

The amplituhedron crossing and winding numbers

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Abstract

In [AHTT18], Arkani-Hamed, Thomas and Trnka formulated two conjectural descriptions of the tree amplituhedron $\mathcal{A}_{n,k,m}$ depending on the parity of m . When m is even, the description involves the winding number and when m is odd the description involves the crossing number. In this paper, we prove that if a point of the amplituhedron is in the image of the positive Grassmannian by the amplituhedron map, then it satisfies the winding or crossing descriptions depending on the parity of m . When $m = 2$, we also prove the other direction: a point satisfying the winding description is inside the amplituhedron.

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1 Introduction

1.1 Overview

The totally nonnegative (TNN) Grassmannian $\text{Gr}_{k,n}^{\geq 0}$ is the subspace of the Grassmannian $\text{Gr}_{k,n}$ given by the k -dimensional vector spaces of \mathbb{R}^n with nonnegative Plücker coordinates. It has relations with many areas of mathematics including integrable systems [KW14] or tropical geometry [SW05].

The (tree) amplituhedron is generalization of the TNN Grassmannian introduced in 2013 by Arkani-Hamed and Trnka [AHT14] to study scattering amplitudes in physics from a geometrical point of view. It has attracted a lot of interest both in high-energy physics and in mathematics. As a consequence, ideas from physics gave rise to beautiful mathematical developments such as the study of the BCFW triangulation [BCF05, BCFW05, EZLT21] or the exploration of the T-duality [LPW20, PSBW21]. It is an intriguing object finding connections with other areas of mathematics such as cluster algebras [LPSV19, PSBW21] or tropical geometry [LPW20]. See also [Wil21] for a recent survey.

The amplituhedron in its original definition [AHT14] is given by the image of the totally nonnegative Grassmannian $\text{Gr}_{k,n}^{\geq 0}$, inside an ambient Grassmannian $\text{Gr}_{k,k+m}$, where $m \leq n - k$, by a (map induced by) linear map. Although being simple, this definition is not fully satisfying. First, this definition is very redundant since the dimension of the source Grassmannian is bigger than the dimension of the ambient Grassmannian. Second, it is very hard to check whether or not a point of ambient Grassmannian $\text{Gr}_{k,k+m}$ belongs to the amplituhedron. To overcome the first point, one could select cells in the source TNN Grassmannian such that they map injectively to the amplituhedron, and such that their images triangulate the amplituhedron. There have been a lot of effort in that direction, see for instance [KW19] when $m = 1$, or [BH19, KWZ20, PSBW21] for $m = 2$ and [EZLT21] when $m = 4$, but this is not the path we follow here.

In [AHTT18] Arkani-Hamed, Thomas and Trnka conjectured three new definitions of the amplituhedron purely in terms of combinatorial and topological data. To understand the new point of view of these definitions, one has to regard the amplituhedron as a generalization of a convex polytope inside the ambient Grassmannian. It is well-know that convex polytopes have two descriptions; either as convex hull of its vertices, or as a finite intersection of half-spaces. The original definition of the amplituhedron uses the first point of view, and the new definitions of [AHTT18] use the second. More precisely, the points of the amplituhedron satisfy

a set of inequalities and one wants to interpret these inequalities as codimension-one faces of the amplituhedron inside the ambient Grassmannian. However, in general, these inequalities do not determine the amplituhedron. To fully determine the amplituhedron, the authors introduce three numbers associated to each point of the ambient Grassmannian: the crossing number (for odd m), the winding number (for even m) and the number of sign flips (for all m). All these numbers are interrelated, and defined according to combinatorial and topological properties of the point of the ambient Grassmannian. Then, Arkani-Hamed, Thomas and Trnka conjecture that a point of the ambient Grassmannian is inside the amplituhedron if and only if it satisfies the boundary inequalities and one of the three topological and combinatorial numbers has a definite value.

This completely new perspective on the amplituhedron offers numerous compelling advantages, and new avenues to tackle the amplituhedron. We give here a few motivations. First, it is an intrinsic definition, that is the amplituhedron is directly defined as a subset of the ambient Grassmannian and does not require external information such as the TNN grassmannian, in particular it not redundant as the original definition. Second, the three numbers: crossing, winding, and sign flips can be computed simply by analyzing the sign configurations of the so-called *twistor coordinates*. Moreover, the boundary inequalities also correspond to a sign configuration of some twistor coordinates. Thus, we can test whether or not a point of the ambient Grassmannian is inside the amplituhedron solely by checking a finite number of sign configurations of its twistor coordinates. Third, these definitions are interesting from the point of view of positivity and cluster algebras. A recurring phenomenon for positive spaces is that they exhibit simpler features than expected. For example, to test if a plane of the Grassmannian is nonnegative it suffices to check the positivity of a strict subset of the set of Plücker coordinates, this is related to the cluster structure of the Grassmannian [Sco06]. This kind of behavior appears in the new definitions of the amplituhedron since they only depend on sign configurations of a strict subset of the set of twistor coordinates. One can see this new approach of the amplituhedron as a (small) step towards understanding the cluster structure of the amplituhedron. Fourth, these three new definitions directly yield triangulations of the amplituhedron, see [AHTT18, Section 7]. The sign flip triangulation is being studied: when $m = 2$, it is proved in [PSBW21] that the sign flip triangulation is a so-called positroidal triangulation of the amplituhedron, and from the perspective of physics, these sign flip cells are used in [KL20] to shed new lights on scattering amplitudes. While the sign flip triangulation has attracted most of the attention, we believe that the other two triangulations deserve a careful consideration. Finally, the sign flip viewpoint contributes to the quest to unveil the yet-to-be discovered dual amplituhedron [KL20, HLTZ21].

Significant progresses have been made on the proof of the equivalence of the sign flip definition with the original one; in [AHTT18] they sketched an argument to justify that if a point is in the amplituhedron then it satisfies the sign flip definition, in [KW19] they prove the same direction for $m = 1$ and in [PSBW21] they prove the equivalence of the definitions for $m = 2$. However no systematic study of the crossing and winding pictures has been made.

The goal of this paper is to prove one direction of the equivalence of the crossing and

winding definitions with the original definition. More precisely, we prove that if a point of the ambient Grassmannian is inside the amplituhedron, then it has the correct winding or crossing number depending on the parity of m . Since it was already observed in [AHTT18] that a point in the amplituhedron satisfies the boundary conditions, this indeed proves one way of the equivalence. In addition, we prove the complete equivalence of the original definition with the winding picture when $m = 2$. Along the way, we also give a new proof that a point of the amplituhedron satisfies the sign flip definition.

A crucial tool in our argument is given by a new set of equations on twistor coordinates, valid for any n, k, m , that we called the C - and \mathcal{Z} - equations. When applied to the points of the amplituhedron, these equations together with the positivity conditions yield interesting constraints on sign patterns of twistor coordinates. For example, the sign flip definition directly follows from them. We hope that these equations will find applications beyond these proofs.

1.2 The amplituhedron

Let n and k be two nonnegative integers such that $0 \leq k \leq n$.

Definition 1.1. The (real) *Grassmannian* $\text{Gr}_{k,n}$ is the space of all k -dimensional vector spaces in \mathbb{R}^n .

We denote by $[n]$ the set $\{1, \dots, n\}$ and by $\binom{[n]}{k}$ the set of lists of elements of $[n]$ of size k sorted in ascending order. Let $I \in \binom{[n]}{k}$ and $V \in \text{Gr}_{k,n}$ represented by a $k \times n$ matrix M , then we denote by $p_I(V)$ the minor of M relative to I . The minors $p_I(V)$ are the *Plücker coordinates* of V ; they do not depend on the choice of M (up to simultaneous rescaling by a nonzero constant).

Definition 1.2. The *totally nonnegative Grassmannian* $\text{Gr}_{k,n}^{\geq 0}$ is the set of k -vector spaces $V \in \text{Gr}_{k,n}$ such that $p_I(V) \geq 0$, for all $I \in \binom{[n]}{k}$.

The *totally positive Grassmannian* $\text{Gr}_{k,n}^{> 0}$ is the set of k -vector spaces $V \in \text{Gr}_{k,n}$ such that $p_I(V) > 0$, for all $I \in \binom{[n]}{k}$.

The space $\text{Mat}_{k,n}^{\geq 0}$ (resp. $\text{Mat}_{k,n}^{> 0}$) is the set of $k \times n$ matrices such that $p_I(V) \geq 0$ (resp. $p_I(V) > 0$), for all $I \in \binom{[n]}{k}$.

Definition 1.3 (Amplituhedron). Let (n, k, m) be a triplet of nonnegative integers such that $k + m \leq n$. Let \mathcal{Z} be an element of $\text{Mat}_{n,k+m}^{> 0}$. This matrix induces a map

$$\begin{aligned} \tilde{\mathcal{Z}} : \text{Gr}_{k,n}^{\geq 0} &\rightarrow \text{Gr}_{k,k+m} \\ \text{Span}(c_1, \dots, c_k) &\rightarrow \text{Span}(\mathcal{Z}c_1, \dots, \mathcal{Z}c_k) \end{aligned}$$

or equivalently, if C is a $k \times n$ matrix representing $\text{Span}(c_1, \dots, c_k)$ in $\text{Gr}_{k,n}^{\geq 0}$ then $\tilde{\mathcal{Z}}(C)$ is represented by $C\mathcal{Z}$. The (*tree*) *amplituhedron* $\mathcal{A}_{n,k,m}$ is the image of $\text{Gr}_{k,n}^{\geq 0}$ by the map $\tilde{\mathcal{Z}}$.

We also denote by $\mathcal{A}_{n,k,m}^{> 0}$ the image of $\text{Gr}_{k,n}^{> 0}$ by $\tilde{\mathcal{Z}}$.

Remark 1.4. The space $\mathcal{A}_{n,k,m}^{>0}$ is a priori different from the interior of the amplituhedron. In [GL20, Section 9] the authors compare these two spaces, in particular they show that $\mathcal{A}_{n,k,m}^{>0}$ is open and thus contained in the interior of the amplituhedron.

Definition 1.5 (Twistor coordinates). Let (n, k, m) be a triplet of nonnegative integers such that $k + m \leq n$. Let $\mathcal{Z} \in \text{Mat}_{n,k+m}^{>0}$ and denote by $\mathcal{Z}_1, \dots, \mathcal{Z}_n \in \mathbb{R}^{k+m}$ its n rows. Let $Y \in \text{Gr}_{k,k+m}$ and denote by $Y_1, \dots, Y_k \in \mathbb{R}^{k+m}$ the k rows of a matrix representing Y . Let (i_1, \dots, i_m) be a list of elements of $[n]$. The *twistor coordinate* (i_1, \dots, i_m) of Y , denoted by

$$\langle Y, \mathcal{Z}_{i_1}, \dots, \mathcal{Z}_{i_m} \rangle,$$

is the determinant of $(Y_1, \dots, Y_k, \mathcal{Z}_{i_1}, \dots, \mathcal{Z}_{i_m})$.

Remark 1.6. The twistor coordinates of Y are defined up to a global nonzero factor, corresponding to the choice of a representative of the k -plane Y , but this factor will not matter.

Notation 1.7. When \mathcal{Z} is understood, we denote the twistor coordinate $\langle Y, \mathcal{Z}_{i_1}, \dots, \mathcal{Z}_{i_m} \rangle$ by $\langle Y, i_1, \dots, i_m \rangle$. We will also denote the determinant of $(\mathcal{Z}_{i_1}, \dots, \mathcal{Z}_{i_{k+m}})$ by $\langle i_1, \dots, i_{k+m} \rangle$, for any list (i_1, \dots, i_{k+m}) of elements of $[n]$.

1.3 Coarse boundary of the amplituhedron

Let $Y \in \mathcal{A}_{n,k,m}$. It follows from the Cauchy-Binet formula and the positivity of the minors that certain twistor coordinates are positive: when m is even we have

$$\begin{cases} \langle Y, I \rangle \geq 0 & \text{for } I = (i_1, i_1 + 1, \dots, i_{\frac{m}{2}}, i_{\frac{m}{2}} + 1) \in \binom{[n]}{m}, \\ (-1)^{k+1} \langle Y, I, n, 1 \rangle \geq 0 & \text{for } I = (i_1, i_1 + 1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1} + 1) \in \binom{[2, n-1]}{m-2}, \end{cases} \quad (1)$$

whereas when m is odd we have

$$\begin{cases} (-1)^k \langle Y, 1, I \rangle \geq 0 & \text{for } I = (i_1, i_1 + 1, \dots, i_{\frac{m+1}{2}-1}, i_{\frac{m+1}{2}-1} + 1) \in \binom{[2, n]}{m-1}, \\ \langle Y, I, n \rangle \geq 0 & \text{for } I = (i_1, i_1 + 1, \dots, i_{\frac{m+1}{2}-1}, i_{\frac{m+1}{2}-1} + 1) \in \binom{[n-1]}{m-1}. \end{cases} \quad (2)$$

See Lemma 2.1 for a proof of these inequalities. Furthermore, these inequalities are strict if $Y \in \mathcal{A}_{n,k,m}^{>0}$. In most of the cases, these inequalities are not refined enough to determine whether or not a point of $\text{Gr}_{k,k+m}$ is in the amplituhedron.

Definition 1.8. We call Eq. (1) and Eq. (2) the *coarse boundary conditions* of the amplituhedron.

Let \mathcal{L} be the locus of points in $\text{Gr}_{k,k+m}$ where at least one of the inequalities of Eq. (1) or Eq. (2), depending on the parity of m , is an equality. We define

$$\mathcal{A}_{n,k,m}^{\text{wcb}} := \mathcal{A}_{n,k,m} \setminus \mathcal{L} \quad \text{and} \quad \text{Gr}_{k,k+m}^{\text{wcb}} := \text{Gr}_{k,k+m} \setminus \mathcal{L},$$

the notation referring to “without coarse boundary”. In particular, we have

$$\mathcal{A}_{n,k,m}^{>0} \subset \mathcal{A}_{n,k,m}^{\text{wcb}} \subset \mathcal{A}_{n,k,m}.$$

1.4 Projection and simplices

Fix a k -plane Y in \mathbb{R}^{k+m} and the n vectors $\mathcal{Z}_1, \dots, \mathcal{Z}_n$ in \mathbb{R}^{k+m} . Denote by V_Y the quotient space \mathbb{R}^{k+m}/Y and by

$$\pi_Y : \mathbb{R}^{k+m} \rightarrow V_Y \simeq \mathbb{R}^m$$

the quotient map.

Notation 1.9. When \mathcal{Z} and Y are understood, we denote $Z_i := \pi_Y(\mathcal{Z}_i)$ for $1 \leq i \leq n$.

Remark 1.10. Let $Y \in \text{Gr}_{k,k+m}$ and let (f_1, \dots, f_m) be a basis of a complement of Y in \mathbb{R}^{k+m} , then $\mathcal{B} = (\pi_Y(f_1), \dots, \pi_Y(f_m))$ is a basis of V_Y . Then, the list

$$\left(\det_{\mathcal{B}}(Z_{i_1}, \dots, Z_{i_m}), 1 \leq i_1, \dots, i_m \leq n \right),$$

where $\det_{\mathcal{B}}$ is the determinant in the basis \mathcal{B} , is equal to the list

$$(\langle Y, \mathcal{Z}_{i_1}, \dots, \mathcal{Z}_{i_m} \rangle, 1 \leq i_1, \dots, i_m \leq n),$$

up to a global nonzero factor. Thus, we interpret twistor coordinates as determinants of Z_i in V_Y .

The definitions of the winding number and the crossing number use simplices in V_Y that we introduce now.

Definition 1.11. For any subset I of $[n]$, we denote $\mathcal{S}(I)$ the simplex in \mathbb{R}^{k+m} given by the convex hull of the points \mathcal{Z}_i , for $i \in I$ in \mathbb{R}^{k+m} . We denote by $S(I)$ the image of the simplex $\mathcal{S}(I)$ by π_Y . We call $S(I)$ a simplex even if its vertices are not necessarily affinely independent. If the vertices Z_i , for $i \in I$, of $S(I)$ are affinely independent, we say that $S(I)$ is *full-dimensional*.

1.5 The winding number

Let m be an even positive integer. Fix k, n such that $n \geq k + m$. Fix also $\mathcal{Z} \in \text{Mat}_{n,k+m}^{>0}$, and denote by $\mathcal{Z}_1, \dots, \mathcal{Z}_n$ its n rows.

