SPACES OF BOUNDED MEASURABLE FUNCTIONS INVARIANT UNDER A GROUP ACTION

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Abstract. In this paper we characterize spaces of L^∞-functions on a compact Hausdorff space that are invariant under a transitive and continuous group action. This work generalizes the author's 2021 results, found in [1], concerning the specific case of unitarily and Möbius invariant spaces of L^∞-functions defined on the unit sphere in $\mathbb{C}^n$.

1. Introduction

To understand this paper's place in the literature, we must first understand the relationships between the papers [2] (Nagel and Rudin, 1976), [1] (Hokamp, 2021), and [3] (Hokamp). Brief descriptions are given below.

In [2], Nagel and Rudin determine the closed unitarily invariant spaces of continuous and L^p-functions on the unit sphere of $\mathbb{C}^n$, for $1 \leq p < \infty$. That is, there exists a collection C of (minimal and invariant) spaces of continuous functions such that each closed unitarily invariant space is the closed direct sum of some subcollection of C. The same result is not shown for L^∞-functions, since for each L^∞-function f, the map $u \mapsto f \circ u$ from the unitary group into the L^∞ -functions need not be continuous under the norm topology.

In [1], the author formulates a result for L^∞-functions on the unit sphere that is analogous to the results of Nagel and Rudin when the L^∞-functions are endowed with the weak*-topology. The conclusion of this paper is that the same collection of (minimal and invariant) spaces of continuous functions described in [2] serves as the "building blocks" of the weak*-closed unitarily invariant spaces of L^∞-functions via closures of direct sums of subcollections.

In [3], the author generalizes the results of Nagel and Rudin in [2] by considering spaces of complex continuous and L^p-functions, for $1 \leq p < \infty$, defined on an arbitrary compact Hausdorff space X, on which a compact group G acts continuously and transitively. In particular, when a collection $\mathscr{G}$ of (minimal and invariant) spaces of continuous functions with certain properties exists, this collection plays the same role in constructing the closed spaces of continuous and L^p-functions on X as the collection from [2].

This paper generalizes the results in [1] and acts as an analogue to [3], in that we explore the case of L^∞-functions defined on X endowed with the weak*-topology. The main result (Theorem 4.1) is that when the same collection $\mathscr{G}$ of continuous functions on X from [3] exists (Definition 2.9), all weak*-closed invariant spaces of L^∞-functions can be constructed by closing the direct sum of some subcollection of $\mathscr{G}$.

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2. Preliminaries

Let X be a compact Hausdorff space and $C(X)$ the space of continuous complex functions with domain X. Let G be a compact group (with Haar measure m) that acts continuously and transitively on X. When we wish to be explicit, the map $\varphi_\alpha : X \to X$ shall denote the action of α on X for each $\alpha \in G$; otherwise, αx denotes the action of $\alpha \in G$ on $x \in X$.

Let μ denote the unique regular Borel probability measure on X that is invariant under the action of G. Specifically,

$$\int_X f\, d\mu = \int_X f \circ \varphi_\alpha \, d\mu, \tag{2.1}$$

for all $f \in C(X)$ and $\alpha \in G$. The existence of such a measure is a result of André Weil from [4], and a construction of μ can be found in [5] (Theorem 6.2). Throughout the paper, μ shall refer to this measure.

The notation $L^p(\mu)$ denotes the usual Lebesgue spaces, for $1 \leq p \leq \infty$. For $Y \subset C(X)$, the uniform closure of Y is denoted $\overline{Y}$, and for $Y \subset L^p(\mu)$, the norm-closure of Y in $L^p(\mu)$ is denoted $\overline{Y}^p$. When $L^\infty(\mu)$ is equipped with the weak*-topology, $\overline{Y}^*$ denotes the weak*-closure of $Y \subset L^\infty(\mu)$.

Remark 2.1. For $Y \subset L^\infty(\mu)$ convex and $1 \leq p < \infty$, we have

$$\overline{Y}^* \subset \overline{Y}^p \cap L^\infty(\mu).$$

This follows from the local convexity of $L^p(\mu)$, and from the fact that the weak*-topology on $L^\infty(\mu)$ is *stronger* than the topology which $L^\infty(\mu)$ inherits from each $L^p(\mu)$ endowed with the weak topology.

The following is an easy consequence of (2.1):

Remark 2.2. Let $1 \leq p < \infty$ and let p' be its conjugate exponent. Then

$$\int_X (f \circ \varphi_\alpha) \cdot g \, d\mu = \int_X f \cdot (g \circ \varphi_{\alpha^{-1}}) \, d\mu,$$

for $f \in L^p(\mu)$, $g \in L^{p'}(\mu)$, and $\alpha \in G$.

