Generalization of tail inequalities for random variables,

using in the martingale theory.

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Abstract.

We generalize a famous tail Doob's inequality, relative two non - negative random variables, arising in the martingale theory, in two directions: on the more general source data and on the random variables belonging to the so - called Grand Lebesgue Spaces.

We bring also several examples in each sections in order to show the exactness of our estimates.

Key words and phrases. Martingale, random variable (r.v.), expectation, generating function, inequalities, upper and lower estimates, moment, dilation operator, Hölder's inequality, event, tail of distribution, Lebesgue - Riesz. Grand Lebesgue and Orlicz spaces and norms, Young - Fenchel transform, Doob's and other inequalities, probability space, examples.

1 Notations. Statement of problem.

Let $(\Omega = {\omega}, \mathcal{B}, P)$ be non-trivial probability space with expectation **E**. The classical Doob's inequality, see e.g. [9], [10], see also [24], tell us that if the non - negative numerical valued r.v. - s (ξ, X) are such that for some positive finite constants β , C

1

$$
t \mathbf{P}(\xi > \beta t) \le C \mathbf{E}[X \ I(\xi > t)], \ t \ge 0,
$$
\n⁽¹⁾

here and further $I(A)$ denotes as ordinary an indicator function of the random event (predicate) A, then

$$
||\xi||_p = [\mathbf{E}|\xi^p|]^{1/p} \le C \frac{p}{p-1} \beta^p |X|_p, \ p > 1.
$$
 (2)

Henceforth $||\eta||_p$ denotes as ordinary the classical Lebesgue - Riesz norm of the r.v. η :

$$
||\eta||_p := [||\mathbf{E}|\eta|^p||^{1/p}, ||p \ge 1.
$$

This inequality (2) play a very important role in particular, in the martingale theory, see $[2]$, $[3]$.

Our aim in this short report is generalization of Doob's inequality in two mentioned directions.

STATEMENT OF PROBLEM.

Given:

$$
h(t) \mathbf{P}(\xi > \beta t) \le C \mathbf{E}[X \ I(\xi > t)], \ t \ge 0,
$$
\n
$$
(3)
$$

where $h = h(t)$ is certain non-negative continuous strictly increasing deterministic function and as before X be non - negative random variable, β , $C = \text{const} \in$ $(0, \infty).$

Let also $g = g(t)$ be some non - negative continuous differentiable deterministic strictly increasing function such that $g(0) = g(0+) = 0$. We intent on the assumption of (3) to find such a function $g = g(t)$ for which the following moment is finite:

$$
\mathbf{E}g(\xi) \le Z(\beta, h, X) < \infty \tag{4}
$$

and moreover to estimate the right - hand side of the relation (4), i.e. the using in practice functional $Z = Z(\beta, h, X)$. We bring also the examples in order to show the exactness of obtained estimates.

2 Main result.

We have

$$
\mathbf{P}(\xi/\beta > t) \le \frac{C}{h(t)} \mathbf{E}\{ X I(\xi > t) \}, t > 0,
$$

where as ordinary $I(A)$ denotes the indicator function for the random event (predicat) A. Hence

$$
\int_0^\infty pt^{p-1} \; \mathbf{P}(\xi/\beta > t) \; dt \le \int_0^\infty \frac{C \; p \; t^{p-1}}{h(t)} \; \mathbf{E} \{ \; X \; I(\xi > t) \; \} \; dt, \; p \ge 1. \tag{5}
$$

The left - hand side of (5) is equal to $L := \mathbf{E} |\xi/\beta|^p = ||\xi||_p^p \beta^{-p}$. Let us investigate the right - hand side R of (5) . We deduce by virtue of Fubini's theorem

$$
R = C \, p \, \mathbf{E} \left[X \, \int_0^{\xi} \frac{t^{p-1}}{h(t)} \, dt \right].
$$

