REGULAR SOLUTIONS OF CHEMOTAXIS-CONSUMPTION SYSTEMS INVOLVING TENSOR-VALUED SENSITIVITIES AND ROBIN TYPE BOUNDARY CONDITIONS

JAEWOOK AHN, KYUNGKEUN KANG, AND JIHOON LEE

Abstract. This paper deals with a parabolic-elliptic chemotaxis-consumption system with tensor-valued sensitivity $S(x,n,c)$ under no-flux boundary conditions for n and Robin-type boundary conditions for c. The global existence of bounded classical solutions is established in dimension two under general assumptions on tensor-valued sensitivity S. One of main steps is to show that $\nabla c(\cdot,t)$ becomes tiny in $L^2(B_r(x)\cap\Omega)$ for every $x\in\overline{\Omega}$ and t when r is sufficiently small, which seems to be of independent interest. On the other hand, in the case of scalar-valued sensitivity $S=\chi(x,n,c)\mathbb{I}$, there exists a bounded classical solution globally in time for two and higher dimensions provided the domain is a ball with radius R and all given data are radial. The result of the radial case covers scalar-valued sensitivity χ that can be singular at $c=0$.

1. Introduction

Chemotaxis-consumption systems are usually studied with scalar-valued chemotactic sensitivities where the chemotactic bacteria partially orient their movement along a gradient of a signal substance which they consume. However, according to recent modeling approaches, we do not necessarily have to assume that the chemotactic sensitivity is a scalar value. It has been suggested, based on the experimental findings [8, 14] (see also [33]), to use more general, tensor-valued and spatially inhomogeneous chemotactic sensitivity. [18, 32, 34]

Taking into account tensor-valued sensitivity, in this paper, we consider the parabolic-elliptic chemotaxis-consumption system

$$\begin{cases} n_t = \nabla\cdot(\nabla n - nS(x,n,c)\cdot\nabla c), & x\in\Omega,\ t>0,\\ 0=\Delta c - nc, & x\in\Omega,\ t>0, \end{cases} \tag{1.1}$$

in a bounded smooth domain $\Omega\subset\mathbb{R}^d$, $d\geq 2$, where the sensitivity $S(x,n,c)$ attains values in $\mathbb{R}^{d\times d}$. Here, the unknowns n and c denote the bacterial population density and the signal concentration, respectively. The boundary conditions posed will be

$$(\nabla n - nS(x,n,c)\cdot\nabla c)\cdot\nu = 0, \qquad \nabla c\cdot\nu = \gamma - c, \qquad x\in\partial\Omega,\ t>0, \tag{1.2}$$

where ν denotes the outward unit normal vector to $\partial\Omega$. We emphasize that the boundary condition for c is of Robin type.

Homogeneous Neumann boundary conditions for c have been often used in mathematical studies regarding (1.1) and its variants. [6, 15, 27, 29, 30] However, in the original version of (1.1) by Tuval et al.[21], certain non-trivial boundary conditions for c are proposed to

2010 *Mathematics Subject Classification.* 35K55, 35Q92, 92C17.

Key words and phrases. Chemotaxis-consumption system; regular solution; Robin-type boundary condition; tensor-valued sensitivity.

take into account the effect of oxygen c at the drop-air interface. Motivated by experimental observations in Tuval et al., [21] it has been suggested in [1] (see also [2]) to use non-homogeneous boundary conditions of the form

$$\nabla c \cdot \nu = (\gamma - c)g \quad \text{on} \quad \partial\Omega. \tag{1.3}$$

As seen in (1.2), we will impose (1.3) with $g \equiv 1$ but results in Theorem 1 are valid for more general g (see Remark 2).

We compare (1.1)–(1.2) to the chemotaxis-consumption system with homogeneous Neumann boundary conditions

$$n_t = \nabla \cdot (\nabla n - \chi(c) n \nabla c), \qquad c_t = \Delta c - nc, \qquad x \in \Omega,\ t > 0, \tag{1.4}$$

$$\nabla n \cdot \nu = \nabla c \cdot \nu = 0, \qquad x \in \partial\Omega,\ t > 0. \tag{1.5}$$

We remark that the c equation should be of parabolic type since an elliptic approximation of the c equation in (1.4)–(1.5) leads to $c \equiv 0$. It is known that solutions of (1.4)–(1.5) satisfy the energy-like inequality (see, e.g. [7, 9, 24, 25, 28])

$$\frac{d}{dt}\left(\int_\Omega n \log n + \frac{1}{2}\int_\Omega \chi(c)\frac{|\nabla c|^2}{c}\right) + \int_\Omega \frac{|\nabla n|^2}{n} + \frac{1}{4}\int_\Omega \frac{c}{\chi(c)}|D^2\rho(c)|^2 \le 0, \tag{1.6}$$

where $\rho(c) = \int_1^c \chi(s)/s\, ds$. The inequality (1.6) is typically deduced via a subtle cancellation caused by nonlinear structure of the system (1.4)–(1.5). In the case of the system (1.1)–(1.2), because of presence of tensor-valued sensitivity, it is not clear whether or not such energy like inequality can be derived due to loss of the cancellation effect. As a variant of (1.6), we refer to [3] for a chemotaxis-consumption-fluid system with constant sensitivity and Robin boundary condition.

As far as we know, there have been relatively few results dealing with tensor-valued sensitivity or Robin type boundary conditions. In particular, the only result in presence of both tensor-valued sensitivities and Robin type boundary conditions we are aware of is that bounded weak solutions to a 3D chemotaxis-Stokes system with nonlinear cell diffusion are known to exist globally in time.[20]

In presence of constant sensitivities and Robin type boundary conditions, smooth solutions to (1.1) are known to exist globally in time for general data and any dimension [10] (see also [3, 31]). However, in presence of tensor-valued sensitivities and Neumann boundary conditions, even when $d = 2$, smooth solutions to the fully parabolic counterpart of (1.1) have been found to exist globally in time only under a smallness assumption on c [15] (see also [6]), or under additional regularizing effects such as nonlinear diffusion enhancement at large densities. [5, 22, 23, 26] Without such additional effects, large data global existence results are so far available only for certain generalized weak solutions when $d \ge 1$ in [27] and when $d = 2$ in [29]. Recently, the eventual smoothness and stabilization of certain generalized solutions are also investigated when $d = 2$ by Winkler. [30]

The main motivation of the present work is to prove that smooth solutions to the two-dimensional chemotaxis-consumption system (1.1)–(1.2) exist globally in time for general tensor-valued sensitivity and arbitrary large initial data. As we mentioned earlier, it is unclear whether or not the energy-like inequality (1.6) can be derived in the case of the system (1.1)–(1.2). Instead, we derive a series of spatially localized estimates (see Proposition 1, Lemma 4, Lemma 5), which will lead to the uniform in time bound of $\int_\Omega n\log n$ (see Corollary 1). Especially in Proposition 1, which may be of independent interest, it is shown that for arbitrary small $\varepsilon>0$, we can find $r>0$, independent of $x\in\overline{\Omega}$ and $t<T_{\max}$, such that

$$\|\nabla c(\cdot,t)\|_{L^2(\Omega\cap B_r(x))}\le\varepsilon.$$

We also consider (1.1)–(1.2) with scalar-valued $S\equiv\chi\mathbb{I}_d$ for higher dimensions under the assumption of radial symmetry. It will turn out that all solutions emanating from bounded radial initial data remain globally bounded, even in the case χ becomes singular at $c=0$ (see Theorem 2).

Now, to formulate our main results, let us specify the precise problem setting. We use the notation $\mathbb{R}_+:=(0,\infty)$. On the tensor-valued sensitivity $S=(S_{ij})_{i,j\in\{1,\cdots,d\}}$, we will impose the conditions

$$\begin{aligned}&S_{ij}\in\mathcal{C}^2(\overline{\Omega}\times\overline{\mathbb{R}_+}\times\overline{\mathbb{R}_+})\quad\text{for all}\quad i,j\in\{1,\cdots,d\}\quad\text{and}\\&|S(x,r,s)|+|\partial_r S(x,r,s)|\le S_0(s)\quad\text{for}\quad(x,r,s)\in\overline{\Omega}\times\overline{\mathbb{R}_+}\times\overline{\mathbb{R}_+}\\&\text{with some } S_0\in\mathcal{C}(\overline{\mathbb{R}_+}).\end{aligned}\tag{1.7}$$

The boundary data γ and the initial condition $n(\cdot,0)=n_0$ are assumed to satisfy

$$\gamma\in\mathbb{R}_+,\qquad 0\le n_0\in L^\infty(\Omega).\tag{1.8}$$

Our first main result is the global existence of regular solutions to the system (1.1)–(1.2) in two dimensions.

Theorem 1. *Let $\Omega\subset\mathbb{R}^2$ be a bounded smooth domain. Then,* (1.1)–(1.2) *subject to* (1.7)–(1.8) *admits a unique non-negative solution (n,c) satisfying*

$$\begin{aligned}&n\in\bigcap_{p\in[1,\infty)}\mathcal{C}([0,\infty);L^p(\Omega))\cap\mathcal{C}^{2,1}(\overline{\Omega}\times(0,\infty))\cap L^\infty(0,\infty;L^\infty(\Omega)),\\&c\in\mathcal{C}^{2,0}(\overline{\Omega}\times(0,\infty))\cap L^\infty(0,\infty;W^{1,\infty}(\Omega)).\end{aligned}\tag{1.9}$$

Remark 1. *The results in Theorem 1 can be extended to higher dimensions $d\ge3$ provided $S(x,n,c)\equiv\chi(c)\mathbb{I}_d$, $\chi\in\mathcal{C}^2(\overline{\mathbb{R}_+})$, and $\chi,\chi'\ge0$. Indeed, a priori L^p-estimate shows that*

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|n^{\frac{p}{2}}\|^2_{L^2(\Omega)}+\frac{4(p-1)}{p^2}\|\nabla n^{\frac{p}{2}}\|^2_{L^2(\Omega)}\\&\quad=\frac{(p-1)}{p}\int_{\partial\Omega}\chi(c)n^p(\gamma-c)-\frac{(p-1)}{p}\int_\Omega n^p\chi'(c)|\nabla c|^2-\int_\Omega n^{p+1}\chi(c)c\\&\quad\le\frac{(p-1)}{p}\|\chi\|_{\mathcal{C}([0,\gamma])}\gamma\|n^{\frac{p}{2}}\|^2_{L^2(\partial\Omega)},\qquad p\ge1,\end{aligned}$$

where we used the non-negativities of χ and χ' in the last inequality. If we further use the trace and interpolation inequalities and a Moser-type iteration argument, then we can obtain a uniform-in-time bound for n (see [10] *for the case $\chi\equiv1$).*

Remark 2. *We remark that the results in Theorem 1 are still valid for more general Robin boundary conditions $\nabla c\cdot\nu=(\gamma-c)g$ with $0<g\in\mathcal{C}^{1+\theta}(\partial\Omega)$ for some $\theta\in(0,1)$. This can be verified by following the same methods of proof for Theorem 1 and thus, for simplicity, all computations are performed for the case $g\equiv1$.*

Remark 3. *In Theorem 1, applying the classical parabolic regularity theory*[13] *to the no-flux boundary problem* $n_t = \nabla \cdot (\nabla n - \vec{a})$ *for* $x \in \Omega$, $t > 0$ *with* $\vec{a} = nS \cdot \nabla c \in L^\infty(0, \infty; L^\infty(\Omega))$, *we can further have Hölder continuity of* n *up to* $t = 0$ *provided that* n_0 *is Hölder continuous. See, e.g.,* [19, Thm. 1.3].