We associate a number, the winding number, to any point $Y \in \text{Gr}_{k,k+m}^{\text{wcb}}$ in the following way. Let $P(Y, \mathcal{Z})$ be the following polyhedron in V_Y

$$P(Y, \mathcal{Z}) := \bigcup_I S(I) \bigcup_J S(J, n, 1) \subset V_Y,$$

where $I = (i_1, i_1 + 1, \dots, i_{\frac{m}{2}}, i_{\frac{m}{2}} + 1)$ is a strictly ascending list of positive integers between 1 and n , and $J = (j_1, j_1 + 1, \dots, j_{\frac{m-1}{2}}, j_{\frac{m-1}{2}} + 1)$ is a strictly ascending list of positive integers

between 2 and $n - 1$. Let S^{k-1} be the $(k - 1)$ -sphere centered in the origin of V_Y of radius 1. We define the map

$$f : P(Y, \mathcal{Z}) \rightarrow S^{k-1}$$

$$x \rightarrow \frac{x}{\|x\|},$$

where $\|\cdot\|$ is a norm on V_Y . Note that since $Y \in \text{Gr}_{k,k+m}^{\text{wcb}}$, the origin of V_Y does not belong to $P(Y, \mathcal{Z})$ and then f is well defined.

Definition 1.12 (Winding number). The *winding number*

$$w_{n,k,m}(Y, \mathcal{Z})$$

is the degree of the map f (i.e. the image of the homology class of $P(Y, \mathcal{Z})$ by the map $f_* : H_{k-1}(P(Y, \mathcal{Z})) \rightarrow H_{k-1}(S^{k-1}) \simeq \mathbb{Z}$).

It corresponds to the number of times that $P(Y, \mathcal{Z})$ winds around the origin of V_Y .

Remark 1.13. For a generic $Y \in \text{Gr}_{k,k+m}$, the ray issued from almost all vectors of V_Y intersects the simplices of $P(Y, \mathcal{Z})$ only in their m -dimensional interior. Pick Y generic enough and such a vector, say Z_* , in V_Y . Then the degree of f is given by the number of simplices intersecting the ray issued from Z_* weighted by a sign corresponding to the orientation of the simplex. We recover in these cases the definition of the winding number of [AHTT18, Section 3].

Theorem 1. *The winding is independent of $Y \in \mathcal{A}_{n,k,m}^{\text{wcb}}$, of $\mathcal{Z} \in \text{Mat}_{n,k+m}^{>0}$, and it equals*

$$w_{n,k,m}(Y, \mathcal{Z}) = \binom{\lfloor \frac{k+m-1}{2} \rfloor}{\frac{m}{2}}. \quad (3)$$

Remark 1.14. It is apparent from Eq. (3) that the winding number is also independent of n .

A point in $\mathcal{A}_{n,k,m}^{\text{wcb}}$ also satisfies the coarse boundary conditions Eq. (1). In [AHTT18, Section 3], the authors also conjectured the inverse implication: a point with the correct winding number and satisfying the coarse boundary condition is in the amplituhedron. This implication is still open.

Open problem 1.15. *Does a point $Y \in \text{Gr}_{k,k+m}^{\text{wcb}}$ satisfying the inequalities of Eq. (1) and satisfying Eq. (3) belong to $\mathcal{A}_{n,k,m}$?*

When $m = 2$, the following proposition gives a positive answer to this problem. It allows to give an alternative definition of the $m = 2$ amplituhedron in terms of the winding number, see Corollary 1.17.

Proposition 1.16. *Fix $m = 2$. Let $Y \in \text{Gr}_{k,k+2}^{\text{wcb}}$ and $\mathcal{Z} \in \text{Mat}_{n,k+2}^{>0}$.*

1. *We have*

$$w_{n,k,2}(Y, \mathcal{Z}) \leq \left\lfloor \frac{k+1}{2} \right\rfloor.$$

2. If $w_{n,k,2}(Y, \mathcal{Z}) = \lfloor \frac{k+1}{2} \rfloor$ and satisfies the coarse boundary conditions, see Eq. (1), then $Y \in \mathcal{A}_{n,k,2}$.

Corollary 1.17. Let $Y \in \text{Gr}_{k,k+2}^{\text{wcb}}$ and $\mathcal{Z} \in \text{Mat}_{n,k+2}^{>0}$. The point Y is in $\mathcal{A}_{n,k,2}$ if and only if $w_{n,k,2}(Y, \mathcal{Z}) = \lfloor \frac{k+1}{2} \rfloor$ and Y satisfies the coarse boundary conditions, see Eq. (1).

1.6 The crossing number

Let m be an odd positive integer, we write $m = 2r - 1$ with r a positive integer. Fix k, n such that $n \geq k + m$. Fix also $\mathcal{Z} \in \text{Mat}_{n,k+m}^{>0}$, and denote by $\mathcal{Z}_1, \dots, \mathcal{Z}_n$ its n rows.

We associate a number, the crossing number, to any point $Y \in \text{Gr}_{k,k+m}$ in the following way. Start from a k -plane Y in \mathbb{R}^{k+m} and the n vectors $\mathcal{Z}_1, \dots, \mathcal{Z}_n$ in \mathbb{R}^{k+m} . Let $(i_1, i_1 + 1, \dots, i_r, i_r + 1)$ be strictly ascending list of $m + 1$ positive integers between 1 and n . The list is composed of r pairs of consecutive integers. We decompose the simplex $S(i_1, \dots, i_r + 1)$ into cells:

- the 0-cells are the vertices $Z_{i_1}, Z_{i_1+1}, \dots, Z_{i_r}$ and Z_{i_r+1} of $S(i_1, \dots, i_r + 1)$,
- let $1 \leq a \leq 2r - 1$. For any choice of $(a + 1)$ affinely independent vertices of $S(i_1, \dots, i_r + 1)$, say $Z_{j_1}, \dots, Z_{j_{a+1}}$ where $\{j_1, \dots, j_{a+1}\} \subset \{i_1, i_1 + 1, i_2 + 1, \dots, i_r + 1\}$, we define the a -cell $\mathfrak{C}(j_1, \dots, j_{a+1})$ to be the relative interior of the convex hull of $\pi_Y(Z_{j_1}), \dots, \pi_Y(Z_{j_{a+1}})$.

Denote by \mathfrak{Cells} the set of all the cells of the simplices $S(i_1, \dots, i_r + 1)$ for any list $(i_1, i_1 + 1, i_2, \dots, i_r + 1)$ of strictly ascending integers between 1 and n .

Definition 1.18. The *crossing number*

$$c_{n,k,m}(Y, \mathcal{Z})$$

is the number of cells in \mathfrak{Cells} containing the origin $\pi_Y(Y)$ of V_Y .

Remark 1.19. Suppose the origin $\pi_Y(Y)$ of V_Y is contained only in m -dimensional cells (i.e. in the interior of the simplices). Then the crossing number counts the number of simplices of type $S(i_1, i_1 + 1, i_2, \dots, i_r + 1)$ containing the origin V_Y . However, the sign of $\langle Y, I \setminus \{i\} \rangle$ determines the relative position of the origin with respect to the hyperplane containing the affine hull of $\{Z_j, j \in I \setminus \{i\}\}$. Thus, in this particular case, the origin belongs to the simplex $S(I)$ if and only if the sequence

$$(\text{sign} \langle Y, I \setminus \{i\} \rangle)_{i \in I}$$

is alternating. Thus, Definition 1.18 of the crossing number agrees with the definition of [AHTT18, Section 4]. Otherwise, the two definitions differ and Theorem 2 only holds using Definition 1.18.

Theorem 2. The crossing number is independent of $Y \in \mathcal{A}_{n,k,m}^{>0}$ and of $\mathcal{Z} \in \text{Mat}_{n,k+m}^{>0}$, and it equals

$$c_{n,k,m}(Y, \mathcal{Z}) = \begin{cases} \frac{2k+m-1}{m+1} \binom{\frac{k+m-2}{2}}{\frac{m-1}{2}} & \text{for } k \text{ odd,} \\ 2 \binom{\frac{k+m-1}{2}}{\frac{m+1}{2}} & \text{for } k \text{ even.} \end{cases} \quad (4)$$

Remark 1.20. It is apparent from Eq. (4) that the crossing number is independent of n .

Remark 1.21. From the explicit formulas of the winding number Eq. (3) and of the crossing number Eq. (4), we obtain the following relation between the crossing and the winding numbers

$$c_{n,k,m}(Y, \mathcal{Z}) = \begin{cases} 2w_{n,k,m+1}(Y, \mathcal{Z}) - w_{n,k,m-1}(Y, \mathcal{Z}) & \text{for } k \text{ odd,} \\ 2w_{n,k,m+1}(Y, \mathcal{Z}) & \text{for } k \text{ even,} \end{cases}$$

for $(Y, \mathcal{Z}) \in \mathcal{A}_{n,k,m}^{>0} \times \text{Mat}_{n,k+m}^{>0}$. This relation can be directly deduced by geometrical reasons. See [AHTT18, Section 4] for an idea of the argument.

A point in $\mathcal{A}_{n,k,m}^{>0}$ also satisfies the coarse boundary conditions Eq. (2). In [AHTT18, Section 4], the authors also conjectured the inverse implication: a point with the correct crossing number and satisfying the coarse boundary condition is in the amplituhedron. This implication is still open.

Open problem 1.22. *Does a point $Y \in \text{Gr}_{k,k+m}$ satisfying Eq. (2) and Eq. (4) belong to $\mathcal{A}_{n,k,m}$?*

1.7 Organization of the paper

In Section 2, we prove Theorem 1. We first prove that the winding number $w_{n,k,m}(Y, \mathcal{Z})$ is constant when $(Y, \mathcal{Z}) \in \mathcal{A}_{n,k,m}^{\text{wcb}} \times \text{Mat}_{n,k+m}^{>0}$. Then we prove that the winding number is also independent of n . Finally, we obtain the explicit expression of the winding number for $n = k + m$. In the last subsection, we prove Proposition 1.16 establishing the equivalence between the winding description and the original description of the amplituhedron when $m = 2$.

In section 3, we prove Theorem 2. The proof follows the same path. We first prove that the crossing number $c_{n,k,m}(Y, \mathcal{Z})$ is constant when $(Y, \mathcal{Z}) \in \mathcal{A}_{n,k,m}^{>0} \times \text{Mat}_{n,k+m}^{>0}$. In this case, this part is more subtle. It requires to first prove two sets of equations on twistor coordinates that we call C -equations and \mathcal{Z} -equations. We emphasize that these equations are valid for every n, k and m . We deduce from these equations that we can avoid unpleasant behavior of the simplices and cells involved in the count of the crossing number, and then prove the constancy of the crossing number in $\mathcal{A}_{n,k,m}^{>0} \times \text{Mat}_{n,k+m}^{>0}$. Then we prove the independence of the crossing number in n , and finally we obtain the explicit expression of the crossing for $n = k + m$.

The two sections are independent.

1.8 Acknowledgments

We are grateful to Ran Tessler for valuable discussions and comments. X. B. was supported by the ISF grant 335/19 in the group of Ran Tessler. J.L. was supported by the Austrian Science Fund (FWF): P 34602 individual project.

2 The winding number

In this section m will denote an even integer.

2.1 Preliminaries

Throughout the text, we use the Cauchy-Binet formula to develop twistor coordinates. A formulation of this development is given by Lemma 3.3 of [PSBW21]. Let us recall the statement.

Lemma 2.1 ([PSBW21]). *Let $\mathcal{Z} \in \text{Mat}_{n,k+m}^{>0}$. Write $Y \in \text{Gr}_{k,k+m}$ as $Y = C\mathcal{Z}$ with $C \in \text{Gr}_{k,n}$. We have*

$$\langle C\mathcal{Z}, i_1, \dots, i_m \rangle = \sum_{J=(j_1 < \dots < j_k) \in \binom{[n]}{k}} p_J(C) \langle j_1, \dots, j_k, i_1, \dots, i_m \rangle,$$

where we used Notation 1.7.

For completeness, we include a proof.

Proof. We have

$$\langle C\mathcal{Z}, i_1, \dots, i_m \rangle = \det \left(\begin{pmatrix} C \\ I_{i_1, \dots, i_m} \end{pmatrix} \mathcal{Z} \right),$$

where I_{i_1, \dots, i_m} is the $m \times n$ matrix such that the l th row has a 1 in position i_l and zeros elsewhere. We then use the Cauchy-Binet formula to obtain the result. \square

In particular, (strict) inequalities of the coarse boundary conditions, Eq. (1) and Eq. (2), follow from the positivity of $\mathcal{Z} \in \text{Mat}_{n,k+m}^{>0}$ and the (strict) positivity of $C \in \text{Gr}_{k,n}^{\geq 0}$.

2.2 Constancy of the winding number in $\mathcal{A}_{n,k,m}^{\text{wcb}} \times \text{Mat}_{n,k+m}^{>0}$

Proposition 2.2. *Fix n, k and m even such that $n \geq k + m$. The winding number $w_{n,k,m} : \mathcal{A}_{n,k,m}^{\text{wcb}} \times \text{Mat}_{n,k+m}^{>0} \rightarrow \mathbb{Z}$ is constant.*

Remark 2.3. We can prove in a similar way that the winding number is constant on each path-connected component of $\text{Gr}_{k,k+m}^{\text{wcb}} \times \text{Mat}_{n,k+m}^{>0}$.

Proof. Let $\mathcal{Z}, \mathcal{Z}' \in \text{Mat}_{n,k+m}^{>0}$ and $Y, Y' \in \mathcal{A}_{n,k,m}^{\text{wcb}}$. Let $C, C' \in \text{Gr}_{k,n}^{\geq 0}$ such that

$$Y = C\mathcal{Z} \quad \text{and} \quad Y' = C'\mathcal{Z}'.$$

Since $\text{Gr}_{k,n}^{\geq 0}, \text{Mat}_{n,k+m}^{>0}$ are path connected (see [Pos06] for the positive Grassmannian) and $\text{Gr}_{k,n}^{\geq 0} = \overline{\text{Gr}_{k,n}^{>0}}$, there exists a path

$$\left(\check{C}, \check{\mathcal{Z}} \right) : [0, 1] \rightarrow \text{Gr}_{k,n}^{\geq 0} \times \text{Mat}_{n,k+m}^{>0}$$

such that $(\check{C} \times \check{Z})(0) = (C, Z)$, $(\check{C} \times \check{Z})(1) = (C', Z')$ and such that $\check{C}(t) \in \text{Gr}_{k,n}^{>0}$ for $t \in]0, 1[$. The winding number

$$w_{n,k,m}(\check{Y}(t), \check{Z}(t)),$$

where $\check{Y}(t) := \check{C}\check{Z}$, is independent of t . Indeed, the winding number may change only as a result of an intersection between the origin $\pi_{\check{Y}(t)}(\check{Y}(t))$ of $V_{\check{Y}}$ and the polytope $P(\check{Y}(t), \check{Z}(t))$. However, for every $t \in [0, 1]$ we have $\check{Y}(t) \in \mathcal{A}_{n,k,m}^{\text{wcb}}$, hence $\check{Y}(t)$ satisfies the strict inequality of the coarse boundary condition Eq. (1), thus the origin $\pi_{\check{Y}(t)}(\check{Y}(t))$ of $V_{\check{Y}(t)}$ does not hit the polytope $P(\check{Y}(t), \check{Z}(t))$. \square

2.3 Independence of the winding number in n

We show in this section that for a specific choice of points in $\mathcal{A}_{n,k,m}^{\text{wcb}}$ and $\text{Mat}_{n,k+m}^{>0}$, the winding number does not depend on n . Since, we proved that the winding number $w_{n,k,m}(Y, Z)$ is independent of $Y \in \mathcal{A}_{n,k,m}^{\text{wcb}}$ and $Z \in \text{Mat}_{n,k+m}^{>0}$, we deduce that the winding number is also independent of n .

Proposition 2.4. *There exist $(C', Z') \in \text{Gr}_{k,n+1}^{>0} \times \text{Mat}_{n+1,k+m}^{>0}$ and $(C, Z) \in \text{Gr}_{k,n}^{>0} \times \text{Mat}_{n,k+m}^{>0}$ such that*

$$w_{n,k,m}(CZ, Z) = w_{n+1,k,m}(C'Z', Z'). \quad (5)$$

Proof. In Step 1, we introduce (C', Z') and (C, Z) . Then, in Step 2 we view the polytopes $P(CZ, Z)$ and $P(C'Z', Z')$ associated to these points as singular chains. We prove that the difference $Q = P(C'Z', Z') - P(CZ, Z)$ is given by a sum of boundaries of singular m -simplices. Finally in Step 3, we use this formula to prove that the winding of Q is zero. This implies that the winding of $P(C'Z', Z')$ is equal to the winding of $P(CZ, Z)$.

