On the maximal abelian subalgebras of the general linear Lie color algebras

Shujuan Wang[1] and Wende Liu[2,*]
[1]Department of Mathematics, Shanghai Maritime University, Shanghai 201306, China
[2]School of Mathematics and Statistics, Hainan Normal University, Haikou 571158, China

Abstract: Let Γ be a finite group and V a finite-dimensional Γ-graded space over an algebraically closed field of characteristic not equal to 2. In the sense of conjugation, we classify all the so-called pre-nil or nil maximal abelian subalgebras for the general linear Lie color algebra $\mathfrak{gl}(V,\Gamma)$. In the situation of Γ being a cyclic group, we determine the minimal dimensions of pre-nil or nil faithful representations for any finite-dimensional abelian Lie color algebra.

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1. Introduction

Lie color algebras, which were introduced by Ree in 1960 (see [14]), play an important role in mathematics and theoretical physics, especially in the conformal field theory and supersymmetries (see [1, 26]). For example, Scheunert and Zhang studied cohomology of Lie color algebras (see [18]); Su, Zhao and Zhu constructed simple Lie color algebras of generalized Witt type and Weyl type (see [20] and [21]).

Let L be a finite-dimensional Lie (color) algebra and write

$$\mu(L) = \min\{\dim V \mid (\rho, V) \text{ is a faithful representation of } L\}.$$

Ado's theorem of Lie (color) version guarantees the existence of $\mu(L)$ (see [17, p.719]). In 1905, I. Schur determined the maximal dimension of abelian subalgebras for the general linear Lie algebra over $\mathbb{C}$ in [19] and then $\mu(L)$ can be determined for any finite-dimensional abelian Lie algebra L (see also [9, 13]). A super-version of Schur's work was given for Lie superalgebras over $\mathbb{C}$ (see [23, 24]). In this paper, we shall offer a color-version of Schur's work for Lie color algebras over an algebraically closed field of characteristic not equal to 2. We also determine the minimal pre-nil or nil faithful representations for any finite-dimensional abelian Lie color algebra with respect to a finite cyclic group.

As is well-known, the function μ plays an important role on affine crystallographic groups and finitely generated torsion-free nilpotent groups. Benoist, Burde and

*Correspondence: wendeliu@ustc.edu.cn (W. Liu)

Grunewald (see [2, 5]) gave an example of a nilpotent Lie algebra L such that $\mu(L) > \dim L + 1$, which is a counterexample of Milnor's conjecture (see [12]). However, it is not easy to compute $\mu(L)$, even the bounds of $\mu(L)$ for a given finite-dimensional Lie (color) algebra. A lot of work on the function μ revolves around nilpotent or (semi)simple Lie (super)algebras as well as their various extensions (see [2–4, 6–11, 13, 15, 16, 19, 22–25] for example).

Throughout $\mathbb{F}$ is an algebraically closed field of characteristic not equal to 2 and all vector spaces and algebras are over $\mathbb{F}$ and of finite dimensions.

1.1. Generalized matrix units. Denote by $\mathrm{M}(m)$ the set consisting of all $m \times m$ matrices over $\mathbb{F}$. We define a total order on the set $\{(i,j) \mid 1 \le i, j \le m\}$ by

$$(i,j) < (k,l) \Longleftrightarrow i < k, \quad \text{or} \quad i = k \quad \text{but} \quad j < l.$$

Denote by $\mathrm{ht}(X) = \min\{(i,j) \mid x_{ij} \neq 0\}$ for $X = (x_{ij}) \in \mathrm{M}(m)$ and $\mathrm{ht}(0) = (\infty, \infty)$. Hereafter a_{mn} is (m,n)-entry of the matrix $A = (a_{ij})$. An element X is called a generalized matrix unit of (i,j)-form, usually written as $u_{i,j}$, if $\mathrm{ht}(X) = (i,j)$ and its ith row is $(0, ..., 0, 1, 0..., 0)$, where 1 is at the j-th position. In general, for given (i,j), there are many generalized matrix units $u_{i,j}$. For a subspace S of $\mathrm{M}(m)$, write

$$\mathrm{ht}(S) := \min\{\mathrm{ht}(X) \mid X \in S\}.$$

Denote by $\mathrm{I}_m = \Sigma_{i=1}^m e_{ii}$ the $m \times m$ identity matrix. Hereafter e_{ij} is the matrix unit having 1 at (i,j)-position and 0 elsewhere. Fix a nonzero element $a \in \mathbb{F}$ and let $\mathrm{T}_{ij}(a)$ or $\mathrm{D}_i(a)$ be $\mathrm{I}_m + ae_{ij}$ or $\mathrm{I}_m + (a-1)e_{ii}$, respectively. As in [24], we introduce the following similar operations of $\mathrm{M}(m)$:

(1) *t-type operators*

$$s_{\mathrm{T}_{ij}(a)} := \mathrm{l}_{\mathrm{T}_{ij}(a)^{-1}} \mathrm{r}_{\mathrm{T}_{ij}(a)} \text{ for } 1 \le i < j \le m \text{ and } a \in \mathbb{F}.$$

(2) *d-type operators*

$$s_{\mathrm{D}_i(a)} := \mathrm{l}_{\mathrm{D}_i(a)^{-1}} \mathrm{r}_{\mathrm{D}_i(a)} \text{ for } 1 \le i \le m \text{ and } a \in \mathbb{F} \backslash \{0\},$$

where, l_A or r_A is the operator of left or right associative multiplication by the matrix A for $A \in \mathrm{M}(m)$, respectively. If $A = (a_{ij})$ and $\mathrm{ht}(A) = (i,j)$, we write

$$\mathrm{h}_A = \sum_{l>j} s_{\mathrm{T}_{jl}(-a_{il})} s_{\mathrm{D}_i(a_{ij})}.$$