Let us introduce the new important random variable (measurable function), if it there exists:

$$
\kappa_p(\xi) = \kappa_p \stackrel{def}{=} \int_0^{\xi} \frac{t^{p-1}}{h(t)} dt,
$$

then

$$
||\xi||_p^p \beta^{-p} \le C \mathbf{E} [X \kappa_p(\xi)].
$$

Denote now

$$
K_p(\theta) := ||\kappa_p(\xi)||_{\theta} < \infty, \ \exists \ \theta > 1; \ \alpha = \alpha(\theta) := \theta/(\theta - 1) = \theta' \ \in (1, \infty).
$$

and assume

$$
\exists \theta > 1 \Rightarrow K_p(\theta) := ||\kappa_p(\xi)||_{\theta} < \infty,
$$
\n(6)

and denote by $\Theta = \Theta(p) = \Theta(p)[h, \xi]$ the set all the values θ for which the value $K_p(\theta)$ is finite:

$$
\Theta = \Theta(p) = \Theta(p)[h,\xi] = \{ \theta, \ \theta \ge 1, \ K_p(\theta) < \infty \}.
$$

We apply as expected the Hölder's inequality for such a values of the auxiliary parameter $\theta \in \Theta$

$$
||\xi||_p^p \le C \beta^p K_p(\theta) ||X||_{\alpha(\theta)}.
$$

We impose also the following condition on the source datum

$$
K_p(\theta) = ||\kappa_p(\xi)||_{\theta} \le v(\theta, p, r) ||\xi||_p^r,
$$
\n(7)

$$
\exists r = r(\theta, p) \in [1, p), \exists v = v(\theta, p, r) < \infty.
$$
\n(8)

Denote by $R = R(\theta, p)$ the set of finiteness of the value r in the relation (7):

$$
R = R(\theta, p) := \{r : 1 < r < p, \ K_p(\theta) < \infty\}.
$$

To summarize:

Theorem 2.1. We deduce under formulated above conditions

$$
||\xi||_p \leq [C \ v(\theta, p) \ \beta^p]^{1/(p-r)} \cdot ||X||_{\alpha(\theta)}^{1/(p-r)}.
$$
 (9)

Of course,

$$
||\xi||_p \le \inf_{r \in R} \inf_{\theta \in \Theta} \left\{ C \ v(\theta, p, r) \ \beta^p \right\}^{1/(p-r)} \cdot ||X||_{\alpha(\theta)}^{1/(p-r)}.
$$
 (10)

Some examples.

Example 2.1. Put now $h(t) = t$, so that

 $t \mathbf{P}(\xi > \beta t) \leq C \mathbf{E}(X \ I(\xi > t)), t, \xi, X \geq 0, p > 1.$

We choose in the conditions of theorem 2.1 $p = 1$, $r = p/(p-1)$, then

$$
v(t) = [p/(p-1)] \cdot t^{p-1}.
$$

We get using the proposition of theorem 2.1, after simple calculations, the classical result, see e.g. [24]

$$
|\xi|_p \le C \frac{p}{p-1} \beta^p |X|_p, \ p > 1. \tag{11}
$$

Example 2.2. A more general case. Suppose as before that ξ , X are non negative r.v. such that

$$
\exists C = \text{const} > 0, \quad \exists \Delta = \text{const} > 1,
$$

and

$$
t^{\Delta} \mathbf{P}(\xi > \beta \ t) \leq C \mathbf{E}(X \ I(\xi > t)), \ t \geq 0, \ p > \Delta. \tag{12}
$$

We get again using the proposition of theorem 2.1 after simple calculations the following estimation

$$
|\xi|_p \le C \frac{p}{p-\Delta} \beta^p |X|_p, \ p > \Delta. \tag{13}
$$

3 Unimprovability of our estimations. Lower bounds.

Let us show the exactness of obtained results, in particular, ones (11), (13). Introduce the following important functionals, which are responsible for the lower estimate.

