

A MULTISSET VERSION OF EVEN-ODD PERMUTATIONS IDENTITY

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ABSTRACT. In this paper, we give a new bijective proof of a multiset analogue of even-odd permutations identity. This multiset version is equivalent to the original coin arrangements lemma which is a key combinatorial lemma in the Sherman's Proof of a conjecture of Feynman about an identity on paths in planar graphs related to combinatorial solution of two dimensional Ising model in statistical physics.

1. INTRODUCTION AND MOTIVATION

The *Ising model* [1] is a theoretical physics model of the *nearest-neighbor* interactions in a crystal structure. In the Ising model, the vertices of a graph $G = (V, E)$ represent *particles* and the edges describe interactions between pairs of particles. The most common example of a two dimensional Ising model is a *planar square lattice* where each particle interacts only with its neighbors. A factor (weight) J_{ij} is assigned to each edge $\{i, j\}$, where this factor describes the nature of the interaction between particles i and j . A *physical state* of the system is an assignment of $\sigma_i \in \{+1, -1\}$ to each vertex i . The *Hamiltonian* (or energy function) of the system is defined as:

$$H(\sigma) = - \sum_{\{i,j\} \in E} J_{ij} \sigma_i \sigma_j.$$

The *distribution* of the physical states over all possible energy levels is encapsulated in the *partition function*:

$$Z(\beta, G) = \sum_{\sigma} e^{-\beta H(\sigma)},$$

where β is changed for $\frac{K}{T}$, in which K is a constant and T is a variable representing the temperature.

Motivated by a generalization of a *cycle* in a graph, a set A of edges is called *even* if each vertex of V is incident with an even number of edges of A . The generating function of even subsets denoted by $\mathcal{E}(G, x)$ can be defined as

$$\mathcal{E}(G, x) = \sum_{A: A \text{ is even}} \prod_{e \in A} x_e.$$

It turns out that the Ising partition function for a graph G may be expressed in terms of the generating function of the even sets of the same graph G .

More precisely, we have the following Van der Waerden's formula [2]

$$Z(G, \beta) = 2^{|V|} \prod_{\{i,j\} \in E} \cosh(\beta J_{ij}) \mathcal{E}(G, x) \Big|_{x^{J_{ij}} = \tanh(\beta J_{ij})}.$$

Now, let $G = (V, E)$ be a planar graph embedded in the plane and for each edge e we associate a formal variable x_e which can be seen as a weight of that edge. Let $A = (V, A(G))$ be an arbitrary orientation of G . If $e \in E$ then a_e will denote the orientation of e in $A(G)$ and a_e^{-1} will be the reversed orientation to a_e . We put $x_{a_e} = x_{a_e^{-1}} = x_e$. A *circular sequence* $p = v_1, a_1, v_2, a_2, \dots, a_n, (v_{n+1} = v_1)$ is called *non-periodic closed walk* if the following conditions are satisfied: $a_i \in \{a_e, a_e^{-1} : e \in E\}$, $a_i \neq a_{i+1}^{-1}$ and $(a_1, \dots, a_n) \neq Z^m$ for some sequence Z and $m > 1$. We also let $X(p) = \prod_{i=1}^n x_{a_i}$. We further let $sign(p) = (-1)^{n(p)}$, where $n(p)$ is a *rotation number* of p ; i.e., the number of integral revolutions of the tangent vector. Finally put $W(p) = sign(p)X(p)$.

There is a natural *equivalence* on non-periodic closed walks; that is, p is equivalent with *reversed* p . Each equivalence class has two elements and will be denoted by $[p]$. We assume $W([p]) = W(p)$ and note that this definition is correct since equivalent walks have the same sign.

The following beautiful formula is due to Feynman who conjectured it, but did not give a proof of it. It was Sherman who gave a proof based on a key *combinatorial lemma on coin arrangements* [3].

Theorem 1.1 (Feynman and Sherman). *Let G be a planar graph. Then*

$$\mathcal{E}(G, x) = \prod [1 - W([p])],$$

where the product is over all equivalence classes of non-periodic closed walks of G .