Step 1. Choose $Z' = (Z_1, \dots, Z_{n+1}) \in \text{Mat}_{n+1,k+m}^{>0}$ and then define $Z = (Z_1, \dots, Z_n) \in \text{Mat}_{n,k+m}^{>0}$. Let $C \in \text{Gr}_{k,n}^{>0}$ and let $Y = CZ \in \mathcal{A}_{n,k,m}^{>0}$. We define the matrix C' by adding a $(n+1)$ th column of zeros to C . It follows from Lemma 2.1 together with the positivity of Z' and C that $Y = C'Z'$ does not belong to the coarse boundary of the amplituhedron:

$$\langle C'Z', I \rangle \neq 0 \quad (6)$$

for any subset $I = (i_1, i_1 + 1, \dots, i_{\frac{m}{2}}, i_{\frac{m}{2}} + 1)$ or $I = (i_1, i_1 + 1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1} + 1, n + 1, 1)$ in $\binom{[n+1]}{m}$. Hence the winding number $w_{n+1,k,m}(Y, (Z_1, \dots, Z_{n+1}))$ is well-defined.

Step 2. Let $P(Y, \mathcal{Z})$ and $P(Y, \mathcal{Z}')$ be the polytopes associated to \mathcal{Z} and \mathcal{Z}' in V_Y . We consider $P(Y, \mathcal{Z})$ as a singular chain given by the sum of the singular simplices

$$\Delta^{m-1} \rightarrow S\left(i_1, i_1 + 1, \dots, i_{\frac{m}{2}} + 1\right) \quad \text{and} \quad \Delta^{m-1} \rightarrow S\left(i_1, i_1 + 1, \dots, i_{\frac{m}{2}-1} + 1, n, 1\right)$$

where Δ^{m-1} is the standard $(m-1)$ -simplex (e_1, \dots, e_m) in \mathbb{R}^m and the first (resp. second) map is the linear map sending the basis (e_1, \dots, e_m) of \mathbb{R}^m to $(Z_{i_1}, Z_{i_1+1}, \dots, Z_{i_{\frac{m}{2}}+1})$ (resp. $(Z_{i_1}, Z_{i_1+1}, \dots, Z_n, Z_1)$). Similarly, we consider $P(Y, \mathcal{Z}')$ as a singular chain. We have

$$\partial P(Y, \mathcal{Z}) = \partial P(Y, \mathcal{Z}') = 0.$$

Write $Q := P(Y, \mathcal{Z}') - P(Y, \mathcal{Z})$. It is then a closed chain and does not intersect the origin, so its winding number is well defined. Moreover, the winding number associated to $P(Y, \mathcal{Z}')$ is equal to the winding number of $P(Y, \mathcal{Z})$ plus the winding number of Q . We will prove that the winding number of Q is zero, hence proving Eq. (5).

We first prove that

$$Q = \sum \partial \sigma_{i_1, i_1+1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1}+1, n, n+1, 1}, \quad (7)$$

where the summation is over strictly ascending lists of integers $(i_1, i_1 + 1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1} + 1)$ between 2 and $n-1$. We used the notation $\sigma(I)$ to denote the singular simplex $\Delta^{|I|-1} \rightarrow S(I)$.

The singular simplices $\sigma_{i_1, i_1+1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1}+1, n, n+1}$ and $\sigma_{i_1, i_1+1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1}+1, n+1, 1}$ in $\partial \sigma_{i_1, i_1+1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1}+1, n, n+1, 1}$ come with a positive sign. These simplices belong to the set of simplices of $P(Y, \mathcal{Z}')$ not in $P(Y, \mathcal{Z})$. Since $i_1 \geq 2$ and $i_{\frac{m}{2}+1} \leq n-1$ they do not constitute the whole set of simplices in $P(Y, \mathcal{Z}')$ and not in $P(Y, \mathcal{Z})$. On the other hand, the simplices $\sigma_{i_1, i_1+1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1}+1, n, 1}$ come with a minus sign and are exactly the simplices of $P(Y, \mathcal{Z})$ not in $P(Y, \mathcal{Z}')$.

The rest of the simplices are of the form $\sigma_{i_1, i_1+1, \dots, \widehat{i_j+\epsilon}, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1}+1, n, n+1, 1}$ for some $1 \leq j \leq \frac{m}{2} - 1$ and $\epsilon \in \{0, 1\}$. The set of indices $\{i_1, i_1 + 1, \dots, \widehat{i_j + \epsilon}, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1} + 1\}$ is uniquely described as disjoint union of intervals $I_1 \sqcup I_2 \sqcup \dots \sqcup I_r$ where each I_k is a sequence of consecutive integers in $\{2, \dots, n-1\}$, and the least element of I_{k+1} is greater by at least 2 from the largest element of I_k . Clearly, there is a unique interval of odd size, we denote it by I_{k_0} . There are now two possibilities; either $\{2\} \notin I_{k_0}$ and $\{n-1\} \notin I_{k_0}$, or exactly one element of $\{2, n-1\}$ belongs to I_{k_0} (if both of them were in I_{k_0} then m must have been 0).

If $2, n-1 \notin I_{k_0}$, then each simplex $\sigma_{i_1, i_1+1, \dots, \widehat{i_j+\epsilon}, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1}+1, n, n+1, 1}$ appears twice in Q with opposite sign. Indeed, let $I_{k_0}^+$ be the extension of I_{k_0} by the least integer greater than all

the elements of I_{k_0} , and let $I_{k_0}^-$ be the extension of I_{k_0} by the largest integer smaller than all the elements of I_{k_0} . Write

$$I^\pm = I_{k_0}^\pm \sqcup \bigsqcup_{i \neq k_0} I_i.$$

Then $\sigma_{i_1, i_1+1, \dots, \widehat{i_j+\epsilon}, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1}+1, n, n+1, 1}$ appears exactly in $\partial\sigma_{I^+, n, n+1, 1}$ and $\partial\sigma_{I^-, n, n+1, 1}$ with opposite sign.

Now suppose that 2 or $n-1$ belongs to I_{k_0} . Suppose first that 2 belongs to I_{k_0} , then $I_{k_0} = I_1$ and $\epsilon = 1$. Then the simplex $\sigma_{i_1, i_1+1, \dots, \widehat{i_j+\epsilon}, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1}+1, n, n+1, 1}$ is the simplex of $P(Y, \mathcal{Z}')$ with the set of indices $\{j_1, j_1+1, \dots, j_{\frac{m}{2}-1}, j_{\frac{m}{2}-1}+1, n, n+1\}$ such that

$$j_1 = 1 \text{ and } I_1 = \{j_1+1, j_2, j_2+1, \dots, j_{l_1}+1\}, \dots, I_r = \{j_{l_{r-1}}, j_{l_{r-1}}+1, \dots, j_{\frac{m}{2}-1}, j_{\frac{m}{2}-1}+1\}.$$

Moreover, this simplex $\sigma_{j_1, j_1+1, \dots, j_{\frac{m}{2}-1}, j_{\frac{m}{2}-1}+1, n, n+1}$ appears in the sum of Eq. (7) with a positive sign, thus it contributes to the simplices of $P(Y, \mathcal{Z}')$ not in $P(Y, \mathcal{Z})$. In the same way, if $n-1 \in I_{k_0}$, we obtain simplices of type $\sigma_{i_1, i_1+1, \dots, i_{\frac{m}{2}}, i_{\frac{m}{2}}+1, n+1, 1$, where $i_{\frac{m}{2}}+1 = n$ contributing with a positive sign to Eq. (7). This completes the list of simplices of $P(Y, \mathcal{Z}')$ not in $P(Y, \mathcal{Z})$.

Putting everything together, we see that all simplices appearing in Eq. (7) cancel in pairs, except exactly those simplices which are simplices $P(Y, \mathcal{Z}')$ but not of $P(Y, \mathcal{Z})$, or of $P(Y, \mathcal{Z})$ but not of $P(Y, \mathcal{Z}')$, and in both of these cases they appear with the correct sign. Thus, Eq. (7) holds.

Step 3. We now show that the winding number of Q is zero. Let

$(i_1, i_1+1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1}+1)$ be a strictly ascending list of integers between 2 and $n-2$. The boundary

$\partial\sigma_{i_1, i_1+1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1}+1, n, n+1, 1}$ is a cycle and this cycle does not touch the origin of V_Y . Indeed, up to modifying C in $\text{Gr}_{k, n}^{>0}$, we have

$$\left\langle C' \mathcal{Z}', i_1, i_1+1, \dots, \widehat{i_j+\epsilon}, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1}+1, n, n+1, 1 \right\rangle \neq 0,$$

for $j \in [\frac{m}{2}]$ and $\epsilon \in \{0, 1\}$. This is possible since by Lemma 2.1 these twistor coordinates only involve the minors of C , and the vanishing of these twistor coordinates is a codimension 1 locus. Moreover, the rest of the twistor coordinates $\langle Y, i_1, i_1+1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1}+1, \widehat{n}, n+1, 1 \rangle$, $\langle Y, i_1, i_1+1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1}+1, n, \widehat{n+1}, 1 \rangle$ and $\langle Y, i_1, i_1+1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1}+1, n, n+1, \widehat{1} \rangle$ do not vanish since $Y = C\mathcal{Z} = C'\mathcal{Z}'$ does not belong to the coarse boundary of the amplituhedron. Hence the winding number of $\partial\sigma_{i_1, i_1+1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1}+1, n, n+1, 1}$ is well defined. Moreover,

it follows from Eq. (7) that the winding number of Q is the sum of the winding numbers of such cycles. We show that the winding number of $\partial\sigma_{i_1, i_1+1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1}+1, n, n+1, 1}$ is zero.

Suppose the winding number of $\partial\sigma_{i_1, i_1+1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1}+1, n, n+1, 1}$ is nonzero, then the simplex $\sigma_{i_1, i_1+1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1}+1, n, n+1, 1}$ is of dimension m and contains the origin of V_Y . As mentioned in Remark 1.19, this implies that the sign of its twistor coordinates is alternating:

$$\begin{aligned} \text{sgn} \left\langle \widehat{i_1}, i_1 + 1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1} + 1, n, n + 1, 1 \right\rangle & \\ &= -\text{sgn} \left\langle i_1, \widehat{i_1 + 1}, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1} + 1, n, n + 1, 1 \right\rangle \\ &= \dots \\ &= \text{sgn} \left\langle i_1, i_1 + 1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1} + 1, \widehat{n}, n + 1, 1 \right\rangle \\ &= -\text{sgn} \left\langle i_1, i_1 + 1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1} + 1, n, \widehat{n + 1}, 1 \right\rangle. \end{aligned}$$

However the last equality cannot hold. Indeed, since $Y = CZ$ and C is positive, the strict coarse boundary conditions give $\text{sign} \left\langle Y, i_1, i_1 + 1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1} + 1, n, \widehat{n + 1}, 1 \right\rangle = (-1)^{k+1}$. On the other hand, since $Y = C'Z'$ it follows from Eq. (6) and the coarse boundary conditions that $\text{sign} \left\langle i_1, i_1 + 1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1} + 1, \widehat{n}, n + 1, 1 \right\rangle = (-1)^{k+1}$. This concludes the proof. \square

2.4 The winding number for $n = k + m$

Suppose m even. We show that there exists $\mathcal{Z} \in \text{Mat}_{n,n}^{>0}$ and $C \in \text{Gr}_{k,k+m}^{>0}$ such that the winding number is

$$w_{n=k+m, k, m}(C\mathcal{Z}, \mathcal{Z}) = \left(\left\lfloor \frac{k+m-1}{2} \right\rfloor \frac{m}{2} \right). \quad (8)$$

Since we showed Proposition 2.2 and Proposition 2.4 that the winding number is independent of $\mathcal{Z} \in \text{Mat}_{n,n}^{>0}$, of $Y \in \mathcal{A}_{n,k,m}^{\text{wcb}}$ and of n , this proves Theorem 1.

Step 1. Since $C \in \text{Gr}_{k,k+m}^{>0}$, it follows from the strict coarse boundary conditions that each simplex involved in $P(Y, \mathcal{Z})$ is of full dimension equal to $m - 1$. Then we choose $0 < \mu \ll 1$ such that the ray issued from the vector

$$Z_* = Z_n + \mu Z_{n-1} + \dots + \mu^{m-1} Z_{n-m+1}$$

avoids the $(m - 2)$ -skeleton of $P(Y, \mathcal{Z})$ (that is $P(Y, \mathcal{Z})$ minus its maximal cells) in V_Y . This is always possible since (Z_n, \dots, Z_{n-m+1}) is a basis of V_Y and the skeleton is of codimension 2. Thus, the ray issued from Z_* can only intersect a simplex of $P(Y, \mathcal{Z})$ in its interior. Hence, the winding number is given by counting the number of simplices intersected in their interior by the ray Z_* , counted with a weight ± 1 depending on the orientation of the simplex. More precisely, let I be either a list $(i_1, i_1 + 1, \dots, i_{\frac{m}{2}}, i_{\frac{m}{2}} + 1)$ of strictly ascending integers between 1 and n ,

or a list $(i_1, i_1 + 1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1} + 1, n, 1)$ of integers such that $i_1, i_1 + 1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1} + 1$ are strictly ascending between 2 and $n - 1$. We define the elementary winding w_I^{ele} to be $+1$ if for every $j \in \{1, m\}$ we have

$$\text{sign} \langle Z_*, I \setminus \{j\text{th element}\} \rangle = (-1)^j \text{sign} \langle I \rangle \quad (9)$$

and 0 otherwise. Thus, the elementary winding number w_I^{ele} is 1 precisely if the ray Z_* hits the simplex $S(I)$ in its interior. Then, we have

$$w_{k,m} = \sum_I \text{sign} \langle I \rangle \times w_I^{ele},$$

where the summation is over the lists described above. Since $C \in \text{Gr}_{k,k+m}^{>0}$, we have $\text{sign} \langle i_1, i_1 + 1, \dots, i_{\frac{m}{2}}, i_{\frac{m}{2}} + 1 \rangle = +1$ and the negative weights in the summation only appear for $I = (i_1, i_1 + 1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1} + 1, n, 1)$ and k even.

Step 2 . In order to compute the winding number, it suffices to compute signs of certain twistor coordinates. Since $n = k + m$, this boils down computing the sign of determinants of square $k + m$ matrices.

If $I = (i_1, i_1 + 1, \dots, i_{\frac{m}{2}}, i_{\frac{m}{2}} + 1)$, we have

$$\text{sign} \langle Z_*, I \setminus \{i_j + \epsilon\} \rangle = (-1)^{k+i_j+(1-\epsilon)}, \quad (10)$$

$j \in \{1, \dots, \frac{m}{2}\}$ and $\epsilon \in \{0, 1\}$. Indeed, we have

$$\left\langle Z_*, i_1, i_1 + 1, \dots, \widehat{i_j + \epsilon}, \dots, i_{\frac{m}{2}}, i_{\frac{m}{2}} + 1 \right\rangle = \det \begin{pmatrix} C \\ V_* \\ I_{i_1, i_1+1, \dots, \widehat{i_j + \epsilon}, \dots, i_{\frac{m}{2}}, i_{\frac{m}{2}}+1} \end{pmatrix} \det(Z),$$

where $V_* = (0, \dots, 0, \mu^{n-m+1}, \dots, \mu^0)$ and $I_{i_1, i_1+1, \dots, \widehat{i_j + \epsilon}, \dots, i_{\frac{m}{2}}, i_{\frac{m}{2}}+1}$ is the $(m - 1) \times (k + m)$ matrix whose l th row has a 1 at the l th index of the list $i_1, i_1 + 1, \dots, \widehat{i_j + \epsilon}, \dots, i_{\frac{m}{2}}, i_{\frac{m}{2}} + 1$ and zeros elsewhere. We first expand the determinant along the rows of I ; the only row possibly contributing with a sign to the determinant is the one with a $+1$ in the position $i_j + (1 - \epsilon)$, since the sign contributions coming from the development of the other rows of I compensate two by two. We obtain

$$\det \begin{pmatrix} C \\ V_* \\ I_{i_1, i_1+1, \dots, \widehat{i_j + \epsilon}, \dots, i_{\frac{m}{2}}, i_{\frac{m}{2}}+1} \end{pmatrix} = (-1)^{k+i_j+(1-\epsilon)} \det \begin{pmatrix} C_{[n] \setminus \{i_1, i_1+1, \dots, \widehat{i_j + \epsilon}, \dots, i_{\frac{m}{2}}, i_{\frac{m}{2}}+1\}} \\ (V_*)_{[n] \setminus \{i_1, i_1+1, \dots, \widehat{i_j + \epsilon}, \dots, i_{\frac{m}{2}}, i_{\frac{m}{2}}+1\}} \end{pmatrix}.$$

Finally, the sign of the determinant on the RHS is determined by the sign of its smallest order in μ since $0 < \mu \ll 1$. There are two cases depending on $M = \max([n] \setminus I)$. If $i_j + 1 < M$ this determinant is given by

$$\mu^{n-M} \underbrace{(-1)^{k+1+1+M}}_{=+1} \det \left(C_{[n] \setminus \{i_1, i_1+1, \dots, \widehat{i_j+\epsilon}, \dots, i_{\frac{m}{2}}, i_{\frac{m}{2}+1\}} \cup \{M\}} \right) + o(\mu^{n-M}),$$

where we used $M \equiv n \equiv k \pmod{2}$ to simplify the sign. If $i_j > M$, then the determinant is given by

$$\mu^{n-(i_j+\epsilon)} \underbrace{(-1)^{k+1+i_j+\epsilon+\epsilon}}_{=+1} \det \left(C_{[n] \setminus \{i_1, i_1+1, \dots, i_{\frac{m}{2}}, i_{\frac{m}{2}+1\}} \right) + o(\mu^{n-(i_j+\epsilon)}),$$

where we used that in this case $i_j \equiv n-1 \equiv k-1 \pmod{2}$. Thus, we deduce Eq. (10) from the positivity of C and \mathcal{Z} .