The following definition has appeared in several sources, such as [3], [6], or [7], but no attribution is given. The last citation is the specific case of the unitary group acting on the unit sphere in $\mathbb{C}^n$.

Definition 2.3. A space of complex functions Y defined on X is **invariant under** G (**G-invariant**) if $f \circ \varphi_\alpha \in Y$ for every $f \in Y$ and every $\alpha \in G$.

Remark 2.4. Since the action is continuous, $C(X)$ is G-invariant. Conversely, if $C(X)$ is G-invariant, then each action φ_α must be continuous.

Remark 2.5. The invariance property (2.1) means $\mu(\alpha E) = \mu(E)$ for every Borel set $E \subset X$ and every $\alpha \in G$. Consequently, (2.1) holds for every L^p-function, and $L^p(\mu)$ is G-invariant for all $1 \leq p \leq \infty$.

Definition 2.6 and Definition 2.7, stated in [3], are generalizations of definitions found in [7] related to the unitary group.

Definition 2.6 (2.3 [3])**.** If Y is G-invariant and T is a linear transformation on Y, we say T **commutes with** G if

$$T(f \circ \varphi_\alpha) = (Tf) \circ \varphi_\alpha$$

for every $f \in Y$ and every $\alpha \in G$.

Definition 2.7 (2.4 [3])**.** A space $Y \subset C(X)$ is **G-minimal** if it is G-invariant and contains no nontrivial G-invariant spaces.

The remaining definitions all come from [3].

Definition 2.8 (4.1 [3])**.** For each $x \in X$, the space $H(x)$ is the set of all continuous functions that are unchanged by the action of any element of G which stabilizes x. That is,

$$H(x) = \{f \in C(X) : f = f \circ \varphi_\alpha, \text{ for all } \alpha \in G \text{ such that } \alpha x = x\}.$$

Definition 2.9 (4.2 [3])**.** Let $\mathscr{G}$ be a collection of spaces in $C(X)$ with these properties:

(1) Each $H \in \mathscr{G}$ is a closed G-minimal space.
(2) Each pair H_1 and H_2 in $\mathscr{G}$ is orthogonal (in $L^2(\mu)$): If $f_1 \in H_1$ and $f_2 \in H_2$, then
$$\int_X f_1 \bar{f}_2 \, d\mu = 0.$$
(3) $L^2(\mu)$ is the direct sum of the spaces in $\mathscr{G}$.

We say $\mathscr{G}$ is a **G-collection** if it also possesses the following property:

($*$) $\dim(H \cap H(x)) = 1$ for each $x \in X$ and each $H \in \mathscr{G}$.

Throughout the paper, $\mathscr{G}$ shall denote a G-collection of $C(X)$, indexed by I, whose elements are denoted H_i, for $i \in I$, and further, *we assume that a G-collection exists for X.*

Remark 2.10. It should be stressed that we are not implying that a G-collection *always* exists for any X and G. However, a collection of spaces in $C(X)$ lacking at most only property ($*$) of Definition 2.9 always exists, as a consequence of the Peter-Weyl theorem from [8]. This collection is necessarily unique.

Definition 2.11 (4.7 [3])**.** We define π_i to be the projection of $L^2(\mu)$ onto H_i.

Remark 2.12. In Theorem 4.5 of [3], it is shown that each π_i commutes with G, and to each $x \in X$ there exists a unique $K_x \in H_i$ such that

$$\pi_i f = \int_X f(x) K_x \, d\mu(x),$$

for all $f \in L^2(\mu)$. The domain of π_i can then be extended to $L^1(\mu)$ by defining $\pi_i f$ to be the above integral for all $f \in L^1(\mu)$.

Definition 2.13 (4.8 [3])**.** For $\Omega \subset I$, E_Ω denotes the direct sum of the spaces H_i for $i \in \Omega$.

Remark 2.14. The G-invariance of each E_Ω is a natural consequence of the definition.