Our second main result states that in the case of scalar-valued sensitivity, (1.1) has a global smooth solution in two and higher dimensions provided the domain is a ball and all given data are radial.

Theorem 2. *Let* $\Omega = B_R(0) \subset \mathbb{R}^d$, $d \geq 2$. *Assume that* (1.8) *holds and* n_0 *is radial. Then,* (1.1)–(1.2) *with the scalar sensitivity* $S(x, n, c) \equiv \chi(x, n, c)\mathbb{I}_d$, $0 \leq \chi \in \mathcal{C}^2(\overline{\Omega} \times \overline{\mathbb{R}_+} \times \mathbb{R}_+)$, *admits a unique non-negative solution* (n, c) *satisfying* (1.9) *provided that for* $x, y \in \overline{\Omega}$, $r \in \overline{\mathbb{R}_+}$, $s \in \mathbb{R}_+$

$$\begin{aligned} &\chi(x, r, s) = \chi(y, r, s) \quad \textit{if} \quad |x| = |y| \quad \textit{and} \\ &\chi(x, r, s) + |\partial_r \chi(x, r, s)| \leq \chi_0(s) \quad \textit{with some} \ \ \chi_0 \in \mathcal{C}(\mathbb{R}_+). \end{aligned}$$

Remark 4. *The proof of Theorem 2 mainly relies on the decay estimate of the cumulative mass distribution* Q *defined in* (4.1) *(see Lemma 7). This is crucially used to obtain the upper bound of* $|\nabla c|$ *and the lower bound of* c.

Remark 5. *We emphasize that unlike the case of Theorem 1, the sensitivity* $\chi(\cdot, \cdot, c)$ *in Theorem 2 may allow singularities at* $c = 0$, *for example,* $\chi(x, n, c) = 1/c$. *Although* χ *can be singular at* $c = 0$, *no singularity, however, occurs since signal concentration* c *is turned out to be bounded below, independent of time, away from zero (see Lemma 9).*

The outline is as follows: in Section 2, the local existence result is established; in Section 3 and Section 4, we prove Theorem 1 and Theorem 2, respectively. Throughout this paper, the surface area of $B_1(0)$ is denoted by σ_d.

2. Local existence

In this section, we prove a local existence result via the Banach fixed point theorem. Our local existence result reads as follows.

Lemma 1. *Let* $\Omega \subset \mathbb{R}^d$, $d \geq 2$, *be a bounded smooth domain. Then, there exists a maximal time of existence,* $T_{\max} \in (0, \infty]$, *such that for* $t < T_{\max}$, *a unique solution* (n, c) *of* (1.1)–(1.2) *subject to* (1.7)–(1.8) *exists and satisfies*

$$\begin{aligned} &n \in \bigcap_{p \in [1, \infty)} \mathcal{C}([0, t); L^p(\Omega)) \cap \mathcal{C}^{2,1}(\overline{\Omega} \times (0, t)) \cap L^\infty(0, t; L^\infty(\Omega)), \\ &c \in \mathcal{C}^{2,0}(\overline{\Omega} \times (0, t)), \end{aligned}$$

$$\int_\Omega n(\cdot, t) = \int_\Omega n_0, \qquad n(x, t) \geq 0, \qquad 0 < c(x, t) < \gamma \qquad \textit{for} \quad x \in \Omega.$$

Moreover, it holds that

$$\textit{either} \quad T_{\max} = \infty \quad \textit{or} \quad \limsup_{t \nearrow T_{\max}} \|n(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{2.1}$$

To obtain Lemma 1, we prepare the following elementary lemma.

Lemma 2. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded smooth domain, and let $p > d$. For any $u, f \in L^p(\Omega)$ with $u \geq 0$ and any constant $\eta \geq 0$, the problem*

$$\begin{cases} -\Delta v + uv = f, & x \in \Omega, \\ \nabla v \cdot \nu + v = \eta, & x \in \partial\Omega \end{cases} \tag{2.2}$$

admits a unique solution $v \in W^{2,p}(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ with the following properties:

(i) *There exists $C = (d, \Omega, p)$ such that*

$$\|v\|_{W^{2,p}(\Omega)} \leq C(\|uv\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \eta(|\partial\Omega|^{\frac{1}{2}} + |\partial\Omega|^{\frac{1}{p}})).$$

(ii) *If $\eta = 0$, then there exists $C = (d, \Omega, p, \|u\|_{L^p(\Omega)})$ such that*

$$\|v\|_{W^{2,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)}.$$

(iii) *If $f \equiv 0$, then $0 \leq v \leq \eta$.*

Proof. To obtain the existence, we let u be approximated in $L^p(\Omega)$ by a sequence of bounded non-negative functions u_l and let $v_l \in W^{2,p}(\Omega)$ be a unique solution of the problem

$$\begin{cases} -\Delta v_l + u_l v_l = f, & x \in \Omega, \\ \nabla v_l \cdot \nu + v_l = \eta, & x \in \partial\Omega, \end{cases} \tag{2.3}$$

which is uniquely solvable by [12, Thm. 2.4.2.6]. Note that the elliptic regularity theory [12, Thm. 2.3.3.6] gives that there exists $C > 0$ satisfying

$$\|v_l\|_{W^{2,p}(\Omega)} \leq C(\|u_l v_l\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \|v_l\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} + \eta|\partial\Omega|^{\frac{1}{p}}).$$

We also note that Hölder's and the Gagliardo–Nirenberg inequalities yield $C > 0$ such that

$$\|u_l v_l\|_{L^p(\Omega)} \leq C\|u_l\|_{L^p(\Omega)}\|v_l\|^{\theta_1}_{L^2(\Omega)}\|v_l\|^{1-\theta_1}_{W^{2,p}(\Omega)}, \qquad \theta_1 = \frac{\frac{2}{d} - \frac{1}{p}}{\frac{2}{d} - \frac{1}{p} + \frac{1}{2}} \in (0,1),$$

and by $W^{1,p}(\Omega) \hookrightarrow W^{1-\frac{1}{p},p}(\partial\Omega)$ and the Gagliardo–Nirenberg inequality, there exists $C > 0$ fulfilling

$$\|v_l\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \leq C\|v_l\|^{\theta_2}_{L^2(\Omega)}\|v_l\|^{1-\theta_2}_{W^{2,p}(\Omega)}, \qquad \theta_2 = \frac{\frac{1}{d}}{\frac{2}{d} - \frac{1}{p} + \frac{1}{2}} \in (0,1).$$

Combining above estimates, after applying Young's inequality, we have that with some $C > 0$,

$$\|v_l\|_{W^{2,p}(\Omega)} \leq C(\|u_l\|^{\frac{1}{\theta_1}}_{L^p(\Omega)}\|v_l\|_{L^2(\Omega)} + \|f\|_{L^p(\Omega)} + \|v_l\|_{L^2(\Omega)} + \eta|\partial\Omega|^{\frac{1}{p}}). \tag{2.4}$$

Now, we multiply the v_l equation by v_l, integrate over Ω, and use integration by parts, Hölder's inequality and $H^1(\Omega) \hookrightarrow L^{\frac{p}{p-1}}(\Omega)$ to find $C > 0$ such that

$$\begin{aligned} \int_\Omega |\nabla v_l|^2 + \int_{\partial\Omega} |v_l|^2 + \int_\Omega u_l v_l^2 &= \int_\Omega f v_l + \int_{\partial\Omega} \eta v_l \\ &\leq C(\|f\|_{L^p(\Omega)}\|v_l\|_{H^1(\Omega)}) + \int_{\partial\Omega} \eta v_l. \end{aligned}$$

Then, the Poincaré inequality with trace term (see e.g. [4]) and Young's inequality yield $C > 0$ satisfying

$$\|v_l\|_{H^1(\Omega)} \leq C(\|f\|_{L^p(\Omega)} + \eta|\partial\Omega|^{\frac{1}{2}}). \tag{2.5}$$

In view of (2.4)–(2.5), v_l is bounded in $W^{2,p}(\Omega)$. The compactness of $W^{2,p}(\Omega) \hookrightarrow \mathcal{C}^1(\overline{\Omega})$ and the weak compactness of bounded sets in $W^{2,p}(\Omega)$ allow us to extract a subsequence v_{l_j} converging in $\mathcal{C}^1(\overline{\Omega})$ that converges weakly in $W^{2,p}(\Omega)$. If we take the limit in the problem for v_{l_j}, then its limit v is in $W^{2,p}(\Omega)\cap\mathcal{C}^1(\overline{\Omega})$ and satisfies (2.2). This concludes the existence result.

To obtain the uniqueness, we let v and $\tilde{v}$ be two solutions. Since a simple integration by parts gives

$$\int_\Omega |\nabla(v-\tilde{v})|^2 + \int_{\partial\Omega} |v-\tilde{v}|^2 + \int_\Omega u|v-\tilde{v}|^2 = 0,$$

we have $v \equiv \tilde{v}$.

Repeating similar computations as above, we can find $C = C(d, \Omega, p)$ such that

$$\|v\|_{W^{2,p}(\Omega)} \leq C\big(\|uv\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \eta(|\partial\Omega|^{\frac{1}{2}} + |\partial\Omega|^{\frac{1}{p}})\big),$$

which yields (i).

Repeating similar computations as (2.4) and (2.5), since $\eta = 0$, there exists $C = C(d, \Omega, p)$ such that

$$\|v\|_{W^{2,p}(\Omega)} \leq C(\|u\|_{L^p(\Omega)}^{\frac{1}{\theta_1}} + 1)\|f\|_{L^p(\Omega)}.$$

This concludes (ii).

