

On some complete monotonic functions

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Abstract

Motivated by open questions in the papers "Refinements and sharpenings of some double inequalities for bounding the gamma function" and "Complete monotonicity and monotonicity of two functions defined by two derivatives of a function involving trigamma function", we confirm among other results and disprove other one.

1 Introduction

Completely monotonic functions have attracted the attention of many authors. Mathematicians have proved many interesting results on this topic. For example, Koumandos [11] obtained upper and lower polynomial bounds for the function $x/(e^x - 1)$, $x > 0$, with coefficients of the Bernoulli numbers B_k . This enabled him to give simpler proofs of some results of H. Alzer and F. Qi et al., concerning complete monotonicity of certain functions involving the functions $\Gamma(x)$, $\psi(x)$ and the polygamma functions $\psi^{(n)}(x)$, $n = 1, 2, \dots$, [6].

A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I which alternate successively in sign, that is, $(-1)^n f^{(n)}(x) \geq 0$ for all $x \in I$ and all $n \in \mathbb{N} \cup \{0\}$. See for example [[32], Chap VIII], [[33], Chap I], and [[34], Chap IV].

A notion of logarithmically completely monotonic is introduced in reference [30] and [31]. A positive function f is said to be logarithmically completely monotonic on an open interval I , if f satisfies $(-1)^n (\log f(x))^{(n)} \geq 0$ for all $x \in I$ and all $n \in \mathbb{N}$. For more informations on completely monotonic functions, see for instance [9, 12, 13, 14, 15, 16, 20, 22, 23, 24] and closely related references.

For $x > 0$, the classical gamma function $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ first introduced by L. Euler, is one of the most important functions in mathematical analysis. It often appears in asymptotic series, hypergeometric series, Riemann zeta function, number theory, and so on.

In this section we give an answer to a problem suggested by Qi in his paper [21]. Which states, for all $m \geq 0$, the function $(-1)^m x^m \Phi^{(m)}(x)$ is completely monotonic, where $\Phi(x) = x\psi'(x) - 1$. We show that this is not true in general. Also, we investigate the q -analog of this problem.

Before proving our results, let us recall the following facts. For $m \geq 1$, we define on $(0, +\infty)$ the function

$$f_m(t) = \frac{d^{m-1}}{dt^m} \left(\frac{t^m}{1 - e^{-t}} \right).$$

Recall the Hardy-Littlewood entire function is defined on the complex plane as follows $H(z) = \sum_{k=1}^{\infty} \frac{1}{k} \sin\left(\frac{z}{k}\right)$, see [4]. We set also

$$s(z) = \frac{1}{2} + \frac{1}{\pi} H\left(\frac{z}{2\pi}\right).$$

In [4] theorem 1.1, it is proved that,

$$\lim_{m \rightarrow \infty} \frac{1}{m!} f_m(z/m) = s(z),$$

the convergence begin uniform on every compact set of the complex plane. Recall the result du to Alzet et al. See for instance [4].

Proposition 1.1 *There are positive constants C and a positive sequence x_{j_k} with $\lim_{k \rightarrow \infty} x_{j_k} = +\infty$ such that for all $k \in \mathbb{N}$,*

$$H(x_{j_k}) \leq -C \sqrt{\log \log x_{j_k}}.$$

Proposition 1.2 *Let*

$$\Phi(x) = x\psi'(x) - 1.$$

There exists $m_0 \in \mathbb{N}$, such that for every $m \geq m_0$ and $\alpha \in \{m, m-1\}$, the functions $(-1)^m x^\alpha \Phi^{(m)}(x)$ are not completely monotonic.

Proof. Using the Liebneitz rule we get for $m \geq 1$

$$\Phi^{(m)}(x) = x\psi^{(m+1)}(x) + m\psi^{(m)}(x),$$

Since,

$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n}{1 - e^{-t}} e^{-xt} dt.$$

Thus,

$$(-1)^m x^m \Phi^{(m)}(x) = \int_0^\infty \frac{t^{m+1}}{1 - e^{-t}} x^{m+1} e^{-xt} dt - m \int_0^\infty \frac{t^m}{1 - e^{-t}} x^m e^{-xt} dt,$$

Integrate by part yields,

$$(-1)^m x^m \Phi^{(m)}(x) = \int_0^\infty (f_{m+1}(t) - m f_m(t)) e^{-xt} dt,$$

where $f_m(t) = \left(t^m/(1 - e^{-t})\right)^{(m)}$.

Since, Therefore,

$$(-1)^m x^m \Phi^{(m)}(x) = \int_0^\infty g_m(t) e^{-xt} dt,$$

where

$$g(t) = t f'_m(t) + f_m(t) = (t f_m(t))'.$$

If $(-1)^m \Phi^{(m)}(x)$ is completely monotonic for all m and $t > 0$. Then, $t f_m(t)$ increases on $(0, +\infty)$, and $f_m(t) \geq 0$. Moreover, By a result of Alzer [4], we saw that $\lim_{m \rightarrow \infty} (1/m!) f_m(t/m) = s(t)$ for all $t > 0$. Which is impossible, due to the result Proposition 1.1.