Here is the original statement of the coin arrangement lemma: Suppose we have a fixed collection of N objects of which m_1 are of one kind, m_2 are of second kind, \dots , and m_n of n -th kind. Let $b_{N,k}$ be the number of exhaustive unordered arrangements of these symbols into k disjoint, nonempty, circularly ordered sets such that no two circular orders are the same and none are periodic. Then, we have

$$\sum_{k=1}^N (-1)^k b_{N,k} = 0, \quad (N > 1).$$

It is worth to note that when the collection of objects constitute a set of n elements, then the numbers $b_{n,k}$ are exactly *Stirling cycle* numbers; that is, the number of permutations of the set $\{1, 2, \dots, n\}$ (or n - permutations) with exactly k cycles in its decompositions into disjoint cycles. It is noteworthy that the coin arrangements lemma in this particular case, can be reformulated as the following well-known identity in combinatorics of permutations.

Proposition 1.2. [*Even-Odd Permutations Identity*] For any integer number $n > 1$, The number of even n -permutations is the same as the number of odd n -permutations.

Our main goal here is to formulate a *weighted version* of the even-odd permutations identity in the multiset setting.

2. BASIC DEFINITIONS AND NOTATION

As Knuth has noted in [7, p.36] , the term *multiset* was suggested by N.G.de Bruijn in a private communication to him. Roughly speaking, a multiset is an unordered collection of elements in which repetition is allowed.

Definition 2.1 (Multiset). Let $\Sigma = \{a_1, \dots, a_n\}$ be a finite alphabet. A multiset M over Σ denoted by $[a_1^{m_1}, a_2^{m_2}, \dots, a_n^{m_n}]$ is a finite collection of elements of Σ with m_1 occurrences of a_1 , m_2 occurrences of a_2 , \dots , and m_n occurrences of a_n . The number $N = m_1 + m_2 + \dots + m_n$ is called the cardinality of M and m_i ($1 \leq i \leq n$) is called the multiplicity of the element a_i .

Definition 2.2 (Permutation of a multiset). Let M be a multiset over a finite alphabet Σ of cardinality N . We also let $i \geq N$ be a given integer. Then an i -permutation of M is defined as an ordered arrangement of i elements of M . In particular, an N -permutation of M is also called a permutation of M .

Example 2.1. For the alphabet $\Sigma = \{a, b, c\}$, the string $\sigma = abcba$ is a permutation of the multiset $M = [a^3, b^2, c^1]$.

It is worth to note that by a simple counting argument, one can obtain that the number of permutations of the multiset $M = [a_1^{m_1}, a_2^{m_2}, \dots, a_n^{m_n}]$ of cardinality N is equal to $\frac{N!}{m_1!m_2!\dots m_n!}$.

In the rest of this section, we quickly review the basics of the combinatorics of words. The reader can consult the reference [5].

Let Σ be a finite alphabet. The elements of Σ are called *letters*. A finite sequence of elements of Σ is called a *word* (or string) over the alphabet Σ . An empty sequence of letters is called an *empty* word and is denoted by λ . The set of all words over the alphabet Σ will be denoted by Σ^* . We also denote the set of non-empty words by Σ^+ . A word u is called a factor (resp. a prefix, resp. a suffix) of a word w , if there exists words w_1 and w_2 such that $w = w_1uw_2$ (resp. $w = uw_2$, resp. $w = w_1u$).

The k -th power of a word w is defined by $w^k = ww^{k-1}$ with the convention that $w^0 = \lambda$. A word $w \in \Sigma^+$ is called *primitive* if the equation $w = u^n$ ($u \in \Sigma^+$) implies $n = 1$. Two words w and u are *conjugate* if there exist two words w_1 and w_2 such that $w = w_1w_2$ and $u = w_2w_1$. It is easy to see that the conjugacy relation is an *equivalence relation*. A *conjugacy class* (or *necklace*) is a class of this equivalence relation.

For an *ordered* alphabet $(\Sigma, <)$, the *lexicographic* order \preceq on $(\Sigma^*, <)$ is defined by letting $w_1 \preceq w_2$ if

- $w_1 = uw_2$, ($u \in \Sigma^*$) or
- $w_1 = ras$, $w_2 = rbt$ $a < b$, for $a, b \in \Sigma$ and $r, s, t \in \Sigma^*$.

In particular, if $w_1 \trianglelefteq w_2$ and w_1 is not a proper prefix of w_2 , we write $w_1 \triangleleft w_2$.