To obtain (iii), we multiply the v equation by $v_- := -\min\{0, v\}$, integrate over Ω, and use integration by parts. Then, we have

$$\int_\Omega |\nabla v_-|^2 + \int_\Omega u|v_-|^2 + \int_{\partial\Omega} |v_-|^2 = -\eta\int_{\partial\Omega} v_-.$$

Since the right-hand-side is non-positive, $v_- \equiv 0$, namely, $v \geq 0$. Using the same argument, we can deduce $v \leq \eta$. □

We are now ready to prove Lemma 1.

Proof of Lemma 1. We fix $p > d+2$ and let $M := 2\|n_0\|_{L^p(\Omega)} + 1$. With a positive number $T < 1$ to be specified below, we introduce the Banach space

$$X_T := \{f \in \mathcal{C}([0,T]; L^p(\Omega)) \mid \|f\|_{L^\infty(0,T;L^p(\Omega))} \leq M, \;\; f \geq 0 \text{ for } t \leq T\}.$$

For any given $\tilde{n} \in X_T$, we note from Lemma 1 that the problem

$$\begin{cases} 0 = \Delta c - \tilde{n}c, & x \in \Omega, \\ \nabla c \cdot \nu = \gamma - c, & x \in \partial\Omega, \end{cases}$$

admits a unique solution $c(\cdot, t) \in W^{2,p}(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ for $t \leq T$ such that $0 \leq c \leq \gamma$. We also note, using Lemma 1 (ii), $W^{2,p}(\Omega) \hookrightarrow \mathcal{C}^1(\overline{\Omega})$, and Lemma 1 (iv), that there exists $C = C(d, \Omega, p) > 0$ satisfying

$$\|c\|_{L^\infty(0,T;\mathcal{C}^1(\overline{\Omega}))} \leq C\gamma\|\tilde{n}\|_{L^\infty(0,T;L^p(\Omega))}. \tag{2.6}$$

With such $c = c(\tilde{n})$, according to [13, III. Thm. 5.1], the linear problem

$$\begin{cases} n_t = \nabla \cdot (\nabla n - nS(x,\tilde{n},c)\cdot \nabla c), & x\in\Omega,\ t>0,\\ (\nabla n - nS(x,\tilde{n},c)\cdot\nabla c)\cdot\nu = 0, & x\in\partial\Omega,\ t>0,\\ n(x,0)=n_0(x), & x\in\Omega \end{cases}$$

has a unique weak solution $n \in \mathcal{C}([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$. If we use the weak formulation with the test function $n_- := -\min\{0,n\}$, then after using (1.7) and Young's and Hölder's inequalities, we have

$$\begin{aligned} \int_\Omega |n_-(\cdot,t)|^2 &\le -2\int_0^t\int_\Omega |\nabla n_-|^2 + 2\int_0^t\int_\Omega |n_-||S_0(c)||\nabla n_-||\nabla c| \\ &\le \frac{1}{2}\|S_0\|^2_{\mathcal{C}([0,\gamma])}\|\nabla c\|^2_{L^\infty(0,T;L^\infty(\Omega))}\int_0^t\int_\Omega |n_-|^2 \quad \text{for} \quad t\le T \end{aligned}$$

and $n \ge 0$ follows by Grönwall's inequality. Moreover, from [17, Thm. VI. 6.40],

$$n \in L^\infty(0,T;L^\infty(\Omega)),$$

and by similar computations as above, we have

$$\begin{aligned} &\int_\Omega n^p(\cdot,t) \\ &\quad\le \int_\Omega n_0^p + \frac{p(p-1)}{4}\|S_0\|^2_{\mathcal{C}([0,\gamma])}\|\nabla c\|^2_{L^\infty(0,T;L^\infty(\Omega))}\int_0^t\int_\Omega n^p \quad \text{for} \quad t\le T. \end{aligned}$$

Using Grönwall's inequality and taking supremum over the time interval, it follows that

$$\|n\|_{L^\infty(0,T;L^p(\Omega))} \le \|n_0\|_{L^p(\Omega)}\exp\left(\frac{(p-1)}{4}\|S_0\|^2_{\mathcal{C}([0,\gamma])}\|\nabla c\|^2_{L^\infty(0,T;L^\infty(\Omega))}T\right).$$

Therefore, if we use (2.6) and take a sufficiently small T, then the mapping Φ given by $\Phi(\tilde{n}) := n$ maps X_T into itself.

Next, for given $\tilde{n}_1, \tilde{n}_2 \in X_T$, we denote $n_i = \Phi(\tilde{n}_i)$, $c_i = c(\tilde{n}_i)$ for $i = 1,2$, and $\delta f = f_1 - f_2$. Note that for any $t \le T$ and $\xi \in L^2(0,T;W^{1,2}(\Omega))$ with $\xi_t \in L^2(0,T;L^2(\Omega))$, we have

$$\begin{aligned} &\int_\Omega \delta n\xi(\cdot,t) - \int_0^t\int_\Omega \delta n\xi_t + \int_0^t\int_\Omega \nabla\delta n\cdot\nabla\xi \\ &\quad= \int_0^t\int_\Omega (\delta n S(x,\tilde{n}_1,c_1)\cdot\nabla c_1 + n_2 Z\cdot\nabla c_1 + n_2 S(x,\tilde{n}_2,c_2)\cdot\nabla\delta c)\cdot\nabla\xi, \end{aligned}$$

where

$$Z = S(x,\tilde{n}_1,c_1) - S(x,\tilde{n}_2,c_1) + S(x,\tilde{n}_2,c_1) - S(x,\tilde{n}_2,c_2).$$

Note also that by the mean value theorem, (1.7) and $c \le \gamma$, there exists $C > 0$ satisfying

$$|Z| \le C(|\delta\tilde{n}| + |\delta c|) \quad \text{a.e.} \quad \text{in } \Omega\times(0,T).$$

Along with this, if we use the above weak formulation with the test function $|\delta n|^{p-2}\delta n$, (1.7), (2.6), and $c \le \gamma$, then with some $C_1 = C_1(d,\Omega,p,M) > 0$, we have

$$
\begin{aligned}
&\frac{1}{p}\int_\Omega |\delta n(\cdot,t)|^p + \frac{4(p-1)}{p^2}\int_0^t\int_\Omega |\nabla|\delta n|^{\frac{p}{2}}|^2 \\
&\le C_1\Big(\int_0^t\int_\Omega |\nabla|\delta n|^{\frac{p}{2}}||\delta n|^{\frac{p}{2}} + \int_0^t\int_\Omega |\nabla|\delta n|^{\frac{p}{2}}||\delta n|^{\frac{p}{2}-1} n_2|\delta\tilde{n}| \\
&\qquad + \int_0^t\int_\Omega |\nabla|\delta n|^{\frac{p}{2}}||\delta n|^{\frac{p}{2}-1} n_2(|\delta c| + |\nabla\delta c|)\Big).
\end{aligned} \tag{2.7}
$$

We apply Young's inequality to the first term on the right-hand-side above to find $C > 0$ satisfying

$$
\int_0^t\int_\Omega |\nabla|\delta n|^{\frac{p}{2}}||\delta n|^{\frac{p}{2}} \le \frac{p-1}{C_1p^2}\int_0^t\int_\Omega |\nabla|\delta n|^{\frac{p}{2}}|^2 + C\int_0^t\int_\Omega |\delta n|^p.
$$

Similarly, applying Young's inequality to the rightmost term, after using $W^{2,p}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ and $n_2 \in X_T$, we observe that with some $C > 0$,

$$
\begin{aligned}
&\int_0^t\int_\Omega \Big(|\nabla|\delta n|^{\frac{p}{2}}||\delta n|^{\frac{p}{2}-1} n_2(|\delta c| + |\nabla\delta c|)\Big) \\
&\le \frac{p-1}{C_1p^2}\int_0^t\int_\Omega |\nabla|\delta n|^{\frac{p}{2}}|^2 + C\left(\int_0^t\int_\Omega |\delta n|^p + \int_0^t \|\delta c\|^p_{W^{2,p}(\Omega)}\right).
\end{aligned}
$$

It remains to estimate the second term on the right-hand-side of (2.7). Note that, due to our choice of p, Hölder's and the Gagliardo-Nirenberg inequalities yield $C > 0$ satisfying

$$
\begin{aligned}
&\int_0^t\int_\Omega |\nabla|\delta n|^{\frac{p}{2}}||\delta n|^{\frac{p}{2}-1} n_2|\delta\tilde{n}| \\
&\le C\int_0^t\Big(\|\nabla|\delta n|^{\frac{p}{2}}\|^{1+\frac{d}{p}}_{L^2(\Omega)}\|\delta n\|^{\frac{p-d-2}{2}}_{L^p(\Omega)} + \|\nabla|\delta n|^{\frac{p}{2}}\|_{L^2(\Omega)}\|\delta n\|^{\frac{p-2}{2}}_{L^p(\Omega)}\Big)\|n_2\|_{L^p(\Omega)}\|\delta\tilde{n}\|_{L^p(\Omega)}.
\end{aligned}
$$

Thus, using Young's inequality and $n_2 \in X_T$, we can find $C > 0$ fulfilling

$$
\begin{aligned}
&\int_0^t\int_\Omega \Big(|\nabla|\delta n|^{\frac{p}{2}}||\delta n|^{\frac{p}{2}-1} n_2|\delta\tilde{n}|\Big) \\
&\le \frac{p-1}{C_1p^2}\int_0^t\int_\Omega |\nabla|\delta n|^{\frac{p}{2}}|^2 + C\left(\int_0^t\int_\Omega |\delta n|^p + \int_0^t \|\delta\tilde{n}\|^p_{L^p(\Omega)}\right).
\end{aligned}
$$

Combining the above computations, we have that with some $C > 0$,

$$
\int_\Omega |\delta n(\cdot,t)|^p \le C\left(\int_0^t\int_\Omega |\delta n|^p + \int_0^t (\|\delta c\|^p_{W^{2,p}(\Omega)} + \|\delta\tilde{n}\|^p_{L^p(\Omega)})\right).
$$

Since applying Lemma 2 (ii) to the problem for δc,

$$
\begin{cases}
-\Delta\delta c + \tilde{n}_1\delta c = -c_2\delta\tilde{n}, & x \in \Omega, \\
\nabla\delta c\cdot\nu + \delta c = 0, & x \in \partial\Omega,
\end{cases}
$$

and using $c_2 \le \gamma$ yields $C > 0$ such that

$$
\|\delta c\|_{L^\infty(0,T;W^{2,p}(\Omega))} \le C\gamma\|\delta\tilde{n}\|_{L^\infty(0,T;L^p(\Omega))},
$$

by Grönwall's inequality, it follows that with some $C = C(d, \Omega, p, M) > 0$,

$$\|\delta n\|_{L^\infty(0,T;L^p(\Omega))} \le CT^{\frac{1}{p}} \exp(CT)\|\delta\tilde{n}\|_{L^\infty(0,T;L^p(\Omega))}.$$

Hence, for a sufficiently small choice of T, the mapping Φ becomes contraction on X_T, and by the Banach fixed point theorem, we have a unique fixed point $n = \Phi(n)$.

