

AN ALGEBRAIC APPROACH TO THE GOLDBACH AND POLIGNAC CONJECTURES

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ABSTRACT. This paper will give both the necessary and sufficient conditions required to find a counter-example to the Goldbach Conjecture by using an algebraic approach where no knowledge of the gaps between prime numbers is needed. To eliminate ambiguity the set of natural numbers, \mathbb{N} , will include zero throughout this paper. Also, for any sufficiently large $a \in \mathbb{N}$ the set \mathcal{P} is the set of all primes $p_i \leq a$. It will be shown there exists a counter-example to the Goldbach Conjecture, given by $2a$ where $a \in \mathbb{N}_{>3}$, if and only if for each prime $p_i \in \mathcal{P}$ there exists some unique $q_i, \alpha_i \in \mathbb{N}$ where $a < q_i < 2a$ and $2a = q_i + p_i$ along with the condition that $\prod_{p_i \in \mathcal{P}} q_i = \prod_{p_i \in \mathcal{P}} p_i^{\alpha_i}$. A substitution of $q_i = 2a - p_i$ for each q_i from each sum gives the product relationship $\prod_{p_i \in \mathcal{P}} (2a - p_i) = \prod_{p_i \in \mathcal{P}} p_i^{\alpha_i}$. Therefore, if a counter-example exists to the Goldbach Conjecture, then there exists a mapping $\mathcal{G}_- : \mathbb{C} \rightarrow \mathbb{C}$ where

$$\mathcal{G}_-(z) = \prod_{p_i \in \mathcal{P}} (z - p_i) - \prod_{p_i \in \mathcal{P}} p_i^{\alpha_i}$$

and $\mathcal{G}_-(2a) = 0$. A proof of the Goldbach Conjecture will be given utilizing Hensel's Lemma to show $2a$ must be of the form $2a = p_i^{\alpha_i} + p_i$ for all primes up to a when $a > 3$. However, this leads to contradiction since $2a < a\#$ for all $a > 4$.

A similar method will be employed to give the necessary and sufficient conditions when an even number is not the difference of two primes with one prime being less than that even number. To begin, let $a \in \mathbb{N}_{>3}$ with the condition that the function $\gamma(a+1)$ is equal to one if $a+1$ is prime and zero otherwise. $2a$ is a counter-example if and only if for each prime $p_i \in \mathcal{P}$ there exists some unique $u_i, \beta_i \in \mathbb{N}$ where $2a < u_i \leq 3a$ and $2a = u_i - p_i$ along with product relationship $\prod_{p_i \in \mathcal{P}} u_i = (a+1)^{\gamma(a+1)} \prod_{p_i \in \mathcal{P}} p_i^{\beta_i}$. A substitution of $u_i = 2a + p_i$ for each u_i from each sum gives $\prod_{p_i \in \mathcal{P}} (2a + p_i) = (a+1)^{\gamma(a+1)} \prod_{p_i \in \mathcal{P}} p_i^{\beta_i}$. Therefore, if a counter-example exists, it is possible to define the mapping $\mathcal{G}_+ : \mathbb{C} \rightarrow \mathbb{C}$ where

$$\mathcal{G}_+(z) = \prod_{p_i \in \mathcal{P}} (z + p_i) - (a+1)^{\gamma(a+1)} \prod_{p_i \in \mathcal{P}} p_i^{\beta_i}$$

and $\mathcal{G}_+(2a) = 0$. A proof will then be given that every even number is the difference of two primes by showing $2a$ must be of the form $2a = p_i^{\beta_i} - p_i$ for all odd primes up to a when $a > 3$ to the equation above, leading to the same contradiction as the Goldbach Conjecture since $2a < a\#$ for $a > 4$. These proofs will have implications for proving the Polignac Conjecture.

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1. INTRODUCTION

The Goldbach Conjecture, page 117 in [8], appeared in a correspondence between Leonard Euler and Christian Goldbach in 1742 where it was suspected that every number greater than two could be written as the sum of three primes. Since the number one was considered a prime, however, no longer is, this conjecture has been split up into a strong and a weak version. The strong version in some texts may be referred to as the "binary" Goldbach Conjecture. The weak version is sometimes named the "ternary" conjecture as it involves three prime numbers.

The strong version of the Goldbach Conjecture states that for every even integer greater than two there will exist two primes whose sum is that even number. Although this conjecture is simple to state all attempts to prove it, or find a counter-example, have failed. With that said, this conjecture has been verified to an astonishing degree. In July of 2000 Jörg Riechstein published a paper [10] using computational techniques showing that the Goldbach Conjecture was valid up to 4×10^{14} . In November of 2013 a paper [4] was published by Thomás Oliveira e Silva, Siegfried Herzog, and Silvo Pardi which also used advances in computational computing proving that the binary form of the Goldbach Conjecture is true up to 4×10^{18} .

The weaker version of the Goldbach Conjecture, or Ternary Conjecture, states that every odd number greater than 7 can be written as the sum of three prime numbers. Much like the strong version, this conjecture has also been verified up to large orders of magnitude. As an example, in 1998 [12] Yannick Saouter proved this conjecture up to 10^{20} . In fact, it was shown that if the generalization of the Reimann Hypothesis were true, that the Ternary Conjecture would follow. This was proven by Hardy and Littlewood [5] in 1923. Since the Generalized Reimann Hypothesis is still an open question, this did not give a definitive answer as to the truth of the Ternary Conjecture, however, it did provide a possible path to follow.

Another breakthrough in the Ternary Conjecture came in 2013 when Herald Helfgott verified in a paper [7] that the Ternary Conjecture was valid up to 10^{30} . Later that year a preprint [6] by Harold Helfgott was placed on the ARXIV claiming that the Ternary Conjecture is true. Although this paper has not been published as of yet, it has been accepted by many in the mathematics community as being true.

2. MOTIVATION FOR PRODUCING A NEW THOUGHT EXPERIMENT

All attempts to prove the Goldbach Conjecture have failed. Many of these attempts rely on an analytic number theory approach such as analyzing the gaps between primes [15]. Another method is to assume a certain hypothesis is true, such as the Generalized Reimann Hypothesis, to show that hypothesis implies one of these conjectures [5]. If that hypothesis can then be proven, the conjecture would follow. There are also experimental [3] along with computational results from [10], [4], and [12], however, these methods will most likely require major breakthroughs in order to proceed. For this reason, a new approach is needed.

The method which will be explored in this paper is a novel technique that will be used to determine algebraically both the necessary and sufficient conditions for a counter-example to the Goldbach Conjecture to be discovered. The advantage of this method lies in the fact that it circumvents two main reasons why a proof of the Goldbach Conjecture has not been discovered. The first of these difficulties in finding a proof is simply that there is no known formula that allows one to determine precisely how many prime numbers there are in a given range. The Prime Number Theorem¹ [13] does give an approximation to the number of primes up to a given value; however, this alone is not sufficient to give strong enough evidence that

¹A good approximation for $\pi(n)$, where $n > 1$, is given by $\frac{n}{\ln(n)}$

the conjectures hold for any value chosen. For this reason most probabilistic arguments about how many prime pairs there could be which sum up to a desired even number will fail.

The second issue is that there is no known parameterization of the prime numbers, or even a computationally efficient way to determine when a number is prime. Wilson's Theorem² [14] does provide both the necessary and sufficient conditions for determining if a number is prime; however, since it is a function of the factorial it is computationally inefficient to use in any practical manner. Because of these two facts, any question about additive properties of the primes has been destined to run into near insurmountable difficulties using current techniques.

To begin laying the foundation for this new method a thought experiment will be given. Suppose one wished to show that the number 20 satisfied the Goldbach Conjecture. A simple way to proceed is to take each prime up to 10, labeled by the sequence $p_1 < p_2 < p_3 < p_4$, and assign to it a unique q_i labeled by the sequence $q_1 > q_2 > q_3 > q_4$ where $20 = q_i + p_i$. This allows for the following set of arithmetic relationships.

$$(2.1) \quad 20 = 18 + 2 = 17 + 3 = 15 + 5 = 13 + 7$$

Assuming that 20 is not the sum of two prime numbers, it then follows from The Fundamental Theorem of Arithmetic [1] that there must exist a unique sequence of $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{N}$ where

$$(2.2) \quad q_1 q_2 q_3 q_4 = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4}.$$

However,

$$(2.3) \quad 18 \times 17 \times 15 \times 13 \neq 2^{\alpha_1} \times 3^{\alpha_2} \times 5^{\alpha_3} \times 7^{\alpha_4}$$

for any sequence of exponents restricted to the natural numbers. Since each q_i on the L.H.S. ranges between 10 and 20 it can be seen that equation 2.3 is true if and only if at least one number on the L.H.S. is not divisible by any prime on the R.H.S., thus proving at least one q_i is a new prime. Therefore, it may be concluded that 20 can be written as the sum of two primes without having any particular knowledge about the distribution of the prime numbers or which prime numbers sum up to 20. All that is needed is the closure property of the integers, page 1 in [8], along with the Fundamental Theorem of Arithmetic. This method can be extended to a general case given by Definition 5.1 in the following section. An analysis of this polynomial, along with the condition where $2a$ is a root will be explored.

This same method will be used to determine if every even number, again given by $2a$, is the difference of two primes where one prime is less than a . Slight modifications need to be made which will be made evident with a similar thought experiment used for the Goldbach Conjecture. To begin, assume that the number 20 was not the difference of two primes where one prime was less than 10. Taking the same approach as in the Goldbach Conjecture shows each prime up to 10, labeled by the sequence $p_1 < p_2 < p_3 < p_4$, may be assigned a unique q_i labeled by the sequence $q_1 < q_2 < q_3 < q_4$ where $20 = q_i - p_i$. This allows for the following

$$(2.4) \quad 20 = 22 - 2 = 23 - 3 = 25 - 5 = 27 - 7.$$

At this point careful attention needs to be given to the fact that the $q_1 = 22$ term is divisible by a prime greater than 10, but composite. Defining $p_5 = 11$ will be useful since 11 is a prime greater than 10. However, it is important to note that this is the only time this can occur since q_1 is the only even term and any odd $20 < q_i \leq 30$ can not be divisible by any primes greater than 10 unless it is itself prime. Assuming that 20 is not the difference of two prime numbers, then The Fundamental Theorem of Arithmetic states that there exists a unique sequence of $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{N}$ where

$$(2.5) \quad q_1 q_2 q_3 q_4 = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4} p_5.$$

However,

$$(2.6) \quad 27 \times 25 \times 23 \times 22 \neq 2^{\alpha_1} \times 3^{\alpha_2} \times 5^{\alpha_3} \times 7^{\alpha_4} \times 11$$

²A number p is a prime only if there is some integer n where $(p-1)! + 1 = pn$.

for any sequence of exponents restricted to the natural numbers. Since each q_i on the L.H.S. ranges between 20 and 30 it can be seen that equation 2.6 is true if and only if at least one number on the L.H.S. is not divisible by any prime on the R.H.S., thus proving at least one q_i is a new prime. Therefore, 20 can be written as the difference of two prime numbers. As with the Goldbach Conjecture, formalizing to a general case will be done in Definition 10.1.

3. A SIMPLER SET OF PROBLEMS

This section will give an idea of how to prove the Goldbach Conjecture, along with its analogue for the differences of primes, based on solving a set of similar problems.

Notation 3.1. Let the set of primes be denoted by the set \mathbb{P} .

Definition 3.2. Let $a \in \mathbb{N} \setminus \{0\}$ where the *Prime Divisor Set* of $2a$ is $\mathcal{D} = \{p_i \in \mathbb{P} : p_i | 2a\}$.

The question to be answered in this section is what form $2a$ must take for there to exist solutions to the equation $\prod_{p_i \in \mathcal{D}} (2a - p_i) = \prod_{p_i \in \mathcal{D}} p_i^{\alpha_i}$ where each $\alpha_i \in \mathbb{N} \cup \{0\}$. As an example it may be seen that $(6 - 2) \times (6 - 3) = 2^2 \times 3$. To begin, note the following lemmas.