A word is called a *Lyndon* word if it is primitive and the *smallest* word with respect to the lexicographic order in its conjugacy class.

Example 2.2. Let $\Sigma = \{1, 2, 3\}$ be an ordered alphabet. Then, $l_1 = 1123$ and $l_2 = 1223$ are Lyndon words but $l_3 = 1131$ is not a Lyndon word.

The following factorization of the words as a *non-increasing* product of Lyndon words is of fundamental importance in the combinatorics of words. From now on, we will denote the set of all Lyndon words by L .

Theorem 2.1 (Lyndon Factorization). *Any word $w \in \Sigma^+$ can be written uniquely as a non-increasing product of Lyndon words:*

$$w = l_1 l_2 \cdots l_h, \quad l_i \in L, \quad l_1 \trianglerighteq l_2 \trianglerighteq \cdots \trianglerighteq l_h.$$

One of the important results about the characterization of Lyndon words is the following.

Proposition 2.2. *A word $w \in \Sigma^+$ is a Lyndon word if and only if $w \in \Sigma$ or $w = rs$ with $r, s \in L$ and $r \triangleleft s$. Moreover, if there exists a pair (r, s) with $w = rs$ such that $s, w \in L$ and s of maximal length, then $r \in L$ and $l \triangleleft rs \triangleleft s$.*

Definition 2.3. For $w \in L \setminus \Sigma$ a Lyndon word consisting of more than a single letter, the pair (r, s) with $w = rs$ such that $r, s \in L$ and s of maximal length is called the *standard factorization* of the Lyndon w .

3. MULTISSET VERSION OF EVEN-ODD PERMUTATIONS

In this section, we first briefly review basics of the combinatorics of permutations. For more detailed introduction see [6]. From now on, we will denote the set $\{1, 2, \dots, n\}$ by $[n]$.

Recall that a *permutation* τ of a set $[n]$ (or simply an n -permutation) is a bijective function $\tau : [n] \mapsto [n]$. A *one-line* representation of τ is denoted by $\tau = \tau(1)\tau(2) \cdots \tau(n)$.

Recall that from abstract algebra, we know that any permutation can be written as a product of *disjoint* cycles. Hence, a representation of a permutation in terms of disjoint cycles is called *cycle* representation.

Example 3.1. *Consider the bijective function*

$$\tau : [5] \mapsto [5], \quad \tau(1) = 3, \quad \tau(2) = 4, \quad \tau(3) = 1, \quad \tau(4) = 5, \quad \tau(5) = 2.$$

A one-line representation of τ is $\tau = 34152$. The cycle representation of τ is equal to $\tau = (13)(245)$.

The set of all permutations of the set $[n]$ will be denoted by S_n .

Definition 3.1 (Cycle Index). Let $\tau = c_1 c_2 \cdots c_k$ be the cycle representation of the permutation $\tau \in S_n$. Then, the number $n - k$ is called the cycle index of τ and will be denoted by $ind_c(\tau)$.

Definition 3.2 (Inversion). Let $\tau \in S_n = \tau(1)\tau(2)\cdots\tau(n)$ be a permutation. We say that $(\tau(i), \tau(j))$ is an inversion of τ if $i < j$ implies $\tau(i) > \tau(j)$.

We will denote the number of inversions of a permutation τ with $inv(\tau)$.

We recall the well-known fact due to Cauchy [8] that for any permutation $\tau \in S_n$, the parity of $inv(\tau)$ and $ind_c(\tau)$ are the same. Therefore, we can divide the class of all permutations S_n into two important subclasses.

Definition 3.3 (Even - Odd Permutations). A permutation $\tau = c_1 c_2 \cdots c_k$ in S_n is called an even (resp. odd) n -permutation if $ind_c(\tau)$ is even (resp. odd).

Example 3.2. For $n = 5$, the permutation $\tau = 13524 = (1)(2354)$ has cycle index equal to 3 and hence τ is an odd permutation, but the cycle index of $\tau' = 21354 = (12)(3)(45)$ is 2 and so the permutation τ' is even.

Considering the above discussions, the coin arrangements lemma in the case that there exists exactly one coin of each type can be restate as follows.

Proposition 3.1 (Set version of coin arrangements). For any integer $n > 1$, the number of even n -permutations is the same as the number of odd n -permutations.