Remark 2.15. Definition 2.9 yields that each $f \in L^2(\mu)$ has a unique expansion $f = \sum f_i$, with each $f_i \in H_i$, which converges unconditionally to f in the L^2-norm. Since π_i is the identity map on H_i and the spaces H_i are pairwise orthogonal, we have $f_i = \pi_i f$ for $i \in I$. Thus,

$$f = \sum \pi_i f.$$

Each π_i is continuous as the orthogonal projection of $L^2(\mu)$ onto the closed subspace H_i. Thus, π_i annihilates a subset of $L^2(\mu)$ if and only if it annihilates its closure. The following is a consequence of this and Remark 2.15:

Remark 2.16. For each set $\Omega \subset I$, we have

$$\overline{E}^2_\Omega = \{f \in L^2(\mu) : \pi_i f = 0 \text{ when } i \notin \Omega\}.$$

Finally, the classical results used in this paper can be found in many texts, with the reference given in each instance being just one such place.

3. Closures of G-Invariant Sets

In this section, we show that G-invariance is preserved by closures in the spaces $C(X)$ and $L^p(\mu)$ for $1 \le p \le \infty$ (Corollaries 3.2 and 3.4). In particular, G induces classes of isometries on $L^p(\mu)$ and on $C(X)$ (Theorem 3.1), as well as a class of weak*-homeomorphisms on $L^\infty(\mu)$ (Theorem 3.3).

Theorem 3.1. *Suppose $\mathscr{X}$ is any of the spaces $C(X)$ or $L^p(\mu)$ for $1 \le p \le \infty$ and $\alpha \in G$. If $L_\alpha : \mathscr{X} \to \mathscr{X}$ is the map given by $L_\alpha f = f \circ \varphi_\alpha$, then L_α is a bijective linear isometry.*

Proof. The bijectivity of each L_α is clear because each has an inverse map $L_{\alpha^{-1}}$. The linearity of each L_α is also clear. Further, the invariance property (2.1) of μ yields that each L_α is an isometry on $L^p(\mu)$ (the case for $L^\infty(\mu)$ follows from Remark 2.5).

To show the same on $C(X)$, we observe that

$$|(L_\alpha f)(x)| = |f(\alpha x)| \le ||f|| \quad \text{and} \quad |f(x)| = |(L_\alpha f)(\alpha^{-1}x)| \le ||L_\alpha f||$$

for all $x \in X$. These inequalities yield that $||L_\alpha f|| = ||f||$. □

Corollary 3.2. *Suppose $\mathscr{X}$ is any of the spaces $C(X)$ or $L^p(\mu)$ for $1 \le p \le \infty$. If $Y \subset \mathscr{X}$ is G-invariant, then the closure of Y in $\mathscr{X}$ is G-invariant.*

Theorem 3.3. *Let $\alpha \in G$. If $L_\alpha : L^\infty(\mu) \to L^\infty(\mu)$ is the map given by $L_\alpha(f) = f \circ \varphi_\alpha$, then L_α is a weak*-homeomorphism.*

Proof. Recall the weak*-topology on $L^\infty(\mu)$ is a weak topology induced by the maps on $L^\infty(\mu)$ of the form

$$\Lambda_g f = \int_X fg \, d\mu,$$

for some $g \in L^1(\mu)$. Thus, L_α is continuous with respect to the weak*-topology if and only if $\Lambda_g \circ L_\alpha$ is continuous for all maps Λ_g.

Fix $g \in L^1(\mu)$. We observe that

$$(\Lambda_g \circ L_\alpha)(f) = \Lambda_g(f \circ \varphi_\alpha) = \int_X (f \circ \varphi_\alpha) \cdot g \, d\mu = \int_X f \cdot (g \circ \varphi_{\alpha^{-1}}) \, d\mu = \Lambda_{g \circ \varphi_{\alpha^{-1}}} f,$$

for every $f \in L^\infty(\mu)$, by Remark 2.2. We conclude L_α is continuous on $L^\infty(\mu)$ with respect to the weak*-topology.

Finally, the map $L_{\alpha^{-1}} : L^\infty(\mu) \to L^\infty(\mu)$ given by $L_{\alpha^{-1}}(f) = f \circ \varphi_{\alpha^{-1}}$ is the inverse of L_α. By a similar argument, $L_{\alpha^{-1}}$ is continuous with respect to the weak*-topology, and thus L_α is a weak*-homeomorphism. □

Corollary 3.4. *If $Y \subset L^\infty(\mu)$ is G-invariant, then $\overline{Y}^*$ is G-invariant.*

Remark 3.5. From Remark 2.14 and Corollary 3.4, each $\overline{E}^*_\Omega$ is a weak*-closed G-invariant subspace of $L^\infty(\mu)$.

4. Characterization of Weak*-Closed G-Invariant Subspaces of $L^\infty(\mu)$

In this section, we state and prove our main result (Theorem 4.1), which shows that the spaces $\overline{E}_\Omega^*$ are the *only* weak*-closed G-invariant subspaces of $L^\infty(\mu)$.