Lemma 3.3. Assume there exists some $a \in \mathbb{N}_{\geq 3}$ where for each $p_i \in \mathcal{D}$ there exists some $\alpha_i \in \mathbb{N}$ where the product relationship $\prod_{p_i \in \mathcal{D}} (2a - p_i) = \prod_{p_i \in \mathcal{D}} p_i^{\alpha_i}$ holds. Then all $\alpha_i > 0$.

Proof. Note from the product $\prod_{p_i \in \mathcal{D}} (2a - p_i) = \prod_{p_i \in \mathcal{D}} p_i^{\alpha_i}$ that all prime $p_i \in \mathcal{D}$ satisfy the condition that all $p_i | 2a$ under Notation 3.1 and Definition 3.2. Hence, any $p_i | (2a - p_i)$ on the L.H.S. of the product and must be a divisor of the R.H.S. showing $p_i | p_i^{\alpha_i}$ where all $\alpha_i > 0$. \square

Lemma 3.4. $GCD(p_i, 2a - p_j) = 1$ for any $p_i \neq p_j, p_j \in \mathcal{D}$ when $\prod_{p_i \in \mathcal{D}} (2a - p_i) = \prod_{p_i \in \mathcal{D}} p_i^{\alpha_i}$.

Proof. Assume $\prod_{p_i \in \mathcal{D}} (2a - p_i) = \prod_{p_i \in \mathcal{D}} p_i^{\alpha_i}$ where all $\alpha_i \in \mathbb{N}$ from Lemma 3.3. Since $p_i | 2a$ from Definition 3.2, it is impossible for $p_i | (2a - p_j)$ for any p_j where $i \neq j$. \square

Theorem 3.5. If $\prod_{p_i \in \mathcal{D}} (2a - p_i) = \prod_{p_i \in \mathcal{D}} p_i^{\alpha_i}$, then $2a = p_i^{\alpha_i} + p_i$ for all $p_i \in \mathcal{D}$.

Proof. Under Definition 3.2 it follows that each $p_i | 2a$ in the product $\prod_{p_i \in \mathcal{D}} (2a - p_i) = \prod_{p_i \in \mathcal{D}} p_i^{\alpha_i}$. Since the lemmas above show that all $\alpha_i > 0$ and no primes $p_i, p_j \in \mathcal{D}$ satisfies the condition that $p_i | (2a - p_j)$ when $i \neq j$, then solutions must be given by $2a - p_i = p_i^{\alpha_i}$ for each $p_i \in \mathcal{D}$. \square

A similar question may be asked about sums of primes with the caveat that $2a + 2$ allows for the possibility of $a + 1$ being a new prime where the following function will be introduced.

Definition 3.6. Let the function $\gamma(a + 1)$ be defined by the conditions

$$(3.1) \quad \gamma(a + 1) = \begin{cases} 1, & \text{if } a + 1 \text{ is prime} \\ 0, & \text{if } a + 1 \text{ is not prime.} \end{cases}$$

Accounting for this fact allows for an inquiry into the form of $2a$ when it is a solution for the equation $\prod_{p_i \in \mathcal{D}} (2a + p_i) = (a + 1)^{\gamma(a+1)} \prod_{p_i \in \mathcal{D}} p_i^{\beta_i}$ where each β_i is assumed to be a natural number. The solutions for $2a$ differ only in sign to the previous ones and will be shown below.

Lemma 3.7. Assume there exists some $a \in \mathbb{N}_{\geq 3}$ where for each $p_i \in \mathcal{D}$ there exists some $\beta_i \in \mathbb{N}$ where the product relationship $\prod_{p_i \in \mathcal{D}} (2a + p_i) = (a + 1)^{\gamma(a+1)} \prod_{p_i \in \mathcal{D}} p_i^{\beta_i}$ holds. Then all $\beta_i > 0$.

Proof. Note from the product $\prod_{p_i \in \mathcal{D}} (2a + p_i) = (a + 1)^{\gamma(a+1)} \prod_{p_i \in \mathcal{D}} p_i^{\beta_i}$ that all prime $p_i \in \mathcal{D}$ satisfy the condition that all $p_i | 2a$ under Notation 3.1 and Definition 3.2. Hence, any $p_i | (2a + p_i)$ on the L.H.S. of the product and must be a divisor of the R.H.S. showing $p_i | p_i^{\beta_i}$ where $\beta_i > 0$. \square

Lemma 3.8. $GCD(p_i, 2a + p_j) = 1$ for any $p_i \neq p_j, p_j \in \mathcal{D}$ when the product relationship $\prod_{p_i \in \mathcal{D}} (2a + p_i) = (a + 1)^{\gamma(a+1)} \prod_{p_i \in \mathcal{D}} p_i^{\beta_i}$ is met.

Proof. Assume $\prod_{p_i \in \mathcal{D}} (2a + p_i) = (a + 1)^{\gamma(a+1)} \prod_{p_i \in \mathcal{D}} p_i^{\beta_i}$ where all $\beta_i \in \mathbb{N}$ from Lemma 3.7. Since $p_i | 2a$ from Definition 3.2, it is impossible for $p_i | (2a + p_j)$ for any p_j where $i \neq j$. \square

Theorem 3.9. *If $\prod_{p_i \in \mathcal{D}} (2a + p_i) = (a + 1)^{\gamma(a+1)} \prod_{p_i \in \mathcal{D}} p_i^{\beta_i}$, then $2a = p_i^{\beta_i} - p_i$ for all $p_i \in \mathcal{D}$.*

Proof. Under Definition 3.2 each $p_i | 2a$ in $\prod_{p_i \in \mathcal{D}} (2a + p_i) = (a + 1)^{\gamma(a+1)} \prod_{p_i \in \mathcal{D}} p_i^{\beta_i}$. Since the lemmas above show that all $\beta_i > 0$ and no primes $p_i, p_j \in \mathcal{D}$ satisfies the condition that $p_i | (2a + p_j)$ when $i \neq j$, then solutions must be given by $2a + p_i = p_i^{\beta_i}$ for each $p_i \in \mathcal{D}$. \square

An example of the above relationship is given by $(6 + 2) \times (6 + 3) = 2^3 \times 3^2$. The method that will be explored in the paper is to see if a similar set of solutions exist to a counter-example for the Goldbach Conjecture and its analogue for differences of primes.

4. THE GOLDBACH CONJECTURE AND GOLDBACH DIFFERENCE CONJECTURE

In order to begin this paper each conjecture will be stated along with the necessary and sufficient conditions required to find a counter-example. Once this is done, it will be shown that a counter-example cannot exist, hence proving the conjectures true. The method for finding counter-examples to each conjecture is very similar.

Conjecture 4.1. Let $a \in \mathbb{N}_{>3}$ and the primes up to a are given by $p_1 < p_2 < \dots < p_{\pi(a)}$. The *Goldbach Conjecture* (G.C.) states there exists two primes q_i, p_i where $2a = q_i + p_i$.

To begin, it is important to expand Definition 3.2. Notation 3.1 will stay the same.

Definition 4.2. Let $a \in \mathbb{N}_{\geq 3}$ where the *Prime Set* of a is $\mathcal{P} = \{p_i \in \mathbb{P} : p_i \leq a\}$.

Theorem 4.3. *Let $a \in \mathbb{N}_{>3}$. Then $2a$ is a counter-example to the G.C. if and only if for each prime $p_i \in \mathcal{P}$ there exists a unique $\alpha_i \in \mathbb{N} \cup \{0\}$ where $\prod_{p_i \in \mathcal{P}} (2a - p_i) = \prod_{p_i \in \mathcal{P}} p_i^{\alpha_i}$.*

Proof. If there exists a counter-example to the G. C. given by $2a$, then for each prime $p_i \in \mathcal{P}$ there exists some unique q_i where $a < q_i < 2a$ and

$$(4.1) \quad 2a = q_i + p_i$$

with q_i being a composition of primes up to a . Therefore, under the Fundamental Theorem of Arithmetic it follows that for each prime $p_i \in \mathcal{P}$ there must exist a unique $\alpha_i \in \mathbb{N} \cup \{0\}$ where

$$(4.2) \quad \prod_{i=1}^{\pi(a)} q_i = \prod_{p_i \in \mathcal{P}} p_i^{\alpha_i}.$$

A substitution of $q_i = 2a - p_i$ for each q_i from 4.1 in equation 4.2 gives

$$(4.3) \quad \prod_{p_i \in \mathcal{P}} (2a - p_i) = \prod_{p_i \in \mathcal{P}} p_i^{\alpha_i}.$$

Conversely, if there exists some $2a > 6$ where equation 4.3 holds for some $\alpha_1, \alpha_2, \dots, \alpha_{\pi(a)} \in \mathbb{N}$, then equations 4.1 and 4.2 are true with the Fundamental Theorem of Arithmetic showing no q_i can be prime in equation 4.1. Therefore, the G.C. would be shown false. \square

Lemma 4.4. *For any α_i it follows that $\alpha_i > 0$ if and only if $p_i | q_i$ or $p_i | q_j$ for some $j \neq i$.*

Proof. From equation 4.2 it can be seen upon inspection that if any $\alpha_i > 0$, then that p_i must be a divisor of the R.H.S. of the equation and $p_i | p_i^{\alpha_i}$. Therefore, that p_i divides the L.H.S. of the equation and must divide its corresponding q_i or some other q_j .

Conversely, from equation 4.2 if any prime p_i is a divisor of its corresponding q_i or some other q_j , then that p_i divides the R.H.S. of equation 4.2 showing $p_i | p_i^{\alpha_i}$ where $\alpha_i > 0$. \square

Lemma 4.5. *For any prime $p_i \in \mathcal{P}$ it follows $p_i | 2a$ if and only if $p_i | q_i$.*

Proof. Under equation 4.1 it is seen upon inspection if $p_i | q_i$, then $p_i | 2a$. Conversely, from equation 4.1 it follows that if any $p_i | 2a$, then $p_i | q_i$. \square

Lemma 4.6. *For any prime $p_i \in \mathcal{P}$, if $p_i | q_i$, then $p_i \nmid q_j$ for any $j \neq i$.*

Proof. Under equation 4.1 of Theorem 4.3 it follows for any primes $p_i, p_j \leq a$

$$(4.4) \quad q_i + p_i = q_j + p_j.$$

If some p_i existed where $p_i|q_i$ and $p_i|q_j$ for some $j \neq i$ in equation 4.4, then $p_i|p_j$. Since both p_i, p_j are primes, then $p_i \nmid p_j$ when $j \neq i$. Thus, if any $p_i|q_i$, then $p_i \nmid q_j$ for any $j \neq i$. \square

Proposition 4.7. *If $p_i|2a$, there exists $n_i, \alpha_i \in \mathbb{N} \setminus \{0\}$ s.t. $2a = n_i p_i^{\alpha_i} + p_i$ and $GCD(p_i, n_i) = 1$.*

Proof. Suppose some $p_i|2a$. From Lemmas 4.4 through 4.6 it can be seen that p_i only divides its corresponding q_i in equations 4.1 and 4.2 showing that there exists some $n_i \in \mathbb{N}$ where $q_i = n_i p_i^{\alpha_i}$. A substitution into equation 4.1 shows $2a = n_i p_i^{\alpha_i} + p_i$ where $GCD(p_i, n_i) = 1$. \square

A similar conjecture to the G.C. can be defined by asking whether or not every even number may be written as the difference of two prime numbers.

Conjecture 4.8. Let $a \in \mathbb{N}_{>3}$. The *Goldbach Difference Conjecture* (G.D.C.) states that for every value of $a > 3$ there exists two primes u_i, p_i such that $2a = u_i - p_i$ and $p_i \in \mathcal{P}$.

Remark 4.9. At this point it is necessary to recall the function in Definition 3.6 that will account for the case where $a + 1$ is prime. This lies in the fact that in Conjecture 4.8 it is possible for $a + 1$ to be prime, but $2a + 2$ to be composite. Since all other $2a < u_i \leq 3a$ every other u_i is either a new prime greater than $a + 1$ or a composition of primes up to a .