Furthermore,

$$(-1)^m x^{m-1} \Phi^{(m)}(x) = \int_0^\infty t f_m(t) e^{-xt} dt,$$

Using again Proposition 1.1, we get the desired result. Namely, there is m_0 , such that for $m \geq m_0$ the functions $(-1)^m x^{m-1} \Phi^{(m)}(x)$ and $(-1)^m x^m \Phi^{(m)}(x)$ are not completely monotonic.

Proposition 1.3 *For all $m \in \mathbb{N}$, the function $(-1)^m x^{m+1} \Phi^{(m)}(x)$ is not completely monotonic.*

Proof. We have,

$$x \psi'(x) = \int_0^\infty h(t) e^{-xt} dt,$$

where, $h(t) = e^{-t}(e^t - 1 - t)/(1 - e^{-t})^2 = \left(t/(1 - e^{-t})\right)'$

$$(-1)^m x^{m+1} \Phi^{(m)}(x) = \int_0^\infty h(t) t^m x^{m+1} e^{-xt} dt.$$

Therefore,

$$\left((-1)^m x^{m+1} \Phi^{(m)}(x)\right)^{(m+1)} = (m+1)! \int_0^\infty h(t) t^m L_{m+1}(xt) e^{-xt} dt.$$

Moreover,

$$t^m h(t) = t^m + \sum_{k=1}^\infty (1 - kt) t^m e^{-kt}.$$

Then,

$$\begin{aligned} \left((-1)^m x^{m+1} \Phi^{(m)}(x)\right)^{(m+1)} &= (m+1)! \int_0^\infty t^m L_{m+1}(xt) e^{-xt} dt \\ &\quad + (m+1)! \sum_{k=1}^\infty \int_0^\infty (t^m - kt^{m+1}) L_{m+1}(xt) e^{-(x+k)t} dt. \end{aligned}$$

So, by the substitution $u = (x+k)t$, we get

$$\left((-1)^m x^{m+1} \Phi^{(m)}(x)\right)^{(m+1)} = (m+1)! \sum_{k=1}^{\infty} \int_0^{\infty} \left(\frac{u^m}{(x+k)^{m+1}} - k \frac{u^{m+1}}{(x+k)^{m+2}}\right) L_{m+1}\left(\frac{x}{x+k}u\right) e^{-u} du.$$

Moreover, $|L_{m+1}(\frac{x}{x+k}u)| \leq e^{xu/2(x+k)}$, so, for $x > 0, m \geq 1$ then $|L_{m+1}(\frac{x}{x+k}u)|e^{-u} \leq e^{-u/2}$, and $\lim_{x \rightarrow 0} \sum_{k=1}^{\infty} \frac{1}{(x+k)^m} L_{m+1}(\frac{x}{x+k}t) = \zeta(m)$. By the dominated convergence theorem, it follows that for $m \geq 1$,

$$\lim_{x \rightarrow 0^+} \left((-1)^m x^{m+1} \Phi^{(m)}(x)\right)^{(m+1)} = (m+1)! (m! \zeta(m+1) - (m+1)! \zeta(m+1)) = -m(m+1)! m! \zeta(m+1).$$

So, $(-1)^{m+1} \left((-1)^m x^{m+1} \Phi^{(m)}(x)\right)^{(m+1)}$ is negative for m odd. And the function $(-1)^m x^{m+1} \Phi^{(m)}(x)$ is not completely monotonic for m odd.

On the other hand by differentiation we get

$$\begin{aligned} & \left((-1)^m x^{m+1} \Phi^{(m)}(x)\right)^{(m+2)} \\ &= (m+1)! \sum_{k=1}^{\infty} \int_0^{\infty} \left(- (m+1) \frac{u^m}{(x+k)^{m+2}} + k(m+2) \frac{u^{m+1}}{(x+k)^{m+3}}\right) L_{m+1}\left(\frac{x}{x+k}u\right) e^{-u} du \\ &+ (m+1)! \sum_{k=1}^{\infty} \int_0^{\infty} \left(\frac{u^m}{(x+k)^{m+1}} - k \frac{u^{m+1}}{(x+k)^{m+2}}\right) \left(-\frac{ku}{(x+k)^2}\right) L'_{m+1}\left(\frac{x}{x+k}u\right) e^{-u} du. \end{aligned}$$

