

# A FAMILY OF MAXIMAL MATHIEU SUBSPACES OF MATRIX ALGEBRAS

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ABSTRACT. Let  $F$  be a field. In this note we give a construction for a family of maximal Mathieu subspaces (or Mathieu-Zhao subspaces) of the matrix algebras  $M_n(F)$  ( $n \geq 2$ ). As an application we also give a classification of Mathieu subspaces of  $M_2(F)$  under the condition that  $F$  is algebraically closed.

## 1. Background and Motivation

Let  $F$  be a field. We start with the definition of a Mathieu subspace for an associative algebra  $\mathcal{A}$  over  $F$ , which was introduced by the second author in [Z2] (see also [Z3]).

**Definition 1.1.** *A  $F$ -subspace  $M$  of  $\mathcal{A}$  is a Mathieu subspace (MS) of  $\mathcal{A}$  if for all  $a, b, c \in \mathcal{A}$  such that  $a^m \in M$  for all  $m \geq 1$ , there exists an  $N$  (depending  $a, b, c$ ) such that  $ba^m c \in M$  for all  $m \geq N$ .*

A MS (Mathieu subspace) is also called a Mathieu-Zhao space or Mathieu-Zhao subspace in the literature (e.g., see [EKC, DEZ]). The introduction of MSs was directly motivated by the Mathieu Conjecture [Mat] and the Image Conjecture [Z1], each of which implies the Jacobian conjecture [Ke, BCW, E]. For example, the Jacobian conjecture will follow if some explicitly given subspaces of multivariate polynomial algebras over  $\mathbb{C}$  can be shown to be MSs of the polynomial algebras. For details, see [Z1, DEZ, EKC].

Note that ideals are MSs, but not conversely. Therefore the concept of MSs can be viewed as a natural generalization of the concept of ideals. However, in contrast to ideals, MSs are currently far from being well-understood. This is even the case for the most of finite rings or

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finite dimensional algebras over a field. For example, the classification of all MSs of the matrix algebras  $M_n(F)$  ( $n \geq 3$ ) is still wide open.

The study of MSs of matrix algebras  $M_n(F)$  was initiated by the second author in [Z3] in which the following two results were proven.

**Theorem 1.2.** [Z3, Thm. 4.2] *Let  $V \subseteq M_n(F)$  be a proper subspace of  $M_n(F)$ . Then  $V$  is a MS of  $M_n(F)$  if and only if it does not contain any nonzero idempotent of  $M_n(F)$ .*

**Theorem 1.3.** [Z3, Thm 5.1] *Let  $H$  be the subspace of  $M_n(F)$  consisting of all trace-zero matrices. Then the following statements holds:*

- i) if  $\text{char. } F = 0$  or  $\text{char. } F > n$ , then  $H$  is the only MS of  $M_n(F)$  of codimension 1;*
- ii) if  $\text{char. } F \leq n$ , then  $M_n(F)$  has no MS of codimension 1.*

Actually, the theorem above holds also for one-sided codimension one MSs of  $M_n(F)$ .

It is easy to see from the two theorems above that every subspace of a MS of  $M_n(F)$  is also a MS of  $M_n(F)$ . Therefore, to classify all MSs of  $M_n(F)$  it suffices to classify all maximal MSs of  $M_n(F)$ .

In particular, when  $\text{char. } F = 0$  or  $\text{char. } F > n$ , every subspace of the trace-zero codimension 1 subspace  $H$  is a MS of  $M_n(F)$ . Conversely, under the same condition on  $\text{char. } F$  and  $n \geq 3$ , A. Konijnenberg [Kon, Cor. 3.5] in his master thesis directed by A. van den Essen proved that every MS of  $M_n(F)$  of codimension 2 is a subspace contained in  $H$ . Furthermore, M. de Bondt [Bo] proved that it is also the case for all the MSs of  $M_n(F)$  of codimension less than  $n$ .

**Theorem 1.4.** [Bo, Thm. 1.4] *Assume  $\text{char. } F = 0$  or  $\text{char. } F \geq n$ . If  $V \subseteq M_n(F)$  is a MS of  $M_n(F)$  of codimension less than  $n$ , then  $V$  is a subspace of  $H$ .*

Therefore, we have a unique maximal MS of  $M_n(F)$  of codimension 1 (with the above restriction on the characteristic of  $F$ ) and there are no maximal MSs of  $M_n(F)$  with codimension between 2 and  $n - 1$ . Note, in [Bo, Prop. 1.2] a MS of  $M_n(F)$  of codimension  $n$ , which is not contained in  $H$ , is given (hence is necessarily maximal) and de Bondt notes, based on [Kon], that “. . . the codimension  $n$  case seems quite difficult”.

