

RAY STRUCTURES ON TEICHMÜLLER SPACE

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ABSTRACT. While there may be many Thurston metric geodesics between a pair of points in Teichmüller space, we find that by imposing an additional energy minimization constraint on the geodesics, thought of as limits of harmonic map rays, we select a unique Thurston geodesic through those points. Extending the target surface to the Thurston boundary yields, for each point Y in Teichmüller space, an “exponential map” of rays from that point Y onto Teichmüller space with visual boundary the Thurston boundary of Teichmüller space.

We first depict harmonic map ray structures on Teichmüller space as a geometric transition between Teichmüller ray structures and Thurston geodesic ray structures. In particular, by appropriately degenerating the source of a harmonic map between hyperbolic surfaces (along “harmonic map dual rays”), the harmonic map rays through the target converge to a Thurston geodesic; by appropriately degenerating the target of the harmonic map, those harmonic map dual rays through the domain converge to Teichmüller geodesics. We then extend this transition to one from Teichmüller disks through Hopf differential disks to stretch-earthquake disks. These results apply to surfaces with boundary, resolving a question on stretch maps between such surfaces.

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1. INTRODUCTION

1.1. Overview. Teichmüller space admits several ray structures, each arrived at through a similar process. One begins with a pair of points in Teichmüller space and decides whether to construe those points as complex structures or hyperbolic structures on that surface. Then one chooses a variational problem to solve, the solution to which defines an auxiliary object on the surface, for example a holomorphic differential or a geodesic lamination and a scaling constant. A family of pairs where the auxiliary object is fixed projectively then defines a path in Teichmüller space.

Probably the most famous example of this process are the Teichmüller geodesics. Here we attempt to minimize the quasiconformal dilatation between a pair of Riemann surfaces, finding a solution that may be described in terms of holomorphic quadratic differentials on each surface; looking for families in which the projective class of such a differential on one of the surfaces is fixed defines a Teichmüller ray.

Thurston [Thu98] developed an analogue of this perspective in the 1980's where he viewed points of Teichmüller space as hyperbolic structures on a surface. Then one sought minimizers of the Lipschitz stretch of maps between the surfaces, obtained by a map with maximal stretch locus a chain-recurrent geodesic lamination. When, for example, that lamination was maximal, one could consider the family of target surfaces for which the lamination was fixed, and this family defined a Thurston geodesic for his (asymmetric) metric.

There are several other families of minimizers, for example Kerckhoff's "lines of minima" [Ker92], but here we focus on ones related to minimization of L^2 Energy. Because of a conformal invariance in two dimensions, this energy depends on the hyperbolic structure of the target but only the conformal (hence complex) structure of the domain, and so occupies a middle ground between the two variational problems we opened with. Here also the solution, a harmonic map, determines and may be understood through the use of a holomorphic quadratic (Hopf) differential, and once again the families of targets who projectively share such a differential foliate Teichmüller space.

As a summary comment, though these ray structures are developed through similar processes, the Teichmüller geodesic rays, the Thurston geodesic rays and the harmonic map Hopf differential rays seem unrelated.

We need one more construction. A somewhat more obscure ray structure comes from seeking a family of Riemann surfaces whose Hopf differentials share one particular (horizontal) projective measured foliation. (See [Tab85].) We can imagine these as defining a type of "dual" harmonic map rays which we will clarify a bit later.

With all this context in place, we may explain our goals in this paper. We show a collection of results that together portray the harmonic maps rays structures as providing a transition between the Teichmüller ray structures and the Thurston ray structures. In particular, we show the following, stated roughly in this overview, and in terms of an initial harmonic map $u : X \rightarrow Y$ and of course in terms of the Hopf differential $\text{Hopf}(X, Y)$.

First, if we allow the domain X to degenerate along appropriate paths, either Teichmüller geodesics or harmonic map dual rays, then the harmonic map rays through Y converge to Thurston geodesics. (See Theorems 1.1, 1.2, and 1.3.)

Second, if we allow the targets Y to degenerate along harmonic map rays, the harmonic map dual rays converge to Teichmüller geodesics. (See Theorem 1.4.)

Third, there are natural ways that the Teichmüller and Thurston rays may be arranged into disks, either as Teichmüller disks in the complex case or into Stretch-Earthquake disks in the hyperbolic case. Of course, the Hopf differentials also admit a \mathbb{C}^* action, and the results above on harmonic map rays defining a transition between Teichmüller and Thurston geodesics extend to the disk setting.

Finally, we take up some applications of these results. We were slightly careful in our description of the Thurston Lipschitz problem to distinguish solutions where the maximally stretched lamination was maximal (so that the complementary regions were ideal triangles) from the more general solutions. Thurston noted that in the non-maximal case, there was no canonical (Thurston) geodesic between the pair of surfaces. There have been a number of proposals for some more canonical geodesic between some pairs of surfaces in some settings (see [PY17, HP19, CF21]). Here we observe that if we additionally require the geodesic between Y and Z to solve an energy-minimization problem in the sense of being a limit of harmonic map rays that proceed from Y to Z , then this results in a uniquely defined geodesic, a “harmonic stretch geodesic”. (See Theorem 1.8.)

Moreover, a simple extension of that technique produces a well-defined Thurston geodesic proceeding from a hyperbolic surface to a point on the Thurston boundary of Teichmüller space, represented by a projective measured foliation. Thus in Theorem 1.11 we show that the harmonic stretch geodesics from a point Y foliate Teichmüller space and accumulate only at their endpoint on the Thurston boundary: thus they provide an “exponential” map for the Thurston metric with visual boundary the Thurston boundary.

This exponential map for the Thurston metric defines one of two distinct versions we may define for the Thurston geodesic flow on the bundle over the Teichmüller space with fibers the space \mathcal{PMF} of projective measured foliations: here the orbit of a hyperbolic surface X and a projective measured foliation $[\eta]$ is the harmonic stretch line which passes through X and converges to $[\eta]$ in the Thurston boundary in the forward direction. The other version of the Thurston geodesic flow is defined such that the orbit of $(X, [\eta])$ is the harmonic stretch line arising as limits of harmonic map rays through X when the domain degenerates along the harmonic map dual ray determined by X and η . The first version of Thurston geodesic flow describes the contracting foliation along its orbits (which is also the endpoint on the Thurston boundary) while the second version describes the stretching lamination. Both of these two flows commute with the mapping class group action, and hence descend to the bundle over moduli space with fiber \mathcal{PMF} .

Along the way, we show some other results, for example an extension (Theorem 14.1) of this theory to surfaces with geodesic boundary, proving that the optimal Lipschitz constants between hyperbolic surfaces with boundaries are always realized by some surjective Lipschitz homeomorphisms. This verifies a conjecture of Alessandrini and Disarlo [AD19].

In several places, we prove not only subconvergence of the approximating rays but actual convergence of the families. The technique here often comes down to regarding a harmonic map as an equivariant minimal graph over a hyperbolic surface with values in an \mathbb{R} -tree, and we prove a uniqueness theorem in that case. (See Theorem 5.6.) The difficulties here are both that the tree typically does not

admit an orientation and the graphs we imagine have infinite diameter, but our result might still be regarded as in the spirit of the Jenkins-Serrin uniqueness theorem (see [JS66]). Having broached this topic, in the appendix we provide some completeness to this topic by also proving a corresponding existence theorem for graphs with values in a real tree and asymptotic boundary values over hyperbolic domains with geodesic boundary. We do not need that existence result for our main results, but the arguments develop themes introduced in other parts of the paper. (Here, by extending the range to trees, we extend the possible domains to surfaces whose polygonal ends are not limited to an even number of sides, as in [JS66] and elsewhere.)

In the next section, we define our terms and state our results somewhat more carefully.

1.2. Harmonic map rays and harmonic-stretch rays. The remainder of this introductory section is devoted to stating our results. We quickly declare some notation: a fuller treatment of the constructions behind those definitions is given in Section 2.

1.2.1. Four old families of rays and one new one. A Teichmüller ray $\mathbf{TR}_{X,\Phi} : [1, \infty) \rightarrow \mathcal{T}(S)$ is defined by a base Riemann surface X and a quadratic differential Φ which is holomorphic on X . It is convenient to also denote that ray by $\mathbf{TR}_{X,\lambda} : [1, \infty) \rightarrow \mathcal{T}(S)$, where λ is a measured (or projective measured) lamination defined by Φ .

A Thurston stretch ray $\mathbf{SR}_{Y,\lambda}(1, \infty)$ is defined with base point a hyperbolic surface Y and a maximal geodesic lamination λ .

A harmonic map ray $\mathbf{HR}_{X,\Phi}(1, \infty)$ is the family of surfaces so that the Hopf differential from X to any member of that family has Hopf differential proportional to Φ , a holomorphic quadratic differential on X . We will also have need of the notation $\mathbf{HR}_{X,Y}(1, \infty)$ which indicates the harmonic map ray beginning at the Riemann surface X that passes through the hyperbolic surface Y .

The most obscure previously defined ray we will consider initially is the harmonic map dual ray $\mathbf{hr}_{Y,\lambda} = \mathbf{hr}_{Y,\lambda}(t)$, defined by the condition that the horizontal measured foliation of the Hopf differential $\text{Hopf}(\mathbf{hr}_{Y,\lambda}(t) \rightarrow Y)$ is measure equivalent to the measured lamination $t\lambda$. Alternatively, $\mathbf{hr}_{Y,\lambda}(t)$ is the conformal structure underlying the minimal surface which is a graph over Y in the product $Y \times T_{t\lambda}$, where $T_{t\lambda}$ is the tree dual to the lamination $t\lambda$.

Eventually we will define a family of “harmonic stretch lines”, a distinguished family of Thurston geodesics arising as limits of harmonic map rays.

1.2.2. Foundational convergence results. We now state our first results on asymptotic relationships between these ray families.

Theorem 1.1. *Let $Y \in \mathcal{T}(S)$ be a hyperbolic surface and λ a measured lamination. Then the harmonic map rays $\mathbf{HR}_{X_t,Y}(1, \infty)$ converge to a Thurston geodesic locally uniformly as X_t diverges along the harmonic map dual ray $\mathbf{hr}_{Y,\lambda}$.*

Moreover, if λ is maximal, then the limit Thurston geodesic is exactly the Thurston stretch ray $\mathbf{SR}_{Y,\lambda}(1, \infty)$ defined by Y and λ .

The conclusion also holds if we let X_t diverge along Teichmüller rays. Namely,

Theorem 1.2. *Let $Y \in \mathcal{T}(S)$ be a hyperbolic surface and λ a measured lamination. Then the harmonic map rays $\mathbf{HR}_{X_t, Y}(1, \infty)$ converge to a Thurston geodesic locally uniformly as X_t diverges along the Teichmüller ray $\mathbf{TR}_{Y, \lambda}$.*

Moreover, if λ is maximal, then the limit Thurston geodesic is exactly the Thurston stretch ray $\mathbf{SR}_{Y, \lambda}(1, \infty)$ defined by Y and λ .

The limit Thurston geodesics in Theorem 1.1 and Theorem 1.2 for non-maximal measured lamination will be characterized later (see Section 8). For now, we extend the definition of stretch lines $\mathbf{SR}_{Y, \lambda}(1, \infty)$ to be the Thurston geodesics through Y which are limits of harmonic map rays through Y whose Hopf differentials have maximal stretch foliation projectively measure equivalent to λ .

We also have the following compactness result.

Theorem 1.3. *For any fixed $Y \in \mathcal{T}(S)$, let $X_n \in \mathcal{T}(S)$ be any divergent sequence. Then the sequence of harmonic rays $\mathbf{HR}_{X_n, Y} : [1, \infty) \rightarrow \mathcal{T}(S)$ contains a subsequence which converges to some Thurston geodesic locally uniformly.*

1.2.3. *Harmonic map dual rays and Teichmüller rays.* Informally, these results we just presented assert that if we look at a family of harmonic map rays $\mathbf{HR}_{X_s, Y}$ through a hyperbolic structure Y and then degenerate the domains X_s appropriately, say along a Teichmüller ray or (better) a harmonic map dual ray, then the rays limit on a Thurston geodesic.

It is natural to ask, what happens to limits of harmonic map rays if we degenerate in the other direction, say by letting the target surface, now Y_s , tend to infinity? We should expect some object to emerge that is defined in terms of the complex structure X . Roughly, the answer is that the relevant rays converge to Teichmüller geodesics through X . More formally, the rays through X to focus on are the “harmonic map dual rays” defined in Subsection 1.2.1. We then show that if we degenerate appropriately, this time letting the target Y_s degenerate along a harmonic map ray, then the harmonic map dual rays through X increasingly approximate Teichmüller geodesics.

Precisely, fix a complex structure $X \in \mathcal{T}(S)$. Let Φ be a holomorphic quadratic differential on X . Let $Y_s := \mathbf{HR}_{X, \Phi}(s)$; we will be degenerating the harmonic map dual rays defined by X and Y_s .

To that end, let $\lambda \in \mathcal{ML}(S)$ be the measured lamination which is measure equivalent to the horizontal foliation of Φ . This data defines a family of harmonic map dual rays through X as follows.

Let $X_{s,t} \in \mathbf{hr}_{Y_s, \sqrt{s}\lambda}$ be such that $\text{Hor}(\text{Hopf}(X_{s,t} \rightarrow Y_s)) = t\sqrt{s}\lambda$ ([Wol98]). Then $X_{s,1} \equiv X$ for all $s > 0$. An alternative, perhaps more geometric, description is that $X_{s,t}$ is the minimal surface graph over Y_s in the product $Y_s \times t\sqrt{s}T_\lambda$, where T_λ is the real tree dual to λ : the parameters s and t refer to where the target hyperbolic surface is along the harmonic map ray and the scaling of the tree T_λ .

Then in that setting, we prove our convergence result.

Theorem 1.4. *The family of harmonic map dual rays*

$$\mathbf{hr}_{Y_s, \sqrt{s}\lambda} : [1, \infty) \rightarrow \mathcal{T}(S)$$

converges locally uniformly to the Teichmüller geodesic ray $\mathbf{TR}_{X, \Phi} : [1, \infty) \rightarrow \mathcal{T}(S)$.

1.3. **Extension of results from Rays to Disks.** So far, we have focused on rays, i.e. sets defined in terms of a single real parameter. Yet an important topic

in Teichmüller theory are Teichmüller disks, i.e. images of Teichmüller maps from X whose holomorphic quadratic differentials (on X) are proportional by a complex number.

On the hyperbolic geometric side, there are also distinguished disks, defined in terms of a measured geodesic lamination λ on a surface Y , and two operations that deform a hyperbolic structure that each use λ . Naturally, one may stretch the structure along the lamination, as we have described throughout this section. One can also perform an earthquake along this lamination. The two operations, defined with data X and a measured lamination λ , together define a real two-dimensional family of hyperbolic surfaces through X , called a *stretch-earthquake disk*.

We state results that assert that the Hopf differential disks well-approximate stretch-earthquake disks for nearly degenerate domains X , and that there is a reasonable notion of dual Hopf differential disks that well-approximate Teichmüller disks for nearly degenerate ranges Y .

1.3.1. Convergence to Thurston stretch-earthquake disks. We begin with the approximation of stretch-earthquake disks.

Let $Y \in \mathcal{T}(S)$ be a hyperbolic surface and $\lambda \in \mathcal{ML}(S)$ a measured foliation/lamination. Let $\mathbf{SR}_{Y,\lambda}$ be the stretch line defined by Y and λ (where we recall that after Theorem 1.1 we extended the definition of these rays to include the limit of harmonic map rays $\mathbf{HR}_{X_t,Y}$ where X_t degenerates along the harmonic map dual ray $\mathbf{hr}_{Y,\lambda}$). Let $\mathcal{E}_{t\lambda}(Y)$ be the surface obtained from Y by earthquake along $t\lambda$. Define the *stretch-earthquake disk* $\mathbf{SED}(Y, \lambda)$ of (Y, λ) to be the set:

$$\bigcup_{0 < t < +\infty} \mathbf{SR}_{\mathcal{E}_{t\lambda}(Y), \lambda}(0, +\infty).$$

We wish to say, roughly, that the images of Hopf differential disks through a hyperbolic surface Y converge to a stretch-earthquake disk through Y – as we let the center X of the disks degenerate appropriately. To state this properly, we quickly introduce a bit more notation. Let $X_t \in \mathbf{hr}_{Y,\lambda}$ be the Riemann surface such that the horizontal foliation of $\text{Hopf}(X_t, Y) = \Phi_t$ is $t\lambda$. Let $Y(t, r, s)$ be the hyperbolic surface such that $\text{Hopf}(X_t, Y(t, r, s)) = re^{i\frac{s}{2t}}\Phi_t$ and $Y_r = \mathbf{SR}_{Y,\lambda}(r) \in \mathbf{SR}_{Y,\lambda}$. For such data (Y, X_t, Φ_t) , the *Hopf differential disk* ($\mathbf{HDD}(X_t, \Phi_t), Y$) comprises surfaces Z satisfying $\text{Hopf}(X_t, Z) = \zeta\Phi_t$ for some $\zeta \in \mathbb{C}$. (Naturally, $Y \in \mathbf{HDD}(X_t, \Phi_t), Y$) for the choice of $\zeta = 1$ in the above.)

In this language, we may state our result on the convergence of disks in the harmonic setting to disks in the hyperbolic geometric setting.

Theorem 1.5. *Let $Y \in \mathcal{T}(S)$ be a hyperbolic surface and $\lambda \in \mathcal{ML}(S)$ a measured foliation/lamination. Then the family of Hopf differential disks ($\mathbf{HDD}(X_t, \Phi_t), Y$) with base point Y locally uniformly converge to the stretch-earthquake disk ($\mathbf{SED}(Y, \lambda), Y$) with base point Y . Namely, for any prescribed $\mathbf{s} > 0$ and $0 < \mathbf{r} < \mathbf{r}'$, the family $Y(t, r, s)$ of surfaces converges to $\mathcal{E}_{s\lambda}(Y_r)$ uniformly in $(r, s) \in [\mathbf{r}, \mathbf{r}'] \times [-\mathbf{s}, \mathbf{s}]$ as $t \rightarrow \infty$.*

1.3.2. Convergence to Teichmüller disks. On the other hand, we may look for a family of disks defined via harmonic maps that well-approximate Teichmüller disks. There are two possibilities for approximates, but we focus in this introduction on just the first: here we consider an S^1 family of harmonic map dual rays defined by a point Y very far from X in Teichmüller space, and then show that this “disk” of dual rays converges to a classical Teichmüller disk as Y diverges.

More precisely, begin with a surface Y and, fixing a domain X , a Hopf differential $\Phi = \text{Hopf}(X, Y)$. Let $Y_{s,\theta}$ be the hyperbolic surface such that $\text{Hopf}(X, Y_{s,\theta}) = se^{2i\theta}\Phi$. In particular, $Y_{1,0} = Y$ and $Y_{s,\theta}$ diverges as $s \rightarrow \infty$ (for any choice of θ). Let

$$\mathbf{hd}_{X,\Phi,s} = \bigcup_{0 \leq \theta \leq \pi} \mathbf{hr}_{Y_{s,\theta},\lambda_\theta}$$

denote the (variable target) harmonic map dual disk, where λ_θ is the horizontal foliation of $e^{2i\theta}\Phi$.

Let $\mathbf{TD}_{X,\Phi}$ be the Teichüller disk determined by X and Φ .

Theorem 1.6. $\mathbf{hd}_{X,\Phi,s}$ locally uniformly converges to the Teichmüller disk $\mathbf{TD}_{X,\Phi}$, as $s \rightarrow +\infty$.

In Section 10, we also provide for a version of convergence to Teichmüller half-disks.

1.4. Piecewise harmonic stretch rays and Harmonic Stretch lines. Theorem 1.1 enables us to generalize Thurston's construction of stretch lines (maps) from maximal laminations to non-maximal ones using harmonic maps.

Theorem 1.7 (piecewise harmonic stretch map). *Let $Y \in \mathcal{T}(S)$ be any closed hyperbolic surface, and let λ be any geodesic lamination. Then for any surjective harmonic diffeomorphism $f : X \rightarrow Y \setminus \lambda$ from some (possibly disconnected) punctured surface X , there is a new hyperbolic surface*

$$Y_t := \text{Stretch}(Y, \lambda, f; t) \in \mathcal{T}(S)$$

depending analytically on $\{t > 0\}$ such that

- (a) the identity map $f_t : X \rightarrow Y_t \setminus \lambda$ is a surjective harmonic map $f_t : X \rightarrow Y_t \setminus \lambda$ with Hopf differential $t\text{Hopf}(f)$;
- (b) for any $0 < s < t$, the identity map $(f_t \circ f_s^{-1})$ is $\sqrt{t/s}$ -Lipschitz with (pointwise) Lipschitz constant strictly less than $\sqrt{t/s}$ in $S - \lambda$, and exactly expands arc length of λ by the constant factor $\sqrt{t/s}$.

A family of hyperbolic structures $\text{Stretch}(g, \lambda; \widehat{\Phi}, \widehat{f}; t)$ constructed above is called a *piecewise harmonic stretch line*. It admits an canonical orientation coming from the orientation of the half positive real ray $\{t > 0\}$. In that orientation, a piecewise harmonic stretch line is a (reparametrized) geodesic in the Thurston metric. Whenever we say a piecewise harmonic stretch line, we mean a directed line. Similarly to Thurston's construction of concatenation of stretch lines, one can construct a geodesic between any two points by a concatenation of piecewise harmonic stretch line segments.

Now, in contrast with the piecewise harmonic stretch lines described in Theorem 1.7 above, an important and distinct object of study for us will be families we call *harmonic stretch lines*. A piecewise harmonic stretch line is called a *harmonic stretch line* if it is the limit of a sequence of harmonic map rays. Given hyperbolic surfaces $Y, Z \in \mathcal{T}(S)$, a harmonic stretch line passing through Y to Z will be a limit of a family of harmonic map rays from base points $X_n \in \mathcal{T}(S)$ that all proceed through Y to Z , as the base points X_n degenerate in $\mathcal{T}(S)$. Of course, as in the case of the piecewise harmonic stretch lines, these harmonic stretch lines are also directed.

Our basic theorem on these harmonic stretch lines is the following.

Theorem 1.8 (Uniqueness of harmonic stretch lines). *For any two distinct hyperbolic surfaces $Y, Z \in \mathcal{T}(S)$, there exists a unique harmonic stretch line proceeding from Y to Z .*

Remark 1.9. The constructions still work if g is a hyperbolic surface with geodesic boundary components. Using the uniqueness result Theorem 1.8, we verify a conjecture (.cf Section 14) of Alessandrini-Disarlo ([AD19, Conjecture 1.8]).

Remark 1.10. From the construction of Thurston stretch lines and piecewise harmonic stretch lines, we see that every Thurston stretch line is a piecewise harmonic stretch line. However, Thurston stretch lines are not necessarily harmonic stretch lines. In fact, a Thurston stretch line is a harmonic stretch line if and only if the its maximally stretched lamination is *chain-recurrent*, see Corollary ??.

1.5. Visual boundary of the Thurston metric and an exponential map.

Finally, we extend the existence/uniqueness theory of harmonic stretch lines to rays whose terminal point is a projective measured lamination, representing a point on the boundary of the Thurston compactification of the Teichmüller space $\mathcal{T}(S)$. Properties of these rays, which are also Thurston geodesics, allow us to construct an exponential map on Teichmüller space from any base point $Y \in \mathcal{T}(S)$, where the boundary $\mathcal{PML}(S)$ of the Thurston compactification appearing as the visual boundary for this family of rays.

Theorem 1.11. *For any $Y \in \mathcal{T}(S)$ and any $[\eta] \in \mathcal{PML}(S)$, there exists a unique harmonic stretch ray starting at Y , which converges to $[\eta] \in \mathcal{PML}(S)$ in the Thurston compactification.*

Moreover, these rays foliate $\mathcal{T}(S)$ if we fix Y and let $[\eta]$ vary in $\mathcal{PML}(S)$, or if we fix $[\eta]$ and let Y vary in $\mathcal{T}(S)$.

Remark 1.12. For any $X \in \mathcal{T}(S)$, the set of unit vectors tangent to Thurston stretch lines (whose stretch lamination is maximal) has Hausdorff measure zero in $T_X^1\mathcal{T}(S)$ ([Thu98, Theorem 10.5]), while the set of unit vectors tangent to harmonic stretch lines is exactly $T_X^1\mathcal{T}(S)$.

1.6. Geodesic flow for the Thurston metric. The Teichmüller geodesic flow on the moduli space has been extensively studied in the literature. However, the notion of geodesic flow does not exist naturally for the Thurston metric. A natural challenge by Rafi ([Su16, Problem 3.10]) is to introduce a suitable notion of geodesic flow for the Thurston metric. Here we respond to this question by defining two versions of the geodesic flow for the Thurston metric.

Theorem 1.11 allows us to define the Thurston geodesic flow

$$\psi_t : \mathcal{T}(S) \times \mathcal{PML}(S) \longrightarrow \mathcal{T}(S) \times \mathcal{PML}(S)$$

such that the orbit through $(Y, [\eta]) \in \mathcal{T}(S) \times \mathcal{PML}(S)$ is the harmonic stretch line determined by Y and $[\eta]$ via Theorem 1.11. Moreover, every harmonic stretch line appears as a (forward) orbit.

There is another version of Thurston geodesic flow

$$\phi_t : \mathcal{T}(S) \times \mathcal{PML}(S) \rightarrow \mathcal{T}(S) \times \mathcal{PML}(S)$$

such that the orbit through $(Y, [\eta]) \in \mathcal{T}(S) \times \mathcal{PML}(S)$ is the stretch line $\mathbf{SR}_{Y, [\eta]}$ obtained from Theorem 1.1.

Both of these flows are mapping class group equivariant, and so they descend to $\mathcal{M}(S) \times \mathcal{PML}(S)$, where $\mathcal{M}(S)$ is the moduli space. Moreover, the earthquake flow

and ϕ_t are compatible in the sense that earthquake translates of ϕ_t -orbits are still ϕ_t -orbits, and hence define an action of the upper-triangular subgroup of $SL(2, \mathbb{R})$, see Proposition 13.10. We imagine it could be interesting to study the dynamical properties (invariant measures, ergodicity, and mixing) of these two versions of the Thurston geodesic flow over the moduli space.

1.7. Minimal graphs. An important tool for us will be conformal harmonic graphs over a hyperbolic domain with values in a real tree. Such maps will enjoy uniqueness properties which we can leverage to prove that the families of harmonic rays, such as those in Theorems 1.1 and 1.2, have a unique limit for any chosen subsequence. The graphs we study have infinite diameter and are extensions to the singular setting of the properly embedded classical minimal surfaces studied by Jenkins-Serrin [JS66] (see also [CR10]). A complication is that because the trees do not, in general, fold, some of the usual arguments only partially generalize. Our principal result is the following theorem.

Theorem 1.13. *Let Y be a crowned hyperbolic surface. Let T be an admissible dual tree of Y . Then for any prescribed order-preserving boundary correspondence, there exists a unique $\pi_1(Y)$ -equivariant minimal graph in $\tilde{Y} \times T$.*

We apply the uniqueness portion of this result in our central results and so prove it within the body of the paper. Having opened the discussion of these sorts of graphs, we treat the existence portion in an appendix that we do not reference in the body of the paper.

1.8. Previous results. A number of authors have studied how harmonic maps might approximate Teichmüller maps, or how Thurston stretch maps might be approached from an analytic perspective.

Before Bers' exposition [Ber60] of a proof of Teichmüller's theorem, Gerstenhaber and Rauch ([GR54b], [GR54a]) began a program to find Teichmüller maps as limits of energy-minimizing maps, optimized over conformal metrics. This program was completed by Mese in [Mes04].

In the other direction, Daskalopoulos and Uhlenbeck [DU20, DU22] focused on developing an analytic approach to finding least stretch maps: they build on literature on least gradient maps (see for example [SWZ92, MRSdL14, SZ93]), but they also develop the analytic foundations of what they term J_p harmonic maps.

In [BMS13] and [BMS15], Bonsante, Mondello and Schlenker develop an S^1 action on $\mathcal{T}(S) \times \mathcal{T}(S)$, the "landslide flow", which limits to the earthquake flow when one of the parameters goes to a measured lamination in the Thurston boundary of $\mathcal{T}(S)$. Certainly, these landslides, as smooth approximates to earthquakes, have many interesting properties. Similarly to our analysis, these landslides $L(h, h^*)$ are defined in terms of the circle action on the Hopf differential disks, here centered at a Riemann surface c that is the "midpoint" of the ray between h and h^* ; the limit to the earthquake flow on the second variable in $\mathcal{T}(S) \times \mathcal{T}(S)$ occurs as the first variable (not the midpoint) decays appropriately in Teichmüller space. In contrast, described in terms of the language of [BMS13] and [BMS15], here we study the limits of the Hopf differential disks as the center c degenerates along a dual ray; these are more specific paths than those studied by these authors, but allow us to show the convergence we require in our setting. The precise relationship between the two constructions is not clear.

There have been a number of studies comparing various rays in Teichmüller space. Choi, Rafi and Series show that short curves along lines of minima coincide with short curves along Teichmüller geodesics (with the same defining laminations) ([CRS08a]), and they prove that every line of minima is a Teichmüller quasi-geodesic ([CRS08b]); Choi, Dumas and Rafi prove that every grafting ray is a Teichmüller quasi-geodesic which stays within a bounded distance of some Teichmüller geodesic ([CDR12]); Gupta proves that every grafting ray is asymptotic to a Teichmüller geodesic ([Gup14, Gup15]); Lenzhen, Rafi and Tao compare the Teichmüller geodesics and Thurston geodesics in [LRT12].

1.9. Organization of the paper. We provide a basic background and define our notation in Sections 2 and 3. Section 4 is devoted to finding some subsequential limits of a family of harmonic maps $f_n : X_n \rightarrow Y$ from a degenerating sequence X_n of Riemann surfaces: crucial to our studies here and elsewhere is Minsky's [Min92] convex regions \mathcal{P}_R , and we include some first results here. In Section 5 we develop a key tool for our uniqueness and convergence claims: a type of Jenkins-Serrin uniqueness theorem for graphs over hyperbolic domains with values in a real tree. Sections 6, 7, and 8 provide a proof of Theorem 1.1: subconvergence is proved in Section 6, we prove Theorem 1.7 in Section 7 and use that result to reach the final convergence result in Section 8.

We change foci in the next two sections. In Sections 9 and 10, we consider limits of harmonic map dual rays when the range of the harmonic map degenerates, and find that Teichmüller rays and disks emerge: we prove Theorems 1.4 and 1.6 (as well as a second version of this convergence).

We then return to limits of the harmonic map rays and disks as the domain surface degenerates for the rest of the paper. In section 11, we prove the convergence result Theorem 11.3 on limits of families of harmonic map rays which define an earthquake-stretch disk. Then in Section 12, we prove our basic existence and uniqueness result for harmonic stretch lines, our refinement of Thurston geodesics for non-maximal laminations. Sections 13 and 14 offer consequences of our study, including a proof of Theorem 1.11 in Section 13.

The paper concludes by resolving a dangling issue: we prove in the Appendix an existence result for our Jenkins-Serrin problem to complement the uniqueness result Theorem 5.6 that we used throughout.

1.10. Acknowledgements. The first author is supported by National Natural Science Foundation of China NSFC 11901241. The second author gratefully acknowledges support from the Simons Foundation and the U.S. NSF grant DMS-2005551. He also benefited from a casual insightful remark of Yair Minsky and a conversation with Jérémy Toulisse about [Tho17].

2. DEFINITIONS AND BACKGROUND STRETCH MAPS AND TEICHMÜLLER MAPS

2.1. Teichmüller space. Let S be an oriented closed surface of genus $g \geq 2$. A *marked Riemann surface* is a pair (X, f) where X is a Riemann surface and $f : S \rightarrow X$ is an orientation preserving homeomorphism. Two marked Riemann surfaces (X, f) and (X', f') are called *equivalent* if there exists a conformal map in the homotopy class of $f' \circ f^{-1}$. The Teichmüller space $\mathcal{T}(S)$ is then defined as the space of equivalence classes of marked Riemann surfaces. Topologically, $\mathcal{T}(S)$ is homeomorphic to \mathbb{R}^{6g-6} .

The Teichmüller space can also be defined via hyperbolic surfaces. By the uniformization theorem, every Riemann surface of genus at least two admits a unique conformal metric of constant Gaussian curvature -1 , the hyperbolic metric. A *marked hyperbolic surface* is a pair (X, f) where X is a hyperbolic surface and $f : S \rightarrow X$ is an orientation preserving homeomorphism. Two marked hyperbolic surfaces (X, f) and (X', f') are called *equivalent* if there exists an isometry in the homotopy class of $f' \circ f^{-1}$. The Teichmüller space $\mathcal{T}(S)$ is then defined as the space of equivalence classes of marked hyperbolic surfaces.

For simplicity, we denote the (equivalence class of) marked Riemann/hyperbolic surface (X, f) by X .

2.2. Quadratic differentials, Measured foliations and measured laminations.

Given a Riemann surface X , a holomorphic quadratic differential Φ is a holomorphic section of K_X^2 . The space of holomorphic quadratic differentials on X , denoted by $H^0(X, K_X^2)$, is a vector space of real dimension $6g - 6$.

Every holomorphic quadratic differential Φ defines two measured foliations, the *horizontal foliation* $\text{Hor}(\Phi)$ by the imaginary part of $\sqrt{\Phi}$ and the *vertical foliation* $\text{Vert}(\Phi)$ by the real part of $\sqrt{\Phi}$. Conversely, every measured foliation on X is realized as the horizontal/vertical foliation of a unique holomorphic quadratic differential on X ([HM79, Wol96]). The space of measured foliations on S , denoted by $\mathcal{MF}(S)$, is homeomorphic to \mathbb{R}^{6g-6} ([FLP12]).

The hyperbolic counterpart of foliations are geodesic laminations. A *geodesic lamination* on a hyperbolic surface X is a closed subset consisting of simple geodesics, called *leaves*. Typical examples are simple closed geodesics. A geodesic lamination λ is said to be *maximal* if the complementary regions on X are ideal triangles. Consider the space of geodesic laminations on S equipped with the Hausdorff topology of the space of closed subsets of X . This is a compact metric space with the Hausdorff distance. Since any two different hyperbolic metrics induce a canonical correspondence between geodesics, the space of geodesic laminations, denoted by $\mathcal{GL}(S)$, depends only on the topology of S . A collection of disjoint simple closed geodesics is called a *multicurve*. A geodesic lamination is called *chain-recurrent* if it can be approximated by multicurves in $\mathcal{GL}(S)$.

A *measured geodesic lamination* (or *measured lamination* for short) is a geodesic lamination equipped with a *transverse invariant measure*, which associates to every arc transverse to the lamination a *Radon measure*. Typical examples are simple closed geodesics with the Dirac measures. The *intersection number* between simple closed geodesics extends naturally to the setting of measured geodesic laminations. Given a measured geodesic lamination λ and a simple closed geodesic α , the intersection number, denoted by $i(\lambda, \alpha)$, is defined to be the mass of α given by the transverse measure of λ . This associates to every measured geodesic lamination a function over the set of simple closed geodesics. Let $\mathcal{ML}(S)$ be the space of measured geodesic laminations, equipped with the weak-* topology of the space of functions over the set of simple closed geodesics. Topologically, $\mathcal{ML}(S)$ is homeomorphic to \mathbb{R}^{6g-6} .

There is a natural correspondence between the space of measured foliations and the space of measured geodesic laminations by straightening leaves of foliations using hyperbolic geodesics ([Lev83]).

2.3. Extremal length and hyperbolic length. Given the Riemann surface X and a simple closed curve α , the *extremal length* $\text{Ext}_X(\alpha)$ is defined as:

$$\text{Ext}_X(\alpha) = \sup_{\rho} \frac{\ell_{\rho}^2(\alpha)}{\text{Area}(\rho)},$$

where the supremum ranges over all conformal metrics ρ on X , and where $\ell_{\rho}(\alpha)$ is the infimum of the length of curves (free) homotopic to α . Alternatively, we can also define the extremal length via the conformal modulus of cylinders. Namely,

$$\text{Ext}_X(\alpha) = \inf_C \frac{1}{\text{Mod}(C)},$$

where the infimum ranges over all embedded cylinders with core curves homotopic to α . In practice, one obtains a lower bound of the extremal length via the first formulation and an upper bound via the second formulation.

From the perspective of hyperbolic geometry, one can define the *hyperbolic length function* $\ell_X(\alpha)$ by associating to α the length of the unique geodesic representative with respect to the hyperbolic metric of X . Both the extremal length function and the hyperbolic length function can be extended to measured laminations and measured foliations ([Ker80, Ker83]).

2.4. Teichmüller maps and Teichmüller rays. The Teichmüller maps between a pair of Riemann surfaces is the solution to the variational problems of finding the minimizing quasiconformal constants between two Riemann surfaces. Let $X \in \mathcal{T}(S)$ be a Riemann surface and Φ a holomorphic quadratic differential on X . Near the regular points of Φ , there are natural coordinates in which one represents Φ as $dz^2 = (dz + idy)^2$. For any $k \geq 1$, consider the quadratic differential Φ_k locally defined by $(k^{1/2}dx + ik^{-1/2}dy)^2$. It defines a unique Riemann surface $X_t \in \mathcal{T}(S)$ with respect to which Φ_t is holomorphic. The ray $\mathbf{TR}_{X,\Phi} : [1, \infty) \rightarrow \mathcal{T}(S)$ sending $k \geq 1$ to X_k is called a *Teichmüller ray*.

With respect the natural coordinates of Φ and Φ_t , the identity map $X \rightarrow X_t$ is a Teichmüller map. Conversely, all Teichmüller maps arise in this way. Namely, for any two Riemann surfaces $X, Y \in \mathcal{T}(S)$, there exist a unique holomorphic quadratic differential Φ on X (the *initial quadratic differential*) and a unique holomorphic quadratic differential Ψ on Y (the *terminal quadratic differential*), such that the (unique) Teichmüller map $X \rightarrow Y$ is locally defined by $(x, y) \mapsto (k^{1/2}x, k^{-1/2}y)$ with respect to the natural coordinates of Φ and Ψ , where k is a positive constant. Consider the Teichmüller ray determined by X and the initial quadratic differential Φ . We denote it by $\mathbf{TR}_{X,Y}$, indicating that it initiates at X and passes through Y .

Given $X, Y \in \mathcal{T}(S)$, the Teichmüller distance d_T on $\mathcal{T}(S)$ is defined as

$$d_T(X, Y) = \frac{1}{2} \log K,$$

where K is the infimum of quasiconformal constants among quasiconformal maps from X to Y in the homotopy class determined by the markings of X and Y . Another formulation is given by Kerckhoff [Ker80] in terms of extremal lengths:

$$d_T(X, Y) = \frac{1}{2} \log \sup_{\alpha} \frac{\text{Ext}_Y(\alpha)}{\text{Ext}_X(\alpha)},$$

where the supremum ranges over all simple closed curves. In this regard, the projective class of the horizontal foliation of the initial quadratic differential of

the Teichmüller map from X to Y is characterized as the unique one realizing the maximum $\max_{\mu \in \mathcal{MF}(S)} \frac{\text{Ext}_Y(\mu)}{\text{Ext}_X(\mu)}$.

Teichmüller lines are geodesics under the Teichmüller metric. Conversely, every geodesic under the Teichmüller metric is a Teichmüller line. Given a Teichmüller line $\mathbf{TR}_{X,\Phi}$, for any two points X_t, X_s in this line, we have $d_T(X_t, X_s) = |\log \sqrt{s/t}|$.

2.5. Remark on ray and disk structures on Teichmüller space. The paper concerns “ray structures” on Teichmüller space, where we imagine a number of different ways of defining the “rays”. Here, by ray structure, we will mean that for every point $Y \in \mathcal{T}(S)$, there is a family of parametrized halflines with initial point at Y which together foliate Teichmüller space. Probably the most famous such ray structure is that of Teichmüller rays from Y parametrized by rays in the vector space $H^0(Y, K_Y^2)$. That complex vector space $H^0(Y, K_Y^2)$ also has complex lines in it, and we imagine a “disk structure” on Teichmüller space $\mathcal{T}(S)$ to be an organization of a ray structure with some naturally defined free S^1 action on the rays.

2.6. Stretch maps. A seminal paper [Thu98] defined a new (Finsler) metric (“Thurston’s asymmetric metric”) on Teichmüller space, characterized it in terms of the variational problem of finding the minimizing Lipschitz stretch between two hyperbolic surfaces, described the solution to that variational problem, and used the solution in describing shortest paths in Teichmüller space for the metric. In addition to Thurston’s original paper, other sources for exposition of these topics, and further refinements of the theory, are [PT07, CR07, LRT12, Wal14, Pan, DLRT20, HOP21, Pap21].

A stretch map refers to a maximal geodesic lamination λ on a hyperbolic surface. Here a geodesic lamination is a closed collection of simple geodesics (either closed or not) and a maximal geodesic lamination has complementary regions which are ideal triangles. An ideal triangle admits a partial foliation by horocyclic arcs centered at the ideal points which both respects the symmetries of the ideal triangle and also meets the complete boundary of the ideal triangle. One defines a K -stretch map of the ideal triangle as a K -Lipschitz map which preserves the horocyclic foliation, fixes the three boundary points common to the foliations about each ideal point and stretches distance along the bounding geodesics by K .

Remarkably, Thurston’s careful analysis shows that these stretch maps of complementary domains of the maximal geodesic lamination extend to K -Lipschitz maps between hyperbolic surfaces and indeed, the Lipschitz constant K for these maps is the least possible Lipschitz constant among any maps between these two surfaces ([Thu98], Corollary 4.2).

We introduce some notation to summarize these constructions. If Y is a hyperbolic surface and λ is a maximal geodesic lamination, then we set $\mathbf{SR}_{Y,\lambda}(t)$ to define the surface constructed from Y and λ by a stretch map of Lipschitz constant $\sqrt{t} \geq 1$ along the lamination. We set $\mathbf{SR}_{Y,\lambda}(1, \infty)$ to be the family of hyperbolic surfaces obtained by allowing t to range over all positive values $[1, \infty)$, and we colloquially refer to that set as a *stretch map ray*.

Given $X, Y \in \mathcal{T}(S)$, the Thurston (asymmetric) distance is defined by

$$d_{Th}(X, Y) := \log L,$$

where L is the infimum of Lipschitz constants of Lipschitz homeomorphisms from X to Y in the homotopy class determined by the markings of X and Y . In particular, for $1 \leq s \leq t$, we see that $d_{Th}(\mathbf{SR}_{Y,\lambda}(s), \mathbf{SR}_{Y,\lambda}(t)) = \log \sqrt{t/s}$. Thurston also characterized this distance in terms of ratios of length functions of simple closed curves:

$$(2.1) \quad d_{Th}(X, Y) = \log \sup_{\alpha} \frac{\ell_Y(\alpha)}{\ell_X(\alpha)},$$

where the supremum ranges over all simple closed curves on X .