Next, we consider more regularity properties of solutions. Since n belongs to $\mathcal{C}([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)) \cap L^\infty(0,T; L^\infty(\Omega))$ and satisfies for every $[t_1, t_2] \subset (0,T]$ and $\xi \in W^{1,2}_{\mathrm{loc}}(0,T; L^2(\Omega)) \cap L^2_{\mathrm{loc}}(0,T; W^{1,2}(\Omega))$,

$$\int_\Omega n\xi(\cdot, t_2) - \int_{t_1}^{t_2}\int_\Omega n\xi_t + \int_{t_1}^{t_2}\int_\Omega (\nabla n - nS(x,n,c)\cdot\nabla c)\cdot\nabla\xi = \int_\Omega n\xi(\cdot, t_1),$$

for any $\eta \in (0,T)$ we have $n \in \mathcal{C}^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[\eta,T])$ with some $\theta \in (0,1)$ by the classical parabolic regularity theory[13] (see, e.g., [19, Thm. 1.3]). Then, $c(\cdot,t) \in \mathcal{C}^{2,\theta}(\overline{\Omega})$ for $t \in [\eta, T]$ by the elliptic regularity theory [[16, Cor. 4.41], and moreover, since we have for $[s,t] \subset [\eta, T]$

$$\begin{cases} -\Delta(c(t)-c(s)) + n(t)(c(t)-c(s)) = -c(s)(n(t)-n(s)), & x \in \Omega, \\ \nabla(c(t)-c(s))\cdot\nu + (c(t)-c(s)) = 0, & x \in \partial\Omega, \end{cases}$$

it follows by [[16, Thm. 2.26] (see also [10, Lem. 2.4]) that with some $C > 0$,

$$\|c(t)-c(s)\|_{\mathcal{C}^{2,\theta}(\overline{\Omega})} \le C\|n(t)-n(s)\|_{\mathcal{C}^{\theta}(\overline{\Omega})} \quad \text{for} \quad \eta \le s \le t \le T.$$

This yields Hölder regularity on the time variable, $c \in \mathcal{C}^{2+\theta,\frac{\theta}{2}}(\overline{\Omega}\times[\eta,T])$, and by the standard parabolic regularity theory, $n \in \mathcal{C}^{2,1}(\overline{\Omega}\times[\eta,T])$. Since $\eta \in (0,T)$ is arbitrary, we have the desired regularity result. Note that the blow-up criteria (2.1) follows by the standard extension argument, the mass conservation property of n is a consequence of integrating the n equation, and $0 < c < \gamma$ is the result of the elliptic maximum principle. □

Remark 6. *We remark that Lemma 1 provides local existence of the solutions in Theorem 1. Moreover, Lemma 1 can be also used to obtain Theorem 2 since in radial case, a priori estimate shows that there exists $c_* > 0$ such that $c \ge c_*$ independent of any regularization of χ keeping non-negative sign, local existence of the solutions in Theorem 2 is also available because singularity of χ at $c = 0$ does not play any role. Since its verification is admissible, the details are omitted.*

3. Case of tensor sensitivity in two dimensions

In this section, we prove Theorem 1 via a series of spatially localized estimates. To this end, we first establish a uniform-in-time smallness of spatially localized L^2-norm of ∇c in the following proposition. We remark that ∇c has a uniform-in-time L^2-norm over Ω from

$$\begin{aligned} \int_\Omega |\nabla c|^2 &\le \int_\Omega |\nabla c|^2 + \int_\Omega nc^2 + \frac{1}{2}\int_{\partial\Omega} c^2 \\ &= \int_{\partial\Omega}\gamma c - \frac{1}{2}\int_{\partial\Omega} c^2 \\ &\le \frac{1}{2}\gamma^2|\partial\Omega|. \end{aligned} \tag{3.1}$$

This bound implies that L^2 norm of ∇c becomes very small in a small neighborhood of each point, but it may not be uniformly small in time. In the next proposition, we prove that it is the case, namely localized norm of ∇c can be uniformly small independent of time.

Proposition 1. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded smooth domain. Let (n, c) be a solution given by Lemma 1. For any given $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ independent of $q \in \overline{\Omega}$ such that*

$$\sup_{t<T_{\max}} \|\nabla c(\cdot,t)\|_{L^2(\Omega\cap B_\delta(q))} \leq \varepsilon \quad \textit{for} \quad \delta \in (0,\delta_\varepsilon).$$

Proof. Let $\eta \in (0, e^{-1})$ and $B_\eta(0) = \{x \in \mathbb{R}^d \,|\, |x| < \eta\}$. We introduce the non-negative radial function

$$\psi_\eta(x) := \begin{cases} \ln(-\ln|x|) - \ln(-\ln\eta), & x \in B_\eta(0)\setminus\{0\}, \\ 0, & \text{otherwise}, \end{cases}$$

and recall that the surface area of $B_1(0)$ is denoted by σ_d. Direct computations show that

$$\begin{aligned} \|\psi_\eta\|^2_{L^2(\mathbb{R}^d)} &= \sigma_d \int_0^\eta |\ln(-\ln r) - \ln(-\ln\eta)|^2 r^{d-1}\,dr \\ &\leq \sigma_d \int_0^\eta |\ln(-\ln r)|^2 r^{d-1} dr \\ &= \sigma_d \int_{\ln\frac{1}{\eta}}^\infty |\ln\rho|^2 e^{-d\rho} d\rho \end{aligned}$$

and

$$\begin{aligned} \|\nabla\psi_\eta\|^2_{L^2(\mathbb{R}^d)} &= \sigma_d \int_0^\eta \frac{1}{|r\ln r|^2} r^{d-1} dr \\ &= \sigma_d \int_{\ln\frac{1}{\eta}}^\infty \frac{1}{\rho^2} e^{-(d-2)\rho} d\rho. \end{aligned}$$

Since the right hand sides above are both finite, $\psi_\eta \in H^1(\mathbb{R}^d)$. Moreover, since $\ln\frac{1}{\eta}$ tends to ∞ as η approach 0,

$$\|\psi_\eta\|_{H^1(\mathbb{R}^d)} \to 0 \quad \text{as} \quad \eta \to 0. \tag{3.2}$$

Fix $q \in \overline{\Omega}$, and we denote $\psi(x) = \psi_\eta(x-q)$ and $B_\eta = B_\eta(q)$. If we test the c equation of (1.1) with $c\psi^2$ and integrate over Ω, then integration by parts gives, due to $\psi = 0$ in $(B_\eta)^c$, that

$$\int_{\Omega\cap B_\eta} nc^2\psi^2 + \int_{\Omega\cap B_\eta} |\nabla c|^2\psi^2 = \int_{\partial\Omega} \nabla c\cdot\nu c\psi^2 - 2\int_{\Omega\cap B_\eta} \nabla c\cdot\nabla\psi c\psi.$$

Using Young's inequality and $c \leq \gamma$, we compute the rightmost term as

$$\left| -2\int_{\Omega\cap B_\eta} \nabla c\cdot\nabla\psi c\psi \right| \leq \frac{1}{2}\int_{\Omega\cap B_\eta} |\nabla c|^2\psi^2 + 2\gamma^2 \int_{\Omega\cap B_\eta} |\nabla\psi|^2.$$

Next, to control the boundary term, we consider two cases. If $B_\eta \subset \Omega$, then $\psi = 0$ on $\partial\Omega$ and thus,

$$\int_{\partial\Omega} \nabla c\cdot\nu c\psi^2 = 0.$$

Otherwise, if $B_\eta \not\subset \Omega$, then since $\psi = 0$ in $(B_\eta)^c$, we have

$$\int_{\partial\Omega} \nabla c \cdot \nu c \psi^2 = \int_{\partial\Omega \cap B_\eta} \nabla c \cdot \nu c \psi^2.$$

Thus, using the boundary condition and $c \leq \gamma$, we can compute

$$\begin{aligned}\left| \int_{\partial\Omega \cap B_\eta} \nabla c \cdot \nu c \psi^2 \right| &= \left| \int_{\partial\Omega \cap B_\eta} (\gamma - c) c \psi^2 \right| \\ &\leq \gamma^2 \int_{\partial\Omega \cap B_\eta} \psi^2 \\ &\leq \gamma^2 \int_{\partial\Omega} \psi^2.\end{aligned}$$

Combining the above estimates, after using $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$, we have that with some $C > 0$ independent of η,

$$\int_{\Omega \cap B_\eta} |\nabla c|^2 \psi^2 \leq C \|\psi\|^2_{H^1(\mathbb{R}^d)}.$$

In view of (3.2), there exists sufficiently small $\eta_0 > 0$ such that the right-hand-side above is less than or equal to ε^2 for $\eta < \eta_0$. Moreover, since there exists $\delta_0 > 0$ satisfying

$$\psi^2 \geq 1 \quad \text{a.e.} \quad \text{in } B_\delta \quad \text{for} \quad \delta \in (0, \delta_0),$$

we can deduce the desired result. □

For further local-in-space estimates, we introduce a smooth cut-off function and its properties (see, e.g. [11]):

Lemma 3. *Let $\delta > 0$. There is a radially decreasing function $\varphi_\delta \in C_0^\infty(\mathbb{R}^d)$ satisfying*

$$\varphi_\delta(x) = \begin{cases} 1, & x \in B_{\frac{\delta}{2}}(0), \\ 0, & x \in \mathbb{R}^d \setminus B_\delta(0), \end{cases}$$

$$0 \leq \varphi_\delta \leq 1 \quad in \quad \mathbb{R}^d,$$

and

$$|\nabla \varphi_\delta| \leq K \varphi_\delta^{\frac{1}{2}} \quad in \quad \mathbb{R}^d,$$

where K is a positive constant of order $\mathcal{O}(\delta^{-1})$.

We now prepare the following lemma which is used to prove Lemma 5. For computational simplicity, we use φ^3 as a test function.