In the rest of this section, we attempt to formulate a *multiset version* of the above well-known result in combinatorics of permutations.

For finding the right formulation of the coin arrangement lemma for multisets, we have to first replace permutations of the set $[n]$ with words of length N defined on the multiset $M = [1^{m_1}, 2^{m_2}, \dots, n^{m_n}]$ of cardinality N . The next step is to find the analogue of the cyclic decomposition of permutations into disjoint cycles. It seems that the *Lyndon factorization* of a word in which all factors are *distinct* is the suitable candidate. Hence, we come up with the following analogue of cycle index.

Definition 3.4 (Lyndon tuple). Let $\Sigma = \{1, 2, \dots, n\}$ be a finite ordered alphabet and $M = [1^{m_1}, 2^{m_2}, \dots, n^{m_n}]$ be a multiset over Σ of cardinality N . We will call any permutation $w = w_1 w_2 \cdots w_N$ of M an N -word over M . If $w = l_1 l_2 \cdots l_k$ is a Lyndon factorization of w in which $l_1 \triangleright l_2 \triangleright \dots \triangleright l_k$, then a tuple $tup(w) = (l_k, \dots, l_2, l_1)$ is called a Lyndon tuple of the word w over M .

Remark 3.1. It is noteworthy to mention that a Lyndon tuple of a word consists of only distinct Lyndon words.

Definition 3.5 (Lyndon index). Let $\Sigma = \{1, 2, \dots, n\}$ be a finite ordered alphabet and $M = [1^{m_1}, 2^{m_2}, \dots, n^{m_n}]$ be a multiset of over Σ cardinality N . For a N -word $w \in \Sigma^*$ over M with $tup(w) = (l_1, l_2, \dots, l_k)$ such that

$l_1 \triangleleft l_2 \triangleleft \dots \triangleleft l_k$, the Lyndon index of w denoted by $i_l(w)$ is defined to be the number $N - k$.

Definition 3.6 (Even-Odd Words). Let $\Sigma = \{1, 2, \dots, n\}$ be a finite ordered alphabet and $M = [1^{m_1}, 2^{m_2}, \dots, n^{m_n}]$ be a multiset of over Σ of cardinality N . A N -word $w \in \Sigma^*$ over M is said to be even (resp. odd) N -word if the Lyndon index $i_l(w)$ of w is even (resp. odd).

Example 3.3. For an ordered alphabet $\Sigma = \{1, 2, 3\}$ and a multiset $M = [1^2, 2, 3]$, the 4-word $w_1 = 2113 = (2)(113)$ has the Lyndon index equals 2 and hence it is an even 4-word. But the Lyndon index of $w_2 = 2131 = (2)(13)(1)$ is 1 and so the 4-word w_2 is odd.

Thus, we finally get the following reformulation of the Sherman's original coin arrangements lemma.

Proposition 3.2 (Multiset version of even-odd permutations identity). Let $\Sigma = \{1, 2, \dots, n\}$ be a finite ordered alphabet and $M = [1^{m_1}, 2^{m_2}, \dots, n^{m_n}]$ be a multiset over Σ of cardinality $N > 1$. Then, the number of even N -words over M is the same as the number of odd N -words over M .

In the next section, we will give a *bijective proof* a weighted version of the above coin arrangements lemma.

4. WEIGHTED COIN ARRANGEMENTS LEMMA

In this section, we will first give a weighted reformulation of the coin arrangements lemma. Then, we present a bijective proof of our main result by constructing a weight-preserving involution on the set of words. But before doing it, for the sake of completeness, we present the original proof of Sherman based on the so called *Witt identity* in the context of *combinatorial group theory* [4].

Proposition 4.1. Let Σ be a finite alphabet of k letters. Let $M(m_1, \dots, m_k)$ be the number of Lyndon words with m_1 occurrences of a_1 , m_2 occurrences of a_2 , \dots , m_k occurrences of a_k . Let x_1, \dots, x_k be commuting variables. Then

$$(4.1) \quad \prod_{m_1, \dots, m_k \geq 0} (1 - x_1^{m_1} \dots x_k^{m_k})^{M(m_1, \dots, m_k)} = 1 - x_1 - \dots - x_k.$$

Proof. By using Lyndon factorization and formal power series identities on words, we have

$$\begin{aligned} \frac{1}{1 - x_1 - \dots - x_k} &= \sum_{w \in \{x_1, \dots, x_k\}^*} \omega = \prod_{l \in L} \frac{1}{1 - l} \\ &= \frac{1}{\prod_{m_1, \dots, m_k \geq 0} (1 - x_1^{m_1} \dots x_k^{m_k})^{M(m_1, \dots, m_k)}}. \end{aligned}$$

□

Now, considering the Witt identity, the proof of the coin arrangements lemma can be simply obtained by equating the coefficients of monomials of the same degree in both sides of the identity.