2.6.1. Maximally stretched laminations. As we mentioned earlier, for any two different Riemann surfaces $X, Y \in \mathcal{T}(S)$, the maximal ratio of extremal lengths $\max_{\mu \in \mathcal{MF}(S)} \frac{\text{Ext}_Y(\mu)}{\text{Ext}_X(\mu)}$ is uniquely realized by the projective class of the horizontal measured foliation of the initial quadratic differential of the Teichmüller map from X to Y . For the ratio of hyperbolic length functions, the uniqueness is no longer true in the setting of measured laminations. If a non-uniquely ergodic measured lamination μ realizes the maximal ratio $\max_{\mu \in \mathcal{ML}(S)} \frac{\ell_Y(\mu)}{\ell_X(\mu)}$, then any measured lamination with the same support as μ also realizes the maximal ratio. Nevertheless, Thurston showed that there is still a well-defined object in this setting, the *maximal maximally stretched chain-recurrent lamination* (or the *maximally stretched lamination* for short). For every two different hyperbolic surfaces X and Y , the maximally stretched lamination, denoted by $\Lambda(X, Y)$, is defined as the union of chain-recurrent laminations to which the restriction of every optimal Lipschitz map from X to Y is an affine map with the optimal Lipschitz constant.

Theorem 2.1 ([Thu98], Theorem 8.4). *Let g and h be any two distinct hyperbolic structures on S . If g_i and h_i are sequences of hyperbolic structures converging to g and h respectively, then $\Lambda(g, h)$ contains any lamination in the limit set of $\Lambda(g_i, h_i)$ in the Hausdorff topology.*

In particular, if $\Lambda(g_i, h_i)$ contains a simple closed curve α_i which converges to a maximal lamination λ , then $\Lambda(g, h) = \lambda$ because the only recurrent lamination contained in λ is λ itself.

2.6.2. Generalized stretch maps and their rays. Not all of the laminations that will arise in our discussions will be maximal; indeed, a focus of this paper is the non-maximal case. Corresponding to the case when a quadratic differential will have higher order zeroes, we will find ourselves considering versions of stretch maps where the lamination will have complementary domains on a surface which are hyperbolic ideal polygons, or more generally, the so-called crowned hyperbolic surfaces. We extend the definition of stretch maps to this case, called *piecewise harmonic stretch maps* see Section 7; we will also define in Section 12 a special class of Thurston geodesics which we call *harmonic stretch rays*.

2.7. Crowned hyperbolic surfaces.

Definition 2.2 (Crown end). A crown \mathcal{C} with $m \geq 1$ boundary cusps is an incomplete hyperbolic surface bounded by a closed geodesic boundary c , and a crown end comprising bi-infinite geodesics $\{\gamma_i\}_{1 \leq i \leq m}$ arranged in a cyclic order, such that the right half-line of the geodesic γ_i is asymptotic to the left half-line of geodesic γ_{i+1} , where $\gamma_{m+1} = \gamma_1$.

Definition 2.3. A crowned hyperbolic surface is obtained by attaching crowns to a compact hyperbolic surface with geodesic boundaries by isometries along some of their closed boundaries. This results in an incomplete hyperbolic metric of finite area on the surface. Topologically, the underlying surface is a compact surface with finitely many points removed from some boundary components.

Remark 2.4. The definition of crowned hyperbolic surfaces here is slightly differently from the definition in [Gup17]. The definition in [Gup17] requires all ends to be crown ends.

A *truncation* of a crown \mathcal{C} with m boundary cusps is obtained from \mathcal{C} by removing a choice of disjoint horocycle neighborhoods U_1, U_2, \dots, U_m at each ideal vertex of \mathcal{C} .

Definition 2.5. The *metric residue* of the hyperbolic crown \mathcal{C} with m boundary cusps is defined to be zero when m is odd, and equal to (the absolute value of) the alternating sum of lengths of geodesic sides of a truncation when m is even.

2.8. Horizontal foliations of meromorphic quadratic differentials. Let Φ be a meromorphic quadratic differential on a punctured X . Consider a complete regular leaf of the horizontal foliation $\text{Hor}(\Phi)$ of Φ . Then there are five possibilities ([Str84, Chapter IV]).

- (i) α is a simple closed geodesic contained in some finite horizontal cylinder.
- (ii) α is precompact but it is not a simple closed curve.
- (iii) α is contained in some half-infinite horizontal cylinder.
- (iv) α a bi-infinite geodesic contained in some bi-infinite horizontal strip.
- (v) α is a bi-infinite geodesic contained in some half-plane.

Finite cylinders in the first type and the closures of leaves in the second type are called *compact components* of $\text{Hor}(\Phi)$. Half-planes, strips and half-infinite cylinders are called *non-compact components* of $\text{Hor}(\Phi)$. Notice that a strip may spiral to a non-horizontal half-infinite cylinder, or be parallel to some half-plane, or both.

3. HARMONIC MAPS

In this section, we provide a brief overview of harmonic maps between surfaces.

3.1. Harmonic maps. Let $(M, \sigma(z)|dz|^2)$ and $(N, \rho(w)|dw|^2)$ be two Riemannian surfaces. A differentiable map $f : M \rightarrow N$ is said to be *harmonic* if it satisfies the *Euler-Lagrangian Equation*

$$f_{z\bar{z}} + (\log \rho)_w f_z f_{\bar{z}} = 0.$$

If M and N are compact, then f is harmonic if and only if it is a critical point of the energy functional:

$$E(f) := \int_M e(z) \sigma(z) dz d\bar{z},$$

where $e(f) := \frac{\rho(f(z))}{\sigma(z)} (|f_z|^2 + |f_{\bar{z}}|^2)$. Notice that the energy depends on the conformal structure on M and the metric on N . The energy of the map $f : X \rightarrow Y$ is labeled by $E(f) = E(f : X \rightarrow Y) = E(X, Y)$ depending on the context that is required.

The basic existence result of harmonic maps was established by Eells and Sampson in [ES64] and by Hamilton in [Ham75], if the target manifold has nonpositive sectional curvature. The uniqueness was obtained by Al'ber [Al'64] and Hartman [Har67] if the target manifold has negative sectional curvature and if the image is not contractible to a point or a geodesic. Moreover, Sampson [Sam78] and Schoen-Yau

[SY78] proved that any harmonic map between compact surfaces which is homotopic to a diffeomorphism is a diffeomorphism, provided that the target surface has nonpositive curvature.

3.2. Hopf differentials. Let X and Y be two hyperbolic surfaces. Let

$$f : (X, \sigma(z)|dz|^2) \rightarrow (Y, \rho(w)|dw|^2)$$

be a (surjective) harmonic diffeomorphism. Consider the pullback of ρ by f :

$$f^*(\rho) = \rho f_z \overline{f_z} dz^2 + e(f) \sigma dz d\bar{z} + \rho \overline{f_z} f_z d\bar{z}^2.$$

The $(2, 0)$ -part of $f^*(\rho)$ is called the *Hopf differential* of f . The harmonicity of f implies that the Hopf differential of f is holomorphic (see [Sch84, Jos84]). The Hopf differential for the map $f : X \rightarrow Y$ is labeled $\text{Hopf}(f) = \text{Hopf}(f : X \rightarrow Y) = \text{Hopf}(X, Y) = \Phi_X(Y) = \Phi$, depending on the context that is required.

Recall that every holomorphic quadratic differential defines two measured foliations on X , the horizontal foliation and the vertical foliation. The leaves of horizontal foliation (resp. vertical foliation) of $\Phi := \text{Hopf}(f)$ are exactly the maximally stretched (resp. minimally stretched) directions of f . Choosing a local coordinate $z = x + iy$ on M such that the leaves of the horizontal foliation (resp. vertical foliation) of $\text{Hopf}(f)$ are tangent to the x -axis (resp. y -axis). Then

$$(3.1) \quad f^*\rho = (e\sigma + 2|\Phi|)dx^2 + (e\sigma - 2|\Phi|)dy^2.$$

In particular, if we choose the coordinate $z = x + iy$ such that $\Phi = dz^2$ and choose σ to be the singular flat metric induced by $|\Phi|$, then $f^*(\rho)$ can be simply expressed as

$$f^*\rho = (e + 2)dx^2 + (e - 2)dy^2.$$

Let $\nu(z) := \frac{f_z d\bar{z}}{f_z dz}$ be the Beltrami differential of f . Set $\mathcal{G}(z) = \log(1/|\nu(z)|)$. By calculation, we see that $\cosh \mathcal{G} = \frac{e\sigma}{2|\Phi|}$. Substituting this into (3.1) yields

$$(3.2) \quad (f^*\rho)(z) = 2|\Phi(z)|(\cosh \mathcal{G}(z) + 1)dx^2 + 2|\Phi(z)|(\cosh \mathcal{G}(z) - 1)dy^2.$$

3.3. Harmonic maps to trees and minimal suspensions. We begin with a harmonic map $u : X \rightarrow Y$ with Hopf differential $\Phi = \text{Hopf}(u)$. We lift the setting to the universal cover with a map $\tilde{u} : \tilde{X} \rightarrow \tilde{Y}$ and a Hopf differential $\tilde{\Phi} = \text{Hopf}(\tilde{u})$. We consider the projection $p : \tilde{X} \rightarrow T_{\tilde{\Phi}}$ from \tilde{X} to the leaf space of the horizontal foliation $\text{Hor}(\tilde{\Phi})$. That leaf space $T = T_{\tilde{\Phi}}$, comprising equivalence classes of connected leaves of $\text{Hor}(\tilde{\Phi})$ (including leaves which branch at zeroes of $T_{\tilde{\Phi}}$) has the structure of a tree, with topology induced from $\text{Hor}(\tilde{\Phi})$. The tree acquires a well-defined distance $d = d_{T_{\tilde{\Phi}}}$ from the push-forward of the measure $\mu_{\Phi, h}$ on arcs transverse to the horizontal foliation of $\tilde{\Phi}$. The metric tree (T, d) is not locally compact when the genus of X is at least two, even if the singularities are typically finite valence.

By construction, the map $p : \tilde{X} \rightarrow (T_{\tilde{\Phi}}, 2d)$ is harmonic in the senses of [Wol96], [DDW98], and hence so is the product map $(\tilde{u}, p) : \tilde{X} \rightarrow \tilde{Y} \times T_{\tilde{\Phi}}$. We call the latter product map the *minimal suspension* of the harmonic map u because, by construction, the harmonic map (\tilde{u}, p) is conformal, and since it is also harmonic, it is therefore minimal. The minimal suspension is stable in an appropriate sense (see [Wol98] for this definition and further details), and indeed, given a tree (T, d) dual to a measured foliation \mathcal{F} , there is a unique minimal suspension $(\tilde{u}, p) : \tilde{X} \rightarrow \tilde{Y} \times (T, 2d)$ so that both \tilde{u} and p are harmonic with Hopf differentials that are

additive inverses (so that (\tilde{u}, p) is conformal and harmonic). A key portion of the present work, principally in Section 5 but also in the Appendix, may be seen as a less technically restrictive approach to case where X is compact and also an extension to the case where X is complete but with punctures. Not formally related to any of these results but partially aligned in spirit is recent work of Markovic [Mar21] in higher codimension.

3.4. Harmonic Map Rays and Dual Rays. This paper centers on harmonic map rays in Teichmüller space as an interpolating structure between Thurston stretch rays and Teichmüller rays. In this subsection, we define these rays in two ways: the first is in terms of the Hopf differential and the second is in terms of the minimal suspension in the previous subsection. The latter definition then suggests a dual construction of a different sort of ray structure which we call harmonic map dual rays.

3.4.1. Harmonic map rays. If $\Phi \in H^0(X, K_X^2)$ is a holomorphic quadratic differential on the Riemann surface X , then we are led to study the ray $s\Phi \subset H^0(X, K_X^2)$ for $s > 0$. By the identification of Teichmüller space $\mathcal{T}(S)$ with $H^0(X, K_X^2)$ via $Y \in \mathcal{T}(S) \mapsto \text{Hopf}(X, Y)$ (cf. [Wol89], [Hit87]), we then obtain a family $Y_s \subset \mathcal{T}(S)$ of surfaces for which $\text{Hopf}(X, Y_s) = s\Phi$ for some non-trivial element $\Phi \in H^0(X, K_X^2)$.

Definition 3.1. A family of hyperbolic surfaces Y_s for which $\text{Hopf}(X, Y_s) = s\Phi$ for some non-trivial element $\Phi \in H^0(X, K_X^2)$ and $s \geq 0$ is called a *harmonic maps ray* and is denoted $\mathbf{HR}_{X, \Phi}(s)$ or sometimes $\mathbf{HR}_{X, Y}(s)$ when we want to describe the ray passing between X and Y .

Note that the ray $Y_s \in \mathbf{HR}_{X, \Phi}(s)$ describes a family of minimal suspensions in $Y_s \times (T_\Phi, 2s^{\frac{1}{2}}d)$ all with the same second factor T_Φ up to a scaling of the metric on the tree by $s^{\frac{1}{2}}$. In particular, we can imagine the ray as determining a change in first factor Y_s so that the minimal surface's conformal structure X is held constant while the second factor is scaled.

Regarding a minimal suspension $X \subset Y \times (T, 2d)$ as having the three variables of the two factors Y and T as well as the conformal structure X , we see that a new ray structure naturally presents itself: we may fix the two factors and let the conformal structure vary.

Definition 3.2. We set $\mathbf{hr}_{Y, \Phi}(t)$ be the family of conformal structures X_t so that $\text{Hor}(\text{Hopf}(X_t \rightarrow Y)) = t\text{Hor}(\Phi)$, i.e. the horizontal foliation of the Hopf differential of the harmonic map from X_t to Y is proportional to that of Φ by a factor of t .

Indeed, the harmonic map dual ray has its parametrization defined only by the scaling of the dual tree to the horizontal measured foliation, say λ , of Φ . Thus, we often also use the notation $\mathbf{hr}_{Y, \lambda}(t)$ where λ is the horizontal measured foliation of Φ .

Equivalently, a harmonic map dual ray is a family of conformal structures defined via the minimal suspensions in a way dual to that of the harmonic map rays. The family $\mathbf{hr}_{Y, \Phi}(t)$ is the family of underlying conformal structures to the minimal surface in $Y \times (T_\Phi, 2td)$ parametrized by t . Here the duality is expressed as follows: for the harmonic maps ray $\mathbf{HR}_{X, \Phi}(s)$, we fix the conformal structure of the minimal surface and vary the surface $Y \in \mathbf{HR}_{X, \Phi}(s)$, while for the dual ray $\mathbf{hr}_{Y, \Phi}(t)$, we fix the surface Y and let the conformal structure $X_t \in \mathbf{hr}_{Y, \Phi}(t)$ vary in $\mathbf{hr}_{Y, \Phi}(t)$.

To see that these rays provide a ray structure for Teichmüller space $\mathcal{T}(S)$, see [Wol98] and [Tab85].

Our first focus in this paper will be the effect on the rays $\mathbf{HR}_{X,\Phi}(s)$ and $\mathbf{hr}_{Y,\Phi}(t)$ through a point on the ray if we allow a defining surface to degenerate along the other ray. More precisely, we fix a point $Y \in \mathcal{T}(S)$ and consider all the harmonic map rays $\mathbf{HR}_{X,\Phi}(s)$ which pass through Y and then study the limits of these as X tends to infinity along a harmonic maps dual ray. Dually, we consider a fixed point X on a family harmonic maps dual rays and study the limits of those dual rays $\mathbf{hr}_{Y,\Phi}(t)$ as Y diverges along a harmonic maps ray.

3.5. Minsky's estimates. For harmonic maps with high energy, the function \mathcal{G} appearing in (3.2) is nearly zero at points which are far away from the zeros of Φ . More precisely,

Lemma 3.3 ([Min92], Lemma 3.2 and Lemma 3.3). *Let $p \in X$ be at a $|\Phi|$ -distance at least d from any zero of Φ . Then*

$$\mathcal{G}(p) \leq \frac{\sinh^{-1}(|\chi(M)|/d^2)}{\exp(d)}.$$

Minsky also defined a family \mathcal{P}_R of regions whose geometry is controlled and in whose complement the harmonic map is nearly a projection. We now summarize what we need of this work.

Theorem 3.4 ([Min92], Theorem 5.1). *Let $s > 0$ and $c_1, \dots, c_3|_{\chi(M)|/2} > 0$ be chosen constants, where M is a closed surface carrying a flat metric induced from some holomorphic quadratic differential Φ . For any $R > 0$ there exists a boundary-convex set $\mathcal{P}_R \subset M$ with the following properties:*

- (i) \mathcal{P}_R contains the R neighbourhood of zeros of Φ .
- (ii) Every component of $\partial\mathcal{P}_R$ is either polygonal or regular geodesics, and the regular geodesic components occur in pairs bounding homotopically distinct flat cylinders.
- (iii) If \mathcal{F}_k is the k th maximal flat cylinder whose subcylinders occur as components of $M - \mathcal{P}_r$ for $r \leq R$, and \mathcal{F}_k is partially contained in \mathcal{P}_R , then $\mathcal{F}_k \cap \mathcal{P}_R$ is a pair of flat cylinders with length at least $r_W + R + c_k R^2/W$, where $W = W(\mathcal{F}_k)$ is the circumference of \mathcal{F}_k .
- (iv) $\ell(\partial\mathcal{P}_R) \leq K_1 R$.
- (v) $\text{Area}(\mathcal{P}_R) \leq A_1 + \left(K_2 + 2 \sum_{i=1}^k c_i\right) R^2$, where k is the number of flat cylinder components of $M - \mathcal{P}_r$ that have occurred for $r \leq R$.
- (vi) Each edge of a polygonal boundary component has length at least $K_3 R$.
- (vii) The polygonal components of $\partial\mathcal{P}_R$ are s -separated. Namely, for any two components γ_1, γ_2 of $\partial\mathcal{P}_R$ (where possibly $\gamma_1 = \gamma_2$), any arc in M with endpoints in γ_1 and γ_2 that can not be deformed (rel endpoints) into \mathcal{P}_R has length greater than $s \max\{\ell(\gamma_1), \ell(\gamma_2)\}$.

In the above, A_1, K_1, K_2 , and K_3 are constants depending only on s and $\chi(M)$.

Theorem 3.5 ([Min92], Theorem 7.1). *Let $f : M \rightarrow N$ be a harmonic diffeomorphism between closed hyperbolic surfaces with Hopf differential Φ . There are choices of constants $s > 0$ and $c_1, \dots, c_3|_{\chi(M)|/2} > 0$ for the construction of the polygonal region \mathcal{P}_R and an $R_0 > 0$, such that in the complement of \mathcal{P}_{R_0} there is a map π from the leaves of $\text{Hor}(\Phi)$ to the lamination corresponding to $f(\text{Hor}(\Phi))$ that*

factors through f , and is a local diffeomorphism on each leaf of $\text{Hor}(\Phi)$, mapping it to the corresponding geodesic representative of its image. For any point p on a leaf in $M - \mathcal{P}_{R_0}$,

$$d_N(f(p), \pi(p)) < a \exp(-bd_{|\Phi|}(p, \mathcal{P}_{R_0})),$$

and the derivative of π along leaves with respect to the $|\Phi|$ -metric satisfies

$$||d\pi| - 2| \leq a \exp(-bd_{|\Phi|}(p, \mathcal{P}_{R_0})),$$

where a and b are positive constants depending only on $\chi(M)$.

Theorem 3.6 ([Min92], Theorem 7.2). *There exists a constant C_0 depending on the topology of S , such that*

$$(3.3) \quad \sup_{\mu \in \mathcal{ML}(S)} \frac{1}{2} \frac{\ell_Y^2(\mu)}{\text{Ext}_X(\mu)} \leq E(X, Y) \leq \frac{1}{2} \frac{\ell_Y^2(\gamma)}{\text{Ext}_X(\gamma)} + C_0$$

where γ is the horizontal measured foliation of the Hopf differential of the harmonic map $f : X \rightarrow Y$.

Remark 3.7. The method of Minsky actually proves a slightly stronger estimate in the following sense. Let $\gamma = \sum_{i=1}^k \gamma_i$ be the component decomposition. Namely, for any $i \neq j$, γ_i and γ_j have disjoint support. Let X_i be the underlying surface of γ_i . Since X_i contains every leaf of γ_i , the construction of Minsky's polygonal region and train-track approximate occurs within each X_i . Then Theorem 3.6 holds on X_i . Namely,

$$(3.4) \quad \sup_{\mu \in \mathcal{ML}(X_i)} \frac{1}{2} \frac{\ell_Y^2(\mu)}{\text{Ext}_{X_i}(\mu)} \leq E(f|_{X_i}) \leq \frac{1}{2} \frac{\ell_Y^2(\gamma_i)}{\text{Ext}_{X_i}(\gamma_i)} + C_0,$$

where $\mathcal{ML}(X_i)$ represents the measured foliations on X which are contained in X_i .

Combining with the relation between energy and norm of Hopf differential $\Phi := \text{Hopf}(X, Y)$, we see that

$$(3.5) \quad 2\|\Phi\|_U \leq E_U(X, Y) \leq 2\|\Phi\|_U + 2\pi|\chi(S)|$$

for any subsurface $U \subset X$ (see [Wol89, Lemma 3.2]). Notice that for the horizontal measured foliation/lamination γ of $\Phi := \text{Hopf}(X, Y)$,

$$(3.6) \quad \text{Ext}_X(\gamma) = \|\Phi\|.$$

Combining (3.3), (3.5), and (3.6), we obtain some basic first estimates for this paper.

Lemma 3.8. *Let $X, Y \in \mathcal{T}(S)$ be two hyperbolic surfaces and $f : X \rightarrow Y$ the unique harmonic map. Let Φ be the Hopf differential of f . Let γ be the measured lamination corresponding to the horizontal measured foliation of Φ . Then there exists a constant $C > 0$ depending on the topology of S such that the following holds.*

$$(3.7) \quad \ell_Y(\gamma) - C \leq 2\|\Phi\| \leq \ell_Y(\gamma) + C$$

and

$$(3.8) \quad \ell_Y(\gamma) - C \leq E(f) \leq \ell_Y(\gamma) + C.$$

Moreover, let $\gamma = \sum_{i=1}^k \gamma_i$ be the component decomposition. Let Φ_i be the restriction of Φ to the underlying subsurface X_i of γ_i . Then

$$(3.9) \quad \ell_Y(\gamma_i) - C \leq 2\|\Phi_i\| \leq \ell_Y(\gamma_i) + C$$

and

$$(3.10) \quad \ell_Y(\gamma_i) - C \leq E(f|_{X_i}) \leq \ell_Y(\gamma_i) + C.$$

Proof. It follows from (3.3), (3.5), and (3.6) that

$$4\|\Phi\|(\|\Phi\| - C_0/2) \leq \ell_Y^2(\gamma) \leq 4\|\Phi\|(\|\Phi\| + \pi|\chi(S)|)$$

which implies (3.7). Inequality (3.8) then follows from (3.5) and (3.7). Replacing (3.3) by (3.4) we obtain (3.9) and (3.10). \square

4. COMPACTNESS OF HARMONIC MAPS TO A FIXED TARGET

The goal of this section is to prove the compactness of harmonic maps to a fixed target (Lemma 4.5). We continue with the notations introduced in the previous section.

4.1. Extension of estimates on Minsky's polygonal regions. We begin with an extension of Minsky's analysis of the polygonal region that we will need in our discussion of compactness of a family of the Riemann surface domains for the harmonic maps.

Lemma 4.1. *Let $Y \in \mathcal{T}(S)$ be a hyperbolic surface. For any R sufficiently large, there exists a positive constant δ depending on R and Y such that for any harmonic map $f : X \rightarrow Y$ with $X \in \mathcal{T}(S)$, each component of the polygonal region \mathcal{P}_R has injectivity radius at least δ with respect to the Hopf differential metric.*

Proof. If \mathcal{P}_R is a topological disk, then the conclusion follows directly from Minsky's construction. Now let us assume that \mathcal{P}_R is not a topological disk.

First, we claim that

Claim 1: *there exists a positive constant δ_1 depending on R and Y such that the extremal length of any simple closed curve α of \mathcal{P}_R is at least δ_1 : $\text{Ext}_{\mathcal{P}_R}(\alpha) \geq \delta_1$.*

By (3.5), we have $E(f|_{\mathcal{P}_R}) \leq 2\|\text{Hopf}(f)\|_{\mathcal{P}_R} + \text{Area}(Y) \leq cR^2 + \text{Area}(Y)$. It then follows from (3.3) that

$$\text{Ext}_{\mathcal{P}_R}(\alpha) \geq \frac{\ell_Y^2(\alpha)}{2E(f|_{\mathcal{P}_R})} \geq \frac{\text{Syst}(Y)^2}{2cR^2 + 2\text{Area}(Y)}.$$

Next, we claim that

Claim 2: *there exists a positive constant δ_2 such that, for each boundary component α of \mathcal{P}_R , we have $\text{Ext}_{\mathcal{P}_R}(\alpha) \leq \delta_2$.*

If α is a polygonal boundary component, the fact that $\partial\mathcal{P}_R$ is s -separated (item (vii) in Theorem 3.4) implies that α admits a (flat) annulus neighbourhood A attached to \mathcal{P}_R of width $\mathbf{w}(A) \geq s\ell(\alpha)$. Notice that the angle of α at each corner in \mathcal{P}_R is $\pi/2$. Therefore, the angle of α at each corner in A is $3\pi/2$. This means that the geodesic curvature of α at each corner with respect to A is $-\pi/2$. Let K_α be the total curvature of α with respect to A . Then the other boundary component α' of A has length $\ell(\alpha') = \ell(\alpha) - K_\alpha \mathbf{w}(A)$. Combining the facts $\ell(\alpha) \leq \ell(\partial\mathcal{P}_R) \leq K_1 R$ (item (iv) in Theorem 3.4) and that each horizontal and vertical segment of α has length at least $K_3 R$ (item (vi) in Theorem 3.4), we see that the number of corners

is at most K_1/K_3 . Then the total curvature of α satisfies: $|K_\alpha| \leq \frac{\pi K_1}{2K_3}$. Combined with [Min92, Theorem 4.5], this yields that the modulus of A satisfies:

$$\text{Mod}(A) \geq \frac{1}{|K_\alpha|} \log \frac{\ell(\alpha')}{\ell(\alpha)} \geq \frac{1}{|K_\alpha|} \log \frac{\ell(\alpha) + |K_\alpha|s\ell(\alpha)}{\ell(\alpha)} \geq \frac{\log s}{|K_\alpha|} \geq \frac{2K_3 \log s}{\pi K_1},$$

where we use the fact that $|K_\alpha| \geq 2\pi > 1$ because α has at least four corners. Therefore,

$$\text{Ext}_{\mathcal{P}_R \cup A}(\alpha) = \frac{1}{\text{Mod}_{\mathcal{P}_R \cup A}(\alpha)} \leq \frac{1}{\text{Mod}(A)} \leq \frac{\pi K_1}{2K_3 \log s}.$$

Recall that $\ell(\alpha) \leq K_1 R$. Then \mathcal{P}_{R+sK_1R} contains $\mathcal{P}_R \cup A$. Now we enlarge R to $R + sK_1R$, the estimate above gives that

$$\text{Ext}_{\mathcal{P}_{R+sK_1R}}(\alpha) \leq \frac{\pi K_1}{2K_3 \log s}.$$

If α is a cylinder boundary component, by the assumption that α admits a flat annulus of width at least $r_W + R + cR^2/\ell(\alpha)$ inside \mathcal{P}_R , we see that the modulus $\text{Mod}_{\mathcal{P}_R}(\alpha)$ of α inside \mathcal{P}_R is at least

$$\frac{r_W + R + cR^2/\ell(\alpha)}{\ell(\alpha)} \geq \frac{cR^2}{\ell(\alpha)^2} \geq c/K_1^2$$

where in the last inequality we use the assumption that $\ell(\alpha) \leq \ell(\partial\mathcal{P}_R) \leq K_1 R$. Then $\text{Ext}_{\mathcal{P}_R}(\alpha) = \frac{1}{\text{Mod}_{\mathcal{P}_R}(\alpha)} \leq \frac{K_1^2}{c}$.

Thirdly, let us consider the double of \mathcal{P}_R , denoted by \mathcal{P}_R^d obtained by gluing a copy of \mathcal{P}_R to \mathcal{P}_R along the corresponding boundary edges. We claim that

Claim 3: *there exists a positive constant δ_3 such that, for any simple closed curve α on \mathcal{P}_R^d , we have $\text{Ext}_{\mathcal{P}_R^d}(\alpha) \geq \delta_3$.*

If α is a boundary component of \mathcal{P}_R or an interior curve of \mathcal{P}_R , then

$$\text{Ext}_{\mathcal{P}_R^d}(\alpha) \geq \frac{1}{2} \text{Ext}_{\mathcal{P}_R}(\alpha) \geq \frac{\delta_1}{2}.$$

If α intersects one of the boundary components of \mathcal{P}_R say β , then

$$\text{Ext}_{\mathcal{P}_R^d}(\alpha) \geq \frac{i(\alpha, \beta)^2}{\text{Ext}_{\mathcal{P}_R^d}(\beta)} \geq \frac{1}{\text{Ext}_{\mathcal{P}_R^d}(\beta)} \geq \frac{1}{\text{Ext}_{\mathcal{P}_R}(\beta)} \geq \frac{1}{\delta_2},$$

where the last inequality follows from the second claim above.

Finally, we claim that there exists $\delta_4 > 0$ such that

Claim 4: *the polygonal region \mathcal{P}_R has injectivity radius at least δ_4 with respect to the Hopf differential metric.*

Notice that the restriction of the Hopf differential to \mathcal{P}_R and its copy gives a meromorphic quadratic differential q^d on \mathcal{P}_R^d . This differential has simple poles at the corners of the boundary components of \mathcal{P}_R ; the number of these points is bounded from above by K_1/K_3 times the number of components of $\partial\mathcal{P}_R$ (see the proof of claim 2), and that number is bounded by $3\text{genus}(X) - 3$. Notice that the genus of \mathcal{P}_R^d is at most $\text{genus}(X) + (3\text{genus}(X) - 2) = 4\text{genus}(X) - 2$. Set $\mathbf{k} = (3\text{genus}(X) - 3)K_1/K_3$ and $\mathbf{g} = 4\text{genus}(X) - 2$. Then

$$\left(\mathcal{P}_R^d, \frac{q^d}{2\|\mathcal{P}_R\|} \right) \in \cup_{g \leq \mathbf{g}, \kappa \leq \mathbf{k}} \mathcal{Q}^\kappa \mathcal{M}_g$$

where $\mathcal{Q}^\kappa \mathcal{M}_g^\kappa$ represents the bundle over the moduli space \mathcal{M}_g whose fiber, over a Riemann surface $M \in \mathcal{M}_g$, is the space $Q^\kappa(M)$ of area one meromorphic quadratic differentials containing κ simple poles. Moreover, the third claim above implies that for each g there exists a compact subset $K_g \subset \mathcal{M}_g$ such that $\mathcal{P}_R^d \subset K_g$, i.e. $(\mathcal{P}_R^d, \frac{q^d}{2\|\mathcal{P}_R\|}) \in \cup_{g \leq \mathbf{g}, \kappa \leq \mathbf{k}} \mathcal{Q}^\kappa K_g$. Then for each pair (g, κ) , there exists a positive constant $\delta_{g, \kappa}$, such that

$$\inf_{(M, q') \in \mathcal{Q}^\kappa K_g(M)} \frac{\text{inj}(q')}{\text{inj}(M)} \geq \delta_{g, \kappa}$$

where $\text{inj}(M)$ represents the injectivity radius of the hyperbolic metric of M and $\text{inj}(q')$ represents the injectivity radius of the singular flat metric induced by q' . Hence for any $M \in K_g$ and any $q' \in Q^\kappa(M)$, we have

$$\text{inj}(q') \geq \delta_{g, \kappa} \min_{M \in K_g} \text{inj}(M).$$

The finiteness of the pairs (g, κ) with $0 < g \leq \mathbf{g}$ and $4 \leq \kappa \leq \mathbf{k}$ then implies that

$$\delta'_4 := \min_{0 < g \leq \mathbf{g}, 4 \leq \kappa \leq \mathbf{k}} \delta_{g, \kappa} \min_{M \in K_g} \text{inj}(M) > 0.$$

In particular, the injectivity radius of $\frac{q^d}{2\|\mathcal{P}_R\|}$ with respect to the singular flat metric $\frac{q^d}{2\|\mathcal{P}_R\|}$ is at least δ'_4 . Hence the injectivity radius of q^d is at least $2\|\mathcal{P}_R\|\delta'_4$, which is at least $3\pi R^2\delta'_4$ because \mathcal{P}_R contains the R neighbourhood of its zeros. Recall that q^d is obtained as a double of \mathcal{P}_R along its boundary edges. Therefore, the injectivity radius of \mathcal{P}_R is at least $\frac{3}{2}\pi R^2\delta'_4$. This finishes the proof of the claim and hence the proof of the lemma. \square

We will need an estimate on the total error that accumulates in using that the harmonic map outside \mathcal{P}_R is a projection

Lemma 4.2. *Let R_0 be the constant from Minsky's estimate. Let $f : X \rightarrow Y$ be a harmonic diffeomorphism between closed hyperbolic surfaces $X, Y \in \mathcal{T}(S)$. Let $e(f)$ be the energy density with respect to the metric induced by the Hopf differential $\text{Hopf}(f)$. Then for any $R > R_0$,*

$$\int_{X \setminus \mathcal{P}_R} |e(f) - 2| dA \leq C |\chi(S)| e^{-R/2}$$

where C is a constant depending on $\chi(X)$, \mathcal{P}_R is the Minsky's polygonal region and dA is the area measure induced by the Hopf differential $\text{Hopf}(f)$.

Proof. Let R_0 be the constant from Minsky's estimate. Then by (3.2) and Lemma 3.3, for any $R > R_0$, and any $z \in X \setminus \mathcal{P}_R$,

$$(4.1) \quad |e(f) - 2| \leq C e^{-|z|},$$

where C is a constant depending on $\chi(X)$, and $|z|$ represents the distance from z to the zero set of $\text{Hopf}(f)$ with respect to the flat metric induced by $\text{Hopf}(f)$.

Consider the Voronoi decomposition of X with respect to $\text{Hopf}(f)$, where the 2-cells are the path components of the set of points which have unique length-minimizing paths to the zero set of $\text{Hopf}(f)$. The number of such 2-cells is exactly the number of zeros of $\text{Hopf}(f)$ counted without multiplicity. Within each 2-cell, consider the horizontal critical segments initiated from the corresponding zero of $\text{Hopf}(f)$. Let κ be the order of this zero. These segments cut the underlying 2-cell

into $\kappa + 2$ sub-cells, each of which can be identified with a subset of the upper or lower half plane in \mathbb{C} . The total number of such sub-cells is at most $3(4g - 4)$, which corresponds to the case where all zeros of $\text{Hopf}(f)$ are simple. Integrating $|e(f) - 2|$ over each sub-cell, we obtain

$$\begin{aligned}
 (4.2) \quad & \int_{X \setminus \mathcal{P}_R} |e(f) - 2| dA \\
 & \leq 3(4g - 4)C \int_{\{z \in \mathbb{C}: |z| \geq R, \text{Im}z \geq 0\}} e^{-|z|} dA \\
 & \leq 6(2g - 2)C\pi e^{-R/2}.
 \end{aligned}$$

□

The proof above also proves the following:

Lemma 4.3. *Let R_0 be the constant from Minsky's estimate. Let $f : X \rightarrow Y$ be a surjective harmonic diffeomorphism from a punctured Riemann surface (possibly disconnected) to a crowned hyperbolic surface Y . Let $e(f)$ be the energy density with respect to the metric induced by the Hopf differential $\text{Hopf}(f)$. Then for any $R > R_0$,*

$$\int_{X \setminus \mathcal{P}_R} |e(f) - 2| dA \leq C|\chi(S)|e^{-R/2}$$

where C is a constant depending on $\chi(X)$, \mathcal{P}_R is the Minsky's polygonal region and dA is the area measure induced by the Hopf differential $\text{Hopf}(f)$.

4.2. Harmonic maps with varying domains. Let $Y \in \mathcal{T}(S)$ be a fixed hyperbolic surface. Let $X_n \in \mathcal{T}(S)$ be an arbitrary divergent sequence of Riemann surfaces, and let Φ_n be the Hopf differential of the harmonic map $f_n : X_n \rightarrow Y$. Let $R_m > 0$ be a sequence of divergent positive real numbers. Since the number of zeros of Φ_n is at most $2|\chi(S)|$, it follows that there exists a positive integer $k \leq 2|\chi(S)|$ such that for sufficiently large m , the polygonal region $\mathcal{P}_{R_m}(\Phi_n)$ contains exactly k components, up to a subsequence of (X_n, Φ_n) . Choose a zero for each component of $\mathcal{P}_{R_m}(\Phi_n)$. Let $\mathbf{p}_n = \{p_{1,n}, \dots, p_{k,n}\}$ be the choice of zeros of Φ_n . Consider the family of pointed singular flat surfaces $(\mathcal{P}_{R_m}(\Phi_n); \mathbf{p}_n)$. By Lemma 4.1, these (sub)converge to some pointed singular flat surface $(Z_m; \mathbf{q})$. Letting $m \rightarrow \infty$ and applying a diagonal argument, we get a nested sequence of singular flat surfaces

$$(Z_1; \mathbf{p}) \subsetneq (Z_2; \mathbf{p}) \subsetneq \dots \subsetneq (Z_m; \mathbf{p}) \subsetneq \dots$$

such that there exists a subsequence of $(\Phi_n; \mathbf{p}_n)$ converging to the pointed singular flat surface $(\cup_{m \geq 1} Z_m; \mathbf{p})$. For simplicity, we still denote this subsequence by $(\Phi_n; \mathbf{p}_n)$. Let X be the Riemann surface underlying $\cup_{m \geq 1} Z_m$. Then there exists a family of quasiconformal embeddings $\iota_{m,n} : \mathcal{P}_{R_m}(\Phi_n) \rightarrow X$ with quasiconformal constant uniformly converging to 1 as $n \rightarrow 1$, whose images exhaust X , that is, $\cup_{m \geq 1} \lim_{n \rightarrow \infty} \iota_{m,n}(\mathcal{P}_{R_m}(\Phi_n)) = X$.

Applying a standard energy estimate, we see that, up to a subsequence if necessary, the sequence $f_n : (X_n; \mathbf{p}_n) \rightarrow Y$ converges to a harmonic map $f : (X; \mathbf{p}) \rightarrow Y$ with Hopf differential Φ in the following sense. Take an arbitrary compact exhaustion $\{\mathcal{K}_j\}$ of X . For each j , the sequence of composition maps $f_n \circ (\iota_{m,n})^{-1}|_{\mathcal{K}_j} : \mathcal{K}_j \rightarrow Y$ converges to $f|_{\mathcal{K}_j} : \mathcal{K}_j \rightarrow Y$ uniformly for sufficiently large m . (For more detailed description and proof of the discussion above, we refer to [Gup19, Section 3 and Section 4]).

In the rest of this paper, for the sake of simplicity, we identify $U \subset X$ with its images $\iota_{m,n}^{-1}(U) \subset X_n$ without mentioning the map $\iota_{m,n}^{-1}$.

We next state a result regarding limits of sequences of harmonic maps and the limit of the associated Hopf differentials. In this direction, we define our notion of limit in this setting.

Definition 4.4. We say that a sequence $f_n : X_n \rightarrow Y$ of harmonic maps f_n from a family of closed Riemann surfaces X_n to a hyperbolic surface Y converges to a harmonic map $f : X_\infty \rightarrow Y$ if (i) each component of X_∞ is a Gromov-Hausdorff limit of X_n for some choice of base points $p_n \in X_n$, (ii) the surface X_∞ contains all pointed Gromov-Hausdorff limits of X_n , and (iii) on each Gromov-Hausdorff component U_∞ of X_∞ , we have that the harmonic map f_∞ is the limit of the harmonic maps f_n on a domain $U_n \subset X_n$, where U_n converges to U_∞ .

Lemma 4.5 (compactness). *Let $Y \in \mathcal{T}(S)$ be a hyperbolic surface. Let $X_n \in \mathcal{T}(S)$ be an arbitrary divergent sequence of Riemann surfaces with $f_n : X_n \rightarrow Y$ being the corresponding harmonic map. Let $\mathfrak{p}_n \subset X_n$ be the choice of marked points defined as in the discussion above. Then there exist a chain-recurrent geodesic lamination λ on Y , and a subsequence $f_{n_m} : (X_{n_m}; \mathfrak{p}_{n_m}) \rightarrow Y$ which converges to a surjective harmonic diffeomorphism $f : X \rightarrow Y \setminus \lambda$ from some (possibly disconnected) punctured surface X . Moreover,*

- the quadratic differential $\text{Hopf}(f)$ has a pole of order at least two at each puncture; and
- $\lambda = \lim_{R \rightarrow \infty} \lim_{m \rightarrow \infty} \overline{f_n(X_{n_m} \setminus \mathcal{P}_R(\text{Hopf}(f_{n_m})))}$.

Proof. We continue using the notations introduced in the beginning of this subsection. In particular, X is the Riemann surface underlying the singular flat metric $\cup_{m \geq 1} Z_m$.

Notice that on each connected component of X , the limiting harmonic map f is a harmonic diffeomorphism onto its image $f(X)$ ([Wol91], proof of Proposition 3.4, following [SY78]). By Theorem 3.4, for sufficiently large n , the image $f_n(X_n \setminus \mathcal{P}_{R_n}(\Phi_n))$ is contained in some ϵ_n neighbourhood of the geodesic lamination corresponding to the horizontal measured foliation of Φ_n , with $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. This implies that $f_n(X_n \setminus \mathcal{P}_{R_n}(\Phi_n))$ converges to some geodesic lamination λ as $n \rightarrow \infty$ in the Hausdorff topology. Recall that every measured lamination is chain-recurrent. Combining with the fact that the Hausdorff limit of a sequence of chain-recurrent laminations is again chain-recurrent ([Thu98, Proposition 6.2]), we see that λ is a chain-recurrent.

We claim that $f(X) \subset Y \setminus \lambda$, which implies that $f(X) = Y \setminus \lambda$. Suppose to the contrary that there exists some $x \in X$ such that $f(x) \in \lambda$. Then there exists a neighbourhood V of $f(x)$ with $V \subset f(X)$. Since $f_n : X_n \rightarrow Y$ converges to $f : X \rightarrow Y$, it follows that there exists small neighbourhoods $U_n \subset X_n$ with $f_n(U_n) \subset V$ and also with U_n approximating some fixed region in X and hence at a uniformly bounded distance from the zeroes of Φ_n . On the other hand, the assumption that λ is the Hausdorff limit of $f_n(X_n \setminus \mathcal{P}_{R_n})$ implies that there exists a sequence of points $p_n \in X_n \setminus \mathcal{P}_{R_n}$ whose distance to the zeros of Φ_n diverges such that $f_n(p_n) \rightarrow f(x) \in \lambda$. In particular, $f_n(p_n) \in f_n(U_n)$ but $p_n \notin U_n$. This contradicts the fact that f_n is a homeomorphism.