Lemma 4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain. Let (n, c) be a solution given by Lemma 1. Assume that $\delta > 0$ and φ_δ is the function introduced in Lemma 3. Denote $\varphi(x) = \varphi_\delta(x - q)$ and $B_\delta = B_\delta(q)$ for $q \in \overline{\Omega}$. Then, there exist two positive constants C_2 and C_3 independent of δ and q such that*

$$\int_\Omega n^2 \varphi^3 \leq C_2 \left(\int_\Omega \frac{|\nabla n|^2}{n} \varphi^3 + \|\varphi^{\frac{3}{2}}\|^2_{W^{1,\infty}(\mathbb{R}^2)} \right), \tag{3.3}$$

$$\int_\Omega |\nabla c|^4\varphi^3$$

$$\le C_3\|\nabla c\|_{L^2(\Omega\cap B_\delta)}\left(\int_\Omega \frac{|\nabla n|^2}{n}\varphi^3 + \|\varphi^{\frac{3}{2}}\|^2_{W^{1,\infty}(\mathbb{R}^2)} + \|\varphi\|^3_{W^{2,\frac{3}{2}}(\mathbb{R}^2)}\right). \tag{3.4}$$

Proof. Since the Sobolev inequality yields $C > 0$ such that

$$\int_\Omega n^2\varphi^3 \le C\left(\|\nabla(n\varphi^{\frac{3}{2}})\|^2_{L^1(\Omega)} + \|n\varphi^{\frac{3}{2}}\|^2_{L^1(\Omega)}\right),$$

after using Hölder's inequality and $\int_\Omega n = \int_\Omega n_0$, we can find $C > 0$, independent of δ and q, satisfying

$$\int_\Omega n^2\varphi^3 \le C\left(\|n_0\|_{L^1(\Omega)}\int_\Omega \frac{|\nabla n|^2}{n}\varphi^3 + \|n_0\|^2_{L^1(\Omega)}\|\varphi^{\frac{3}{2}}\|^2_{W^{1,\infty}(\Omega)}\right).$$

This gives (3.3).

Next, using the Hölder inequality, direct computations, $(a+b)^3 \le 4(a^3+b^3)$ for $a, b \ge 0$, and $c \le \gamma$, we note that

$$\begin{aligned}\int_\Omega |\nabla c|^4\varphi^3 &\le \|\nabla c\|_{L^2(\Omega\cap B_\delta)}\left(\int_\Omega |\nabla c|^6\varphi^6\right)^{\frac{1}{2}}\\ &= \|\nabla c\|_{L^2(\Omega\cap B_\delta)}\|\nabla(c\varphi) - c\nabla\varphi\|^3_{L^6(\Omega)}\\ &\le \|\nabla c\|_{L^2(\Omega\cap B_\delta)}(\|c\varphi\|_{W^{1,6}(\Omega)} + \|c\nabla\varphi\|_{L^6(\Omega)})^3\\ &\le 4\|\nabla c\|_{L^2(\Omega\cap B_\delta)}(\|c\varphi\|^3_{W^{1,6}(\Omega)} + \gamma^3\|\nabla\varphi\|^3_{L^6(\Omega)}).\end{aligned} \tag{3.5}$$

Since

$$\nabla(c\varphi)\cdot\nu = \nabla c\cdot\nu\varphi + c\nabla\varphi\cdot\nu = (\gamma - c)\varphi + c\nabla\varphi\cdot\nu \quad \text{ on } \partial\Omega,$$

using $W^{2,\frac{3}{2}}(\Omega) \hookrightarrow W^{1,6}(\Omega)$ and the elliptic regularity theory [12, Thm. 2.3.3.6], we can find $C > 0$, independent of δ and q, such that

$$\begin{aligned}&\|c\varphi\|_{W^{1,6}(\Omega)}\\ &\le C(\|\Delta(c\varphi)\|_{L^{\frac{3}{2}}(\Omega)} + \|c\varphi\|_{L^{\frac{3}{2}}(\Omega)} + \|(\gamma - c)\varphi\|_{W^{\frac{1}{3},\frac{3}{2}}(\partial\Omega)} + \|c\nabla\varphi\cdot\nu\|_{W^{\frac{1}{3},\frac{3}{2}}(\partial\Omega)}).\end{aligned}$$

Using direct computations, Hölder's inequality, and $c \le \gamma$, we compute the first term on the right-hand-side above as

$$\begin{aligned}\|\Delta(c\varphi)\|_{L^{\frac{3}{2}}(\Omega)} &\le \|\Delta c\varphi\|_{L^{\frac{3}{2}}(\Omega)} + 2\|\nabla c\cdot\nabla\varphi\|_{L^{\frac{3}{2}}(\Omega)} + \|c\Delta\varphi\|_{L^{\frac{3}{2}}(\Omega)}\\ &\le \|nc\varphi\|_{L^{\frac{3}{2}}(\Omega)} + 2\|\nabla c\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^6(\Omega)} + \gamma\|\Delta\varphi\|_{L^{\frac{3}{2}}(\Omega)}.\end{aligned}$$

Since the trace inequality and the smoothness of Ω yield $C > 0$ satisfying

$$\|(\gamma - c)\varphi\|_{W^{\frac{1}{3},\frac{3}{2}}(\partial\Omega)} \le C\|(\gamma - c)\varphi\|_{W^{1,\frac{3}{2}}(\Omega)},$$

and

$$\|c\nabla\varphi\cdot\nu\|_{W^{\frac{1}{3},\frac{3}{2}}(\partial\Omega)} \le C\|c\nabla\varphi\|_{W^{1,\frac{3}{2}}(\Omega)},$$

using $c \le \gamma$ and Hölder's inequality, we have that with some $C > 0$, independent of δ and q,

$$\|(\gamma-c)\varphi\|_{W^{\frac{1}{3},\frac{3}{2}}(\partial\Omega)}+\|c\nabla\varphi\cdot\nu\|_{W^{\frac{1}{3},\frac{3}{2}}(\partial\Omega)} \le C(\gamma\|\varphi\|_{W^{2,\frac{3}{2}}(\Omega)}+\|\nabla c\|_{L^2(\Omega)}\|\varphi\|_{W^{1,6}(\Omega)}).$$

Note that by repeating the computations used to derive (2.5), we can find $C>0$ such that

$$\|c\|_{H^1(\Omega)} \le C\gamma|\partial\Omega|^{\frac{1}{2}}.$$

Combining above estimates gives, after using $c\le\gamma$, that with some $C>0$, independent of δ and q,

$$\|c\varphi\|_{W^{1,6}(\Omega)} \le C(\gamma\|n\varphi\|_{L^{\frac{3}{2}}(\Omega)}+\gamma|\partial\Omega|^{\frac{1}{2}}\|\varphi\|_{W^{1,6}(\Omega)}+\gamma\|\varphi\|_{W^{2,\frac{3}{2}}(\Omega)}).$$

Plugging it into (3.5), since Hölder's inequality and $\int_\Omega n=\int_\Omega n_0$ imply

$$\|n\varphi\|_{L^{\frac{3}{2}}(\Omega)} \le \|n_0\|_{L^1(\Omega)}^{\frac{1}{3}}\left(\int_\Omega n^2\varphi^3\right)^{\frac{1}{3}}, \tag{3.6}$$

it follows that there exists $C>0$, independent of δ and q, satisfying

$$\int_\Omega|\nabla c|^4\varphi^3 \le C\|\nabla c\|_{L^2(\Omega\cap B_\delta)}\left(\int_\Omega n^2\varphi^3+\|\varphi\|^3_{W^{1,6}(\Omega)}+\|\varphi\|^3_{W^{2,\frac{3}{2}}(\Omega)}\right).$$

Therefore, by (3.3) and $W^{2,\frac{3}{2}}(\Omega)\hookrightarrow W^{1,6}(\Omega)$, we can conclude (3.4). □

The spatially localized $L\log L$-norm of n is bounded uniformly in time:

Lemma 5. *Let $\Omega\subset\mathbb{R}^2$ be a bounded smooth domain. Let (n,c) be a solution given by Lemma 1. Assume that $\delta>0$ and φ_δ is the function introduced in Lemma 3. Denote $\varphi(x)=\varphi_\delta(x-q)$ and $B_\delta=B_\delta(q)$ for $q\in\overline{\Omega}$. Then, there exist $\delta_*>0$ independent of q such that if $\delta<\delta_*$, then there exists $C=C(\delta)>0$ independent of q satisfying*

$$\sup_{t<T_{\max}}\int_\Omega n\log n(\cdot,t)\varphi^3 \le C.$$

Proof. We begin by noting that due to Proposition 1, there exists $\delta_*>0$ independent of $q\in\overline{\Omega}$ such that

$$\sup_{t<T_{\max}}(\|S_0\|_{\mathcal{C}([0,\gamma])}C_2^{\frac{1}{4}}C_3^{\frac{1}{4}}\|\nabla c\|^{\frac{1}{4}}_{L^2(\Omega\cap B_\delta)}+\frac{1}{4}C_3\|\nabla c\|_{L^2(\Omega\cap B_\delta)}) \le \frac{1}{5} \quad \text{for} \quad \delta<\delta_*, \tag{3.7}$$

where S_0 is the function given in (1.7), and C_2 and C_3 are the positive numbers given in Lemma 4.

Let $\delta<\delta_*$. From the n equation and the no-flux condition, we observe that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega n\log n\varphi^3-\frac{d}{dt}\int_\Omega n\varphi^3\\
&=-\int_\Omega\nabla(\log n\varphi^3)\cdot[\nabla n-nS(x,n,c)\cdot\nabla c]\\
&=-\int_\Omega\frac{|\nabla n|^2}{n}\varphi^3+\int_\Omega\nabla n\cdot(S(x,n,c)\cdot\nabla c)\varphi^3\\
&\quad-3\int_\Omega\log n\nabla n\cdot\nabla\varphi\varphi^2+3\int_\Omega n\log n(S(x,n,c)\cdot\nabla c)\cdot\nabla\varphi\varphi^2.
\end{aligned}$$

By (1.7), $c \le \gamma$, and Hölder's inequality, we have

$$\begin{aligned}
&\int_\Omega \nabla n \cdot (S(x,n,c)\cdot \nabla c)\,\varphi^3 \\
&\le \|S_0\|_{\mathcal{C}([0,\gamma])} \left(\int_\Omega \frac{|\nabla n|^2}{n}\varphi^3\right)^{\frac12} \left(\int_\Omega n^2\varphi^3\right)^{\frac14} \left(\int_\Omega |\nabla c|^4 \varphi^3\right)^{\frac14}.
\end{aligned}$$

This gives, by Lemma 4, Young's inequality, and (3.1), that there exists $M = M(\delta) > 0$, independent of q, satisfying

$$\begin{aligned}
&\int_\Omega \nabla n \cdot (S(x,n,c)\cdot \nabla c)\,\varphi^3 \\
&\le \|S_0\|_{\mathcal{C}([0,\gamma])} C_2^{\frac14} C_3^{\frac14} \|\nabla c\|_{L^2(\Omega\cap B_\delta)}^{\frac14} \int_\Omega \frac{|\nabla n|^2}{n}\varphi^3 + \frac15 \int_\Omega \frac{|\nabla n|^2}{n}\varphi^3 + M.
\end{aligned}$$