If $I = (i_1, i_1+1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1}+1, n, 1)$, we obtain in a similar way

$$\text{sign} \langle Z_*, I \setminus \{i_j + \epsilon\} \rangle = (-1)^{i_j+(1-\epsilon)}, \quad (11)$$

where $j \in \{1, \frac{m}{2} - 1\}$ and $\epsilon \in \{0, 1\}$. We also have

$$\text{sign} \langle Z_*, I \setminus \{n\} \rangle = +1 \quad \text{and} \quad \text{sign} \langle Z_*, I \setminus \{1\} \rangle = (-1)^{k+1}. \quad (12)$$

Step 3. We now count the number of simplices contributing to the winding number. There are two types of simplices: simplices of type (a) associated to lists $(i_1, i_1+1, \dots, i_{\frac{m}{2}}, i_{\frac{m}{2}+1})$ of strictly ascending integers between 1 and n , and simplices of type (b) associated to lists $(i_1, i_1+1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1}+1, n, 1)$ of integers such that $i_1, i_1+1, \dots, i_{\frac{m}{2}-1}, i_{\frac{m}{2}-1}+1$ are strictly ascending between 2 and $n-1$.

If k is odd, it follows from Eq. (12) that a simplex of type (b) cannot satisfy Eq. (9). However, it follows from Eq. (10) that simplex of type (a) contributes by $+1$ to the winding number if and only if

$$i_1 \equiv i_2 \equiv \dots \equiv i_{\frac{m}{2}} \equiv 0 \pmod{2},$$

where $1 \leq i_1 \leq \dots \leq i_{\frac{m}{2}} \leq n-1$. Thus, the winding number is equal to the number of such sequences $i_1, \dots, i_{\frac{m}{2}}$, there are

$$\binom{\frac{n-1}{2}}{\frac{m}{2}} = \binom{\frac{k+m-1}{2}}{\frac{m}{2}}$$

possibilities. This ends the proof of Eq. (8) for k odd.

If k is even, a simplex of type (a) contributes by $+1$ to the winding number if and only if

$$i_1 \equiv i_2 \equiv \dots \equiv i_{\frac{m}{2}} \equiv 1 \pmod{2},$$

where $1 \leq i_1 \leq \dots \leq i_{\frac{m}{2}} \leq n-1$. There are $\binom{\frac{n}{2}}{\frac{m}{2}}$ possibilities. Moreover, a simplex of type (b) contributes by -1 to the winding number if and only if

$$i_1 \equiv i_2 \equiv \dots \equiv i_{\frac{m}{2}-1} \equiv 0 \pmod{2},$$

where $2 \leq i_1 \leq \dots \leq i_{\frac{m}{2}-1} \leq n-2$. There are $\binom{\frac{n-2}{2}}{\frac{m}{2}}$ possibilities. Thus, the winding number is equal to

$$\binom{\frac{n}{2}}{\frac{m}{2}} - \binom{\frac{n-2}{2}}{\frac{m}{2}-1} = \binom{\frac{n}{2}-1}{\frac{m}{2}},$$

that is Eq. (8) for $n = k + m$ and k even.

This ends the proof of Theorem 1.

2.5 Maximality of the winding number for $m = 2$

In this subsection, we prove Proposition 1.16 that we recall now. Fix $Y \in \text{Gr}_{k,k+2}^{\text{wcb}}$ and $\mathcal{Z} \in \text{Mat}_{n,k+2}^{>0}$. We prove in the first part that

$$w_{n,k,2}(Y, \mathcal{Z}) \leq \left\lfloor \frac{k+1}{2} \right\rfloor,$$

and in the second part that if $w_{n,k,2}(Y, \mathcal{Z}) = \lfloor \frac{k+1}{2} \rfloor$ and satisfies the coarse boundary conditions Eq. (1), then $Y \in \mathcal{A}_{n,k,2}$.

Proof of part 1. First, there exists $Y' \in \text{Gr}_{k,k+2}^{\text{wcb}}$ in the connected component of Y in $\text{Gr}_{k,k+2}^{\text{wcb}}$ such that:

- $Z'_i = \pi_{Y'}(\mathcal{Z}_i)$ is nonzero in $V_{Y'}$,
- any two vectors (Z'_i, Z'_j) , for $i \neq j$, are non-collinear,
- $w_{n,k,2}(Y', \mathcal{Z}) = w_{n,k,2}(Y, \mathcal{Z})$.

Indeed, each connected component of $\text{Gr}_{k,k+2}^{\text{wcb}}$ is of full-dimension $2k$ since the locus \mathcal{L} defined in Section 1.3 is of dimension 1. Thus, we can choose a point in the connected component of Y in $\text{Gr}_{k,k+2}^{\text{wcb}}$ generic enough to satisfy the first two conditions. Moreover, any two points in the same connected component of $\text{Gr}_{k,k+2}^{\text{wcb}}$ have the same winding number (see Remark 2.3). We now compute the winding number of Y . Since the winding numbers of Y and of Y' are equal, up to redefining $Y := Y'$, we suppose that Y satisfies the two first conditions.

Denote by s the number of sign flips of $(\langle Y, 1, i \rangle)_{i \in [n]}$. We show that

$$\begin{cases} 2w_{n,k,2}(Y, \mathcal{Z}) = s + 1 & \text{for } s \text{ odd,} \\ 2w_{n,k,2}(Y, \mathcal{Z}) = s & \text{for } s \text{ even.} \end{cases}$$

Then, it follows from Remark 3.8 that $s \leq k$. Thus we deduce that $w_{n,k,2}(Y, \mathcal{Z}) \leq \lfloor \frac{k+1}{2} \rfloor$.

Choose Z_* in $V_Y \setminus \{0\}$ in a small neighborhood of Z_1 such that the line l generated by Z_* avoids all the points Z_i , for $1 \leq i \leq n$. This is always possible since $\cup_{1 \leq i \leq n} \text{span}(Z_i)$ is of codimension 1 in V_Y . Moreover, since no pair of vector (Z_i, Z_j) , for $i \neq j$, is collinear, the line l can only intersect a simplex $S(i, i+1)$ (resp. $S(1, n)$) transversally and in its relative interior. In particular, the line intersects the simplex $S(i, i+1)$, for $1 \leq i \leq n-1$, if and only if $\text{sign} \langle Y, Z_*, Z_i \rangle = -\text{sign} \langle Y, Z_*, Z_{i+1} \rangle$ (resp. the simplex $S(1, n)$ if and only if $\text{sign} \langle Y, Z_*, Z_n \rangle = -\text{sign} \langle Y, Z_*, Z_1 \rangle$). We say that this intersection is positive or negative if in addition we have $\text{sign} \langle Y, i, i+1 \rangle$ is positive or negative (resp. $\text{sign} \langle Y, 1, n \rangle$ is positive or negative). Then, by the definition of the winding number, we have

$$2w_{n,k,2}(Y, \mathcal{Z}) = \left| \sum_{x \in l \cap P(Y, \mathcal{Z})} \epsilon(x) \right|,$$

where $\epsilon(x)$ is $+1$ if the intersection of l and $P(Y, \mathcal{Z})$ at x is positive and -1 if it is negative. In particular, we have

$$2w_{n,k,2}(Y, \mathcal{Z}) \leq \sum_{x \in l \cap P(Y, \mathcal{Z})} 1.$$

Moreover, the number of intersections of l with $P(Y, \mathcal{Z})$ is s or $s+1$ depending on the parity of s . Indeed, by choosing Z_* close enough to Z_1 , we get that the number of sign flips s of $(\langle Y, 1, i \rangle)_{i \in [n]}$ counts the number of intersections of l with the simplices $S(i, i+1)$ for $1 \leq i \leq n-1$. If s is even, then the list $(\langle Y, 1, i \rangle)_{i \in [n]}$ has an even number of sign flips, and then $\text{sign} \langle Y, Z_1, Z_2 \rangle = \text{sign} \langle Y, Z_1, Z_n \rangle$. By choosing Z_* close enough to Z_1 , we also get $\text{sign} \langle Y, Z_*, Z_2 \rangle = \text{sign} \langle Y, Z_*, Z_n \rangle$. We can in addition choose Z_* such that Z_1 and Z_2 belong to the same half plane of $\mathbb{R}^2 \setminus l$. Then, l does not intersect $S(1, 2)$ and $S(1, n)$, so the number of intersections of l with $P(Y, \mathcal{Z})$ is s . Similarly, if s is odd, then with the same choice of Z_* we see that l intersects $S(1, n)$ and then the number of intersections of l with $P(Y, \mathcal{Z})$ is $s+1$.

Proof of part 2. Now suppose that $Y \in \text{Gr}_{k,k+2}^{\text{wcb}}$, satisfies the coarse boundary conditions and $w_{n,k,2}(Y, \mathcal{Z}) = \lfloor \frac{k+1}{2} \rfloor$. This implies by the first part of the proof that the number of sign flips s of $(\langle Y, 1, i \rangle)_{i \in [n]}$ is maximal and equal to k . Hence, it follows from Theorem 5.1 of [PSBW21] that $Y \in \mathcal{A}_{n,k,2}$.

3 The crossing number

The goal of this section is to prove Theorem 2. The structure of the proof is similar to the case of the winding number: we first prove in Subsections 3.1, 3.2, 3.3 and 3.4 that the crossing number is independent of the point in $\mathcal{A}_{n,k,m}^{>0} \times \text{Mat}_{n,k+m}^{>0}$, then we prove in Subsection 3.5 that the crossing number is independent of n and finally we obtain in Subsection 3.6 the explicit expression of the crossing number for $n = k + m$.

The first part of the proof, that is the constancy of the crossing number in $\mathcal{A}_{n,k,m}^{>0} \times \text{Mat}_{n,k+m}^{>0}$, is structured as follows. In Subsection 3.1 we derive two types of equations on the twistor coordinates: the C -equations and the \mathcal{Z} -equations. These equations involve the Plücker coordinates of C and \mathcal{Z} , and are obtained from the Plücker relations. We emphasize that they are valid for any n, k, m . From the C - and \mathcal{Z} - equations, together with the positivity of the minors of C and \mathcal{Z} , we deduce in Subsection 3.2 nontrivial constraints on the twistor coordinates. These constraints are used in Subsection 3.3 to exclude some configurations of simplices relative to the origin in the projected space. These are precisely the configurations where the crossing number can change. We then conclude in Subsection 3.4 that the crossing number is constant in $\mathcal{A}_{n,k,m}^{>0}$.

3.1 The C -equations and the \mathcal{Z} -equations

In this section, we derive two sets of equations involving the twistor coordinates: the C -equations and the \mathcal{Z} -equations. These equations are obtained in the same way; they follow from Plücker relations in $\text{Gr}_{k,n}$ and in $\text{Gr}_{k+m,n}$. In particular, the positivity of the minors of C and \mathcal{Z} is not necessary. Moreover, these equations are valid for any m , even or odd.

3.1.1 The C -equations

We recall that $\binom{[n]}{k}$ denotes the set of lists of elements of $[n]$ of size k sorted in ascending order. If A and B are two lists of elements of $[n]$, we denote by A, B the concatenation of the two lists, and by $A \cup B$ the list obtained by sorting in ascending order the elements of A and of B . We also recall that, if $I \in \binom{[n]}{k}$ and $V \in \text{Gr}_{k,n}$ is represented by a $k \times n$ matrix M , then the Plücker coordinates $p_I(V)$ denotes the minor of M relative to I , and they do not depend on the choice of M up to simultaneous rescaling by a nonzero constant.

Proposition 3.1 (C -equations). *Let (n, k, m) be a triplet of nonnegative integers such that $k + m \leq n$. Let \mathcal{Z} be a $n \times (k + m)$ matrix of rank $k + m$, let $C \in \text{Gr}_{k,n}$ and $Y = \tilde{\mathcal{Z}}(C)$. Let $A \in \binom{[n]}{k-1}$ and $B \in \binom{[n]}{m-1}$. The C -equations are*

$$\sum_{i=1}^n p_{A,i}(C) \langle Y, B, i \rangle = 0, \quad (13)$$

where we used Notation 1.7 for the twistor coordinate $\langle Y, B, i \rangle$.

Remark 3.2. Both the vector of Plücker coordinates $(p_{A,i}(C))_{A,i}$ and the vector of twistor coordinates $(\langle B, i \rangle)_{B,i}$ are defined up to a non zero multiplicative scalar, corresponding to the choice of the matrix representing C and Y . However these scalars do not affect Eq. (13).

We now prove the C -equations.

Proof. From Lemma 2.1 we get

$$\sum_{i=1}^n p_{A,i}(C) \langle Y, B, i \rangle = \sum_{i=1}^n \sum_{J=(j_1 < \dots < j_k) \in \binom{[n]}{k}} p_{A,i}(C) p_J(C) \langle J, B, i \rangle. \quad (14)$$

Fix $L = \{l_1 < \dots < l_{k+1}\} \in \binom{[n]}{k+1}$. We now extract the coefficient of the determinant $\langle l_1, \dots, l_{k+1}, B \rangle$ in the RHS of Eq. (14). To do so, the index i must be in L and once it is fixed, we have $J = L \setminus \{i\}$. Taking care of the sign given by the antisymmetry of the determinant, we write the coefficient of $\langle l_1, \dots, l_{k+1}, B \rangle$ in the RHS of Eq. (14) as

$$(-1)^{k+m} \left(\sum_{\alpha=1}^{k+1} (-1)^\alpha p_{A, l_\alpha}(C) p_{l_1, \dots, \widehat{l_\alpha}, \dots, l_{k+1}}(C) \right).$$

This expression vanishes by the Plücker relations (see for instance [GH14]). This ends the proof. \square

3.1.2 The \mathcal{Z} -equations

Proposition 3.3 (\mathcal{Z} -equations). *Let (n, k, m) be a triplet of nonnegative integers such that $k + m < n$. Let \mathcal{Z} be a $n \times (k + m)$ matrix of rank $k + m$ and denote by $W = \mathcal{Z}^T$ the $(k + m)$ -plane in \mathbb{R}^n generated by the columns of \mathcal{Z} . Let $C \in \text{Gr}_{k,n}$ and $Y = \tilde{\mathcal{Z}}(C)$. Let $A \in \binom{[n]}{k+m+1}$ and $B \in \binom{[n]}{m-1}$. The \mathcal{Z} -equations are*

$$\sum_{i \in A} (-1)^{\#i} p_{A \setminus i}(W) \langle Y, B, i \rangle = 0, \quad (15)$$

where $\#i$ is the position of i in the list A .

Proof. Using Lemma 2.1 we get

$$\sum_{i \in A} (-1)^{\#i} p_{A \setminus i}(W) \langle Y, B, i \rangle = \sum_{i \in A} \sum_{J \in \binom{[n]}{k}} (-1)^{\#i} p_J(C) p_{A \setminus i}(W) \langle J, B, i \rangle.$$

Fix $J = (j_1 < \dots < j_k) \in \binom{[n]}{k}$, the coefficient of $p_J(C)$ is

$$\sum_{i \in A} (-1)^{\#i} p_{A \setminus i}(W) \langle J, B, i \rangle,$$

and we can write this expression as

$$\sum_{i \in A} (-1)^{\#i} p_{A \setminus i}(W) p_{J, B, i}(W).$$

This last expression vanishes by Plücker relations (see [GH14]) in $\text{Gr}_{k+m,n}$. This proves the \mathcal{Z} -equations. \square

3.2 Consequences of the C - and \mathcal{Z} - equations on twistor coordinates

From now on, we suppose that $m = 2r - 1$ for $r \geq 1$. The purpose of this section is to deduce from the C - and \mathcal{Z} - equations, together with positivity of the minors of C and \mathcal{Z} , the following constraints on twistor coordinates.