Since, $L'_{m+1}(0) = -(m+1)$. It follows that

$$\begin{aligned} \lim_{x \rightarrow 0^+} \left((-1)^m x^{m+1} \Phi^{(m)}(x)\right)^{(m+2)} &= (m+1)! \sum_{k=1}^{\infty} \int_0^{\infty} \left(- (m+1) \frac{u^m}{k^{m+2}} + (m+2) \frac{u^{m+1}}{k^{m+2}}\right) e^{-u} du \\ &+ (m+1)! \sum_{k=1}^{\infty} \int_0^{\infty} \left(\frac{u^m}{k^{m+1}} - \frac{u^{m+1}}{k^{m+1}}\right) (m+1) \frac{u}{k} e^{-u} du. \end{aligned}$$

$$\lim_{x \rightarrow 0^+} \left((-1)^m x^{m+1} \Phi^{(m)}(x)\right)^{(m+2)} = ((m+1)!)^2 (\zeta(m+2) - (m+2)\zeta(m+2) + \zeta(m+1) - (m+2)\zeta(m+1)).$$

$$\lim_{x \rightarrow 0^+} \left((-1)^m x^{m+1} \Phi^{(m)}(x)\right)^{(m+2)} = -((m+1)!)^2 (m+1) (\zeta(m+2) + \zeta(m+1)).$$

So,

$$\lim_{x \rightarrow 0^+} (-1)^{m+2} \left((-1)^m x^{m+1} \Phi^{(m)}(x)\right)^{(m+2)} = (-1)^{m+1} ((m+1)!)^2 (\zeta(m+2) + \zeta(m+1)).$$

The right hand side is negative for m even. Thus, the function $(-1)^m x^{m+1} \Phi^{(m)}(x)$ is not completely monotonic for m even. Which implies that the function $(-1)^m x^{m+1} \Phi^{(m)}(x)$ is not completely monotonic for all $m \geq 0$. Thus, $\deg_{cm}^x((-1)^m \Phi^{(m)}(x)) < m+1$.

Proposition 1.4 For all $m \in \mathbb{N}$, the function $(-1)^m x^{m-2} \Phi^{(m)}(x)$ is completely monotonic.

Proof. Following the proof of Proposition 1.2, we have

$$(-1)^m x^{m-2} \Phi^{(m)}(x) = \int_0^\infty \Theta_m(t) e^{-xt} dt,$$

where

$$\Theta_m(t) = \int_0^t u f_m(u) du.$$

We saw that

$$f_m(t) = m! + m! \sum_{k=1}^{\infty} e^{-kt} L_m(kt),$$

and by the fact that, $|L_m(x)| \leq e^{\frac{x}{2}}$ for all $x > 0$. It then follows that

$$\left| \sum_{k=1}^{\infty} e^{-kt} L_{m-1}(kt) \right| \leq \frac{e^{-\frac{t}{2}}}{1 - e^{-\frac{t}{2}}},$$

hence,

$$m! \left(1 - \frac{e^{-\frac{t}{2}}}{1 - e^{-\frac{t}{2}}}\right) \leq f_m(t),$$

It is easy seeing that $1 - \frac{e^{-\frac{t}{2}}}{1 - e^{-\frac{t}{2}}} \geq \frac{1}{2}$ if and only if $t \geq 2 \log 3$. Then, for all $t \geq 2 \log 3 \simeq 2.19$

$$f_m(t) \geq \frac{m!}{2} t.$$

Moreover, H. Alzer et al. [4] (p.113) showed that for all $u \in (-2\pi, 2\pi)$,

$$f_m(u) = \int_0^\infty s(ux) x^m e^{-x} dx,$$

where,

$$s(x) = \frac{1}{2} + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \sin\left(\frac{x}{2k\pi}\right).$$

Let $t \in [0, 2\pi)$, then,

$$\int_0^t u f_m(u) du = \int_0^t \left(\int_0^\infty s(ux) x^m e^{-x} dx \right) u du.$$

then,

$$\begin{aligned} \int_0^t u f_m(u) du &= \int_0^\infty \left(\frac{t^2}{4} + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \int_0^t u \sin\left(\frac{xu}{2k\pi}\right) du \right) x^m e^{-x} dx \\ &= 2 \int_0^\infty \left(\frac{t^2}{8} + \sum_{k=1}^{\infty} \left(\sin\left(\frac{tx}{2k\pi}\right) - \frac{tx}{2\pi k} \cos\left(\frac{tx}{2k\pi}\right) \right) \right) x^{m-2} e^{-x} dx. \end{aligned}$$

$$2 \int_0^\infty \left(\sum_{k=1}^\infty \left(\frac{3(tx)^2}{(2k\pi)^2} + \sin\left(\frac{tx}{2k\pi}\right) - \frac{tx}{2\pi k} \cos\left(\frac{tx}{2k\pi}\right) \right) \right) x^{m-2} e^{-x} dx.$$

Since, the function $3x^2 - x \cos x + \sin x$ increases on $[0, +\infty)$ and then $3x^2 - x \cos x + \sin x \geq 0$. Therefore, for all $t \geq 0$, $\int_0^t u f_m(u) du \geq 0$. This completes the proof.