In Section 2 of this note we give a construction of a family of maximal MSs of  $M_n(F)$ , which for all  $n \geq 2$  contains some maximal MSs of codimension  $n$ . In Section 3 we give a classification for all the (maximal) MSs of  $M_2(F)$  under the condition that  $F$  is algebraically closed.

## 2. A Class of Maximal Mathieu Subspaces of $M_n(F)$

Throughout this section  $F$  stands for a field of arbitrary characteristic, whose algebraic closure is denoted by  $\bar{F}$ . We make use of the non-degenerate Killing form on  $M_n(F)$  given by

$$\langle b, c \rangle = \text{Tr}(bc) \text{ for all } b, c \in M_n(F).$$

Hence  $H$ , the subspace of trace zero matrices in  $M_n(F)$ , is equal to  $I_n^\perp = \{b \in M_n(F) : \text{Tr}(b) = 0\}$ . More generally, for any set  $S$ , we write  $S^\perp = \{b \in M_n(F) : \text{Tr}(bx) = 0 \text{ for all } x \in S\}$  for the orthogonal complement of  $S$  with respect to the Killing form.

Now let  $T \subseteq GL_n(F)$  be a (split, but not necessarily maximal) torus acting on  $M_n(F)$  by conjugation and

$$Z(T, M_n) := \{a \in M_n(F) : \tau a \tau^{-1} = a \text{ for all } \tau \in T\}$$

be the centralizer of  $T$  in  $M_n(F)$ . Denote by  $X(T)$  the character group of  $T$  and let  $W(T, M_n) \subseteq X(T)$  be the set of weights of  $T$  in  $M_n(F)$ . Since  $X(T)$  is an abelian group, we will use  $+$  as the binary operation of  $X(T)$  and denote the identity element by  $0$ . For each  $\omega \in W(T, M_n)$ , let  $M_\omega \subseteq M_n(F)$  be the weight space corresponding to  $\omega$ . So  $M_0 = Z(T, M_n)$ . Then we have the following direct sum composition:

$$M_n(F) = \bigoplus_{\omega \in W(T, M_n)} M_\omega.$$

For each  $\omega \in W(T, M_n)$  we let  $\pi_\omega : M_n(F) \rightarrow M_\omega$  be the corresponding projection. We will use the observation that for any  $a \in M_n(F)$  we have  $\text{Tr}(a) = \text{Tr}(\pi_0(a))$ . Finally, let  $X_\mathbb{Q} = X(T) \otimes_\mathbb{Z} \mathbb{Q}$  and let  $f : X_\mathbb{Q} \rightarrow \mathbb{Q}$  be a linear functional such that  $f(\omega) \neq 0$  for all  $0 \neq \omega \in W(T, M_n)$ . Define  $W_f^+ = \{\omega \in W(T, M_n) : f(\omega) > 0\}$ .

**Theorem 2.1.** *Let  $\Lambda \in T$  be such that  $\text{Tr}(\Lambda e) \neq 0$  for all nonzero idempotents  $e \in M_0$ . Let  $V$  be a subspace of  $M_n(F)$  such that both  $V$  and  $\pi_0(V)$  are contained in  $\Lambda^\perp$ . If  $\pi_\omega(V)\pi_{-\omega'}(V) = 0$  for all  $\omega, \omega' \in W_f^+$ , then  $V$  is a MS of  $M_n(F)$ .*

*Proof.* Assume that  $V$  contains a nonzero idempotent  $e \in V$ . Then  $\pi_0(e) = \pi_0(e^2)$  and we can write

$$\pi_0(e^2) = \pi_0(e)^2 + \sum_{\omega \in W_f^+} (\pi_\omega(e)\pi_{-\omega}(e) + \pi_{-\omega}(e)\pi_\omega(e)).$$

Since by assumption  $\pi_\omega(e)\pi_{-\omega'}(e) = 0$  for all  $\omega, \omega' \in W_f^+$ , the above equation reduces to

$$(2.1) \quad \pi_0(e) = \pi_0(e^2) = \pi_0(e)^2 + \sum_{\omega \in W_f^+} \pi_{-\omega}(e)\pi_\omega(e).$$

Set  $c := \sum_{\omega \in W_f^+} \pi_{-\omega}(e)\pi_\omega(e)$ . Then using the above assumption again, we have  $c^2 = 0$ . Therefore, by eq. (2.1),  $\pi_0(e)$  must have only 0 and 1 as eigenvalues in  $\bar{F}$  with at least one nonzero eigenvalue. Hence there exists a nonzero idempotent  $e' \in M_0$  such that  $\pi_0(e) = e' + w$  for some nilpotent  $w \in M_0$ . Furthermore, since  $\Lambda w$  is nilpotent, we have  $\text{Tr}(\Lambda e) = \text{Tr}(\Lambda \pi_0(e)) = \text{Tr}(\Lambda e') \neq 0$ . But this contradicts our assumption that  $V \subseteq \Lambda^\perp$ . Therefore  $V$  does not contain any nonzero idempotent and is a MS of  $M_n(F)$  by Theorem 1.2.  $\square$