$$
Y[C,\beta,\Delta](\xi,X,p) \stackrel{def}{=} \left[\frac{|\xi|_p}{C \ p \ (p-\Delta)^{-1} \ \beta^p \ |X|_p} \right],
$$

$$
U = U[C,\beta,\Delta,p] = \stackrel{def}{=} \sup_{p>\Delta} \sup_{\xi \in L_p} \sup_{X \in L_p} Y[C,\beta,\Delta](\xi,X,p), \tag{14}
$$

where all the supremums are calculated over the r.v. - s ξ , X satisfying the condition (12) and when $p > \Delta$, $\Delta = \text{const} > 1$.

Proposition 3.1.

$$
U(1, 1, \Delta) = 1.\t(15)
$$

Proof. The upper estimate $U(C, \beta, \Delta) \leq 1$ is contained in (13). In order to deduce the lower one, we bring an example.

Let us choose $C, \beta = 1$ and bring as the variables ξ , X the following: $X_0 = \xi_0$ and let the random positive variable ξ_0 , as well as one X_0 , has a standard exponential distribution

$$
\mathbf{P}(\xi_0 > t) = \mathbf{P}(X_0 > t) = e^{-t}, \ \xi_0, \ X_0 > 0, \ t > 0;
$$

then the natural generating function for these r.v.- s has a form

$$
\nu(p) \stackrel{\text{def}}{=} |\xi_0|_p = |X_0|_p = \Gamma^{1/p}(p+1), \ p \ge 1,
$$

where as ordinary $\Gamma(\cdot)$ is Euler's Gamma function.

Note that as $p \to \infty \Rightarrow \nu(p) \sim p/e$. Note also that the condition (1) is satisfied.

We have

$$
Y[C, \beta, \Delta](\xi_0, X_0, p) = \frac{p - \Delta}{p}, \ p > \Delta.
$$

Following,

$$
U(1,1,\Delta) \ge \sup_{p>\Delta} \left\{ \frac{p-\Delta}{p} \right\}.
$$
 (16)

Our proposition (15) follows immediately from the equality

$$
\lim_{p \to \infty} \left\{ \frac{p - \Delta}{p} \right\} = 1.
$$

Remark 3.1. The cases $C, \beta \neq 1$ may be considered quite analogously.

4 Generalization on the Grand Lebesgue Spaces approach. Examples.

We intent in this section to extend the previous results upon the so - called Grand Lebesgue Spaces (GLS) of the random variables.

Let $(a, b) = \text{const}, 1 \le a < b \le \infty$. Let also $\psi = \psi(p), p \in (a, b)$ be certain numerical valued measurable strictly positive: $\inf_{p\in (a,b)} \psi(p) > 0$ function, not necessary to be bounded. Denotation: $Dom(\psi) \stackrel{\text{def}}{=} \{ p : \psi(p) < \infty \}$,

$$
(a, b) := \text{supp}(\psi); \ \Psi(a, b) := \{ \ \psi : \ \text{supp}(\psi) = (a, b) \ \},
$$

$$
\Psi \stackrel{def}{=} \cup_{(a, b)} \Psi(a, b).
$$

Definition 4.1., see e.g. [19], [11], [17]. Let the function $\psi = \psi(p)$, $p \in (a, b)$ belongs to the set $\Psi(a, b)$: $\psi(\cdot) \in \Psi(a, b)$, which is named as *generating function* for introduced after space. The so-called *Grand Lebesgue Space* $G\psi$ is defined as a set of all random variables (measurable functions) τ having a finite norm

$$
||\tau||G\psi \stackrel{def}{=} \sup_{p \in (a,b)} \left\{ \frac{||\tau||L_p(\Omega)}{\psi(p)} \right\} = \sup_{p \in (a,b)} \left\{ \frac{||\tau||_p}{\psi(p)} \right\}.
$$
 (17)

The particular case of these spaces and under some additional restrictions on the generating function $\psi = \psi(p)$ correspondent to the so-called *Yudovich spaces*, see [25], [26]. These spaces was applied at first in the theory of Partial Differential Equations (PDE), see [6], [7].