Proposition 1.5 *For all $m \in \mathbb{N}$, the function $(-1)^m x^{m+1} \Phi^{(m)}(x)$ decreases on $(0, +\infty)$. The double inequality*

$$\frac{m!}{2x^{m+1}} < (-1)^m \Phi^{(m)}(x) < \frac{m!}{x^{m+1}}.$$

is valid on $(0, +\infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers respectively.

Proof. Applying Liebnitz rule we get $m \geq 1$

$$\Phi^{(m)}(x) = x\psi^{(m+1)} + m\psi^{(m)}(x), \quad (1.1)$$

Differentiate the function $x^{m+1}\Phi^{(m)}(x)$ yields

$$\left((-1)^m x^{m+1} \Phi^{(m)}(x) \right)' = x^m (-1)^m \left(m(m+1)\psi^{(m)} + 2(m+1)x\psi^{(m+1)}(x) + x^2\psi^{(m+2)}(x) \right). \quad (1.2)$$

Using the integral representation

$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \phi_n(t) e^{-xt} dt,$$

where $\phi_n(t) = t^n/(1 - e^{-t})$. It follows, that

$$\begin{aligned} & (-1)^m (m(m+1)\psi^{(m)} + 2(m+1)x\psi^{(m+1)}(x) + x^2\psi^{(m+2)}(x)) \\ &= \int_0^\infty \left(2(m+1)\phi'_{m+1}(t) - \phi''_{m+2}(t) - m(m+1)\phi_m(t) \right) e^{-xt} dt. \end{aligned}$$

By some algebra we get

$$2(m+1)\phi'_{m+1}(t) - \phi''_{m+2}(t) - m(m+1)\phi_m(t) = \frac{t^{1+m}e^t}{(e^t - 1)^3} (e^t(2-t) - t - 2).$$

Set $\Theta(t) = e^t(2-t) - t - 2$, then $\Theta'(t) = e^t(1-t) - 1$, and $\Theta''(t) = -te^t$, moreover, $\Theta'(0) = 0$ and $\Theta(0) = 0$. Therefore, $\Theta(t) < 0$ for all $t > 0$. From equation (1.2), one deduces that the function $(-1)^m x^{m+1} \Phi(x)$ is strictly increasing on $(0, +\infty)$. On the other hand, we saw that

$$\psi^{(m)}(x+1) = \psi^{(m)}(x) + \frac{(-1)^m m!}{x^{m+1}}, \quad (1.3)$$

and for x large enough

$$\psi^{(m)}(x)(x) = (-1)^{m-1} \left(\frac{(m-1)!}{x^m} + \frac{m!}{2x^{m+1}} + R_m(x) \right), \quad (1.4)$$

with $\lim_{x \rightarrow \infty} x^{m+1} R_m(x) = 0$. By equation (1.1), we get

$$x^{m+1} \Phi^{(m)}(x) = x^{m+2} \psi^{(m+1)}(x+1) - (-1)^{m+1} (m+1)! + x^{m+1} \psi^{(m)}(x+1) - (-1)^m m(m)!,$$

and

$$\lim_{x \rightarrow 0} (-1)^m x^{m+1} \Phi^{(m)}(x) = m!.$$

Moreover, using (1.4) we obtain

$$x^{m+1} \Phi^{(m)}(x) = (-1)^m (m!x + \frac{(m+1)!}{2} + x^{m+2} R_{m+1}(x)) + m(-1)^{m-1} ((m-1)!x + \frac{m!}{2x} + x^{m+1} R_m(x)).$$

Hence,

$$\lim_{x \rightarrow +\infty} (-1)^m x^{m+1} \Phi^{(m)}(x) = \frac{m!}{2}.$$

This completes the proof.

Proposition 1.6 For $q \in (0, 1)$, let define on $(0, +\infty)$ the function

$$\Phi_q(x) = \frac{q^x - 1}{\log q} \psi'_q(x) - q^x,$$

For all $n \geq 0$, the function $(-1)^n \Phi_q^{(n)}(x)$ is completely monotonic.

For $q > 1$ the function

$$\phi_q(x) = \frac{q^x - 1}{q^x \log q} \psi'_q(x) - 1,$$

is completely monotonic on $(0, +\infty)$.