**Example 2.2.** Let  $e_1, \dots, e_t \in M_n(F)$  be a set of orthogonal idempotents so that  $I_n = e_1 + e_2 + \dots + e_t$ . Let  $T = F^\times e_1 + \dots + F^\times e_t$  and let  $n_i$  be the rank of  $e_i$  for  $1 \leq i \leq t$ . If  $\varepsilon_i \in X(T, M_n)$  is the character given by  $\varepsilon_i(\alpha_1 e_1 + \dots + \alpha_t e_t) = \alpha_i$  for all  $\alpha_1 e_1 + \dots + \alpha_t e_t \in T$ , then  $\{\varepsilon_i : 1 \leq i \leq t\}$  forms a basis of  $X(T, M_n)$  as a  $\mathbb{Z}$ -lattice and  $W(T, M_n) = \{\varepsilon_i - \varepsilon_j : 1 \leq i, j \leq t\}$ . Let  $f : X(T, M_n) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \mathbb{Q}$  be given by  $f(\varepsilon_i) = i$  for all  $1 \leq i \leq t$ . Let  $\Lambda = \sum_{i=1}^t \sigma_i e_i \in T$  be such that  $\sum_{i=1}^t \sigma_i k_i \neq 0$  for all integer  $t$ -tuples  $\vec{0} \neq (k_1, k_2, \dots, k_t) \in \prod_{i=1}^t [0, n_i]$ . Then by Theorem 2.1 we see that the subspace

$$V := \left( M_0 \oplus \bigoplus_{\omega \in W_f^+} M_\omega \right) \cap \Lambda^\perp = (M_0 \cap \Lambda^\perp) \oplus \bigoplus_{\omega \in W_f^+} M_\omega.$$

is a MS of  $M_n(F)$ .

Before proceeding further we fix the following notation in the setting of Example 2.2 above. For any  $1 \leq i, j \leq n$  we set  $M_{i,j} = e_i M_n(F) e_j$ . Then  $M_n(F) = \bigoplus_{1 \leq i, j \leq n} M_{i,j}$  and hence, each  $a \in M_n(F)$  can be written as  $a = \sum_{1 \leq i, j \leq n} a_{ij}$  for some  $a_{ij} \in M_{i,j}$ . In this case we also write  $a$  formally as a  $t \times t$  matrix  $(a_{ij})$ . We call this matrix **the block matrix form** of  $a$ . Then, in terms of the block matrix forms the MS  $V$  in Example 2.2 above is the subspace formed by all the matrices whose block matrix forms are lower triangular with the blocks  $(a_{11}, a_{22}, \dots, a_{tt})$  on the diagonal such that  $a_{ii} \in M_{i,i}$  and  $\sum_{i=1}^t \sigma_i \text{Tr}(a_{ii}) = 0$ .

When  $F$  is algebraically closed, a much more concrete way to look at the MS  $V$  above is as follows. Up to conjugations we may assume that the idempotent  $e_i$  ( $1 \leq i \leq t$ ) is the  $n \times n$  matrix with blocks  $(0_{n_1 \times n_1}, \dots, 0_{n_{i-1} \times n_{i-1}}, I_{n_i \times n_i}, 0_{n_{i+1} \times n_{i+1}}, \dots, 0_{n_t \times n_t})$  on the diagonal and zero elsewhere. In this case the weight space  $M_{i,j} =$

$e_i M_n(F) e_j$  is canonically isomorphic to the space of  $n_i \times n_j$  matrices over  $F$ . The MS  $V$  in Example 2.2 is formed by all the matrices which are in the (usual) lower triangular block form with the matrices  $(a'_{11}, a'_{22}, \dots, a'_{tt})$  on the diagonal such that  $a'_{ii} \in M_{n_i \times n_i}(F)$  and  $\sum_{i=1}^t \sigma_i \text{Tr}(a'_{ii}) = 0$ .

In the rest of this section we will freely use the notations fixed in and after Example 2.2. We will show in Corollary 2.6 that the MSs in Example 2.2 when  $t = 2$  are actually maximal MSs of  $M_n(F)$ . However, when  $t \geq 3$ , the MSs in Example 2.2 are not maximal, which can be seen from the following example.