These spaces are complete Banach functional rearrangement invariant; they are investigated in many works, see e.g. [12], [11], [16], [17], [18], [13], [14], [15], [19], $[20], [21], [22], [23].$ It is important for us in particular to note that there is exact of course up to finite multiplicative constant interrelations under certain natural conditions on the generating function between belonging the r.v. τ to this space and it tail behavior. Indeed, assume for the definiteness that $\tau \in G\psi$ and moreover $||\tau||G\psi = 1;$ then

$$
T_{\tau}(t) \le \exp\{-h^*(\ln t)\}, t \ge e,\tag{18}
$$

where $h(p) = h[\psi](p) := p \ln \psi(p)$ and $h^*(\cdot)$ is the famous Young - Fenchel (Legendre) transform of the function $h(\cdot)$:

$$
h^*(u) \stackrel{def}{=} \sup_{p \in \text{Dom}(\psi)} (pu - h(p)).
$$

Inversely, let the tail function $T_{\tau}(t)$, $t \geq 0$ be given. Introduce the following so - called *natural function* generated by τ

$$
\psi_{\tau}(p) \stackrel{def}{=} \left[p \int_0^{\infty} t^{p-1} T_{\tau}(t) dt \right]^{1/p} = ||\tau|| L_p(\Omega), \tag{19}
$$

if it is finite for some value $b \in (a, \infty]$, following, it is finite at last for all the values $p \in (a, b).$

As long as

$$
\mathbf{E}|\tau|^p = p \int_0^\infty t^{p-1} T_\tau(t) dt = \psi_\tau^p(p), \ p \in [1, b),
$$

we conclude that if the last *natural* for the r.v. τ function $\psi_{\tau}(p)$ is finite inside some non - trivial segment $p \in [1, b)$, $1 < b \leq \infty$, then

$$
\tau \in G\psi_{\tau}; \quad ||\tau||G\psi_{\tau} = 1.
$$

Further, let the estimate (18) be given. Suppose in addition that the generating function $\psi = \psi(p)$, $p \in Dom(\psi)$ is continuous and suppose in the case when $b = \infty$

$$
\lim_{p \to \infty} \frac{\psi(p)}{p} = 0. \tag{20}
$$

Then the r.v. τ belongs to the Grand Lebesgue Space $G\psi$:

$$
||\tau||G\psi \le K[\psi] < \infty,\tag{21}
$$

see e.g. [21].

These conditions on the generating function $\psi(\cdot)$ are satisfied for example for the functions $\psi_{m,L}(p)$ of the form

$$
\psi_{m,L}(p) \stackrel{\text{def}}{=} p^{1/m} L(p), \ m = \text{const} > 1, \ b = \infty,
$$
 (22)

where $L = L(p)$ be some continuous strictly positive *slowly varying* at infinity function such that

$$
\forall \theta > 0 \Rightarrow \sup_{p \ge 1} \left[\frac{L(p^{\theta})}{L(p)} \right] = C(\theta) < \infty. \tag{23}
$$

For instance, $L(p) = [\ln(p+1)]^r$, $r \in R$.

We conclude that under formulated restrictions the r.v. τ belongs to the space $G\psi_{m,L}$:

$$
\sup_{p\geq 1} \left\{ \frac{||\tau||_{p,\Omega}}{\psi_{m,L}(p)} \right\} = C(m,L) < \infty \tag{24}
$$

if and only if

$$
T_{\tau}(u) \le \exp\left(-C_2(m, L) \, u^m/L(u)\right), \ u \ge e, \ \exists \ C_2(m, L) > 0. \tag{25}
$$

