence Theorem 1.6 shows that Σ_1^p relations are the correct notion of formula to consider in the enumeration setting. We see further evidence for this in Section 2 when r.i.p.e. formulas are used to define the positive jump of a structure. Theorem 2.11 supports the claim that the positive jump is the proper enumeration jump for structures as it behaves well with the regular enumeration jump of sets.

The authors in [5] showed that when comparing classes of structures with respect to enumeration reducibility, positive enumerable functors are the correct effectivization of functors to consider as they preserve enumeration degree spectra. The equivalence given by Theorem 3.8 between enumeration adjointness and positive bi-interpretability thus justifies the choice of positive bi-interpretability as the enumeration analogue to bi-interpretability.

We strongly believe that r.i.p.e. relations are a valuable addition to the field of computable structure theory. Developing the idea of definability by positive formulas further a Lopez-Escobar theorem for positive infinitary formulas is proven in upcoming work by Bazhenov, Fokina, Rossegger, Soskova, Soskova and Vatev: The sets of structures definable by Σ_α^p formulas are precisely the Σ_α^0 sets in the Scott topology on the space of structures. This result as well as the results in this article show promising signs of being useful to answer questions in other areas, such as for instance algorithmic learning theory.

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