Theorem 4.10. *Let $a \in \mathbb{N}_{>3}$. Then $2a$ is a counter-example to the G.D.C. iff for each prime $p_i \in \mathcal{P}$ there exists a unique $\beta_i \in \mathbb{N} \cup \{0\}$ where $\prod_{p_i \in \mathcal{P}} (2a + p_i) = (a + 1)^{\gamma(a+1)} \prod_{p_i \in \mathcal{P}} p_i^{\beta_i}$.*

Proof. If there exists a counter-example to the G.D.C. given by $2a$, then for each prime $p_i \in \mathcal{P}$ there exists some unique u_i where $2a < u_i \leq 3a$ and

$$(4.5) \quad 2a = u_i - p_i$$

with u_i being a composition of primes up to a . Therefore, under the Fundamental Theorem of Arithmetic it follows that for each prime $p_i \in \mathcal{P}$ there must exist a unique $\beta_i \in \mathbb{N}$

$$(4.6) \quad \prod_{i=1}^{\pi(a)} u_i = (a + 1)^{\gamma(a+1)} \prod_{p_i \in \mathcal{P}} p_i^{\beta_i}.$$

Substituting $u_i = 2a + p_i$ from equation 4.5 into 4.6 gives

$$(4.7) \quad \prod_{p_i \in \mathcal{P}} (2a + p_i) = (a + 1)^{\gamma(a+1)} \prod_{p_i \in \mathcal{P}} p_i^{\beta_i}.$$

Conversely, if there exists some $2a > 6$ where equation 4.7 holds, then under the Fundamental Theorem of Arithmetic equations 4.5 and 4.6 are true with no u_i being prime in equation 4.5. A substitution of each u_i from 4.5 into equation 4.6 shows $2a$ is a solution to equation 10.1 and it must be a counter-example to the G.D.C. \square

Lemma 4.11. *For any β_i it follows that $\beta_i > 0$ if and only if $p_i|u_i$ or $p_i|u_j$ for some $j \neq i$.*

Proof. From equation 4.6 it can be seen upon inspection that if any $\beta_i > 0$, then that p_i must be a divisor of the R.H.S. of the equation. Therefore, that p_i divides the L.H.S. of the equation and must divide its corresponding u_i or some other u_j .

Conversely, from equation 4.6 if any prime p_i is a divisor of its corresponding u_i or some other u_j , then that p_i divides the R.H.S. of equation 4.6 showing $p_i|p_i^{\beta_i}$ where $\beta_i > 0$. \square

Lemma 4.12. *For any prime $p_i \in \mathcal{P}$ it follows $p_i|2a$ if and only if $p_i|u_i$.*

Proof. Under equation 4.5 it is seen upon inspection if $p_i|u_i$, then $p_i|2a$. Conversely, from equation 4.5 it follows that if any $p_i|2a$, then $p_i|u_i$. \square

Lemma 4.13. *For any prime $p_i \in \mathcal{P}$, if $p_i|u_i$, then $p_i \nmid u_j$ for any $j \neq i$.*

Proof. Under equation 4.5 of Theorem 4.10 it follows for any primes $p_i, p_j \leq a$

$$(4.8) \quad u_i - p_i = u_j - p_j.$$

If some p_i existed where $p_i|u_i$ and $p_i|u_j$ for some $j \neq i$ in equation 4.8, then $p_i|p_j$. Since both p_i, p_j are primes, then $p_i \nmid p_j$ when $j \neq i$. Thus, if any $p_i|u_i$, then $p_i \nmid u_j$ for any $j \neq i$. \square

Proposition 4.14. *If $p_i|2a$, there exists $m_i, \beta_i \in \mathbb{N} \setminus \{0\}$ s.t. $2a = m_i p_i^{\beta_i} - p_i$, $GCD(p_i, m_i) = 1$.*

Proof. Suppose some $p_i|2a$. From Lemmas 4.11 through 4.13 it can be seen that p_i only divides its corresponding u_i in equations 4.5 and 4.6 showing there exists some non-zero $m_i \in \mathbb{N}$ where $u_i = m_i p_i^{\beta_i}$. A substitution into equation 4.5 shows $2a = m_i p_i^{\alpha_i} - p_i$ where $GCD(p_i, m_i) = 1$. \square

5. CONSTRUCTION OF THE GOLDBACH POLYNOMIAL TYPE I AND ITS PROPERTIES

The G.C. 4.1 will be proven by assuming there exists a counter-example, given by $2a$ where $a \in \mathbb{N}_{>3}$ along with the condition that $2a$ is a solution to equation 4.3 in Theorem 4.3. Using Theorem 4.3 it is possible to define a polynomial based on the behavior of equation 4.3. It will then be shown no counter-examples exist as they would lead to contradiction.

Definition 5.1. It was shown under Theorem 4.3 that a counter-example to the G.C. 4.1 is given by $2a$ where $a \in \mathbb{N}_{>3}$ if and only if it is possible to assign to each prime $p_i \in \mathcal{P}$ some unique $\alpha_i \in \mathbb{N}$ where equation 4.3 holds. Assume $2a$ is a counter-example. To construct the *Goldbach Polynomial Type I* (G.P.I.), it is possible to define the mapping $\mathcal{G}_- : \mathbb{C} \rightarrow \mathbb{C}$ where

$$(5.1) \quad \mathcal{G}_-(z) = \prod_{p_i \in \mathcal{P}} (z - p_i) - \prod_{p_i \in \mathcal{P}} p_i^{\alpha_i}$$

and $2a$ is a root. This shows $\mathcal{G}_-(2a) = 0$ produces equation 4.3 in Theorem 4.3.

Definition 5.2. The Fundamental Theorem of Algebra ensures that there exists, with reciprocity allowed, *Goldbach Polynomial Roots* where the set $G = \{r_k \in \mathbb{C} : \mathcal{G}_-(r_k) = 0\}$. It then follows that equation 5.1 may be written as

$$(5.2) \quad \mathcal{G}_-(z) = \prod_{r_k \in G} (z - r_k).$$

where Definition 5.1 in conjunction with the above factorization shows

$$(5.3) \quad \prod_{r_k \in G} (z - r_k) = \prod_{p_k \in \mathcal{P}} (z - p_k) - \prod_{p_k \in \mathcal{P}} p_k^{\alpha_k}.$$

Lemma 5.3. *There exists some root unique $r_i \in G$ where $r_i \equiv 0 \pmod{p_i}$ iff $\alpha_i > 0$.*

Proof. Assume $\alpha_i > 0$. From equation 5.3 in Definition 5.2 it follows for any integer $1 \leq \mu \leq \alpha_i$

$$(5.4) \quad \prod_{r_k \in G} (z - r_k) \pmod{p_i^{\alpha_i - \mu + 1}} \equiv \prod_{p_k \in \mathcal{P}} (z - p_k) \pmod{p_i^{\alpha_i - \mu + 1}}$$

where it can be seen that there is a singular root at $z \equiv 0 \pmod{p_i}$ proving if $\alpha_i > 0$, there must exist a unique root $r_i \in G$ where $r_i \equiv 0 \pmod{p_i}$.

Alternatively, assume some $\alpha_i = 0$. From equation 5.3 in Definition 5.2

$$(5.5) \quad \prod_{r_k \in G} (z - r_k) \pmod{p_i} \equiv \prod_{p_k \in \mathcal{P}} (z - p_k) - \prod_{p_k \in \mathcal{P}} p_k^{\alpha_k} \pmod{p_i}.$$

Evaluating the equation above at $z = p_i$ gives $\prod_{p_k \in \mathcal{P}} (p_i - p_k) = 0$ showing

$$(5.6) \quad \prod_{r_k \in G} (p_i - r_k) \pmod{p_i} \equiv - \prod_{p_k \in \mathcal{P}} p_k^{\alpha_k} \pmod{p_i}.$$

Since the R.H.S. of the equation cannot be 0 because $p_i^{\alpha_i} = p_i^0 = 1$, it must follow that the L.H.S. is also not zero. Thus, if $\alpha_i = 0$, there exists no $r_i \equiv 0 \pmod{p_i}$. \square

Example 5.4. Let $a = 3$. There exists a G.P.I., $\mathcal{G}_-(z) = (z-2)(z-3) - 2^2 \times 3$ and $\mathcal{G}_-(6) = 0$. Note $6 = 2^2 + 2 = 3 + 3$ in accordance with equations 4.1 and 4.2. There exists another root $r_2 = -1$ in accordance with Definition 5.2. Thus, the roots are given by the set $G = \{-1, 6\}$.

Definition 5.5. The *Goldbach Polynomial Type I Derivative* is given by

$$(5.7) \quad \mathcal{G}'_-(z) = \prod_{p_i \in \mathcal{P}} (z - p_i) \left[\frac{1}{z - p_1} + \frac{1}{z - p_2} + \cdots + \frac{1}{z - p_{\pi(a)}} \right]$$

and follows directly from equation 5.1.

Definition 5.6. The *Goldbach Polynomial Type I Coefficients* are produced by Vietas Formulas [14] for equation 5.1 of Definition 5.1. It is possible to write out each constant term $c_{\pi(a)}, c_{\pi(a)-1}, \dots, c_0$ for the G.P.I. in equations 5.1 and 5.7 in Definitions 5.1 and 5.5 in terms of the primes $p_i \in \mathcal{P}$. For this paper the only constants of importance are given by $c_1 = \mathcal{G}'_-(0)$ and $c_0 = \mathcal{G}_-(0)$ below.

$$(5.8) \quad \mathcal{G}'_-(0) = (-1)^{\pi(a)-1} a\# \left(\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_{\pi(a)}} \right)$$

and

$$(5.9) \quad \mathcal{G}_-(0) = (-1)^{\pi(a)} a\# - \prod_{p_i \in \mathcal{P}} p_i^{\alpha_i}$$

Corollary 5.7. For any prime $p_i \in \mathcal{P}$ the $GCD(p_i, c_1) = 1$.

Proof. The c_1 term in equation 5.8 of Definition 5.6 is the sum of $\pi(a)$ products consisting of $\pi(a) - 1$ primes. Since any prime $p_i \in \mathcal{P}$ is only absent from one product in the sum, $p_i \nmid c_1$ \square

Corollary 5.8. If $\alpha_i > 1$, then $p_i^2 \nmid \mathcal{G}_-(0)$

Proof. This is a direct consequence of equation 5.9. \square

Proposition 5.9. $\alpha_i > 0$ if and only if $\mathcal{G}_-(0) \equiv 0 \pmod{p_i}$.

Proof. This follows directly From equation 5.9 where $\mathcal{G}_-(0) = (-1)^{\pi(a)} a\# - \prod_{p_i \in \mathcal{P}} p_i^{\alpha_i}$. \square

Proposition 5.10. For all prime $p_i \in \mathcal{P}$ the $\mathcal{G}'_-(0) \not\equiv 0 \pmod{p_i}$.

Proof. This is a direct consequence of equation 5.8 of Definition 5.6. \square

Corollary 5.11. For any prime $p_i \in \mathcal{P}$ the $\mathcal{G}'_-(0) \equiv (-1)^{\pi(a)-1} \frac{a\#}{p_i} \pmod{p_i}$.

Proof. Using equation 5.8 proves the corollary since only one term is not divisible by p_i . \square

Proposition 5.12. for any $p_i, p_j \in \mathcal{P}$ it follows $\mathcal{G}_-(p_i) = \mathcal{G}_-(p_j)$.

Proof. For any $p_k \in \mathcal{P}$ equation 5.1 shows $\mathcal{G}_-(p_k) = - \prod_{p_i \in \mathcal{P}} p_i^{\alpha_i}$, proving the proposition. \square

6. PRELIMINARY ANALYSIS OF $\mathcal{G}_-(z)$ WHEN $\mathcal{G}_-(2a) = 0$

Proposition 6.1. Under equation 5.1 of Definition 5.1 it follows that $2a | \mathcal{G}_-(0)$.

Proof. With it assumed $\mathcal{G}_-(2a) = 0$ in 5.1, it then follows that $2a | \mathcal{G}_-(0)$ since $2a$ is a root. \square

Proposition 6.2. The $GCD(2a, \frac{\mathcal{G}_-(0)}{2a}) = 1$.