**Example 2.3.** *Let  $V$  be as in the Example 2.2 with  $t \geq 3$ . Fix  $u \in e_1 M_n(F) e_2$  and  $w \in e_2 M_n(F) e_3$  such that  $uw \neq 0$ . Then let  $U = F(u + w) + V$ . We show below that  $U$  does not contain any nonzero idempotent of  $M_n(F)$ . Hence  $U$  by Theorem 1.2 is a MS of  $M_n(F)$ , which contains  $V$  as a proper subspace.*

*Assume that  $U$  contains a nonzero idempotent  $e$ . Then  $e = \alpha(u + w) + v$  for some  $\alpha \in F$  and  $v \in V$ . Since  $e_1 U e_3 = 0$ , we have  $0 = e_1 e e_3 = \alpha^2 u w$ . Therefore,  $\alpha = 0$  and  $e \in V$ , which is a contradiction, since by Theorem 1.2  $V$  does not contain any nonzero idempotent.*

Next we consider the following family of MSs of  $M_n(F)$ , which coincides the case  $t = 2$  of Example 2.2 when  $e_1$  or  $e_2 = 0$ .

**Example 2.4.** *Let  $e_1, e_2, e_3$  be as in Example 2.2 with  $t = 3$ , except we here do not assume that all  $e_i \neq 0$ . Let  $T, \varepsilon_i$  ( $1 \leq i \leq 3$ ), and  $X(T, M_n)$  be as in Example 2.2 with  $t = 3$ . Then  $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$  is a basis of  $X(T, M_n)$ . Let  $T' \subseteq T$  be the subtorus defined by  $T' = F^\times(e_1 + e_2) + F^\times e_3$  and let  $\varepsilon'_1 = \varepsilon_1|_{T'}$ . We define  $f : X(T', M_n) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \mathbb{Q}$  by setting  $f(\varepsilon'_1) = 1$  and  $f(\varepsilon_3) = 3$ . Hence  $W(T', M_n) = \{\omega, 0, -\omega\}$  where  $\omega = \varepsilon_3 - \varepsilon'_1$ . Let  $\Lambda = \sigma_1(e_1 + e_2) + \sigma_2 e_3 \in T'$  be such that  $\sigma_1 \neq \sigma_2$  and  $k_1 \sigma_1 + k_2 \sigma_2 \neq 0$  for all integer ordered pairs  $\vec{0} \neq (k_1, k_2) \in [0, n_1 + n_2] \times [0, n_3]$ . Then it follows from Theorem 2.1 that*

$$\begin{aligned} V &= \left( Z(T', M_n) + e_1 M_n(F) e_3 + e_3 M_n(F) e_2 \right) \cap \Lambda^\perp \\ (2.2) \quad &= \left( Z(T', M_n) \cap \Lambda^\perp \right) + e_1 M_n(F) e_3 + e_3 M_n(F) e_2 \end{aligned}$$

*is a MS of  $M_n(F)$  of codimension  $(n_1 + n_2)n_3 + 1$ .*

The main result of this section is the following theorem.

**Theorem 2.5.** *The MSs in Example 2.4 are maximal MSs of  $M_n(F)$ .*

*Proof.* Let  $V$  be a MS of the type given in Eq. (2.2) and let  $e_i, \varepsilon_i$  ( $i = 1, 2, 3$ ) and  $\Lambda = \sigma_1(e_1 + e_2) + \sigma_2 e_3$  be as in Example 2.4. Fix any

$w \notin V$ . We show below that there exists a nonzero idempotent  $Q$  in  $V + Fw$ . Hence  $V + Fw$  by Theorem 1.2 is not a MS of  $M_n(F)$ , and  $V$  is a maximal MS of  $M_n(F)$ .

Note that we may assume  $w \in Z(T', M_n) + e_3M_n(F)e_1 + e_2M_n(F)e_3$ . Furthermore, if  $w \in Z(T', M_n)$ , then  $w \notin \Lambda^\perp$ , and we may choose  $Q = I_n$ , for  $I_n \in Z(T', M_n) \subseteq Fw + V$ . Therefore, we may assume further  $w \notin Z(T', M_n)$ . Let  $w_0 \in Z(T', M_n)$ ,  $w_1 \in e_3M_n(F)e_1$  and  $w_2 \in e_2M_n(F)e_3$  such that  $w = w_0 + w_1 + w_2$ . Then  $w_1$  and  $w_2$  are not both zero. We now divide the proof into two cases.