Next, we use Young's inequality, $a|\log a|^2 \le 16e^{-2}a^{\frac32} + 4e^{-2}$ for $a \ge 0$, Lemma 3, and (3.6) to compute

$$\begin{aligned}
-3&\int_\Omega \log n \nabla n \cdot \nabla\varphi \varphi^2 \\
&\le \frac18 \int_\Omega \frac{|\nabla n|^2}{n}\varphi^3 + 18\int_\Omega n|\log n|^2 \varphi |\nabla \varphi|^2 \\
&\le \frac18 \int_\Omega \frac{|\nabla n|^2}{n}\varphi^3 + 18\int_\Omega (16e^{-2}n^{\frac32} + 4e^{-2})\varphi |\nabla \varphi|^2 \\
&\le \frac18 \int_\Omega \frac{|\nabla n|^2}{n}\varphi^3 + 18\cdot 16e^{-2}K^2 \int_\Omega n^{\frac32}\varphi^{\frac32} + 18\cdot 4e^{-2}K^2\int_\Omega \varphi^2 \\
&\le \frac18 \int_\Omega \frac{|\nabla n|^2}{n}\varphi^3 + 18\cdot 16e^{-2}K^2 \|n_0\|_{L^1(\Omega)}^{\frac12} \left(\int_\Omega n^2\varphi^3\right)^{\frac12} + 18\cdot 4e^{-2}K^2\int_{\mathbb{R}^2} \varphi^2.
\end{aligned}$$

It follows by Young's inequality and (3.3) that with some $M = M(\delta) > 0$,

$$-3\int_\Omega \log n \nabla n \cdot \nabla\varphi\varphi^2 \le \frac15 \int_\Omega \frac{|\nabla n|^2}{n}\varphi^3 + M.$$

Similarly, if we use (1.7), Young's inequality, $a^{\frac{4}{3}}|\log a|^{\frac{4}{3}} \le 16e^{-\frac{4}{3}}a^{\frac{3}{2}}+e^{-\frac{4}{3}}$ for $a \ge 0$, Lemma 3, and (3.6), then we have

$$\begin{aligned}
&3\int_\Omega n\log n(S(x,n,c)\cdot\nabla c)\cdot\nabla\varphi\varphi^2\\
&\le 3\|S_0\|_{\mathcal{C}([0,\gamma])}\int_\Omega n|\log n||\nabla c||\nabla\varphi|\varphi^2\\
&\le \frac{1}{4}\int_\Omega|\nabla c|^4\varphi^3+\frac{3}{4}(3\|S_0\|_{\mathcal{C}([0,\gamma])})^{\frac{4}{3}}\int_\Omega n^{\frac{4}{3}}|\log n|^{\frac{4}{3}}\varphi^{\frac{5}{3}}|\nabla\varphi|^{\frac{4}{3}}\\
&\le \frac{1}{4}\int_\Omega|\nabla c|^4\varphi^3+\frac{3}{4}(3\|S_0\|_{\mathcal{C}([0,\gamma])})^{\frac{4}{3}}K^{\frac{4}{3}}\int_\Omega(16e^{-\frac{4}{3}}n^{\frac{3}{2}}+e^{-\frac{4}{3}})\varphi^{\frac{3}{2}}\\
&\le \frac{1}{4}\int_\Omega|\nabla c|^4\varphi^3+12(3\|S_0\|_{\mathcal{C}([0,\gamma])})^{\frac{4}{3}}K^{\frac{4}{3}}e^{-\frac{4}{3}}\|n_0\|_{L^1(\Omega)}^{\frac{1}{2}}\left(\int_\Omega n^2\varphi^3\right)^{\frac{1}{2}}\\
&\qquad+\frac{3}{4}(3\|S_0\|_{\mathcal{C}([0,\gamma])})^{\frac{4}{3}}K^{\frac{4}{3}}e^{-\frac{4}{3}}\int_{\mathbb{R}^2}\varphi^{\frac{3}{2}}.
\end{aligned}$$

Thus, by Lemma 4 and Young's inequality, we can find $M = M(\delta) > 0$ such that

$$\begin{aligned}
&3\int_\Omega n\log n(S(x,n,c)\cdot\nabla c)\cdot\nabla\varphi\varphi^2\\
&\qquad\le \frac{1}{4}C_3\|\nabla c\|_{L^2(\Omega\cap B_\delta)}\int_\Omega\frac{|\nabla n|^2}{n}\varphi^3+\frac{1}{5}\int_\Omega\frac{|\nabla n|^2}{n}\varphi^3+M.
\end{aligned}$$

Combining above estimates gives that with some $M = M(\delta) > 0$,

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega n\log n\varphi^3-\frac{d}{dt}\int_\Omega n\varphi^3+\frac{2}{5}\int_\Omega\frac{|\nabla n|^2}{n}\varphi^3\\
&\le\left(\|S_0\|_{\mathcal{C}([0,\gamma])}C_2^{\frac{1}{4}}C_3^{\frac{1}{4}}\|\nabla c\|_{L^2(\Omega\cap B_\delta)}^{\frac{1}{4}}+\frac{1}{4}C_3\|\nabla c\|_{L^2(\Omega\cap B_\delta)}\right)\int_\Omega\frac{|\nabla n|^2}{n}\varphi^3+M. \qquad (3.8)
\end{aligned}$$

We note that using $a\log a \le 2e^{-1}a^{\frac{3}{2}}$ for $a \ge 0$, (3.6), (3.3), and Young's inequality, we can find $M = M(\delta) > 0$ such that

$$\begin{aligned}
\int_\Omega n\log n\varphi^3-\int_\Omega n\varphi^3 &\le \int_\Omega 2e^{-1}n^{\frac{3}{2}}\varphi^{\frac{3}{2}}\\
&\le 2e^{-1}\|n_0\|_{L^1(\Omega)}^{\frac{1}{2}}\left(\int_\Omega n^2\varphi^3\right)^{\frac{1}{2}}\\
&\le \frac{1}{5}\int_\Omega\frac{|\nabla n|^2}{n}\varphi^3+M. \qquad (3.9)
\end{aligned}$$

Thus, adding both sides of (3.8) by

$$\int_\Omega n\log n\varphi^3-\int_\Omega n\varphi^3$$

and using (3.7), and (3.9), we can deduce that with some $M = M(\delta) > 0$,

$$\frac{d}{dt}\mathcal{F}(t)+\mathcal{F}(t)\le M,$$

where

$$\mathcal{F}(t) = \int_\Omega n \log n(\cdot,t)\varphi^3 - \int_\Omega n(\cdot,t)\varphi^3.$$

Since solving this ordinary differential inequality gives

$$\mathcal{F}(t) \le \mathcal{F}(0)e^{-t} + M(1-e^{-t}),$$

with $\int_\Omega n\varphi^3 \le \int_\Omega n_0$, we can conclude the desired estimate. □

As a direct consequence, $L\log L$-norm of n over Ω is also bounded.

Corollary 1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain. There exists $C > 0$ such that*

$$\sup_{t<T_{\max}} \int_\Omega n \log n(\cdot,t) \le C.$$

Proof. Let $\delta > 0$, and let φ_δ be a function given in Lemma 3. Denote $\varphi(x) = \varphi_\delta(x-q)$ and $B_\delta = B_\delta(q)$ for $q \in \overline{\Omega}$. Using Lemma 5, $a\log a + e^{-1} \ge 0$ for $a \ge 0$, and $\varphi = 1$ in $B_{\frac{\delta}{2}}$, we can find $\delta > 0$ and $C > 0$ both independent of q such that

$$\sup_{t<T_{\max}} \int_{\Omega\cap B_{\frac{\delta}{2}}(q)} (n\log n(\cdot,t) + e^{-1}) \le \sup_{t<T_{\max}} \int_\Omega (n\log n(\cdot,t)+e^{-1})\varphi^3 \le C.$$

Since the open covering $\bigcup_{q\in\overline{\Omega}} B_{\frac{\delta}{2}}(q)$ of compact set $\overline{\Omega}$ has a finite subcovering $\bigcup_{i=1}^N B_{\frac{\delta}{2}}(q_i)$, $q_i \in \overline{\Omega}$, we have that with some $C > 0$,

$$\sup_{t<T_{\max}} \int_\Omega (n\log n(\cdot,t) + e^{-1}) \le \sum_{i=1}^N \sup_{t<T_{\max}} \int_{\Omega\cap B_{\frac{\delta}{2}}(q_i)} (n\log n(\cdot,t)+e^{-1}) \le C.$$

This gives the desired bound. □

To obtain higher integrability of n, we prepare the following lemma which can be seen as a generalization of [11, Lem. 2.4].

Lemma 6. *Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain. There exists $C = C(\Omega) > 0$ such that for any $p \ge 1$, $s > 1$, $\varepsilon > 0$, and non-negative $f \in \mathcal{C}^1(\overline{\Omega})$,*

$$\int_\Omega f^{p+1} \le C\frac{(p+1)^2}{\log s}\int_\Omega (f\log f + e^{-1})\int_\Omega f^{p-2}|\nabla f|^2 + (4C)^{1+\frac{\varepsilon}{2}}\left(\int_\Omega f^{\frac{\varepsilon}{2}\frac{p+1}{1+\varepsilon}}\right)^{\frac{2(1+\varepsilon)}{\varepsilon}} + 6s^{p+1}|\Omega|.$$

Proof. We recall from [11, (2.1)–(2.5)] that there exists $C = C(\Omega) > 0$ such that

$$\int_\Omega f^{p+1} \le C\frac{(p+1)^2}{2\log s}\int_\Omega (f\log f+e^{-1})\int_\Omega f^{p-2}|\nabla f|^2 + 2C\|w\|^2_{L^1(\Omega)} + 3s^{p+1}|\Omega|,$$

where

$$w = \max\{f^{\frac{p+1}{2}} - s^{\frac{p+1}{2}}, 0\}.$$

Using a direct computation, and Hölder's and Young's inequalities, we compute

$$\begin{aligned}\|w\|_{L^1(\Omega)}^2 \le & \left(\int_{\{f>s\}} f^{\frac{p+1}{2}}\right)^2 \le \left(\int_\Omega f^{\frac{p+1}{2+\varepsilon}} f^{\frac{\varepsilon}{2}\frac{p+1}{2+\varepsilon}}\right)^2 \\ & \le \left(\int_\Omega f^{p+1}\right)^{\frac{2}{2+\varepsilon}} \left(\int_\Omega f^{\frac{\varepsilon}{2}\frac{p+1}{1+\varepsilon}}\right)^{\frac{2(1+\varepsilon)}{2+\varepsilon}} \\ & \le \frac{1}{4C}\int_\Omega f^{p+1} + \left(\frac{8C}{2+\varepsilon}\right)^{\frac{\varepsilon}{2}} \frac{\varepsilon}{2+\varepsilon}\left(\int_\Omega f^{\frac{\varepsilon}{2}\frac{p+1}{1+\varepsilon}}\right)^{\frac{2(1+\varepsilon)}{\varepsilon}}.\end{aligned}$$

Since $\left(\frac{8C}{2+\varepsilon}\right)^{\frac{\varepsilon}{2}} \frac{\varepsilon}{2+\varepsilon} \le (4C)^{\frac{\varepsilon}{2}}$, we can deduce the desired result. □

We are ready to prove Theorem 1.