Proof. From Definition 5.6 and Corollary 5.7, it follows that the $GCD(2a, c_1) = 1$. With $\mathcal{G}_-(2a) = 0$ the equation 5.1 becomes

$$(6.1) \quad 2a \left[(2a)^{\pi(a)-1} + c_{\pi(a)-1} (2a)^{\pi(a)-2} + c_{\pi(a)-2} (2a)^{\pi(a)-3} + \cdots + c_1 \right] = -c_0$$

where the only term in the brackets not multiplied by $2a$ is the c_1 term. Therefore, the $GCD(2a, \frac{c_0}{2a}) = 1$ is a consequence of $\mathcal{G}_-(2a) = 0$ in equation 5.1 and Corollary 5.7. \square

Corollary 6.3. $2a$ is not a repeated root when $\mathcal{G}_-(2a) = 0$.

Proof. This follows directly from Proposition 6.2 and the Rational Root Theorem. \square

Corollary 6.4. Any rational root other than $2a$ is odd when $\mathcal{G}_-(2a) = 0$.

Proof. Under Proposition 6.2 it was shown that $GCD(2a, \frac{c_0}{2a}) = 1$. Taking all roots with multiplicity to be $r_1, r_2, \dots, r_{\pi(a)} \in \mathbb{C}$ with $r_1 = 2a$, it then follows that $GCD(2a, \frac{r_2 \cdots r_{\pi(a)}}{2a}) = 1$. Thus, W.L.O.G. assume there exist a rational root r_2 . With the G.P. being monic, it follows that all rational roots are integers. Therefore, $r_2 \in \mathbb{Z}$. Since $r_2 | c_0$ and $c_0 \in \mathbb{Z}$, there exists some integer x where $\frac{c_0}{2a} = r_2 x$ and the $GCD(2a, r_2 x) = 1$, proving that $GCD(2a, r_2) = 1$. \square

Corollary 6.5. If $\mathcal{G}_-(2a) = 0$, then there are no other rational roots to the G.P. for $\pi(a) > 2$.

Proof. W.L.O.G. let the G.R. $r_1 = 2a$ and assume for the sake of contradiction there exists some other rational root r_2 . Since the G.P. is monic, the Rational Root Theorem shows that $r_2 \in \mathbb{Z}$. Proposition 6.2 ensures $2a$ can not be a repeated root, and Corollary 6.4 ensures $2 \nmid r_2$. Therefore, it must follow from equation 5.1 with $\mathcal{G}_-(r_2) = 0$ that

$$(6.2) \quad \prod_{p_i \in \mathcal{P}} (r_2 - p_i) = \prod_{p_i \in \mathcal{P}} p_i^{\alpha_i}.$$

With $2 \nmid r_2$, only one term of the L.H.S. above is not divisible by 2. Hence,

$$(6.3) \quad 2^{\alpha_1} \geq 2^{\pi(a)-1}.$$

Knowing the minimum value for α_1 , equation 4.1 allows for again to write

$$(6.4) \quad 2a = q_1 + 2.$$

It follows from Corollary 4.6 that q_1 is the only even q_i corresponding to the root $2a$, as all other $q_{i>1}$ must be odd since all other prime $p_{i>1}$ are odd. Hence, from equation 6.3 in conjunction with equation 6.4 it may be seen that $q_1 \geq 2^{\pi(a)-1}$. Therefore, it follows

$$\begin{aligned} 2a &\geq 2^{\pi(a)-1} + 2 \\ a - 1 &\geq 2^{\pi(a)-2}. \end{aligned}$$

However, when $a = 19$ and $\pi(19) = 8$ a substitution of $a = 19$ in the equations above gives

$$18 \not\geq 2^6$$

producing a contradiction. Furthermore, Chebyshev's Theorem says that there for any $a \in \mathbb{N}$ where $a > 1$ there is always some prime p such that $a \leq p < 2a$. Therefore, whenever $a \geq 19$ then $a \not\geq 2^{\pi(a)-2}$. This result can even be made stronger given there are no solutions to the G.P. where $2a$ is a G.R. and $2 < \pi(a) \leq 19$. Therefore, the corollary is true. \square

7. HENSEL'S LEMMA

A proof of Hensel's Lemma [2] is given below.

Theorem 7.1. *Hensel's Lemma.*

Proof. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ where $f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ and $f(z) \in \mathbb{Z}[z]$. It is possible to use a Taylor Series $f(z_0 + t_\mu p_i^\mu) \pmod{p_i^{\mu+1}} \equiv f(z_0) + t_\mu p_i^\mu f'(z_0) \pmod{p_i^{\mu+1}}$. If $f(z_0 + t_\mu p_i^\mu) \equiv 0 \pmod{p_i^{\mu+1}}$ and $f'(z_0) \not\equiv 0 \pmod{p_i}$, then $t_\mu p_i^\mu f'(z_0) \equiv -f(z_0) \pmod{p_i^{\mu+1}}$ where lifting allows $z_0 = t_0 + t_1 p_i + \dots + t_{\mu-1} p_i^{\mu-1} + t_\mu p_i^\mu$ for a unique set $t_0, \dots, t_{\mu-1}, t_\mu \in \mathbb{Z}_{p_i}$. \square

Corollary 7.2. From Propositions 5.9 and 5.10 Hensel's Lemma may be used for any $p_i^{\alpha_i} > 1$.

Proof. Hensel's Lemma 7.1 states that for any $s \in \mathbb{Z}_p$ where $\mathcal{G}_-(s) \equiv 0 \pmod{p}$ along with condition $\mathcal{G}'_-(s) \not\equiv 0 \pmod{p}$, then there exists a unique $t \in \mathbb{Z}_p$ where $\mathcal{G}_-(t) \equiv 0 \pmod{p}$ and $s \equiv t \pmod{p}$. From Proposition 5.9 and 5.10, Hensel's Lemma is satisfied for any $p_i^{\alpha_i} > 1$. \square

8. A DERIVATION OF THE HENSEL-GOLDBACH EQUATIONS

Given everything so far it is actually possible to construct solutions for the Goldbach Polynomial. The strategy is to Use Hensel's Lemma 7.1 to construct solutions for $2a$ based on the Lemmas 4.4, 4.5, and 4.6.

Lemma 8.1. *Any $\alpha_i > 0$ if and only if $2a \equiv 0 \pmod{p_i}$ or $2a \equiv p_j \pmod{p_i}$ for some $j \neq i$.*

Proof. This follows directly from equations 4.1 and 4.2 in Theorem 4.3 and Lemma 4.4. \square

The strategy now is to use Hensel's Lemma 7.1 in conjunction with Lemma 8.1 to construct the form of $2a$ in terms of its prime divisors. The Hensel-Goldbach Equations are defined below.

Definition 8.2. From Proposition 5.9 it was shown any $\alpha_i > 0$ if and only if 0 is a root of $\mathcal{G}_-(z) \pmod{p_i}$. It was also shown from Proposition 5.10 that for any prime $p_i \in \mathcal{P}$ that $\mathcal{G}'_-(0) \not\equiv 0 \pmod{p_i}$ where Corollary 5.11 shows $\mathcal{G}'_-(0) \equiv (-1)^{\pi(a)-1} \frac{a\#}{p_i} \pmod{p_i}$. Therefore, for any $\alpha_i > 0$ Theorem 7.1 and Corollary 7.2 show there exists some unique root $r_i \in G$ from Definition 5.2 where $r_i \equiv 0 \pmod{p_i}$ and $\mathcal{G}'_-(0) \not\equiv 0 \pmod{p_i}$. This allows for the production of the *Hensel-Goldbach Equations* for any integer $0 \leq \mu \leq \alpha_i$ below

$$(8.1) \quad -t_{\alpha_i-\mu} p_i^{\alpha_i-\mu} \mathcal{G}'_-(r_i) \equiv \mathcal{G}_-(r_i) \pmod{p_i^{\alpha_i-\mu+1}}.$$

Therefore, for any $\alpha_i > 0$, Definition 5.2 allows the following relation for all integers $1 \leq \mu \leq \alpha_i$

$$(8.2) \quad \prod_{p_k \in \mathcal{P}} p_k^{\alpha_k} \pmod{p_i^{\alpha_i-\mu+1}} \equiv 0 \pmod{p_i^{\alpha_i-\mu+1}},$$

showing equation 5.1 may be simplified greatly to

$$(8.3) \quad \begin{cases} -t_{\alpha_i-\mu} p_i^{\alpha_i-\mu} \mathcal{G}'_-(r_i) \equiv \prod_{p_k \in \mathcal{P}} (r_i - p_k) \pmod{p_i^{\alpha_i-\mu+1}} \text{ for all integers } 1 \leq \mu \leq \alpha_i \\ -t_{\alpha_i+\mu} p_i^{\alpha_i+\mu} \mathcal{G}'_-(r_i) \equiv \prod_{p_k \in \mathcal{P}} (r_i - p_k) - \prod_{p_k \in \mathcal{P}} p_k^{\alpha_k} \pmod{p_i^{\alpha_i+\mu+1}} \text{ for all integers } \mu \geq 0 \end{cases}$$

where

$$(8.4) \quad r_i = t_0 + t_1 p_i + \dots + t_{\alpha_i-1} p_i^{\alpha_i-1} + t_{\alpha_i} p_i^{\alpha_i} + \dots$$

for unique $t_0, \dots, t_{\alpha_i-1}, t_{\alpha_i}, \dots, \in \mathbb{Z}_{p_i}$ and $r_i \equiv t_0 \pmod{p_i}$.

Example 8.3. Let $\mathcal{G}_-(z) = (z-2)(z-3) - 2^2 \times 3$ where $\mathcal{G}'_-(z) = 2z - 5$. It can be seen that $\mathcal{G}_-(0) \equiv 0 \pmod{2}$ and $\mathcal{G}'_-(0) \equiv -5 \pmod{2} \equiv 1 \pmod{2}$. Using the Hensel-Goldbach Equations gives the 2-adic root approximation $r_k = 0 + t_1 2 + t_2 2^2 + \dots$ with the next iteration

$$\begin{aligned} -t_1 \times 2 \times \mathcal{G}'_-(0) &\equiv \mathcal{G}_-(0) \pmod{2^2} \\ -t_1 \times 2 \times (2 \times 0 - 5) &\equiv (0 - 2) \times (0 - 3) - 2^2 \times 3 \pmod{2^2} \\ t_1 &\equiv 1 \pmod{2} \end{aligned}$$

where the root is $r_k = 0 + 2 + t_2 2^2 + \dots$ where the next iteration gives

$$\begin{aligned} -t_2 \times 2^2 \times \mathcal{G}'_-(2) &\equiv \mathcal{G}_-(2) \pmod{2^3} \\ -t_2 \times 2^2 \times (2 \times 2 - 5) &\equiv (2 - 2) \times (2 - 3) - 2^2 \times 3 \pmod{2^3} \\ t_2 &\equiv 1 \pmod{2} \end{aligned}$$

where $r_k = 0 + 2 + 2^2 + \dots$. However, note that $\mathcal{G}_-(6) = (6-2)(6-3) - 2^2 \times 3 = 0$ identically. Therefore any lifts of 2 in the 2-adic root will vanish. It can also be seen that $6 = 2 \times 3$ is 3-adic and is also a root derived working in $\pmod{3}$.