Finally, we show that $f : X \rightarrow Y$ is globally injective. Recall that for each component X^i of X , the restriction $f|_{X^i}$ is injective. In particular, f is open. To

prove that $f : X \rightarrow Y$ is globally injective, it suffices to show that for $i \neq j$, we have $f(X^i) \cap f(X^j) \neq \emptyset$. Suppose to the contrary that there exist $x^i, x^j \in X$ such that $f(x^i) = f(x^j)$. Then there exist disjoint neighbourhoods U^i of x^i and U^j of x^j such that $f(U^i) = f(U^j)$. It follows that for n sufficiently large, $f_n(U^i) \cap f_n(U^j) \neq \emptyset$. Again this contradicts the fact that f_n is a homeomorphism for every n . \square

5. THE GENERALIZED JENKINS-SERRIN PROBLEM

In this section, we consider minimal surfaces in $M \times T$ where M is a hyperbolic surface and T is a tree satisfying some conditions. The story begins with the Scherk's example, which is a minimal graph over a square in \mathbb{R}^2 with boundary values plus or minus infinity alternatively, the Dirichlet problem with infinite boundary values. This was generalized to minimal graphs over $2n$ -gons in \mathbb{R}^2 in [JS66] and over ideal $2n$ -gons in \mathbb{H}^2 in [NR02]. (In [JS66] and [NR02, NR07], the domains also allow strictly convex arcs as part of the boundary.) In these cases, the surface M is a polygon with an even number of edges and the tree T is simply the real axis \mathbb{R} . For our purpose, we need to consider the case where M is the universal cover of a "hyperbolic crowned surface" and T is a tree dual to some measured foliation. In particular, by extending to trees (which may not admit a folding to a real line), we extend the scope of the results to include the case where M is a hyperbolic ideal polygon with an odd number of edges. This section concerns the uniqueness problem of minimal graphs (see Theorem 5.6). The existence problem will be addressed in Appendix A (see Theorem A.1).

5.1. Minimal graphs over domains in \mathbb{H}^2 . In this subsection, we collect some results about the minimal graphs over domains in \mathbb{H}^2 . For more details, we refer to [NR02]. Consider the unit disk model of \mathbb{H}^2 . Denote by (x_1, x_2, x_3) the coordinates on the product $\mathbb{H}^2 \times \mathbb{R}$. The metric on $\mathbb{H}^2 \times \mathbb{R}$ is

$$d\sigma^2 = \frac{dx_1^2 + dx_2^2}{F} + dx_3^2$$

where

$$F = \left(\frac{1 - x_1^2 - x_2^2}{2} \right)^2.$$

Let $D \subset \mathbb{H}^2$ be a hyperbolic domain. The graph of a function $u : D \subset \mathbb{H}^2 \rightarrow \mathbb{R}$ is *minimal* if and only if satisfies the *minimal surface equation*:

$$(5.1) \quad \operatorname{div} \left(\frac{\nabla u}{\tau_u} \right) = 0,$$

where $\tau_u = \sqrt{1 + |\nabla u|^2}$, and ∇u and div are the gradient and divergence with respect to \mathbb{H}^2 .

Let u be a solution of the minimal surface equation. It is clear that the differential

$$(5.2) \quad du^* := \frac{F(u_1 dx_2 - u_2 dx_1)}{\tau_u}$$

is closed on D , where $u_i := \frac{\partial u}{\partial x_i}$. Locally, we may then define a function u^* on D , uniquely up to an additive constant, the *conjugate function* of u . Geometrically, we

can interpret du^* as follows. Let $U \subset D$ be a subdomain and $\alpha \subset \partial U$ a boundary arc with arc length parametrization s such that the domain is on the left. Then

$$\int_{\alpha} du^* = \int_{\alpha} \left\langle \frac{\nabla u}{\tau_u}, \nu \right\rangle ds$$

where ν is the outward unit conormal to U along α . (The integral $\int_{\alpha} du^*$ is called the *flux* of u across α .) In particular,

$$(5.3) \quad \int_{\alpha} |du^*| \leq |\alpha|,$$

where $|\alpha|$ is the hyperbolic length of α .

Suppose $\gamma \subset \mathbb{H}^2$ is a geodesic segment. Consider the infinite strip bounded by the two geodesics which are orthogonal to γ and which pass through the endpoints of γ . For each $\epsilon > 0$, there are two level curves contained in this strip each of which consists of points of distance ϵ to γ . Each of these two level curves is said to be an ϵ -translate of γ . We need the following estimate from [NR02, Lemma 1], which we state slightly differently here. Geometrically, this lemma says that the tangent planes near “divergence points” are almost vertical.

Theorem 5.1 ([NR02] Lemma 1, see also [CR10] Flux theorem). *Let $D \subset \mathbb{H}^2$ be a convex domain. Let $\alpha \subset \partial D$ be a compact geodesic arc. Let $\alpha_{\epsilon} \subset D$ be an ϵ -translate of α . Let $u : D \rightarrow \mathbb{R}$ be a solution of the minimal surface equation (5.1).*

- If $u|_{\alpha} = +\infty$, then
 - (1) $\left\langle \frac{\nabla u}{\tau_u}, \nu \right\rangle$ converges uniformly to 1 as $\epsilon \rightarrow 0$, where ν is the unit field normal to α_{ϵ} and pointing toward α ;
 - (2) $\lim_{\epsilon \rightarrow 0} \int_{\alpha_{\epsilon}} du^* = |\alpha|$, where $|\alpha|$ is the hyperbolic length of α .
- If $u|_{\alpha} = -\infty$, then
 - (3) $\left\langle \frac{\nabla u}{\tau_u}, \nu \right\rangle$ converges uniformly to -1 as $\epsilon \rightarrow 0$, where ν is the unit field normal to α_{ϵ} and pointing toward α ;
 - (4) $\lim_{\epsilon \rightarrow 0} \int_{\alpha_{\epsilon}} du^* = -|\alpha|$, where $|\alpha|$ is the hyperbolic length of α .

5.2. Minimal graphs in $M \times T$.

Definition 5.2 (Admissible foliations). Let Y be a crowned hyperbolic surface. A measured foliation F on Y is said to be *admissible* if (i) each end of a leaf either accumulates on a non-trivial arc of a boundary curve or is asymptotic to an ideal point of a crown, and (ii) if the foliation has parallel leaves to the crown or the boundary, then the transverse measure of an arc to the frontier is infinite.

Remark 5.3. Let Y be a crowned hyperbolic surface whose boundary consists of m closed geodesics $\{\alpha_1, \dots, \alpha_m\}$ and n crowned ends C_1, \dots, C_n with ideal geodesics $\{\gamma_{ij} : 1 \leq i \leq n, 1 \leq j \leq k_i\}$ where $k_i \geq 3$. Let F be an admissible measured foliation of Y . Then the leaves of F have ends which either (i) are simple closed curves homotopic to α_i , or (ii) spiral around α_i with accumulation set the whole α_i or (iii) are asymptotic to γ_{ij} .

Definition 5.4 (Admissible dual trees). Let Y be a crowned hyperbolic surface. The (metric) tree dual to the lift of an admissible measured foliation on Y to the universal cover \tilde{Y} is said to be an *admissible dual tree*.

Remark 5.5. Note that the lift of a proper path to a boundary curve or a crown projects to a half-infinite path in the tree.

Let T be the dual tree of some admissible measured foliation F on Y . Let $\iota : \tilde{Y} \rightarrow T$ be a projection map along leaves of \tilde{F} . Then ι induces a boundary correspondence $\partial\iota : \partial\tilde{Y} \rightarrow \partial T$, where a boundary referred to here is the Gromov boundary. It is well known that $\partial\iota$ is independent of the choice of ι . A homeomorphism $\partial\tilde{Y} \rightarrow \partial T$ is said to be an *admissible boundary correspondence* if it is the boundary map of some projection map $\tilde{Y} \rightarrow T$ along the leaves of an admissible measured foliation.

Our main result in this section is the following uniqueness result, which we will rely on in places in order to show the uniqueness of some limits of sequences.

Theorem 5.6. *Let Y be a crowned hyperbolic surface. Let T be an admissible dual tree. Then there exists at most one $\pi_1(Y)$ -equivariant minimal graph in $\tilde{Y} \times T$ with a prescribed admissible boundary correspondence.*

We briefly describe the organization of the proof. Of course, we want to compare the minimal graphs of two maps from \tilde{Y} to T , say $u : \tilde{Y} \rightarrow T$ and $v : \tilde{Y} \rightarrow T$. The proof is divided into two steps. In the first step, we prove that the distance function $\text{dist}(u(\cdot), v(\cdot))$ is bounded and the supremum is realized at some point. The idea of this step is to fix a point $p \in \tilde{Y}$ which is not a zero of any Hopf differential, and then, for any q , consider the distances

$$\mathbf{u}(q) := u(q) - u(p), \quad \mathbf{v}(q) := v(q) - v(p).$$

These distances are not well-defined in a tree, but we are principally interested in a form $\tilde{\Psi} = (\mathbf{u} - \mathbf{v})d(\mathbf{u}^* - \mathbf{v}^*)$, and we find that this form is well-defined on a neighbourhood of $\partial\tilde{Y}$ and descends. Analyzing the level sets of $\text{dist}(u(\cdot), v(\cdot))$ near $\partial\tilde{Y}$, we also find that $d\mathbf{u}^* - d\mathbf{v}^*$ is “nearly well defined” on simply connected domains near $\partial\tilde{Y}$ (regardless of the number of ideal geodesics of each crown end), and on cylindrical domains near crown ends which have an even number of ideal geodesics, which will then turn out to be the only case which remains. Estimating these forms, and applying Stokes theorem and the (strong) maximum principle, we prove that $\text{dist}(u(\cdot), v(\cdot))$ is bounded and the supremum is realizable. In the second step, we prove that $\text{dist}(u(\cdot), v(\cdot))$ is identically zero, using the (strong) maximum principle again and the fact that harmonic maps are conformal at zeros of Hopf differentials.

This concludes the outline. We now describe the argument more precisely.

Remark 5.7. The existence problem about the equivariant minimal graphs in $\tilde{Y} \times T$ will be addressed in the Appendix, see Theorem A.1. As a direct consequence, we are able to parametrize surjective harmonic diffeomorphisms from the complex plane to any ideal hyperbolic n -gon, using trees with n boundary edges. (An edge of a tree is said to be a boundary edge if one of its endpoints is of valence one.) Let P be an ideal hyperbolic n -gon. Let $\text{ADT}(P)$ be the set of admissible dual trees of P . Then there is a bijection between $\text{ADT}(P)$ and the set of surjective harmonic diffeomorphisms from \mathbb{C} to P . It is clear that $\text{ADT}(P)$ is homeomorphic to \mathbb{R}^{n-3} .

For $u : \tilde{Y} \rightarrow T$ the equivariant map from \tilde{Y} to T , we let \tilde{X} be the Riemann surface underlying the graph $(z, u(z))$, and we let $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ be the equivariant projection map $(z, u(z)) \mapsto z$ from \tilde{X} to \tilde{Y} , and we set $f : X \rightarrow Y$ its descent.

Let $\{a_i, c_j\}$ be the set of punctures of X labeled in such a way that a_i corresponds to the closed geodesic boundary α_i of Y while c_j corresponds to the crowned end C_j of Y . Then we have the following.

Lemma 5.8. *(i) Each puncture a_i is a pole of order two, with residue of non-vanishing real part, of $\text{Hopf}(f)$ and c_j is a pole of order k_j of $\text{Hopf}(f)$, where $k_j \geq 3$ is the number of ideal geodesics contained in the boundary of the crowned end C_j .*

(ii) The tree T is the dual tree to the horizontal measured foliation of $\text{Hopf}(\tilde{f})$.

(iii) Let γ be an arbitrary ideal geodesic in ∂Y . Then for any lift $\tilde{\gamma}$ and any compact subsegment $I \subset \tilde{\gamma}$, there is a convex domain $U \subset \tilde{Y}$ with $I \subset \partial U$ such that $u(U)$ is contained in a single half-infinite edge of T and that $u(p) \rightarrow \infty$ as $p \rightarrow I$.

(iv) Let α be an arbitrary closed geodesic loop in ∂Y . Then for any lift $\tilde{\alpha}$ and any compact subsegment $I \subset \tilde{\alpha}$, there is a convex domain $U \subset \tilde{Y}$ with $I \subset \partial U$ and a geodesic ray $r : [0, +\infty) \rightarrow T$ such that $u(U)$ is contained in the image of r and that $u(p) \rightarrow \infty$ as $p \rightarrow I$.

Here the phrasing that $u(p) \rightarrow \infty$ as $p \rightarrow I$ means that $u(p)$ leaves all compact sets in the half-infinite (half-closed) edge of T as $p \rightarrow I$.

Proof. The first two items follows from [Gup17] except for the statement that the residue at a_i has non-vanishing real part. Suppose that the residue at some a_i is purely imaginary then the half-infinite cylinder corresponding to a_i is vertical. Let ω_d be the core curve whose distance to the compact boundary of this half-infinite cylinder is exactly d . By Theorem 3.5, the length of image $f(\omega_d)$ under the harmonic map $f : X \rightarrow Y$ converges to zero as $d \rightarrow \infty$. This contradicts the fact that the length of non-trivial simple closed curves on Y have a uniform lower bound away from the zero. Hence the residue can not be purely imaginary.

For the third item, let $U' \subset \tilde{X}$ be the half-plane (cf also [Gup17]) corresponding to $\tilde{\gamma}$. Then $\tilde{\gamma} \subset \partial \tilde{f}(U')$ and $u(\tilde{f}(U'))$ is contained in the half-infinite edge of T corresponding to $\tilde{\gamma}$. We may choose U to be a convex domain of $\tilde{f}(U')$ with $I \subset \partial U$.

It remains to show the fourth item. By the first item of this lemma, we know that the puncture corresponding to α is a second order pole of Φ . (Here, from the first statement, the residue of this second order pole has non-vanishing real part. In the following argument, the non-vanishing of the real part of the residue will play no role; note that the assumption that α has positive length precludes the case of a purely imaginary residue.) This gives a half-infinite cylinder C in the flat metric $|\Phi|$ (which is horizontal if and only Φ has purely real residue at this puncture). Let \tilde{C} be a lift of C corresponding to $\tilde{\alpha}$. Then $\tilde{f}(\tilde{C})$ is a simply connected domain of \tilde{Y} with $\tilde{\alpha} \subset \partial \tilde{f}(\tilde{C})$. Moreover, $u(\tilde{f}(\tilde{C}))$ is a geodesic ray with $u(p) \rightarrow \infty$ as $p \in \tilde{f}(\tilde{C})$ approaches $\tilde{\alpha}$. We may choose U to be a convex domain of $\tilde{f}(\tilde{C})$ with $I \subset \partial U$. \square

5.3. Two differentials. Recall that $u : \tilde{Y} \rightarrow T$ and $v : \tilde{Y} \rightarrow T$ are two $\pi_1(Y)$ -equivariant minimal graphs in $\tilde{Y} \times T$ with the same admissible boundary correspondence.

Let \tilde{X}_u be the graph of $u : \tilde{Y} \rightarrow T$ in $\tilde{Y} \times T$, and $\tilde{f}_u : \tilde{X} \rightarrow \tilde{Y}$ the equivariant projection map which is harmonic. Then T is dual to the horizontal measured

foliation of $\text{Hopf}(\tilde{f}_u)$. Let f_u be the projection map descended from \tilde{f}_u . Consider the map $v : \tilde{Y} \rightarrow T$. Define \tilde{X}_v and \tilde{f}_v similarly. Then T is also dual to the horizontal measured foliation of $\text{Hopf}(\tilde{f}_v)$. Let $X_u = \tilde{X}_u/\pi_1(Y)$ $X_v = \tilde{X}_v/\pi_1(Y)$ be the quotient surfaces. Then the horizontal measured foliations $\text{Hor}(\text{Hopf}(f_u))$ and $\text{Hor}(\text{Hopf}(f_v))$ of $\text{Hopf}(f_u)$ and $\text{Hopf}(f_v)$ respectively are topologically equivalent. Let $F_u = f_u(\text{Hor}(\text{Hopf}(f_u)))$ and $F_v = f_v(\text{Hor}(\text{Hopf}(f_v)))$ be the associated measured foliations on Y . Since f_u and f_v are both homotopic to the identity, it follows that F_u and F_v differ by an isotopy and Whitehead moves. Let $\text{Sing}(u)$ be the set of singular points of F_u and $\text{Crit}(u)$ be the union of critical leaves of F_u . Let $\text{Reg}(u)$ be the complement of $\text{Crit}(u) \cup \text{Sing}(u)$ in Y . Let $\widetilde{\text{Reg}}(u), \widetilde{\text{Crit}}(u), \widetilde{\text{Sing}}(u)$ be respectively the lifts to \tilde{Y} of $\text{Reg}(u), \text{Crit}(u), \text{Sing}(u)$. Then

- $\widetilde{\text{Reg}}(u)$ is the subset of points $p \in \tilde{Y}$ such that $u(p)$ is not a vertex of T ;
- $\widetilde{\text{Crit}}(u)$ is the subset of points $p \in \tilde{Y}$ such that $u(p)$ is a vertex of T but admit a neighbourhood whose image under u is a geodesic segment;
- $\widetilde{\text{Sing}}(u)$ is the subset of points $p \in \tilde{Y}$ such that $u(p)$ is vertex and admit a neighbourhood whose image under u is a star with $m \geq 3$ edges/prongs.

Let $\text{Sing}(v), \text{Crit}(v), \text{Reg}(v), \widetilde{\text{Sing}}(v), \widetilde{\text{Crit}}(v)$, and $\widetilde{\text{Reg}}(v)$ be similarly defined.

We now begin our analysis of how the horizontal measured foliations F_u and F_v align, and the implications for the maps u and v .

Lemma 5.9. *Let $p \in \widetilde{\text{Reg}}(u) \cap \widetilde{\text{Reg}}(v)$. Then there exists a neighbourhood $U \subset \tilde{Y}$ of p such that the convex hull of $u(U) \cup v(U)$ is a geodesic.*

Proof. Let U be a neighbourhood of p such that $u(U)$ and $v(U)$ are both geodesic segments. Let $\text{hull}(U)$ be the convex hull of $u(U) \cup v(U)$ in T . If $\text{hull}(U)$ is a geodesic segment, then we are done. Otherwise, the assumption that neither $u(p)$ nor $v(p)$ is a vertex of T implies that there exists a subdomain $U' \ni p$ such that $u(U') \cup v(U')$ avoids the (discrete set of) points of $\text{hull}(U)$ which have valence at least three. It then follows that the convex hull of $u(U') \cup v(U')$ is a geodesic segment in T . \square

Consider the minimal graphs $u : \tilde{Y} \rightarrow T$ and $v : \tilde{Y} \rightarrow T$. If there exists a folding $\xi : T \rightarrow \mathbb{R}$ then we can compare u and v by considering the difference $\xi \circ u - \xi \circ v$. Theorem 5.6 then follows directly from the argument in the proof of step 6 (uniqueness) of [NR02, Theorem 3]. But such a folding does not exist in general. Nevertheless, the observation below allows us to get past this obstruction.

Let $p \in \widetilde{\text{Reg}}(u) \cap \widetilde{\text{Reg}}(v)$. Let $U \subset \tilde{Y}$ be a neighbourhood of p such that the convex hull of $u(U) \cup v(U)$ is a geodesic, which is denoted by \mathbf{L} , (Lemma 5.9). We notice that this geodesic admits two orientations; we pick one of these. Now, let us define two functions \mathbf{u} and \mathbf{v} on U as follows. For each $q \in U$, let

$$\mathbf{u}(q) := u(q) - u(p), \quad \mathbf{v}(q) := v(q) - v(p)$$

be respectively the oriented distance from $u(p)$ to $u(q)$ and from $v(p)$ to $v(q)$; here we may measure distance along the geodesic we have constructed. Let $\mathbf{u}^*, \mathbf{v}^*, d\mathbf{u}^*$ and $d\mathbf{v}^*$ be defined as in (5.2). Notice that both $\mathbf{u} - \mathbf{v}$ and $d(\mathbf{u}^* - \mathbf{v}^*)$ are independent on the particular choice of $p \in U$. If we reverse the orientation of the geodesic \mathbf{L} , then $\mathbf{u} - \mathbf{v}$ differs by a negative sign, so does $d(\mathbf{u}^* - \mathbf{v}^*)$. This implies that the differential $(\mathbf{u} - \mathbf{v})d(\mathbf{u}^* - \mathbf{v}^*)$ is independent on the choice of the orientation of \mathbf{L} . Therefore, we obtain a well defined differential $\tilde{\Psi}$ on $\widetilde{\text{Reg}}(u) \cap \widetilde{\text{Reg}}(v) =$

$\tilde{Y} - u^{-1}(u(\widetilde{\text{Sing}}(u))) \cup v^{-1}(v(\widetilde{\text{Sing}}(v)))$, the complementary region of the union of the preimages of the vertices of T by u and v . Locally, the differential $\tilde{\Psi}$ can be represented as $(\mathbf{u} - \mathbf{v})d(\mathbf{u}^* - \mathbf{v}^*)$.

Notice that $\tilde{\Psi}$ is not closed in general. In fact, we have

$$(5.4) \quad \begin{aligned} d\Psi_M &= \left[(\mathbf{u}_1 - \mathbf{v}_1) \left(\frac{\mathbf{u}_1}{\tau_{\mathbf{u}}} - \frac{\mathbf{v}_1}{\tau_{\mathbf{v}}} \right) + (\mathbf{u}_2 - \mathbf{v}_2) \left(\frac{\mathbf{u}_2}{\tau_{\mathbf{u}}} - \frac{\mathbf{v}_2}{\tau_{\mathbf{v}}} \right) \right] dx_1 dx_2 \\ &= \left(\frac{\tau_{\mathbf{u}} + \tau_{\mathbf{v}}}{2} \right) \left[\left(\frac{\mathbf{u}_1}{\tau_{\mathbf{u}}} - \frac{\mathbf{v}_1}{\tau_{\mathbf{v}}} \right)^2 + \left(\frac{\mathbf{u}_2}{\tau_{\mathbf{u}}} - \frac{\mathbf{v}_2}{\tau_{\mathbf{v}}} \right)^2 + \frac{1}{F} \left(\frac{1}{\tau_{\mathbf{u}}} - \frac{1}{\tau_{\mathbf{v}}} \right)^2 \right] dx_1 dx_2 \end{aligned}$$

where $\mathbf{u}_i = \frac{\partial \mathbf{u}}{\partial x_i}$, $\mathbf{v}_i = \frac{\partial \mathbf{v}}{\partial x_i}$; the second identity is taken from [NR02, Page 280].

We first extend the domain of definition of $\tilde{\Psi}$ to the lift of a cylindrical neighbourhood of ∂Y .

Lemma 5.10. *Let C be a closed geodesic loop boundary component or crown end of Y . There exists a cylindrical neighbourhood U of C such that the following holds.*

- (i) *The differential $\tilde{\Psi}$ is well defined on the lift of U to the universal cover and descends to a differential Ψ on U .*
- (ii) *For any simple arc η of U which cuts U into a simply connected domain, the restriction of u and v to $U \setminus \eta$ are minimal graphs in $\mathbb{H}^2 \times \mathbb{R}$.*

Proof. Notice that Y may have geodesic boundary components or crown ends. To prove the Lemma, it suffices to find a cylindrical neighbourhood for each boundary component which satisfies the two mentioned properties.

Let α be an arbitrary geodesic boundary loop of Y . Let $I \subset \tilde{\alpha}$ be a segment of a lift $\tilde{\alpha}$ of α to \tilde{Y} with $\ell(I) > \ell(\alpha)$. Then by item (iv) in Lemma 5.8, we see that there exists a convex neighbourhood $U_I \subset \tilde{Y}$ of I with $I \subset \partial U_I$ and a geodesic ray $r : [0, +\infty) \rightarrow T$ such that

- both $u(U_I)$ and $v(U_I)$ are contained in the image of r ,
- $u(p) \rightarrow \infty$ and $v(p) \rightarrow \infty$, as $p \rightarrow I$.

By the definition of $\tilde{\Psi}$, we see that $\tilde{\Psi}$ is well defined on any point in U_I . Any cylindrical subdomain of the quotient of U_I , which admits α as a boundary component, satisfies the conditions (i,ii).

Now we move on to consider the case of crown ends. Let C be a crown end with ideal geodesic boundary arcs $\gamma_1, \dots, \gamma_k$ labelled cyclically. Let \tilde{X}_u and \tilde{X}_v be respectively the minimal graphs of $u : \tilde{Y} \rightarrow T$ and $v : \tilde{Y} \rightarrow T$. Let $f_u : \tilde{X}_u \rightarrow \tilde{Y}$ and $f_v : \tilde{X}_v \rightarrow \tilde{Y}$ be the harmonic projection maps described at the outset of this subsection. Consider the horizontal foliations $\text{Hor}(\text{Hopf}(f_u))$ and $\text{Hor}(\text{Hopf}(f_v))$ of the Hopf differentials of f_u and f_v , respectively. Recall that $\text{Crit}(u) \subset Y$ is the image under f_u of the critical leaves of $\text{Hor}(\text{Hopf}(f_u))$, which is also the preimage of the vertices of T under u . Of course, $\text{Crit}(v)$ is defined similarly. For the cusp P_i bounded by γ_i and γ_{i+1} , there exists a cusp neighbourhood W_i such that $W_i \cap (\text{Crit}(u) \cup \text{Crit}(v))$ consists of half-infinite simple arcs approaching the cusp P_i . Observe that these arcs are contained in the interior of W_i . We may take smaller cusp neighbourhoods so that for any $i \neq j$, we have that W_i and W_j are disjoint. We may then take U to be a cylindrical neighbourhood of the crown C such that

- (a) $\partial U \cap \text{int}(W_i)$ is connected and nonempty,
- (b) $(\text{Crit}(u) \cap U) \subset \cup_i W_i$ and $(\text{Crit}(v) \cap U) \subset \cup_i W_i$.

In particular, this proves item (ii). It remains to prove (i). Let \widetilde{W}_i be a connected component of the lift of W_i to the universal cover \widetilde{Y} . Then for each W_i , we have that $u(W_i) = v(W_i)$, which is isometric to the real line \mathbb{R} (by applying Lemma 5.8(iii) twice). By the definition of $\widetilde{\Psi}$, we see that it is well defined on \widetilde{W}_i , and so $\widetilde{\Psi}$ descends to a differential Ψ on W_i .

Consider the complementary components of $\text{Crit}(u) \cup \text{Crit}(v)$ in U . For each γ_i , there exists exactly one of these components, say R_i , which contains γ_i in the boundary. Let \widetilde{R}_i be a connected component of the lift of R_i to the universal cover \widetilde{Y} . Let $\widetilde{\gamma}_i$ be a lift of γ_i which is contained in $\partial\widetilde{R}_i$. Let e_i be the half-infinite edge of T corresponding to $\widetilde{\gamma}_i$. Then both $u(\widetilde{R}_i)$ and $v(\widetilde{R}_i)$ are contained in e_i . By the definition of $\widetilde{\Psi}$, we see that it is well defined on \widetilde{R}_i , which descends to a differential Ψ on R_i . Since $U = \cup_i(R_i \cup W_i)$, it follows that Ψ is well defined on U , and that $\widetilde{\Psi}$ is well defined on \widetilde{U} . This finishes the proof. \square

If a crown end C consists of an even number of ideal geodesic arcs $\gamma_1, \dots, \gamma_{2n}$, then we can say a bit more about u and v as follows. Let $\text{Hor}(\text{Hopf}(f_u))$ and $\text{Hor}(\text{Hopf}(f_v))$ be as defined in the above proof (see also the beginning of Section 5.3). Let U and R_i also be as defined in the above proof. Then the restriction to U of both $F_u := f_u(\text{Hor}(\text{Hopf}(f_u)))$ and $F_v := f_v(\text{Hor}(\text{Hopf}(f_v)))$ are orientable. We choose one orientation which induces the same orientation on $R_i \subset U$. Each choice of orientation yields two 1-forms ω_u and ω_v on U , defined by $F_u|_U$ and $F_v|_U$, respectively. For each cusp P_i of C , recalling section 2.8, consider the half-infinite strip $\text{Strip}_i(u)$ of F_u (resp. $\text{Strip}_i(v)$ of F_v) approaching P_i . The heights of $\text{Strip}_i(u)$ and $\text{Strip}_i(v)$ are determined by T , hence are equal. Let $h_i \geq 0$ be the height. Then for any oriented simple closed curve α in U , we have

$$\int_{\alpha} \omega_u = \int_{\alpha} \omega_v = a \sum_{i=1}^{2n} (-1)^i h_i,$$

where $a \in \{1, -1\}$ depends on the orientation we select for the foliations on U as well as the orientation of α . In other words, both u and v induce functions from U to $\mathbb{R}/(h\mathbb{Z})$, where $h = \sum_{i=1}^{2n} (-1)^i h_i$. In particular, this implies that both $u - v$ and $du^* - dv^*$ are well defined on U . We summarize the discussion below.

Lemma 5.11. *Suppose that C is a crown end of Y which consists of an even number of ideal geodesic arcs. Let U be the cylindrical neighbourhood of C obtained from Lemma 5.10. Let F_u and F_v be defined as above. Then there are compatible choices of orientations of the restriction $F_u|_U$ and $F_v|_U$ so that both $u - v$ and $du^* - dv^*$ are well defined on U .*

For crown ends with k ideal geodesic arcs, where k is not necessarily an even number, we have the following more general result. We continue with the notations as above. Recall that h_i is the height of the strip of $F_u|_U$ approaching the cusp P_i , which is also the height of the strip of $F_v|_U$ approaching the cusp P_i . Let \mathbf{K} be a k -gon in the Euclidean plane with clockwise-labelled vertices Q_1, Q_2, \dots, Q_k , such that the edge $\overline{Q_i Q_{i+1}}$ has length h_i . At each vertex Q_i , we attach an half-infinite edge E_i to \mathbf{K} at Q_i so that E_i sits in the noncompact component of $\mathbb{R}^2 \setminus \mathbf{K}$. Let \mathbf{G} be the resulting (non-self-intersecting) graph. We now consider the measured foliations F_u . Let $\text{HP}_i(u)$ be the half plane corresponding to γ_i . Let $\text{Strip}_i(u)$ be the half-infinite strip of F_u approaching P_i . Consider the leaf space of $F_u|_{\cup_i(\text{HP}_i(u) \cup \text{Strip}_i(u))}$,

i.e. the restriction of F_u to $\cup_i(\overline{\text{HP}}_i \cup \text{Strip}_i(u))$. The orientation of Y induces a cyclic order of edges incident at a common vertex. It follows that this space is isometric to \mathbf{G} , and that the cyclic order of edges incident at vertices are the same (up to possibly relabeling the vertices of \mathbf{K} counterclockwise). Moreover, the map $u : \tilde{Y} \rightarrow T$ descends to a map $u_U : U \rightarrow \mathbf{G}$. Considering the foliation F_v similarly, we get a similarly defined map $v_U : U \rightarrow \mathbf{G}$. In summary, we have the following.

Lemma 5.12. *Let C be a crown end with k ideal geodesic arcs $\gamma_1, \dots, \gamma_k$. Let \mathbf{G} be the graph defined as above. Then u and v descend to maps $u_U : U \rightarrow \mathbf{G}$ and $v_U : U \rightarrow \mathbf{G}$, respectively.*

5.4. Boundedness of $\text{dist}(u(\cdot), v(\cdot))$. We next apply the structure theory of the previous subsection to find that, for the Jenkins-Serrin maps we are studying, any pair of them have their maximum distance realized at (the lift of) an interior point of Y . Here we see the importance of ensuring that Ψ is well-defined in Lemma 5.10, as we are allowed to then use Stokes theorem, and especially the sign of the expression for $d\Psi$ in (5.4) in our computations. The subsection is devoted to a proof of the following Lemma.

Lemma 5.13. *Let $u : \tilde{Y} \rightarrow T$ and $v : \tilde{Y} \rightarrow T$ be two minimal graphs sharing the same admissible boundary correspondence. Then there exists a point $p \in \tilde{Y}$ which realizes the supremum of the distance function on \tilde{Y} , i.e.*

$$\text{dist}(u(p), v(p)) = \sup_{q \in \tilde{Y}} \text{dist}(u(q), v(q)).$$

Proof. Let U be the union of the cylinder neighbourhoods of ∂Y obtained in Lemma 5.10. Set $K := Y \setminus U$. Then K is a compact subset. Notice that the distance function $\text{dist}(u(\cdot), v(\cdot))$ descends to Y . Suppose to the contrary that the conclusion (of the lemma) does not hold. Then we have

$$(5.5) \quad \max_{p \in K} \text{dist}(u(p), v(p)) < \sup_{p \in Y} \text{dist}(u(p), v(p)).$$

Let M be an arbitrary constant such that

$$\max_{p \in K} \text{dist}(u(p), v(p)) < M < \sup_{p \in Y} \text{dist}(u(p), v(p)).$$

In particular, $M > 0$. Consider a component Ω of

$$\{p \in Y : \text{dist}(u(p), v(p)) > M\}.$$

It is clear that $\Omega \subset U$, as the points in the complement of U in Y have images at distance less than M . In the following, we shall show that $\text{dist}(u(p), v(p)) \equiv M$ on Ω . The arbitrariness of M then implies that

$$\max_{p \in K} \text{dist}(u(p), v(p)) = \sup_{p \in Y} \text{dist}(u(p), v(p)),$$

which contradicts (5.5).

Consider the topology of Ω . There are three possibilities:

- (i) Ω is simply connected and $\overline{\Omega}$ is contained in the interior of U ,
- (ii) Ω is simply connected but $\overline{\Omega}$ is not contained in the interior of U ,
- (iii) Ω is multiconnected.

Case (i): Ω is simply connected and $\bar{\Omega}$ is contained in the interior of U . Then we may take u and v as functions valued in \mathbb{R} with signs chosen so that $u - v = M$ on $\partial\Omega$. Therefore, $\int_{\partial\Omega} \Psi = M \int_{\partial\Omega} (du^* - dv^*) = 0$ since both du^* and dv^* are locally well-defined closed differentials. Combining (5.4) and the Stokes theorem, we see that $\nabla u = \nabla v$ on Ω . Hence $u - v$ is a constant function over Ω , which is exactly M since $u - v = M$ on $\partial\Omega$.

Case (ii): Ω is simply connected but $\bar{\Omega}$ is not contained in the interior of U . We approximate Ω by a domain $\Omega^{\delta,\epsilon}$ which is slightly separated from ∂Y , and we then prove that the error used in the approximation is negligible. Notice that any consecutive pair of ideal geodesics γ_i, γ_{i+1} in the crown end \mathcal{C} are asymptotic, and hence determines a point at infinity which we denoted by P_i . Let δ be a small positive constant to be determined. Let F_i^δ be a neighbourhood of P_i which is bounded by segments in γ_i, γ_{i+1} , and a horocycle arc centered at P_i with length δ . Let $\gamma_i^\delta := \gamma_i \cap \partial(\Omega - (\cup_i F_i^\delta))$. Let ϵ be another small constant. Let $\Omega^{\delta,\epsilon} \subset \Omega - (\cup_i F_i^\delta)$ be the subsurface in Ω consisting of points whose distance to the boundary of Y is at least ϵ . The boundary $\partial\Omega^{\delta,\epsilon}$ consists of arcs $\gamma_i^{\delta,\epsilon}$ corresponding to γ_i , arcs $h_i^{\delta,\epsilon}$ contained in the horocycle boundary of F_i^δ , and a subarc $\omega^{\delta,\epsilon}$ of $\partial\Omega \cap U$. In particular,

$$(5.6) \quad |h_i^{\delta,\epsilon}| < \delta$$

for each $i = 1, 2, \dots, k$. Since both u and v are well-defined functions over Ω with values in \mathbb{R} ,

$$(5.7) \quad \int_{\partial\Omega \cap U} \Psi = M \int_{\partial\Omega \cap U} (du^* - dv^*), \quad \int_{\omega^{\delta,\epsilon}} \Psi = M \int_{\omega^{\delta,\epsilon}} (du^* - dv^*).$$

Combining with the fact that both du^* and dv^* are closed differentials on Ω , we see that

$$(5.8) \quad \begin{aligned} & \int_{\omega^{\delta,\epsilon}} (du^* - dv^*) + \sum_{i=1}^k \int_{\gamma_i^{\delta,\epsilon}} (du^* - dv^*) + \sum_{i=1}^k \int_{h_i^{\delta,\epsilon}} (du^* - dv^*) \\ &= \int_{\Omega^{\delta,\epsilon}} d(du^* - dv^*) \\ &= 0. \end{aligned}$$

First, we consider the integration over $h_i^{\delta,\epsilon}$,

$$(5.9) \quad \begin{aligned} \left| \int_{h_i^{\delta,\epsilon}} (du^* - dv^*) \right| &\leq \int_{h_i^{\delta,\epsilon}} (|d\mathbf{u}^*| + |d\mathbf{v}^*|) \\ &\leq \int_{h_i^{\delta,\epsilon}} 2ds \quad (\text{by (5.3)}) \\ &= 2|h_i^{\delta,\epsilon}| \\ &< 2\delta, \quad (\text{by (5.6)}) \end{aligned}$$

We move on to consider $\int_{\gamma_i^{\delta,\epsilon}} (du^* - dv^*)$. Notice that u and v can be viewed as minimal graphs in $\mathbb{H}^2 \times \mathbb{R}$ with $u|_{\gamma_i} = +\infty$ and $v|_{\gamma_i} = +\infty$. Combining this with Theorem 5.1, we see that for any $\eta > 0$, we may choose ϵ small enough so that

$$(5.10) \quad \left| \left\langle \frac{\nabla u}{\tau_u}, \nu \right\rangle - 1 \right| < \eta \quad \text{and} \quad \left| \left\langle \frac{\nabla v}{\tau_v}, \nu \right\rangle - 1 \right| < \eta$$

on the lift $\gamma_i^{\delta, \epsilon}$, where ν is the unit normal field of $\gamma_i^{\delta, \epsilon}$ pointing toward γ_i . Hence

$$\begin{aligned}
\left| \int_{\gamma_i^{\delta, \epsilon}} (du^* - dv^*) \right| &= \left| \int_{\gamma_i^{\delta, \epsilon}} \left(\left\langle \frac{\nabla u}{\tau_u}, \nu \right\rangle - \left\langle \frac{\nabla v}{\tau_v}, \nu \right\rangle \right) ds \right| \\
&= \left| \int_{\gamma_i^{\delta, \epsilon}} \left(\left(\left\langle \frac{\nabla u}{\tau_u}, \nu \right\rangle - 1 \right) - \left(\left\langle \frac{\nabla v}{\tau_v}, \nu \right\rangle - 1 \right) \right) ds \right| \\
&\leq \int_{\gamma_i^{\delta, \epsilon}} \left(\left| \left\langle \frac{\nabla u}{\tau_u}, \nu \right\rangle - 1 \right| + \left| \left\langle \frac{\nabla v}{\tau_v}, \nu \right\rangle - 1 \right| \right) ds \\
&\leq \int_{\gamma_i^{\delta, \epsilon}} 2\eta \quad \text{by (5.10)} \\
(5.11) \quad &\leq 2\eta(|\gamma_i^\delta| + 2\epsilon).
\end{aligned}$$

Combining (5.7), (5.8), (5.9) (5.11), we get

$$\left| \int_{\omega^{\delta, \epsilon}} \Psi \right| \leq \sum_{i=1}^k 2M\delta + \sum_{i=1}^k 2M\eta(|\gamma_i^\delta| + 2\epsilon).$$

Letting $\epsilon \rightarrow 0$, we have

$$\lim_{\epsilon \rightarrow 0} \left| \int_{\omega^{\delta, \epsilon}} \Psi \right| \leq \sum_{i=1}^k 2M\eta|\gamma_i^\delta| + \sum_{i=1}^k 2M\delta.$$

The arbitrariness of η then implies that

$$\lim_{\epsilon \rightarrow 0} \left| \int_{\omega^{\delta, \epsilon}} \Psi \right| \leq 2kM\delta.$$

Therefore,

$$\left| \int_{\partial\Omega \cap U} \Psi \right| = \lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \left| \int_{\omega^{\delta, \epsilon}} \Psi \right| = 0.$$

It remains to show that the distance function is constant. Let M' be a constant with $M < M' < \sup_{p \in \Omega} \text{dist}(u(p), v(p))$. Consider a component Ω' of $\{p \in \Omega : \text{dist}(u(p), v(p)) > M'\}$. If Ω' admits a boundary component which encloses a disc in Ω , then by subcase (i), we see that the distance function is a constant on Ω' . Combined with the maximum principle, this implies that the distance function is a constant on Ω . If Ω' admits a boundary component $\partial\Omega' \cap U$ which intersects $\cup_i \gamma_i$, then by the discussion in this subcase we know that $\int_{\partial\Omega'} \Psi = 0$. Consider the region A in Ω bounded by $\partial\Omega \cap U$ and $\partial\Omega' \cap U$. Approximating $\partial A \cap (\cup_i \gamma_i)$ similarly as above and applying the Stokes theorem to the approximating regions using (5.4), we see that the distance function on A is a constant. Applying the strong maximum principle, we find that the distance function is constant on Ω .

Case (iii): Ω is multiconnected. If some boundary component of Ω encloses a simply connected domain, the conclusion follows from the discussion in subcase (i). Now we assume that any simple closed boundary component of Ω is homotopically nontrivial. Let ζ be such a boundary component. Since U is a cylinder, it follows that ζ is homotopic to the core curve of U . By Lemma 5.12, we see that u and v descend to two maps, say $u_U : U \rightarrow \mathbf{G}$ and $v_U : U \rightarrow \mathbf{G}$, from U to the graph \mathbf{G} .

Notice that the cyclic order of the vertices $\{Q_i\}$ of \mathbf{G} induces a cyclic order of all edges of \mathbf{G} as follows:

$$(5.12) \quad E_1, \overline{Q_1Q_2}, E_2, \overline{Q_2Q_3}, \dots, E_k, \overline{Q_kQ_1}, E_1.$$

If some edge $\overline{Q_iQ_{i+1}}$ has length zero, then we remove it from the above list, meaning that E_i and E_{i+1} are now consecutive. Consider the restrictions $u_U : \zeta \rightarrow \mathbf{G}$ and $v_U : \zeta \rightarrow \mathbf{G}$.

For any point p in the cusp neighbourhood W_i (defined in the proof of Lemma 5.11), we see that both $u(p)$ and $v(p)$ are contained in the union $E_i \cup \overline{Q_iQ_{i+1}} \cup E_{i+1}$: this union is isometric to \mathbb{R} . Therefore, $d_{\mathbf{G}}(u_U(p), v_U(p)) = \text{dist}(u(p), v(p))$ for any $p \in W_i$. Recall from last paragraph of the proof of Lemma 5.10 that there is a component R_i of $U \setminus (\text{Crit}(u) \cup \text{Crit}(v))$ which contains γ_i in the boundary. In particular, for any point $p \in R_i$, we see that both $u(p)$ and $v(p)$ are contained in the half-infinite edge E_i . This implies that $d_{\mathbf{G}}(u_U(p), v_U(p)) = \text{dist}(u(p), v(p))$ for any $p \in R_i$. Hence $d_{\mathbf{G}}(u_U(p), v_U(p)) = \text{dist}(u(p), v(p))$ for any $p \in U$ because $U = \cup_i (R_i \cup W_i)$. In particular, this equation holds on ζ . For each i , let p_i be a point in $R_i \cap \zeta$. Combining the discussion above and the assumption that $\text{dist}(u(\cdot), v(\cdot))$ is identically $M > 0$ on ζ and the fact that $d_{\mathbf{G}}(u_U(\cdot), v_U(\cdot)) = \text{dist}(u(\cdot), v(\cdot))$ on ζ , we see that $d_{\mathbf{G}}(u_U(p_i), Q_i) - d_{\mathbf{G}}(v_U(p_i), Q_i)$ is well-defined and has signs which are alternatively $+$ and $-$ with i .

In particular, this implies that \mathbf{G} has even number of edges. Correspondingly, the crown end C in consideration also has an even number of ideal geodesic arcs.