Proof. Since,

$$\psi'_q(x) = \int_0^\infty \frac{t}{1 - e^{-t}} e^{-xt} \gamma_q(dt),$$

where γ_q is the positive discrete measure

$$\gamma_q(t) = \begin{cases} -\log q \sum_{k=1}^{\infty} \delta_{t+k \log q} & \text{if } q \in (0, 1) \\ = t & \text{if } q = 1 \end{cases}.$$

By differentiation with respect to x , we get

$$\left(\frac{q^x - 1}{\log q} e^{-xt} \right)^{(n)} = -\frac{1}{\log q} (-1)^n t^n e^{-xt} + \sum_{k=0}^n C_n^k q^x (\log q)^{k-1} (-1)^{n-k} t^{n-k} e^{-xt}. \quad (1.5)$$

Since,

$$\left(\frac{q^x - 1}{\log q} e^{-xt}\right)^{(n)} = -\frac{1}{\log q} (-1)^n (t^n - q^x (t - \log q)^n) e^{-xt}.$$

Thus,

$$(-1)^n \Phi_q^{(n)}(x) = -\frac{1}{\log q} \int_0^\infty \frac{t}{1 - e^{-t}} (t^n - q^x (t - \log q)^n) e^{-xt} \gamma_q(dt) - (-\log q)^n q^x. \quad (1.6)$$

We split the integral on $(0, x_0)$ and $(x_0, +\infty)$ where $x_0 = -(q^{\frac{x}{n}} \log q)/(1 - q^{\frac{x}{n}})$. Easy computations reveal that the function $U : t \mapsto t/(1 - e^{-t})$ is strictly increasing on $(0, +\infty)$, with $\lim_{t \rightarrow 0} U(t) = 1$. Let $n \geq 1$, and employing (1.6). We get

$$(-1)^n \Phi_q^{(n)}(x) \geq -\frac{U(x_0)}{\log q} \int_0^\infty (t^n - q^x (t - \log q)^n) e^{-xt} \mu_q(dt) - (-\log q)^n q^x.$$

Moreover, by using the fact that

$$\Theta_q(x) := \log q \frac{q^x}{q^x - 1} = \int_0^\infty e^{-xt} \gamma_q(dt),$$

we get

$$\begin{aligned} -\frac{1}{\log q} \int_0^\infty (t^n - q^x (t - \log q)^n) e^{-xt} \gamma_q(dt) &= -\frac{(-1)^n}{\log q} \Theta_q^{(n)}(x) + \sum_{k=0}^n C_n^k q^x (\log q)^{k-1} (-1)^k \Theta_q^{(n-k)}(x). \\ -\frac{1}{\log q} \int_0^\infty (t^n - q^x (t - \log q)^n) e^{-xt} \mu_q(dt) &= -\frac{(-1)^n}{\log q} \Theta_q^{(n)}(x) + \frac{(-1)^n}{\log q} (q^x \Theta_q(x))^{(n)} = (-1)^n (\log q)^n q^x. \end{aligned}$$

So, for all $n \geq 1$,

$$(-1)^n \Phi_q^{(n)}(x) \geq (-\log q)^n q^x (U(x_0) - 1) \geq 0.$$

Furthermore, one deduces that $\Phi_q(x)$ decreases and we have $\lim_{x \rightarrow +\infty} \Phi_q(x) = 0$ for $q \in (0, 1)$. Thus, $\Phi_q(x) \geq 0$.

The second item follows from the fact that for $q > 1$, $\phi_q(x) = \Phi_{\frac{1}{q}}(x)$.

Which completes the proof.

Corollary 1.7 *The previous result is a generalization of the following one, the function $x\psi'(x) - 1$ is completely monotonic. It corresponds to the case $q = 1$.*

In [10], it has been posed the following conjecture. The function $x^{\psi(x) - \log x - \gamma}$, is logarithmically completely monotonic on $(0, +\infty)$, where γ is the Euler constant. In the next proposition we prove a more general result.

Proposition 1.8 *The function*

$$f_\alpha(x) = x^{x(\psi(x) - \log x) - \alpha},$$

is logarithmically completely monotonic on $(0, +\infty)$ for $\alpha \geq -1/4$.

Recall the following results, see for instance [33], [Theorem 3.6, p 19]

Lemma 1.9 *Let $h : (0, +\infty) \rightarrow (0, +\infty)$ and g be two functions.*

1. *If h' and g are completely monotonic, then $g \circ h$ is also completely monotonic.*
2. *If $(-\log(h))'$ is completely monotonic, then h is also completely monotonic.*