A very popular example of these spaces forms the so - called subgaussian space Sub = Sub(Ω); it consists on the subgaussian random variables, for which $\psi(p)$ = $\psi_2(p) := \sqrt{p}$:

$$
||\tau||\text{Sub} = ||\tau||\text{G}\psi_2 \stackrel{\text{def}}{=} \sup_{p \ge 1} \left[\frac{||\tau||_{p,\Omega}}{\sqrt{p}} \right]. \tag{26}
$$

The r.v. τ belongs to the subgaussian space Sub(Ω) iff

$$
\exists C > 0 \Rightarrow T_{\tau}(u) \le \exp(-Cu^2), \ u \ge 0. \tag{27}
$$

Example 4.1. Introduce the following $G\Psi$ function

$$
\nu[\gamma](p) = \nu(p) := \exp(0.5\gamma \, p), \ p \ge 1, \ \gamma = \text{const} > 0. \tag{28}
$$

If the r.v. ζ belongs to the space $G\nu[\gamma]$ and has therein an unit norm: $||\zeta||G\nu[\gamma] = 1$, then

$$
T_{\zeta}(t) \le \exp\left(-0.5 \, \gamma^{-1} \left(\ln^2 t\right)\right), \ t \ge e. \tag{29}
$$

Conversely, let the estimation (29) holds true for some r.v. ζ ; then this r.v. ζ belongs to the Grand Lebesgue Space $G\nu : ||\zeta||G\nu[\gamma] \leq C_1(\gamma) < \infty$.

Remark 4.1. As a rule, on the the r.v. τ from the spaces $G\psi_{m,L}$ is imposed the condition of *centering*: $E\tau = 0$.

Another examples. Suppose that the r.v. τ be such that

$$
T_{\tau}(t) \leq T^{\beta,\gamma,L}(t), \ \beta > 1, \ \gamma > -1, \ L = L(t),
$$

where

$$
T^{\beta,\gamma,L}(t) \stackrel{def}{=} t^{-\beta} (\ln t)^{\gamma} L(\ln t), \ t \ge e
$$

and $L = L(t)$, $t \ge e$ be as before slowly varying at infinity positive continuous function. It is known [21] that as $p \in [1, \beta)$

$$
\psi_{\tau}(p) = ||\tau||_p \le C_1(\beta, \gamma, L) \ (\beta - p)^{-(\gamma + 1)/\beta} \ L^{1/\beta}(1/(\beta - p)), \tag{30}
$$

and conversely, if the relation (30) there holds, then

 $T_{\tau}(t) \leq C_{7}(\beta, \gamma, L) T^{\beta, \gamma + 1, L}(t).$

Herewith both this estimations are unimprovable.

Let us return to the formulated above in this section problem. Indeed, we assume that the r.v. X belongs to the certain Grand Lebesgue Space (GLS) $G\psi = G\psi(a, b)$:

$$
||X||G\psi = ||X||G\psi(a,b) < \infty; \quad 1 \le a < b \le \infty.
$$

Of course, this generating function $\psi(\cdot)$ may be choosed as a natural for the r.v. $X: \psi_0(p) := |X|_p$, if it is finite.

Let $\Delta = \text{const} \in [a, b]$; we introduce a new generating function

$$
\psi_{\Delta,\beta}(p) = \psi_{\Delta,\beta}[\psi](p) \stackrel{def}{=} \frac{p}{p-\Delta} \beta^p \psi(p), \ \Delta < p \le b, \ \beta > 1. \tag{31}
$$

so that $\psi_{\Delta,\beta}(\cdot) \in \Psi(\Delta, b)$.

Proposition 4.1. One has in these notations, definitions and under our condition (12)

$$
||\xi||G\psi_{\Delta,\beta} \le C \, ||X||G\psi,\tag{32}
$$

with correspondent tail estimation (18). Herewith the last estimation (32) is in general case essentially non -improvable.