**Case 1:** Assume  $w_0 \in \Lambda^\perp$ . Then  $w_0 \in Z(T', M_n) \cap \Lambda^\perp \subseteq V$ . Replacing  $w$  by  $w - w_0$  we can assume  $w_0 = 0$  and  $w = w_1 + w_2$ . Without loss of generality, assume  $w_1 \neq 0$  is of rank  $r$ . Then there exists  $v \in e_1M_n(F)e_3 \subseteq V$  such that  $w_1v$  and  $vw_1$  are both idempotents of rank  $r$  with  $w_1vw_1 = w_1$  and  $vw_1v = v$ . The existence of  $v$  follows from the following general fact: *for each  $a \in M_{m \times k}(F)$  of rank  $r$ , there exists  $b \in M_{k \times m}(F)$  such that  $ab$  and  $ba$  are both idempotents of rank  $r$ , which satisfy the equations  $aba = a$  and  $bab = b$ .* Indeed, we can find  $P \in GL_m(F)$  and  $Q \in GL_k(F)$  such that  $a = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}_{m \times k} Q$ . Then

$$\text{we let } b = Q^{-1} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}_{k \times m} P^{-1}.$$

Now we claim that, for any  $\beta \neq -1$ , we have that

$$Q := (1 + \beta)^{-1}((e_3 + w_2 + \beta v)(e_3 + w_1) + \beta(e_3 - w_1v))$$

is an idempotent with trace equal to  $\text{rank}(e_3) = n_3$ . To show  $Q$  is an idempotent, we write  $Q$  in the block matrix form as:

$$Q = \frac{1}{1 + \beta} \begin{bmatrix} \beta v w_1 & 0 & \beta v \\ w_2 w_1 & 0 & w_2 \\ w_1 & 0 & (1 + \beta)e_3 - \beta w_1 v \end{bmatrix}$$

where the  $(i, j)$ -block represents an element in  $e_iM_n(F)e_j$ . Then it is straightforward to check  $Q^2 = Q$  using the identities  $vw_1v = v$  and  $w_1vw_1 = w_1$ .

Next, note that  $\text{Tr}(\Lambda(Q - w)) = \text{Tr}(\Lambda Q) = \sigma_2 n_3 + \frac{(\sigma_1 - \sigma_2)\beta r}{\beta + 1}$ . Note also that by the assumption on  $\sigma_1$  and  $\sigma_2$  we have  $r\sigma_1 + (n_3 - r)\sigma_2 \neq 0$ , since  $n_3 - r \geq 0$  as  $w_1 \in e_3M_n(F)e_1$  so  $r = \text{rank}(w_1) \leq \text{rank}(e_3) = n_3$ .

Choose  $\beta = -\frac{\sigma_2 n_3}{r\sigma_1 + (n_3 - r)\sigma_2}$ . Since  $\sigma_1 \neq \sigma_2$ , we get  $\beta \neq -1$  and  $\text{Tr}(\Lambda(Q - w)) = 0$ . Hence we have  $Q - w \in V$  and  $Q \in Fw + V$ .

**Case 2:** Assume  $w_0 \notin \Lambda^\perp$ . Then  $Fw_0 + (\Lambda^\perp \cap Z(T', M_n)) = Z(T', M_n)$ , whence there exist  $0 \neq \alpha \in F$  and  $x_0 \in (\Lambda^\perp \cap Z(T', M_n)) \subseteq$

$V$  such that  $\alpha w_0 + x_0 = e_3$ . Set

$$Q := (\alpha w + x_0 + \alpha^2 w_2 w_1) = (e_3 + \alpha w_1 + \alpha w_2 + \alpha^2 w_2 w_1)$$

which in the block matrix form is

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ \alpha^2 w_2 w_1 & 0 & \alpha w_2 \\ \alpha w_1 & 0 & e_3 \end{bmatrix}.$$

Then it follows that  $Q^2 = Q$ . Since  $w_2 w_1 \in e_2 M_n(F) e_1 \subseteq \Lambda^\perp \cap Z(T', M_n) \subseteq V$ , we get  $Q \in Fw + V$ .  $\square$

Note that by letting  $e_2 = 0$  in Example 2.4 and Theorem 2.5 it is easy to check that we have also the following corollary.

**Corollary 2.6.** *Let  $e_1 \in M_n(F)$  be a rank  $r$  idempotent with  $0 < r < n$  and  $\sigma_1 \neq \sigma_2 \in F$  such that  $k_1 \sigma_1 + k_2 \sigma_2 \neq 0$  for all integer ordered pairs  $\vec{0} \neq (k_1, k_2) \in [0, r] \times [0, n - r]$ . Set  $e_2 = I_n - e_1$ . Then*

$$V = \left( \sigma_1 e_1 + \sigma_2 e_2 + e_1 M_n(F) e_3 \right)^\perp,$$

which in the block matrix form is

$$V = \left( \begin{array}{cc} \sigma_1 e_1 & e_1 M_n(F) e_2 \\ 0 & \sigma_2 e_2 \end{array} \right)^\perp$$

is a maximal MS of  $M_n(F)$  of codimension  $(rn - r^2 + 1)$ .

When  $r = 1$  the MSs in the corollary above have codimension  $n$ . The example given in [Bo, Prop. 1.2] is one of these MSs of this type.