It then follows from Lemma 5.11 that there are compatible choices of orientations of the restrictions $F_u|_U$ and $F_v|_U$ so that both $u - v$ and $du^* - dv^*$ are well-defined on U . We choose an orientation of $F_u|_U$ so that $u - v = M$ on ζ . Applying a similar argument as in subcase (ii) we see that $\int_{\zeta} (du^* - dv^*) = 0$. Hence $\int_{\zeta} \Psi = M \int_{\zeta} (du^* - dv^*) = 0$.

To prove that the distance function is a constant over Ω , let $M'' > M$ be a constant which is close enough to M such that $\Omega'' := \{p \in \Omega : \text{dist}(u(p), v(p)) > M''\}$ admits a boundary component say $\partial\Omega'' \cap \Omega$ which is homotopic to $\partial\Omega \cap U$. The discussion in the previous paragraphs yields that $\int_{\partial\Omega'' \cap \Omega} \Psi = 0$. Applying Stokes theorem to the cylinder region bounded by $\partial\Omega \cap U$ and $\partial\Omega'' \cap \Omega$ and using (5.4), we see that the distance function is a constant on this cylinder region. The strong maximum principle then implies that the distance function is a constant over the whole Ω . \square

5.5. Generalized maximum principle. We continue with the notations $X_u, X_v, f_u : X_u \rightarrow Y, f_v : X_v \rightarrow Y, \text{Sing}(u), \text{Crit}(u), \text{Reg}(u), \widetilde{\text{Sing}}(u), \widetilde{\text{Crit}}(u), \widetilde{\text{Reg}}(u), \text{Sing}(v), \text{Crit}(v), \text{Reg}(v), \widetilde{\text{Sing}}(v), \widetilde{\text{Crit}}(v),$ and $\widetilde{\text{Reg}}(v)$ introduced in the beginning of Section 5.3.

Recall that

- $\widetilde{\text{Reg}}(u)$ is the subset of points $p \in \widetilde{Y}$ such that $u(p)$ is not a vertex of T ;
- $\widetilde{\text{Crit}}(u)$ is the subset of points $p \in \widetilde{Y}$ such that $u(p)$ is a vertex of T but admit a neighbourhood whose image under u is a geodesic segment;
- $\widetilde{\text{Sing}}(u)$ is the subset of points $p \in \widetilde{Y}$ such that $u(p)$ is vertex and admit a neighbourhood whose image under u is a star with $m \geq 3$ edges/prongs.

Lemma 5.14. *Let $p \in Y$ be a point.*

- (a) If $p \in Y \setminus \text{Sing}(u)$, then there exists a neighbourhood Ω of p such that the component through p of the projection of $u^{-1}(u(p))$ into Ω cuts Ω into two sectors, each of which bounds an angle of π at p .
- (b) If $p \in \text{Sing}(u)$ is a singular point such that $u(p)$ is a vertex of valence $m \geq 3$. Then there is a neighbourhood $\Omega \subset Y$ of p such that the component through p of the projection of $u^{-1}(u(p))$ into Ω cuts Ω into m sectors, each of which bounds an angle of $2\pi/m$ at p .

Similar conclusion also holds for $\text{Sing}(v)$.

Proof. Let $f_u : X_u \rightarrow Y$ be the harmonic map from the (closed) minimal graph to Y . Consider the horizontal foliation of the Hopf differential $\text{Hopf}(f_u)$ near $f_u^{-1}(p)$.

For statement (a), there exists a neighbourhood $\Omega' \subset X_u$ of $f_u^{-1}(p)$ such that the component of $f_u^{-1} \circ u^{-1}(u(p)) \cap \Omega'$ containing $f_u^{-1}(p)$ is a smooth curve crossing Ω' . Since f_u is a diffeomorphism, we see that $\Omega := f_u(\Omega')$ is cut out by the component of $u^{-1}(u(p)) \cap \Omega$ containing p into two sectors, each of which bounds an angle of π at p .

For statement (b), there exists a neighbourhood Ω' of $f_u^{-1}(p)$ cut out by the component of $f_u^{-1} \circ u^{-1}(u(p)) \cap \Omega'$ containing $f_u^{-1}(p)$ into m sectors, each of which bounds an angle of $2\pi/m$ at $f_u^{-1}(p)$ ([Str84, Section 6]). Since f_u is conformal at $f_u^{-1}(p)$ (because the Beltrami differential of f_u is zero at $f_u^{-1}(p)$), it follows that $\Omega := f_u(\Omega')$ is cut out by the component of $u^{-1}(u(p)) \cap \Omega$ containing p into m sectors, each of which bounds an angle of $2\pi/m$ at p . \square

Lemma 5.15 (generalized maximum principle). *Let $u : \tilde{Y} \rightarrow T$ and $v : \tilde{Y} \rightarrow T$ be two minimal graphs. If there exists a point $p \in \tilde{Y}$ which realizes the supremum of the distance function on \tilde{Y} , i.e.*

$$\text{dist}(u(p), v(p)) = \sup_{q \in \tilde{Y}} \text{dist}(u(q), v(q))$$

then $u = v$.

Remark 5.16. We will apply this lemma in the case when the two maps u and v share a boundary correspondence, but, as we do not need that hypothesis in the proof, we state the result in a more general form.

Proof. Let q be an arbitrary point on Y . Consider the images of neighbourhood of q under u and v . There are two possibilities.

- (a) for any neighbourhood Ω of q , the convex hull of $u(\Omega) \cup v(\Omega)$ is not a geodesic.
- (b) q admits a neighbourhood Ω such that the convex hull of $u(\Omega) \cup v(\Omega)$ is a geodesic segment.

Notice that any singular point $q \in \text{Sing}(u) \cup \text{Sing}(v)$ satisfies the condition (a). Moreover, the subset of points satisfying the condition (b) is an proper open subset of Y , while the subset of points satisfying the condition (a) is a nonempty closed subset of Y .

Case (a): for any neighbourhood Ω of p , the convex hull of $u(\Omega) \cup v(\Omega)$ is not a geodesic. Let Ω be a small neighbourhood of p such that both $u(\Omega)$ and $v(\Omega)$ are stable, meaning that smaller neighbourhoods share the same image as Ω , up to isotopies in T fixing $u(p)$ and $v(p)$ respectively. Suppose that $u(\Omega)$ and $v(\Omega)$ are respectively stars with $m_u \geq 2$ and $m_v \geq 2$ edges, where we take respectively $u(p)$ and $v(p)$ as the vertices of the stars. Let $\{\Omega_i(u) : 1 \leq i \leq m_u\}$ be the set of

complementary sectors in Ω of the connected locus of $u^{-1}(u(p)) \cap \Omega$ containing p . Let $\{\Omega_j(v) : 1 \leq j \leq m_v\}$ be defined similarly. There is an one-to-one correspondence between $\{\Omega_i(u)\}$ (resp. $\{\Omega_j(v)\}$) and the edges of $u(\Omega)$ (resp. $v(\Omega)$).

Suppose $u(p) \neq v(p)$ and $m_u = m_v = 2$. There are two subcases depending on whether the convex hull of $u(\Omega) \cup v(\Omega)$ is a tripod or not. If it is not a tripod, then any point q in the nonempty set $(\Omega_1(u) \cup \Omega_2(u)) \cap (\Omega_1(v) \cup \Omega_2(v))$ satisfies $d(u(q), v(q)) > d(u(p), v(p))$, contradicting the assumption that p attains the supremum of $\text{dist}(u(\cdot), v(\cdot))$ over Y . If the convex hull of $u(\Omega) \cup v(\Omega)$ is a tripod, then exactly one of $u(p)$ and $v(p)$ is a vertex of valence three of this tripod (since $v(p) \neq u(p)$). Without loss of generality, we may assume that $u(p)$ is the vertex. Then there exists one of $\{\Omega_1(v), \Omega_2(v)\}$, say $\Omega_1(v)$, such that any point q in $\Omega_1(v) \cap (\Omega_1(u) \cup \Omega_2(u))$, which is nonempty by Lemma 5.14, satisfies $d(u(q), v(q)) > d(u(p), v(p))$, contradicting the assumption that p attains the supremum of $\text{dist}(u(\cdot), v(\cdot))$ over Y .

Suppose $u(p) \neq v(p)$ and either m_u or m_v is bigger than 2. Without loss of generality, we may assume that $m_u > 2$. Consider the stars $u(\Omega)$ and $v(\Omega)$. There exists at most one (open) edge of $u(\Omega)$, say $u(\Omega_1)$, such that any point of this edge to $v(\Omega)$ is strictly less than $d(u(p), v(p))$. Similarly, there exists at most one (open) edge of $v(\Omega)$, say $v(\Omega_1)$, such that the distance of any point of this edge to $u(\Omega)$ is strictly less than $d(u(p), v(p))$. In particular, if

$$(5.13) \quad (\cup_{2 \leq i \leq m_u} \Omega_i(u)) \cap (\cup_{2 \leq j \leq m_v} \Omega_j(v)) \neq \emptyset,$$

then any point in this set satisfies $d(u(q), v(q)) > d(u(p), v(p))$, contradicting the assumption that p attains the supremum of $\text{dist}(u(\cdot), v(\cdot))$ over Y . Next, we shall prove (5.13). By Lemma 5.14, the set $\Omega_1(u)$ bounds an angle of $2\pi/m_u \leq 2\pi/3$ while $\Omega_1(v)$ bounds an angle of $2\pi/m_v \leq \pi$. It follows that the closures (in Ω) $\overline{\cup_{2 \leq i \leq m_u} \Omega_i(u)}$ and $\overline{\cup_{2 \leq i \leq m_v} \Omega_i(v)}$ are sectors based at p of angles at least $\frac{4\pi}{3}$ and π , respectively. Consequently, the intersection set

$$\overline{\cup_{2 \leq i \leq m_u} \Omega_i(u)} \cap \overline{\cup_{2 \leq j \leq m_v} \Omega_j(v)}$$

has positive measure. On the other hand, the complement of $\overline{\cup_{2 \leq i \leq m_u} \Omega_i(u)}$ in $\overline{\cup_{2 \leq i \leq m_u} \Omega_i(u)}$ is contained in $\text{Crit}(u) \cap \Omega$, and hence has measure zero. Similarly, the complement of $\overline{\cup_{2 \leq i \leq m_v} \Omega_i(v)}$ in $\overline{\cup_{2 \leq i \leq m_v} \Omega_i(v)}$ also has measure zero. Therefore,

$$\overline{\cup_{2 \leq i \leq m_u} \Omega_i(u)} \cap \overline{\cup_{2 \leq j \leq m_v} \Omega_j(v)} \neq \emptyset,$$

completing the proof of (5.13).

The discussion above excludes the possibility that $u(p) \neq v(p)$. Therefore, $u(p) = v(p)$. In particular, $d(u(p), v(p)) = 0$. The assumption that p attains the supremum of $\text{dist}(u(\cdot), v(\cdot))$ over Y implies that $\text{dist}(u(\cdot), v(\cdot)) \equiv 0$ on Y . Hence $u = v$.

Case (b): p admits a neighbourhood Ω such that the convex hull of $u(\Omega) \cup v(\Omega)$ is a geodesic segment. We notice that this geodesic admits two orientations; we pick one of these. This allows us to define two functions \mathbf{u} and \mathbf{v} on Ω as follows:

$$\mathbf{u}(q) := u(q) - u(p), \quad \mathbf{v}(q) := v(q) - v(p), \quad \forall q \in \Omega.$$

In other words, both u and v , restricted to Ω , may be viewed as minimal graphs in $\mathbb{H}^2 \times \mathbb{R}$ over a bounded simply connected domain. The (strong) maximum principle then implies that the distance function $\text{dist}(u(\cdot), v(\cdot))$ is a constant on Ω . Then, again by the strong maximum principle, we see that the distance function $\text{dist}(u(\cdot), v(\cdot))$ is a constant over the whole component, say Θ , of the subset of points

satisfying condition (b) which contains p . By continuity, the distance function is also a constant over the closure $\overline{\Theta}$ of Θ . On the other hand, any point in the (interior) boundary $\partial\Theta \cap Y$, which is nonempty, satisfies the condition (a). It then follows from **Case (a)** that $u = v$. This completes the proof. \square

5.6. Finishing the proof of Theorem 5.6.

Proof of Theorem 5.6. The theorem now follows from Lemma 5.13 and Lemma 5.15. \square

6. SUBCONVERGENCE OF HARMONIC MAPS RAYS

The goal of this section is to begin the proof of Theorem 1.1. Roughly that theorem asserts the convergence of some harmonic map rays to a Thurston geodesic. In this section, we prove the *subconvergence* of that family (see Theorem 1.3). We will be left to prove the (full) convergence, whose proof will occupy sections 7 and 8. Looking ahead, that proof of convergence will involve two steps: in section 8, we will show that the harmonic maps rays defined from our degenerating family X_t converge to a harmonic maps ray from a punctured surface X_∞ ; This will be combined with the work of section 7 that will show that such a harmonic map ray from X_∞ , whose range is naturally a family of crowned surfaces, extends to a Thurston geodesic through closed surfaces.

6.1. Uniformly Lipschitz property of harmonic map rays. we begin by showing that the parametrization of a harmonic maps ray captures a Lipschitz bound.

Lemma 6.1 (Uniformly Lipschitz). *Let $X \in \mathcal{T}(S)$ and $\Phi \in Q(X)$. Let*

$$\mathbf{HR}_{X,\Phi}(\cdot) : [0, \infty) \longrightarrow \mathcal{T}(S)$$

be the harmonic maps ray. Let $f_t : X \longrightarrow \mathbf{HR}_{X,\Phi}(t)$ be the harmonic map at a fixed time t . Then the map $f_s \circ f_t^{-1} : \mathbf{HR}_{X,\Phi}(t) \longrightarrow \mathbf{HR}_{X,\Phi}(s)$ is $\sqrt{s/t}$ -Lipschitz for all $s \geq t > 0$.

Proof. By [Wol89, Proposition 4.3], it follows that for all $p \in M$ with $\Phi(p) \neq 0$, $|\nu(p, t)|$ is an increasing function of $t \in (0, \infty)$, where $\nu(p, t)$ is the Beltrami differential of f_t .

Let $\mathcal{G}(p, t) = \log(1/|\nu(p, t)|)$. Then \mathcal{G} is a decreasing function of $t \in (0, \infty)$ at p with $\Phi(p) \neq 0$. Let $z = x + iy$ be the canonical coordinate charts of Φ near p . Then by (3.2), the pullback metric of $Y_t := \mathbf{HR}_{X,\Phi}(t)$ by f_t on X is:

$$(6.1) \quad f_t^* Y_t = 2t(\cosh \mathcal{G}(z, t) + 1)dx^2 + 2t(\cosh \mathcal{G}(z, t) - 1)dy^2.$$

Then

$$\left. \frac{f_s^*(Y_s)}{f_t^*(Y_t)} \right|_p \leq \frac{s}{t}, \quad \forall s \geq t > 0.$$

This implies that $f_s \circ (f_t)^{-1}$ is $\sqrt{(s/t)}$ -Lipschitz outside the zero locus of Φ . By continuity, $f_s \circ (f_t)^{-1}$ is $\sqrt{(s/t)}$ -Lipschitz on the whole surface Y . \square

For $X, Y \in \mathcal{T}(S)$, let $\mathbf{HR}_{X,Y} : [1, +\infty) \rightarrow \mathcal{T}(S)$ be the harmonic map ray such that $\text{Hopf}(X, \mathbf{HR}_{X,Y}(s)) = s\text{Hopf}(X, Y)$. A direct consequence of Lemma 6.1 is the following corollary.

Corollary 6.2. *Let Y be a fixed hyperbolic surface. For any $s > 1$, the family of maps $\{\mathbf{HR}_{X,Y} : [1, s] \rightarrow \mathcal{T}(S)\}_{X \in \mathcal{T}(S)}$ is uniformly bounded and equi-continuous.*

6.2. Subconvergence of harmonic map rays. We first show that harmonic map rays are almost geodesics with respect to the Thurston metric. (We refer to the notion of a length of a foliation defined in subsection 2.3.)

Proposition 6.3. *For any $\epsilon > 0$, there exists $\mathbf{T} = \mathbf{T}(\epsilon) > 0$, such that for any harmonic map ray $\mathbf{HR}_{X,\Phi}$ with $\|\Phi\| = 1$ and for all $s > t \geq \mathbf{T}$,*

$$(6.2) \quad \sqrt{\frac{s}{t}} \cdot (1 - \epsilon) \leq \frac{\ell_{Y_s}(\text{Hor}(\Phi))}{\ell_{Y_t}(\text{Hor}(\Phi))} \leq \sqrt{\frac{s}{t}}$$

and

$$(6.3) \quad \log \sqrt{\frac{s}{t}} - \epsilon \leq d_{\text{Th}}(Y_t, Y_s) \leq \log \sqrt{\frac{s}{t}},$$

where $Y_t = \mathbf{HR}_{X,\Phi}(t)$ and $Y_s = \mathbf{HR}_{X,\Phi}(s)$.

Proof. Let $Y_s := \mathbf{HR}_{X,\Phi}(s)$ and $\lambda := \text{Hor}(\Phi)$ the horizontal measured foliation of Φ . From Lemma 3.8, we see that

$$\lim_{s \rightarrow \infty} \frac{\ell_{Y_s}(\sqrt{s}\lambda)}{2\|s\Phi\|} = 1.$$

Then for any $\epsilon > 0$, there exists $\mathbf{T} > 0$, such that for all $s > t \geq \mathbf{T}$,

$$(6.4) \quad \begin{aligned} \log \frac{\ell_{X_s}(\lambda)}{\ell_{X_t}(\lambda)} &= \log \left(\frac{\sqrt{t}}{\sqrt{s}} \cdot \frac{\ell_{X_s}(\sqrt{s}\lambda)}{\ell_{X_t}(\sqrt{t}\lambda)} \right) \\ &\geq \log \left(\frac{\sqrt{t}}{\sqrt{s}} \cdot \frac{2\|s\Phi\|}{2\|t\Phi\|} (1 - \epsilon) \right) \\ &\geq \log \sqrt{\frac{s}{t}} - 2\epsilon. \end{aligned}$$

Therefore

$$\begin{aligned} d_{\text{Th}}(X_t, X_s) &= \log \sup_{\mu \in \mathcal{ML}(S)} \frac{\ell_{X_s}(\mu)}{\ell_{X_t}(\mu)} \\ &\geq \log \frac{\ell_{X_s}(\lambda)}{\ell_{X_t}(\lambda)} \\ &\geq \log \sqrt{\frac{s}{t}} - 2\epsilon. \end{aligned}$$

On the other hand, by Lemma 6.1, we see that

$$(6.5) \quad d_{\text{Th}}(Y_t, Y_s) \leq \log \sqrt{s/t}, \quad \forall s > t \geq 1.$$

□

We now consider the compactness of family of harmonic map rays passing through a fixed hyperbolic surface. The first part of the following theorem is a restatement of Theorem 1.3.

Theorem 6.4. *For any fixed $Y \in \mathcal{T}(S)$, let $X_n \in \mathcal{T}(S)$ be any divergent sequence. Then the sequence of harmonic map rays $\mathbf{HR}_{X_n, Y} : [1, \infty) \rightarrow \mathcal{T}(S)$ contains a subsequence which converges to some (reparametrized) Thurston geodesic locally uniformly.*

If moreover, $\text{Hor}(\text{Hopf}(X_n, Y))$ converges to some $\lambda \in \mathcal{PML}(S)$ as $n \rightarrow \infty$, then the limit geodesic has λ as a maximally stretched lamination.

Proof. By Corollary 6.2, we see that $\mathbf{HR}_{X_n, Y} : [1, \infty) \rightarrow \mathcal{T}(S)$ contains subsequence, still denote by $\mathbf{HR}_{X_n, Y}$ for simplicity, which converges locally uniformly to some continuous map: $\mathbf{R} : [1, \infty) \rightarrow \mathcal{T}(S)$. Let $\Phi_n := \text{Hopf}(X_n, Y)$, where $Y_{n, s} := \mathbf{HR}_{X_n, Y}(s)$. Then $\mathbf{R}(s) = \lim_{n \rightarrow \infty} Y_{n, s}$ and $\|\Phi_n\| \rightarrow \infty$, as $n \rightarrow \infty$. By Proposition 6.3, we see that for any $\epsilon > 0$, there exists N_ϵ , such that for all $n > N_\epsilon$ and all $s > t > 1$,

$$\log \sqrt{\frac{s}{t}} - \epsilon \leq d_{\text{Th}}(Y_{n, t}, Y_{n, s}) \leq \log \sqrt{\frac{s}{t}}.$$

By letting $n \rightarrow \infty$, we see that

$$\log \sqrt{\frac{s}{t}} - \epsilon \leq d_{\text{Th}}(\mathbf{R}(t), \mathbf{R}(s)) \leq \log \sqrt{\frac{s}{t}}$$

By the arbitrariness of ϵ , we see that

$$d_{\text{Th}}(\mathbf{R}(t), \mathbf{R}(s)) = \log \sqrt{\frac{s}{t}}.$$

This proves that \mathbf{R} is a Thurston geodesic.

It remains to show that \mathbf{R} maximally stretches λ . Let $\lambda_n := \text{Hor}(\Phi_n)$. Notice that $\|\Phi_n\| \rightarrow \infty$ as $n \rightarrow \infty$. It then follows from Equation (6.2) that for any $\epsilon > 0$, there exists N_ϵ , such that for all $n > N_\epsilon$ and all $s > t > 1$,

$$\sqrt{\frac{s}{t}} \cdot (1 - \epsilon) \leq \frac{\ell_{X_{n, s}}(\lambda_n)}{\ell_{X_{n, t}}(\lambda_n)} \leq \sqrt{\frac{s}{t}}.$$

Letting $n \rightarrow \infty$, we see that

$$(6.6) \quad \sqrt{\frac{s}{t}} \cdot (1 - \epsilon) \leq \frac{\ell_{X_s}(\lambda)}{\ell_{X_t}(\lambda)} \leq \sqrt{\frac{s}{t}}.$$

where $X_s = \mathbf{R}(s)$ and $X_t = \mathbf{R}(t)$. The arbitrariness of ϵ then implies that

$$\frac{\ell_{X_s}(\lambda)}{\ell_{X_t}(\lambda)} = \sqrt{\frac{s}{t}}.$$

□

We end this subsection with a similar estimate for harmonic map dual rays.

Proposition 6.5. *For any $\epsilon > 0$, there exists $\mathbf{T}(\epsilon) > 0$, such that for any harmonic map dual ray $\mathbf{hr}_{Y, \lambda}$ with $\ell_Y(\lambda) = 1$, and for all $s > t \geq \mathbf{T}$,*

$$(6.7) \quad d_T(\mathbf{hr}_{Y, \lambda}(t), \mathbf{hr}_{Y, \lambda}(s)) \geq \log \sqrt{\frac{s}{t}} - \epsilon.$$

Proof. Let $X_t := \mathbf{hr}_{Y, \lambda}(t)$ and $\Psi_t := \text{Hopf}(X_t, Y)$. Then the horizontal measured foliation/lamination of Ψ_t is $t\lambda$. By Lemma 3.8, we see that

$$\lim_{t \rightarrow \infty} \frac{\ell_Y(t\lambda)}{2\text{Ext}_{X_t}(t\lambda)} = \frac{\ell_Y(t\lambda)}{2\|\Psi_t\|} = 1.$$

Then for any $\epsilon > 0$, there exists $\mathbf{T} > 0$, such that for all $s > t \geq \mathbf{T}$,

$$\begin{aligned}
d_T(X_s, X_t) &= \frac{1}{2} \log \sup_{\mu \in \mathcal{ML}(S)} \frac{\text{Ext}_{X_t}(\mu)}{\text{Ext}_{X_s}(\mu)} \\
&\geq \frac{1}{2} \log \frac{\text{Ext}_{X_t}(\lambda)}{\text{Ext}_{X_s}(\lambda)} \\
&= \frac{1}{2} \log \left(\frac{s^2 \text{Ext}_{X_t}(t\lambda)}{t^2 \text{Ext}_{X_s}(s\lambda)} \right) \\
&\geq \frac{1}{2} \log \left(\frac{s^2 \ell_Y(t\lambda)}{t^2 \ell_Y(s\lambda)} (1 - \epsilon) \right) \\
&\geq \log \sqrt{\frac{s}{t}} - \epsilon.
\end{aligned}$$

□

7. CONSTRUCTION OF PIECEWISE HARMONIC STRETCH MAPS

The goal of this section is to prove Theorem 1.7, which will be used in Section 8 to finish the proof of Theorem 1.1.

It is perhaps worth taking a moment to recall how this passage will sit in our general theory. In this section, we prove the existence of “piecewise harmonic stretch rays”. These are paths that generalize Thurston’s construction of concatenation of stretch lines when the maximally stretched lamination λ is not maximal. Here instead of concatenating stretch maps with maximal laminations that extend λ , we use harmonic maps to define the stretch line on the complementary regions that are not ideal triangles.

On the other hand, we will later, in Section 12, construct a family of maps that extend the Thurston theory for these general laminations in a different way. In that section, given hyperbolic surfaces $Y, Z \in \mathcal{T}(S)$, a harmonic stretch ray will be a limit of a family of harmonic map rays from base points $X_n \in \mathcal{T}(S)$ that all proceed through Y to Z , as the base points X_n degenerate in $\mathcal{T}(S)$. In some sense, we will pick out, from some possible ways of stretching from Y to Z , a canonical path that minimizes a particular energy.

With that context in mind, we now begin the discussion of the piecewise harmonic stretch maps.

7.1. Thurston’s construction of stretch maps for maximal geodesic lamination. Given a maximal lamination, Thurston constructs stretch maps in two steps.

- Step 1. The first step is to define a change in the hyperbolic structure of each ideal triangle such that the sides of each triangle are expanded by a factor of e^t . The change is realized by a built-in Lipschitz map having Lipschitz constant e^t on the boundary and at most e^t in the interior. This is the only place where Thurston uses the assumption of λ being maximal. These changes match up along λ so that they change the arc length of the leaves of λ by a factor of e^t as well.
- Step 2. The second step is to extend the (new) hyperbolic structures on the complement of λ over a neighbourhood \mathcal{N} of λ by describing its developing map as an infinite product in the group of isometries of the hyperbolic plane,

with the help of an induced measured foliation $F_{\mathcal{N}}(\lambda)$ on \mathcal{N} transverse to λ . In this step, Thurston makes assumptions neither about the existence of transverse measures for λ , nor about the topological or combinatorial type of the complement of λ . In other words, this second step works for all geodesic laminations. The hyperbolic structure on \mathcal{N} is determined by the transverse measured foliation $F_{\mathcal{N}}(\lambda)$ and a *sharpness function* in a neighbourhood of each ideal vertex (spike), which records the length of each horocycle leaf inside each spike of $Y - \lambda$. The hyperbolic structure on $Y - \lambda$ constructed in Step one provides a unique sharpness function. So the construction results in a unique hyperbolic structure on X with a built-in Lipschitz homeomorphism. This step is carried out in detail in [PT07, Section 3] and [Bon96, Section 5] (See also [CF21]).

To generalize the construction from the case of maximal laminations to non-maximal ones, all we need to do is to change the hyperbolic structure on each component of $Y - \lambda$ in a suitable way. Here we provide an approach to doing this by considering harmonic maps from punctured surfaces to crowned hyperbolic surfaces (possibly with simple closed geodesic boundary components).

7.2. Harmonic maps from punctured surfaces to crowned hyperbolic surfaces. The change of hyperbolic metric on $Y - \lambda$ is based on the following theorem.

Theorem 7.1 ([Gup17] Theorem 1.2). *Let \widehat{X} be a closed Riemann surface with a set of marked points D with fixed coordinate disks around them. For a collection of principal parts $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$ having poles of orders $n_i \geq 2$ for $i = 1, 2, \dots, k$,*

- *let $Q(\widehat{X}, D, \mathcal{P})$ be the space of meromorphic quadratic differentials on \widehat{X} with principal part P_i at p_i , and*
- *$\mathcal{T}(\mathcal{P})$ be the space of marked crowned hyperbolic surfaces homeomorphic to $\widehat{X} \setminus D$ with k crowns, each having $(n_i - 2)$ boundary cusps, with metric residues (see Definition 2.5) compatible with the residues of the principal parts P_i .*

Then we have a homeomorphism

$$\widehat{\Psi} : \mathcal{T}(\mathcal{P}) \rightarrow Q(\widehat{X}, D, \mathcal{P})$$

that assigns, to any marked crowned hyperbolic surface \widehat{Y} , the Hopf differential of the unique harmonic map from $\widehat{X} \setminus D$ to \widehat{Y} with prescribed principal parts.

Remark 7.2. The original first statement in [Gup17, Theorem 1.2] is for $n_i \geq 3$, but the method also works for $n_i = 2$, so we state the result more generally.

7.3. Deformation of $Y - \lambda$. We start with the following lemma which is an analogue of the Lemma 6.1.

Lemma 7.3 (Uniformly Lipschitz). *Let $\widehat{f}_t : \widehat{X} \setminus D \rightarrow \widehat{Y}_t$ be a sequence of surjective harmonic diffeomorphisms from a punctured Riemann surface $\widehat{X} \setminus D$ to crowned hyperbolic surfaces \widehat{Y}_t with Hopf differential $\widehat{\Phi}_t = t\widehat{\Phi}$. Then for any $0 < t < s$, the composition map $f_s \circ (f_t)^{-1} : \widehat{Y}_t \rightarrow \widehat{Y}_s$ is a Lipschitz map having (pointwise) Lipschitz constant strictly less than $\sqrt{s/t}$. Moreover, it extends to an affine map of factor $\sqrt{s/t}$ from ∂Y_t^i to ∂Y_s^i .*

We postpone the proof of Lemma 7.3 to the end of this section. We now sketch the construction of piecewise harmonic stretch lines based on Lemma 7.3, proving Theorem 1.7.

Proof of Theorem 1.7. The proof is similar to the proof of [Thu98, Corollary 4.2], except that we use Lemma 7.3 to deform the hyperbolic metrics on each component of $S - \lambda$. Let $Y \in \mathcal{T}(S)$ be a closed hyperbolic surface and λ a geodesic lamination. Let Y^1, \dots, Y^m be the (connected) crowned hyperbolic surfaces of $Y \setminus \lambda$. Let F be the transverse measured foliation in a neighbourhood of λ induced from Y . For each $1 \leq i \leq k$, let us choose a closed Riemann surface X^i with marked points D^i and a surjective harmonic diffeomorphism $f^i : X^i \rightarrow Y^i$ with Hopf differentials denoted by Φ^i . By Theorem 7.1, this gives a family of crowned hyperbolic surfaces Y_t^i and surjective harmonic diffeomorphisms $f_t^i : X^i \rightarrow Y_t^i$ with $\text{Hopf}(f_t^i) = t\text{Hopf}(f^i) = t\Phi^i$. By Lemma 7.3, we see that for any $0 < t < s$, the composition map $u_{t,s}^i := f_s^i \circ (f_t^i)^{-1} : Y_t^i \rightarrow Y_s^i$ is a Lipschitz map having (pointwise) Lipschitz constant strictly less than $\sqrt{s/t}$ and extends to an affine map of factor $\sqrt{s/t}$ from ∂Y_t^i to ∂Y_s^i . Let Y_t be closed hyperbolic surface obtained by gluing Y_t^1, \dots, Y_t^k using λ and the transverse measured foliation tF . Then for any $0 < t < s$, the maps $u_{t,s}^1, \dots, u_{t,s}^k$ extends to a $\sqrt{s/t}$ -Lipschitz homeomorphism from Y_t to Y_s which is an affine map of factor $\sqrt{s/t}$ on λ and which has (pointwise) Lipschitz constant strictly less $\sqrt{s/t}$ in $Y \setminus \lambda$. □

7.4. Limits of piecewise harmonic stretch rays. Recall that every piecewise harmonic stretch line is directed. We next prove a proposition that will prove useful when we define canonical harmonic stretch lines from given base points to points on the Thurston boundary of Teichmüller space in Section 13.

Proposition 7.4. *Let $Y \in \mathcal{T}(S)$ be a closed hyperbolic surface and λ a geodesic lamination on Y . Let $f : X \rightarrow Y \setminus \lambda$ be a surjective harmonic diffeomorphism from some punctured surface X . Let β be the pushforward to Y of the vertical foliation of $\text{Hopf}(f)$. Then the piecewise harmonic stretch line determined by (Y, λ, X, f) converges to $[\beta] \in \mathcal{PM}\mathcal{L}(S)$ in the Thurston compactification.*

Proof. Let **PSR** be the piecewise harmonic stretch line determined by (Y, λ, X, f) . Let \widehat{Y}_t be the (possibly disconnected) crowned hyperbolic determined by the pair $(X, t\text{Hopf}(f))$ by Theorem 7.1. Let $Y_t \in \mathbf{PSR}$ be the hyperbolic surface such that $Y_t \setminus \lambda = \widehat{Y}_t$. Let α be an arbitrary simple closed curve on Y . Consider the pushforward to Y of the horizontal and vertical foliations of $\text{Hopf}(f)$. For any $\epsilon > 0$, let α^* be a representative satisfying the following:

- α^* consists of segments of the pushforward of horizontal and vertical foliations of $\text{Hopf}(f)$ alternatively,
- α^* avoids the zeros of $\text{Hopf}(f)$,
- α^* almost realizes the (minimal) intersection number with β , the vertical foliation of $\text{Hopf}(f)$, namely

$$(7.1) \quad \left| \int_{\alpha^*} |\text{Re} \sqrt{\text{Hopf}(f)}| - i(\alpha, \beta) \right| < \epsilon.$$

Let $d > 0$ be the distance of $f^{-1}(\alpha^*)$ to the zeros of $\text{Hopf}(f)$. Let K_v be the number of vertical segments of α^* . Then by (3.2) and Lemma 3.3, for $t > 0$

sufficiently large, the total length $\text{Length}_{X_t}(v\alpha^*)$ of vertical segments of α^* with respect to the hyperbolic metric $Y_t \in \mathbf{PSR}$ satisfies:

$$\text{Length}_{Y_t}(v\alpha^*) \leq K_v \int_{td}^{\infty} \sqrt{2 \sinh^{-1}(\chi(Y)/s^2) \exp(-s)} ds \rightarrow 0, \text{ as } t \rightarrow +\infty.$$

and the total length $\text{Length}_{X_t}(h\alpha^*)$ of horizontal segments of α^* with respect to the hyperbolic metric $Y_t \in \mathbf{PSR}$ satisfies:

$$\begin{aligned} \text{Length}_{Y_t}(h\alpha^*) &\leq 2 \int_{\alpha^*} |\text{Re} \sqrt{t \text{Hopf}(f_\infty)}| (1 + \exp(-td) \sinh^{-1}(\chi(Y)/(td)^2)) \\ &\rightarrow 2\sqrt{t} \int_{\alpha^*} |\text{Re} \sqrt{\text{Hopf}(f_\infty)}|, \text{ as } t \rightarrow +\infty. \end{aligned}$$

Therefore, the length $\text{Length}_{Y_t}(\alpha^*)$ of α^* on X_t satisfies:

$$\begin{aligned} \limsup_{t \rightarrow +\infty} \frac{\text{Length}_{Y_t}(\alpha^*)}{\sqrt{t}} &= \limsup_{t \rightarrow +\infty} \frac{\text{Length}_{Y_t}(v\alpha^*) + \text{Length}_{Y_t}(h\alpha^*)}{\sqrt{t}} \\ &\leq 2 \int_{\alpha^*} |\text{Re} \sqrt{\text{Hopf}(f)}|. \end{aligned}$$

Hence the length of the geodesic representative of α on Y_t satisfies:

$$(7.2) \quad \limsup_{t \rightarrow +\infty} \frac{\ell_{Y_t}(\alpha)}{\sqrt{t}} \leq \limsup_{t \rightarrow +\infty} \frac{\text{Length}_{Y_t}(\alpha^*)}{\sqrt{t}} \leq 2 \int_{\alpha^*} |\text{Re} \sqrt{\text{Hopf}(f)}|.$$

On the other hand, by (3.2),

$$\ell_{Y_t}(\alpha) \geq 2 \int_{\alpha^*} |\text{Re} \sqrt{t \text{Hopf}(f_\infty)}| = 2\sqrt{t} \int_{\alpha^*} |\text{Re} \sqrt{\text{Hopf}(f)}|.$$

Combined with (7.2), this implies that

$$\lim_{t \rightarrow +\infty} \frac{\ell_{Y_t}(\alpha)}{\sqrt{t}} = 2 \int_{\alpha^*} |\text{Re} \sqrt{\text{Hopf}(f)}|.$$

It then follows from (7.1) and the arbitrariness of ϵ that

$$\lim_{t \rightarrow +\infty} \frac{\ell_{X_t}(\alpha)}{\sqrt{t}} = 2i(\alpha, \beta).$$

This proves that Y_t converges to $[\beta] \in \mathcal{PML}(S)$ as $t \rightarrow +\infty$. \square

7.5. Generalized maximum principle. To prove Lemma 7.3, we need the following generalized maximum principle from [CY75].

Theorem 7.5 ([CY75] Theorem 3 and Theorem 8). *Let M be complete noncompact manifold with Ricci curvature bounded from below by $-K$ for some constant $K \geq 0$. Let u be a C^2 function on M .*

(I) *If u is bounded from above, then there exists a sequence of points $p_k \in M$ such that*

$$\lim_{k \rightarrow \infty} u(p_k) = \sup_{p \in M} u(p), \quad \lim_{k \rightarrow \infty} |\nabla u(p_k)| = 0, \quad \limsup_{k \rightarrow \infty} \Delta u(p_k) \leq 0.$$

(II) *If u satisfies the differential inequality $\Delta u \geq f(u)$, where f is a function on \mathbb{R} with the property that there exists a continuous non-decreasing function g positive on some interval $[a, \infty)$ such that:*

$$\liminf_{t \rightarrow \infty} \frac{f(t)}{g(t)} > 0$$

and

$$\int_b^\infty \left(\int_a^t g(\tau) d\tau \right)^{-1/2} dt < \infty \text{ for some } b \geq a.$$

Then u is bounded from above. Furthermore, if f is lower semi-continuous, $f(\sup u) \leq 0$.

7.6. Geometry of harmonic map rays from degenerated surfaces. To prove Lemma 7.3, we need a generalization of [Wol89, Proposition 4.3]. Before that we need to introduce some notations. Let $(M, \sigma|dz|^2)$ be the hyperbolic plane or the complex plane and $(\mathbb{H}^2, \rho|dw|^2)$ the hyperbolic plane. Let $f_t : M \rightarrow (\mathbb{H}^2, \rho|dw|^2)$ be a harmonic diffeomorphism with Hopf differential $t\Phi$. Define

$$\mathcal{H}_t := \frac{\rho(f_t)}{\sigma} \left| \frac{\partial f_t}{\partial z} \right|^2, \quad \mathcal{L}_t := \frac{\rho(f_t)}{\sigma} \left| \frac{\partial f_t}{\partial \bar{z}} \right|^2,$$

and

$$\Delta_\sigma := \frac{4}{\sigma} \frac{\partial^2}{\partial z \partial \bar{z}}.$$

Then

$$(7.3) \quad t^2 |\Phi|^2 / \sigma^2 = \mathcal{H}_t \mathcal{L}_t$$

$$(7.4) \quad \nu_t = \partial f_t \partial \bar{z} / \partial f_t \partial z = \bar{t} \bar{\Phi} / \sigma \mathcal{H}_t$$

$$(7.5) \quad |\nu_t|^2 = \mathcal{L}_t / \mathcal{H}_t$$

$$(7.6) \quad \Delta_\sigma \log \mathcal{H}_t = 2\mathcal{H}_t - 2\mathcal{L}_t + 2K(\sigma), \text{ (where } \mathcal{H} \neq 0)$$

$$(7.7) \quad \Delta_\sigma \log \mathcal{L}_t = 2\mathcal{L}_t - 2\mathcal{H}_t + 2K(\sigma), \text{ (where } \mathcal{L} \neq 0)$$

$$(7.8) \quad t\Phi = \sigma \mathcal{H}_t \bar{\nu}_t$$

where (7.6) and (7.7) can be found in [SY78].

Proposition 7.6. *Let $(M, \sigma|dz|^2)$ be the hyperbolic plane \mathbb{H}^2 or the complex plane \mathbb{C} . Let Φdz^2 be a holomorphic quadratic differential on M , which is a polynomial if M is the complex plane. Let $f_t : (M, \sigma|dz|^2) \rightarrow \mathbb{H}^2$ be a family of harmonic diffeomorphisms onto the corresponding images with Hopf differentials $\Phi_t = t\Phi$, where $t > 0$. Then the following statements hold, where we use primes to indicate differentiation with respect to t .*

(i) For any $p \in M$, $\mathcal{H}'_t(p) \geq 0$.

(ii) For any $p \in M$, $\mathcal{L}'_t(p) \geq 0$.

(iii) For any $p \in M$, $|\nu_t|'(p) \geq 0$, and $|\nu_t|'(p) > 0$ if $\Phi(p) \neq 0$.

(iv) The function $\frac{|\nu_t|'}{|\nu_t|}(p)$ extends to a bounded positive analytic function on M .

In particular, for any compact subset N of M , there exists a constant $\epsilon > 0$, such that $\frac{|\nu_t|'}{|\nu_t|}(p) > \epsilon$ for any $p \in N$.

Proof. The idea is to use the maximum principle and generalized maximum principle.

Notice that for any fixed $t \geq 0$,

$$(7.9) \quad \inf_{p \in M} \mathcal{H}_t(p) > 0, \quad \forall p \in M.$$

In fact, if M is the hyperbolic plane, then it follows from [Wan92, Theorem 12 and Proposition 10] that $\mathcal{H}_t(p) \geq 1$ for all $t \geq 0$ and all $p \in M$. If M is the complex plane, since Φ is a nonconstant polynomial, then by (7.3), since $\mathcal{H}_t(p) \geq \mathcal{L}_t(p)$ (notice f_t is orientation-preserving, we have that $\mathcal{H}_t(p) \geq t|\Phi(p)| \rightarrow \infty$, as $p \rightarrow \infty$).

In particular, there exists $R > 0$, such that $\mathcal{H}_t(p) > 1$ for all $|p| > R$. On the other hand on $\{p \in \mathbb{C} : |p| \leq R\}$, we have that $\mathcal{H}_t(p)$ is positive and continuous. Therefore, in both cases, (7.9) holds.