Proof of Theorem 1. Let (n, c) be a solution given by Lemma 1. Once we have a uniform-in-time bound for $\|n\|_{L^p(\Omega)}$ with some $p > d = 2$, then by Lemma 2 (i), $W^{2,p}(\Omega) \hookrightarrow \mathcal{C}^1(\overline{\Omega})$, (1.7) and $c \le \gamma$, we have a uniform-in-time bound of $S(x, n, c)\nabla c$. Then, applying a Moser-type iteration argument to

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega n^p + \frac{4(p-1)}{p^2}\int_\Omega |\nabla n^{\frac{p}{2}}|^2 \\ &\quad = \frac{2(p-1)}{p}\int_\Omega n^{\frac{p}{2}}\nabla n^{\frac{p}{2}} \cdot (S(x,n,c)\cdot\nabla c) \\ &\quad \le \frac{2(p-1)}{p^2}\int_\Omega |\nabla n^{\frac{p}{2}}|^2 + \frac{(p-1)^2}{p}\|S(x,n,c)\nabla c\|_{L^\infty(\Omega)}^2 \int_\Omega n^p, \qquad p\ge 1,\end{aligned}$$

we can find uniform-in-time bound for n and Theorem 1 follows by Lemma 1. Thus, it is enough to show that there exists $C > 0$ satisfying

$$\sup_{t<T_{\max}} \int_\Omega n^3(\cdot,t) \le C. \tag{3.10}$$

To this end, multiplying the n equation by n^2 and integrating over Ω, we observe that

$$\frac{1}{3}\frac{d}{dt}\int_\Omega n^3 + \frac{8}{9}\int_\Omega |\nabla n^{\frac{3}{2}}|^2 = \frac{4}{3}\int_\Omega n^{\frac{3}{2}}\nabla n^{\frac{3}{2}}\cdot(S(x,n,c)\cdot\nabla c).$$

Using (1.7), $c \le \gamma$, and Hölder's inequality, we compute the right-hand-side as

$$\begin{aligned}&\frac{4}{3}\int_\Omega n^{\frac{3}{2}}\nabla n^{\frac{3}{2}}\cdot(S(x,n,c)\cdot\nabla c) \\ &\quad \le \frac{4}{3}\|S_0\|_{\mathcal{C}([0,\gamma])}\int_\Omega n^{\frac{3}{2}}|\nabla n^{\frac{3}{2}}||\nabla c| \\ &\quad \le \frac{4}{3}\|S_0\|_{\mathcal{C}([0,\gamma])}\|n\|_{L^4(\Omega)}^{\frac{3}{2}}\|\nabla n^{\frac{3}{2}}\|_{L^2(\Omega)}\|c\|_{W^{1,8}(\Omega)}.\end{aligned}$$

Then, we use the Gagliardo-Nirenberg interpolation inequality,

$$\|f\|_{W^{1,8}(\Omega)} \le C(\|f\|_{L^\infty(\Omega)}^{\frac{1}{2}}\|f\|_{W^{2,4}(\Omega)}^{\frac{1}{2}} + \|f\|_{L^\infty(\Omega)}) \quad \text{for all} \quad f\in\mathcal{C}^2(\overline{\Omega}),$$

Lemma 2 (i), and $c \le \gamma$ to find $C > 0$ such that

$$\|c\|_{W^{1,8}(\Omega)} \le C(\|n\|_{L^4(\Omega)}^{\frac{1}{2}} + 1).$$

Since Lemma 6 with $(f, p, \varepsilon) = (n, 3, 1)$ and Corollary 1 yield $C > 0$ independent of $s > 1$ satisfying

$$\int_\Omega n^4 \le \frac{C}{\log s}\int_\Omega |\nabla n^{\frac{3}{2}}|^2 + (4C)^{\frac{3}{2}}\left(\int_\Omega n_0\right)^4 + 6s^4|\Omega|,$$

combining above estimates, after using Young's inequality, we can find $C > 0$ independent of $s > 1$ such that

$$\begin{aligned}
&\frac{1}{3}\frac{d}{dt}\int_\Omega n^3 + \frac{8}{9}\int_\Omega |\nabla n^{\frac{3}{2}}|^2 \\
&\quad\le \frac{4}{3}\|S_0\|_{\mathcal{C}([0,\gamma])}\|n\|_{L^4(\Omega)}^{\frac{3}{2}}\|\nabla n^{\frac{3}{2}}\|_{L^2(\Omega)}\|c\|_{W^{1,8}(\Omega)} \\
&\quad\le C\left(\frac{1}{\sqrt{\log s}}\|\nabla n^{\frac{3}{2}}\|_{L^2(\Omega)} + 1 + s^2\right)\|\nabla n^{\frac{3}{2}}\|_{L^2(\Omega)}.
\end{aligned}$$

If we take sufficiently large s and use Young's inequality, then with some $C > 0$,

$$\frac{d}{dt}\int_\Omega n^3 + \int_\Omega |\nabla n^{\frac{3}{2}}|^2 \le C.$$

This implies, by the Gagliardo-Nirenberg type inequality,

$$\|f\|_{L^3(\Omega)}^3 \le \|\nabla f^{\frac{3}{2}}\|_{L^2(\Omega)}^2 + C\|f\|_{L^1(\Omega)}^3 \quad \text{for all} \quad f \in \mathcal{C}^1(\overline{\Omega}),$$

and $\int_\Omega n = \int_\Omega n_0$, that with some $C > 0$,

$$\frac{d}{dt}\int_\Omega n^3 + \int_\Omega n^3 \le C.$$

Therefore, we can deduce (3.10). □

4. Case of scalar sensitivity in general dimensions

Throughout this section, let $\Omega = B_R(0) = \{x \in \mathbb{R}^d \,|\, r = |x| < R\}$, and n_0 be radial.

In the radially symmetric setting, two equations of (1.1) can be written as

$$n_t = r^{1-d}\left(r^{d-1}n_r\right)_r - r^{1-d}\left(r^{d-1}n\chi(r,n,c)c_r\right)_r, \quad r^{1-d}\left(r^{d-1}c_r\right)_r = nc.$$

Thus, the cumulative mass distribution Q defined by

$$Q(r,t) := \int_{B_r(0)} n(x,t)\,dx = \sigma_d\int_0^r \rho^{d-1}n(\rho,t)\,d\rho \tag{4.1}$$

satisfies

$$Q_t = r^{d-1}\left(r^{1-d}Q_r\right)_r - Q_r\chi(r,n,c)c_r, \qquad r < R, t < T_{\max}. \tag{4.2}$$

We note that

$$Q(R,t) = \|n_0\|_{L^1(\Omega)},$$

and

$$Q_r \ge 0, \qquad c_r = r^{1-d}\int_0^r \rho^{d-1}nc\,d\rho \ge 0, \qquad r < R, t < T_{\max}. \tag{4.3}$$

The non-negativities (4.3) and $\chi \ge 0$ yield an upper bound for Q stated below.

Lemma 7. *Let Q be the cumulative mass distribution defined in* (4.1)*. Then, there exists* $M_0 = M_0(d, R, \|n_0\|_{L^1(\Omega)}, \|n_0\|_{L^\infty(\Omega)}) \geq 0$ *such that*

$$Q(r,t) \leq M_0 r^d \quad \textit{for} \quad r < R, t < T_{\max}.$$

Proof. We use a comparison argument. Due to (4.3) and $\chi \geq 0$, it follows from (4.2) that

$$Q_t \leq r^{d-1}\left(r^{1-d}Q_r\right)_r.$$

Define

$$M_0 := \max\left\{\frac{1}{R^d}\|n_0\|_{L^1(\Omega)}, \frac{\sigma_d}{d}\|n_0\|_{L^\infty(\Omega)}\right\},$$

and

$$W(r) := M_0 r^d.$$

Then, $Q(R,t) \leq W(R)$, $Q(r,0) \leq W(r)$, and

$$0 = r^{d-1}\left(r^{1-d}W_r\right)_r.$$

Let $\varepsilon > 0$ be given, and we now show that F defined by

$$F(r,t) := (Q(r,t) - W(r,t))\exp(-t)$$

can not attain value ε as long as solution exists. Note that $F(0,t) = 0$, $F(R,t) \leq 0$, $F(r,0) \leq 0$, and

$$\begin{aligned} F_t &= (Q_t - W_t)\exp(-t) - F \\ &\leq r^{d-1}\left(r^{1-d}F_r\right)_r - F \\ &= F_{rr} + (1-d)r^{-1}F_r - F. \end{aligned}$$

Assume to the contrary that $F(r_1,t_1) = \varepsilon$ for the first time $t_1 < T_{\max}$. Then, $r_1 \neq 0$ or R and

$$\begin{aligned} 0 \leq F_t(r_1,t_1), &\qquad F_{rr}(r_1,t_1) \leq 0, \\ (1-d)r_1^{-1}F_r(r_1,t_1) = 0, &\qquad -F(r_1,t_1) = -\varepsilon < 0, \end{aligned}$$

which leads to a contradiction. Since $\varepsilon > 0$ is arbitrary, $F \leq 0$ and the desired bound follows. □

Due to Lemma 7, for each $t < T_{\max}$, there exist a radius $r_t \in [0, R]$ and a number $m_0 > 0$ satisfying $c(r_t, t) \geq m_0$:

Lemma 8. *Let (n, c) be the solution given by Lemma 1, and let M_0 be a number given in Lemma 7. Then, for each $t < T_{\max}$, there exists a radius $r_t \in [0, R]$ such that*

$$c(r_t,t) \geq m_0 := \frac{\gamma}{2}\left(\frac{M_0 R}{\sigma_d} + 1\right)^{-1}. \tag{4.4}$$