Theorem 8.4. *There exists a unique root $r_i \in G$ to equations 8.3 of Definition 8.2 where $r_i \equiv t_{\alpha_i} p_i^{\alpha_i} + p_i \pmod{p_i^{\alpha_i+1}}$ and $t_{\alpha_i} \in \mathbb{Z}_{p_i}$ is non-zero if and only if $\alpha_i > 0$.*

Proof. Assume $\alpha_i > 0$. Using the Hensel-Goldbach Equation in 8.3 gives the set of equations

$$(8.5) \quad -t_0 \mathcal{G}'_-(\bar{r}_i) \equiv \prod_{p_k \in \mathcal{P}} (\bar{r}_i - p_k) \pmod{p_i}$$

$$(8.6) \quad \vdots$$

$$(8.7) \quad -t_{\alpha_i-1} p_i^{\alpha_i-1} \mathcal{G}'_-(r_i) \equiv \prod_{p_k \in \mathcal{P}} (r_i - p_k) \pmod{p_i^{\alpha_i}}$$

$$(8.8) \quad -t_{\alpha_i} p_i^{\alpha_i} \mathcal{G}'_-(r_i) \equiv \prod_{p_k \in \mathcal{P}} (r_i - p_k) - \prod_{i=k}^{\pi(a)} p_k^{\alpha_k} \pmod{p_i^{\alpha_i+1}}.$$

From Proposition 5.9 Corollary 5.11 it follows that $0 \pmod{p_i}$ is a root to the Hensel-Goldbach Equations above where plugging in the root zero to equation 8.5 shows $t_0 \equiv 0 \pmod{p_i}$. To solve for the next term it is possible to use the fact that r_i may be written as the p_i -adic series

$$(8.9) \quad r_i = 0 + t_1 p_i + t_2 p_i^2 + \cdots + t_{\alpha_i-1} p_i^{\alpha_i-1} + t_{\alpha_i} p_i^{\alpha_i} + \cdots$$

where all $t \in \mathbb{Z}_{p_i}$. Moving to the second iteration and plugging in $r_i = 0 \pmod{p_i}$ gives

$$\begin{aligned} -t_1 p_i \mathcal{G}'_-(0) &\equiv \mathcal{G}'_-(0) \pmod{p_i^2} \\ -t_1 p_i \mathcal{G}'_-(0) &\equiv (-1)^{\pi(a)} a\# \pmod{p_i^2} \end{aligned}$$

where a substitution from Corollary 5.11 for $\mathcal{G}'_-(0)$ on the L.H.S. shows

$$\begin{aligned} -t_1 p_i \mathcal{G}'_-(0) &\equiv (-1)^{\pi(a)} a\# \pmod{p_i^2} \\ -t_1 \mathcal{G}'_-(0) &\equiv (-1)^{\pi(a)} \frac{a\#}{p_i} \pmod{p_i} \\ -t_1 (-1)^{\pi(a)-1} \frac{a\#}{p_i} &\equiv (-1)^{\pi(a)} \frac{a\#}{p_i} \pmod{p_i} \\ t_1 &\equiv 1 \pmod{p_i} \end{aligned}$$

where the root becomes

$$(8.10) \quad r_i = 0 + p_i + t_2 p_i^2 + \cdots + t_{\alpha_i-1} p_i^{\alpha_i-1} + t_{\alpha_i} p_i^{\alpha_i} + \cdots$$

This allows equation above to be used in the next iteration to give

$$\begin{aligned} -t_2 p_i^2 \mathcal{G}'_-(p_i) &\equiv \prod_{p_k \in \mathcal{P}} (p_i - p_k) \pmod{p_i^3} \\ -t_2 p_i^2 \mathcal{G}'_-(p_i) &\equiv (p_i - p_1)(p_i - p_2) \cdots (p_i - p_i) \cdots (p_i - p_{\pi(a)}) \pmod{p_i^3} \\ -t_2 p_i^2 \mathcal{G}'_-(p_i) &\equiv 0 \pmod{p_i^3} \\ t_2 &\equiv 0 \pmod{p_i} \end{aligned}$$

where

$$(8.11) \quad r_i = 0 + p_i + 0 \times p_i^2 + \cdots + t_{\alpha_i-1} p_i^{\alpha_i-1} + t_{\alpha_i} p_i^{\alpha_i} + \cdots$$

Note the root above shows the R.H.S. of equations 8.5 to 8.7 must be 0 and gives the value for all $t_{1 < k < \alpha_i} = 0$ where the roots becomes

$$(8.12) \quad r_i = p_i + t_{\alpha_i} p_i^{\alpha_i} + \cdots .$$

Moving to the final iteration and plugging in the appropriate value for r_i into 8.8 shows

$$\begin{aligned} -t_{\alpha_i} p_i^{\alpha_i} \mathcal{G}'_-(p_i) &\equiv \prod_{p_k \in \mathcal{P}} (p_i - p_k) - \prod_{i=k}^{\pi(a)} p_k^{\alpha_k} \pmod{p_i^{\alpha_i+1}} \\ -t_{\alpha_i} p_i^{\alpha_i} \mathcal{G}'_-(p_i) &\equiv (p_i - p_1)(p_i - p_2) \cdots (p_i - p_i) \cdots (p_i - p_{\pi(a)}) - \prod_{p_k \in \mathcal{P}} p_k^{\alpha_k} \pmod{p_i^{\alpha_i+1}} \\ -t_{\alpha_i} p_i^{\alpha_i} \mathcal{G}'_-(p_i) &\equiv - \prod_{p_k \in \mathcal{P}} p_k^{\alpha_k} \pmod{p_i^{\alpha_i+1}} \end{aligned}$$

where a simplification by dividing each side by $p_i^{\alpha_i}$ and using the appropriate substitution from Corollary 5.11 for $\mathcal{G}'_-(0)$ on the L.H.S. shows

$$(8.13) \quad t_{\alpha_i} (-1)^{\pi(a)-1} \frac{a\#}{p_i} \equiv \prod_{p_k \in \mathcal{P} \setminus \{p_i\}} p_k^{\alpha_k} \pmod{p_i}.$$

Since $p_i^{\alpha_i}$ was cancelled from both sides above, the R.H.S. of the equation above is never 0. Hence, $t_{\alpha_i} \neq 0$ where equation 8.9 shows $2a \equiv t_{\alpha_i} p_i^{\alpha_i} + p_i \pmod{p_i^{\alpha_i+1}}$ for some $t_{\alpha_i} \in \mathbb{Z}_{p_i} \setminus \{0\}$.

Alternatively, assume that $\alpha_i = 0$. From Proposition 5.9 there is no root $\bar{r}_i \equiv 0 \pmod{p_i}$. \square

9. CONSTRUCTING THE ROOT $2a$ WHEN $\mathcal{G}_-(2a) = 0$

It was shown in Proposition 6.5 that $2a$ is the only rational root to the G.P. in Definition 5.1. Under Theorem 8.4 it was shown that there exists a root of the form $r_i \equiv t_{\alpha_i} p_i^{\alpha_i} + p_i \pmod{p_i^{\alpha_i+1}}$ iff $\alpha_i > 0$. The final step to show no solutions exist when $\deg(\mathcal{G}_-) > 2$ and $\mathcal{G}_-(2a) = 0$. This can be accomplished because it was shown in Lemma 8.1 that $2a$ is related directly to each α_i . The final question that needs to be answered is whether repeated roots emerge if $2a \equiv p_j \pmod{p_i}$ for some $p_j \in \mathcal{P}$ where $i \neq j$. It is possible to show there are not. The reason this is important lies in the fact that when $\alpha_i = 1$, that prime may only divide one q in equation 4.2. Similarly, it was shown under Corollaries 4.5 and 4.6 that if a prime is a divisor of $2a$, it only divides one q . The question seeking an answer is if this pattern continues for any $\alpha_i > 1$ where p_i does not divide $2a$. This will be the focus of the following theorem and corollary.

Lemma 9.1. *If a $p_i \in \mathcal{P}$ exists in 4.1 where $p_i|q_j$ and $p_i|q_k$, then $2a \equiv p_j \equiv p_k \pmod{p_i}$.*

Proof. Assume there exists some $p_i \in \mathcal{P}$ where $p_i|q_j$ and $p_i|q_k$. From equation 4.1

$$(9.1) \quad q_j + p_j = q_k + p_k$$

where $p_j \equiv p_k \pmod{p_i}$. Since $2a = q_j + p_j = q_k + p_k$ it follows that $2a \equiv p_j \equiv p_k \pmod{p_i}$. \square

Proposition 9.2. *For any $p_j \in \mathcal{P}$ the $\mathcal{G}'_-(p_j) = \prod_{p_k \in \mathcal{P} \setminus \{p_j\}} (p_j - p_k)$*

Proof. This follows directly from equation 5.7 in Definition 5.5. \square

Corollary 9.3. *If $p_i \in \mathcal{P}$ where $p_i|q_j$ and $p_i|q_k$, then $\mathcal{G}'_-(p_j) \equiv 0 \pmod{p_i}$.*

Proof. This follows directly from Proposition 9.2 and Lemma 9.1. \square

Proposition 9.4. *If $\alpha_i > 1$, there are no $p_j, p_k \in \mathcal{P}$ where $p_j \equiv p_k \pmod{p_i}$.*

Proof. Suppose there exists some $\alpha_i > 1$ and $2a \not\equiv 0 \pmod{p_i}$. From Lemma 8.1 it can be seen that there must exist some $p_j \in \mathcal{P}$ where $2a \equiv p_j \pmod{p_i}$. The question now is if the Hensel-Goldbach Equations 8.3 may be used for the root approximation for $2a \equiv p_j \pmod{p_i}$. Recall from Proposition 5.12 that $\mathcal{G}_-(p_i) = \mathcal{G}_-(p_j)$ where it follows $\mathcal{G}_-(p_i) \equiv \mathcal{G}_-(p_j) \pmod{p_i^k}$ for any integer $k > 0$. This allows for $\mathcal{G}_-(0) \equiv \mathcal{G}_-(p_j) \pmod{p_i}$. Since α_i is assumed to be greater than one, Corollary 5.8 and Propositions 5.9, 5.10 show that $\mathcal{G}_-(0)$ has a singular root $\pmod{p_i}$, thus proving that $\mathcal{G}_-(p_j)$ is a singular root $\pmod{p_i}$. Hence, it may be concluded that there exists no prime $p_k \in \mathcal{P}$ where $p_j \equiv p_k \pmod{p_i}$. From Lemma 9.1 it may be seen that no prime $p_i \in \mathcal{P}$ divides any q_j and q_k . \square

Corollary 9.5. For any q_i, q_j in equations 4.1 and 4.2 in Theorem 4.3, the $GCD(q_i, q_j) = 1$.

Proof. Under Proposition 9.4 it was shown that no two q_j, q_k are divisible by the same prime $p_i \in \mathcal{P}$. Since all q 's must be divisible by some prime in \mathcal{P} , the $GCD(q_i, q_j) = 1$. \square

The final step is to use the proof Catalan's Conjecture [9] which showed that the largest solutions to the Diophantine Equation of the form $x^\mu - y^\nu = 1$ is given by $3^2 - 2^3 = 1$.

Theorem 9.6. There are no solutions to equation 5.1 where $\mathcal{G}_-(2a) = 0$ and $a > 3$.

Proof. From Corollary 9.5 it was shown that no prime p_i divides any two q 's. Therefore, each q is a perfect prime power and for any $p_i \in \mathcal{D}$ from 3.2 it follows that

$$(9.2) \quad 2a = p_i^{\alpha_i} + p_i \text{ for all prime } p_i \in \mathcal{D}.$$

Since, there are $\pi(a)$ q 's that share no primes, it follows from the Pigeon Hole Principle that all $\alpha_i > 0$. Using the transitive property from equation 4.1, and the fact that $2|2a$, it follows that there exists some prime power where $2^{\alpha_1} + 2 = p_j^{\alpha_j} + 3$. However, the proof of Catalan's Conjecture ensures that the largest values this equation has is $2^2 + 2 = 3 + 3$. Therefore, that the largest value for $2a$ satisfying $\mathcal{G}_-(2a) = 0$ is when $a = 3$. Under Theorem 4.3 no counter-examples to the Goldbach Conjecture exists and it must be true. \square

Theorem 9.7. The Ternary Conjecture³ is true.

Proof. For any odd $n \in \mathbb{N}$ such that $n \geq 7$ there exists some even $m \in \mathbb{N}$ where $n = 3 + m$. Under Theorem 9.6 for any even $m > 2$ there exists $p_2, p_3 \in \mathbb{P}$ where $m = p_1 + p_2$. Thus, for any odd $n \geq 7$ there exists $p_1, p_2, p_3 \in \mathbb{P}$ where $n = p_1 + p_2 + p_3$. \square

Corollary 9.8. Every prime larger than 7 is the sum of three odd primes.