Theorem 4.1. *If Y is a weak*-closed G-invariant subspace of $L^\infty(\mu)$, then $Y = \overline{E}_\Omega^*$ for some $\Omega \subset I$.*

This result is an analogue to Theorem 5.1 of [3], which is used in its proof:

Theorem 4.2 (5.1 [3])**.** *Let $\mathscr{X}$ be any of the spaces $C(X)$ or $L^p(\mu)$ for $1 \le p < \infty$. If Y is a closed G-invariant subspace of $\mathscr{X}$, then Y is the closure of E_Ω for some $\Omega \subset I$.*

The set Ω from Theorem 4.2 is the set $\{i \in I : \pi_i Y \neq 0\}$. The proof of Theorem 4.1 further requires Lemma 4.3, which we prove in Section 5.

Lemma 4.3. *Let $Y \subset L^\infty(\mu)$ be a G-invariant space. Then for $g \in L^\infty(\mu)$, we have that $g \notin \overline{Y}^2$ whenever $g \notin \overline{Y}^*$.*

Remark 4.4. From Remark 2.1 and Lemma 4.3, for any G-invariant space $Y \subset L^\infty(\mu)$,

$$\overline{Y}^* = \overline{Y}^2 \cap L^\infty(\mu).$$

Remark 4.5. Remark 4.4 and Remark 2.16 give a description of the sets $\overline{E}_\Omega^*$:

$$\overline{E}_\Omega^* = \overline{E}_\Omega^2 \cap L^\infty(\mu) = \{f \in L^\infty(\mu) : \pi_i f = 0 \text{ when } i \notin \Omega\}.$$

Proof of Theorem 4.1. Let $Y \subset L^\infty(\mu)$ be a weak*-closed G-invariant space. Then

$$Y = \overline{Y}^* = \overline{Y}^2 \cap L^\infty(\mu)$$

from Remark 4.4. Since Y is G-invariant, so is $\overline{Y}^2$ from Corollary 3.2. By Theorem 4.2,

$$\overline{Y}^2 = \overline{E}_{\Omega'}^2,$$

where $\Omega' = \{i \in I : \pi_i \overline{Y}^2 \neq 0\}$.

We define $\Omega = \{i \in I : \pi_i Y \neq 0\}$. Then, Remark 4.5 yields

$$\overline{E}_\Omega^2 \cap L^\infty(\mu) = \overline{E}_\Omega^*.$$

We have $\Omega = \Omega'$ by the continuity of each π_i, and thus $\overline{E}_\Omega^2 = \overline{E}_{\Omega'}^2$. □

5. Proof of Lemma 4.3.

In this section, we prove Lemma 4.3, which we note is an analogue to Lemma 5.4 of [3], as well as a generalization of Lemma 4.2 from [1].

Lemma 5.1. *Let $g \in L^\infty(\mu)$. Then the map $\phi : G \to L^\infty(\mu)$ given by $\phi(\alpha) = g \circ \varphi_\alpha$ is weak*-continuous.*

Proof. To show that ϕ is weak*-continuous, we verify each $\Lambda_h \circ \phi$ is continuous, where Λ_h is the map $L^\infty(\mu) \to \mathbb{C}$ given by integration against the function $h \in L^1(\mu)$.

Observe the map $\Lambda_h \circ \phi$ is given by

$$(\Lambda_h \circ \phi)(\alpha) = \int_X (g \circ \varphi_\alpha) \cdot h \, d\mu.$$

From Lemma 6.2 of [3], the map $\alpha \mapsto h \circ \varphi_\alpha$ is continuous from G into $L^1(\mu)$. Thus, the map $\alpha \mapsto g \cdot (h \circ \varphi_{\alpha^{-1}})$ is continuous. We apply Remark 2.2 to get that

$$\alpha \mapsto \int_X g \cdot (h \circ \varphi_{\alpha^{-1}})\, d\mu = \int_X (g \circ \varphi_\alpha) \cdot h\, d\mu$$

is continuous, as desired. □

Proof of Lemma 4.3. Suppose $g \in L^\infty(\mu)$ and $g \notin \overline{Y}^*$. Then there exists a weak*-continuous linear functional Γ on $L^\infty(\mu)$ such that $\Gamma f = 0$ for $f \in Y$, and $\Gamma g = 1$, due to the Hahn-Banach theorem (Theorem 3.5 [9]). Since each weak*-continuous linear functional on $L^\infty(\mu)$ is induced by an element of $L^1(\mu)$, there exists $h \in L^1(\mu)$ such that $\Gamma F = \int_X Fh\, d\mu$ for $F \in L^\infty(\mu)$.