Proof. It is established in [3] that the function $\theta_\beta(x) = x^\beta(\log x - \psi(x))$ is strictly completely monotonic on $(0, +\infty)$ if and only if $\beta \leq 1$. Moreover,

$$\theta_1(x) = \frac{1}{2} + \int_0^\infty h(t)e^{-xt} dt, \quad (1.7)$$

where $h(t) = 1/t^2 - e^{-t}/(1 - e^{-t})^2$. In [18], it is proved that h is strictly decreasing on $(0, +\infty)$ with $\lim_{x \rightarrow +\infty} h(t) = 0$ and $\lim_{x \rightarrow 0} h(t) = 1/12$. Hence, the function $\theta_1(x) - 1/2$ is completely monotonic, and,

$$\frac{1}{2} \leq \theta_1(x) \leq \frac{1}{2} + \frac{1}{12x}. \quad (1.8)$$

Let $\varphi_\alpha(x) = (-\log f_\alpha(x))'$. Then,

$$\varphi_\alpha(x) = (\alpha \log x + \theta_1(x) \log x)', \quad (1.9)$$

and

$$\varphi_\alpha(x) = \frac{\alpha + \frac{1}{2}}{x} + \frac{1}{x}(\theta_1(x) - \frac{1}{2}) + \theta_1'(x) \log x \quad (1.10)$$

We show that $(-1)^n \varphi_\alpha^{(n)}(x) > 0$ for $n \in \mathbb{N}$ and $x > 0$. Since,

$$\begin{aligned} (-1)^{(n)} \varphi^{(n)}(x) &= \frac{n!(\alpha + \frac{1}{2})}{x^{n+1}} + (-1)^n \theta_1^{(n+1)}(x) \log x + \sum_{k=1}^n C_n^k \frac{(k-1)!}{x^k} (-1)^{n-k+1} \theta_1^{(n-k+1)}(x) \\ &\quad + (-1)^n \left(\frac{1}{x} (\theta_1(x) - \frac{1}{2}) \right)^{(n)}. \end{aligned}$$

For every $x > 0$

$$(-1)^n \left(\frac{1}{x} (\theta_1(x) - \frac{1}{2}) \right)^{(n)} + \sum_{k=1}^n C_n^k \frac{(k-1)!}{x^k} (-1)^{n-k+1} \theta_1^{(n-k+1)}(x) \geq 0,$$

and $(-1)^n \theta_1^{(n+1)}(x) \leq 0$. So, $(-1)^n \varphi_\alpha^{(n)}(x) \geq 0$ for $n \geq 0$ and $x \in (0, 1)$

Let us assume, $x \geq 1$. It remains to show that for $n \geq 1$, $x \geq 1$ and $\alpha \geq -1/4$,

$$\frac{n!(\alpha + \frac{1}{2})}{x^{n+1}} + (-1)^n \theta_1^{(n+1)}(x) \log x \geq 0. \quad (1.11)$$

It sufficient to prove that for $n \geq 1$ and $x \geq 1$,

$$\frac{n!}{4x^{n+1}} + (-1)^n \theta_1^{(n+1)}(x) \log x \geq 0.$$

On the first hand,

$$(-1)^n (\theta_1(x))^{(n+1)} \log x = (-1)^n (x \log x)^{(n+1)} \log x + (-1)^{n+1} (x\psi(x))^{(n+1)} \log x,$$

and

$$(-1)^n (x \log x)^{(n+1)} = (-1)^n \left(\frac{(-1)^n n!}{x^n} + (n+1) \frac{(-1)^{n-1} (n-1)!}{x^n} \right) = -\frac{(n-1)! \log x}{x^n}.$$

Then,

$$(-1)^n (\theta_1(x))^{(n+1)} \log x = ((-1)^{n+1} (x\psi(x))^{(n+1)} - \frac{(n-1)!}{x^n}) \log x,$$

Let us denote

$$g_n(x) = (-1)^n \theta_1^{(n+1)}(x) \log x + \frac{n!}{4x^{n+1}}.$$

$$g_n(x) = ((-1)^{n+1} (x\psi^{(n+1)}(x) + (n+1)\psi^{(n)}(x)) - \frac{(n-1)!}{x^n}) \log x + \frac{n!}{4x^{n+1}},$$