Proof. This follows trivially from Theorem 9.7 since all primes greater than seven are odd. \square

Definition 9.9. Let $a \in \mathbb{N}$ such that $a > 1$. Since

$$(9.3) \quad 2a = (a + b) + (a - b)$$

for any $b \in \mathbb{N}$, a *Prime Reflective Point (P.R.P.)* is any $b_R \in \mathbb{N}$ where $a \pm b_R \in \mathbb{P}$ and $b_R < a$.

Theorem 9.6 along with Definition 9.9 allow for a slightly stronger conjecture than the G.C. if it can be shown that there are no solutions to the G.P. when $a > 3$ and $2a$ is a G.R.

Theorem 9.10. Every $a \in \mathbb{N}$ where $a > 3$ has some non-zero P.R.P.

Proof. Since no solutions exist to Theorem 4.3 when $2a > 6$, this must also hold when a is prime. This would allow for a cancellation of a from both sides of equation 4.2. Since solutions would still not exist, another q_i must be prime in equation 4.1. Thus, since every prime has a non-zero P.R.P. and any composite a must also have a non-zero P.R.P., the theorem is true. \square

The next section will follow a nearly identical approach to this section in order to prove that there are no counter-examples to the G.D.C. 4.8

10. CONSTRUCTION OF THE GOLDBACH POLYNOMIAL TYPE II

Using a nearly identical procedure as was used in the previous section it is also important to expand this idea to Theorem 4.10 below.

Definition 10.1. It was shown under Theorem 4.10 that a counter-example to the G.D.C. is given by $2a$ where $a \in \mathbb{N}_{>3}$ if and only if it is possible to assign to each prime $p_i \in \mathcal{P}$ some unique $\beta_i \in \mathbb{N}$ where equation 4.7 holds. Assume $2a$ is a counter-example. To construct the *Goldbach Polynomial Type II (G.P.II)*, it is possible to define the mapping $\mathcal{G}_+ : \mathbb{C} \rightarrow \mathbb{C}$

$$(10.1) \quad \mathcal{G}_+(z) = \prod_{p_i \in \mathcal{P}} (z + p_i) - (a + 1)^{\gamma(a+1)} \prod_{p_i \in \mathcal{P}} p_i^{\beta_i}.$$

³Harald Helfgott's 2013 work is generally accepted as sufficient for proving this conjecture.

and $2a$ is a root. This shows $\mathcal{G}_+(2a) = 0$ produces equation 4.7 in Theorem 4.10.

Definition 10.2. The Fundamental Theorem of Algebra ensures that there exists, with reciprocity allowed, *Goldbach Difference Polynomial Roots* where $G' = \{r'_k \in \mathbb{C} : \mathcal{G}_+(r'_k) = 0\}$. It then follows that equation 10.1 may be written as

$$(10.2) \quad \mathcal{G}_+(z) = \prod_{r'_k \in G'} (z - r'_k).$$

where Definition 10.1 in conjunction with the above factorization shows

$$(10.3) \quad \prod_{r'_k \in G'} (z - r'_k) = \prod_{p_k \in \mathcal{P}} (z + p_k) - (a + 1)^{\gamma(a+1)} \prod_{p_k \in \mathcal{P}} p_k^{\beta_k}.$$

Lemma 10.3. *There exists some root unique $r'_i \in G'$ where $r'_i \equiv 0 \pmod{p_i}$ iff $\beta_i > 0$.*

Proof. Assume $\beta_i > 0$. From equation 10.3 in Definition 10.2, for any integer $1 \leq \mu \leq \beta_i$

$$(10.4) \quad \prod_{r'_k \in G'} (z - r'_k) \pmod{p_i^{\beta_i - \mu + 1}} \equiv \prod_{p_k \in \mathcal{P}} (z + p_k) \pmod{p_i^{\beta_i - \mu + 1}}$$

where it can be seen that there is a singular root at $z \equiv 0 \pmod{p_i}$ proving if $\beta_i > 0$, there must exist a unique root $r'_i \in G'$ where $r'_i \equiv 0 \pmod{p_i}$.

Alternatively, assume some $\beta_i = 0$. From equation 10.3 in Definition 10.2

$$(10.5) \quad \prod_{r'_k \in G'} (z - r'_k) \pmod{p_i} \equiv \prod_{p_k \in \mathcal{P}} (z + p_k) - (a + 1)^{\gamma(a+1)} \prod_{p_k \in \mathcal{P}} p_k^{\beta_k} \pmod{p_i}.$$

Evaluating the equation above at $z = p_i$ gives $\prod_{p_k \in \mathcal{P}} (p_i + p_k) \equiv 0 \pmod{p_i}$ showing

$$(10.6) \quad \prod_{r'_k \in G'} (p_i - r'_k) \pmod{p_i} \equiv -(a + 1)^{\gamma(a+1)} \prod_{p_k \in \mathcal{P}} p_k^{\alpha_k} \pmod{p_i}.$$

Since the R.H.S. of the equation cannot be 0 because $p_i^{\beta_i} = p_i^0 = 1$ and Definition 3.6 ensures $p_i \nmid (a + 1)$, the L.H.S. is not zero. Thus, if $\beta_i = 0$, there exists no $r'_i \equiv 0 \pmod{p_i}$. \square

Example 10.4. Let $a = 3$. There exists a G.P.II, $\mathcal{G}_+(z) = (z + 2)(z + 3) - 2^3 \times 3^2$ since $\mathcal{G}_+(6) = 0$. Note $6 = 2^3 - 2 = 3^2 - 3$ in accordance with equations 4.5 and 4.6. There exists another root $r'_2 = -11$ in accordance with Definition 10.2 the roots are given by $G' = \{-11, 6\}$.

Definition 10.5. The *Goldbach Polynomial Type II Derivative* is given by

$$(10.7) \quad \mathcal{G}'_+(z) = \prod_{p_i \in \mathcal{P}} (z + p_i) \left[\frac{1}{z + p_1} + \frac{1}{z + p_2} + \dots + \frac{1}{z + p_{\pi(a)}} \right]$$

and follows directly from equation 10.1.

Definition 10.6. The *Goldbach Polynomial Type II Difference Coefficients* are produced by Vietas Formulas [14] in the same manner as equations 5.1 and 5.5 of Definitions 5.1 and 5.5. It is possible to write out each constant term $d_{\pi(a)}, d_{\pi(a)-1}, \dots, d_0$ for the G.P.II in equations 10.1 and 10.7 in Definitions 10.1 and 10.5 in terms of the primes $p_i \in \mathcal{P}$. For this paper the only constants of importance are given by $d_1 = \mathcal{G}'_+(0)$ and $d_0 = \mathcal{G}_+(0)$ below. The only difference in these equations from previous sections are the minus signs and $a + 1$ term below.

$$(10.8) \quad \mathcal{G}'_+(0) = a\# \left(\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_{\pi(a)}} \right)$$

and

$$(10.9) \quad \mathcal{G}_+(0) = a\# - (a + 1)^{\gamma(a+1)} \prod_{p_i \in \mathcal{P}} p_i^{\beta_i}.$$

Corollary 10.7. *For any prime $p_i \in \mathcal{P}$ the $GCD(p_i, d_1) = 1$.*

Proof. The d_1 term in equation 10.8 of Definition 10.6 is the sum of $\pi(a)$ products consisting of $\pi(a) - 1$ primes. Since any prime $p_i \in \mathcal{P}$ is only absent from one product in the sum, $p_i \nmid d_1$ \square

Corollary 10.8. *If $\beta_i > 1$, then $p_i^2 \nmid \mathcal{G}_+(0)$*

Proof. This is a direct consequence of equation 10.9. \square

Proposition 10.9. *$\beta_i > 0$ if and only if $\mathcal{G}_+(0) \equiv 0 \pmod{p_i}$.*

Proof. This follows From Definition 10.1 where $\mathcal{G}_+(0) = a\# - (a+1)^{\gamma(a+1)} \prod_{p_i \in \mathcal{P}} p_i^{\beta_i}$. \square

Proposition 10.10. *For all prime $p_i \in \mathcal{P}$ the $\mathcal{G}'_+(0) \not\equiv 0 \pmod{p_i}$.*

Proof. This is a direct consequence of equation 10.8 of Definition 10.6. \square

It is actually possible to calculate an explicit value for $\mathcal{G}'_+(0)$ using Definition 10.6

Corollary 10.11. *For any prime $p_i \in \mathcal{P}$ the $\mathcal{G}'_+(0) \equiv \frac{a\#}{p_i} \pmod{p_i}$.*

Proof. Using equation 10.8 proves the corollary since only one term is not divisible by p_i . \square

Proposition 10.12. *for any $p_i, p_j \in \mathcal{P}$ it follows $\mathcal{G}_+(-p_i) = \mathcal{G}_+(-p_j)$.*

Proof. For any $p_k \in \mathcal{P}$ equation 10.1 shows $\mathcal{G}_+(-p_k) = -(a+1)^{\gamma(a+1)} \prod_{p_i \in \mathcal{P}} p_i^{\beta_i}$. \square

It is possible to use this same line of reasoning to analyze the structure of $2a$ when $\mathcal{G}_+(2a) = 0$.

11. PRELIMINARY ANALYSIS OF $\mathcal{G}_+(z)$ WHEN $\mathcal{G}_+(2a) = 0$

Proposition 11.1. *Under equation 10.1 of Definition 10.1 it follows that $2a \mid \mathcal{G}_+(0)$.*

Proof. With it assumed $\mathcal{G}_+(2a) = 0$ in 10.1, it then follows that $2a \mid \mathcal{G}_+(0)$ since $2a$ is a root. \square

Proposition 11.2. *The $GCD(2a, \frac{\mathcal{G}_+(0)}{2a}) = 1$.*

Proof. From Definition 10.6 Corollary 10.7 any prime $p_i \in \mathcal{P}$ has $GCD(p_i, d_1) = 1$. Hence, it follows that the $GCD(2a, d_1) = 1$. With $\mathcal{G}_+(2a) = 0$ the equation 10.1 becomes

$$(11.1) \quad 2a \left[(2a)^{\pi(a)-1} + d_{\pi(a)-1} (2a)^{\pi(a)-2} + d_{\pi(a)-2} (2a)^{\pi(a)-3} + \dots + d_1 \right] = -d_0$$

where the only term in the brackets not multiplied by $2a$ is the d_1 term. Therefore, the $GCD(2a, \frac{\mathcal{G}_+(0)}{2a})$ is a consequence of $\mathcal{G}_+(2a) = 0$ in equation 10.1 and Corollary 10.7. \square

Corollary 11.3. *$2a$ is not a repeated root when $\mathcal{G}_+(2a) = 0$.*

Proof. This follows directly from Proposition 11.2 and the Rational Root Theorem. \square

Corollary 11.4. *Any rational root other than $2a$ is odd when $\mathcal{G}_+(2a) = 0$.*

Proof. Under Proposition 11.2 it was shown that $GCD(2a, \frac{d_0}{2a}) = 1$. Taking all roots with multiplicity to be $r'_1, r'_2, \dots, r'_{\pi(a)} \in \mathbb{C}$ with $r'_1 = 2a$, it then follows that $GCD(2a, \frac{r'_2 \dots r'_{\pi(a)}}{2a}) = 1$. Thus, W.L.O.G. assume there exist a rational root r'_2 . With the G.P. being monic, it follows that all rational roots are integers. Therefore, $r'_2 \in \mathbb{Z}$. Since $r'_2 \mid d_0$ and $d_0 \in \mathbb{Z}$, there exists some integer x where $\frac{d_0}{2a} = r'_2 x$ and the $GCD(2a, r'_2 x) = 1$, proving that $GCD(2a, r'_2) = 1$. \square

Corollary 11.5. *If $\mathcal{G}_+(2a) = 0$, then there are no other rational roots to the G.P. for $\pi(a) > 2$.*