To obtain a weighted generalization of the coin arrangements lemma, we first associate a formal variable u_a with each letter a of alphabet Σ which can be viewed as a weight of that letter. For any Lyndon word $l = i_1 i_2 \cdots i_h$, we define the weight $wt(l)$ of the Lyndon word $l \in L$ as the product of weights of its letters. That is, $wt(l) = u_{i_1} u_{i_2} \cdots u_{i_h}$. The weight of an N -word $w \in \Sigma^*$, is defined as $wt(w) = \prod_{l \in tup(w)} wt(l)$. From now on, we will denote the set of all even (resp. odd) N -words over M by E (resp. O). Thus, a weighted version of the coin arrangement lemma can be read as follows.

Theorem 4.2. [*Weighted Coin Arrangements Lemma*] For any multiset M of cardinality $N > 1$, the weighted sum of even N -words over M is the same as the weighted sum of odd N -words over M . In other words, we have

$$(4.2) \quad \sum_{w \in E} wt(w) = \sum_{w \in O} wt(w).$$

The following lemma is the key in the proof of the above theorem.

Lemma 4.3. **i:** Let $l = rs$ where $r, s \in L$ with $r \triangleleft s$ and let r be a single letter Lyndon word. Then, $l = (r, s)$ is the standard factorization of l .

ii: Let $l = rs$ where $r, s \in L$ $r \triangleleft s$ and let $r = (r_1, s_1)$ be the standard factorization of r with $r_1 \triangleleft s_1$. Then, $l = (r, s)$ is the standard factorization of l .

Proof. **i:** In this case, it is obvious that s is of maximal length. Hence by Definition 2.3, the result is immediate.

ii: Assume in contrary that s is not of maximal length. Then there exists a Lyndon word $s' = s'_1 s$ ($s' \triangleleft s$) where s' is of maximal length and $l = r'_1 s'_1$ with $r'_1 \in L$. Now if $s_1 \triangleleft s'_1$, since $s'_1 \triangleleft s$ it implies that $s_1 \triangleleft s$ which is a contradiction. On the other hand, since $r = (r_1, s_1)$ is the standard factorization of r , s'_1 must be a proper right factor of s_1 . But we already know that every Lyndon word is smaller than its any proper right factor. Thus we get $s_1 \triangleleft s'_1$, which is again a contradiction. □

The Proof of Theorem 4.2. For a given N -word w with Lyndon tuple $tup(w) = (l_1, l_2, \dots, l_k)$, we call the Lyndon word l_1 splittable, if l_1 is not a single letter and the standard factorization of $l_1 = (r_1, s_1)$ satisfies $s_1 \triangleleft l_1$. Now, one of the following cases may happen:

- The Lyndon word l_1 is splittable. Then, a mapping

$$f : E \mapsto O, \quad w' = f(w), \quad tup(w') = (r_1, s_1, l_2, \dots, l_k)$$

is a well-defined weight-preserving mapping (because $r_1 \triangleleft s_1 \triangleleft l_2$ and $wt(l_1) = wt(r_1)wt(s_1)$) .

- The Lyndon word l_1 is not splittable. Then, a mapping

$$g : O \mapsto E, \quad w' = f(w), \quad tup(w') = (l_0, l_3, \dots, l_k)$$

with $l_0 = l_1 l_2$, is a well-defined weight-preserving mapping (because $l_0 \in L$ and $l_0 \triangleleft l_2 \triangleleft l_3$ with $wt(l_1) = wt(r_1)wt(s_1)$) .

Clearly the mappings f and g are inverse of one another. Thus, the function f is a weight-preserving bijection from the set of even N -words to the set odd N -words and the conclusion immediately follows. \square

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