### 3. Classification of Mathieu Subspaces of $M_2(F)$

Throughout this section  $F$  stands for an algebraically closed field of arbitrary characteristic. In this section we give a classification for all MSs of  $M_2(F)$ . As pointed out in Section 1, by Theorems 1.2 and 1.3 we only need to classify all maximal MSs of  $M_2(F)$ . We start with the following lemma.

**Lemma 3.1.** *Let  $V$  be a MS of  $M_2(F)$  and  $a \in V$  with  $\text{Tr}(a) \neq 0$ . Then the following two statements hold:*

- i)  $a$  is invertible;*
- ii) if  $\dim_F V = 2$ , then  $a$  has distinct (nonzero) eigenvalues.*

*Proof.* *i)* Assume that  $a$  is singular. Then  $a$  is of rank 1 and  $a^2 = \text{Tr}(a)a$  by the Cayley-Hamilton Theorem. Hence  $\text{Tr}(a)^{-1}a$  is a nonzero idempotent in  $V$ , and, by Theorem 1.2,  $V$  is not a MS of  $M_2(F)$ , contradicting our assumption on  $V$ .

*ii)* Let  $\lambda_1$  and  $\lambda_2$  be the eigenvalues of  $a$ . Then  $\lambda_1\lambda_2 \neq 0$  by *i)*. Let  $c \in I_2^\perp$  so that  $\{a, c\}$  is a basis of  $V$ . Note that for all  $x \in F$  we have  $\text{Tr}(a + xc) = \text{Tr}(a) \neq 0$ . Then by *i)*  $\det(a + xc) \neq 0$  for all  $x \in F$ . Since  $F$  is algebraically closed, this is true if and only if  $a^{-1}c$  has no nonzero eigenvalue in  $F$ . Therefore,  $a^{-1}c$  must be nilpotent, which gives us  $a^{-1}ca^{-1}c = 0 \Rightarrow ca^{-1}c = 0$ . Hence,  $\det c = 0$ , so  $c$  is nilpotent of degree 2 by the Cayley-Hamilton Theorem since we have  $\text{Tr}(c) = 0$  as well. Viewing  $c$  as a nonzero linear endomorphism of  $F^2$  we have  $\ker(c) = \text{Im}(c) = \ker(ca^{-1})$  (as one-dimensional subspaces of  $F^2$ ). Fix any  $0 \neq \vec{v} \in \ker(c)$ . Then  $\vec{v}$  is simultaneously an eigenvector of both  $a$  and  $c$  since  $a\vec{v} \in \ker(ca^{-1}) = \ker(c)$ . Without loss of generality, assume  $a\vec{v} = \lambda_1\vec{v}$ .

Assume  $\lambda_1 = \lambda_2$ . Note that  $a - \lambda_1 I_2 \neq 0$  (otherwise  $I_2$  would be in  $V$ ) and is nilpotent. Therefore we have

$$\ker(a - \lambda_1 I_2) = \text{Im}(a - \lambda_1 I_2) = F\vec{v} = \text{Im}(c) = \ker(c).$$

Hence there exists  $\beta \in F$  such that  $a - \lambda_1 I_2 = \beta c$ . But this implies  $I_2 \in V$ , which by Theorem 1.2 is a contradiction. Therefore  $\lambda_1 \neq \lambda_2$  and statement *ii)* holds.  $\square$

In his master thesis ([Kon, Thm. 3.10]) directed by A. van den Essen, A. Konijnberg characterized all the codimension 2 subspaces of  $M_2(F)$  when  $F$  is algebraically closed and  $\text{char. } F \neq 2$ . Now we can give a different proof for this result without the condition  $\text{char. } F \neq 2$ .

**Corollary 3.2.** *Let  $V \subseteq M_2(F)$  be a MS with  $\dim_F V = 2$ . Then either  $V \subseteq I_2^\perp$  or there exist nonzero idempotents  $e_1, e_2 \in M_2(F)$  such that  $e_1 + e_2 = I_2$  and  $V = (\lambda_1 e_1 + \lambda_2 e_2) + e_1 M_2(F) e_2$  for some distinct nonzero  $\lambda_1, \lambda_2 \in F$  with  $\lambda_1 + \lambda_2 \neq 0$ .*

*In other words, either  $V \subseteq I_2^\perp$  or  $V$  is conjugate to the subspace  $\left\{ \begin{pmatrix} \lambda_1 s & t \\ 0 & \lambda_2 s \end{pmatrix} \mid s, t \in F \right\}$  for some distinct nonzero  $\lambda_1, \lambda_2 \in F$  with  $\lambda_1 + \lambda_2 \neq 0$ .*

*Proof.* Assume  $V \not\subseteq I_2^\perp$ . Then there exists an  $a \in V$  with  $\text{Tr}(a) \neq 0$ . Let  $\lambda_1$  and  $\lambda_2$  be eigenvalues of  $a$  in  $F$ . Then  $\lambda_1 + \lambda_2 \neq 0$ , and by Lemma 3.1, we also have  $\lambda_1\lambda_2 \neq 0$  and  $\lambda_1 \neq \lambda_2$ .