Proof. W.L.O.G. let the G.R. $r'_1 = 2a$ and assume for the sake of contradiction there exists some other rational root r'_2 . Since the G.P. is monic, the Rational Root Theorem shows $r'_2 \in \mathbb{Z}$. Proposition 11.2 ensures $2a$ can not be a repeated root, and Corollary 11.4 ensures $2 \nmid r'_2$. Therefore, it must follow from equation 10.1 with $\mathcal{G}_+(r'_2) = 0$ that

$$(11.2) \quad \prod_{p_i \in \mathcal{P}} (r'_2 + p_i) = (a+1)^{\gamma(a+1)} \prod_{p_i \in \mathcal{P}} p_i^{\beta_i}.$$

With $2 \nmid r'_2$, only one term of the L.H.S. above is not divisible by 2. Hence,

$$(11.3) \quad 2^{\beta_1} \geq 2^{\pi(a)-1}.$$

Knowing the minimum value for β_1 , equation 4.5 allows for again to write

$$(11.4) \quad 2a = u_1 - 2.$$

It follows from Corollary 4.13 u_1 is the only even u_i corresponding to the root $2a$, as all other $u_{i>1}$ must be odd since all other prime $p_{i>1}$ are odd. Hence, from equation 11.3 in conjunction with equation 11.4 it may be seen that $u_1 \geq 2^{\pi(a)-1}$. Therefore, it follows

$$\begin{aligned} 2a &\geq 2^{\pi(a)-1} - 2 \\ a + 1 &\geq 2^{\pi(a)-2}. \end{aligned}$$

However, when $a = 19$ and $\pi(19) = 8$ a substitution of $a = 19$ in the equations above gives

$$20 \not\geq 2^6$$

producing a contradiction. Furthermore, Chebyshev's Theorem says that there for any $a \in \mathbb{N}$ where $a > 1$ there is always some prime p such that $a \leq p < 2a$. Therefore, whenever $a \geq 19$ then $a \not\geq 2^{\pi(a)-2}$. This result can even be made stronger given there are no solutions to the G.D.P. where $2a$ is a G.D.R. and $2 < \pi(a) \leq 19$. Therefore, the corollary is true. \square

Corollary 11.6. *From Propositions 10.9 and 10.11 Hensel's Lemma may be used for $p_i^{\beta_i} > 1$.*

Proof. Hensel's Lemma 7.1 states that for any $s \in \mathbb{Z}_p$ where $\mathcal{G}_+(s) \equiv 0 \pmod{p}$ along with condition $\mathcal{G}'_+(s) \not\equiv 0 \pmod{p}$, then there exists a unique $t \in \mathbb{Z}_p$ where $\mathcal{G}_+(t) \equiv 0 \pmod{p}$ and $s \equiv t \pmod{p}$. From Proposition 10.9 and 10.11, Hensel's Lemma is satisfied if $p_i^{\beta_i} > 1$. \square

The strategy now is to Use Hensel's Lemma to construct $2a$ based on Lemmas 4.11 - 4.13.

12. A DERIVATION OF THE HENSEL-GOLDBACH DIFFERENCE EQUATIONS

Lemma 12.1. $\beta_i > 0$ if and only if $2a \equiv 0 \pmod{p_i}$ or $2a \equiv -p_j \pmod{p_i}$ for some $j \neq i$.

Proof. This follows directly from equations 4.5 and 4.6 in Theorem 4.10 and Lemma 4.11. \square

Definition 12.2. From Proposition 10.9 it was shown that any $\beta_i > 0$ if and only if p_i is a root of $\mathcal{G}_+(z) \pmod{p_i}$. It was also shown from Proposition 10.10 that for for any prime $p_i \in \mathcal{P}$ that $\mathcal{G}'_+(0) \not\equiv 0 \pmod{p_i}$ where Corollary 10.11 shows $\mathcal{G}'_+(0) \equiv \frac{a\#}{p_i} \pmod{p_i}$. Therefore, for any $\beta_i > 0$ Theorem 7.1 and Corollary 11.6 show there exists some unique root $r'_i \in G'$ from Definition 10.2 where $r'_i \equiv 0 \pmod{p_i}$ and $\mathcal{G}'_+(0) \not\equiv 0 \pmod{p_i}$. This root allows for the *Hensel-Goldbach Difference Equations* given below

$$(12.1) \quad -t'_{\beta_i - \mu} p_i^{\beta_i - \mu} \mathcal{G}'_+(r'_i) \equiv \mathcal{G}_+(r'_i) \pmod{p_i^{\beta_i - \mu + 1}}$$

for any $0 \leq \mu \leq \beta_i$. Therefore, for any $\beta_i > 0$, Definition 10.1 allows the following relation for all integers $1 \leq \mu \leq \beta_i$

$$(12.2) \quad (a + 1)^{\gamma(a+1)} \prod_{p_k \in \mathcal{P}} p_k^{\beta_k} \pmod{p_i^{\beta_i - \mu + 1}} \equiv 0 \pmod{p_i^{\beta_i - \mu + 1}}$$

where using equation 10.1 and equation 12.1 may be simplified greatly to

$$(12.3) \quad \begin{cases} -t'_{\beta_i - \mu} p_i^{\beta_i - \mu} \mathcal{G}'_+(r'_i) \equiv \prod_{p_k \in \mathcal{P}} (r'_i + p_k) \pmod{p_i^{\beta_i - \mu + 1}} : \text{for all } 1 \leq \mu \leq \beta_i \\ -t'_{\beta_i} p_i^{\beta_i} \mathcal{G}'_+(r'_i) \equiv \prod_{p_k \in \mathcal{P}} (r'_i + p_k) - (a + 1)^{\gamma(a+1)} \prod_{p_k \in \mathcal{P}} p_k^{\beta_k} \pmod{p_i^{\beta_i + 1}} \end{cases}$$

where it is now possible to use these equations to find roots for the G.P.II given by

$$(12.4) \quad r'_i = t'_0 + t'_1 p_i + t'_{\beta_i - 1} p_i^{\beta_i - 1} + t'_{\beta_i} p_i^{\beta_i} + \dots$$

for unique $t'_0, \dots, t'_{\beta_i - 1}, t'_{\beta_i} \in \mathbb{Z}_{p_i}$ and $r'_i \equiv t'_0 \pmod{p_i}$.

Example 12.3. Let $\mathcal{G}_+(z) = (z+2)(z+3) - 2^3 \times 3^2$ where $\mathcal{G}'_+(z) = 2z+5$. It can be seen that $\mathcal{G}_+(0) \equiv 0 \pmod{3}$ and $\mathcal{G}'_+(0) \equiv 5 \pmod{3} \equiv 2 \pmod{3}$. Using the Hensel-Goldbach Difference Equations allows for the 3-adic root $r'_k = 0 + t'_1 3 + t'_2 3^2 + \dots$ where the next iteration gives

$$\begin{aligned} -t'_1 \times 3 \times \mathcal{G}'_+(0) &\equiv \mathcal{G}_+(0) \pmod{3^2} \\ -t'_1 \times 3 \times (2 \times 0 + 5) &\equiv (0+2) \times (0+3) - 2^3 \times 3^2 \pmod{3^2} \\ -t'_1 \times 3 \times (2 \times 0 + 5) &\equiv (0+2) \times (0+3) \pmod{3^2} \\ -t'_1 \times 5 &\equiv 2 \pmod{3} \\ t'_1 &\equiv 2 \pmod{3} \end{aligned}$$

At this point it is best to write $2 = (3-1)$ where the root is $r_k = 0 + (3-1) \times 3 + t_2 3^2 + \dots$. However, note that $\mathcal{G}_+(6) = (6+2)(6+3) - 2^3 \times 3^2 = 0$ identically. Therefore any higher lifts in the 3-adic root will vanish. Notice also that $6 = (3-1) \times 3 = 3^2 - 3$ in accordance with Proposition 4.14 and will have relevance in the following theorem.

Theorem 12.4. *There exists a unique root r'_i to the Hensel-Goldbach Difference Equations of Definition 12.2 where $r'_i \equiv t'_{\beta_i} p_i^{\beta_i} - p_i \pmod{p_i^{\beta_i+1}}$ and $t'_{\beta_i} \in \mathbb{Z}_{p_i} \setminus \{0\}$ iff $\beta_i > 0$.*

Proof. Assume $\beta_i > 0$. Using the Hensel-Goldbach Difference Equations in 12.3 gives

$$(12.5) \quad -t'_0 \mathcal{G}'_+(r'_i) \equiv \prod_{p_k \in \mathcal{P}} (r'_i + p_k) \pmod{p_i}$$

$$(12.6) \quad \vdots$$

$$(12.7) \quad -t'_{\beta_i-1} p_i^{\beta_i-1} \mathcal{G}'_+(r'_i) \equiv \prod_{p_k \in \mathcal{P}} (r'_i + p_k) \pmod{p_i^{\beta_i}}$$

$$(12.8) \quad -t'_{\beta_i} p_i^{\beta_i} \mathcal{G}'_+(r'_i) \equiv \prod_{p_k \in \mathcal{P}} (r'_i + p_k) - (a+1)^{\gamma(a+1)} \prod_{p_k \in \mathcal{P}} p_k^{\beta_k} \pmod{p_i^{\beta_i+1}}.$$

From Proposition 10.9 Corollary 10.11 it follows that $0 \pmod{p_i}$ is a root to the Hensel-Goldbach Difference Equations above where plugging in the root zero to equation 12.5 shows $t'_0 \equiv 0 \pmod{p_i}$. To solve for the next term it is possible to use the fact that r'_i may be written as a series of the prime powers p_i below

$$(12.9) \quad r'_i = 0 + t'_1 p_i + t'_2 p_i^2 + \dots + t'_{\beta_i-1} p_i^{\beta_i-1} + t'_{\beta_i} p_i^{\beta_i} + \dots$$

where all $t' \in \mathbb{Z}_{p_i}$. Moving to the next iteration and plugging in the root $\bar{r}_i = 0$ gives

$$(12.10) \quad -t'_1 p_i \mathcal{G}'_+(0) \equiv \mathcal{G}_+(0) \pmod{p_i^2}$$

$$(12.11) \quad -t'_1 p_i \mathcal{G}'_+(0) \equiv a\# \pmod{p_i^2}$$

where a simplification of the R.H.S. and a substitution from Corollary 10.11 for $\mathcal{G}'_+(0)$ on the L.H.S. shows $t_1 \equiv -1 \pmod{p_i}$ showing $t_1 = p_i - 1$. This allows equation 12.9 to be written as

$$(12.12) \quad r'_i = 0 + (p_i - 1)p_i + t'_2 p_i^2 + \dots + t'_{\beta_i-1} p_i^{\beta_i-1} + t'_{\beta_i} p_i^{\beta_i} + \dots$$

where $r'_i = p_i^2 - p_i \pmod{p_i^3}$. Moving to the next iteration to solve for t_2 gives

$$\begin{aligned}
-t'_2 p_i^2 \mathcal{G}'_+(p_i^2 - p_i) &\equiv \prod_{p_k \in \mathcal{P}} (p_i^2 - p_i + p_k) \pmod{p_i^3} \\
-t'_2 p_i^2 \mathcal{G}'_+(p_i^2 - p_i) &\equiv (p_i^2 - p_i + p_1)(p_i^2 - p_i + p_2) \cdots (p_i^2 - p_i + p_i) \cdots (p_i^2 - p_i + p_{\pi(a)}) \pmod{p_i^3} \\
-t'_2 p_i^2 \mathcal{G}'_+(p_i^2 - p_i) &\equiv p_i^2 \prod_{p_k \in \mathcal{P} \setminus \{p_i\}} (p_i^2 - p_i + p_k) \pmod{p_i^3} \\
-t'_2 \mathcal{G}'_+(p_i^2 - p_i) &\equiv \prod_{p_k \in \mathcal{P} \setminus \{p_i\}} (p_i^2 - p_i + p_k) \pmod{p_i} \\
-t'_2 \mathcal{G}'_+(0) &\equiv \frac{a\#}{p_i} \pmod{p_i}
\end{aligned}$$

where Corollary 10.11 gives $t'_2 \equiv -1 \pmod{p_i}$ showing $t'_2 = p_i - 1$. Plugging into 12.12 gives

$$\begin{aligned}
r'_i &= 0 + (p_i - 1)p_i + (p_i - 1)p_i^2 + \cdots + t'_{\beta_i - 1} p_i^{\beta_i - 1} + t'_{\beta_i} p_i^{\beta_i} + \cdots \\
r'_i &= p_i^3 - p_i + t'_3 p_i^3 + \cdots + t'_{\beta_i - 1} p_i^{\beta_i - 1} + t'_{\beta_i} p_i^{\beta_i} + \cdots
\end{aligned}$$