First, we show that $\mathcal{H}'_t(p) \geq 0$ for all $p \in M$. To this end, it is equivalent to show that for any fixed $p \in M$, $\mathcal{H}_t(p)$ is an increasing function of $t > 0$. For any $0 < t < s$, let $\mathcal{W}_t = \frac{1}{2} \log \mathcal{H}_t$, then by (7.6), we have the Bochner-type equations:

$$(7.10) \quad \begin{aligned} \Delta_\sigma \mathcal{W}_t &= e^{2\mathcal{W}_t} - \frac{t^2 |\Phi|^2}{\sigma^2} e^{-2\mathcal{W}_t} + K(\sigma), \\ \Delta_\sigma \mathcal{W}_s &= e^{2\mathcal{W}_s} - \frac{s^2 |\Phi|^2}{\sigma^2} e^{-2\mathcal{W}_s} + K(\sigma). \end{aligned}$$

By subtracting both sides, we get

$$\Delta_\sigma (\mathcal{W}_t - \mathcal{W}_s) = (e^{2\mathcal{W}_t} - e^{2\mathcal{W}_s}) - \frac{t^2 |\Phi|^2}{\sigma^2} e^{-2\mathcal{W}_t} + \frac{s^2 |\Phi|^2}{\sigma^2} e^{-2\mathcal{W}_s}.$$

Let $\tilde{\sigma} = \tilde{\sigma}(s, z) := \sigma e^{2\mathcal{W}_s} |dz|^2$ and $\eta := \mathcal{W}_t - \mathcal{W}_s$, then

$$(7.11) \quad \begin{aligned} \Delta_{\tilde{\sigma}} \eta &= e^{2\eta} - 1 - \frac{t^2}{s^2} |\nu_s|^2 e^{-2\eta} + |\nu_s|^2 \quad (\text{by (7.8)}) \\ &\geq e^{2\eta} - e^{-2\eta} - 1 \quad (\text{since } |\nu_s| \leq 1). \end{aligned}$$

Recall that $\tilde{\sigma} = \tilde{\sigma}(s)$ is a complete metric with curvature bounded from below: indeed, for fixed s ,

$$\begin{aligned} K(\tilde{\sigma}) &= -2\Delta_{\tilde{\sigma}} \log \tilde{\sigma} \\ &= -\frac{2\sigma}{\tilde{\sigma}} \Delta_\sigma (\log \sigma + 2\mathcal{W}_s) \\ &= -e^{-2\mathcal{W}_s} (2\Delta_\sigma \log \sigma + 4\Delta_\sigma \mathcal{W}_s) \\ &\stackrel{\text{by (7.3, 7.6)}}{=} -e^{-2\mathcal{W}_s} \left(-K(\sigma) + 4e^{2\mathcal{W}_s} - 4\frac{s^2 |\Phi|^2}{\sigma^2} e^{-2\mathcal{W}_s} - 4K(\sigma) \right) \\ &\stackrel{\text{by (7.8)}}{=} 5K(\sigma)e^{-2\mathcal{W}_s} - 4 + 4|\nu_s|^2 \\ &\geq -5e^{-2\mathcal{W}_s} - 4 + 4|\nu_s|^2 \quad (\text{since } K(\sigma) \in \{0, -1\}) \\ &= -\frac{5}{\mathcal{H}_s} - 4 + 4|\nu_s|^2 \\ &\geq -\frac{5}{\mathcal{H}_s} - 4 \\ &\geq -\frac{5}{\inf_{p \in M} \mathcal{H}_s(p)} - 4 > -\infty. \quad (\text{by (7.9)}) \end{aligned}$$

It then follows from Theorem 7.5 by taking $f(t) = e^{2t} - e^{-2t} - 1$ and $g(t) = e^{2t}$, that $\bar{\eta} := \sup_{p \in M} \eta(p) < +\infty$. Moreover, again by Theorem 7.5, there exists a sequence $p_k \in M$, such that

$$\lim_{k \rightarrow \infty} \eta(p_k) = \bar{\eta} = \sup_{p \in M} \eta(p), \quad \limsup_{k \rightarrow \infty} \Delta_{\tilde{\sigma}} \eta(p_k) \leq 0.$$

By taking a subsequence if necessary, we can assume that

$$\lim_{k \rightarrow \infty} |\nu_s(p_k)|^2 = a \in [0, 1].$$

It then follows from (7.11) that

$$\begin{aligned}
0 &\geq \limsup_{k \rightarrow \infty} \Delta_{\bar{\sigma}} \eta(p_k) \\
&= \limsup_{k \rightarrow \infty} \left(e^{2\eta(p_k)} - 1 - \frac{t^2}{s^2} |\nu_s(p_k)|^2 e^{-2\eta(p_k)} + |\nu_s(p_k)|^2 \right) \\
&= e^{2\bar{\eta}} - \frac{t^2}{s^2} a e^{-2\bar{\eta}} - 1 + a \\
&\geq e^{2\bar{\eta}} - a e^{-2\bar{\eta}} - 1 + a \quad (\text{since } 0 < t < s)
\end{aligned}$$

which implies that $\bar{\eta} \leq 0$. Therefore,

$$\mathcal{W}_t(p) - \mathcal{W}_s(p) = \eta(p) \leq \bar{\eta} \leq 0$$

Hence, $\mathcal{H}_t(p) \leq \mathcal{H}_s(p)$ for all $p \in M$. Equivalently, $\mathcal{H}'_t(p) \geq 0$ for all $p \in M$.

Next, we show that $\mathcal{L}'_t(p) \geq 0$ for all $p \in M$. Notice that $t^2 |\Phi|^2 / \sigma^2 = \mathcal{H}_t \mathcal{L}_t$ and $s^2 |\Phi|^2 / \sigma^2 = \mathcal{H}_s \mathcal{L}_s$. Then

$$(7.12) \quad \frac{\mathcal{L}_t}{\mathcal{L}_s}(p) = \frac{t^2}{s^2} \frac{\mathcal{H}_s}{\mathcal{H}_t}(p), \quad \Phi(p) \neq 0$$

extends to a well-defined analytic function on the whole surface M , still denoted by $\frac{\mathcal{L}_t}{\mathcal{L}_s}(p)$ for simplicity. Let $\delta(p) = \frac{1}{2} \log \frac{\mathcal{L}_t}{\mathcal{L}_s}(p)$. The equation above implies that $\delta(p) = \log \frac{t}{s} + \frac{1}{2} \log \frac{\mathcal{H}_s}{\mathcal{H}_t}(p)$. It then follows that

$$(7.13) \quad \Delta_{\sigma} \delta = \frac{1}{2} \Delta_{\sigma} \log \mathcal{H}_s - \frac{1}{2} \Delta_{\sigma} \log \mathcal{H}_t$$

$$(7.14) \quad = \mathcal{H}_s - \mathcal{H}_t - \frac{s^2 |\Phi|^2}{\sigma^2} \mathcal{H}_s^{-1} + \frac{t^2 |\Phi|^2}{\sigma^2} \mathcal{H}_t^{-1}.$$

Let $\hat{\sigma} := \sigma \mathcal{H}_t |dz|^2$, then

$$\begin{aligned}
\Delta_{\hat{\sigma}} \delta &= \frac{\mathcal{H}_s}{\mathcal{H}_t} - 1 - \frac{s^2 |\Phi|^2}{\sigma^2 \mathcal{H}_s \mathcal{H}_t} + \frac{s^2 |\Phi|^2}{\sigma^2 \mathcal{H}_t^2} \\
&= \frac{\mathcal{H}_s}{\mathcal{H}_t} - 1 - \frac{t^2 |\Phi|^2}{\sigma^2 \mathcal{H}_t^2} \frac{s^2}{t^2} \frac{\mathcal{H}_t}{\mathcal{H}_s} + \frac{s^2 |\Phi|^2}{\sigma^2 \mathcal{H}_t^2} \\
&\stackrel{\text{by (7.8)}}{=} \frac{\mathcal{H}_s}{\mathcal{H}_t} - 1 - |\nu_t|^2 \frac{s^2}{t^2} \frac{\mathcal{H}_t}{\mathcal{H}_s} + |\nu_t|^2 \\
&\stackrel{\text{by (7.12)}}{=} \frac{s^2}{t^2} e^{2\delta} - 1 - |\nu_t|^2 e^{-2\delta} + |\nu_t|^2 \\
&\geq \frac{s^2}{t^2} e^{2\delta} - 1 - e^{-2\delta}.
\end{aligned}$$

By applying the argument similar as in the case of η , we see that $\sup_{p \in M} \delta < 0$. In particular, $\mathcal{L}_t < \mathcal{L}_s$ for all $0 \leq t < s$ and all $p \in M$ with $\Phi(p) \neq 0$. This implies that $\mathcal{L}'_t(p) \geq 0$ for all $t \geq 0$ and all $p \in M$ with $\Phi(p) \neq 0$. It then follows from continuity of \mathcal{L}'_t that $\mathcal{L}'_t(p) \geq 0$ for all $t \geq 0$ and all $p \in M$.

It remains to show item (iii) and item (iv). Suppose that $\Phi(p) \neq 0$. Taking derivatives respect to t for (7.3) and (7.8), we see that

$$(7.15) \quad \frac{\mathcal{H}'_t}{\mathcal{H}_t}(p) + \frac{\mathcal{L}'_t}{\mathcal{L}_t}(p) = \frac{2}{t},$$

$$(7.16) \quad \frac{\mathcal{H}'_t}{\mathcal{H}_t}(p) + \frac{|\nu|'_t}{|\nu|_t}(p) = \frac{1}{t}.$$

Therefore, for any fixed $t > 0$, both $\frac{\mathcal{L}'_t}{\mathcal{L}_t}$ and $\frac{|\nu|'_t}{|\nu|_t}$ extend to be bounded non-negative analytic functions on the whole of M . We next improve this bound on $\frac{\mathcal{L}'_t}{\mathcal{L}_t}$.

Applying Theorem 7.5 to $\frac{\mathcal{H}'_t}{\mathcal{H}_t}$, which is a bounded non-negative analytic function by (7.15), we see that there exists a sequence of points $p_k \in M$ with $p_k \rightarrow \infty$, such that

$$(7.17) \quad \lim_{k \rightarrow \infty} \frac{\mathcal{H}'_t}{\mathcal{H}_t}(p_k) = \sup_{p \in M} \frac{\mathcal{H}'_t}{\mathcal{H}_t}(p), \quad \limsup_{k \rightarrow \infty} \Delta_\sigma \frac{\mathcal{H}'_t}{\mathcal{H}_t}(p_k) \leq 0.$$

Therefore,

$$\begin{aligned} & 2 \sup_{p \in M} \frac{\mathcal{H}'_t}{\mathcal{H}_t}(p) - \frac{2}{t} \\ &= \lim_{k \rightarrow \infty} \left(2 \frac{\mathcal{H}'_t}{\mathcal{H}_t}(p_k) - \frac{2}{t} \right) \\ &= \lim_{k \rightarrow \infty} \left(\frac{\mathcal{H}'_t}{\mathcal{H}_t}(p_k) - \frac{\mathcal{L}'_t}{\mathcal{L}_t}(p_k) \right) \\ &\leq \lim_{k \rightarrow \infty} \left(\frac{\mathcal{H}'_t}{\mathcal{H}_t}(p_k) - \frac{\mathcal{L}'_t}{\mathcal{H}_t}(p_k) \right) \quad (\text{since } \mathcal{H}_t > \mathcal{L}_t \geq 0 \text{ and } \mathcal{L}'_t \geq 0) \\ &= \lim_{k \rightarrow \infty} \frac{\frac{1}{2} \Delta_\sigma \frac{\mathcal{H}'_t}{\mathcal{H}_t}(p_k)}{\mathcal{H}_t(p_k)} \quad (\text{by (7.6)}) \\ &\leq 0. \quad (\text{by (7.9), (7.17)}) \end{aligned}$$

Consequently,

$$(7.18) \quad \sup_{p \in M} \frac{\mathcal{H}'_t}{\mathcal{H}_t}(p) \leq \frac{1}{t}.$$

Next, we claim that

$$(7.19) \quad \frac{\mathcal{H}'_t}{\mathcal{H}_t}(p) < \frac{1}{t}$$

holds for any $p \in M$. Suppose to the contrary that there exists some $p_0 \in M$ such that $\frac{\mathcal{H}'_t}{\mathcal{H}_t}(p_0) = \frac{1}{t}$. Combined with (7.18), this implies that

$$\Delta_\sigma \frac{\mathcal{H}'_t}{\mathcal{H}_t}(p_0) \leq 0.$$

It then follows from (7.6) that

$$\mathcal{H}'_t(p_0) - \mathcal{L}'_t(p_0) = \Delta_\sigma \frac{\mathcal{H}'_t}{\mathcal{H}_t}(p_0) \leq 0,$$

which implies that $\mathcal{H}'_t(p_0) \leq \mathcal{L}'_t(p_0)$. Combined with (7.15) and the fact that $\mathcal{H}_t(p_0) > \mathcal{L}_t(p_0)$, this yields that

$$\frac{\mathcal{H}'_t(p_0)}{\mathcal{H}_t(p_0)} < \frac{1}{2} \left(\frac{\mathcal{H}'_t(p_0)}{\mathcal{H}_t(p_0)} + \frac{\mathcal{L}'_t(p_0)}{\mathcal{L}_t(p_0)} \right) = \frac{1}{t}.$$

This contradicts the assumption that $\frac{\mathcal{H}'_t}{\mathcal{H}_t}(p_0) = \frac{1}{t}$. Hence

$$\frac{\mathcal{H}'_t}{\mathcal{H}_t}(p) < \frac{1}{t}, \forall p \in M \text{ and } \forall t > 0.$$

It then follows from (7.16) that

$$\frac{|\nu_t|'}{|\nu_t|}(p) = \frac{1}{t} - \frac{\mathcal{H}'_t}{\mathcal{H}_t}(p) > 0, \quad \forall p \in M \text{ and } \forall t > 0.$$

Combined with the continuity of $\frac{|\nu_t|'}{|\nu_t|}$, this proves (iii) and (iv). \square

Proof of Lemma 7.3. Let $p \in \widehat{X}$ be an arbitrary point with $\Phi(p) \neq 0$. By the third item of Proposition 7.6, we see that $|\nu_t|_p$ is a strictly increasing function of $t \in (0, \infty)$. Therefore, the function $\mathcal{G}(p, t) := \log(1/|\nu_t(p)|)$ is a strictly decreasing function of $t \in [0, \infty)$ at p with $\widehat{\Phi}(p) \neq 0$. Recall that with respect to the canonical coordinate charts $z = x + iy$ of $\widehat{\Phi}$ near p , the pullback metric of \widehat{Y}_t by f_t is:

$$(7.20) \quad f_t^* = 2t(\cosh \mathcal{G}(p, t) + 1)dx^2 + 2t(\cosh \mathcal{G}(p, t) - 1)dy^2.$$

Then for any $s > t > 0$, the Lipschitz constant of $f_s \circ (f_t)^{-1}$ at $f_t(p)$ satisfies

$$\text{Lip}(f_s \circ (f_t)^{-1})|_{f_t(p)} \leq \max \left\{ \sqrt{\frac{2s(\cosh \mathcal{G}(p, s) + 1)}{2t(\cosh \mathcal{G}(p, t) + 1)}}, \sqrt{\frac{2s(\cosh \mathcal{G}(p, s) - 1)}{2t(\cosh \mathcal{G}(p, t) - 1)}} \right\} < \sqrt{\frac{s}{t}}.$$

Combined with Lemma 3.3, this implies that as p approaches $\partial \widehat{X}$, we have that $\text{Lip}(f_s \circ (f_t)^{-1})|_{f_t(p)}$ converges uniformly to $\sqrt{\frac{s}{t}}$.

We now consider Lipschitz constants of $f_s \circ f_t^{-1}$ at zeros of Φ . Let p_0 be an arbitrary zero of Φ . Then by (7.4), $\nu_t(p_0) = 0$ for all $t \geq 0$. Hence

$$(7.21) \quad \mathcal{G}(p_0, t) = +\infty \text{ and } \lim_{p \rightarrow p_0} \mathcal{G}(p, t) = +\infty.$$

Let U be a small neighbourhood of p_0 . By item (iv) of Proposition 7.6, for any $0 < t < s$, there exists $\epsilon > 0$, such that $\frac{|\nu_s|'}{|\nu_s|}(p) > \epsilon$ for any $p \in U$ and any $r \in [t, s]$. It then follows that

$$(7.22) \quad \frac{|\nu_s|}{|\nu_t|}(p) = \exp(\log |\nu_s|(p) - \log |\nu_t|(p)) > \exp(\epsilon(s - t))$$

holds for any $p \in U$. Therefore, the Lipschitz constant of $f_s \circ (f_t)^{-1}$ at $f_t(p_0)$ satisfies

$$\begin{aligned}
& \text{Lip}(f_s \circ (f_t)^{-1})|_{f_t(p_0)} \\
&= \lim_{p \rightarrow p_0} \text{Lip}(f_s \circ (f_t)^{-1})|_{f_t(p)} \\
&\leq \lim_{p \rightarrow p_0} \max \left\{ \sqrt{\frac{2s(\cosh \mathcal{G}(p, s) + 1)}{2t(\cosh \mathcal{G}(p, t) + 1)}}, \sqrt{\frac{2s(\cosh \mathcal{G}(p, s) - 1)}{2t(\cosh \mathcal{G}(p, t) - 1)}} \right\} \\
&= \lim_{p \rightarrow p_0} \sqrt{\frac{s \frac{1}{|\nu_s|(p)}}{t \frac{1}{|\nu_t|(p)}}} \quad (\text{by (7.21)}) \\
&= \lim_{p \rightarrow p_0} \sqrt{\frac{s |\nu_t|(p)}{t |\nu_s|(p)}} \\
&\leq \sqrt{\frac{s}{t} \exp(-\epsilon(s-t))} \quad (\text{by (7.22)}) \\
&< \sqrt{\frac{s}{t}}.
\end{aligned}$$

This completes the proof. \square

8. CONVERGENCE OF HARMONIC MAPS RAYS

In this subsection, we complete the proof of Theorem 1.1. To prove the theorem, it will suffice by Theorem 1.7 to prove that the family of harmonic maps $f_t : X_t \rightarrow Y$ converges in the sense of Section 4.2, as X_t degenerates along the harmonic map dual ray $\mathbf{hr}_{Y, \lambda}$. To this end, by Lemma 4.5, it suffices to show that every convergent sequence of the family $\{f_t : X_t \rightarrow Y\}_{t \geq 0}$ shares the same limit. Let $f_{t_n} : X_{t_n} \rightarrow Y$ be an arbitrary convergent sequence with limit surjective harmonic diffeomorphism $f_\infty : X_\infty \rightarrow Y - \lambda_\infty$ for some geodesic lamination λ_∞ .

Lemma 8.1. *With the notations introduced as above, we have $\lambda_\infty = \lambda$.*

Proof. Let P_{R_n, t_n} be the Minsky's polygonal region on X_{t_n} with $R_n \rightarrow \infty, t_n \rightarrow \infty$ as $n \rightarrow \infty$. Then by Theorem 3.5 the complement $f_{t_n}(X - P_{R_n, t_n})$ is contained in an ϵ_n neighbourhood of λ on Y , where $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Hence $\lambda_\infty \subset \lambda$ and $f_\infty(X_\infty) \supset Y - \lambda$. For the other direction, consider the minimal components of λ , denoted by $\lambda^i, \dots, \lambda^k$. For each λ^i , choose a horizontal leaf of $\text{Hopf}(f_{t_n})$ whose image under f_{t_n} is straightened to a leaf contained in λ^i . For n sufficiently large, we see that $\lambda = \lambda^1 \cup \dots \cup \lambda^k$ is also contained in a small neighbourhood of $f_{t_n}(X - P_{R_n, t_n})$. Hence $\lambda_\infty \supset \lambda$. Therefore, $\lambda_\infty = \lambda$. \square

A direct consequence of the above lemma is that X_∞ and $Y - \lambda$ are homeomorphic. Let Y^1, \dots, Y^m be the components of $Y \setminus \lambda$. Let $X_\infty = X^1 \cup X^2 \cup \dots \cup X^m$ with X^i homeomorphic to Y^i . Set $f_i := f_\infty|_{X^i}$.

For each holomorphic or meromorphic quadratic differential Φ , the union of the zeros of Φ and the horizontal critical leaves is called the *critical graph* of Φ .

Lemma 8.2. *The critical graph of the Hopf differential $\text{Hopf}(f^i)$ is connected. Moreover, the complement of the critical graph of $\text{Hopf}(f^i)$ consists of half-infinite cylinders and half-planes corresponding to the boundary of Y^i .*

Proof. Let λ^i be the horizontal measured foliation of $\text{Hopf}(f^i)$.

Claim 1: If λ contains a simple closed α , then the corresponding (maximal) cylinder component A_{t_n} of the horizontal foliation of $\text{Hopf}(f_{t_n})$ limits on two half-infinite cylinders on $\text{Hopf}(f_\infty)$. As $n \rightarrow \infty$, the height of A_{t_n} on X_{t_n} goes to infinity. Combining with the Minsky's estimate, we see that the circumference of A_{t_n} converges to the hyperbolic length of the geodesic representative of α on Y , which is both finite and positive. Therefore the cylinder A_{t_n} converges to two half-infinite cylinders on X_∞ with circumferences equal to the hyperbolic length of the geodesic representative of α on Y .

Claim 2: for each i , the measured foliation λ^i has no compact components (see section 2.8 for definition). Otherwise, suppose that λ^i contains a compact component λ_0^i . Since the horizontal foliation of $\text{Hopf}(f_t)$ is simply $t\lambda$, we see that λ_0^i has to be a component of $t\lambda$. Hence it is a component of λ with transverse measure $\frac{1}{t}$ times the transverse measure of λ_0^i , which goes to zero as t goes to infinity. As a consequence, λ_0^i is not a component of λ , which contradicts that λ_0^i is a component of $t\lambda$.

By claim 2, we see that each component of the complement of the critical graph of $\text{Hopf}(f^i)$ in X^i is either a half-infinite cylinder, a half-plane, or an infinite strip.

Claim 3: the complement of the critical graph of $\text{Hopf}(f^i)$ contains no infinite-strip components (see section 2.8 for definition). Suppose to the contrary that the complement of the critical graph of $\text{Hopf}(f^i)$ contains a component C which is an infinite strip. Let β be a saddle connection in C which connects two zeros z^+, z^- of $\text{Hopf}(f^i)$ belonging to different boundary components of C . Let U be a δ -neighbourhood of β on X^i under the flat metric $|\text{Hopf}(f^i)|$ which contains no zeros of $\text{Hopf}(f^i)$ other than z^\pm . Since flat surface $(X_{t_n}, \text{Hopf}(f_{t_n}))$ converges to $(X_\infty, \text{Hopf}(f_\infty))$, there exists a sequence of maps $\eta_{t_n} : U \rightarrow X_{t_n}$ homeomorphic to the corresponding images such that

- $\eta_{t_n}(z^\pm)$ are zeros of $\text{Hopf}(f_{t_n})$, and
- the pullback metrics $\eta_{t_n}^*(|\text{Hopf}(f_{t_n})|)$ converges to the restriction of $|\text{Hopf}(f^i)|$ to U .

The assumption that U contains no zeros of $\text{Hopf}(f^i)$ other than z^\pm implies that as n goes to infinity, every zero contained in the image $\eta_{t_n}(U)$ would collapse to z^+ or z^- . Since β is a geodesic segment which contains no zeros of $\text{Hopf}(f_\infty)$ in the interior, we may modify η_{t_n} so that the geodesic representative of $\eta_{t_n}(\beta)$ is also a saddle connection for large enough n . Let us fix such a large enough \bar{n} . It then follows from the recurrence property of leaves of measured foliations on closed surfaces that there exists a closed curve γ which is a concatenation of a horizontal leaf segment $\zeta_{t_{\bar{n}}}$ and a subset segment $\xi_{t_{\bar{n}}}$ of $\eta_{t_{\bar{n}}}(\beta)$ centered near the midpoint of $\eta_{t_{\bar{n}}}(\beta)$ and with length $0 < |\xi_{t_{\bar{n}}}| < |\beta|/2$ (that $0 < |\xi_{t_{\bar{n}}}|$ follows from claim 1). In particular, the intersection number of γ and the horizontal measured foliation of $\text{Hopf}(f_{t_{\bar{n}}})$, which is equivalent to $t_{\bar{n}}\lambda$, satisfies:

$$0 < i(\gamma, t_{\bar{n}}\lambda) < |\xi_{t_{\bar{n}}}| < |\beta|/2.$$

Therefore,

$$(8.1) \quad i(\gamma, t_n\lambda) = \frac{t_n}{t_{\bar{n}}} i(\gamma, t_{\bar{n}}\lambda) \rightarrow \infty, \text{ as } n \rightarrow \infty \text{ because } \frac{t_n}{t_{\bar{n}}} \rightarrow \infty.$$

Notice that for each $n > \bar{n}$ the (topological) horizontal foliation of $\text{Hopf}(f_{t_n})$ can be obtained from that of $\text{Hopf}(f_{t_{\bar{n}}})$ by a homeomorphism followed by a sequence

of Whitehead moves. This implies that γ can also be realized on X_{t_n} as a concatenation of a horizontal leaf segment ζ_{t_n} and a subset segment ξ_{t_n} of $\eta_{t_n}(\beta)$, which gives

$$i(\gamma, t_n \lambda) < |\eta_{t_n}(\beta)| \rightarrow |\beta|, \text{ as } n \rightarrow \infty.$$

This contradicts (8.1).

The lemma then follows from claim 1, claim 2, and claim 3. \square

As a direct consequence of Lemma 8.2, we have

Lemma 8.3. *The dual tree of the lift of the horizontal measured foliation of the Hopf differential $\text{Hopf}(f^i)$ to the universal cover consists of exactly one vertex and countably many half-infinite edges corresponding to the boundary of \widetilde{Y}^i .*

Combining Lemma 8.2 and Theorem 3.5, we see that

Lemma 8.4 (Train-track approximation). *For any $\epsilon > 0$, there exists $T_0 > 0$ and $R_0 > 0$ such that for any $t > T_0$ and any $R > R_0$, the train track $\tau_{t,R}$ in Y corresponding to $X_t - \mathcal{P}_R(\Phi_t)$ is an ϵ train track which carries λ .*

Proof. By Lemma 8.2, there exists $T_0 > 0$ and $R_0 > 0$ such that every non-critical horizontal leaf $\mathcal{P}_R(\Phi_t)$ is

- either a closed leave homotopic to one of the boundary component of $\mathcal{P}_R(\Phi_t)$, or
- contained in a maximal leaf which is homotopic *rel* $\partial_v \mathcal{P}_R(\Phi_t)$ to a horizontal boundary leaf of $\mathcal{P}_R(\Phi_t)$, where $\partial_v \mathcal{P}_R(\Phi_t)$ is the vertical boundary segments of $\partial_v \mathcal{P}_R(\Phi_t)$.

In both cases, the image of every non-critical horizontal leaf of $\mathcal{P}_R(\Phi_t)$ is carried by the train-track $\tau_{t,R}$ corresponding to $X_t - \mathcal{P}_R(\Phi_t)$. Therefore, the train track $\tau_{t,R}$ carries λ for all $t > T_0$ and $R > R_0$. \square

Remark 8.5. Note that for Hopf differentials in less restrictive settings than we are considering here, the train track τ in Y corresponding to the complement of \mathcal{P}_R is not sufficient to carry λ , because it may not carry those bi-infinite leaves which are contained in \mathcal{P}_R .

8.1. Proofs of Theorem 1.1 and 1.2.

Proof of Theorem 1.1. Let $X_t = \mathbf{hr}_{Y,\lambda}(t)$. Consider the family of harmonic maps $f_t : X_t \rightarrow Y$. By Lemma 4.5, any sequence of maps $f_{t_n} : X_{t_n} \rightarrow Y$ contains a convergent subsequence. By Lemma 8.3, the dual tree of the horizontal foliation of each component of any limit Hopf differential consists of one vertex and countably many half-infinite edges corresponding to the boundary of $\widetilde{Y} \setminus \widetilde{\lambda}$. By Theorem 5.6, the limit harmonic maps of all convergent sequences of $f_t : X_t \rightarrow Y$ are the same, say $f_\infty : X_\infty \rightarrow Y \setminus \lambda$.

Correspondingly, we can apply the Thurston construction described as ‘‘Step 2’’ in Section 7 to the family of harmonic map rays $\mathbf{HR}_{X_t,Y}$ to show that that family converges to the piecewise harmonic stretch line. To see this, note that for any fixed s , and any limit, say Y_s , of $\mathbf{HR}_{X_\infty,Y}(s)$ of $\mathbf{HR}_{X_t,Y}(s)$ as $t \rightarrow \infty$, we see that the measured foliation orthogonal to λ in the neighbourhood of λ on Y_s is the restriction of the vertical foliation of $s \lim_{t \rightarrow \infty} \text{Hopf}(X_t, Y)$ which is $s \text{Hopf}(X_\infty, Y)$. This orthogonal field to λ is Lipschitz as in ‘‘Step 2’’ because the limiting image of

the horizontal foliation is a geodesic lamination, as required in that step. That the convergence is locally uniform follows from Theorem 1.3. \square

Proof of Theorem 1.2. Let $X_t \in \mathbf{TR}_{Y,\lambda}$ be a point in the Teichmüller ray. By Lemma 4.5, any sequence of maps $f_{t_n} : X_{t_n} \rightarrow Y$ contains a convergent subsequence. By [Gup19, Lemma 4.10], we know that the limit of any convergent subsequence of $f_t : X_t \rightarrow Y$ is a harmonic map $f_\infty : X_\infty \rightarrow Y$ with Hopf differential obtained by attaching half-planes and cylinders to the critical graph of the holomorphic quadratic differential determining the Teichmüller ray. Equivalently, this means that for any pair of convergent subsequences, the dual trees (of the pair of horizontal foliations of the limiting Hopf differentials) coincide. The uniqueness of minimal graph (Theorem 5.6) then implies that $f_t : X_t \rightarrow Y$ converges (instead of subconverges). \square

9. CONVERGENCE TO TEICHMULLER RAYS

Let X be fixed, let $\mathbf{HR}_{X,\Phi}$ be the harmonic ray, let $Y_s = \mathbf{HR}_{X,\Phi}(s)$, and let T_h be the \mathbb{R} -tree dual to the horizontal measured foliation of $\tilde{\Phi}$, the lift of Φ to the universal cover \tilde{X} . Then \tilde{X} is the minimal surface in $\tilde{Y}_s \times (T_h, 2s^{1/2}d)$, also a rescaled minimal surfaces in $s^{-1/2}\tilde{Y}_s \times (T_h, 2d)$. As $s \rightarrow \infty$, \tilde{X} is exactly the minimal surface in $T_v \times T_h$. Let $X_{s,t} \in \mathbf{hr}_{Y_s, \sqrt{s}\lambda}$ be such that $\text{Hor}(\text{Hopf}(X_{s,t} \rightarrow Y_s)) = t\sqrt{s}\lambda$. Then $X_{s,1} \equiv X$ for all $s > 0$. The goal of this section is to prove the following:

Theorem 9.1. *For any fixed X and λ , the family of harmonic map dual rays*

$$\mathbf{hr}_{Y_s, \sqrt{s}\lambda}([1, \infty))$$

locally uniformly converge to the Teichmüller geodesic ray $\mathbf{TR}_{X,\Phi}$ with $\text{Hor}(\Phi) = \lambda$.

Convention. In the remainder of this section, to simplify the notation, we will denote the dual tree $(T_\eta, 2d)$ by T_η .

9.1. Minimal surfaces in the product of trees. Let α and β be a pair of transverse measured foliations. Let T_α and T_β be respectively the dual trees of α and β . Define the energy map $E(\bullet, T_\alpha \times T_\beta) : \mathcal{T}(S) \rightarrow \mathbb{R}$ which associates to $Z \in \mathcal{T}(S)$ the (equivariant) energy of the equivariant harmonic map $\tilde{Z} \rightarrow T_\alpha \times T_\beta$. Let Ψ be the quadratic differential whose horizontal and vertical measured foliations are α and β respectively.

Lemma 9.2. *The function $E(\bullet, T_\alpha \times T_\beta) : \mathcal{T}(S) \rightarrow \mathbb{R}$ is proper. Moreover,*

$$E(Z, T_\alpha \times T_\beta) \geq 4\|\Psi\|,$$

where the equality holds if and only if Z is the underlying Riemann surface of the quadratic differential Ψ .

Proof. Recall that $E(Z, T_\alpha \times T_\beta) = 2\text{Ext}_Z(\alpha) + 2\text{Ext}_Z(\beta)$. The properness then follows from the fact that $\text{Ext}_Z(\alpha) + \text{Ext}_Z(\beta)$ is a proper function over $\mathcal{T}(S)$. Moreover,

$$\begin{aligned} & \text{Ext}_Z(\alpha) + \text{Ext}_Z(\beta) \\ & \geq 2\sqrt{\text{Ext}_Z(\alpha) \cdot \text{Ext}_Z(\beta)} \\ & \geq 2i(\alpha, \beta) \quad (\text{by [GM91, Theorem 5.1]}) \\ & = 2\|\Psi\|, \end{aligned}$$

where the equalities hold if and only if

- $\text{Ext}_Z(\alpha) = \text{Ext}_Z(\beta)$,
- Z is the underlying Riemann surface of Ψ .

□

9.2. Estimating the energy of harmonic maps to trees. For each measured foliation μ , we denote by $E(Z, T_\mu)$ the equivariant energy of the harmonic map from \tilde{Z} to the dual tree of the lift of μ . It is clear that $E(Z, T_\mu) = 2\text{Ext}_Z(\mu)$.

Lemma 9.3. *Let $Y_s \in \mathbf{HR}_{X, \Phi}$. Let $\bar{\lambda}$ be the vertical measured lamination of Φ . Then for any $Z \in \mathcal{T}(S)$, $E(Z, T_{\bar{\lambda}}) \leq E(Z, s^{-1}Y_s)$.*

Proof. we define a family of equivariant projection maps:

$$j_s : s^{-1}\tilde{Y}_s \longrightarrow T_{\bar{\lambda}}$$

as follows. Recall that on the natural coordinates of Φ on X , the hyperbolic metric on Y_s can be written as:

$$f_s^*Y_s = 2s(\cosh \mathcal{G}(z, s) + 1)dx^2 + 2s(\cosh \mathcal{G}(z, s) - 1)dy^2,$$

where $f_s : X \rightarrow Y_s$ is the unique harmonic map. Let $j_s : s^{-1}\tilde{Y}_s \longrightarrow T_{\bar{\lambda}}$ be the projection map along the vertical leaves of $\tilde{\Phi}$, the lift of Φ to \tilde{X} . Then j_s collapses the vertical leaves while it expands the horizontal leaves by a factor of $\frac{1}{\cosh \mathcal{G}(z, s) + 1} < 1$ at $f_s(z) \in \tilde{Y}_s$ (with respect to the hyperbolic metric on \tilde{Y}_s).

Let $F_s : \tilde{Z} \rightarrow s^{-1}\tilde{Y}_s$ be the harmonic map, then

$$E(Z, T_{\bar{\lambda}}) \leq E(j_s \circ F_s) \leq E(F_s) = E(Z, s^{-1}Y_s).$$

□

9.3. Proof of Theorem 9.1.

Proof of Theorem 9.1. Let $\bar{\lambda}$ be the vertical measured lamination of Φ . Let $\Phi_{s,t}$ be the Hopf differential of $X_{s,t} \rightarrow Y_s$. Then the Hopf differential of $\tilde{X}_{s,t} \rightarrow T_{\sqrt{st}\lambda}$ is $-\tilde{\Phi}_{s,t}$, the lift of $-\Phi_{s,t}$. Consider the equivariant harmonic map $\tilde{X}_{s,t} \rightarrow T_{\bar{\lambda}} \times tT_{\lambda}$. By Lemma 9.3, we have

$$\begin{aligned}
& E(X_{s,t}, T_{\bar{\lambda}} \times T_{t\lambda}) \\
&= E(X_{s,t}, T_{\bar{\lambda}}) + E(X_{s,t}, T_{t\lambda}) \\
&\leq E(X_{s,t}, s^{-1}Y_s) + E(X_{s,t}, T_{t\lambda}) \quad (\text{by Lemma 9.3}) \\
&= s^{-1}E(X_{s,t}, Y_s) + s^{-1}E(X_{s,t}, T_{\sqrt{st}\lambda}) \\
&= s^{-1}E(X_{s,t}, Y_s) + 2s^{-1}\|\Phi_{s,t}\| \\
&\leq s^{-1}(\ell_{Y_s}(\sqrt{st}\lambda) + C) + s^{-1}(\ell_{Y_s}(\sqrt{st}\lambda) + C) \quad (\text{by Lemma 3.8}) \\
&= 2s^{-1}(t\ell_{Y_s}(\sqrt{s}\lambda) + C) \\
&\leq 2s^{-1}(2t\|s\Phi\| + tC) + C \quad (\text{by Lemma 3.8}) \\
(9.1) \quad &= 4t\|\Phi\| + 2s^{-1}C(t+1).
\end{aligned}$$

Combined with Lemma 9.2, this implies that for any fixed t , $E(X_{s,t}, T_{\bar{\lambda}} \times T_{t\lambda}) \rightarrow 4t\|\Phi\| = 4i(\bar{\lambda}, t\lambda)$, as $s \rightarrow \infty$. It then follows from Lemma 9.2 that $X_{s,t} \rightarrow \mathbf{TR}_{\Phi}(t)$, as $s \rightarrow \infty$.

To see the locally uniform convergence, consider the function $\psi : [0, \infty) \rightarrow [0, \infty)$ defined as following. Let $X_{\infty, t} \in \mathbf{TR}_{X, \Phi}$ be the Riemann surface underlying the

quadratic differential whose horizontal and vertical measured foliations are $t\lambda$ and $\bar{\lambda}$, respectively. Then for any $Z \in \mathcal{T}(S)$, we have $E(Z, T_{\bar{\lambda}} \times T_{t\lambda}) \geq E(X_{\infty, t}, T_{\bar{\lambda}} \times T_{t\lambda})$. For any $t_0 \geq 1$, let

$$(9.2) \quad \psi_{t_0}(r) = \max_{1 \leq t \leq t_0} \max\{d_T(Z, X_{\infty, t}) : E(Z, T_{\bar{\lambda}} \times T_{t\lambda}) - E(X_{\infty, t}, T_{\bar{\lambda}} \times T_{t\lambda}) \leq r\}.$$

Then ψ is continuous and increasing in r . By (9.1), we see that,

$$\begin{aligned} & E(X_{s, t}, T_{\bar{\lambda}} \times T_{t\lambda}) - E(X_{\infty, t}, T_{\bar{\lambda}} \times T_{t\lambda}) \\ & \leq 2s^{-1}C(t_0 + 1) \rightarrow 0, \text{ as } s \rightarrow \infty. \end{aligned}$$

Then

$$d_T(X_{s, t}, X_{\infty, t}) \leq \psi_{t_0}(2s^{-1}C(t_0 + 1)) \rightarrow 0$$

uniformly in $t \in [1, t_0]$. \square

10. CONVERGENCE TO TEICHMULLER DISKS

In this section, we introduce two models of harmonic dual disks, both of which will converge to Teichmüller disks.

M1. Let $Y_{s, \theta}$ be the hyperbolic surface such that $\text{Hopf}(X, Y_{s, \theta}) = se^{2i\theta}\Phi$. Let

$$\mathbf{HD}_{X, \Phi, s} = \bigcup_{0 \leq \theta \leq \pi} \mathbf{hr}_{Y_{s, \theta}, \lambda_{\theta}}$$

denote the (variable target) harmonic map dual disk.

M2. Let λ_{θ} be the horizontal foliation of $e^{2i\theta}\Phi$. Let

$$\widehat{\mathbf{HD}}_{X, \Phi, s} = \bigcup_{-\pi/2 < \theta < \pi/2} \mathbf{hr}_{Y_s, \lambda_{\theta}}$$

denote the (fixed target) harmonic map dual disk.

In the first version **M1**, the family of terminal points $Y_{s, \theta}$ changes with θ , while in the second version **M2**, there is a single target surface Y_s , and the dependence on θ is only in the lamination.

Let $\mathbf{TD}_{X, \Phi}$ be the Teichmüller disk determined by X and Φ . Let $h : \mathcal{QT}(S) \rightarrow \mathcal{QT}(S)$ be the horocyclic flow on the quadratic differential bundle $\mathcal{QT}(S) \rightarrow \mathcal{T}(S)$ corresponding to the lower triangular matrix

$$\left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} : a \in \mathbb{R} \right\}.$$

The horocycle flow acts in the standard way on \mathbb{R}^2 once we define charts from the Riemann surface to \mathbb{R}^2 by using natural coordinates. Let

$$\mathbf{TD}_{X, \Phi}^h := \bigcup_{a \in \mathbb{R}} h_a(\mathbf{TR}_{X, \Phi})$$

be the half Teichmüller disk which is the union of the horocyclic translates of $\mathbf{TR}_{X, \Phi}$.

Theorem 10.1. $\mathbf{HD}_{X, \Phi, s}$ locally uniformly converges to the Teichmüller disk $\mathbf{TD}_{X, \Phi}$. $\widehat{\mathbf{HD}}_{X, \Phi, s}$ locally uniformly converges to the half Teichmüller disk $\mathbf{TD}_{X, \Phi}^h$.

Before proving Theorem 10.1, we state a lemma we will need, deferring its proof to the end of this section.

Lemma 10.2. *Let $Y_s \in \mathbf{HR}_{X,\Phi}$ and let λ_θ be the horizontal foliation of $e^{2i\theta}\Phi$. Let $\delta > 0$ be a constant which is smaller than half of the shortest distance between zeros of Φ . Then there exists $s_0 > 0$, which depends only on δ , such that for $s > s_0$, we have*

$$\ell_{Y_s}(\lambda_\theta) \leq 2\sqrt{s}i(\lambda_{\pi/2}, \lambda_\theta) + 96(g-1)\delta^2\sqrt{s} + \|\Phi\|.$$

Proof of Theorem 10.1. The proof is almost the same as the proof of Theorem 9.1.

(1) Convergence of $\mathbf{HD}_{X,\Phi,s}$. Let $X_{s,t,\theta}$ be the Riemann surface such that the maximal stretch lamination of the harmonic map $X_{s,t,\theta} \rightarrow Y_{s,\theta}$ is $s^{1/2}t\lambda_\theta$. Let $\Phi_{s,t,\theta}$ be the Hopf differential of $X_{s,t,\theta} \rightarrow Y_{s,\theta}$. Then the Hopf differential of $\tilde{X}_{s,t,\theta} \rightarrow T_{\sqrt{st},\theta}$ is $-\tilde{\Phi}_{s,t,\theta}$, the lift of $-\Phi_{s,t,\theta}$. Therefore,

$$\begin{aligned} & E(X_{s,t,\theta}, T_{\lambda_{\theta+\pi/2}} \times T_{t\lambda_\theta}) \\ &= E(X_{s,t,\theta}, T_{\lambda_{\theta+\pi/2}}) + E(X_{s,t,\theta}, T_{t\lambda_\theta}) \\ &\leq E(X_{s,t,\theta}, s^{-1}Y_{s,\theta}) + E(X_{s,t,\theta}, T_{t\lambda_\theta}) \quad (\text{by Lemma 9.3}) \\ &= s^{-1}E(X_{s,t,\theta}, Y_{s,\theta}) + s^{-1}E(X_{s,t,\theta}, T_{\sqrt{st}\lambda_\theta}) \\ &= s^{-1}E(X_{s,t,\theta}, Y_{s,\theta}) + 2s^{-1}\|\Phi_{s,t,\theta}\| \\ &\leq s^{-1}(\ell_{Y_{s,\theta}}(\sqrt{st}\lambda_\theta) + C) + s^{-1}(\ell_{Y_{s,\theta}}(\sqrt{st}\lambda_\theta) + C) \quad (\text{by Lemma 3.8}) \\ &= 2s^{-1}t\ell_{Y_{s,\theta}}(\sqrt{s}\lambda_\theta) + 2s^{-1}C \\ &\leq 2s^{-1}t(2\|\Phi_{1,s,\theta}\| + C) + 2s^{-1}C \quad \text{by Lemma 3.8} \\ &= 4t\|\Phi_{1,1,\theta}\| + 2s^{-1}C(t+1) \\ &= 4t\|\Phi\| + 2s^{-1}C(t+1). \end{aligned}$$

Therefore, $\lim_{s \rightarrow \infty} E(X_{s,t,\theta}, T_{\lambda_{\theta+\pi/2}} \times T_{t\lambda_\theta}) \leq 4t\|\Phi\| = 4i(\lambda_{\theta+\pi/2}, t\lambda_\theta)$. It then follows from Lemma 9.2 that $X_{s,t,\theta}$ converges to the Riemann surface $X_{\infty,t,\theta} \in \mathbf{TD}_{X,\Phi}$ underlying the quadratic differential $\Psi_{t,\theta}$ whose horizontal and vertical measured foliations are $t\lambda_\theta$ and $\lambda_{\theta+\pi/2}$.