$$\begin{aligned}
& g_n(x+1) - g_n(x) \\
&= ((-1)^{n+1} \left((x+1)\psi^{(n+1)}(x+1) + (n+1)\psi^{(n)}(x+1) \right) - \frac{(n-1)!}{(x+1)^n} \log(x+1) + \frac{n!}{4(x+1)^{n+1}} \\
&\quad - ((-1)^{n+1} \left(x\psi^{(n+1)}(x) + (n+1)\psi^{(n)}(x) \right) - \frac{(n-1)!}{x^n} \log x - \frac{n!}{4x^{n+1}} \\
&= ((-1)^{n+1} \left((x+1)\psi^{(n+1)}(x+1) + (n+1)\psi^{(n)}(x+1) \right) - \frac{(n-1)!}{(x+1)^n} \log(1+1/x) \\
&\quad + \left((-1)^{n+1} \left((x+1)\psi^{(n+1)}(x+1) + (n+1)\psi^{(n)}(x+1) - x\psi^{(n+1)}(x) - (n+1)\psi^{(n)}(x) \right) \right. \\
&\quad \left. - \frac{(n-1)!}{(x+1)^n} + \frac{(n-1)!}{x^n} \right) \log x + \frac{1}{4} \left(\frac{n!}{(x+1)^{n+1}} - \frac{n!}{x^{n+1}} \right) \\
&= ((-1)^{n+1} \left((x+1)\psi^{(n+1)}(x+1) + (n+1)\psi^{(n)}(x+1) \right) - \frac{(n-1)!}{(x+1)^n} \log(1+1/x) \\
&\quad + ((-1)^{n+1} \left(\frac{(-1)^{n+1}(n+1)!}{x^{n+1}} + \psi^{(n+1)}(x+1) + \frac{(-1)^n(n+1)!}{x^{n+1}} \right) - \frac{(n-1)!}{(x+1)^n} + \frac{(n-1)!}{x^n}) \log x \\
&\quad + \frac{1}{4} \left(\frac{n!}{(x+1)^{n+1}} - \frac{n!}{x^{n+1}} \right) \\
&= ((-1)^{n+1} \left((x+1)\psi^{(n+1)}(x+1) + (n+1)\psi^{(n)}(x+1) \right) - \frac{(n-1)!}{(x+1)^n} \log(1+1/x) \\
&\quad + \left((-1)^{n+1} \psi^{(n+1)}(x+1) - \frac{(n-1)!}{(x+1)^n} + \frac{(n-1)!}{x^n} \right) \log x + \frac{1}{4} \left(\frac{n!}{(x+1)^{n+1}} - \frac{n!}{x^{n+1}} \right),
\end{aligned}$$

where we used in the third equation the following equation

$$\psi^{(n+1)}(x+1) = \psi^{(n)}(x) \frac{(n-1)!}{x^n} + \frac{(-1)^n n!}{x^n},$$

Therefore,

$$\begin{aligned}
g_n(x+1) - g_n(x) &= ((-1)^{n+1} ((x+1)\psi(x+1))^{(n+1)} - \frac{(n-1)!}{(x+1)^n} \log(1+1/x) \\
&\quad + ((-1)^{n+1} \psi^{(n+1)}(x+1) - \frac{(n-1)!}{(x+1)^n} + \frac{(n-1)!}{x^n}) \log x \\
&\quad + \frac{1}{4} \left(\frac{n!}{(x+1)^{n+1}} - \frac{n!}{x^{n+1}} \right)
\end{aligned}$$

We will show that $g_n(x+1) - g_n(x) \leq 0$. Using the inequality due to Alzer for $x > 0$ and $n \geq 1$

$$(-1)^{n+1} (x\psi(x))^{(n+1)} < \frac{(n-1)!}{x^n}.$$

It sufficient to prove that

$$\left((-1)^{n+1} \psi^{(n+1)}(x+1) - \frac{(n-1)!}{(x+1)^n} + \frac{(n-1)!}{x^n} \right) \log x + \frac{1}{4} \left(\frac{n!}{(x+1)^{n+1}} - \frac{n!}{x^{n+1}} \right) \leq 0.$$

On the first hand,

$$(-1)^{n+1}\psi^{(n+1)}(x+1) - \frac{(n-1)!}{(x+1)^n} + \frac{(n-1)!}{x^n} = \int_0^\infty \left(1 - e^{-t} - \frac{t^2}{e^t - 1}\right) t^{n-1} e^{-xt} dt.$$

Moreover, the derivative of the function $1 - e^{-t} - \frac{t^2}{e^t - 1}$ is given by $\frac{e^{-t}(1+te^t - e^t)^2}{(e^t - 1)^2}$ and $\lim_{t \rightarrow 0} 1 - e^{-t} - t^2/(e^t - 1) = 0$. Therefore, the function $(-1)^{n+1}\psi^{(n+1)}(x+1) - \frac{(n-1)!}{(x+1)^n} + \frac{(n-1)!}{x^n}$ is completely monotonic.

Let us denote,

$$J(x) = \left((-1)^{n+1}\psi^{(n+1)}(x+1) - \frac{(n-1)!}{(x+1)^n} + \frac{(n-1)!}{x^n} \right) x + \frac{1}{4} \left(\frac{n!}{(x+1)^{n+1}} - \frac{n!}{x^{n+1}} \right),$$

and

$$j(x) = \left((-1)^{n+1}\psi^{(n+1)}(x+1) - \frac{(n-1)!}{(x+1)^n} + \frac{(n-1)!}{x^n} \right) \log x + \frac{1}{4} \left(\frac{n!}{(x+1)^{n+1}} - \frac{n!}{x^{n+1}} \right),$$