Proposition 3.4. *Let $Y \in \mathcal{A}_{n,k,m}^{>0}$ and let $B = (b_1, b_1 + 1, \dots, b_{r-1}, b_{r-1} + 1)$ be a list of $2r - 2 = m - 1$ integers in $[n]$.*

1. *The list $(\langle Y, B, i \rangle)_{i \in [n] \setminus B}$ cannot contain three consecutive twistor coordinates $\langle Y, B, i_0^- \rangle, \langle Y, B, i_0 \rangle, \langle Y, B, i_0^+ \rangle$, such that*

$$\langle Y, B, i_0 \rangle = 0 \quad \text{and} \quad \text{sign}(\langle Y, B, i_0^- \rangle) = \text{sign}(\langle Y, B, i_0^+ \rangle) \neq 0,$$

where the indices (i_0^-, i_0, i_0^+) satisfy: $2 \leq i_0 \leq n - 1$ and

$$i_0^- = \max([1, \dots, i_0 - 1] \cap B^c),$$

$$i_0^+ = \min([i_0 + 1, \dots, n] \cap B^c),$$

where B^c is the complement of B in $[n]$.

2. *The list $(\langle Y, B, i \rangle)_{i \in [n] \setminus B}$ cannot contain two consecutive zeros.*

We first establish, in Section 3.2.1, a sign flip property of the twistor coordinates. Then, in Section 3.2.2, we deduce from this property, together with the \mathcal{Z} -equations, the proof of Proposition 3.4.

3.2.1 A sign flip property

The following lemma was stated in [AHTT18, Section 5] and a sketch of a proof was given. It will be used in what follows. For completeness we provide another proof, which is based upon the C -, \mathcal{Z} -equations and the positivity of the minors.

Lemma 3.5. *Let $Y \in \mathcal{A}_{n,k,m}^{>0}$ and let $B = (b_1, b_1 + 1, \dots, b_{r-1}, b_{r-1} + 1)$ be a list of $m - 1$ integers in $[n]$. Then, the number of times the list of numbers $(\langle Y, B, i \rangle)_{i \in [n]}$ changes sign (ignoring the zeros) is exactly k .*

Remark 3.6. If $m = 2r$ is even, we can prove in the same way that if $B = (1, b_1, b_1 + 1, \dots, b_{r-1}, b_{r-1} + 1)$ or $B = (b_1, b_1 + 1, \dots, b_{r-1}, b_{r-1} + 1, n)$, then the number of times the list $(\langle Y, B, i \rangle)_{i \in [n]}$ changes sign is exactly k .

We now prove the sign flip property.

Proof of Lemma 3.5. The proof splits in two steps.

Step 1. In this step we use the C -equations and the positivity of C , to deduce that the list $(\langle Y, B, i \rangle)_{i \in [n]}$, either changes sign at least k times, or it is a list of zeros. The second possibility, that $(\langle Y, B, i \rangle)_{i \in [n]}$ is a list of zeros, is ruled out since the first and last element of the list do not satisfy the strict inequalities of the coarse boundary conditions (see Eq. (2)). We emphasize that a more general sign flip statement has already been proven using different tools in [KW19, Corollary 3.21], more details are provided in Remark 3.7.

Suppose $(\langle Y, B, i \rangle)_{i \in [n]}$ changes sign q times with $q < k$. Denote by $s_1 < \dots < s_q$ the positions of the sign flips. More precisely, let s_0 be the smallest integer such that $\langle Y, B, s_0 \rangle \neq 0$ and set

$$s_i = \min(s \in [n] \mid s > s_{i-1} \text{ and } \text{sign}(s) = -\text{sign}(s_{i-1})).$$

Define $A = (a_1, \dots, a_{k-1})$ by

$$a_j = \max(j, s_{j-(k-q)+1}), \quad \text{for } 1 \leq j \leq k-1,$$

with the convention that $s_j = 0$ if $j \leq 0$. More explicitly, let j_0 be the smallest index such that $a_{j_0} \neq j_0$, in this case $a_j = s_{j-(k-q)+1}$ for $j \geq j_0$, and the list A is

$$A = (1, \dots, j_0 - 1, s_{j_0-(k-q)+1}, \dots, s_q).$$

Thus, the C -equation associated to A and B is

$$\sum_{i=a_{j_0}+1}^{a_{j_0+1}-1} p_{A,i}(C) \langle Y, B, i \rangle + \sum_{i=a_{j_0+1}+1}^{a_{j_0+2}-1} p_{A,i}(C) \langle Y, B, i \rangle + \dots + \sum_{i=a_{k-1}+1}^n p_{A,i}(C) \langle Y, B, i \rangle = 0, \quad (16)$$

where the summation can be empty if two sign flips are successive. The LHS is a sum of terms which are all nonnegative (or all nonpositive). Indeed, since a_j corresponds to the position of a sign flip for $j \geq j_0$, the twistor coordinates $\langle Y, B, i \rangle$ for $i \in]a_j, a_{j+1}[$ and $j \geq j_0$ are all nonnegative (resp. all nonpositive). Moreover, the twistor coordinates $\langle Y, B, i \rangle$ for $i \in]a_{j+1}, a_{j+2}[$ and for $i \in]a_{j-1}, a_j[$ are all nonpositive (resp. all nonnegative). Since all the minors of C are positive, the sign coming from the antisymmetry of the determinant $p_{A,i}(C)$ exactly compensates the change of sign of the twistor coordinates. Thus all the terms of the LHS of Eq. (16) vanish, and since the Plücker coordinates of C are nonzero, we obtain

$$\langle Y, B, i \rangle = 0, \quad \text{for } i \in [n] \setminus A.$$

Then, fix $j_0 \in [n] \setminus A$. For each $a \in A$, we define the list \tilde{A} from A by first replacing a by j_0 and then sorting the list in ascending order. The C -equation associated to \tilde{A} and B reads

$$p_{\tilde{A},a}(C) \langle Y, B, a \rangle = 0.$$

Since $p_{\tilde{A},a}(C) \neq 0$, we deduce that $\langle Y, B, a \rangle = 0$ for $a \in A$. Thus $(\langle Y, B, i \rangle)_{i \in [n]}$ is a list of zeros.

Step 2. In this step we use the \mathcal{Z} -equations and the positivity of the minors of \mathcal{Z} , to show that the list $(\langle Y, B, i \rangle)_{i \in [n]}$ changes sign at most k times.

The \mathcal{Z} -equations are valid for $n > k + m$; we first consider the case $k + m = n$. In this case, the list $(\langle Y, B, i \rangle)_{i \in [n]}$ has the same number of sign flips as $(\langle Y, B, i \rangle)_{i \in [n] \setminus B}$ since we only remove the zeros $\langle Y, B, i \rangle = 0$, $i \in B$. Then the list $(\langle Y, B, i \rangle)_{i \in [n] \setminus B}$ is of length $k + 1$ so its maximum number of sign flips is k .

Now suppose $n > k + m$ and $(\langle Y, B, i \rangle)_{i \in [n]}$ changes sign q times with $q > k$. As in Step 1, we denote s_0 the smallest integer such that $\langle Y, B, s_0 \rangle \neq 0$, and by $s_1 < \dots < s_q$ the positions of the sign flips. We introduce the list

$$A = (s_0, \dots, s_{k+1}) \cup B.$$

Once again, the union of two lists is defined by first taking the union of their elements and then sorting them in ascending order. Since $\langle Y, B, i \rangle = 0$ when $i \in B$, an element of B cannot be equal to s_i for $i = 0, \dots, k + 1$, thus the list A is of length $k + m + 1$. Now the \mathcal{Z} -equation associated to A is

$$\sum_{i \in A} (-1)^{\#i} p_{A \setminus i}(W) \langle Y, B, i \rangle = 0. \quad (17)$$

This is a sum of numbers all nonnegative or all nonpositive. Indeed, first the matrix \mathcal{Z} , and hence W , has positive maximal minors thus $p_{A \setminus i}(W)$ is positive. Moreover, the terms of the sum vanish for $i \in B$ since in that case $\langle Y, B, i \rangle = 0$. Now, let i go through $A \setminus B$, since the elements of B come by pairs of consecutive integers it follows that the sign $(-1)^{\#i}$ changes at each element of $A \setminus B$, however this change of sign is exactly compensated by the sign flip of $\langle Y, B, i \rangle$ at each element of $A \setminus B = (s_1, \dots, s_{k+1})$. We conclude that the sum of Eq (17) is a sum of terms of the same sign and thus they all vanish. Since the Plücker coordinates of W are nonzero, we obtain

$$\langle Y, B, a \rangle = 0, \quad \text{for } a \in A.$$

This contradicts the assumption $\langle Y, B, s_i \rangle \neq 0$ for $i = 0, \dots, k + 1$. This ends the proof. \square

Remark 3.7. The assertion and proof of Step 1 is actually valid in a more general context: B can be any element of $\binom{[n]}{m-1}$, but in this case $(\langle Y, B, i \rangle)_{i \in [n]}$ can also be a list of zeros. We emphasize that Karp and Williams already proved a more general statement: this sign flip property is valid for any point of the interior of the amplituhedron, see [KW19, Corollary 3.21]. We recall that $\mathcal{A}_{n,k,m}^{>0}$ is included into the interior of the amplituhedron, see [GL20, Lemma 9.4].

Remark 3.8. The assertion and proof of Step 2 is still correct for $Y \in \text{Gr}_{k,k+m}$. Indeed, we do not use the positivity of C in the proof (or in the proof of the \mathcal{Z} -equations) and any element of $Y \in \text{Gr}_{k,k+m}$ can be written $Y = \tilde{C}\mathcal{Z}$ for $\tilde{C} \in \text{Gr}_{k,n}$.

3.2.2 Proving Proposition 3.4

We begin with the first statement. According to Lemma 3.5, the list $(\langle Y, B, i \rangle)_{i \in [n]}$ has exactly k sign flips. Denote by $S = (s_1 < \dots < s_k)$ the positions of the sign flips, defined as in the

proof of Lemma 3.5. We define the set A of size $k+m+1$ to be the union of S, B and $\{i_0, i_0^+\}$, where i_0 is as in the statement of the proposition. The \mathcal{Z} -equation associated to A and B is

$$\sum_{i \in A} (-1)^{\#i} p_{A \setminus i}(W) \langle Y, B, i \rangle = 0.$$

We now justify that all the terms of the sum are nonnegative or nonpositive. Since the matrix \mathcal{Z} , and hence W , is positive, the Plücker coordinates $p_{A \setminus i}(W)$ have a constant sign for $i \in A$. We show that the change of sign $(-1)^{\#i}$ exactly compensates the change of sign of $\langle Y, B, i \rangle$. First, by definition of i_0^-, i_0 and i_0^+ the twistor coordinates $\langle Y, B, i \rangle$ for $i \in [i_0^-, i_0^+]$ are all nonnegative or all nonpositive. Thus, $[i_0^-, i_0^+]$ is contained between two successive sign flips, say s_j and s_{j+1} . More precisely, we have $[i_0^-, i_0^+] \subset [s_j, s_{j+1}[$ with $\text{sign}(s_j) = \text{sign}(i_0^-) = \text{sign}(i_0^+)$. Since $\langle Y, B, i \rangle = 0$ for $i \in B$, we deduce that when i goes through A , the sign of $\langle Y, B, i \rangle$ only changes at $i = s_1, \dots, s_k$. Moreover, between two successive sign flips, there always are an even number of elements of A ; it can be pairs of successive elements of B or the pair (i_0, i_0^+) . Hence the terms on the LHS of the \mathcal{Z} -equation are all nonnegative or all nonpositive. Thus, they all vanish. Moreover, by the positivity of W , the Plücker coordinates of W are nonzero and we deduce that

$$\langle Y, B, a \rangle = 0, \text{ for } a \in A.$$

Once again, we modify the elements of A one by one; let $j \in [n] \setminus A$ and define \tilde{A} from A by first replacing a given element $a \in A$ by j and then sorting the list in ascending order. Then, the \mathcal{Z} -equation associated to \tilde{A} and B reads (up to a sign)

$$p_{\tilde{A} \setminus j}(W) \langle Y, B, j \rangle = 0.$$

Then, simplifying by the nonzero Plücker coordinate, we obtain $\langle Y, B, j \rangle = 0$. Thus, $(\langle Y, B, i \rangle)_{i \in [n]}$ is a list of zeros, this contradicts $\langle Y, B, i_0^- \rangle \neq 0$.

We now prove the second statement. Denote by i_0 and $i_0 + 1$ the positions of two of the consecutive zeros and by $S = (s_1 < \dots < s_k)$ the positions of the sign flips. Similarly to the previous item, we use the \mathcal{Z} equation with $A = S \cup B \cup \{i_0, i_0 + 1\}$ and deduce that $(\langle Y, B, i \rangle)_{i \in [n]}$ is the zero list. This is impossible by the strict coarse boundary conditions given in Eq. (2).

3.3 Properties of simplices containing the origin of V_Y

The purpose of this section is to use the constraints on the twistor coordinates obtained in the previous section to exclude some configurations of simplices containing the origin $\pi_Y(Y)$ of the quotient space $V_Y = \mathbb{R}^{k+m}/Y$ defined in Section 1.6. The first two sections introduce some terminology for cells and vertices. Then, in Section 3.3.3, we obtain two fundamental lemmas describing the behavior of the simplices containing the origin. At this stage, we give an idea of the proof of Theorem 2 in a simplified context. After some preparatory lemmas in Section 3.3.4, we establish a refined version of these lemmas in Section 3.3.5 which describes the configuration of cells around a cell containing the origin.

Abuse of terminology. In the following, we will only use simplices involved in the count of the crossing number, that is simplices of type

$$S(i_1, i_1 + 1, \dots, i_r, i_r + 1),$$

for $(i_1, i_1 + 1, \dots, i_r, i_r + 1)$ a list of strictly ascending integers between 1 and n . The terminology simplex will only refer to these simplices.

3.3.1 Cells terminology

Definition 3.9 (Descendant and ancestor cells. Lineage). Let \mathfrak{C} be a cell as defined in Section 1.6, it is uniquely associated to the set I of indices of its vertices. A *descendant* of \mathfrak{C} is a cell associated to a strict subset of I . If S is a simplex, then a descendant of S is a descendant of the m -cell of S .

A cell \mathfrak{C}' is an *ancestor* of \mathfrak{C} if \mathfrak{C} is a descendant of \mathfrak{C}' . A simplex S is an *ancestor* of \mathfrak{C} if \mathfrak{C} is a descendant of S .

The *lineage* of a cell \mathfrak{C} is the set containing the descendent cells of \mathfrak{C} , the ancestor cells of \mathfrak{C} and \mathfrak{C} .

Notation 3.10. Let \mathfrak{C} be a cell. Denote by $S_{\mathfrak{C}}$ the set of simplices which are ancestors of \mathfrak{C} . Denote by $L_{\mathfrak{C}}$ the set of cells of the simplices in $S_{\mathfrak{C}}$. Denote by $U_{\mathfrak{C}} \subset V_Y$ the set of points of the cells in $L_{\mathfrak{C}}$.

Definition 3.11 (Boundary cells. Internal cells). A *boundary cell* is either

- the cell $\mathfrak{C}(1, i_1, i_1 + 1, \dots, i_{r-1}, i_{r-1} + 1)$, where $(1, i_1, i_1 + 1, \dots, i_{r-1}, i_{r-1} + 1)$ is a list of strictly ascending integers smaller or equal to n , or
- the cell $\mathfrak{C}(i_1, i_1 + 1, \dots, i_{r-1}, i_{r-1} + 1, n)$, where $(i_1, i_1 + 1, \dots, i_{r-1}, i_{r-1} + 1, n)$ is a list of strictly ascending integers greater or equal to 1, or
- a descendant of one of these cells.

An *internal cell* is a cell which is not a boundary cell.

It follows from the coarse boundary conditions, Eq. (2), that if $Y \in \mathcal{A}_{n,k,m}^{>0}$, then the origin $\pi_Y(Y)$ cannot belong to a boundary cell. Hence the crossing number counts the number of internal cells containing the origin of V_Y .

Example 3.12. When $m = 3$, the vertex Z_i for $i = 1, \dots, n$ is a boundary cell, the cell $\mathfrak{C}(i, i + 1)$ for $i = 1, \dots, n - 1$ is a boundary cell and the cell $\mathfrak{C}(i, j)$, with $j > i + 1$, is an internal cell whenever $j < n - 1$ and $i > 2$.