By considering a function similar to the one defined in (9.2), we see that for any fixed $t > 0$, the convergence is uniform in $(t, \theta) \in [1, t_0] \times [0, 2\pi]$.

(2) Convergence of $\widehat{\mathbf{HD}}_{X,\Phi,s}$. Let $\widehat{X}_{s,t,\theta}$ be the Riemann surface such that the maximal stretch lamination of the harmonic map $\widehat{X}_{s,t,\theta} \rightarrow Y_s$ is $s^{1/2}t\lambda_\theta$. Let $\widehat{\Phi}_{s,t,\theta}$ be the Hopf differential of $\widehat{X}_{s,t,\theta} \rightarrow Y_s$. Then

$$\begin{aligned} & E(\widehat{X}_{s,t,\theta}, T_{\lambda_{\pi/2}} \times T_{t\lambda_\theta}) \\ &= E(\widehat{X}_{s,t,\theta}, T_{\lambda_{\pi/2}}) + E(\widehat{X}_{s,t,\theta}, T_{t\lambda_\theta}) \\ &\leq E(\widehat{X}_{s,t,\theta}, s^{-1}Y_s) + E(\widehat{X}_{s,t,\theta}, T_{t\lambda_\theta}) \quad (\text{by Lemma 9.3}) \\ &= s^{-1}E(\widehat{X}_{s,t,\theta}, Y_s) + s^{-1}E(\widehat{X}_{s,t,\theta}, T_{\sqrt{st}\lambda_\theta}) \\ &= s^{-1}E(\widehat{X}_{s,t,\theta}, Y_s) + 2s^{-1}\|\widehat{\Phi}_{s,t,\theta}\| \\ &\leq s^{-1}(\ell_{Y_s}(\sqrt{st}\lambda_\theta) + C) + s^{-1}(\ell_{Y_s}(\sqrt{st}\lambda_\theta) + C) \quad (\text{by Lemma 3.8}) \\ &= 2s^{-1/2}t\ell_{Y_s}(\lambda_\theta) + 2s^{-1}C(t+1). \end{aligned}$$

Combined with Lemma 10.2, this implies that for any δ which is smaller than half of the shortest distance between zeros of Φ , there exists $s_0(\delta) > 0$, which depends

only on δ , such that for $s > s_0$, we have

$$E(\widehat{X}_{s,t,\theta}, T_{\lambda_{\pi/2}} \times T_{t\lambda_\theta}) \leq 4i(\lambda_{\pi/2}, t\lambda_\theta) + 192(g-1)\delta^2 t + 2ts^{-1/2}\|\Phi\| + 2s^{-1}C(t+1).$$

Combining Lemma 9.2 and the arbitrariness of δ , we see that as $s \rightarrow \infty$, we have that $E(\widehat{X}_{s,t,\theta}, T_{\lambda_{\pi/2}} \times T_{t\lambda_\theta})$ converges to $4i(\lambda_{\pi/2}, t\lambda_\theta)$ uniformly in $(t, \theta) \in [1, t_0] \times (-\pi/2, \pi/2)$. Let $\widehat{X}_{\infty,t,\theta} \in \mathbf{TD}_{X,\Phi}$ be the Riemann surface underlying the quadratic differential $\widehat{\Phi}_{t,\theta}$ whose horizontal and vertical measured foliations are $t\lambda_\theta$ and $\lambda_{\pi/2}$. It then follows from Lemma 9.2 that $\widehat{X}_{s,t,\theta}$ converge to $\widehat{X}_{\infty,t,\theta} \in \mathbf{TD}_{X,\Phi}$. By considering a function similar to the one defined in (9.2), we see that for any fixed $t_0 > 0$, the convergence is uniform in $(t, \theta) \in [1, t_0] \times (-\pi/2, 2\pi/2)$. \square

Proof of Lemma 10.2. Let $\delta > 0$ be a fixed constant which is smaller than half of the shortest distance between zeros of Φ . For a zero z_j of $e^{2i\theta}\Phi$ which is of order n_j , let \mathcal{P}_j be a horizontal-vertical $(2n_j)$ -polygon of $e^{2i\theta}\Phi$ around z_j such that every horizontal segment and every vertical segment of $\partial\mathcal{P}$ have the same $|\Phi|$ -length 2δ . In particular, the $|\Phi|$ -distance from z_j to $\partial\mathcal{P}_j$ is δ . Next, we decompose the horizontal foliation F_θ of $e^{2i\theta}\Phi$ outside the union $\cup_j \mathcal{P}_j$ into rectangles \mathcal{R}_i . Therefore,

$$F_\theta = (\cup_i (F_\theta \cap \mathcal{R}_i)) \cup (\cup_j (F_\theta \cap \mathcal{P}_j)).$$

The hyperbolic length of leaves of $(F_\theta \cap \mathcal{P}_j)$ on Y_s is not convenient to estimate. To overcome this, we need a modification. Notice that the critical leaves of $F_\theta \cap \mathcal{P}_j$ decompose \mathcal{P}_j into several rectangles $\{R_{jk}\}_{1 \leq k \leq n_j}$, where n_j is the order of the zero z_j of Φ . We homotope (relative to its endpoints) each non-critical leaf L of $F_\theta \cap \mathcal{P}_i$ to a curve L' which is contained in the boundary $\partial R_{ji} \cap \partial \mathcal{P}_i$. Equivalently, $L \cup L'$ is the boundary of the component of $R_{ji} \setminus L$ whose closure does not contain the zero z_j of Φ . Then the lengths of L' and $\partial R_{ji} \cap \partial \mathcal{P}_j$ satisfy

$$\text{Length}_{Y_s}(L') \leq \text{Length}_{Y_s}(\partial R_{ji} \cap \partial \mathcal{P}_j)$$

where we use the fact that $L' \subset \partial R_{ji} \cap \partial \mathcal{P}_i$.

Let $\text{Length}_{Y_s}(F_\theta \cap \mathcal{R}_i)$ be the hyperbolic length of the leaves of the foliation F_θ restricted to \mathcal{R}_i , integrated over the induced transverse measure of the family of leaves. By definition, $\ell_{Y_s}(\lambda_\theta)$ is the Y_s -length of geodesic representatives of leaves of F_θ , integrated over the induced transverse measure of family of leaves. Hence,

$$\begin{aligned} \ell_{Y_s}(\lambda_\theta) &\leq \sum_i \text{Length}_{Y_s}(F_\theta \cap \mathcal{R}_i) + \sum_j \text{Length}_{Y_s}(F_\theta \cap \mathcal{P}_j) \\ &= \sum_i \text{Length}_{Y_s}(F_\theta \cap \mathcal{R}_i) + \sum_j \sum_{1 \leq k \leq n_j} \text{Length}_{Y_s}(F_\theta \cap R_{jk}) \\ &\leq \sum_i \text{Length}_{Y_s}(F_\theta \cap \mathcal{R}_i) + \sum_j \delta \sum_{1 \leq k \leq n_j} \text{Length}_{Y_s}(\partial R_{jk} \cap \partial \mathcal{P}_j) \\ (10.1) \quad &= \sum_i \text{Length}_{Y_s}(F_\theta \cap \mathcal{R}_i) + \delta \sum_j \text{Length}_{Y_s}(\partial \mathcal{P}_j). \end{aligned}$$

Recall the with respect to the canonical coordinate of Φ , the hyperbolic metric on Y_s can be expressed as:

$$\begin{aligned} ds_Y^2 &= 2s(\cosh \mathcal{G}(z, s) + 1)dx^2 + 2s(\cosh \mathcal{G}(z, s) - 1)dy^2 \\ (10.2) \quad &= 4sdx^2 + 2s(\cosh \mathcal{G}(z, s) - 1)(dx^2 + dy^2), \end{aligned}$$

where $\mathcal{G}(z, s) = \log(1/|\nu(z, s)|)$ and $\nu(z, s)$ is the Beltrami differential of the harmonic map $X \rightarrow Y_s$. By Lemma 3.3, we see that there exists $\mathbf{s}_0 > 1$, such that for every $s > \mathbf{s}_0$ and every $z \in X \setminus (\cup_j \mathcal{P}_j)$, we have

$$(10.3) \quad 2s(\cosh \mathcal{G}(z, s) - 1) < 1.$$

Let w_i and h_i be respectively of the horizontal width and vertical height of \mathcal{R}_i with respect to $e^{2i\theta}\Phi$. Combining (10.2) and (10.3), we see that

$$(10.4) \quad \begin{aligned} \text{Length}_{Y_s}(F_\theta \cap \mathcal{R}_i) &\leq (2\sqrt{s}w_i \cos \theta + w_i) h_i \\ &= 2\sqrt{s}i(F_{\pi/2} \cap \mathcal{R}_i, F_\theta \cap \mathcal{R}_i) + w_i h_i \\ &= 2\sqrt{s}i(F_{\pi/2} \cap \mathcal{R}_i, F_\theta \cap \mathcal{R}_i) + \|\Phi\|_{\mathcal{R}_i}. \end{aligned}$$

Recall that $\partial \mathcal{P}_j$ consists of n_j horizontal arcs and n_j vertical arcs with respect to $e^{2i\theta}\Phi$. Moreover, every such arc has the same $|\Phi|$ -length δ . Hence by (10.2) and (10.3),

$$(10.5) \quad \begin{aligned} \text{Length}_{Y_s}(\partial \mathcal{P}_j) &\leq n_j(2\sqrt{s}\delta \cos \theta + 2\delta) + n_j(2\sqrt{s}\delta \sin \theta + 2\delta) \\ &\leq 8n_j\sqrt{s}\delta. \end{aligned}$$

Combining (10.1), (10.4), and (10.5), we have

$$\begin{aligned} \ell_{Y_s}(\lambda_\theta) &\leq 2\sqrt{s}i(F_{\pi/2}, F_\theta) + \|\Phi\| + \sum_j 8n_j\delta^2\sqrt{s} \\ &\leq 2\sqrt{s}i(F_{\pi/2}, F_\theta) + \|\Phi\| + 96(g-1)\delta^2\sqrt{s}, \end{aligned}$$

where the last inequality follows from the fact $\sum_j n_j \leq 12(g-1)$. \square

11. CONVERGENCE TO STRETCH-EARTHQUAKE DISKS

Definition 11.1 (Stretch-earthquake disk). Let $Y \in \mathcal{T}(S)$ be a hyperbolic surface and $\lambda \in \mathcal{ML}(S)$ a measured foliation/lamination. Let $\mathbf{SR}_{Y,\lambda}$ be the stretch line defined by Y and λ (where we recall that after Theorem 1.1 we extended the definition of these rays to include the limit of harmonic map rays $\mathbf{HR}_{X_t,Y}$ where X_t degenerates along the harmonic map dual ray $\mathbf{hr}_{Y,\lambda}$). Let $\mathcal{E}_{t\lambda}(Y)$ be the surface obtained from Y by acting by an earthquake along $t\lambda$. Define the stretch-earthquake disk $\mathbf{SED}(Y, \lambda)$ of (Y, λ) to be the set:

$$\bigcup_{0 < t < +\infty} \mathbf{SR}_{\mathcal{E}_{t\lambda}(Y), \lambda}(0, +\infty).$$

Definition 11.2 (Hopf differential disk). Define the Hopf differential disk $\mathbf{HDD}(X, \Phi)$ of (X, Φ) to be the set

$$\bigcup_{-\pi \leq \theta \leq \pi} \mathbf{HR}_{X, e^{i\theta}\Phi}(0, \infty).$$

Let $X_t \in \mathbf{hr}_{Y,\lambda}$ be the Riemann surface such that the horizontal foliation of $\text{Hopf}(X_t, Y) = \Phi_t$ is $t\lambda$. Let $Y(t, r, s)$ be the hyperbolic surface such that $\text{Hopf}(X_t, Y(t, r, s)) = re^{t\frac{\theta}{2r}}\Phi_t$ and $Y_r = \mathbf{SR}_{Y,\lambda}(r) \in \mathbf{SR}_{Y,\lambda}$.

Theorem 11.3. *Let $Y \in \mathcal{T}(S)$ be a hyperbolic surface and $\lambda \in \mathcal{ML}(S)$ a measured foliation/lamination. Let $X_t \in \mathbf{hr}_{Y,\lambda}$ be as described above.*

Then the family of Hopf differential disks $(\mathbf{HDD}(X_t, \Phi_t), Y)$ with base point Y locally uniformly converges to the stretch-earthquake disk $(\mathbf{SED}(Y, \lambda), Y)$ with base

point Y . Namely, for any prescribed $\mathbf{s} > 0$ and $0 < \mathbf{r} < \mathbf{r}'$, the point $Y(t, r, s)$ converges to $\mathcal{E}_{s\lambda}(Y_r)$, uniformly in $(r, s) \in [\mathbf{r}, \mathbf{r}'] \times [-\mathbf{s}, \mathbf{s}]$, as $t \rightarrow \infty$.

Recall that the family of harmonic maps $f_t : X_t \rightarrow Y$ converges to the harmonic map $f_\infty : X_\infty \rightarrow Y$ with $\text{Hopf}(f_\infty) = \Phi_\infty$, where Φ_∞ is the union of half-infinite cylinders and half-planes.

The proof of Theorem 11.3 relies on a generalization of shearing coordinates of $\mathcal{T}(S)$, namely the shear-shape coordinates of $\mathcal{T}(S)$ developed by Calderon-Farre ([CF21]).

11.1. Shear-shape coordinates of the Teichmüller space. Let λ be a geodesic lamination. The shear-shape coordinates of $\mathcal{T}(S)$ comprise two parts: the shape of complement of λ and the shearing data near λ . We sketch some details.

11.1.1. *The orthogeodesic foliations and weighted arc system.* Let $Y \in \mathcal{T}(S)$ be a hyperbolic surface and λ a geodesic lamination. The *orthogeodesic foliation* $\mathcal{O}_\lambda(Y)$ is the foliation of Y whose leaves are fibers of the nearest point projection to λ . For each $x \in Y$, let $\text{val}(x)$ be the number of points on λ realizing the distance $d(x, \lambda)$. Let $\text{Sp}(Y)$ be the set of points of Y with valence at least 2. The spine $\text{Sp}(Y)$ is a properly embedded, piecewise geodesic 1-complex, possibly with some vertices of valence 1 removed if λ is not a multicurve. Let $\pi : Y \setminus \lambda \rightarrow \text{Sp}(Y)$ be the retraction along the orthogeodesic foliation. Let $\text{Sp}_2(Y) \subset \text{Sp}(Y)$ be the set of points of Y with valence 2. For x and y in the same component of $\text{Sp}_2(Y)$, the leaves $\pi^{-1}(x)$ and $\pi^{-1}(y)$ are freely homotopic on $Y \setminus \lambda$. This gives an isotopy class of arc denoted by α_e for each component e of $\text{Sp}_2(Y)$. Let λ_e and λ'_e be the two leaves of λ which intersect $\pi^{-1}(e)$. If e is precompact on Y then there is a unique geodesic representative in the isotopy class α_e which connects the two leaves λ_e and λ'_e orthogonally. For convenience, we denote this geodesic representative also by α_e . The union

$$\underline{\alpha}(Y \setminus \lambda) = \bigcup_e \alpha_e$$

is called the *dual arc system* of Y with respect to λ , where e ranges over the precompact components of $\text{Sp}_2(Y)$. For each precompact component e of $\text{Sp}_2(Y)$, we associate the length c_e of either component of $\pi^{-1}(e) \cap \lambda$. Define the weighted dual arc system as the formal sum:

$$\underline{A}(Y \setminus \lambda) := \sum_e c_e \alpha_e,$$

where e ranges over all precompact components of $\text{Sp}_2(Y)$. The weighted arc system $\underline{A}(Y \setminus \lambda)$ uniquely determines the crowned hyperbolic surface $Y \setminus \lambda$.

An arc system on $Y \setminus \lambda$ is said to be *filling* if it cuts $Y \setminus \lambda$ into topological discs. The *weighted filling arc complex* $|\mathcal{A}_{\text{fill}}(Y \setminus \lambda, \lambda)|_{\mathbb{R}}$ of $Y \setminus \lambda$ *rel boundary* is the set of all weighted arc systems: $\underline{A} = \sum c_i \alpha_i$ where $\underline{\alpha} = \cup \alpha_i$ is a filling arc system and $c_i > 0$. Let $\mathcal{B}(Y \setminus \lambda) \subset |\mathcal{A}_{\text{fill}}(Y \setminus \lambda, \lambda)|_{\mathbb{R}}$ be the subset of weighted filling arc systems satisfying some residue conditions (we refer to [CF21, Section 7.3] for precise description of residue conditions).

11.1.2. *Shear-shape cocycle.* The weighted arc system uniquely determines the shape of $Y \setminus \lambda$. We next describe how to glue these pieces together along λ .

Definition 11.4 ([CF21], Definition 7.2). Let $\lambda \in \mathcal{ML}(S)$. A *shear-shape cocycle* for λ is a pair (σ, \underline{A}) where \underline{A} is a weighted filling arc system $\underline{A} = \sum c_i \alpha_i \in \mathcal{B}(Y \setminus \lambda)$ and σ is a function which assigns to every arc k transverse to λ and disjoint from $\underline{\alpha} := \cup \alpha_i$ a real number $\sigma(k)$, satisfying the following axioms:

- (1) (support): If k does not intersect λ , then $\sigma(k) = 0$.
- (2) (transverse invariance): If k and k' are isotopic transverse to λ and disjoint from $\underline{\alpha}$ then $\sigma(k) = \sigma(k')$.
- (3) (finite additivity): If $k = k_1 \cup k_2$ where k_i have disjoint interiors then $\sigma(k) = \sigma(k_1) + \sigma(k_2)$.
- (4) (\underline{A} -compatibility): Suppose that k is transverse to λ and isotopic rel endpoints to some arc which may be written as $t_i \cup l$, where t_i is a standard transversal to $\underline{\alpha}$ and l is disjoint from $\underline{\alpha}$. Then the loop $k \cup t_i \cup l$ encircles a unique point p of $\lambda \cap \underline{\alpha}$, and

$$\sigma(k) = \sigma(l) + \epsilon c_i$$

where ϵ denotes the winding number of $k \cup t_i \cup l$ about p (where the loop is oriented so that the edges are traversed k then t_i then l).

Let $\mathcal{SH}(\lambda)$ be the set of shear-shape cocycles of λ . Let $\mathcal{SH}^+(\lambda) \subset \mathcal{SH}(\lambda)$ be the subset of positive shear-shape cocycles (we refer to [CF21, Definition 8.4] for the definition of positive shear-shape cocycles). There is a natural way to associate to a hyperbolic surface a shear-shape cocycle, the *geometric shear-shape cocycle*. Moreover,

Theorem 11.5 ([CF21], Theorem 12.1). *The map $\sigma_\lambda : \mathcal{T}(S) \rightarrow \mathcal{SH}^+(\lambda)$ that associates to a hyperbolic surface its geometric shear-shape cocycle is a stratified real-analytic homeomorphism.*

We now sketch the construction of the geometric shear-shape cocycles. Notice that for a hyperbolic surface $Y \in \mathcal{T}(S)$, the geodesic arcs $\underline{\alpha}(Y) = \cup_e \alpha_e$ decompose $Y \setminus \lambda$ into geodesic polygons each of which contains exactly one singularity of the orthogeodesic foliation $\mathcal{O}_\lambda(Y)$. Let \mathcal{H} be the set of such polygons, which is also the set of singularities of $\mathcal{O}_\lambda(X)$. According to Calderon-Farre, elements of \mathcal{H} are called *hexagons*, no matter their shapes. Consider the universal cover \tilde{Y} of Y . Let $\tilde{\lambda}$, $\widetilde{\mathcal{O}_\lambda(Y)}$, and $\widetilde{\mathcal{H}}$ be respectively the lifts of λ , $\mathcal{O}_\lambda(Y)$, \mathcal{H} .

We now define a family of base points associated to the boundary leaves $\partial(\tilde{Y} \setminus \tilde{\lambda})$ of $\tilde{\lambda}$. Let $H_v \in \widetilde{\mathcal{H}}$ be a hexagon with a singular point v of $\widetilde{\mathcal{O}_\lambda(Y)}$. For a boundary leaf g of $\tilde{\lambda}$ intersecting ∂H_v , define p_v to be the projection of v to g along $\widetilde{\mathcal{O}_\lambda(Y)}$. The pair (g, p_v) is called a *pointed geodesic*. For any pair of hexagons $H_v, H_w \in \widetilde{\mathcal{H}}$ belonging to different components of $\tilde{Y} \setminus \tilde{\lambda}$, there is a unique geodesic g_v^w intersecting ∂H_v that separates v from w . Symmetrically, there is a unique geodesic g_w^v intersecting ∂H_w that separates w from v . For sufficiently close H_v and H_w , the critical leaf L_v of $\widetilde{\mathcal{O}_\lambda(Y)}$ containing v meets g_w^v in some point, say F_w^v . Then the shear $\sigma_\lambda(v, w)$ among H_v and H_w is defined to be the signed distance from $F_w^v \in g_w^v$ to $p_w \in g_w^v$, which is positive if p_w sits to the right side of L_v oriented from v to F_w^v and negative otherwise.

Let τ be an ϵ train track of λ , let $\tau_{\underline{\alpha}}$ be a *standard smoothing*¹ train track of $\tau \cup \underline{\alpha}(Y)$. The shears among nearby hexagons defined as above define a weighted system of τ as following:

- To each branch corresponding to a component α of $\underline{\alpha}$, assign the weight to be $i(\mathcal{O}_\lambda(Y), e_\alpha)$ where e_α is the edge of $\text{Sp}(Y)$ dual to α .
- To each branch b which is not a component of $\underline{\alpha}$, choose a lift \tilde{b} of b . Let $H_v, H_w \in \mathcal{H}$ be the pair of hexagons adjacent to \tilde{b} and set the weight to be $\sigma_\lambda(Y)(v, w)$.

This weight system on $\tau_{\underline{\alpha}}$ defines a positive shear-shape cocycle $\sigma_\lambda(Y)$, which is independent of the choice of τ . Moreover, those finite collection of pairs of hexagons gives a local coordinate of $\sigma_\lambda(\mathcal{T}(S)) = \mathcal{SH}^+(\lambda)$. This local coordinatization will be used later in the proof of Lemma 11.6. (We refer to Section 13 of [CF21] for a detailed description.)

Shear-shape coordinates. The proof of Theorem 11.3 is divided into two lemmas.

Lemma 11.6 (shape convergence). *$Y(t, r, s) \setminus \lambda$ converges to $Y_r \setminus \lambda$, locally uniformly in $(r, s) \in [\mathbf{r}, \mathbf{r}'] \times [-\mathbf{s}, \mathbf{s}]$, as $t \rightarrow \infty$.*

Proof. Recall that $\text{Hopf}(X_t, Y(t, r, s)) = e^{i\frac{s}{2t}} \text{Hopf}(X_t, Y(t, r, 0)) = r e^{i\frac{s}{2t}} \Phi_t$, where $\Phi_t = \text{Hopf}(X_t, Y)$. By (7.3, 7.6), we see that the energy density \mathbf{e} , which is defined to be $\mathcal{H} + \mathcal{L}$ in the notations introduced in (7.3, 7.6), satisfies:

$$\mathbf{e}(X_t, Y(t, r, s)) = \mathbf{e}(X_t, Y(t, r, 0)).$$

The pullback of the hyperbolic metric $\rho(t, r, s)$ of $Y(t, r, s)$ to X_t via the harmonic map $f_{t,r,s} : X_t \rightarrow Y(t, r, s)$ is:

$$(f_{t,r,s})^* \rho(t, r, s) = r e^{i\frac{s}{2t}} \Phi_t + \overline{r e^{i\frac{s}{2t}} \Phi_t} + \sigma_t \mathbf{e}(X_t, Y(t, r, s)),$$

where σ_t is the singular flat metric on X_t induced by Φ_t . This gives

$$(f_{t,r,s})^* \rho(t, r, s) - (f_{t,r,0})^* \rho(t, r, 0) = r(e^{i\frac{s}{2t}} - 1) \Phi_t + \overline{r(e^{i\frac{s}{2t}} - 1) \Phi_t}.$$

Combining this with the fact that Φ_t converges to Φ_∞ , uniformly on compact subsets of $X_\infty \xrightarrow{\text{homeo}} Y - \lambda$, as $t \rightarrow \infty$, we see that on any compact subset of $Y - \lambda$, we have $Y(t, r, s)$ converges to $Y(t, r, 0)$ uniformly in $(r, s) \in [\mathbf{r}, \mathbf{r}'] \times [-\mathbf{s}, \mathbf{s}]$ as $t \rightarrow \infty$. □

Lemma 11.7 (shear convergence). *The transverse cocycles $\sigma_\lambda(Y(t, r, s))$ converge to $\sigma_\lambda(\mathcal{E}_{s,\lambda}(Y_t))$ uniformly in $(r, s) \in [\mathbf{r}, \mathbf{r}'] \times [-\mathbf{s}, \mathbf{s}]$ as $t \rightarrow \infty$.*

Proof. Notice that the set of dual arc systems $\{\underline{\alpha}(Y_r - \lambda) : r \in [\mathbf{r}, \mathbf{r}']\}$ is a finite set. The locally uniform convergence of $Y(t, r, s)$ to Y_r in $(r, s) \in [\mathbf{r}, \mathbf{r}'] \times [-\mathbf{s}, \mathbf{s}]$ as $t \rightarrow \infty$ then implies that

$$\{\underline{\alpha}(Y(t, r, s)) : t > T_0, r \in [\mathbf{r}, \mathbf{r}'], s \in [-\mathbf{s}, \mathbf{s}]\}$$

¹The orientation of Y induces an orientation for each component of $Y \setminus \tau$, which in turn gives an orientation to the boundary of each component. A *standard smoothing* is a smoothing at each intersection point $\tau \cap \underline{\alpha}(Y)$ so that the incoming tangent vector corresponding to $\underline{\alpha}Y$ points in the positive direction with respect to the boundary orientation of $Y \setminus \tau$. see [CF21, Section 9.1] for the definition of standard smoothing.

is also finite. Therefore, to prove the lemma, we may assume that the set above contains exactly one element, say the dual arc system $\underline{\alpha}$.

Next, we calculate the transverse cocycle $\sigma_\lambda(Y(t, r, s))$ of $Y(t, r, s)$ with respect to λ , using the train-track coordinates of transverse cocycles. We begin with a preliminary discussion in the setting where $s = 0$, and then extend the analysis to the case where $s \neq 0$.

By Lemma 8.4, for any $\epsilon > 0$, there exists $T_0 > 0$ and $R_0 > 0$ such that for any $t > T_0$ and any $R > R_0$, the train track $\tau_{r,R}$ in Y corresponding to $X_t - \mathcal{P}_R(\Phi_t)$ is an ϵ train track which carries λ . Lift everything to the universal cover. Together with the dual arc system $\underline{\alpha}$, this train-track gives a finite coordinatization of $\sigma_\lambda(Y(t, r, s))$, which consists of a finite collection of pairs of (sufficiently close) pointed geodesics, denoted by \mathcal{G} . Let $((g_1, p_1), (g_2, p_2)) \in \mathcal{G}$ be such a pair with $p_i \in g_i$ being the marked point. Let R_t be a divergent sequence of positive numbers. For $t > T_0$, let $k_{12} = k_{12}(t)$ be a Φ_t -polygonal curve on \tilde{Y} with endpoints p_1 and p_2 ; here we may choose $k_{12}(t)$ in the complement of $\mathcal{P}_{R_t}(X_t)$ as λ is in the complement of the image of $\mathcal{P}_{R_t}(X_t)$. Then, from Theorem 3.5, because the leaves of λ are well-approximated by images of Φ_t -horizontal arcs, with distances along λ well-approximated by $4\Phi_t$ -horizontal lengths (outside of the polygonal region $\mathcal{P}_{R_t}(X_t)$ for large enough R_t), we may estimate that

$$\begin{aligned} \sigma_\lambda(Y(t, r, 0))((g_1, p_1), (g_2, p_2)) &= \sqrt{r}\sigma_\lambda(Y(t, 0, 0))(k_{12}) \\ &= 2\sqrt{r}i(k_{12}, \text{Vert}(\Phi_t))(1 + O(\exp(-brR_t))), \end{aligned}$$

where $\epsilon_i \in \{\pm 1\}$ and b is the constant from Theorem 3.5.

We now consider the effect of a non-zero ‘‘rotation factor’’ $s \neq 0$. Notice that for any fixed $s > 0$ (and as usual, for t sufficiently large), the complement $X_t - \mathcal{P}_{R_t}(e^{is/2t}\Phi_t)$ contains $X_t - \mathcal{P}_{2R_t}(\Phi_t)$ for $-s \leq s \leq s$. Combining with the fact that the image of $X_t - \mathcal{P}_{2R_t}(\Phi_t)$ under the harmonic map $X_t \rightarrow Y(t, r, s)$ is an ϵ -train track approximation of $\text{Hor}(\Phi_t) = t\lambda$, this implies that the image of $X_t - \mathcal{P}_{R_t}(e^{is/2t}\Phi_t)$ under the harmonic map $X_t \rightarrow Y(t, r, s)$ is also an ϵ -train track carrying λ . Therefore,

$$\begin{aligned} &\sigma_\lambda(Y(t, r, s))((g_1, p_1), (g_2, p_2)) \\ &= \sigma_\lambda(Y(t, 0, s), k_{12}) \\ &= 2\sqrt{r} \left(i(k_{12}, \text{Vert}(\Phi_t)) \cos \frac{s}{2t} + ti(k_{12}, \text{Hor}(\Phi_t)) \sin \frac{s}{2t} \right) (1 + O(\exp(-brR_t))), \end{aligned}$$

which converges to $2\sqrt{r}i(k_{12}, \text{Vert}(\Phi_t)) + s\sqrt{r}i(k_{12}, \text{Hor}(\Phi_t))$ as $t \rightarrow \infty$. This means that

$$\sigma_\lambda(Y(t, r, s)) \rightarrow \mathcal{E}_{s\lambda}(\sigma_\lambda(Y_r))$$

as $t \rightarrow \infty$. The finiteness of the pairs of oriented geodesics then imply that the convergence is locally uniform in $(r, s) \in [\mathbf{r}, \mathbf{r}'] \times [-\mathbf{s}, \mathbf{s}]$. □

Proof of Theorem 11.3. The theorem now follows from Lemma 11.6 and Lemma 11.7. □

12. HARMONIC-STRETCH LINES BETWEEN HYPERBOLIC SURFACES

The goal of this section is, in outline, to find for every ordered pair (Y, Z) of distinct hyperbolic surfaces in $\mathcal{T}(S)$, a harmonic stretch line proceeding from Y

to Z , i.e. a unique Thurston geodesic proceeding from Y to Z determined only by Y and Z , together with an extra side condition resulting from a requirement of minimizing a variational quantity. The main result is stated in Theorem 1.8 which characterizes those stretch lines in terms of “admissible triples”. We give that definition in the first subsection before proceeding to state and prove Theorem 1.8 in the following subsections.

12.1. Harmonic stretch lines and admissible triples. Recall that (Theorem 1.7) a piecewise harmonic stretch line is constructed from a closed hyperbolic surface U , a geodesic lamination λ on U , and a surjective harmonic diffeomorphism $X \rightarrow U \setminus \lambda$ from some (possibly disconnected) punctured Riemann surface.

Definition 12.1. We say that a piecewise harmonic stretch line is a *harmonic stretch line*, if it is a limit of a sequence of harmonic map rays.

Let Y and Z be two fixed hyperbolic surfaces in $\mathcal{T}(S)$. Let λ be the maximally stretched chain-recurrent lamination from Y to Z (Section 2.6.1). We will write, for example, $L^{-2}Z$ to indicate the constant curvature metric obtained by scaling the hyperbolic metric Z by the constant L^{-2} .

Definition 12.2. A triple (X, f, h) is said to be *admissible* if the following hold:

- (i) X is a punctured Riemann surface (possibly disconnected);
- (ii) $f : X \rightarrow Y \setminus \lambda$ and $h : X \rightarrow L^{-2}Z \setminus \lambda$ are surjective harmonic diffeomorphisms with meromorphic Hopf differentials satisfying $\text{Hopf}(f) = \text{Hopf}(h)$;
- (iii) the piecewise harmonic stretch line defined by $f : X \rightarrow Y \setminus \lambda$ passes through Z ;
- (iv) the piecewise harmonic stretch line defined by $f : X \rightarrow Y \setminus \lambda$ is a harmonic stretch line.

Remark 12.3. We emphasize that the final condition in the definition requires that (.cf Definition 12.1) that the piecewise harmonic stretch line is a *limit* of harmonic map rays.

By Theorem 1.7 and the construction of piecewise harmonic stretch lines, we see that $h \circ f^{-1}$ extends to a 1-Lipschitz homeomorphism from Y to $L^{-2}Z$.

In section 13, we will extend this definition to the case where the terminal surface Z is replaced by an \mathbb{R} -tree. (Naturally, we will have to lift all of the definitions to spaces on which $\pi_1(S)$ acts equivariantly by automorphisms.) See Definition 13.4 for that analogous construction.

12.2. Harmonic Stretch lines between points in Teichmüller space. With these definitions in hand, we may now state our main goal for this section, a restatement of Theorem 1.8.

Theorem 12.4. *For any two distinct hyperbolic surfaces $Y, Z \in \mathcal{T}(S)$, there is a unique harmonic stretch line proceeding from Y to Z .*

The remainder of this section is devoted to proving Theorem 12.4.

12.3. Equivalent admissible triples. We start with the following definition.

Definition 12.5. Two admissible triples (X, f, h) and $(\hat{X}, \hat{f}, \hat{h})$ of (Y, Z) are said to be *equivalent* if there exists a conformal map $\eta : X \rightarrow \hat{X}$ such that $f = \hat{f} \circ \eta$.

Lemma 12.6. *Let (X, f, h) and $(\hat{X}, \hat{f}, \hat{h})$ be two equivalent admissible triples of (Y, Z) . Then the harmonic stretch lines defined by them coincide.*

Proof. Let $\mathbf{HSR} = \mathbf{HSR}(t)$ be the harmonic stretch line defined by (X, f, h) such that the harmonic map $f_t : X \rightarrow \mathbf{HSR}(t) \setminus \lambda$ has Hopf differential $\text{Hopf}(f_t) = t\text{Hopf}(f)$. In particular $\mathbf{HSR}(1) = Y$. Moreover, by Remark 12.3, we see that $Z = \mathbf{HSR}(L^2)$. Define $\widehat{\mathbf{HSR}}(t)$ and \hat{f}_t similarly.

The assumption that (X, f, h) and $(\hat{X}, \hat{f}, \hat{h})$ are equivalent implies that the composition map $\eta := \hat{f}^{-1} \circ f$ from X to \hat{X} is conformal. Hence $\hat{f}_t \circ \eta : X \rightarrow \widehat{\mathbf{HSR}}(t) \setminus \lambda$ is also harmonic. Moreover,

$$\begin{aligned} \text{Hopf}(\hat{f}_t \circ \eta) &= \eta^*(\text{Hopf}(\hat{f}_t)) = \eta^*(t\text{Hopf}(\hat{f})) \\ &= t\text{Hopf}(\hat{f} \circ \eta) = t\text{Hopf}(f) = \text{Hopf}(f_t). \end{aligned}$$

By Theorem 7.1, there is a unique crowned hyperbolic surfaces with prescribed Hopf differential on X . In particular, $\widehat{\mathbf{HSR}}(t) = \mathbf{HSR}(t)$ holds for all t . This completes the proof. \square

The following uniqueness result is key to the proof of Theorem 12.4.

Proposition 12.7. *Let $(Y, Z) \in \mathcal{T}(S) \times \mathcal{T}(S)$ be a pair of distinct hyperbolic surfaces. Then all admissible triples of (Y, Z) are equivalent.*

We prove this proposition in subsection 12.6.

12.4. Energy difference. Note that both $f : X \rightarrow Y$ and $h : X \rightarrow L^{-2}Z$ have infinite energy. Nevertheless, we are able to talk about the energy difference of admissible triples in the following sense. Let (X, f, h) be an admissible triple of (Y, Z) . Let $e(f)$ and $e(h)$ be respectively the energy densities of f and h . It then follows from Lemma 6.1 that we have the pointwise estimate $e(h) \leq e(f)$. Combined with Lemma 4.3, this implies that for any compact exhaustion $\{\mathcal{K}_j\}$ of X , the limit $\lim_{j \rightarrow \infty} E(f|_{\mathcal{K}_j}) - E(h|_{\mathcal{K}_j})$ exists and is always a non-negative real number. Moreover, again by Lemma 4.3, the limit is independent of the choice of the compact exhaustion. Set $E(f) - E(h) := \lim_{j \rightarrow \infty} E(f|_{\mathcal{K}_j}) - E(h|_{\mathcal{K}_j})$. It is clear that $E(f) - E(h) \geq 0$.

Definition 12.8. The energy difference $E(X, f, h)$ of an admissible triple (X, f, h) is defined by setting $E(X, f, h) := E(f) - E(h)$.

As we will see, the energy difference plays a key role in establishing the uniqueness of admissible triples.

12.5. Existence of admissible triples. Let $Y, Z \in \mathcal{T}(S)$ be two hyperbolic surfaces with the optimal Lipschitz constant $L := \exp(d_{Th}(Y, Z))$. Then for any $X \in \mathcal{T}(S)$, we have

$$(12.1) \quad E(X, Z) \leq L^2 E(X, Y).$$

For any $0 \leq r < L^{-2}$, consider the energy difference

$$\begin{aligned} F_r : \mathcal{T}(S) &\longrightarrow \mathbb{R} \\ X &\longmapsto E(X, Y) - E(X, rZ). \end{aligned}$$

By Equation (12.1), we see that for any $r \in [0, L^{-2})$,

$$F_r(X) \geq E(X, Y) - rE(X, Y) = (1 - r)E(X, Y).$$

In particular, F_r is positive and proper. Therefore, F_r has at least one critical point. Let X_r be such a critical point. Then $0 = dF_r|_{X_r} = \text{Hopf}(X_r, Y) - r\text{Hopf}(X_r, Z)$ (Using quadratic differentials to represent the differential of the energy functional over $\mathcal{T}(S)$ is a classic result, see [Jos91, Tro92, Wol98, Wen07] for example). Hence $\text{Hopf}(X_r, Y) = r\text{Hopf}(X_r, Z)$.

Theorem 12.9 (Tholozan [Tho17]). *For any $r \in [0, L^{-2})$, there exists a unique Riemann surface X_r in $\mathcal{T}(S)$ such that $\text{Hopf}(X_r, Y) = r\text{Hopf}(X_r, Z)$. Moreover, $X_r \rightarrow \infty$ as $r \rightarrow L^{-2}$.*

Using the family of harmonic map rays $\mathbf{HR}_{X_r, Y}$, we have the following existence result.

Proposition 12.10. *Let Y and Z be two distinct hyperbolic surfaces in $\mathcal{T}(S)$. Then there exists a harmonic stretch line passing through Y to Z . Consequently, there exists an admissible triple of (Y, Z) .*

Proof. For any $r \in [0, L^{-2})$, let X_r be the Riemann surface obtained from Theorem 12.9. Consider the family of harmonic map rays \mathbf{HR}_r which starts at X_r and passes through Y and Z . By Theorem 1.3, there exist a sequence $r_n \rightarrow L^{-2}$ such that \mathbf{HR}_{r_n} converges to a Thurston geodesic, say \mathbf{HSR} . Then \mathbf{HSR} is a harmonic stretch line passing through Y to Z . This proves the Lemma. \square

12.6. Uniqueness of admissible triples. In this subsection, we shall prove Proposition 12.7. The idea of the proof is to show that all admissible triples of (Y, Z) have the same energy difference. We start with a preliminary result.

Lemma 12.11. *Let (X, f, h) be an admissible triple of (Y, Z) obtained from Proposition 12.10, and $(\hat{X}, \hat{f}, \hat{h})$ an arbitrary admissible triple of (Y, Z) . Then*

$$\lim_{R \rightarrow \infty} E(\hat{h} \circ \hat{f}^{-1} \circ f|_{\mathcal{P}_R}) - E(h|_{\mathcal{P}_R}) \geq 0,$$

where \mathcal{P}_R is the Minsky polygonal region of $\text{Hopf}(f)$.

Proof. We define the elements used in the construction of (X, f, h) (described in Proposition 12.10) as follows.