Proof. One can take without loss of generality $||X||G\psi = 1$; then

$$
\forall p \in (\Delta, b) \Rightarrow |X|_p \le \psi(p).
$$

We apply the estimation (13) for these values p :

$$
|\xi|_p \le C \frac{p}{p-\Delta} \beta^p \psi(p) = C \psi_{\Delta,\beta}(p),
$$

or equally by means of the direct definition of the norm in the Grand Lebesgue Space $G\psi_{\Delta}$

$$
||\xi||G\psi_{\Delta,\beta} \leq C = C ||X||G\psi.
$$

The non - improvability of obtained estimate may be ground as before, as in the proposition 3.1, by means of considering at the example ones analogously as in the example 4.1:

$$
T_{\xi}(t) = \exp\left(-0.5 \, \gamma^{-1} \left(\ln^2 t\right)\right), \ t \ge e,\tag{33}
$$

where $\gamma = 2 \ln \beta$, $\beta = \text{const} > 1$; and $X = 1$.

In detail, it is easily to verify that the inequality (12) for these variables is satisfied for all the values $\beta > 1$. It remains to note as before that

$$
\sup_{\Delta > 1} \frac{\|\xi\| G \psi_{\Delta,\beta}}{\|X\| G \psi} = 1.
$$

See for details the relation (13).

5 Multivariate case.

We extend obtained results on the multidimensional case. Denotations and restrictions:

$$
d = \dim t = \dim \vec{t} = \dim \xi = \dim \vec{\xi} = 2, 3, \dots; \beta, C, \Delta = \text{const} > 0;
$$

$$
\vec{t} = \{ t_1, t_2, \dots, t_d \}, \quad \vec{\xi} = \{ \xi_1, \xi_2, \dots, \xi_d \};
$$

$$
\forall j = 1, 2, \dots, d \Rightarrow t_j, \ \xi_j \ge 0; \quad \vec{\xi} > \vec{t} \Leftrightarrow \forall j \ \xi_j > t_j;
$$

$$
|\vec{t}| \stackrel{def}{=} \sqrt{\sum_{j=1}^d t_j^2}, \quad |\vec{\xi}|_p \stackrel{def}{=} \left[\sum_{j=1}^d |\xi_j|_p^p \right]^{1/p}, \ p \ge 1.
$$

Given:

$$
|\vec{t}|^{\Delta} \mathbf{P}(\vec{\xi} > \beta \vec{t}) \le C \mathbf{E}[X \ I(\vec{\xi} > \vec{t})]. \tag{34}
$$

Proposition 5.1. We state for the values $p > \max(1, \Delta)$

$$
|\vec{\xi}|_p \le C \frac{p}{p-\Delta} d^{1/p} \beta^p |X|_p. \tag{35}
$$

Proof. We have for the values $j = 1, 2, ..., d$

$$
t_j \ \mathbf{P}(\xi_j > \beta \ t_j) \leq C \mathbf{E}[\ XI(\xi_j > t_j)].
$$

It follows from the one - dimensional estimates

$$
|\xi_j|_p \le C \frac{p}{p-\Delta} \beta^p |X|_p.
$$

Further,

$$
|\vec{\xi}|_p = \left[\sum_{j=1}^d |\xi_j|_p^p\right]^{1/p} = \left[\sum_{j=1}^d |\xi_j|_p^p\right]^{1/p} \le
$$

$$
\left[d\left(\frac{Cp}{p-\Delta}\right)^p \beta^{p^2} |X|_p^p\right]^{1/p} = C \ d^{1/p} \frac{p}{p-\Delta} \beta^p |X|_p,
$$

Q.E.D.

Remark 5.1. One can use instead the classical Euclidean norm for the vector $t : |t|$ arbitrary another one, for instance, l_s one:

$$
|t|_s := (\sum_{j=1}^d |t_j|^s)^{1/s}, \ s = \text{const} \ge 1.
$$

6 Concluding remarks.

It is interest in our opinion to apply obtained in this preprint estimates, in particular, in the martingale theory.

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