3.3.2 Conjugate vertex

Lemma 3.13. *Let $I \in \binom{[n]}{m}$ and $i \in [n]$ be such that $S(I, i)$ is a simplex, and such that its descendent $(m - 1)$ -cell $\mathfrak{C}(I)$ is nonempty and is internal. Then, there exists a unique index $\bar{i} \neq i$ such that $S(I, \bar{i})$ is an ancestor simplex of $\mathfrak{C}(I)$.*

Notation 3.14. When \mathcal{Z} and Y are understood, the vertex i refers to the vertex Z_i .

Definition 3.15 (Conjugate vertex). The vertex \bar{i} is called the *conjugate vertex of i relative to the simplex $S(I, i)$* .

Example 3.16. In Figure 1, the conjugate vertex of i relative to $S(i, i + 1, j, j + 1)$ is $\bar{i} = i + 2$.

Proof. We first relabel the simplices of $S(I, i)$ by pairs of consecutive indices: let j_1, \dots, j_r in $[n]$ such that $\{j_1, j_1 + 1, \dots, j_r, j_r + 1\} = I \cup \{i\}$.

- If $i = j_a$ for $1 \leq a \leq r$. Then

$$\bar{i} := \min \{I^c \cap \{j_a + 2, \dots, n\}\},$$

where I^c is the complement of I in $[n]$, is the conjugate vertex to i relative to $S(I, i)$. Indeed, first \bar{i} exists since $\mathfrak{C}(I)$ is an internal cell. Then, $S(I, \bar{i})$ is a simplex: for any $1 \leq a \leq r$, introduce

$$\begin{aligned} k_a &:= j_a + 1 && \text{if } j_a \in [i, \bar{i}], \\ k_a &:= j_a && \text{otherwise,} \end{aligned}$$

then $\{k_1, k_1 + 1, \dots, k_r, k_r + 1\}$ are the indices of vertices of $S(I, \bar{i})$. Finally, no other simplex can be an ancestor of $\mathfrak{C}(I)$ since it is a $(m - 1)$ -cell, hence \bar{i} is unique.

- If $i = j_a + 1$ for $1 \leq a \leq r$. Then

$$\bar{i} := \max(\{1, \dots, i_a - 1\} \cap I^c)$$

is, for similar reasons, the conjugate vertex to i relative to $S(I, i)$.

□

3.3.3 First properties of simplices containing the origin of V_Y

The purpose of the section is to deduce from the constraints on the twistor coordinates obtained in Section 3.2 two properties of the simplices containing the origin. From these two properties we can already understand why the crossing number is constant on $\mathcal{A}_{n,k,m}^{>0}$ as explained at the end of the section.

Lemma 3.17 (Full-dimensional simplex). *Let $Y \in \mathcal{A}_{n,k,m}^{>0}$. Each simplex with a cell containing $\pi_Y(Y)$ is full dimensional.*

In particular, for an element $Y \in \mathcal{A}_{n,k,m}^{>0}$ the crossing number corresponds to the number of simplices containing the origin $\pi_Y(Y)$ with the convention that if $\pi_Y(Y)$ is in a cell belonging to several simplices, it is counted once.

Proof. Suppose that $S(i_1, i_1 + 1, \dots, i_r, i_r + 1)$ is a flat simplex containing the origin $\pi_Y(Y)$. Then, by Remark 1.10, for any list L of m elements of $\{i_1, i_1 + 1, \dots, i_r, i_r + 1\}$, we have $\langle Y, L \rangle = 0$. In particular, let $B = (i_1, i_1 + 1, \dots, i_{r-1}, i_{r-1} + 1)$, then

$$\langle Y, B, i_r \rangle = 0 = \langle Y, B, i_r + 1 \rangle.$$

This is forbidden by the second assertion of Proposition 3.4. \square

Let \mathfrak{C} be a $(m - 1)$ internal cell. Denote by I the list of indices of vertices of \mathfrak{C} . By definition of a $(m - 1)$ cell, it is the descendant of a least one simplex, say $S(I, i)$, for $i \in [n] \setminus I$. Then by Lemma 3.13, \mathfrak{C} is a descendant cell of exactly two simplices $S(I, i)$ and $S(I, \bar{i})$. Furthermore, the hyperplane $H_{\mathfrak{C}}$ containing \mathfrak{C} divides V_Y into two open half-spaces.

Lemma 3.18 (Main lemma). *Let $Y \in \mathcal{A}_{n,k,m}^{>0}$. If $\pi_Y(Y)$ is contained in \mathfrak{C} or a descendant of \mathfrak{C} , then Z_i and $Z_{\bar{i}}$ must belong to different open half-spaces relative to $H_{\mathfrak{C}}$.*

An example of configurations of simplices for $m = 3$ is given in Figure 1. In this example $I = \{i + 1, j, j + 2\}$ and $\bar{i} = i + 2$. If the origin belongs to $\mathfrak{C}(i + 1, j, j + 1)$ or a descendant of this cell (in red on the figures), then the simplices can only be in the configuration of Figure 1b.

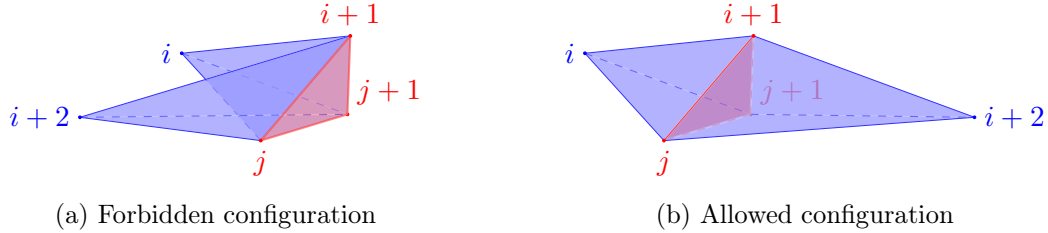


Figure 1: Two configurations of the simplices $S(i, i + 1, j, j + 1)$ and $S(i + 1, i + 2, j, j + 1)$ ancestors of the cell $\mathfrak{C}(i + 1, j, j + 1)$ in $m = 3$. In Figure 1a, the vertices i and $i + 2$ belong to the same side of the plane generated by the cell $\mathfrak{C}(i + 1, j, j + 1)$; in Figure 1b these vertices belong to different sides.

Suppose the origin $\pi_Y(Y)$ belongs to the cell $\mathfrak{C}(i + 1, j, j + 1)$. When Y moves continuously in $\mathcal{A}_{n,k,m}^{>0}$, then the points $Z_i = \pi_Y(Z_i)$ for $1 \leq i \leq n$ move continuously. If the configuration of Figure 1a was allowed, then after a small modification of Y , the crossing number could jump by $+1$ if the origin jumps into the two cells $\mathfrak{C}(i, i + 1, j, j + 1)$ and $\mathfrak{C}(i + 1, i + 2, j, j + 1)$, or by -1 if the origin jumps out of any cell of the two simplices. This bad behavior of the crossing number does not happen in the configuration of Figure 1b. More generally, we prove in this way that if $\pi_Y(Y)$ avoids cells of codimension 2 or more, then the crossing number is constant in $\mathcal{A}_{n,k,m}^{>0}$. The rest of the Section 3.3 is devoted to making this

argument correct if $\pi_Y(Y)$ belongs to any internal cell. To do so, we will only use the main lemma and the lemma about full-dimensional simplices.

Proof. First Z_i (resp. $Z_{\bar{i}}$) cannot belong to $H_{\mathfrak{C}}$, otherwise the ancestor simplex $S(I, i)$ (resp. $S(I, \bar{i})$) of \mathfrak{C} is not full-dimensional, which is forbidden by Lemma 3.17.

Now, suppose that Z_i and $Z_{\bar{i}}$ belong to the same open half-space with respect to $H_{\mathfrak{C}}$. Say $i < \bar{i}$, then it follows from the construction of the conjugate vertex in Lemma 3.13 that $[i + 1, \bar{i} - 1] \subseteq I$ is nonempty. Pick i_0 in this sequence and define $B = I \setminus \{i_0\}$. Since $\pi_Y(Y)$ belongs to $H_{\mathfrak{C}}$, we have

$$\langle Y, B, i_0 \rangle = 0.$$

Moreover, since Z_i and $Z_{\bar{i}}$ belong to the same open half-space with respect to $H_{\mathfrak{C}}$, we have

$$\text{sign}(\langle Y, B, i \rangle) = \text{sign}(\langle Y, B, \bar{i} \rangle) \neq 0.$$

But this situation is impossible according to Proposition 3.4 (with $i_0^- = i$ and $i_0^+ = \bar{i}$). The situation is similar for $\bar{i} < i$. This proves the lemma. \square

3.3.4 Ancestor simplices of an internal cell

The goal of this section is to prove the following proposition which describes the ancestor simplices of an internal cell in terms of one simplex and its conjugate vertices. This proposition will only be used for the proof of Proposition 3.22 in the next section.

Proposition 3.19. *Let \mathfrak{C} be an internal cell of dimension $d < m$, and let $I = (i_0, \dots, i_d)$ be the set of indices of its vertices. Let j_1, j_2, \dots, j_{m-d} in $[n]$ be such that $S(I, j_1, j_2, \dots, j_{m-d})$ is an ancestor simplex of \mathfrak{C} . Introduce, for each $1 \leq a \leq m - d$, the conjugate vertex \bar{j}_a of j_a relative to $S(I, j_1, j_2, \dots, j_{m-d})$. Then, for each choice of $(\alpha_1, \dots, \alpha_{m-d})$ in $\{j_1, \bar{j}_1\} \times \dots \times \{j_{m-d}, \bar{j}_{m-d}\}$ such that the elements of $(\alpha_1, \dots, \alpha_{m-d})$ are all pairwise distinct, the simplex*

$$S(I, \alpha_1, \dots, \alpha_{m-d})$$

is an ancestor simplex of \mathfrak{C} , and moreover each ancestor simplex of \mathfrak{C} is obtained in this way.

To prove this proposition, we first establish two lemmas which can be safely forgotten once the proposition is proved. We use the notations of Proposition 3.19 for these two lemmas.

We first introduce the following terminology. Let S be a simplex. Its vertices are partitioned into r couples with consecutive indices. Each couple is called a *pair of vertices of S* and the corresponding couple of indices is called a *pair of indices of S* . The following lemma explains how indices of an internal cell \mathfrak{C} can be paired in an ancestor of \mathfrak{C} .

Lemma 3.20. *Let \mathfrak{C} be an internal cell. Let $L = (i_a, \dots, i_b)$, with $0 \leq a \leq b \leq d$, be a sequence of consecutive isolated indices of \mathfrak{C} (i.e. the vertex $i_a - 1$ and the vertex $i_b + 1$ are not vertices of \mathfrak{C}). Then,*

1. if the length of L is odd, the vertices of L can appear paired in two different ways in a simplex of $S_{\mathfrak{C}}$:
 - (a) the vertex i_a is paired with the vertex $i_a - 1$ and the rest of the vertices of L are paired together,
 - (b) the vertex i_b is paired with $i_b + 1$ and the rest of the vertices are paired together,
2. if the length of this sequence is even, then in any simplex of $S_{\mathfrak{C}}$ the vertices of (i_a, \dots, i_b) are paired together, i.e. i_a with $i_a + 1$, i_{a+2} with $i_{a+2} + 1$...

Proof. The first assertion follows from the definition of a simplex. We prove the second assertion by contradiction: suppose S is an ancestor simplex of \mathfrak{C} and suppose that the pairings of the indices of S containing an element of L are

$$P_1 = (i_a - 1, i_a), P_2 = (i_a + 1, i_a + 2), \dots, P_{\frac{b-a+3}{2}} = (i_b, i_b + 1).$$

Then, let \mathfrak{C}' be the $(m - 2)$ -cell with the same pairings as S except $P_1, \dots, P_{\frac{b-a+1}{2}}$ which are shifted by $+1$ (i.e replace P_1 by $(i_a, i_a + 1)$ and so on), and $P_{\frac{b-a+3}{2}}$ which is forgotten. But \mathfrak{C}' is a boundary cell, indeed let J be the indices of the vertices of \mathfrak{C}' and J^c its complement in $[n]$, so one can add to \mathfrak{C}' the vertex $\max(J^c)$ to obtain a cell of type $\mathfrak{C}(k_1, k_1 + 1, \dots, k_r, k_r + 1, n)$. Moreover \mathfrak{C}' is an ancestor cell of \mathfrak{C} , hence \mathfrak{C} is a boundary cell, which is a contradiction. \square

The second lemma is the following.

Lemma 3.21. *We have that:*

- the number of vertices of \mathfrak{C} satisfies $d + 1 \geq r$,
- the number $n_{\mathfrak{C}}$ of sequences of isolated indices of odd length in I is exactly $2r - (d + 1) = m - d$.

Proof. First, since \mathfrak{C} is an internal cell, we have $d + 1 \geq r$. Otherwise, we can always complete the vertices of \mathfrak{C} to form a boundary cell of dimension $(m - 1)$.

We then prove by induction on the number of vertices $d + 1$ of \mathfrak{C} that $n_{\mathfrak{C}} = 2r - (d + 1)$.

Initialization $d + 1 = r$. If $n_{\mathfrak{C}} > 2r - (d + 1) = r$, then \mathfrak{C} cannot be a descendant of a simplex because a simplex contains at most r couples of isolated indices. If $n_{\mathfrak{C}} < r$, then two indices of I are consecutive and \mathfrak{C} has r vertices but in this case it is easy to verify that \mathfrak{C} cannot be an internal cell.

Heredity $d + 1 > r$. Suppose that $n_{\mathfrak{C}}$ is greater (resp. lower) than $2r - (d + 1)$. Then pick one odd sequence of consecutive indices and select the maximal index of this sequence. Remove the vertex of \mathfrak{C} corresponding to this index and denote by \mathfrak{C}' the corresponding cell. Since \mathfrak{C} is an internal cell with $(d + 1)$ vertices, then \mathfrak{C}' is an internal cell with d vertices. Moreover $n_{\mathfrak{C}'} = n_{\mathfrak{C}} - 1$ is greater (resp. lower) than $2r - s - 1$, but this is impossible by induction. \square

We now prove the proposition.

Proof of Proposition 3.19. We deduce from these two lemmas that since $S(I, j_1, j_2, \dots, j_{m-d})$ is an ancestor simplex of \mathfrak{C} , then each sequence of isolated indices of odd length of \mathfrak{C} is followed or preceded by j_a for $1 \leq a \leq m - d$. Moreover \bar{j}_a is, by construction, the predecessor or successor of this same sequence. It is then a consequence of Lemma 3.20 that a simplex is an ancestor of \mathfrak{C} if and only if it is associated to the vertices $(I, \alpha_1, \dots, \alpha_{m-d})$ for a choice of $\alpha_1, \dots, \alpha_{m-d}$ in $(j_1, \bar{j}_1) \times \dots \times (j_{m-d}, \bar{j}_{m-d})$ pairwise disjoint. \square

3.3.5 Configuration of ancestor simplices of a cell containing the origin

The following proposition explains how ancestor simplices of a cell \mathfrak{C} containing the origin of V_Y “triangulate” a neighborhood of \mathfrak{C} , i.e. cells of these simplices are pairwise distinct and they cover a neighborhood of \mathfrak{C} . We recall that $S_{\mathfrak{C}}$ is the set of ancestor simplices of \mathfrak{C} , $L_{\mathfrak{C}}$ is the set of cells of the simplices of $S_{\mathfrak{C}}$, and $U_{\mathfrak{C}} \subset V_Y$ is the set of points of the cells in $L_{\mathfrak{C}}$. We also recall that since the origin of V_Y belongs to \mathfrak{C} , then each simplex of $S_{\mathfrak{C}}$ is full-dimensional (Lemma 3.17).

Proposition 3.22 (Configuration of adjacent cells). *Let $Y \in \mathcal{A}_{n,k,m}^{>0}$. Let \mathfrak{C} be a cell containing the origin $\pi_Y(Y)$ of V_Y . Then,*

1. *the set $U_{\mathfrak{C}}$ is a neighborhood of \mathfrak{C} in V_Y , that is $U_{\mathfrak{C}}$ contains an open set of V_Y containing the cell \mathfrak{C} ,*
2. *each point of $U_{\mathfrak{C}}$ belongs to a unique cell of $L_{\mathfrak{C}}$.*

Example 3.23. Suppose $\mathfrak{C} = \mathfrak{C}(i, j)$ with $j > i + 2$, $j < n$, $i > 1$ and $m = 3$. Then \mathfrak{C} is a descendant of 4 simplices. Following Proposition 3.22, these 4 simplices form a neighborhood of \mathfrak{C} and their cells cannot intersect. These 4 simplices must be in the configuration of Figure 2.

In particular, it follows from this proposition that for a small and continuous modification of Y , the number of cells of $L_{\mathfrak{C}}$ containing $\pi_Y(Y)$ is one.