Let $0 < r_n < L^{-2}$ be a sequence in which r_n converges to L^{-2} as $n \rightarrow \infty$. Let $X_n \in \mathcal{T}(S)$ be the unique Riemann surface satisfying (Theorem 12.9):

$$E(X_n, Y) - r_n E(X_n, Z) = \min_{X \in \mathcal{T}(S)} (E(X, Y) - r_n E(X, Z)).$$

Let $f_n : X_n \rightarrow Y$ and $h_n : X_n \rightarrow r_n Z$ be the corresponding harmonic maps. Then $\text{Hopf}(f_n) = \text{Hopf}(h_n)$. We rewrite the above equation as:

$$(12.2) \quad E(f_n) - E(h_n) = \min_{X \in \mathcal{T}(S)} (E(X, Y) - r_n E(X, Z)).$$

Finally, we obtain (X, f, h) from (X_n, f_n, h_n) as in Proposition 12.10. Then from Remark 12.3, we notice that $\hat{h} \circ \hat{f}^{-1} : Y \setminus \lambda \rightarrow L^{-2}Z \setminus \lambda$ extends to a 1-Lipschitz

map from Y to $L^{-2}Z$. Let $\Phi_n = \text{Hopf}(f_n) = \text{Hopf}(h_n)$. Then

$$\begin{aligned}
(12.3) \quad & \int_{X_n \setminus \mathcal{P}_R(\Phi_n)} (e(\hat{h} \circ \hat{f}^{-1} \circ f_n) - e(h_n)) dA_{\Phi_n} \\
&= \int_{X_n \setminus \mathcal{P}_R(\Phi_n)} (e(\hat{h} \circ \hat{f}^{-1} \circ f_n) - 2) + \int_{X_n \setminus \mathcal{P}_R(\Phi_n)} (2 - e(h_n)) dA_{\Phi_n} \\
&\leq \int_{X_n \setminus \mathcal{P}_R(\Phi_n)} (e(f_n) - 2) dA_{\Phi_n} + \int_{X_n \setminus \mathcal{P}_R(\Phi_n)} (2 - e(h_n)) dA_{\Phi_n} \\
&\quad \text{(since } \hat{h} \circ \hat{f}^{-1} \text{ is 1-Lipschitz)} \\
&\leq \int_{X_n \setminus \mathcal{P}_R(\Phi_n)} |e(f_n) - 2| dA_{\Phi_n} + \int_{X_n \setminus \mathcal{P}_R(\Phi_n)} |2 - e(h_n)| dA_{\Phi_n} \\
&\leq 2C|\chi(S)| \exp(-R/2). \quad \text{(by Lemma 4.2)}
\end{aligned}$$

Since $h_n : X_n \rightarrow r_n Z$ is an (energy-minimizing) harmonic map between closed surfaces and that $\hat{h} \circ \hat{f}^{-1} \circ f_n : X_n \rightarrow L^{-2}Z$ and $h_n : X_n \rightarrow rZ$ are homotopic, it follows that $rL^2 E(\hat{h} \circ \hat{f}^{-1} \circ f_n) \geq E(h_n)$. The assumption that $0 < r < L^{-2}$ then implies that $E(\hat{h} \circ \hat{f}^{-1} \circ f_n) \geq E(h_n)$. Therefore,

$$\int_{X_n} (e(\hat{h} \circ \hat{f}^{-1} \circ f_n) - e(h_n)) dA_{\Phi_n} = E(\hat{h} \circ \hat{f}^{-1} \circ f_n) - E(h_n) \geq 0.$$

Combined with (12.3), this implies that

$$\int_{\mathcal{P}_R(\Phi_n)} (e(\hat{h} \circ \hat{f}^{-1} \circ f_n) - e(h_n)) dA_{\Phi_n} \geq -2C|\chi(S)| \exp(-R/2).$$

Letting $n \rightarrow \infty$ gives

$$\int_{\mathcal{P}_R(\Phi)} (e(\hat{h} \circ \hat{f}^{-1} \circ f) - e(h)) dA_{\Phi} \geq -2C|\chi(S)| \exp(-R/2),$$

where Φ is the Hopf differential of both f and h . Hence

$$\begin{aligned}
& \lim_{R \rightarrow \infty} E(\hat{h} \circ \hat{f}^{-1} \circ f |_{\mathcal{P}_R(\Phi)}) - E(h |_{\mathcal{P}_R(\Phi)}) \\
&= \lim_{R \rightarrow \infty} \int_{\mathcal{P}_R(\Phi)} (e(\hat{h} \circ \hat{f}^{-1} \circ f) - e(h)) dA_{\Phi} \\
&\geq - \lim_{R \rightarrow \infty} 2C|\chi(S)| \exp(-R/2) = 0,
\end{aligned}$$

which completes the proof. \square

Lemma 12.12. *Let (X, f, h) be an admissible triple of (Y, Z) obtained from Proposition 12.10, and $(\hat{X}, \hat{f}, \hat{h})$ an arbitrary admissible triple of (Y, Z) . Then*

$$E(f) - E(h) \geq E(\hat{f}) - E(\hat{h}),$$

where the equality holds if and only if $\hat{f}^{-1} \circ f : X \rightarrow \hat{X}$ is conformal.

Proof. By the definition of admissible triples, we have

$$(12.4) \quad \text{Hopf}(f) = \text{Hopf}(h), \quad \text{Hopf}(\hat{f}) = \text{Hopf}(\hat{h}).$$

Let $\sigma|dz|^2$ be a conformal metric on \hat{X} , let $\eta := \hat{f}^{-1} \circ f : X \rightarrow \hat{X}$, let $\mathbf{e} := e(\eta^{-1})$, and let $\Psi := \text{Hopf}(\eta^{-1})$ (not necessary holomorphic). Now applying Tholozan's argument ([Tho17, Lemma 2.5]) to the harmonic maps

$$f : X \rightarrow Y, \quad h : X \rightarrow L^{-2}Z,$$

and

$$\hat{f} : \hat{X} \rightarrow Y, \quad \hat{h} : \hat{X} \rightarrow L^{-2}Z,$$

we see that for any compact subset \hat{K} of \hat{X} ,

$$\begin{aligned} & E(f|_{\eta^{-1}(\hat{K})}) - E(\hat{h} \circ \eta|_{\eta^{-1}(\hat{K})}) \\ &= \int_{\hat{K}} \frac{1}{\sqrt{1-4|\Psi|^2/\sigma^2 e^2}} (e(\hat{f}|_{\hat{K}}) - e(\hat{h}|_{\hat{K}})) d\sigma \\ (12.5) \quad & \geq E(\hat{f}|_{\hat{K}}) - E(\hat{h}|_{\hat{K}}), \end{aligned}$$

where the equality holds if and only if $\Psi \equiv 0$, i.e. η is conformal.

Let \mathcal{P}_R be the Minsky's polygonal region of the Hopf differential of f . Then

$$\begin{aligned} & E(f) - E(h) \\ &= \lim_{R \rightarrow \infty} (E(f|_{\mathcal{P}_R}) - E(h|_{\mathcal{P}_R})) \quad (\text{by definition}) \\ &= \lim_{R \rightarrow \infty} \left((E(f|_{\mathcal{P}_R}) - (E(\hat{h} \circ \eta|_{\mathcal{P}_R}))) \right) \\ & \quad + \lim_{R \rightarrow \infty} \left((E(\hat{h} \circ \eta|_{\mathcal{P}_R}) - E(h|_{\mathcal{P}_R})) \right) \\ & \geq \lim_{R \rightarrow \infty} \left(E(f|_{\mathcal{P}_R}) - (E(\hat{h} \circ \eta|_{\mathcal{P}_R})) \right) \quad (\text{by Lemma (12.11)}) \\ & \geq \lim_{R \rightarrow \infty} \left(E(\hat{f}|_{\eta(\mathcal{P}_R)}) - E(\hat{h}|_{\eta(\mathcal{P}_R)}) \right) \quad (\text{by (12.5)}) \\ & = E(\hat{f}) - E(\hat{h}), \quad (\text{by definition}) \end{aligned}$$

where the equality holds if and only if η is conformal. \square

Next, we use the energy difference minimization property of those admissible triples from Proposition 12.10 to show that all admissible triples have the same energy difference.

Lemma 12.13. *Let (X, f, h) be an admissible triple of (Y, Z) from Proposition 12.10, and $(\hat{X}, \hat{f}, \hat{h})$ an arbitrary admissible triple of (Y, Z) . Then*

$$(12.6) \quad E(\hat{f}) - E(\hat{h}) = E(f) - E(h).$$

Proof. By definitions of admissible triples and harmonic stretch lines, there exists a sequence of harmonic rays \mathbf{HR}_n converging to the (piecewise) harmonic stretch line determined by $(\hat{X}, \hat{f}, \hat{h})$. Let $X_n \in \mathcal{T}(S)$ be the initial point of \mathbf{HR}_n .

Let $0 < r_m < L^{-2}$ be a sequence which converges to L^{-2} as $m \rightarrow \infty$, such that $f_{r_m} : X_{r_m} \rightarrow Y$ and $h_{r_m} : X_{r_m} \rightarrow r_m Z$ converges to f and h respectively, where X_{r_m} is the Riemann surface obtained from Theorem 12.9. By Lemma 6.1, $h_{r_m} \circ (f_{r_m})^{-1} : Y \rightarrow r_m Z$ is 1-Lipschitz. Combined with Lemma 4.2, this implies that $\lim_{m \rightarrow \infty} (E(X_{r_m}, Y) - r_m E(X_{r_m}, Z)) = E(f) - E(h)$. Let $\epsilon > 0$ be an arbitrary positive real number. Since $E(X_{r_m}, Y) - r_m E(X_{r_m}, Z) = \min_{X \in \mathcal{T}(S)} (E(X, Y) - r_m E(X, Z))$, we may assume that, up to a subsequence,

$$(12.7) \quad \left| \min_{X \in \mathcal{T}(S)} (E(X, Y) - r_m E(X, Z)) - (E(f) - E(h)) \right| < \epsilon.$$

Notice that for any fixed $0 < r < L^{-2}$,

$$\lim_{\substack{Y' \rightarrow Y \\ Z' \rightarrow Z}} \min_{X \in \mathcal{T}(S)} (E(X, Y') - r E(X, Z')) = \min_{X \in \mathcal{T}(S)} (E(X, Y) - r E(X, Z)).$$

For each m , choose $X_{n_m}, \mathbf{HR}_{n_m}, Y_{n_m} \in \mathbf{HR}_{n_m}$, and $Z_{n_m} \in \mathbf{HR}_{n_m}$ such that

- (a) $Y_{n_m} \rightarrow Y$ and $Z_{n_m} \rightarrow Z$ as $n \rightarrow \infty$;
- (b) $\text{Hopf}(X_{n_m}, Z_{n_m}) = \frac{1}{r_m} \text{Hopf}(X_{n_m}, Y_{n_m})$, i.e. X_{n_m} realizes the minimum

$$\min_{X \in \mathcal{T}(S)} (E(X, Y_{n_m}) - r_m E(X, Z_{n_m}));$$

- (c) $r_m < \text{Lip}(Y_{n_m}, Z_{n_m})^{-2}$;
- (d)

$$\left| \min_{X \in \mathcal{T}(S)} (E(X, Y_{n_m}) - r_m E(X, Z_{n_m})) - \min_{X \in \mathcal{T}(S)} (E(X, Y) - r_m E(X, Z)) \right| < \epsilon.$$

Combining the items (c), (d) with (12.7), we see that

$$(12.8) \quad |(E(X_{n_m}, Y_{n_m}) - r_m E(X_{n_m}, Z_{n_m})) - (E(f) - E(h))| < 2\epsilon,$$

where we use the assumption (b) that X_{n_m} solves the energy difference minimization problem.

Let $\hat{f}_{n_m} : X_{n_m} \rightarrow Y_{n_m}$ and $\hat{h}_{n_m} : X_{n_m} \rightarrow r_m Z_{n_m}$ be the corresponding harmonic maps. We may then rewrite equation (12.8) as

$$(12.9) \quad |(E(\hat{f}_{n_m}) - E(\hat{h}_{n_m})) - (E(f) - E(h))| < 2\epsilon.$$

Now, by Lemma 6.1, $\hat{h}_{n_m} \circ (\hat{f}_{n_m})^{-1}$ is 1-Lipschitz. Combining this with Lemma 4.2, we see that

$$\left| \lim_{m \rightarrow \infty} (E(\hat{f}_{n_m}) - E(\hat{h}_{n_m})) - (E(\hat{f}) - E(\hat{h})) \right| \leq 2\epsilon.$$

The arbitrariness of ϵ then implies that

$$\lim_{m \rightarrow \infty} (E(\hat{f}_{n_m}) - E(\hat{h}_{n_m})) = E(\hat{f}) - E(\hat{h}).$$

Combined with (12.9), this yields

$$E(\hat{f}) - E(\hat{h}) = E(f) - E(h).$$

□

Proof of Proposition 12.7. The proposition follows directly from Lemma 12.13 and Lemma 12.12. □

Proof of Theorem 12.4. The theorem now follows directly from Proposition 12.7 and Proposition 12.10. □

12.7. Continuity of harmonic stretch lines. By Theorem 12.4 we see that for any two distinct hyperbolic surfaces $Y, Z \in \mathcal{T}(S)$, there is a unique harmonic stretch line $\mathbf{HSR}_{Y,Z}$ proceeding from Y to Z . A natural question one might ask is how stretch lines depend on the prescribed points Y and Z . In this regard, we have the following continuity:

Proposition 12.14. *Let Y and Z be two distinct points in $\mathcal{T}(S)$. Assume that $Y_n, Z_n \in \mathcal{T}(S)$ such that $\lim_{n \rightarrow \infty} Y_n = Y$ and $\lim_{n \rightarrow \infty} Z_n = Z$. Then \mathbf{HSR}_{Y_n, Z_n} converges to $\mathbf{HSR}_{Y,Z}$ locally uniformly as $n \rightarrow \infty$.*

Proof. By the definition of harmonic stretch lines, we see that for every fixed n , there exists a sequence of harmonic map rays $\mathbf{HR}_{n,m}$ which converges to \mathbf{HSR}_{Y_n, Z_n} locally uniformly as $m \rightarrow \infty$. Let

$$r_n := \max\{d_{Th}(Y, Y_n), d_{Th}(Y_n, Y), d_{Th}(Z, Z_n), d_{Th}(Z_n, Z)\}.$$

By assumption, we have $\lim_{n \rightarrow \infty} r_n = 0$. Now for each n , we choose a harmonic map ray \mathbf{HR}_{n,m_n} whose $2r_n$ -neighbourhood contains both Y and Z . This implies that the limiting ray of any convergent subsequence of $\{\mathbf{HR}_{n,m_n}\}_{n \geq 1}$ proceeds from Y to Z . Moreover, by Lemma 6.1 and Definition 12.1, any sublimit is a harmonic stretch line proceeding from Y to Z . It then follows from the uniqueness part of Theorem 12.4 that \mathbf{HR}_{n,m_n} converges to $\mathbf{HR}_{Y,Z}$ locally uniformly as $n \rightarrow \infty$. This implies that \mathbf{HSR}_{Y_n, Z_n} converges to $\mathbf{HR}_{Y,Z}$ locally uniformly as $n \rightarrow \infty$. \square

12.8. A characterization of admissible triples. We end this section with a characterization of harmonic stretch lines among piecewise harmonic stretch lines, in terms of their defining harmonic maps. We say that a triple $(\hat{X}, \hat{f}, \hat{h})$ is quasi-admissible if it satisfies the assumptions (i),(ii), (iii) in the definition of admissible triples.

Proposition 12.15. *A quasi-admissible triple of (Y, Z) is admissible if and only if it has the maximal energy difference among all quasi-admissible triples of (Y, Z) .*

Proof. Let (X, f, h) be an admissible triple of (Y, Z) obtained from Proposition 12.10, and $(\hat{X}, \hat{f}, \hat{h})$ an arbitrary admissible triple of (Y, Z) . Looking at the proof of Lemma 12.11 and Lemma 12.12, we see that the assumptions about $(\hat{X}, \hat{f}, \hat{h})$ that we really use is that $(\hat{X}, \hat{f}, \hat{h})$ is a quasi-admissible triple. In particular, Lemma 12.11 and Lemma 12.12 still hold if (X, f, h) is an admissible triple obtained from Proposition 12.10 while $(\hat{X}, \hat{f}, \hat{h})$ is a quasi-admissible triple. \square

The computation at the end of the proof of Lemma 12.12 and this final result shows that the energy difference of an admissible triple is at least the energy difference of a quasi-admissible triple, with equality only if the quasi-admissible triple is actually (or conformal to) the admissible triple. Returning to the definitions of quasi-admissible and admissible triples, in the end we see that (non-admissible) quasi-admissible triples do not arise as limits of harmonic maps (or else these maps would have the same energy difference as an admissible triple and hence be admissible, acquiring the condition (iv) that the corresponding piecewise harmonic stretch line is in fact a harmonic stretch line). It may be worth noting that a distinction between the two criteria is that an admissible triple will necessarily have identical residues of the Hopf differentials at the paired punctures at a node (where the differential has a second order pole).

Note that while the energy difference of an admissible triple is *at least as large* as that for any quasi-admissible triple, every admissible triple is equivalent to the those arising as the subsequential limit of harmonic maps to Y and Z , for which the energy difference $E(\cdot, Y) - E(\cdot, rZ)$ is a *minimum* for each choice of r and declines as r tends to L^{-2} (where L is the optimal Lipschitz constant from Y to Z).

13. EXPONENTIAL MAP OF THURSTON'S METRIC

The goal of this section is to consider the “visual boundary ” of the Thurston metric (Theorem 1.11), and define two distinct versions of Thurston geodesic flow.

Recall that a harmonic stretch line is a limit of a sequence of harmonic map rays. By Theorem 12.4, for any two distinct hyperbolic surfaces $Y, Z \in \mathcal{T}(S)$, there is a unique harmonic stretch line proceeding from Y to Z . Here we extend this to the following, where a *harmonic stretch ray* is a ray contained in some harmonic stretch line with the induced orientation (Recall that a harmonic stretch line admits an canonical orientation).

Theorem 13.1. *For any hyperbolic surface $Y \in \mathcal{T}(S)$ and any projective measured lamination $[\beta] \in \mathcal{PM}\mathcal{L}(S)$, there is a unique harmonic stretch ray from Y which converges to $[\beta]$ in the Thurston compactification.*

Convention. In the remainder of this section, to simplify the notation, we will denote the dual tree $(T_\beta, 2d)$ by T_β .

13.1. Optimal equivariant Lipschitz maps to trees. Let β be a representative of $[\beta] \in \mathcal{PM}\mathcal{F}(S) = \mathcal{PM}\mathcal{L}(S)$. Let T_β be the dual tree of the universal cover of β . Let L be the least Lipschitz constant of equivariant (surjective) maps from the universal cover \tilde{Y} to T_β . Then

$$(13.1) \quad L \geq \sup_{\mu \in \mathcal{ML}(S)} \frac{2i(\beta, \mu)}{\ell_Y(\mu)}.$$

where the convention on the metric on the (dual) tree T_β implies that lengths on the tree are measured by the intersection number $2i(\beta, \mu)$. (Later we will see that the two quantities are actually equal.) For each $0 < t < L^{-1}$, consider the energy difference function $E(\cdot, Y) - t^2 E(\cdot, T_\beta)$ on $\mathcal{T}(S)$. Tholozan's argument (cf. the proof of Lemma 12.12) gives a unique minimiser $X_t \in \mathcal{T}(S)$. Moreover, for each $0 < r < L^{-1}$, the vertical foliation of $\text{Hopf}(X_r, Y)$ is exactly $r\beta$. Letting $r \rightarrow L^{-1}$ and using Lemma 4.5, we obtain a convergent subsequence $f_n : X_{r_n} \rightarrow Y$ which converges to a surjective harmonic diffeomorphism $f_\infty : X_\infty \rightarrow Y \setminus \lambda$ for some chain-recurrent lamination λ . Correspondingly, the push forward of the vertical foliation of $\text{Hopf}(f_\infty)$ via f_∞ extends to a measured foliation on Y which is exactly $L^{-1}\beta$, viewed as the limit of $r_n\beta$ as $n \rightarrow \infty$. Therefore, by Proposition 7.4, the piecewise harmonic stretch line determined by f_∞ converges to $[\beta] \in \mathcal{PM}\mathcal{L}(S)$.

Recall that in terms of the natural coordinates of $\text{Hopf}(f_\infty)$ ($\text{Hopf}(f_\infty) = dz^2$), the pullback of the hyperbolic metric on $Y \setminus \lambda$ via f_∞ to X_∞ is

$$2(\cosh \mathcal{G} + 1)dx^2 + 2(\cosh \mathcal{G} - 2)dy^2$$

where $\mathcal{G} = \log(1/|\nu|)$ and ν is the Beltrami differential of f_∞ (see (3.2)). Let $\pi : \widetilde{X}_\infty \rightarrow L^{-1}T_\beta$ be the projection map along leaves of vertical foliations of $\text{Hopf}(f_\infty)$. By the definition of T_β , the pullback metric of T_β on \widetilde{X}_∞ via π is exactly $4dx^2$. Then the composition map $\pi \circ \widetilde{f_\infty}^{-1} : \widetilde{Y} \setminus \widetilde{\lambda} \rightarrow T_\beta$ is a Lipschitz map with (pointwise) Lipschitz constant $\sqrt{2/(\cosh \mathcal{G} + 1)} < 1$. Moreover, by Lemma 3.3, the (pointwise) Lipschitz constant of $\pi \circ \widetilde{f_\infty}^{-1}$ along horizontal leaves tends to 1 as the distance to zeros of $\text{Hopf}(f_\infty)$ goes to infinity. Therefore, $\pi \circ \widetilde{f_\infty}^{-1} : \widetilde{Y} \setminus \widetilde{\lambda} \rightarrow T_\beta$ extends to a Lipschitz map $\widetilde{Y} \rightarrow T_\beta$ whose restriction to the geodesic lamination $\widetilde{\lambda}$ is an affine map of factor L and which has (pointwise) Lipschitz constant strictly less than L outside $\widetilde{\lambda}$. Since λ , the projection of $\widetilde{\lambda}$ to Y , is chain-recurrent, there

exists a sequence of multicurves μ_n whose support converges to λ in the Hausdorff topology. Consequently,

$$\lim_{n \rightarrow \infty} \frac{2i(\beta, \mu_n)}{\ell_Y(\mu_n)} = L.$$

Combining this with (13.1), we see that

$$L = \sup_{\mu \in \mathcal{ML}(S)} \frac{2i(\beta, \mu)}{\ell_Y(\mu)}.$$

Moreover, this implies that every optimal Lipschitz map from \tilde{Y} to T_β would maximally stretch $\tilde{\lambda}$. We define $\tilde{\lambda}$ to be the *maximally stretched lamination* from Y to T_β , denoted by $\Lambda(Y, T_\beta)$.

Using the argument analogous to the proof of Proposition 12.10, we see that the family of harmonic map rays $\mathbf{HR}_{X_r, Y}$ determined by the harmonic map $f_r : X_r \rightarrow Y$ contains a sequence which converges to a harmonic stretch line, as r approaches L^{-2} .

In the remainder of this section, we set $L := \sup_{\mu \in \mathcal{MF}(S)} \frac{2i(\beta, \mu)}{\ell_Y(\mu)}$. We summarize the above discussion in the first two statements in the following, while the third statement follows from Proposition 7.4.

Proposition 13.2. *Let $Y \in \mathcal{T}(S)$ and $\beta \in \mathcal{MF}(S)$. Let T_β be the tree dual to the lift of β to the universal cover \tilde{S} . Then we have the following.*

- *There exists a harmonic stretch ray \mathbf{ESR}_Y^β which starts at Y and which maximally stretches along the maximally stretched lamination $\Lambda(Y, T_\beta)$.*
- *There exists an equivariant Lipschitz map $f : \tilde{Y} \rightarrow T_\beta$ whose restriction to the maximally stretched lamination $\Lambda(Y, T_\beta)$ is an affine map of factor L and which has (pointwise) Lipschitz constant strictly less than L outside $\Lambda(Y, T)$.*
- *The ray \mathbf{ESR}_Y^β converges to $[\beta]$ in Thurston's compactification.*

Remark 13.3. In [Tab85], Tabak proved that the push forward of $\text{Hopf}(f_t : X_t \rightarrow Y)$ is a subsonic ρ -holomorphic quadratic differential on Y with $\rho : Y \times [0, 1/4] \rightarrow \mathbb{R}$ defined by $\rho(y, r) = (1 - 4r)^{-1/2}$. In [SS70], Sibner-Sibner proved a nonlinear Hodge-De Rham theorem which states that for any measured foliation $\beta \in \mathcal{MF}(S)$, there exists a threshold value $t_0 > 0$ such that for all $0 < t < t_0$, there exists a unique subsonic ρ -holomorphic quadratic differential whose vertical foliation is $t\beta$. The discussion above gives a different proof of Sibner-Sibner's result for the very specific ρ defined above and describes explicitly L^{-1} as the threshold value of β .

13.2. Admissible triples of (Y, T_β) . Having established the maximally stretched lamination $\tilde{\lambda}$ from \tilde{Y} to T_β , we are now in a position to consider admissible triples for \tilde{Y} and T_β , in the same way as in Section 12.1. Combining the construction of harmonic stretch lines and the discussion in Section 13.1, we see that every harmonic stretch ray which starts at Y and which converges to $[\beta] \in \mathcal{PM}\mathcal{L}(S)$ in Thurston's compactification gives an optimal equivariant Lipschitz map $f : \tilde{Y} \rightarrow T_\beta$ whose restriction to $\tilde{\lambda}$ is an affine map of factor L and which has (pointwise) Lipschitz constant strictly less than L outside $\tilde{\lambda}$.

Definition 13.4. A triple $(X, \tilde{f}, \tilde{h})$ is said to be *admissible* if the following hold:

- (i) X is a punctured Riemann surface (possibly disconnected);
- (ii) $\tilde{f} : \tilde{X} \rightarrow \tilde{Y} \setminus \tilde{\lambda}$ is an equivariant surjective harmonic diffeomorphism and $\tilde{h} : \tilde{X} \rightarrow L^{-1}T_\beta$ is an equivariant surjective harmonic map with meromorphic Hopf differentials satisfying $\text{Hopf}(\tilde{f}) = \text{Hopf}(\tilde{h})$;
- (iii) the piecewise harmonic stretch line defined by the quotient map $f : X \rightarrow Y \setminus \lambda$ of $\tilde{f} : \tilde{X} \rightarrow \tilde{Y} \setminus \tilde{\lambda}$ is a harmonic stretch line.

Remark 13.5. By Proposition 7.4, item (ii) implies that the harmonic stretch line defined by $f : X \rightarrow Y \setminus \lambda$ converges to $[\beta] \in \mathcal{PML}(S)$. Moreover, the composition map $\tilde{h} \circ \tilde{f}^{-1}$ extends to an equivariant 1-Lipschitz map from \tilde{Y} to T_β .

Definition 13.6. Two admissible triples $(\tilde{X}, \tilde{f}, \tilde{h})$ and $(\tilde{X}', \tilde{f}', \tilde{h}')$ of (Y, Z) are said to be *equivalent* if there exists a conformal map $\eta : \tilde{X} \rightarrow \tilde{X}'$ such that $\tilde{f} = \tilde{f}' \circ \eta$.

By Proposition 13.2, there exists at least one admissible triple. Applying the same argument as in Section 12.6, we have

Proposition 13.7. *Let $Y \in \mathcal{T}(S)$ and $\beta \in \mathcal{MF}(S)$. Let T_β be the tree dual to the lift of β to the universal cover \tilde{S} . Then all admissible triples of (\tilde{Y}, T_β) are equivalent.*

Proof. The proof is similar to that of Proposition 12.7, with (Y, Z) replaced by (\tilde{Y}, T_β) . \square

Proof of Theorem 13.1. The existence part follows from Proposition 13.2. The uniqueness part follows from Lemma 12.6 and Proposition 13.7. \square

Proof of Theorem 1.11. The first part follows from Theorem 13.1.

For the second part, we need to show that these harmonic stretch rays are either disjoint (away from their origin Y) or coincide, and we must also show that the union of the harmonic stretch rays cover Teichmüller space. That the union of the harmonic stretch rays covers $\mathcal{T}(S)$ follows basically from Theorem 1.8: given a point $Z \in \mathcal{T}(S)$, we take the harmonic stretch segment from Y to Z and extend it to a proper ray in $\mathcal{T}(S)$ by scaling the Hopf differential on the corresponding domain X_∞ (.cf as found via an admissible triple). This harmonic stretch line converges to a unique point on $\mathcal{PML}(S)$ by Proposition 7.4, and hence is a leaf of the foliation. That the harmonic stretch rays from Y either coincide or are disjoint away from Y is the content of Theorem 1.8, since if two harmonic stretch lines from Y intersected at a point $Z \in \mathcal{T}(S)$, they would coincide on $[Y, Z]$ and hence extend beyond Z identically.

Finally, the third statement, that the harmonic stretch rays terminating at a point $[\eta] \in \mathcal{PML}(S)$ also foliate if we let the initial point Y vary in $\mathcal{T}(S)$, follows easily from the disjointness and surjectivity arguments of the previous paragraph. \square

13.3. Continuity of the exponential map rays. Using the same argument as in the proof of Proposition 12.14, we have the following:

Proposition 13.8. *Let $Y \in \mathcal{T}(S)$ and $[\eta] \in \mathcal{PML}(S)$. Assume that $Y_n \in \mathcal{T}(S)$ and $[\eta_n] \in \mathcal{PML}(S)$ such that $\lim_{n \rightarrow \infty} Y_n = Y$ and $\lim_{n \rightarrow \infty} [\eta_n] = [\eta]$. Then the exponential map ray $\mathbf{ESR}_{Y_n, [\eta_n]}$ converges to the exponential map ray $\mathbf{ESR}_{Y, [\eta]}$ locally uniformly as $n \rightarrow \infty$.*

As a direct consequence, we obtain the following analog of Theorem 2.1:

Proposition 13.9. *Let $Y \in \mathcal{T}(S)$ and $\eta \in \mathcal{ML}(S)$. Assume that $Y_n \in \mathcal{T}(S)$ and $\eta_n \in \mathcal{ML}(S)$ such that $\lim_{n \rightarrow \infty} Y_n = Y$ and $\lim_{n \rightarrow \infty} \eta_n = \eta$. Let $\Lambda(Y_n, \eta_n)$ (resp. $\Lambda(Y, \eta)$) be the maximally stretched lamination from Y_n to T_{η_n} (resp. from Y to T_η). Then $\Lambda(Y, T_\eta)$ contains any geodesic lamination in the limit set of $\Lambda(Y_n, T_{\eta_n})$ in the Hausdorff topology.*

Proof. Let Z_n be a point in the exponential map ray $\mathbf{ESR}_{Y_n, [\eta_n]}$ such that $d_{Th}(Y_n, Z_n) = 1$ and that $\mathbf{ESR}_{Y_n, [\eta_n]}$ proceeds from Y_n to Z_n . Let Z be a point in the exponential map ray $\mathbf{ESR}_{Y, [\eta]}$ such that $d_{Th}(Y, Z) = 1$ and that $\mathbf{ESR}_{Y, [\eta]}$ proceeds from Y to Z . By Proposition 13.8, we see that Z_n converges to Z as $n \rightarrow \infty$. Notice that $\Lambda(Y_n, T_{\eta_n}) = \Lambda(Y_n, Z_n)$ and $\Lambda(Y, T_\eta) = \Lambda(Y, Z)$. The proposition now follows from Theorem 2.1. \square

13.4. Two versions of geodesic flow of Thurston metric. In this subsection, we define two versions of the geodesic flow of the Thurston metric. The first version is defined using the exponential map obtained in this section. The second version is defined using Theorem 1.1.

The exponential map we obtained in this section allows us to define a Thurston geodesic flow

$$\psi_t : \mathcal{T}(S) \times \mathcal{PM}\mathcal{L}(S) \longrightarrow \mathcal{T}(S) \times \mathcal{PM}\mathcal{L}(S)$$

as follows. For each pair $(Y, [\lambda]) \in \mathcal{T}(S) \times \mathcal{PM}\mathcal{L}(S)$, let $\mathbf{ESR}_Y^{[\lambda]} : [1, \infty) \rightarrow \mathcal{T}(S)$ be the harmonic stretch ray which starts at Y and converges to $[\lambda] \in \mathcal{PM}\mathcal{L}(S)$. Then we define the flow ψ_t by setting $\psi_t(Y, [\lambda]) := (\mathbf{ESR}_Y^{[\lambda]}(e^{2t}), [\lambda])$: that this flow is well-defined follows from the second paragraph in Theorem 1.11. In particular, every ψ -orbit is a harmonic stretch line. Moreover, it follows from Theorem 1.11 that every harmonic stretch line appears as a ψ -orbit.

We next define a second version of Thurston geodesic flow

$$\phi_t : \mathcal{T}(S) \times \mathcal{PM}\mathcal{L}(S) \rightarrow \mathcal{T}(S) \times \mathcal{PM}\mathcal{L}(S).$$

For each pair $(Y, [\lambda]) \in \mathcal{T}(S) \times \mathcal{PM}\mathcal{L}(S)$, let $\mathbf{SR}_{Y, [\lambda]} : [1, \infty) \rightarrow \mathcal{T}(S)$ be the stretch ray obtained as the limit of harmonic map rays $\mathbf{HR}_{X_t, Y}$ where X_t degenerates along the harmonic map dual ray $\mathbf{hr}_{Y, \lambda}$ (Theorem 1.1). We then define the flow ϕ_t by setting $\phi_t(Y, [\lambda]) = (\mathbf{SR}_{Y, [\lambda]}(e^{2t}), [\lambda])$. In particular, every ϕ -orbit is a stretch line, a special type of harmonic stretch lines. However, ϕ -orbits (equivalent, stretch lines) are “rare” in the following sense. For any non-uniquely ergodic lamination λ , the projection to $\mathcal{T}(S)$ of the orbit of $(Y, [\lambda])$ under ϕ_t is independent of the transverse measure of λ (by Lemma 8.3 and Theorem 5.6). Moreover, by [Thu98, Theorem 10.7], for any Y and any simple closed curve λ , the set $\mathcal{Z}_{Y, \lambda}$, which consists of surfaces $Z \in \mathcal{T}(S)$ such that the maximally stretched lamination from Y to Z is λ , is an open subset of $\mathcal{T}(S)$. Consequently, for any fixed $Y \in \mathcal{T}(S)$, the union of projection of the ϕ orbits of $(Y, [\lambda])$, as $[\lambda]$ varies in $\mathcal{PM}\mathcal{L}(S)$, is a proper subset of $\mathcal{T}(S)$.

How does the earthquake flow interact with the second version ϕ_t of the Thurston geodesic flow? Do they define an action of the upper triangular subgroup of $SL(2, \mathbb{R})$? From the construction of stretch lines in Section 7, we know that the translates of harmonic stretch lines by the earthquake flow are stretch lines. Here we show that they are also harmonic stretch lines.

Proposition 13.10. *Let \mathbf{R} be a harmonic stretch line in $\mathcal{T}(S)$ which maximally stretches along a measured geodesic lamination λ . Let $\mathcal{E}_\lambda(\mathbf{R})$ be a translate of \mathbf{R} by an earthquake directed by λ . Then $\mathcal{E}_\lambda(\mathbf{R})$ is also a harmonic stretch line.*

Proof. Let $Y, Z \in \mathbf{R}$ be two hyperbolic surfaces such that \mathbf{R} proceeds from Y to Z . Then by Theorem 1.8, there exists a sequence of harmonic maps $f_n : X_n \rightarrow Y$ with $X_n \in \mathcal{T}(S)$ which converges to a surjective harmonic map $f : X \rightarrow Y \setminus \lambda$ from a punctured surface X and that (X, f) defines \mathbf{R} in the sense of Theorem 1.7. Let λ_n be the horizontal measured foliation of $\text{Hopf}(f_n)$. Let X'_n be the unique Riemann surface in $\mathcal{T}(S)$ such that the Hopf differential of the harmonic map $f'_n : X'_n \rightarrow \mathcal{E}_\lambda(Y)$ is also λ_n ([Wol98]). By Lemma 4.5, the sequence of maps $\{f'_n\}$ contains a convergent subsequence, still denoted by $\{f'_n\}$ for simplicity. Let $f' : X' \rightarrow \mathcal{E}_\lambda(Y) \setminus \lambda'$ be the limit surjective harmonic diffeomorphism. Then, since λ_n limits on both λ and λ' , we have that $\lambda = \lambda'$ (as geodesic laminations). Moreover, the horizontal foliations of $\text{Hopf}(f)$ and $\text{Hopf}(f')$ are the same. It then follows from Theorem 5.6 that $X' = X$ and $f' = f$. In particular, $\mathcal{E}_\lambda(\mathbf{R})$ is a harmonic stretch line defined by f' . \square

Remark 13.11. As a direct consequence, the earthquake flow and the ‘‘Thurston geodesic flow’’ ϕ_t are compatible, and hence define an action of the upper triangular subgroup of $SL(2, \mathbb{R})$. More precisely, let

$$a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad h_r = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}.$$

We define $a_t(\lambda, Y) := \phi_t(\lambda, Y)$ and $h_r(\lambda, Y) := (\lambda, \mathcal{E}_r\lambda(Y))$.

14. CONCLUDING REMARKS

14.1. Constructing geodesics in the Teichmüller space of hyperbolic surfaces with geodesic boundary. Let $S_{g,b}$ be an orientable surface of genus g with b boundary components. Let $\mathcal{T}(S_{g,b})$ be the Teichmüller space of hyperbolic surfaces with b geodesic boundary components. There are presently three versions of a Thurston-type metric on $\mathcal{T}(S_{g,n})$ defined as follows. The first one is the so-called *arc metric/distance* introduced by Liu-Papadopoulos-Su-Th  ret[LPST10]. Let \mathcal{C} be the set of isotopy classes of simple closed curves on $S_{g,n}$ and \mathcal{A} the set of isotopy classes (rel $\partial S_{g,n}$) of (essential) simple arcs on $S_{g,n}$ with endpoints on $\partial S_{g,n}$. The *arc distance* is defined as:

$$d_A(X, Y) := \log \sup_{\alpha \in \mathcal{C} \cup \mathcal{A}} \frac{\ell_Y(\alpha)}{\ell_X(\alpha)}.$$

The other two versions, introduced in [AD19], are defined via Lipschitz maps. For $X, Y \in \mathcal{T}(S_{g,n})$, Let $\mathcal{L}(X, Y)$ be the set of Lipschitz maps from X to Y that commute with the marking up to homotopy. Let $\text{Lip}(\phi)$ be the Lipschitz constant of $\phi \in \mathcal{L}(X, Y)$. Define

$$\begin{aligned} d_{L\partial}(X, Y) &:= \log \inf \{ \text{Lip}(\phi) : \phi \in \mathcal{L}(X, Y) \text{ with } \phi(\partial X) \subset \partial Y \}, \\ d_{Lh}(X, Y) &:= \log \inf \left\{ \text{Lip}(\phi) : \begin{array}{l} \phi \in \mathcal{L}(X, Y) \text{ with } \phi \\ \text{a surjective homeomorphism} \end{array} \right\}. \end{aligned}$$

Alessandrini-Disarlo[AD19] showed that $d_A = d_{L\partial}$ on $\mathcal{T}(S_{g,n})$ and conjectured that [AD19, Conjecture 1.8] that $d_A = d_{Lh}$. Using harmonic stretch lines we verify this conjecture.

Theorem 14.1. *With notations as above, for any $X, Y \in \mathcal{T}(S_{g,n})$, we have $d_A = d_{Lh}$. Moreover, the optimal Lipschitz constant from X to Y is always realized by a homeomorphism.*

Proof. It is clear that $d_A \leq d_{Lh}$. To prove the theorem, it suffices to show that the optimal Lipschitz constant from X to Y is realized by a homeomorphism from X to Y .

Let X^d (resp. Y^d) be the double of X (resp. Y), obtained by gluing respectively the orientation-reversing isometric copy of X (resp. Y) to X along ∂X (resp. to Y along ∂Y). Consider the harmonic stretch line $[X^d, Y^d]$ in the Teichmüller space of the double of $S_{g,b}$. The doubling process induces an involution, denoted by ι , on both X^d and Y^d . The uniqueness of harmonic stretch segment (Theorem 12.4) then implies the Lipschitz map ϕ^d from X^d to Y^d induced by any admissible triple of (X^d, Y^d) is symmetric about this involution. Hence ϕ^d descends to a surjective Lipschitz homeomorphism ϕ from X to Y with the same Lipschitz constant as ϕ^d . This implies that $d_{Lh}(X, Y) \leq \log \text{Lip}(\phi) = d_{Th}(X^d, Y^d)$. On the other hand, the double of any (surjective) L -Lipschitz homeomorphism from X to Y gives an L -Lipschitz homeomorphism from X^d to Y^d . Hence, using that the doubling operation provides candidate maps for the minimization problem for closed surfaces from candidates for the minimization problem for the surfaces-with-boundary X and Y , we see that $d_{Lh}(X, Y) \geq d_{Th}(X^d, Y^d) = \log \text{Lip}(\phi)$. Consequently, $d_{Lh}(X, Y) = d_{Th}(X^d, Y^d) = \log \text{Lip}(\phi)$. Combining with the fact $d_A(X, Y) = d_{Th}(X^d, Y^d)$ proved by Liu-Papadopoulos-Su-Théret [LPST10], we know that $d_A(X, Y) = d_{Lh}(X, Y) = \log \text{Lip}(\phi)$. □

14.2. Relation to orthogonal foliation introduced by Choi-Dumas-Rafi and Calderon-Farre. Let $Y \in \mathcal{T}(S)$ be a hyperbolic surface and λ a measured geodesic lamination. The harmonic stretch ray obtained in Theorem 1.1 is determined by a (possibly disconnected) punctured Riemann surface X and a surjective harmonic diffeomorphism $f : X \rightarrow Y \setminus \lambda$. The pushforward of the vertical foliation of the Hopf differential $\text{Hopf}(f)$ extends to a measured foliation on Y which is transverse to λ . On the other hand, there is an orthogonal measured foliation associated to λ constructed in [CDR12, CF21]. If λ is maximal, then these two measured foliations coincide. A natural question is to consider the relationship between these two types of transverse measured foliations of λ on Y for non-maximal λ .

14.3. Optimal Lipschitz map from hyperbolic surface Y to negatively curve surface Z . In this paper, we have considered the sequence of minimizers of the energy difference function: $E(X, Y) - tE(X, Z)$ for constant curvature surfaces Y and Z . It is natural to reflect on how these results might generalize to the case of negatively curved surfaces Y and Z for $0 < t < \text{Lip}(Y, Z)^{-2}$.

APPENDIX A. EXISTENCE FOR THE GENERALIZED JENKINS-SERRIN PROBLEM

A principal tool in this paper was an extension of the Jenkins-Serrin theory for minimal graphs in Euclidean three-space with asymptotic boundary values to minimal graphs over hyperbolic surfaces with boundary which took values in real trees, also with asymptotic boundary values. The uniqueness theory of such graphs was described in Theorem 5.6 and used throughout the paper. In this appendix, we

prove the existence part of this general Jenkins-Serrin problem; while this does not play a role in the proofs of our main results, we include a proof here to complete the discussion of this problem. The proof is somewhat involved, but as it turns on the structures of the foliations of the Hopf differentials, it also provides for some further development of the more technical themes of this paper. We begin with the statement of the main result; recall Definition 5.4 of “admissible dual tree” from Section 5.

Theorem A.1. *Let Y be a crowned hyperbolic surface. Let T be an admissible dual tree. Then there exists a $\pi_1(Y)$ -equivariant minimal graph in $\tilde{Y} \times T$ with a prescribed order-preserving boundary correspondence.*

The idea of the proof is to first construct a sequence of harmonic maps $X_n \rightarrow W$ to a fixed closed surface W consisting of copies of Y which have been glued together, with controlled horizontal foliations of the associated Hopf differentials. The domains X_n for the approximating harmonic maps are found by demanding that the Hopf differentials have approximating maximal stretch laminations. We then prove that the limit of any convergent subsequence gives an equivariant minimal graph in $\tilde{Y} \times T$ (and in fact a unique one by Theorem 5.6). The proof will be divided into two cases:

Case I: Y has no crown ends, and the measured foliation defining T consists of half-infinite cylinders corresponding to boundary components of Y and compactly supported subfoliations;

Case II: the general case.

Here the division is entirely for expositional reasons: the basic structure of the argument will be apparent in the technically simpler Case I. The second Case II extends the technique to a more complicated setting.

Now, the foliations that arise in minimal surfaces over crowned surfaces have several qualitative types: (i) they can be closed curves, or (ii) they can be part of a half-plane adjacent to a boundary leaf of a crown, or (iii) they can be a strip of bounded width with an end tending to an ideal point in a crown or an end spiralling around a boundary curve, or (iv) the leaves can be none of these: in the latter case, the leaves may be collected into a compact portion of the surface. Of course, the technical heart of the proof comprises checking that the limiting Hopf differential on the limiting surface X_∞ has the desired trajectory structure. One crucial estimate is a bound on the diameter of the “compact part”, but one also needs to make sure that the leaves, in each end, limit in the expected way.

A.1. Case I: Geodesic boundary. Let F be the measured foliation on Y whose lift to the universal cover defines T . Suppose that F consists of a half-infinite cylinders $\{A_i\}_{1 \leq i \leq a}$ with core curves $\{\alpha_i\}_{1 \leq i \leq a}$ – which are also the boundary components of Y – and b compactly supported subfoliations $\{B_i\}_{1 \leq i \leq b}$. (In this particular model case, we may take $b = 1$, but we retain the notation for the more general case.) Let β_i be the measured lamination corresponding to B_i .