Note, that this pattern continues in the R.H.S. of all equations 12.5 to 12.7 where it must follow that all $t'_{1 < k < \alpha_i} = p_i - 1$. This allows for

$$\begin{aligned}
r'_i &= 0 + (p_i - 1)p_i + (p_i - 1)p_i^2 + \cdots + (p_i - 1)p_i^{\beta_i - 1} + (p_i - 1)p_i^{\beta_i - 1} + t'_{\beta_i} p_i^{\beta_i} + \cdots \\
r'_i &= p_i^{\beta_i} - p_i + t'_{\beta_i} p_i^{\beta_i} + \cdots
\end{aligned}$$

where it is possible to substitute in $r'_i = p_i^{\beta_i} - p_i$ into equation 12.8 to find t'_{β_i} below.

$$\begin{aligned}
-t'_{\beta_i} p_i^{\beta_i} \mathcal{G}'_+(p_i^{\beta_i} - p_i) &\equiv \prod_{p_k \in \mathcal{P}} (p_i^{\beta_i} - p_i + p_k) - (a+1)^{\gamma(a+1)} \prod_{p_k \in \mathcal{P}} p_k^{\beta_k} \pmod{p_i^{\beta_i + 1}} \\
-t'_{\beta_i} p_i^{\beta_i} \mathcal{G}'_+(p_i^{\beta_i} - p_i) &\equiv (p_i^{\beta_i} - p_i + p_1) \cdots (p_i^{\beta_i} - p_i + p_{\pi(a)}) - (a+1)^{\gamma(a+1)} \prod_{p_k \in \mathcal{P}} p_k^{\beta_k} \pmod{p_i^{\beta_i + 1}} \\
-t'_{\beta_i} p_i^{\beta_i} \mathcal{G}'_+(p_i^{\beta_i} - p_i) &\equiv p_i^{\beta_i} \prod_{p_k \in \mathcal{P} \setminus \{p_i\}} (p_i^{\beta_i} - p_i + p_k) - (a+1)^{\gamma(a+1)} p_i^{\beta_i} \prod_{p_k \in \mathcal{P} \setminus \{p_i\}} p_k^{\beta_k} \pmod{p_i^{\beta_i + 1}} \\
-t'_{\beta_i} \mathcal{G}'_+(p_i^{\beta_i} - p_i) &\equiv \prod_{p_k \in \mathcal{P} \setminus \{p_i\}} (p_i^{\beta_i} - p_i + p_k) - (a+1)^{\gamma(a+1)} \prod_{p_k \in \mathcal{P} \setminus \{p_i\}} p_k^{\beta_k} \pmod{p_i} \\
-t'_{\beta_i} \mathcal{G}'_+(0) &\equiv \frac{a\#}{p_i} - (a+1)^{\gamma(a+1)} \prod_{p_k \in \mathcal{P} \setminus \{p_i\}} p_k^{\beta_k} \pmod{p_i}
\end{aligned}$$

From Corollary 10.11 the equation above becomes

$$(12.13) \quad -t'_{\beta_i} \frac{a\#}{p_i} \equiv \frac{a\#}{p_i} - (a+1)^{\gamma(a+1)} \prod_{p_k \in \mathcal{P} \setminus \{p_i\}} p_k^{\beta_k} \pmod{p_i}$$

where a simplification gives

$$(12.14) \quad -(t'_{\beta_i} + 1) \frac{a\#}{p_i} \equiv -(a+1)^{\gamma(a+1)} \prod_{p_k \in \mathcal{P} \setminus \{p_i\}} p_k^{\beta_k} \pmod{p_i}$$

From Definition 3.6 there are no solutions to 12.14 where $t'_{\beta_i} \equiv -1 \pmod{p_i}$ since the R.H.S. is never divisible by p_i . Therefore, it must follow that $0 \leq t'_{\beta_i} < p_i - 1$ showing $t'_{\beta_i} + 1 \in \mathbb{Z}_{p_i}$. A rescaling of $t'_{\beta_i} \mapsto t'_{\beta_i} + 1$ shows $r'_i \equiv t'_{\beta_i} p_i^{\beta_i} - p_i \pmod{p_i^{\beta_i + 1}}$ where $t'_{\beta_i} \in \mathbb{Z}_{p_i}$.

Alternatively, assume that $\beta_i = 0$. Then, from Proposition 10.9 it can be seen that there is no root where $r'_i \equiv 0 \pmod{p_i}$. \square

Following a similar approach to the one in this section it is possible to derive a set of equations that will allow for the construction of roots to the G.P.II in Definition 10.1

13. CONSTRUCTING THE ROOT $2a$ WHEN $\mathcal{G}_+(2a) = 0$

It was shown in Proposition 11.5 that $2a$ is the only rational root to the G.P. in Definition 10.1. Under Theorem 12.4 it was shown that there exists a root of the form $r'_i \equiv t'_{\beta_i} p_i^{\beta_i} + p_i \pmod{p_i^{\beta_i+1}}$ iff $\beta_i > 0$. The final step to show no solutions exist when $\deg(\mathcal{G}_+) > 2$ when $\mathcal{G}_+(2a) = 0$. This can be accomplished because it was shown in Lemma 12.1 that $2a$ is related directly to each β_i . The final question that needs to be answered is whether repeated roots emerge if $2a \equiv -p_j \pmod{p_i}$ for some $p_j \in \mathcal{P}$ where $i \neq j$. It is possible to show there are not. The reason this is important lies in the fact that when $\alpha_i = 1$, that prime may only divide on u in equation 4.6. Similarly, it was shown under Corollaries 4.11 and 4.12 that if a prime is a divisor of $2a$, it only divides one u . The question seeking an answer is if this pattern continues for any $\beta_i > 1$ where p_i does not divide $2a$. This will be the focus of this section.

Lemma 13.1. *If a $p_i \in \mathcal{P}$ exists in 4.5 where $p_i|u_j$ and $p_i|u_k$, then $2a \equiv -p_j \equiv -p_k \pmod{p_i}$.*

Proof. Assume there exists some $p_i \in \mathcal{P}$ where $p_i|q_j$ and $p_i|q_k$. From equation 4.5

$$(13.1) \quad u_j - p_j = u_k - p_k$$

where $p_j \equiv p_k \pmod{p_i}$. Since $2a = u_j - p_j = u_k - p_k$ it follows $2a \equiv -p_j \equiv -p_k \pmod{p_i}$. \square

Proposition 13.2. *For any $p_j \in \mathcal{P}$ the $\mathcal{G}'_+(-p_j) = \prod_{p_k \in \mathcal{P} \setminus \{p_j\}} (p_k - p_j)$*

Proof. This follows directly from equation 10.7 in Definition 10.7. \square

Corollary 13.3. *If $p_i \in \mathcal{P}$ where $p_i|u_j$ and $p_i|u_k$, then $\mathcal{G}'_+(-p_j) \equiv 0 \pmod{p_i}$.*

Proof. This follows directly from Proposition 13.2 and Lemma 13.1. \square

Proposition 13.4. *If $\beta_i > 1$, there are no $p_j, p_k \in \mathcal{P}$ where $p_j \equiv p_k \pmod{p_i}$.*

Proof. Suppose there exists some $\beta_i > 1$ and $2a \not\equiv 0 \pmod{p_i}$. From Lemma 12.1 it can be seen that there must exist some $p_j \in \mathcal{P}$ where $2a \equiv -p_j \pmod{p_i}$. The question now is if the Hensel-Goldbach Difference Equations 12.3 may be used for the root approximation for $2a \equiv -p_j \pmod{p_i}$. Recall from Proposition 10.12 that $\mathcal{G}_+(-p_i) = \mathcal{G}_+(-p_j)$ where it follows $\mathcal{G}_+(-p_i) \equiv \mathcal{G}_+(-p_j) \pmod{p_i^k}$ for any integer $k > 0$. This allows for $\mathcal{G}_+(0) \equiv \mathcal{G}_+(-p_j) \pmod{p_i}$. Since $\beta_i > 1$ is assumed to be greater than one, Corollary 10.8 and Propositions 10.9, 10.10 show that $\mathcal{G}_+(0)$ has a singular root $\pmod{p_i}$, thus proving that $\mathcal{G}_+(-p_j)$ has a singular root $\pmod{p_i}$. Hence, it may be concluded from above that there exists no prime $p_k \in \mathcal{P}$ where $p_j \equiv p_k \pmod{p_i}$. From Lemma 13.1 it may be seen no prime $p_i \in \mathcal{P}$ divides any q_j and q_k . \square

Corollary 13.5. *For any u_i, u_j in equations 4.1 and 4.2 in Theorem 4.10, the $GCD(u_i, u_j) = 1$.*

Proof. Under Proposition 13.4 it was shown that no two u_j, u_k are divisible by the same prime $p_i \in \mathcal{P}$. Since all u 's must be divisible by some prime in \mathcal{P} , the $GCD(u_i, u_j) = 1$. \square

Theorem 13.6. *There are no solutions to equation 10.1 where $\mathcal{G}_+(2a) = 0$ and $a > 3$.*

Proof. From Corollary 13.5 it was shown that no prime p_i divides any two u 's. Therefore, each u is a perfect prime power and for any $p_i \in \mathcal{D}$ from 3.2 it follows that

$$(13.2) \quad 2a = p_i^{\beta_i} - p_i \text{ for all prime } p_i \in \mathcal{D}.$$

Since, there are $\pi(a)$ u 's that share no primes, it follows from the Pigeon Hole Principle that all $\beta_i > 0$. Using the transitive property from equation 4.5, and the fact that $2|2a$, it follows that there exists some prime power where $2^{\beta_1} - 2 = p_j^{\beta_j} - 3$. However, the proof of Catalan's Conjecture ensures that the largest values this equation has is $2^3 - 2 = 3^2 - 3$. Therefore, that the largest value for $2a$ satisfying $\mathcal{G}_+(2a) = 0$ is when $a = 3$. Under Theorem 4.10 no counter-examples to the Goldbach Difference Conjecture exists and it must be true. \square

Definition 13.7. Let $a \in \mathbb{N}$ such that a . Since

$$(13.3) \quad 2a = (a + b) + (a - b)$$

for any $b \in \mathbb{N}$, a *Prime Difference Point (P.D.P.)* is any $b_D \in \mathbb{N}$ where $a \pm b_D \in \mathbb{P}$ and $b_d > a$.

Theorem 13.8. *Every $a \in \mathbb{N}$ where $a > 3$ has some non-zero P.D.P.*

Proof. Since no solutions exist to Theorem 4.10 when $2a > 6$, this must also hold when a is prime. This would allow for a cancellation of a from both sides of equation 4.7. Since solutions would still not exist, another u_i must be prime in equation 4.5. Thus, since every prime has a non-zero P.D.P. and any composite a must also have a non-zero P.D.P., the theorem is true. \square

Theorem 13.9. *The Polginac Conjecture is true.*

Proof. Under Theorems 9.6 and 13.6 it follows for all even $m, n \in \mathbb{N}$, with $m \geq 6$, there exists odd $p_4, p_3, p_2, p_1 \in \mathbb{P}$, where $p_4 - p_3 = m + n$, and $p_2 + p_1 = m$. Allowing n to be fixed for some even number, and m to cycle through all of the positive even numbers greater than 4 gives an infinite set of equations for n of the form $p_4 - (p_3 + p_2 + p_1) = n$. If the Polignac Conjecture were false for some n , there would be only finitely many primes that were the sum of three odd, prime numbers. Theorem 9.7, and Euclid's proof for the infinitude of the primes, shows this cannot be the case, proving the Polignac Conjecture is true. \square

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