From Lemma 5.1, there exists a neighborhood N of the identity in G such that

$$\mathrm{Re} \int_X (g \circ \varphi_\alpha) \cdot h\, d\mu > \frac{1}{2}$$

for $\alpha \in N$. We choose a continuous map $\psi : G \to [0, \infty)$ such that $\int \psi\, dm = 1$ and the support of ψ is contained in N (recall m denotes the Haar measure on G).

We now define a map Λ on $L^\infty(\mu)$ by

$$\Lambda F = \int_X h(x) \int_G \psi(\alpha) \cdot F(\alpha x)\, dm(\alpha)\, d\mu(x), \text{ for } F \in L^\infty(\mu).$$

We fix $F \in L^\infty(\mu)$ and $x \in X$ and define the map $\mathscr{F}_x : G \to \mathbb{C}$ by $\alpha \mapsto F(\alpha x)$. Since

$$\int_G |\mathscr{F}_x|^2\, dm = \int_G |F(\alpha x)|^2\, dm(\alpha) = \int_X |F|^2\, d\mu = ||F||_2^2 < \infty,$$

we get $\mathscr{F}_x \in L^2(G)$, and since ψ is continuous, we have that $\psi \in L^2(G)$. Further,

$$\Big| \int_G \psi \mathscr{F}_x\, dm \Big| \le \Big(\int_G |\psi|^2\, dm \Big)^{\frac{1}{2}} \Big(\int_G |\mathscr{F}_x|^2\, dm \Big)^{\frac{1}{2}} = ||\psi||_2 \cdot ||F||_2,$$

so that for $F \in L^\infty(\mu)$,

$$|\Lambda F| \le ||\psi||_2 \cdot ||F||_2 \int_X |h|\, d\mu = ||\psi||_2 \cdot ||F||_2 \cdot ||h||_1.$$

The linearity of Λ on $L^\infty(\mu)$ is clear. Thus, Λ defines an L^2-continuous linear functional on $L^\infty(\mu)$, and hence extends to an L^2-continuous linear functional Λ_1 on $L^2(\mu)$ by the Hahn-Banach theorem (Theorem 3.6 [9]). By interchanging the integrals in the definition of Λ, we see that Λ_1 annihilates Y, since Y is G-invariant. Further,

$$\mathrm{Re}\, \Lambda_1 g = \int_G \psi(\alpha) \Big(\mathrm{Re} \int_X g(\alpha x) \cdot h(x)\, d\mu(x) \Big)\, dm(\alpha) > \int_N \psi(\alpha) \cdot \frac{1}{2}\, dm(\alpha) = \frac{1}{2}.$$

We conclude that $g \notin \overline{Y}^2$. □

6. Future Questions

(1) Does a G-collection exist for all groups G acting continuously and transitively on X? What conditions might exist on G or X that yield a collection lacking $(*)$?

(2) Under what conditions can the restrictions on X, G, and the action of G on X be loosened? Can the compactness of X and G be substituted with local compactness? Can the continuity of the action be substituted with separate continuity?

(3) Suppose H is a subgroup of G and $\mathscr{H}$ is a collection of closed H-minimal spaces satisfying the same conditions as $\mathscr{G}$. What is the relationship between $\mathscr{H}$ and $\mathscr{G}$? What if $\mathscr{H}$ and $\mathscr{G}$ lack $(*)$? The uniqueness of μ shows that H does not induce a new H-invariant measure on X. Further, G-invariance implies H-invariance (of a space).

We note that (3) is prompted from the study of $\mathscr{M}$-invariant and $\mathscr{U}$-invariant spaces of continuous functions on the unit sphere of $\mathbb{C}^n$ from [2], in which it is shown that there are infinitely many $\mathscr{U}$-invariant spaces and only six $\mathscr{M}$-invariant spaces. These six $\mathscr{M}$-invariant spaces are found by combining the $\mathscr{U}$-minimal spaces in a specific way (see Lemma 13.1.2 of [7]), and we are curious if this method can be generalized.

(4) Under what conditions can a G-collection characterize the closed G-invariant *algebras* of continuous functions? We note that the case for the unitary group acting on the unit sphere of $\mathbb{C}^n$ is discussed in [10] and is also summarized in [7].

7. Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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