Let W be a closed hyperbolic surface obtained by gluing an isometric copy Y' of Y to Y along ∂Y in a way that preserves the orientation of the two copies. (This choice of orientation is not important for this case, but will be important in Case II, so we introduce it here.) Let β' be the copy of $\beta \in \mathcal{MF}(Y)$ on Y' . Consider the measured foliation $\mu_n := 2n \sum_i \alpha_i + \beta + \beta' \in \mathcal{MF}(W)$. Let $X_n \in \mathcal{T}(W)$ be the Riemann surface such that the horizontal foliation of the Hopf differential of

the harmonic map $f_n : X_n \rightarrow W$ is μ_n ; as usual, this is guaranteed by [Wol98]. Equivalently, the universal cover \widetilde{X}_n of X_n is the unique minimal graph in $\widetilde{W} \times T_n$ where T_n is the tree dual to $\widetilde{\mu}_n$.

Next we decompose X_n as $A_{1,n} \cup \dots \cup A_{k,n} \cup B_n \cup B'_n$ according to the horizontal foliation of $\Phi_n := \text{Hopf}(f_n)$, where $A_{i,n}$ is the maximal flat cylinder corresponding to the curve α_i and B_n (resp. B'_n) is the (precompact) subsurface corresponding to β (resp. β'). In particular, the width of $A_{i,n}$ is $2n$ for all i . By Lemma 3.8, we see that for each $A_{j,n}$ and B_n , we have

$$(A.1) \quad \begin{aligned} 2n\ell_W(\alpha_i) - C &\leq 2\|\Phi_n|_{A_{j,n}}\| \leq 2n\ell_W(\alpha_i) + C, \\ \ell_W(\beta) - C &\leq 2\|\Phi_n|_{B_n}\| \leq \ell_W(\beta) + C, \\ \ell_W(\beta') - C &\leq 2\|\Phi_n|_{B'_n}\| \leq \ell_W(\beta') + C, \end{aligned}$$

where C is a constant depending on the topology of W .

Let $\overline{B_n} \subset X_n$ be the closure of B_n .

Lemma A.2. *There exists a constant $D > 0$ depending on W such that for any $n > 1$, the diameters of $\overline{B_n}$ and $\overline{B'_n}$ are at most D with respect to the $|\Phi_n|$ -metric.*

Proof. We demonstrate the proof for $\{\overline{B_n}\}$. The proof of $\{\overline{B'_n}\}$ is similar. Suppose to the contrary that there exists a subsequence of $\{\overline{B_n}\}_{n \geq 1}$ whose diameter goes to infinity. Without loss of generality, we may assume this subsequence is $\overline{B_n}$ itself.

The area bound (A.1) of B_n implies that the injectivity radius of every point of B_n is at most D_1 for some constant D_1 depending on W . Recall that Φ_n has at most $2|\chi(W)|$ zeros. The assumption that the diameter of $\overline{B_n}$ goes to infinity implies that there exists $p_n \in B_n$ such that

$$d_n := \text{dist}(p_n, \mathbf{Sing}(\Phi_n)) \rightarrow \infty,$$

where $\mathbf{Sing}(\Phi_n)$ is the zero set of Φ_n and $\text{dist}(\cdot, \cdot)$ represents the $|\Phi_n|$ -distance. By [MS91, Lemma 5.1], the point p_n is contained in some smooth closed regular geodesic η_n with length $\ell(\eta_n) < 2D_1$. Then any point of η_n has distance at least $d_n - 2D_1$ from $\mathbf{Sing}(\Phi_n)$. By [MS91, Lemma 5.2], we see that the (maximal) cylinder C'_n containing η_n has width at least $2d_n - 4D_1$. Therefore, there exists a subcylinder $C_n \subset C'_n$, with $p_n \in C_n$, of width $2d_n - 4D_1 - 2R$ such that every point of C_n is of distance at least R to $\mathbf{Sing}(\Phi_n^d)$. (Of course, the constant R here will refer to the constant in Minsky's region \mathcal{P}_R .)

Next, we claim that $C_n \subset B_n$. Otherwise, suppose to the contrary that C_n is not contained in B_n , then C_n would intersect some $A_{i,n}$ because $\cup_i A_{i,n}$ separates B_n and B'_n . Since C_n intersects both B_n and $A_{i,n}$, it follows that C_n is not horizontal. Let $q \in A_{i,n} \cap C_n$. Then the closed geodesic $\gamma_q \subset A_{i,n}$ which contains q (and is a core curve of $A_{i,n}$) would cross C_n because C_n is not horizontal. This implies the $|\Phi_n|$ -length γ_q is at least the width of C_n which goes to infinity. Thus, the circumference of $A_{i,n}$ also goes to infinity. On the other hand, recall that, by the construction of the measured foliation μ_n , the width of $A_{i,n}$ is $2n$. Let $\delta_{i,n}$ be the core curve of $A_{i,n}$ which is of distance n from the boundary of $A_{i,n}$. Then by Minsky's estimate together with the fact that $A_{i,n}$ is horizontal, we see that the $|\Phi_n|$ -length of $\delta_{i,n}$ converges to $\ell_W(\alpha_i)/2$ as $n \rightarrow \infty$. This implies that the circumference of $A_{i,n}$ also converges to $\ell_W(\alpha_i)/2$ as $n \rightarrow \infty$, which yields a contradiction.

The extremal length of η_n on B_n satisfies:

$$\text{Ext}_{B_n}(\eta_n) \leq \frac{1}{\text{Mod}(C_n)} \leq \frac{2D_1}{2d_n - 4D_1 - 2R}.$$

By Theorem 3.6, we see that

$$\begin{aligned} E(f_n|_{B_n}) &\geq \frac{1}{2} \frac{\ell_W^2(\eta_n)}{\text{Ext}_{B_n}(\eta_n)} \\ &\geq \frac{(\text{Syst}(W))^2(2d_n - 4D_1 - 2R)}{4D_1}, \end{aligned}$$

which goes to infinity as $n \rightarrow \infty$. This contradicts (A.1) because $E(f_n|_{B_n}) \leq 2\|\Phi_n|_{B_{1,n}}\| + 2\pi|\chi(W)|$. \square

Proof of Theorem A.1 Case I. Let z_n be a zero of Φ_n in B_n . By Lemma A.2, the subset B_n is contained in the Minsky polygonal region P_R of X_n for $R > D$. By Lemma 4.1, we see that the sequence of pointed flat surfaces (X_n, Φ_n, z_n) contains a subsequence which converges to pointed singular flat surface $(X_\infty, \Phi_\infty, z_\infty)$; here the flat metric $|\Phi_\infty|$ is induced by a meromorphic quadratic differential Φ_∞ which has a pole of order two at the pinching locus of α_i , for each i . Moreover, by Lemma A.2, the horizontal foliation of Φ_∞ consists of a half-infinite cylinders $\{A_{i,\infty}\}_{1 \leq i \leq a}$ with core curve $\{\alpha_i\}_{1 \leq i \leq a}$, and the compactly supported measured foliation β . Namely, the horizontal foliation of Φ_∞ (on the limit surface X_∞) is equivalent to that of $\Phi := \text{Hopf}(f)$, whose dual tree is exactly T .

Up to a subsequence, we see that $\tilde{f}_n : (\tilde{X}_n, \tilde{p}_n) \rightarrow \tilde{W}$ converges to a harmonic map $\tilde{f}_\infty : (\tilde{X}_\infty, \tilde{p}) \rightarrow \tilde{W}$. Notice that the image of a core curve of A_i under $f_\infty : X \rightarrow W$ arbitrarily closely approximates the geodesic representative of α_i on W (as the distance of the core curve from the compact part grows large). Hence the image $\tilde{f}_\infty(\tilde{X}_\infty)$ is exactly the lift \tilde{Y} of Y . Namely, \tilde{X}_∞ is an equivariant minimal graph in the product $\tilde{Y} \times T$. \square

A.2. Case II: general case. Let F be the measured foliation on Y whose universal cover defines T by duality. Suppose that F comprises a half-infinite cylinders $\{A_i\}_{1 \leq i \leq a}$ foliated by closed leaves, k half-planes $\{G_j\}_{1 \leq j \leq k}$, m bi-infinite strips $\{V_j\}_{1 \leq j \leq m}$, and b compactly support subfoliations $\{B_i\}_{1 \leq i \leq b}$. Notice that a bi-infinite strip may spiral around some non-horizontal half-infinite cylinder; this corresponds to a second order pole of Φ with non-real residue. Let $\{A_{a+i}\}_{1 \leq i \leq s}$ be the set of non-horizontal half-infinite cylinders of Φ . Correspondingly, Y contains $a + s$ geodesic boundary components α_i corresponding to half-infinite cylinders A_i , and k ideal geodesic boundary arcs γ_j corresponding to half-planes G_j , as well as m ideal geodesic arcs ξ_j corresponding to V_j , and b compactly supported measured laminations β_i corresponding to B_i .

We start with the construction of a closed hyperbolic surface which consists of four copies of Y . We first take a copy Y' of Y and then glue it back to Y with a "shear" of amount $t \neq 0$ along every ideal geodesic arc γ_i . This yields a hyperbolic surface \tilde{Y} with $2a + 2s + k$ boundary components $\alpha_1, \alpha'_1, \dots, \alpha_{a+s}, \alpha'_{a+s}$ and $\delta_1, \dots, \delta_k$, where α'_i is the geodesic boundary component of the copy Y' corresponding to α_i , and where δ_i is the boundary component obtained by the shearing along the ideal geodesic arc γ_i of $Y \subset \tilde{Y}$. Moreover, the t -shearing along each ideal geodesic yields that all of δ_i have the same length $|2t|$ (all that is important here

is the length of each boundary component of \tilde{Y} is positive). We then get a closed hyperbolic surface W which is obtained by gluing an isometric copy \tilde{Y}' of \tilde{Y} to \tilde{Y} in an orientation-preserving way along each geodesic boundary component. Notice that W consists of four copies of Y . We denote these four copies by Y, Y', Y'', Y''' such that the projection map $W \rightarrow \tilde{Y}$ sends Y and Y''' (resp. Y' and Y'') to Y (resp. Y'). Each of $\alpha_i, \gamma_i, \xi_i, \beta_i$ gives a copy on Y', Y'', Y''' , denoted by $\alpha'_i, \alpha''_i, \alpha'''_i$; $\gamma'_i, \gamma''_i, \gamma'''_i$; $\xi'_i, \xi''_i, \xi'''_i$ and $\beta'_i, \beta''_i, \beta'''_i$. Because of the gluing process, each of the pairs $\{\gamma_i, \gamma'_i\}$, $\{\gamma''_i, \gamma'''_i\}$, $\{\alpha_i, \alpha'_i\}$, $\{\alpha''_i, \alpha'''_i\}$ are identified on W . Let $\gamma_i, \tilde{\gamma}_i, \alpha_i, \alpha'_i$ be the resulting geodesics on W . Consider the geodesic lamination μ :

$$\mu := \left(\bigcup_{1 \leq i \leq a+s} (\alpha_i \cup \alpha'_i) \right) \cup \left(\bigcup_{1 \leq i \leq k} (\gamma_i \cup \tilde{\gamma}_i) \right) \cup \left(\bigcup_{1 \leq i \leq k} \delta_i \right) \\ \cup \left(\bigcup_{1 \leq j \leq m} (\xi_j \cup \xi'_j \cup \xi''_j \cup \xi'''_j) \right) \cup \left(\bigcup_{1 \leq j \leq b} (\beta_j \cup \beta'_j \cup \beta''_j \cup \beta'''_j) \right).$$

The sublamination

$$\left(\bigcup_{1 \leq i \leq a+s} (\alpha_i \cup \alpha'_i) \right) \cup \left(\bigcup_{1 \leq i \leq k} (\gamma_i \cup \tilde{\gamma}_i) \right) \cup \left(\bigcup_{1 \leq i \leq k} \delta_i \right)$$

cuts W into four components which are exactly Y, Y', Y'', Y''' .

Now we construct a sequence of measured laminations on W whose supports converge to that of μ . Notice that both γ_i and $\tilde{\gamma}_i$ spiral around two of $\{\delta_1, \dots, \delta_k\}$, say δ_{i1} and δ_{i2} . Let ${}^\perp\gamma_i$ (resp. ${}^\perp\tilde{\gamma}_i$) be the (open) geodesic arc on $W \setminus (\delta_1 \cup \dots \cup \delta_k)$ whose closure is orthogonal to both δ_{i1} and δ_{i2} and which is freely homotopic to γ_i (resp. $\tilde{\gamma}_i$). (Here, in the universal cover, we have that a lift of ${}^\perp\gamma_i$ connects lifts of δ_{i1} and δ_{i2} while a lift of γ_i might be the common asymptotic to those lifts of δ_{i1} and δ_{i2} .) We then close up ${}^\perp\gamma_i$ and ${}^\perp\tilde{\gamma}_i$ using subarcs of δ_{i1} and δ_{i2} . Denote the resulting simple closed curve by $\gamma_{i,0}$. Applying the same operation to the pairs $\{\xi_i, \xi'_i\}$ and $\{\xi''_i, \xi'''_i\}$, we get simple closed curves $\xi_{i,0}$ and $\xi'_{i,0}$. The closing up process is chosen so that the geodesic representatives of $\{\gamma_{i,0}, \xi_{j,0}, \xi'_{j,0}\}_{1 \leq i \leq k, 1 \leq j \leq m}$ are pairwise disjoint.

For each of δ_l, α_{a+r} , and α'_{a+r} , let $T_{\delta_l}^n, T_{\alpha_{a+r}}^n$ and $T_{\alpha'_{a+r}}^n$ be respectively the n times right or left Dehn twists about δ_l, α_{a+r} , and α'_{a+r} , where the direction right or left is chosen according to the spiralling direction of $\{\gamma_i, \xi_j\}_{1 \leq i \leq k, 1 \leq j \leq m}$ around δ_l . Let

$$\begin{aligned} \gamma_{i,n} &:= T_{\alpha'_{a+s}}^n \circ \dots \circ T_{\alpha'_{a+1}}^n \circ T_{\alpha_{a+s}}^n \circ \dots \circ T_{\alpha_{a+1}}^n \circ T_{\delta_k}^n \circ \dots \circ T_{\delta_1}^n(\gamma_{i,0}), \\ \xi_{j,n} &:= T_{\alpha'_{a+s}}^n \circ \dots \circ T_{\alpha'_{a+1}}^n \circ T_{\alpha_{a+s}}^n \circ \dots \circ T_{\alpha_{a+1}}^n \circ T_{\delta_k}^n \circ \dots \circ T_{\delta_1}^n(\xi_{j,0}), \\ \xi'_{j,n} &:= T_{\alpha'_{a+s}}^n \circ \dots \circ T_{\alpha'_{a+1}}^n \circ T_{\alpha_{a+s}}^n \circ \dots \circ T_{\alpha_{a+1}}^n \circ T_{\delta_k}^n \circ \dots \circ T_{\delta_1}^n(\xi'_{j,0}). \end{aligned}$$

Then as $n \rightarrow \infty$, the geodesic representatives of $\gamma_{i,n}, \xi_{j,n}$, and $\xi'_{j,n}$ converge respectively to the closures of $\gamma_i \cup \tilde{\gamma}_i, \xi_j \cup \xi'_j$, and $\xi''_j \cup \xi'''_j$. (This is elementary hyperbolic geometry, as the lifts of the curves $\gamma_{i,n}, \xi_{j,n}$, and $\xi'_{j,n}$ to the hyperbolic plane have unique limits, hence the ones specified.) In particular, as $n \rightarrow \infty$, the geodesic

representative of $\sum_{i=1}^k \gamma_{i,n} + \sum_{j=1}^m (\xi_{j,n} + \xi'_{j,n})$ converges to

$$\left(\bigcup_{1 \leq i \leq s} (\alpha_{a+i} \cup \alpha'_{a+i}) \right) \cup \left(\bigcup_{1 \leq i \leq k} (\gamma_i \cup \widehat{\gamma}_i) \right) \cup \left(\bigcup_{1 \leq i \leq k} \delta_i \right) \\ \cup \left(\bigcup_{1 \leq j \leq m} (\xi_j \cup \xi'_j \cup \xi''_j \cup \xi'''_j) \right).$$

The desired sequence of measured laminations μ_n on W is defined as:

$$\mu_n := n \sum_{1 \leq i \leq a} (\alpha_i + \alpha'_i) + n \sum_{1 \leq i \leq k} \gamma_{i,n} \\ + \mathbf{w}_j \sum_{1 \leq j \leq m} (\xi_{j,n} + \xi'_{j,n}) + \sum_{1 \leq j \leq b} (\beta_j + \beta'_j + \beta''_j + \beta'''_j),$$

where \mathbf{w}_j is the width of the horizontal strip I_j of Φ corresponding to ξ_j . Compare the constructions of μ and μ_n . It is clear that as $n \rightarrow \infty$, the support of μ_n converges to that of μ .

Let $X_n \in \mathcal{T}(W)$ be the Riemann surface such that the horizontal measured foliation of the Hopf differential Φ_n of the harmonic map $f_n : X_n \rightarrow W$ is equivalent to μ_n . (Again, this follows from [Wol98], cf subsection 3.3.) We decompose X_n according to the realization of components of μ_n by the horizontal foliation of Φ_n as in Table 1, where we list the subsurfaces in the second row corresponding to the subfoliations in the first row.

subfoliations of μ_n	α_i, α'_i	$\gamma_{i,n}$	$\xi_{j,n}, \xi'_{j,n}$	$\beta_j, \beta'_j, \beta''_j, \beta'''_j$
subsurfaces of X_n	$A_{i,n}, A'_{i,n}$	$G_{i,n}$	$V_{j,n}, V'_{j,n}$	$B_{j,n}, B'_{j,n}, B''_{j,n}, B'''_{j,n}$

TABLE 1. Correspondence between subfoliations of μ_n and subsurfaces of X_n .

The remainder of this subsection considers the convergence of the family of harmonic maps $f_n : X_n \rightarrow W$, in a certain sense. We start with the following lemma, which is an analogue of Lemma A.2.

Lemma A.3. *There exists a constant $D > 0$ depending on W such that for any $n > 1$ and any $1 \leq j \leq b$, the diameter of each of $\overline{B_{j,n}}, \overline{B'_{j,n}}, \overline{B''_{j,n}}, \overline{B'''_{j,n}}$ is at most D with respect to the $|\Phi_n|$ -metric.*

Proof. We demonstrate the proof for the family $\{\overline{B_{j,n}}\}_{n \geq 1}$. The proofs for the other three families are exactly the same. Suppose to the contrary that there exists a subsequence of $\{\overline{B_{j,n}}\}_{n \geq 1}$ whose diameter goes to infinity, say $\{\overline{B_{1,n}}\}_{n \geq 1}$. Without loss of generality, we may assume this subsequence is $\overline{B_{1,n}}$ itself. As in the proof of Lemma A.2, there exists a sequence of flat cylinders C_n (not necessary maximal) meeting $\overline{B_{1,n}}$ such that

- the circumference of C_n is less than some constant D_1 ;
- the width of C_n is bigger than $2d_n - 4D_1 - 2R$;
- every point of C_n is of distance at least R from the zero set of Φ_n ,

where $\{d_n\}$ is a divergent positive sequence and R is a sufficiently large constant.

Next, we claim that $C_n \subset B_{1,n}$ for sufficiently large n . Otherwise the boundary of $B_{1,n}$, being a union of saddle connections, would cross C_n . Then $\partial B_{1,n} \cap C_n$ has Φ_n -length at least the width of C_n which is bigger than $2d_n - 4D_1 - 2R$. On the other hand, by the third property of C_n mentioned above, every point of $\partial B_{1,n} \cap C_n$ is of distance at least R from the zero set of Φ . By Minsky's estimate, the image of $\partial B_{1,n} \cap C_n$ is very close to the geodesic representative $[\partial B_{1,n}]$ of $\partial B_{1,n}$ on W . Hence by (3.1),

$$\ell_W([\partial B_{1,n}]) \geq |\partial B_{1,n} \cap C_n|_{|\Phi_n|} \geq 2d_n - 4D_1 - 2R$$

which goes to infinity as $n \rightarrow \infty$. But the fact $B_{1,n}$ is topologically the supporting subsurface of β_1 on W implies the geodesic representative of $\partial B_{1,n}$ on W has a fixed length. This is a contradiction. Therefore, $C_n \subset B_{1,n}$.

The remaining part of the proof is exactly the same as that of Lemma A.2. \square

Proof of Theorem A.1 Case II. Let $z_{i,n}$ be a zero of Φ_n in $B_{i,n}$. By Lemma A.3, the compact set $B_{i,n}$ is contained in the polygonal region P_R of X_n for $R > D$. By Lemma 4.1 and [McM89, Theorem A.3.1], we see that the sequence of the pointed flat surfaces $(X_n, \Phi_n, z_{i,n})$ contains a subsequence which converges to a singular flat surface $(X_{i,\infty}, \Phi_{i,\infty}, z_{i,\infty})$ induced by some meromorphic quadratic differential $\Phi_{i,\infty}$. Moreover, the sequence of harmonic maps $f_n : X_n \rightarrow W$ (sub)converges to a limit harmonic map $f_{i,\infty} : X_{i,\infty} \rightarrow W$ with Hopf differential $\Phi_{i,\infty}$.

Step 1: possible contributions to $(X_{i,\infty}, \Phi_{i,\infty})$ from $A_{j,n}, A'_{j,n}, G_{j,n}, V_{j,n}, V'_{j,n}$. Since the height of $A_{j,n}$ goes to infinity for each j as $n \rightarrow \infty$, the circumference of $A_{j,n}$ converges to $\ell_W(\alpha_j)/2$ by Minsky's estimate. This means that the possible contribution of $A_{j,n}$ to $(X_{i,\infty}, \Phi_{i,\infty})$ is a half-infinite horizontal cylinder. Similarly, the possible contribution of each of $A'_{j,n}$ to $(X_{i,\infty}, \Phi_{i,\infty})$ is also a half-infinite horizontal cylinder.

Notice that since the height of $G_{j,n}$ also goes to infinity as $n \rightarrow \infty$, the image of the core curve $\gamma_{j,n}$ under $f_n : X_n \rightarrow W$ is approximately the geodesic representative of $\gamma_{j,n}$ on W . Recall that as $n \rightarrow \infty$, $\gamma_{j,n}$ converges to the union of $\gamma_j \cup \hat{\gamma}_j$ and the two closed curves from $\{\delta_i\}$ to which they spiral. Therefore, as $n \rightarrow \infty$, the circumference of $G_{j,n}$ for each j converges to $\ell_W(\gamma_j)/2$ which is infinite. This means that the possible contribution of $G_{j,n}$ to $(X_{i,\infty}, \Phi_{i,\infty})$ is a half-plane.

As for $V_{j,n}$, by Lemma 3.8, we get

$$2\|\Phi_n|_{V_{j,n}}\| \geq \ell_W(\xi_{j,n}) - C,$$

which diverges as $n \rightarrow \infty$. Combining this with the fact that the height of $V_{j,n}$ is always \mathbf{w}_j , we see that for each i , the circumference $\ell(V_{j,n})$ of $V_{j,n}$ diverges as $n \rightarrow \infty$. This implies that the possible contribution of $V_{j,n}$ to $(X_{i,\infty}, \Phi_{i,\infty})$ is an infinite strip with height \mathbf{w}_j . Similarly, the possible contribution of $V'_{j,n}$ to $(X_{i,\infty}, \Phi_{i,\infty})$ is also an infinite strip of height \mathbf{w}_j .

Step 2: $f_{i,\infty}(X_{i,\infty})$ is disjoint from any of

$$\{\alpha_l, \alpha'_l, \gamma_j, \hat{\gamma}_j : 1 \leq l \leq a + s, 1 \leq j \leq k\}.$$

We first show that $\alpha_l \cap f_{i,\infty}(X_{i,\infty}) = \emptyset$ for $1 \leq l \leq a$. (As a reminder, these first a geodesics come from purely horizontal cylinders – there is no spiraling of the horizontal foliation.) Let $L_{l,n}$ be the central core curve of $A_{l,n}$, namely the

core curve whose distance to $\partial A_{i,n}$ is $n/2$. Then by Theorem 3.5, we see that $f_n(L_{l,n})$ converges to α_l as $n \rightarrow \infty$. Now suppose that $\alpha_l \cap f_{i,\infty}(X_{i,\infty}) \neq \emptyset$. Let p be a point in the intersection set. Then there exists a neighbourhood U of p with $U \subset f_{i,\infty}(X_{i,\infty})$. Since $f_n : X_n \rightarrow W$ converges to $f_{i,\infty} : X_{i,\infty} \rightarrow W$, it follows that there exists small neighbourhoods $U_n \subset X_n$ with $f_n(U_n) \subset U$ and also with U_n approximating some fixed region in $X_{i,\infty}$ and hence at a uniformly bounded distance from the zeroes of Φ_n . On the other hand, the discussion above implies that there exists a sequence of points $p_n \in L_{l,n}$ whose distance to the zeros of Φ_n diverges such that $f_n(p_n) \rightarrow p \in \alpha_l$. In particular, $f_n(p_n) \in f_n(U_n)$ but $p_n \notin U_n$. This contradicts the fact that f_n is a homeomorphism, proving that $\alpha_l \cap f_{i,\infty}(i, \infty) = \emptyset$ for $1 \leq l \leq a$. Taking similar core curves of $A'_{l,n}$ and $G_{j,n}$ we see that $\alpha'_l \cap f_{i,\infty}(i, \infty) = \emptyset$ for $1 \leq l \leq a$ and $(\gamma_j \cup \widehat{\gamma}_j) \cap f_{i,\infty}(i, \infty) = \emptyset$ for $1 \leq j \leq k$.

It remains to show that $(\alpha_{a+l} \cup \alpha'_{a+l}) \cap f_{i,\infty}(X_{i,\infty}) = \emptyset$ for $1 \leq l \leq s$. Recall that, by the way we constructed the curves $\xi_{j,n}$ and $\xi'_{j,n}$, the Hausdorff limit of $\bigcup_{j=1}^m (\xi_{j,n} \cup \xi'_{j,n})$ contains $\bigcup_{l=1}^s (\alpha_{a+l} \cup \alpha'_{a+l})$. This implies that for each α_{a+l} and α'_{a+l} with $1 \leq l \leq s$, there exists $1 \leq j_l \leq k$ such that α_{a+l} and α'_{a+l} are respectively contained in the Hausdorff limit of $\xi_{j_l,n}$ and $\xi'_{j_l,n}$ as $n \rightarrow \infty$. Consider the corresponding cylinders $V_{j_l,n}$ and $V'_{j_l,n}$. By Step 1 we know that both of their circumferences diverge to infinity as $n \rightarrow \infty$. Now, combining Theorem 3.4, Lemma 4.1 and the fact that $V_{j_l,n}$ and $V'_{j_l,n}$ have fixed width w_{j_l} , we see that for any $R > R_0$, there exists $c > 0$ such that each of $V_{j_l,n}$ and $V'_{j_l,n}$ crosses the Minsky's polygonal region \mathcal{P}_R at most c times. Consider the intersection $\mathcal{P}_R \cap V_{j_l,n}$ (resp. $\mathcal{P}_R \cap V'_{j_l,n}$). By Theorem 3.4, the area of \mathcal{P}_R is at most CR^2 , and the horizontal segments of $\partial \mathcal{P}_R$ has length at most K_1R . Combining with the fact that both $V_{j_l,n}$ and $V'_{j_l,n}$ has fixed width w_{j_l} , we see that the lengths of horizontal leaves of every component of $\mathcal{P}_R \cap V_{j_l,n}$ (resp. $\mathcal{P}_R \cap V'_{j_l,n}$) are bounded above by some constant C_1 . (Here we apply the area bound to substrips of $V_{j_l,n}$ that are contained in the interior of \mathcal{P}_R and the boundary estimate to substrips of $V_{j_l,n}$ that only meet the boundary $\partial \mathcal{P}_R$; the two bounds together yield the uniform bound C_1 .) Therefore, the central core curve of $V_{j_l,n}$ ($V'_{j_l,n}$) has total length at least $\ell(V_{j_l,n}) - cC_1$ (resp. $\ell(V'_{j_l,n}) - cC_1$) outside \mathcal{P}_R , where $\ell(V_{j_l,n})$ and $\ell(V'_{j_l,n})$ are respectively the Φ_n -circumferences of $V_{j_l,n}$ and $V'_{j_l,n}$. This then implies that for sufficiently large n , the central core curve of $V_{j_l,n}$ (reps. $V'_{j_l,n}$) contains a subsegment of length at least $\frac{\ell(V_{j_l,n})}{c} - C_1$ (resp. $\frac{\ell(V'_{j_l,n})}{c} - C_1$) which is contained in the complement $X_n \setminus \mathcal{P}_R$ of \mathcal{P}_R . By Theorem 3.5, the images of these segments are contained in the ϵ_R -neighbourhood of, and are nearly parallel to, $\xi_{j_l,n} \cup \xi'_{j_l,n}$ which converges to $\xi_{j_l} \cup \xi'_{j_l} \cup \xi''_{j_l} \cup \xi'''_{j_l} \cup \alpha_{a+l} \cup \alpha'_{a+l}$ as $n \rightarrow \infty$: here we have chosen R so that ϵ_R is sufficiently small. Since both $\ell(V_{j_l,n})$ and $\ell(V'_{j_l,n})$ diverge, it follows that $\alpha_{a+l} \cup \alpha'_{a+l}$ is contained in the Hausdorff limit of the images of these segments. Now suppose that $(\alpha_{a+l} \cup \alpha'_{a+l}) \cap f_{i,\infty}(X_{i,\infty}) \neq \emptyset$. Similarly to our previous argument, let p be a point in the intersection set. Then there exists a neighbourhood U of p with $U \subset f_{i,\infty}(X_{i,\infty})$. Since $f_n : X_n \rightarrow W$ converges to $f_{i,\infty} : X_{i,\infty} \rightarrow W$, it follows that there exists small neighbourhoods $U_n \subset X_n$ with $f_n(U_n) \subset U$ but with U_n at uniformly bounded distance from the zeroes of Φ_n . On the other hand, the discussion above implies that there exists a sequence of points p_n in the union of the central core curves of $V_{j_l,n}$ and $V'_{j_l,n}$ whose distance to the zeros of Φ_n diverges such that $f_n(p_n) \rightarrow p \in \alpha_l \cup \alpha'_l$. In

particular, $f_n(p_n) \in f_n(U_n)$ but $p_n \notin U_n$. This contradicts the fact that f_n is a homeomorphism, proving that $(\alpha_{a+l} \cup \alpha'_{a+l}) \cap f_{i,\infty}(X_{i,\infty}) = \emptyset$ for $1 \leq l \leq s$.

Similarly, we see that $f_{i,\infty}(X_{i,\infty}) \cap \gamma_j = \emptyset$ and $f_{i,\infty}(X_{i,\infty}) \cap \widehat{\gamma}_j = \emptyset$ for $1 \leq j \leq k$.

Step 3: the image $f_{i,\infty}(X_{i,\infty})$ is exactly one of Y, Y', Y'' , and Y''' . Recall that $\Phi_{i,\infty}$ is a meromorphic differential with infinite area. Let p be a pole of $\Phi_{i,\infty}$ of order at least two.

If p has order at least three, then there exists a neighbourhood $U(p)$ near p such that $(X_{i,\infty}, \Phi_{i,\infty})$ is realized as a union of half-planes and strips. It follows from Step 1 that every such half-plane is a limit of $\{G_{j,n}\}_{g \geq 1, 1 \leq j \leq k}$ and that every strip is a limit of $\{V_{j,n}, V'_{j,n}\}_{n \geq 1, 1 \leq j \leq m}$. By Theorem 3.5, the neighbourhood $U(p)$ is mapped by $f_{i,\infty}$ to a crowned end whose ideal geodesic arcs are a subset of $\{\gamma_j\}_{1 \leq j \leq m}$.

If p is a second order pole with real residue, then a neighborhood of p defines a horizontal half-infinite cylinder $C(p)$ near p . By the analysis in Step 1, we see that this cylinder is a limit of $\{A_{j,n}, A'_{j,n}\}_{n \geq 1, 1 \leq j \leq a}$. By Theorem 3.5, the image $f_{i,\infty}(C(p))$ is an one-sided neighbourhood of some curve in $\{\alpha_l, \alpha'_l\}_{1 \leq l \leq a}$.

If p is a second order pole with non-real complex residue, this provides for a non-horizontal half-infinite cylinder C near p . Let $\omega_d \subset C$ be the core curve whose distance to the compact boundary of C is d . If C is vertical, then by Theorem 3.5 we see that the length of $f_{i,\infty}(\omega_d) \subset W$ converges to zero as $d \rightarrow \infty$. This contradicts the fact that W is a closed hyperbolic surface with shortest closed geodesic having positive length.

We are left with the case that C is neither horizontal nor vertical. Let s be the slope of C . Then $0 < |s| < \infty$. Since C is not horizontal, there exists some horizontal infinite strip crossing it. By step 1, every such strip is the limit of $\{V_{j,n}, V'_{j,n}\}_{n \geq 1}$. Without loss of generality, we assume that it is the limit of $\{V_{j,n}\}_{n \geq 1}$. Let $\ell(C)$ be the circumference of C . Recall that $(X_{i,\infty}, \Phi_{i,\infty}, z_{i,\infty})$ is a limit of (X_n, Φ_n, z_n) . There exists a non-horizontal cylinder C_n on X_n satisfying the following:

- The slope s_n of C_n converges to s as $n \rightarrow \infty$.
- The circumference $\ell(C_n)$ of C_n converges to $\ell(C)$ as $n \rightarrow \infty$.
- The width w_n goes to infinity as $n \rightarrow \infty$.
- The core curve of C_n is homotopic to that of C (because f_n is homotopic to the identity).
- The cylinder $V_{j,n}$ crosses C_n for sufficiently large n (because the limit of $V_{j,n}$ crosses C).

Combining the five properties of C_n mentioned above and Theorem 3.5, we see that for large n , the image $f_n(C_n)$ on W meets a neighbourhood of the closed geodesic on W homotopic to the core curve of C . Moreover, the image $f_n(\xi_{j,n})$ spirals around this geodesic nearly $\frac{w_n}{\ell(C_n)/|s_n|}$ times (recall that $\xi_{j,n}$ is the core curve of $V_{j,n}$). As $n \rightarrow \infty$, the limit of $f_n(\xi_{j,n})$ spirals infinitely many times around the geodesic homotopic to the core curve of C . Since f_n is homotopic to the identity, the limit of $\xi_{j,n}$ also spirals infinitely many times around the geodesic homotopic to the core curve of C . On the other hand, by the construction of $\xi_{j,n}$ we know that the only curves around which $\xi_{j,n}$ spirals infinitely many times is some curve in $\{\alpha_{a+l}, \alpha'_{a+l}, \delta_j\}_{1 \leq l \leq s, 1 \leq j \leq k}$. Therefore, the core curve of C is homotopic to some curve in $\{\alpha_{a+l}, \alpha'_{a+l}, \delta_j\}_{1 \leq l \leq s, 1 \leq j \leq k}$. It then follows from Theorem 3.5 that $f_{i,\infty}(C)$ is a one-sided neighbourhood of this curve.

Summarizing the above discussion in this step, we see that the image $f_{i,\infty}(X_{i,\infty})$ is a crowned surface bounded by some curves from $\{\alpha_l, \alpha'_l, \gamma_j, \widehat{\gamma}_j, \delta_j : 1 \leq l \leq a+s, 1 \leq j \leq k\}$. Combining this with Step 2, we see that $f_{i,\infty}(X_{i,\infty})$ is one of Y, Y', Y'' and Y''' , the four components of the complement of

$$\left(\bigcup_{1 \leq i \leq a+s} (\alpha_i \cup \alpha'_i) \right) \cup \left(\bigcup_{1 \leq i \leq k} (\gamma_i \cup \widehat{\gamma}_i) \right) \cup \left(\bigcup_{1 \leq i \leq k} \delta_i \right).$$

Step 4. horizontal foliation of Φ_∞ . For any $1 \leq i \leq b$, by Step 1 and Step 3, we see that $\Phi_{i,\infty}$ contains a compactly supported foliation B_i corresponding to the lamination β_i and that the image of $f_{i,\infty} : (X_{i,\infty}, \Phi_{i,\infty}) \rightarrow W$ is Y since $\beta_i \subset Y$ (and not one of the copies Y', Y'' or Y'''). Since $\beta_i \subset Y$ (and, again, not one of the copies Y', Y'' or Y''') for all $i = 1, 2, \dots, b$, we then see that we may apply a diagonal argument and choose a sequence of points $z_n \in \cup_{1 \leq i \leq b} \overline{B_{i,n}} \subset X_n$ so that there exists a subsequence of $f_n : (X_n, \Phi_n, z_n) \rightarrow W$ which converges to a harmonic map $f_\infty : (X_\infty, \Phi_\infty, z_\infty) \rightarrow Y$ such that the horizontal foliation of the Hopf differential Φ_∞ contains all of the laminations β_1, \dots, β_b . By Step 1, we see that the horizontal foliation of Φ_∞ contains a half-infinite cylinders corresponding to $\{A_{i,n}\}_{1 \leq i \leq a}$, k half-planes corresponding to $\{G_{j,n}\}_{1 \leq j \leq k}$, and b compactly supported foliations corresponding to $\{B_{j,n}\}_{1 \leq j \leq b}$. It remains to consider the contributions from $\{V_{j,n}, V'_{j,n}\}_{1 \leq j \leq m}$ and $\{B'_{j,n}, B''_{j,n}, B'''_{j,n}\}_{1 \leq j \leq b}$.

By the construction of $\xi_{j,n}$ for each $1 \leq j \leq m$, (the geodesic representative of) the image of the core curve $\xi_{j,n}$ of $V_{j,n}$ converges to the union of $\xi_j \cup \xi'_j$ and the closed geodesics to which they spiral; this becomes clear when one lifts the families to \mathbb{H}^2 . Combined with the fact that $\xi_j \subset Y$, this means that some portion of $V_{j,n}$ survives in Φ_∞ . By Step 1, the contribution from $\{V_{j,n}\}_{n \geq 1}$ is an infinite strip. Consequently, the horizontal foliation of Φ_∞ contains m strips corresponding to $\{V_{j,n}\}_{1 \leq j \leq m}$. On the other hand, if Φ_∞ contains some contribution from $\{V'_{j,n}\}_{n \geq 1}$, then by Step 1 this contribution is an infinite strip whose image on W contains curves homotopic to γ'_i or γ'''_i ; recall that we have identified γ'''_i and γ''_i . But neither γ'_i nor γ'''_i is homotopic to some curve contained in Y . Hence the horizontal foliation of Φ_∞ contains no contribution from any of $\{V'_{j,n}\}_{1 \leq j \leq m, n \geq 1}$.

Finally, suppose that the horizontal foliation of Φ_∞ contains some contribution from $\{\beta'_j, \beta''_j, \beta'''_j\}_{1 \leq j \leq b, n \geq 1}$, say β'_j . Then by Lemma A.3, it contains the whole of β'_j . Correspondingly, $Y = f_\infty(X_\infty)$ contains $f_\infty(\beta'_j)$, which is homotopic to β'_j (on Y'). This contradicts the fact that β'_j is contained in Y' instead of Y . Therefore, the horizontal foliation of Φ_∞ does not contain any contribution from $\{\beta'_j, \beta''_j, \beta'''_j\}_{1 \leq j \leq b, n \geq 1}$.

Summarizing the discussion above, we see that the horizontal foliation of $\Phi_{i,\infty}$ consists of a half-infinite cylinders corresponding to $\{\alpha_i\}_{1 \leq i \leq a}$, k half-planes corresponding to $\{\gamma_j\}_{1 \leq j \leq k}$, m bi-infinite strips corresponding to $\{\xi_j\}_{1 \leq j \leq m}$ and b compactly supported foliations corresponding to $\{\beta_j\}_{1 \leq j \leq b}$.

Step 5. minimal graph. By Step 4, we know that the horizontal foliation of $\Phi_{i,\infty}$ consists of a half-infinite cylinders corresponding to $\{\alpha_i\}_{1 \leq i \leq a}$, k half-planes corresponding to $\{\gamma_j\}_{1 \leq j \leq k}$, m bi-infinite strips corresponding to $\{\xi_j\}_{1 \leq j \leq m}$ and b compactly supported foliations corresponding to $\{\beta_j\}_{1 \leq j \leq b}$. In other words, the horizontal foliation of Φ_∞ is the same as that of Φ . Therefore, X_∞ is a minimal graph in $\widetilde{Y} \times T$. This completes the proof. \square

Remark A.4. In the above proof, we implicitly assume that the horizontal foliation of Φ has non-trivial compact components. If the horizontal foliation of Φ has no compact component (i.e. the foliation comprises only half-infinite cylinders, half-planes and infinite strips), then we simply take z_n to be an arbitrary zero of Φ_n .

The ideas in this section also lead to the following characterization of Thurston stretch lines.

Corollary A.5. *Let $Y \in \mathcal{T}(S)$ be a hyperbolic surface and λ be a maximal geodesic lamination. Let $\mathbf{SR}_{Y,\lambda}$ be the Thurston stretch line determined by Y and λ . Then $\mathbf{SR}_{Y,\lambda}$ is a harmonic stretch line if and only if λ is chain-recurrent.*

Proof. Suppose that $\mathbf{SR}_{Y,\lambda}$ is a harmonic stretch line. Then by definition, there exists a sequence of harmonic map rays $\mathbf{HR}_{X_n,Y}$ for some $X_n \in \mathcal{T}(S)$.

By Lemma 4.5, the sequence of harmonic maps $f_n : X_n \rightarrow Y$ (sub)converges to a surjective harmonic diffeomorphism $f_\infty : X_\infty \rightarrow Y \setminus \mu$ for some chain-recurrent lamination $\mu \subset \lambda'$. Since by assumption $\mathbf{SR}_{Y,\lambda}$ is the limit of $\mathbf{HR}_{X_n,Y}$, we see by the second paragraph in the proof of Theorem 1.1 that $\mathbf{SR}_{Y,\lambda}$ is the piecewise harmonic stretch line constructed from $f_\infty : X_\infty \rightarrow Y \setminus \mu$ in Theorem 1.7. In particular, the harmonic stretch line $\mathbf{SR}_{Y,\lambda}$ maximally stretches exactly μ . Therefore, $\mu = \lambda$. This implies that λ is chain-recurrent.

Now we turn to the other direction. Suppose that λ is chain-recurrent. Then there exists a sequence of multicurves α_n which converges to λ in the Hausdorff topology (on the set of geodesic laminations on Y), as $n \rightarrow \infty$. Let X_n be the Riemann surface such that the horizontal foliation of the Hopf differential of the harmonic map $X_n \rightarrow Y$ is $n\alpha_n$. Let $\mathbf{HR}_{X_n,Y}$ be the corresponding harmonic map ray. By Lemma 4.5, we see that $\{\mathbf{HR}_{X_n,Y}\}_{n \geq 1}$ contains a subsequence which converges to some harmonic stretch line \mathbf{HSR} . Applying the idea of the proof of Theorem A.1, we conclude that the limiting harmonic stretch line \mathbf{HSR} maximally stretches along λ . The assumption that λ is maximal then implies that $\mathbf{HSR} = \mathbf{SR}_{Y,\lambda}$. (This follows from [DLRT20, Corollary 2.3]. In terms of the perspectives in this paper, it also follows from the construction of Thurston stretch lines and piecewise harmonic stretch lines outlined in Section 7.1.)

□

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