Proof. Suppose that (4.4) is false. Then, there exists $T < T_{\max}$ such that

$$c(r,T) < m_0 \quad \text{for all} \quad r \in [0,R].$$

Fix $t = T$. Using the c equation and Lemma 7, we can estimate

$$c_r = r^{1-d}\int_0^r \rho^{d-1} nc\, d\rho < \frac{M_0 m_0}{\sigma_d} r \quad \text{for all} \quad r \in (0,R].$$

If we take $r = R$, then from the boundary condition and $c < m_0$, we have

$$\gamma - m_0 < \gamma - c(R) = c_r(R) < \frac{M_0 m_0}{\sigma_d} R.$$

This leads to a contradiction because $m_0 < \gamma \left(\frac{M_0 R}{\sigma_d} + 1 \right)^{-1}$. □

As a consequence, c has the lower bound which is uniform in space and time:

Lemma 9. *Let (n, c) be the solution given by Lemma 1, and let M_0 and m_0 be numbers given in Lemma 7 and Lemma 8, respectively. Then, it holds that*

$$\min_{r \in [0,R]} c(r,t) \geq c_* := m_0 \exp\left(-\frac{1}{2} \frac{M_0 R^2}{\sigma_d} \right) \quad \text{for} \quad t < T_{\max}.$$

Proof. Note that by Lemma 1,

$$c > 0,$$

and by $c_r \geq 0$ in (4.3),

$$\min_{r \in [0,R]} c(r,t) = c(0,t).$$

Since for each $t < T_{\max}$, there exists $r_t \in [0, R]$ satisfying (4.4), in view of Lemma 7 and

$$r^{1-d} \left(r^{d-1} (\log c)_r \right)_r = \Delta \log c = n - |\nabla \log c|^2 \leq n,$$

we have that

$$(\log c)_r = r^{1-d} \int_0^r \left(\rho^{d-1} (\log c)_\rho \right)_\rho \, d\rho \leq r^{1-d} \int_0^r \rho^{d-1} n \, d\rho \leq \frac{M_0}{\sigma_d} r.$$

If we integrate it from 0 to r_t, then

$$\log \frac{c(r_t, t)}{c(0,t)} \leq \int_0^{r_t} \frac{M_0}{\sigma_d} \rho \, d\rho = \frac{M_0}{2\sigma_d} r_t^2.$$

Therefore, using (4.4) and $r_t \leq R$, we can deduce the desired result. □

We are ready to prove Theorem 2.

Proof of Theorem 2. Let (n, c) be the solution given by Lemma 1. From (4.3) and $Q(r,t) \leq M_0 r^d$ in Lemma 7, we have

$$0 \leq c_r = r^{1-d} \int_0^r \rho^{d-1} n c \, d\rho \leq \frac{\gamma M_0}{\sigma_d} r \quad \text{for} \quad r \in (0, R].$$

Thus, ∇c has uniform-in-time pointwise bounds. Moreover, by $c_* \leq c \leq \gamma$ and $\chi(x, n, c) \leq \chi_0(c) \in \mathcal{C}(\mathbb{R}_+)$, we have that $\chi(x, n, c)$ is bounded uniformly in time. Then, the standard parabolic regularity theory gives a uniform-in-time bound for n. This concludes Theorem 2 from Lemma 1. □

Acknowledgment

The authors express sincere gratitude to the anonymous referees for their helpful remarks and their careful reading of our manuscript. J. Ahn was supported by National Research Foundation (NRF) of Korea (Grant No. NRF-2021R1F1A1064209). K. Kang was supported by NRF-2019R1A2C1084685. J. Lee was supported by Samsung Science and Technology Foundation under Project No. SSTF-BA1701-05.

References

[1] M. Braukhoff, Global (weak) solution of the chemotaxis-Navier-Stokes equations with non-homogeneous boundary conditions and logistic growth, Ann. Inst. Henri Poincaré, Anal. Non Linéaire **34** (2017) 101–1039.

[2] M. Braukhoff, J. Lankeit, Stationary solutions to a chemotaxis-consumption model with realistic boundary conditions for the oxygen, Math. Models Methods Appl. Sci. **29** (2019) 2033–2062.

[3] M. Braukhoff, B. Q. Tang, Global solutions for chemotaxis-Navier-Stokes system with Robin boundary conditions, J. Differential Equations **269** (2020) 10630–10669.

[4] D. Bucur, A. Giacomini, P. Trebeschi, Best constant in Poincaré inequalities with traces: a free discontinuity approach, Ann. Inst. H. Poincaré Anal. Non Linéaire **36** (2019) 1959–1986.

[5] X. Cao, S. Ishida, Global-in-time bounded weak solutions to a degenerate quasilinear Keller–Segel system with rotation, Nonlinearity **27** (2014) 1899–913.

[6] X. Cao, J. Lankeit, Global classical small-data solutions for a three-dimensional chemotaxis Navier–Stokes system involving matrix-valued sensitivities, Calc. Var. Partial Differ. Equ. **55** (2016) 39 pp.

[7] M. Chae, K. Kang, J. Lee, Existence of smooth solutions to coupled chemotaxis-fluid equations, Discrete Contin. Dyn. Syst. **33** (2013) 2271–2297.

[8] W. R. DiLuzio, L. Turner, M. Mayer, P. Garstecki, D. B. Weibel, H. C. Berg, G. M. Whitesides, Escherichia coli swim on the right-hand side, Nature **435** (2005) 1271–1274.

[9] R. Duan, A. Lorz, P. Markowich, Global solutions to the coupled chemotaxis-fluid equations, Comm. Part. Differ. Eq. **35** (2010) 1635–1673.

[10] M. Fuest, J. Lankeit, M. Mizukami, Long-term behaviour in a parabolic-elliptic chemotaxis-consumption model. Journal of Differential Equations **271** (2021) 254–279.

[11] K. Fujie, T. Senba. Global existence and boundedness of radial solutions to a two dimensional fully parabolic chemotaxis system with general sensitivity. Nonlinearity **29** (2016) 2417–2450.

[12] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Pitman, London (1985).

[13] O. Ladyžhenskaya, V. Solonnikov, N. Ural'ceva, Linear and Quasilinear Equations of Parabolic Type, Transl. Math. Monogr., vol. 23, AMS, Providence, RI, 1988.

[14] E. Lauga, W. R. DiLuzio, G. M. Whitesides, H. A. Stone, Swimming in circles: motion of bacteria near solid boundaries, Biophys. J. **90** (2006) 400–412.

[15] T. Li, A. Suen, C. Xue, M. Winkler, Global small-data solutions of a two-dimensional chemotaxis system with rotational flux terms, Math. Models. Methods. Appl. Sci. **25** (2015) 721–46.

[16] G. M. Lieberman, Oblique Derivative Problems for Elliptic Equations, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013.

[17] G. M. Lieberman, Second order parabolic differential equations, World Scientific Publishing Co. Inc., River Edge, NJ, 1996.

[18] H. G. Othmer, T. Hillen, The diffusion limit of transport equations ii: chemotaxis equations, SIAM J. Appl. Math. **62** (2002) 1222–1250

[19] M. M. Porzio, V. Vespri, Hölder estimates for local solutions of some doubly nonlinear degenerate parabolic equations, J. Differential Equations **103** (1993) 146–178.

[20] Y. Tian, Z. Xiang, Global solutions to a 3D chemotaxis-Stokes system with nonlinear cell diffusion and Robin signal boundary condition, J. Differential Equations **269** (2020) 2012–178.

[21] I. Tuval, L. Cisneros, C. Dombrowski, C. W. Wolgemuth, J. O. Kessler, R. E. Goldstein, Bacterial swimming and oxygen transport near contact lines, Proc. Nat. Acad. Sci. USA **102** (2005) 2277–2282.

[22] Y. Wang, X. Cao, Global classical solutions of a 3D chemotaxis-Stokes system with rotation, Discrete Contin. Dyn. Syst. Ser. B **20** (2015) 3235–54.

[23] Y. Wang, X. Li, Boundedness for a 3D chemotaxis–Stokes system with porous medium diffusion and tensor-valued chemotactic sensitivity, Z. Angew. Math. Phys. **68** (2017) 23 pp.

[24] M. Winkler, Global large-data solutions in a chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops, Comm. Part. Differ. Eq. **37** (2012) 319–351.

[25] M. Winkler, Stabilization in a two-dimensional chemotaxis-Navier-Stokes system. Arch. Ration. Mech. Anal. **211** (2014) 455–487.

[26] M. Winkler, Boundedness and large time behavior in a three-dimensional chemotaxis–Stokes system with nonlinear diffusion and general sensitivity, Calc. Var. Partial Differ. Equ. **54** (2015) 3789–828.
[27] M. Winkler, Large-data global generalized solutions in a chemotaxis system with tensor-valued sensitivities, SIAM J. Math. Anal. **47** (2015) 3092–115
[28] M. Winkler, How far do chemotaxis-driven forces influence regularity in the Navier–Stokes system? Trans. Amer. Math. Soc. **369** (2017) 3067–3125.
[29] M. Winkler, Global mass-preserving solutions in a two-dimensional chemotaxis–Navier–Stokes system with rotational flux components, J. Evol. Equ. **18** (2018) 1267–89.
[30] M. Winkler, Can rotational fluxes impede the tendency toward spatial homogeneity in nutrient taxis(-Stokes) systems?, IMRN (2021) 8106–8152.
[31] C. Wu, Z. Xiang, Asymptotic dynamics on a chemotaxis-Navier-Stokes system with nonlinear diffusion and inhomogeneous boundary conditions, Math. Models. Methods. Appl. Sci. **30** (2020) 1325–1374.
[32] C. Xue, Macroscopic equations for bacterial chemotaxis: integration of detailed biochemistry of cell signaling, J. Math. Biol. **70** (2015) 1–44.
[33] C. Xue, E. O. Budrene, H. G. Othmer, Radial and spiral streamformation in proteus mirabilis colonies, PLoS Comput. Biol. **7** e1002332 (2011)
[34] C. Xue, H. G. Othmer, Multiscale models of taxis-driven patterning in bacterial populations, SIAM J. Appl. Math. **70** (2009) 133–167.

Department of Mathematics, Dongguk University, Seoul, 04620, Republic of Korea
Email address: jaewookahn@dgu.ac.kr

School of Mathematics & Computing (Mathematics), Yonsei University, Seoul 03722, Republic of Korea
Email address: kkang@yonsei.ac.kr

Department of Mathematics, Chung-Ang University, Seoul 06974, Republic of Korea
Email address: jhleepde@cau.ac.kr