The rest of this section is devoted to the proof of this Proposition.

The case $\dim(\mathfrak{C}) = m$. If $\dim(\mathfrak{C}) = m$, then the ancestor simplex S of \mathfrak{C} is full-dimensional and the cells of $L_{\mathfrak{C}}$ are the cells of S . In particular, they form a neighborhood of \mathfrak{C} and they cannot pairwise intersect, otherwise S is flattened and this is forbidden by Lemma 3.17.

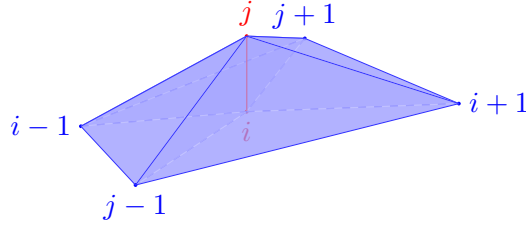


Figure 2: Configuration of the 4 ancestor simplices to a cell $\mathfrak{C}(i, j)$ containing the origin when $j > i + 2$, $j < n$, $i > 1$ and $m = 3$.

The case $\dim(\mathfrak{C}) < m$. In order to prove assertion 1 and 2 when $\dim(\mathfrak{C}) < m$, we need to establish some preparatory lemmas.

We define the cone \mathcal{C}_S associated to a S simplex of $S_{\mathfrak{C}}$ to be the convex cone generated by the vertices of S in V_Y , that is the set of linear combinations of vertices of S with nonnegative coefficients. Since S is a convex simplex containing the origin, its relation with the cone \mathcal{C}_S is particularly simple as explained by the following lemma.

Lemma 3.24. *Let r be a ray of the convex cone \mathcal{C}_S associated to a simplex S of $S_{\mathfrak{C}}$. Then there exists a point x of the ray r , different from the origin, such that the segment $[0, x]$ is included in S .*

Proof. Let r be a ray of \mathcal{C}_S . Since \mathcal{C}_S is generated by S , then S contains at least one point x of the ray r which is different from the origin. Now, since S is convex and contains the origin, the simplex S contains all the points of the segments $[0, x]$. \square

Definition 3.25. Let L be the list of indices of the common vertices of S_1 and S_2 . Then the two simplices S_1 and S_2 intersect trivially if $S_1 \cap S_2$ is contained in $\text{Span}(Z_i, \text{ for } i \in L) \subset V_Y$.

Similarly, the two cones \mathcal{C}_{S_1} and \mathcal{C}_{S_2} associated to S_1 and S_2 intersect trivially if $\mathcal{C}_{S_1} \cap \mathcal{C}_{S_2}$ is contained in $\text{Span}(Z_i, \text{ for } i \in L)$.

Example 3.26. The two simplices of the main lemma intersect trivially. On the other hand, the two simplices of Figure 1a do not intersect trivially.

Lemma 3.27. *Two simplices S_1 and S_2 of $S_{\mathfrak{C}}$ intersect trivially if and only if \mathcal{C}_{S_1} and \mathcal{C}_{S_2} intersect trivially.*

Proof. The forward direction is clear. The backward direction follows from Lemma 3.24. \square

The main ingredient of the proof of assertion 1 and 2 when $\dim(\mathfrak{C}) < m$ is the following lemma.

Lemma 3.28. *Let \mathfrak{C} be a cell containing the origin of V_Y . Then, any two simplices of $S_{\mathfrak{C}}$ intersect trivially.*

Proof. Suppose \mathfrak{C} is of dimension d . Let $1 \leq a \leq m - d$, then we prove by induction on a that: each couple (S, S') of simplices of $S_{\mathfrak{C}}$ with $m + 1 - a$ common vertices intersect trivially. Note that S and S' cannot have less than $d + 1$ common vertices since they are ancestors of \mathfrak{C} and thus they contain the $d + 1$ vertices of \mathfrak{C} .

Initialization. If $a = 1$, then S and S' intersect trivially by the main lemma.

Heredity. Let a be such that $1 \leq a \leq m - d$. Let S and S' be two simplices of $S_{\mathfrak{C}}$ sharing exactly $m + 1 - a$ vertices, and let L be the list of indices of these vertices. Let (L, i_1, \dots, i_a) be the list of indices of vertices of S . Denote by \bar{i}_j , for $1 \leq j \leq a$, the conjugate vertex of i_j relative to S . It follows from Proposition 3.19 that the list of vertices of S' is $(L, \bar{i}_1, \dots, \bar{i}_a)$.

We now replace the study of intersection of simplices by the intersection of cones. We first associate to any simplex $S(L, \alpha_1, \dots, \alpha_a)$ of $S_{\mathfrak{C}}$, where $\alpha_j \in \{i_j, \bar{i}_j\}$, the cone $\mathcal{C}(L, \alpha_1, \dots, \alpha_a)$ of \mathbb{R}^m generated by the vertices of $S(L, \alpha_1, \dots, \alpha_a)$. Furthermore, let $p : \mathbb{R}^m \rightarrow \mathbb{R}^a$ be the projection relative to the vector space $\text{Span}(Z_i, \text{ for } i \in L)$. We associate to $S(L, \alpha_1, \dots, \alpha_a)$ the projection of its corresponding cone, we denote this a -dimensional convex cone by

$$\mathcal{C}(\alpha_1, \dots, \alpha_a) := p(\mathcal{C}(L, \alpha_1, \dots, \alpha_a)) \subset \mathbb{R}^a.$$

Clearly, $\mathcal{C}(\alpha_1, \dots, \alpha_a)$ is the convex cone of \mathbb{R}^a generated by $p(Z_{\alpha_1}), \dots, p(Z_{\alpha_a})$. Since each simplex of $S_{\mathfrak{C}}$ is full-dimensional (Lemma 3.17), the cone $\mathcal{C}(\alpha_1, \dots, \alpha_a)$ is indeed of dimension a . The ray generated by $p(Z_{\alpha_j})$ of $\mathcal{C}(\alpha_1, \dots, \alpha_a)$ is called an extreme ray of index α_j . We naturally define the trivial intersection of projected cones by: two projected cones $\mathcal{C}(\alpha_1, \dots, \alpha_a)$ and $\mathcal{C}(\beta_1, \dots, \beta_a)$, where α_j and β_j belong to $\{i_j, \bar{i}_j\}$, with \tilde{L} being the list of indices of their common extreme rays, intersect trivially if $\mathcal{C}(\alpha_1, \dots, \alpha_a) \cap \mathcal{C}(\beta_1, \dots, \beta_a) \subset \text{Span}(p(Z_i), \text{ for } i \in \tilde{L})$.

Claim 3.29. Two simplices S_1 and S_2 of $S_{\mathfrak{C}}$ intersect trivially if and only if the projected cones $p(\mathcal{C}_{S_1})$ and $p(\mathcal{C}_{S_2})$ intersect trivially.

Indeed, by Lemma 3.27 it suffices to show that \mathcal{C}_{S_1} and \mathcal{C}_{S_2} trivially intersect if and only if $p(\mathcal{C}_{S_1})$ and $p(\mathcal{C}_{S_2})$ intersect trivially, which is clear from the definitions.

We are going to show that two projected cones associated to two simplices of $S_{\mathfrak{C}}$ intersect trivially. This will prove the lemma.

Claim 3.30. The cone $\mathcal{C}(i_1, \dots, i_a)$ (resp. $\mathcal{C}(\bar{i}_1, \dots, \bar{i}_a)$) cannot contain an extreme ray of $\mathcal{C}(\bar{i}_1, \dots, \bar{i}_a)$ (resp. $\mathcal{C}(i_1, \dots, i_a)$).

Indeed, suppose $p(Z_{i_j})$ belongs to $\mathcal{C}(\bar{i}_1, \dots, \bar{i}_a)$, for some $1 \leq j \leq a$. Then the cone $\mathcal{C}(\bar{i}_1, \dots, \hat{i}_j, \dots, \bar{i}_a, i_j)$ can either be of dimension strictly lower than a or it intersects nontrivially with $\mathcal{C}(\bar{i}_1, \dots, \bar{i}_a)$. Both cases are forbidden, the first one because the corresponding simplex would be flattened and the second one because $S(L, \bar{i}_1, \dots, \bar{i}_a)$ and $S(L, \bar{i}_1, \dots, \hat{i}_j, \dots, \bar{i}_a, i_j)$

would intersect nontrivially, which is forbidden by the induction hypothesis. This proves Claim 3.30.

Let $x \in \mathcal{C}(i_1, \dots, i_a) \cap \mathcal{C}(\bar{i}_1, \dots, \bar{i}_a)$ and suppose x is not the origin. Since $\mathcal{C}(i_1, \dots, i_a)$ is convex, the segment $[p(Z_{i_1}), x]$ is contained in $\mathcal{C}(i_1, \dots, i_a)$. It follows from Claim 3.30 that the segment $[p(Z_{i_1}), x]$ cannot be contained in $\mathcal{C}(\bar{i}_1, \dots, \bar{i}_a)$, otherwise $p(Z_{i_1}) \in \mathcal{C}(\bar{i}_1, \dots, \bar{i}_a)$. Thus, there is a point y in $[p(Z_{i_1}), x]$ belonging to the boundary of $\mathcal{C}(\bar{i}_1, \dots, \bar{i}_a)$. Then y belongs to a facet (that is a codimension-one face) of $\mathcal{C}(\bar{i}_1, \dots, \bar{i}_a)$, we denote this facet by F . Let \bar{I} be the set of indices of the vertices generating F , i.e. $F = \text{Span}_{\geq 0}(p(Z_{\bar{i}}), \bar{i} \in \bar{I})$. Let \bar{i}_j , for $1 \leq j \leq a$, be the index such that $\{\bar{i}_1, \dots, \bar{i}_a\} = \bar{I} \cup \{\bar{i}_j\}$. Then $\mathcal{C}(\bar{i}_1, \dots, \bar{i}_a)$ and $\mathcal{C}(\bar{I}, \bar{i}_j)$ intersect in y . Moreover, this intersection is nontrivial since otherwise $y \in \text{Span}_{> 0}(p(Z_{\bar{i}_j}))$, but then $p(Z_{\bar{i}_j})$ belongs to $\mathcal{C}(\bar{i}_1, \dots, \bar{i}_a)$, which is forbidden by Claim 3.30. The nontrivial intersection of $\mathcal{C}(i_1, \dots, i_a)$ and $\mathcal{C}(\bar{I}, \bar{i}_j)$ is forbidden by the induction hypothesis. Hence $\mathcal{C}(i_1, \dots, i_a) \cap \mathcal{C}(\bar{i}_1, \dots, \bar{i}_a) = \{0\}$, that is their intersection is trivial. This ends the proof of the induction and thus of the lemma. \square

Proving assertion 2 when $\dim(\mathfrak{C}) < m$. The cell \mathfrak{C} is an internal cell since $\pi_Y(Y)$ belongs to \mathfrak{C} . We prove that two cells of $L_{\mathfrak{C}}$ cannot intersect. Suppose that \mathfrak{C}_1 and \mathfrak{C}_2 are two cells of $L_{\mathfrak{C}}$ with a common point, then \mathfrak{C}_1 and \mathfrak{C}_2 cannot belong to the same simplex, otherwise this simplex is flattened and this is forbidden by Lemma 3.17. Hence \mathfrak{C}_1 and \mathfrak{C}_2 are descendant cells of two different simplices of $S_{\mathfrak{C}}$, say S_1 and S_2 . Hence, if \mathfrak{C}_1 and \mathfrak{C}_2 intersect, then S_1 and S_2 intersect nontrivially. This is excluded by Lemma 3.28.

Proving assertion 1 when $\dim(\mathfrak{C}) < m$. Let S_i be a simplex of $S_{\mathfrak{C}}$. We recall that \mathcal{C}_{S_i} is the convex cone of $V_Y \simeq \mathbb{R}^m$ generated by the vertices of S_i .

Claim 3.31. The set $U_{\mathfrak{C}} = \bigcup_{S_i \in S_{\mathfrak{C}}} S_i$ is a neighborhood of \mathfrak{C} if and only if $\bigcup_{S_i \in S_{\mathfrak{C}}} \mathcal{C}_{S_i} = \mathbb{R}^m$.

If $\bigcup_{S_i \in S_{\mathfrak{C}}} S_i$ is a neighborhood of \mathfrak{C} , it is also a neighborhood of the origin, then the cone generated by this set is $\bigcup_{S_i \in S_{\mathfrak{C}}} \mathcal{C}_{S_i} = \mathbb{R}^m$. This proves the forward direction. To prove the backward direction, suppose that there exists a point p of \mathfrak{C} in the boundary of $U_{\mathfrak{C}}$. First, this point p cannot be the origin, since we can use Lemma 3.24 to construct a neighborhood of the origin in $U_{\mathfrak{C}}$. Then, if $p \in \mathfrak{C}$ is in the boundary of $U_{\mathfrak{C}}$, we also have that the segment $[0, p]$ is in the boundary of $U_{\mathfrak{C}}$. Indeed, otherwise the segment $]0, p]$ is cut by a facet (codimension one face) of a simplex S . But S is also an ancestor of \mathfrak{C} . Thus the cell \mathfrak{C} intersects another cell of S , and thus S is flattened, which is ruled out by Lemma 3.17.

We now show that the cone $\mathcal{C}_{U_{\mathfrak{C}}} := \bigcup_{S_i \in S_{\mathfrak{C}}} \mathcal{C}_{S_i}$ has no facet (codimension one face). Since $\mathcal{C}_{U_{\mathfrak{C}}} \neq \{0\}$ (it is the union of m -dimensional cones) we deduce that $\mathcal{C}_{U_{\mathfrak{C}}} = \bigcup_{S_i \in S_{\mathfrak{C}}} \mathcal{C}_{S_i} = \mathbb{R}^m$. A point in the relative interior of a facet of $\mathcal{C}_{U_{\mathfrak{C}}}$ is a point in the relative interior of a facet of \mathcal{C}_{S_i} , for some $S_i \in S_{\mathfrak{C}}$. Suppose that such a point, say p , exists. Since two cones intersect trivially (Lemma 3.27 and Lemma 3.28), when a path leaves \mathcal{C}_{S_i} through p , it lands in the conjugate cone relative to this facet. Thus, the point p does not belong to the boundary of $\mathcal{C}_{U_{\mathfrak{C}}}$, and thus it does not exist. This ends the proof.

3.4 Constancy of the crossing number in $\mathcal{A}_{n,k,m}^{>0} \times \text{Mat}_{n,k+m}^{>0}$

Since $\text{Gr}_{k,n}^{>0}$ and $\text{Mat}_{n,k+m}^{>0}$ are path-connected (see [Pos06] for the positive Grassmannian), there exists a continuous path

$$(C, \mathcal{Z}) : [0, 1] \rightarrow \text{Gr}_{k,n}^{>0} \times \text{Mat}_{n,k+m}^{>0}$$

between any two couples of points in $\text{Gr}_{k,n}^{>0} \times \text{Mat}_{n,k+m}^{>0}$. We also denote $Y(t) = C(t) \mathcal{Z}(t)$. We prove that the crossing number $c_{n,k,m}(Y(t), \mathcal{Z}(t))$ is constant along these paths, hence proving the first part of Theorem 2. For every $t \in [0, 1]$, we denote by $\mathcal{L}(t)$ the list of cells containing the origin $\pi_{Y(t)}(Y(t))$ of $V_{Y(t)}$. In the following, we prove that for any $t_0 \in [0, 1]$, there exists $\epsilon > 0$ such that $c_{n,k,m}(Y(t), \mathcal{Z}(t)) = \text{Card}(\mathcal{L}(t))$ is constant for $t \in]t_0 - \epsilon, t_0 + \epsilon[\cap [0, 1]$. Since $[0, 1]$ is compact, we can extract a finite number of such balls to cover it, hence we deduce that the crossing number is constant on $[0, 1]$.

Step 1. Fix $t_0 \in [0, 1]$ and let $\mathcal{L}(t_0) = \{\mathfrak{C}_1, \dots, \mathfrak{C}_{n_0}\}$ be the list of cells containing the origin at t_0 . When (Y, \mathcal{Z}) moves continuously in $\mathcal{A}_{n,k,m}^{>0} \times \text{Mat}_{n,k+m}^{>0}$, then the vertices $Z_i(t) = \pi_{Y(t)}(\mathcal{Z}_i)$ and thus the cells move continuously. By applying Proposition 3.22 and Lemma 3.17 to \mathfrak{C}_i at t_0 we deduce by continuity that there exists ϵ_i such that for every $t \in]t_0 - \epsilon_i, t_0 + \epsilon_i[$:

- the set $U_{\mathfrak{C}_i}$ is still a neighborhood of \mathfrak{C}_i ,
- the cells of $L_{\mathfrak{C}_i}$ do not intersect,
- the origin $\pi_{Y(t)}(Y(t))$ belongs to a unique cell of $L_{\mathfrak{C}_i}$,
- each ancestor simplex of \mathfrak{C}_i is full-dimensional.