Let  $c$  and  $\vec{v}$  be as in the proof above for Lemma 3.1, *ii)*. Then by the arguments there we have  $ca^{-1}c = 0$ ;  $c\vec{v} = 0$  and  $a\vec{v} = \lambda_1\vec{v}$ . Let  $\vec{u} \in F^2$  be the eigenvector of  $a$  such that  $a\vec{u} = \lambda_2\vec{u}$ . Let  $e_1, e_2 \in M_2(F)$  be the idempotents corresponding to the decomposition  $F^2 = F\vec{v} \oplus F\vec{u}$ . Then it is easy to see that  $V$  is of the desired form.  $\square$



**Remark 3.3.** *i) For the subspace  $V$  in Corollary 3.2 such that  $V \not\subseteq I_2^\perp$  we let  $\sigma_1 = -\lambda_2$  and  $\sigma_2 = \lambda_1$ . Then  $V$  is a subspace of the type given in Corollary 2.6 by which  $V$  is indeed a MS of  $M_2(F)$ .*

*ii) If  $\text{char. } F \neq 2$ , then  $I_2^\perp$  does not contain any nonzero idempotent. Hence each subspace  $V \subseteq I_2^\perp$  by Theorem 1.2 is a MS of  $M_2(F)$ . Therefore, Corollary 3.2 in this case coincides with [Kon, Thm. 3.10] which characterizes all MSs of codimension 2 of  $M_2(F)$ .*

**Corollary 3.4.** *i) If  $\text{char. } F \neq 2$ , then the dimensional 1 maximal MSs of  $M_2(F)$  are exactly the subspaces of the form  $F(I_2 + c)$ , where  $c \in M_2(F)$  is nilpotent.*

*ii) If  $\text{char. } F = 2$ , then there is no dimension 1 maximal MS in  $M_2(F)$ .*

*Proof.* Let  $V$  be a dimensional 1 maximal MS of  $M_2(F)$  with  $V = Fa$  for some  $a \in M_2(F)$ . Then by Theorem 1.2  $a^2 \neq \beta a$  for any nonzero  $\beta \in F$ , otherwise  $\beta^{-1}a$  would be a nonzero idempotent in  $V$ . If  $\text{Tr}(a) = 0$ , then  $V \subseteq I_2^\perp$  which would obviously not be maximal. Therefore  $\text{Tr}(a) \neq 0$  and, by Lemma 3.1,  $a$  is invertible.

Let  $\lambda_1, \lambda_2$  be the nonzero eigenvalues of  $a$ . Then  $\lambda_1 + \lambda_2 \neq 0$ . If  $\lambda_1 \neq \lambda_2$ , then  $V$  is a proper subspace of  $(\lambda_1 e_1 + \lambda_2 e_2) + e_1 M_2(F) e_2$  in Corollary 3.2, which as pointed out in Remark 3.3, *i)* is a MS of  $M_2(F)$ , whence  $V$  would not be maximal. Therefore we have  $\lambda_1 = \lambda_2$ , which we may assume are both equal to 1. Let  $c = a - I_2$ . Then  $c$  is nonzero and nilpotent, and  $V = F(I_2 + c)$ .

Conversely, for any nonzero nilpotent  $c \in M_2(F)$ ,  $I_2 + c$  is invertible. The subspace  $F(I_2 + c)$  does not contain any nonzero idempotents, as the only invertible idempotent of  $M_2(F)$  is  $I_2$ . Then by Theorem 1.2  $F(I_2 + c)$  is a MS of  $M_2(F)$ .

Now assume  $\text{char. } F \neq 2$ . To show statement *i)*, we only need to show that, for any nonzero nilpotent  $c \in M_2(F)$ , the subspace  $F(I_2 + c)$  is maximal among MSs of  $M_2(F)$ . Assume otherwise. Then  $V$  is strictly contained in a proper MS  $W$  of  $M_2(F)$ . If  $\dim_F W = 3$ , then  $W = I_2^\perp$  by Theorem 1.3, which is a contradiction of  $\text{Tr}(I_2 + c) = 2 \neq 0$ . Therefore  $\dim_F W = 2$ . Then by Lemma 3.1 the eigenvalues of  $I_2 + c$  must be distinct, which is a contradiction. Therefore  $F(I_2 + c)$  is a maximal MS of  $M_2(F)$ , and statement *i)* follows.