By the previous remark, we get $j(x) \leq J(x)$ for $x > 0$. On the other hand, and by using again the relation $n!/x^n = \int_0^\infty t^{n-1} e^{-xt} dt$, it follows that

$$J(x) = \int_0^\infty \left(\frac{e^{-t}(1+te^t - e^t)^2}{(e^t - 1)^2} + \frac{1}{4} t(e^{-t} - 1) \right) t^{n-1} e^{-xt} dt.$$

Let us denote

$$\theta(t) = (1 + te^t - e^t)^2 - \frac{1}{4} t(e^t - 1)^3.$$

By successive differentiation we get

$$\theta'(t) = \frac{1}{4} (1 + e^{2t} 3 - 2t + 8t^2 + 2(-2 + t) \cosh t + (2 - 8t) \sinh t),$$

$$\theta''(t) = \frac{1}{2} e^{2t} (2 + 6t + 8t^2 - 2(1 + t) \cosh t - (4 + 7t) \sinh t) = \frac{1}{2} e^{2t} \theta_1(t),$$

$$\theta_1'(t) = 6 + 16t - (6 + 7t) \cosh t - (9 + 2t) \sinh t,$$

Using the know inequalities $\cosh t \geq 1$ and $\sinh t \geq t$, we get $\theta_1'(t) \leq -2t^2 < 0$ for $t > 0$. Moreover, $\theta_1(0) = 0$ Then, θ'' is negative. Since, $\theta'(0) = 0$ and $\theta(0) = 0$, therefore, $\theta(t) \leq 0$, for $t \geq 0$

$$J(x) = \int_0^\infty \frac{e^{-t}}{(e^t - 1)^2} \theta(t) e^{-xt} dt.$$

Hence, $j(x) \leq J(x) < 0$. Put all this together we get for all $x > 0$, $g_n(x+1) < g_n(x)$ and by induction, for all $k \in \mathbb{N}$ and all $x > 0$,

$$g_n(x+k) < g_n(x). \quad (1.12)$$

Recall that

$$g_n(x) = ((-1)^{n+1} (x\psi^{(n+1)}(x) + (n+1)\psi^{(n)}(x)) - \frac{(n-1)!}{x^n}) \log x + \frac{n!}{4x^{n+1}},$$

Using the double inequality

$$\frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} \leq (-1)^{n+1}\psi^{(n)}(x) \leq \frac{(n-1)!}{x^n} + \frac{n!}{x^{n+1}},$$

we get for all $n \geq 1$, $\lim_{x \rightarrow \infty} g_n(x) = 0$. Letting k to infinity in equation (1.12) we get $g_n(x) \geq 0$ for all $x \geq 1$. This proves $(-1)^n \varphi^{(n)}(x) \geq 0$ for all $x \geq 1$ and all $n \geq 1$. Since,

$$\varphi(x) = \frac{\alpha}{x} + \frac{1}{x}(\theta_1(x) - \frac{1}{2}) + \theta_1'(x) \log x.$$

Moreover,

$$\theta_1(x) = \int_0^\infty h(t)e^{-xt} dt.$$

Differentiate yields

$$\theta_1'(x) = - \int_0^\infty th(t)e^{-xt} dt = -\frac{1}{x} \int_0^\infty (th(t))' e^{-xt} dt.$$

Since, $(th(t))' = -1/t^2 + 1/(2 - 2 \cosh t) + (1/8)t \sinh t / \sinh^4(t/2)$ and one shows that $(th(t))'$ is bounded on $(0, +\infty)$ by a some positive constant C , say, indeed, $\lim_{t \rightarrow 0} (th(t))' = 1/12$ and $\lim_{t \rightarrow \infty} (th(t))' = 0$. Let $x > 0$, then $|\theta_1'(x)| \leq C/x^2$ and $\lim_{x \rightarrow \infty} \theta_1'(x) \log x = 0$. Moreover, $\lim_{x \rightarrow \infty} \frac{\alpha}{x} + \frac{1}{x}(\theta_1(x) - \frac{1}{2}) = 0$ and φ decreases on $(0, +\infty)$. All this together imply that $\varphi(x) \geq 0$ for all $x > 0$. This completes the proof.

Corollary 1.10 *The function*

$$f_\alpha(x) = x^{x(\psi(x) - \log x) - \alpha},$$

is completely monotonic on $(0, +\infty)$ for $\alpha \geq -1/4$.

Proof. Firstly, $f_\alpha(x) > 0$. Applying Lemma 1.9 and Proposition 1.8, we get that for $\alpha \geq -1/4$, f_α is completely monotonic.

Corollary 1.11 *For all $n \geq 2$, $x > 1$, and $\alpha \geq -1/4$,*

$$0 < (-1)^n \theta_1^{(n)}(x) \leq \frac{(n-1)! (\alpha + \frac{1}{2})}{x^n \log x}$$

Remark 1.12 *It remains an open problem to prove that the function $f_\alpha(x)$ is completely monotonic on the range $(-1/4, -1/2]$*

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