We denote by $\mathfrak{c}_i(t)$ the cell of $L_{\mathfrak{C}_i}$ containing the origin at time t . It is such that $\mathfrak{c}_i(t_0) = \mathfrak{C}_i$. We define such ϵ_i for every $1 \leq i \leq n_0$ and let $\epsilon := \min\{\epsilon_1, \dots, \epsilon_{n_0}\}$. Hence we defined n_0 paths

$$\begin{aligned}]t_0 - \epsilon, t_0 + \epsilon[&\rightarrow \mathfrak{Cells} \\ t &\rightarrow \mathfrak{c}_i(t) \end{aligned}$$

which are pairwise disjoint at t_0 . Each path has a finite number of discontinuous points corresponding to the origin jumping from one cell to another. We redefine $\epsilon > 0$ small enough such that each path is continuous on $]t_0 - \epsilon, t_0 + \epsilon[$ except possibly at t_0 .

Step 2. We prove that, up to choosing $\epsilon > 0$ small enough, any two paths $t \rightarrow \mathfrak{c}_i(t)$ constructed in Step 1 do not intersect on $]t_0 - \epsilon, t_0 + \epsilon[$. We first analyze the situation when $t \in [t_0, t_0 + \epsilon[$.

Suppose that the path $t \rightarrow \mathfrak{c}_i$, for $t \in [t_0, t_0 + \epsilon[$, is discontinuous at t_0 . Denote by \mathfrak{C}_i^+ the value of $t \rightarrow \mathfrak{c}_i(t)$ on $]t_0, t_0 + \epsilon[$.

Claim 3.32. The cell \mathfrak{C}_i^+ is an ancestor cell of \mathfrak{C}_i .

Indeed, we first recall (Step 1) that each ancestor simplex of \mathfrak{C}_i is full-dimensional when $t \in]t_0 - \epsilon, t_0 + \epsilon[$. According to the type of discontinuity of $t \rightarrow \mathfrak{c}_i(t)$ at t_0 , we deduce that the origin $\pi_{Y(t_0)}(Y(t_0))$ must belong to the closure of \mathfrak{C}_i^+ at t_0 . Moreover, the closure of a full-dimensional cell equals the disjoint union of this cell and its descendent cells. Thus, if $\pi_{Y(t_0)}(Y(t_0))$ belongs to the closure of \mathfrak{C}_i^+ at t_0 , then \mathfrak{C}_i^+ is in the lineage of \mathfrak{C}_i , and even more \mathfrak{C}_i^+ must be an ancestor of \mathfrak{C}_i . This ends the proof of the claim.

Fix $i \neq j$ and suppose that the paths $t \rightarrow \mathfrak{c}_i(t)$ and $t \rightarrow \mathfrak{c}_j(t)$ intersect on $]t_0, t_0 + \epsilon[$:

- if $t \rightarrow \mathfrak{c}_i(t)$ and $t \rightarrow \mathfrak{c}_j(t)$ are discontinuous at t_0 , then $\mathfrak{C}_i^+ = \mathfrak{C}_j^+ =: \mathfrak{C}$, and then \mathfrak{C} is an ancestor cell of \mathfrak{C}_i and of \mathfrak{C}_j , so \mathfrak{C}_i and \mathfrak{C}_j belong to the same simplex. But then, at t_0 the origin belongs to two cells of the same simplex, hence this simplex is flattened, which is impossible by Lemma 3.17.
- if $t \rightarrow \mathfrak{c}_i(t)$ is discontinuous and $t \rightarrow \mathfrak{c}_j(t)$ is continuous at t_0 , then $\mathfrak{C}_i^+ = \mathfrak{C}_j$ and then \mathfrak{C}_i and \mathfrak{C}_j are in the same lineage. But then at t_0 the origin is contained in two cells in the same lineage, which is impossible since by definition the intersection of two cells in the same lineage is empty. The situation is similar if we exchange i and j .

In the last case, when $t \rightarrow \mathfrak{c}_i(t)$ and $t \rightarrow \mathfrak{c}_j(t)$ are continuous at t_0 , it suffices to shorten the interval $]t_0, t_0 + \epsilon[$ by choosing $\epsilon > 0$ small enough to avoid the intersection. By doing a similar reasoning for $t \in]t_0 - \epsilon, t_0]$, we prove that there exists $\epsilon > 0$ such that the paths $t \rightarrow \mathfrak{c}_i(t)$, for $1 \leq i \leq n_0$ do not intersect on $]t_0 - \epsilon, t_0 + \epsilon[$.

Step 3. We now prove that for $t \in]t_0 - \epsilon, t_0 + \epsilon[$, we have

$$\mathcal{L}(t) = \{\mathfrak{c}_1(t), \dots, \mathfrak{c}_{n_0}(t)\},$$

up to choosing $\epsilon > 0$ small enough. Since we showed that the paths $t \rightarrow \mathfrak{c}_i(t)$ do not intersect, this concludes the proof. By construction, we have $\mathcal{L}(t_0) = \{\mathfrak{c}_1(t_0), \dots, \mathfrak{c}_{n_0}(t_0)\}$ and the inclusion $\{\mathfrak{c}_1(t), \dots, \mathfrak{c}_{n_0}(t)\} \subseteq \mathcal{L}(t)$ for $t \in]t_0 - \epsilon, t_0 + \epsilon[$. Suppose that the inclusion is strict, then up to choosing $\epsilon > 0$ small enough, this amounts to assuming that there exist $\mu > 0$ and a cell $\tilde{\mathfrak{C}}$ not in the set $\{\mathfrak{c}_1(t), \dots, \mathfrak{c}_{n_0}(t)\}$, for $t \in]t_0, t_0 + \epsilon[$ such that the origin is contained in $\tilde{\mathfrak{C}}$ when $t \in]t_0, t_0 + \mu[$. There are two possibilities.

- If $\tilde{\mathfrak{C}}$ is a cell of a simplex \tilde{S} which is not an ancestor of a cell of $\mathcal{L}(t)$ for $t \in]t_0, t_0 + \epsilon[$. Then, the cell $\tilde{\mathfrak{C}}$ belongs to the boundary of \tilde{S} , which is a closed set of $V_{Y(t)}$. But this is impossible since the origin is contained in $\tilde{\mathfrak{C}}$ when $t \in]t_0, t_0 + \mu[$, and the interval $]t_0, t_0 + \mu[$ is open on the left side.
- If $\tilde{\mathfrak{C}}$ is a cell of an ancestor simplex of a cell, say \mathfrak{C}_i , of $\mathcal{L}(t)$ for $t \in]t_0, t_0 + \epsilon[$. Then, for t close enough to t_0 the origin belongs to two cells of the same simplex: $\tilde{\mathfrak{C}}$ and \mathfrak{C}_i . But then this simplex is flattened and this is ruled out by Lemma 3.17.

By doing a similar reasoning for $t < t_0$, we conclude that there exists ϵ such that $\mathcal{L}(t) = \{\mathfrak{c}_1(t), \dots, \mathfrak{c}_{n_0}(t)\}$ for $t \in]t_0 - \epsilon, t_0 + \epsilon[$. This ends the proof.

3.5 Independence of the crossing number in n

Let $m = 2r - 1$ for some positive integer r . We use the same approach as for the winding number. We showed that the crossing number is independent of the point in $\mathcal{A}_{n,k,m}^{>0} \times \text{Mat}_{n,k+m}^{>0}$. We now show that for two specific choices of couples, one in $\mathcal{A}_{n+1,k,m}^{>0} \times \text{Mat}_{n+1,k+m}^{>0}$ and the other one in $\mathcal{A}_{n,k,m}^{>0} \times \text{Mat}_{n,k+m}^{>0}$ the crossing number is the same. Thus the crossing number is independent of n .

Proposition 3.33. *There exist $(Y, \mathcal{Z}') \in \mathcal{A}_{n+1,k,m}^{>0} \times \text{Mat}_{n+1,k+m}^{>0}$ and $(Y, \mathcal{Z}) \in \mathcal{A}_{n,k,m}^{>0} \times \text{Mat}_{n,k+m}^{>0}$ such that*

$$c_{n+1,k,m}(Y, \mathcal{Z}') = c_{n,k,m}(Y, \mathcal{Z}).$$

Proof. Let $\mathcal{Z}' = (\mathcal{Z}_1, \dots, \mathcal{Z}_{n+1}) \in \text{Mat}_{n+1,k+m}^{>0}$ and then define $\mathcal{Z} = (\mathcal{Z}_1, \dots, \mathcal{Z}_n) \in \text{Mat}_{n,k+m}^{>0}$. We first show that $\mathcal{A}_{n,k,m}^{>0} \cap \mathcal{A}_{n+1,k,m}^{>0} \neq \emptyset$. Indeed, let $C \in \text{Gr}_{k,n}^{>0}$ so that $Y = CZ \in \mathcal{A}_{n,k,m}^{>0}$. We define the matrix C' by adding a $(n+1)$ th column of zeros to C , hence we have $Y = CZ = C'\mathcal{Z}'$, and thus $Y \in \mathcal{A}_{n,k,m}^{>0} \cap \mathcal{A}_{n+1,k,m}^{>0}$. Now, $\mathcal{A}_{n,k,m}^{>0}$ and $\mathcal{A}_{n+1,k,m}^{>0}$ are open sets of $\text{Gr}_{k,k+m}$ (see the proof of Lemma 9.4 in [GL20] for an argument), moreover since the closure of $\text{Gr}_{k,n+1}^{>0}$ equals $\text{Gr}_{k,n+1}^{\geq 0}$ it follows that the closure of $\mathcal{A}_{n+1,k,m}^{>0}$ is the amplituhedron $\mathcal{A}_{n+1,k,m}$. Hence we conclude that $\mathcal{A}_{n,k,m}^{>0} \cap \mathcal{A}_{n+1,k,m}^{>0} \neq \emptyset$.

Then, choose Y_2 in $\mathcal{A}_{n,k,m}^{>0} \cap \mathcal{A}_{n+1,k,m}^{>0}$ such that

$$\langle Y_2, i_1, i_1 + 1, \dots, \widehat{i_j + \epsilon}, \dots, i_r, i_r + 1 \rangle \neq 0,$$

where $(i_1, i_1 + 1, \dots, i_r, i_r + 1) \in \binom{[n+1]}{m+1}$, $1 \leq j \leq r$ and $\epsilon \in (0, 1)$. This is always possible since the dimension of $\mathcal{A}_{n,k,m}^{>0} \cap \mathcal{A}_{n+1,k,m}^{>0}$ is km since it is an intersection of open sets, and the vanishing locus of the twistor coordinates $\langle Y_2, i_1, i_1 + 1, \dots, \widehat{i_j + \epsilon}, \dots, i_r, i_r + 1 \rangle = 0$ is of codimension 1. Thus, if the origin of V_{Y_2} is contained in a simplex, it is contained in its m -dimensional cell.

In order to prove that the crossing number $c_{n,k,m}(Y_2, \mathcal{Z})$ and $c_{n+1,k,m}(Y_2, \mathcal{Z}')$ is the same it suffices to prove that the origin cannot belong to a cell

$$C(i_1, i_1 + 1, \dots, i_{r-1} + 1, n, n + 1),$$

where $(i_1, i_1 + 1, \dots, i_{r-1} + 1) \in \binom{[n-1]}{m-1}$. However, if such a cell contains the origin then

$$\text{sign} \langle Y_2, i_1, i_1 + 1, \dots, i_{r-1} + 1, \widehat{n}, n + 1 \rangle = -\text{sign} \langle Y_2, i_1, i_1 + 1, \dots, i_{r-1} + 1, n, \widehat{n+1} \rangle$$

and it follows from the strict coarse boundary conditions for $Y_2 \in \mathcal{A}_{n+1,k,m}^{>0}$ and for $Y_2 \in \mathcal{A}_{n,k,m}^{>0}$ that the twistor coordinates on LHS and on RHS are positive, and thus the equality cannot hold. \square

3.6 The crossing number for $n = k + m$

Let $m = 2r - 1$ for some positive integer r . We show that there exists $\mathcal{Z} \in \text{Mat}_{n,n}^{>0}$ and $C \in \text{Gr}_{k,k+m}^{>0}$ such that the crossing number

$$c_{n=k+m,k,m}(C\mathcal{Z}, \mathcal{Z}) = \begin{cases} \frac{2k+m-1}{m+1} \binom{\frac{k+m-2}{2}}{\frac{m-1}{2}} & \text{for } k \text{ odd,} \\ 2 \binom{\frac{k+m-1}{2}}{\frac{m+1}{2}} & \text{for } k \text{ even.} \end{cases} \quad (18)$$

Since we proved that the crossing number is independent of \mathcal{Z}, Y and n , this ends the proof of Theorem 2.

Step 1. Each simplex containing the origin of V_Y is full-dimensional by Lemma 3.17, so we can choose $C \in \text{Gr}_{k,n}^{>0}$ such that the origin of V_Y only intersects these simplices in their interior. Equivalently, we pick $C \in \text{Gr}_{k,n}^{>0}$ such that each twistor coordinate

$\langle C\mathcal{Z}, i_1, i_1 + 1, \dots, \widehat{i_j + \epsilon}, \dots, i_r, i_r + 1 \rangle$, for $\epsilon \in \{0, 1\}$ and $(i_1, i_1 + 1, \dots, i_r, i_r + 1) \in \binom{[n]}{m+1}$, which is always possible since the locus of points with one vanishing twistor coordinate is of codimension 1 in $\text{Gr}_{k,n}^{>0}$.

Step 2. Let $(i_1, i_1 + 1, \dots, i_r, i_r, i_r + 1) \in \binom{[n]}{m+1}$, then the origin belongs to the simplex $S(i_1, i_1 + 1, \dots, i_r, i_r + 1)$ if and only if it satisfies the following two conditions:

$$\text{sign} \langle i_1, i_1 + 1, \dots, \widehat{i_j + 1}, \dots, i_r, i_r + 1 \rangle = -\text{sign} \langle i_1, i_1 + 1, \dots, \widehat{i_j}, \dots, i_r, i_r + 1 \rangle, \quad (\text{Condition } (i))$$

for $1 \leq j \leq r$, and

$$\text{sign} \langle i_1, i_1 + 1, \dots, \widehat{i_j}, \dots, i_r, i_r + 1 \rangle \text{ is independent of } j. \quad (\text{Condition } (ii))$$

Since $n = k + m$ the twistor coordinates are given by determinants of square matrices, so we have

$$\langle i_1, i_1 + 1, \dots, \widehat{i_j + \epsilon}, \dots, i_r, i_r + 1 \rangle = \det \begin{pmatrix} C \\ I_{i_1, i_1+1, \dots, \widehat{i_j + \epsilon}, \dots, i_r, i_r+1} \end{pmatrix} \det(\mathcal{Z}),$$

where $I_{i_1, i_1+1, \dots, \widehat{i_j + \epsilon}, \dots, i_r, i_r+1}$ is the $m \times (k + m)$ matrix whose l th row has a 1 at the l th index of the list $i_1, i_1 + 1, \dots, \widehat{i_j + \epsilon}, \dots, i_r, i_r + 1$ and zeros elsewhere. Using the standard expansion of the determinant we get

$$\det \begin{pmatrix} C \\ I_{i_1, i_1+1, \dots, \widehat{i_j + \epsilon}, \dots, i_r, i_r+1} \end{pmatrix} = (-1)^{k+i_j-\epsilon} \det \left(C_{[n] \setminus \{i_1, i_1+1, \dots, \widehat{i_j + \epsilon}, \dots, i_r, i_r+1\}} \right).$$

Hence, since $\mathcal{Z} \in \text{Mat}_{n,n}^{>0}$ and $C \in \text{Gr}_{k,k+m}^{>0}$ we obtain

$$\text{sign} \langle i_1, i_1 + 1, \dots, \widehat{i_j + \epsilon}, \dots, i_r, i_r + 1 \rangle = (-1)^{k+i_j+\epsilon}.$$

Thus, condition (i) is always satisfied and condition (ii) is satisfied precisely if

$$i_1 = i_2 = \cdots = i_r \pmod{2}. \quad (19)$$

Thus the crossing number is equal to the number of sequences (i_1, \dots, i_r) between 1 and $n - 1$ such that $i_{j+1} > i_j + 1$ satisfying Eq. (19). This is exactly the expression of the crossing number given in Eq. (18).

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