Now assume  $\text{char. } F = 2$ . To show statement *ii)* it suffices to show that, for all nonzero nilpotent  $c \in M_2(F)$ , the subspace  $F(I_2 + c)$  is not maximal among MSs of  $M_2(F)$ . Note that  $\dim_F I_2^\perp = 3$  and the only nonzero idempotent contained in  $I_2^\perp$  is  $I_2$ . Hence there exists a dimensional 2 subspace  $U$  of  $I_2^\perp$  such that  $I_2 + c \in U$  and  $I_2 \notin U$ . Then by Theorem 1.2,  $U$  is a MS of  $M_2(F)$  which strictly contains  $V$ . Therefore

$V$  is not a maximal MS of  $M_2(F)$ , and statement *ii*) follows.  $\square$

Now we give a classification for maximal MSs of  $M_2(F)$  in the next two theorems.

**Theorem 3.5.** *Assume  $\text{char. } F \neq 2$ . Then the maximal MSs of  $M_2(F)$  are exactly the followings:*

- i)  $I_2^\perp$ ;*
- ii) the subspaces of the form  $(\lambda_1 e_1 + \lambda_2 e_2) + e_1 M_2(F) e_2$  for some nonzero idempotents  $e_1, e_2$  such that  $e_1 + e_2 = I_2$  and some distinct nonzero  $\lambda_1, \lambda_2 \in F$  with  $\lambda_1 + \lambda_2 \neq 0$ ;*
- iii) the subspaces of the form  $F(I_2 + c)$  for some nonzero nilpotent  $c \in M_2(F)$ .*

*Proof.* By Theorem 1.2 the only MS of dimension 3 of  $M_2(F)$  is  $I_2^\perp$ . Consequently, all dimensional 2 MSs  $V$  of  $M_2(F)$  with  $V \not\subseteq I_2^\perp$  are maximal. Then, by Corollary 3.2 and Remark 3.3, *i*) we see that maximal MSs of  $M_2(F)$  of dimension 2 are exactly those given in *ii*). Finally, by Corollary 3.4 the maximal MSs of  $M_2(F)$  of dimension 1 are exactly those given in *iii*).  $\square$

**Theorem 3.6.** *Assume  $\text{char. } F = 2$ . Then the maximal MSs of  $M_2(F)$  are exactly the followings:*

- i) the dimensional 2 subspaces  $V$  of  $I_2^\perp$  such that  $I_2 \not\subseteq V$ ;*
- ii) the subspaces of the form  $(\lambda_1 e_1 + \lambda_2 e_2) + e_1 M_2(F) e_2$  for some nonzero idempotents  $e_1, e_2$  such that  $e_1 + e_2 = I_2$  and some distinct nonzero  $\lambda_1, \lambda_2 \in F$  such that  $\lambda_1 + \lambda_2 \neq 0$ .*

*Proof.* First, since the only nonzero idempotent contained in  $I_2^\perp$  is  $I_2$ , the subspaces given in *i*) are MSs of  $M_2(F)$ . By Remark 3.3, *i*) the subspaces given in *ii*) are also MSs of  $M_2(F)$ . Since  $M_2(F)$  by Theorem 1.3 has no MS of dimension 3, all MSs of dimension 2 are maximal. Therefore the theorem follows immediately from Corollaries 3.2 and 3.4.  $\square$

We end this section with the following example to show that Theorems 3.5 and 3.6 do not hold if the base field  $F$  is not algebraically closed. Furthermore, the example shows also that in general MSs are not preserved under base field extensions.

**Example 3.7.** *Let  $K$  be a field such that there exists  $s \in K$  with  $s \neq 0, \pm 1$  and  $s \neq t^2$  for all  $t \in K$ . Let  $a = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}$ ,  $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $V = Ka + Kb$ . Since  $\det(a + xb) = s - x^2 \neq 0$  for all  $x \in K$ , all*

nonzero elements of  $V$  are invertible. Hence  $V$  by Theorem 1.2 is a MS of  $M_2(K)$  (for  $I_2 \notin V$ ), and by Theorem 1.3 is maximal (for  $V \not\subseteq I_2^\perp$ ). However,  $V$  cannot be any of MSs of dimension 2 given in Theorem 3.5 or Theorem 3.6, since each of those MSs that is not contained in  $I_2^\perp$  contains some nonzero nilpotent elements but  $V$  does not.

Furthermore, let  $L$  be a field that contains  $K$  and  $\sqrt{s}$ . Set  $U := L \otimes_K V$  and  $c := \frac{1}{1+s}(a + \sqrt{s}b)$ . Then  $c$  is a nonzero idempotent in  $U$ , whence  $U$  by Theorem 1.2 is not a MS of  $M_2(L) = L \otimes_K M_2(K)$ .

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