1.2. Lie color algebras. Let Γ be an abelian group and $V = \oplus_{\alpha \in \Gamma} V_\alpha$ a Γ-graded space. The elements in $\cup_{\alpha \in \Gamma} V_\alpha$ are said to be homogenous. For a homogeneous element $v \in V_\alpha, \alpha \in \Gamma$, we set $|v| = \alpha$, the degree of v. In addition, the symbol $|x|$ implies that x is homogeneous. By definition, a Γ-graded algebra $\mathfrak{g}$ is a Γ-graded space $\mathfrak{g} = \oplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha$ with a bilinear multiplication consistent with the Γ-gradation. Let $\mathfrak{g}$ be a Γ-graded algebra. If the multiplication of $\mathfrak{g}$ is trivial, we say $\mathfrak{g}$ to be abelian; if the multiplication of $\mathfrak{g}$ satisfies the associativity law, we say $\mathfrak{g}$ to be associative.

Let Γ be an abelian group. A bi-character on Γ is a map $\varepsilon : \Gamma \times \Gamma \longrightarrow \mathbb{F}\backslash\{0\}$ such that

$$\varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha)=1, \quad \varepsilon(\alpha\beta,\gamma)=\varepsilon(\alpha,\gamma)\varepsilon(\beta,\gamma),$$

where $\alpha,\beta,\gamma \in \Gamma$. Let Γ be an abelian group of a bi-character ε and $\mathfrak{g}$ a Γ-graded algebra, whose multiplication is denoted by $[\ ,\ ]$. If

$$[x,y]+\varepsilon(|x|,|y|)[y,x]=0,$$

$$\varepsilon(|z|,|x|)[x,[y,z]]+\varepsilon(|x|,|y|)[y,[z,x]]+\varepsilon(|y|,|z|)[z,[x,y]]=0,$$

where $x,y,z\in\mathfrak{g}$, then $\mathfrak{g}$ is called a Lie color algebra. Lie (super)algebras are Lie color algebras (see [21, p.525]). If $\mathfrak{A}=\oplus_{\alpha\in\Gamma}\mathfrak{A}_\alpha$ is a Γ-graded associative algebra, by introducing a new multiplication

$$[x,y]=xy-\varepsilon(|x|,|y|)yx, \quad x,y\in\mathfrak{A},$$

$\mathfrak{A}$ becomes a Lie color algebra, which is denoted by $\mathfrak{A}^-$.

1.3. The general linear Lie color algebras. Let Γ be an abelian group of a bi-character ε and $V=\oplus_{\alpha\in\Gamma}V_\alpha$ a Γ-graded space. Then $\mathrm{End}_{\mathbb{F}}(V)$ is a Γ-graded associative algebra and $\mathrm{End}_{\mathbb{F}}(V)^-$ is a Lie color algebra in the usual fashion, which is called the general linear Lie color algebra and denoted by $\mathfrak{gl}(V,\Gamma)$ (see [28, p.447]). Let $\mathfrak{a}$ be a subalgebra of $\mathfrak{gl}(V,\Gamma)$. If each element in $\mathfrak{a}$ is nilpotent, we say $\mathfrak{a}$ to be nil; if each element in $\cup_{\alpha\in\Gamma^*}\mathfrak{a}_\alpha$ is nilpotent, we say $\mathfrak{a}$ to be pre-nil. Hereafter Γ^* denotes the set $\Gamma\backslash\{0\}$, where 0 is the identity of Γ. If $\mathfrak{a}$ is a (pre-nil or nil) abelian subalgebra of the maximal dimension for $\mathfrak{gl}(V,\Gamma)$, we say $\mathfrak{a}$ to be (pre-nil or nil) maximal abelian. Let $L=\oplus_{\alpha\in\Gamma}L_\alpha$ be a Lie color algebra and $\rho: L\longrightarrow\mathfrak{gl}(V,\Gamma)$ a linear map of degree 0. If $\rho([x,y])=[\rho(x),\rho(y)]$ for $x,y\in L$, we call (ρ,V) to be a representation of L. Let (ρ,V) be a representation of L. If $\ker\rho=0$, we say (ρ,V) to be faithful; if each element in $\rho(V)$ is nilpotent, we say (ρ,V) to be nil; if each element in $\cup_{\alpha\in\Gamma^*}\rho(V)_\alpha$ is nilpotent, we say (ρ,V) to be pre-nil. Write

$$\mu_{\mathrm{nil}}(L)=\min\{\dim V \mid (\rho,V) \text{ is a nil faithful representation of } L\},$$
$$\mu_{\text{pre-nil}}(L)=\min\{\dim V \mid (\rho,V) \text{ is a pre-nil faithful representation of } L\}.$$

1.4. The matrix-version of $\mathfrak{gl}(V,\Gamma)$. Hereafter, we make a convention that the symbol Γ always denotes a finite abelian group of a bi-character ε, whose all elements are $\alpha_1,\dots,\alpha_k$. For the fixed order of $\alpha_1,\dots,\alpha_k$, if $V=\oplus_{i=1}^k V_{\alpha_i}$ is a Γ-graded space and $\dim V_{\alpha_i}=m_i$, we say that $(m_1,\dots,m_k)$ is Γ-dimension of V. Let V be of Γ-dimension $(m_1,\dots,m_k)$ and $m=\sum_{i=1}^k m_i$. A rearrangement of

$$(\overbrace{\alpha_1,\dots,\alpha_1}^{m_1},\overbrace{\alpha_2,\dots,\alpha_2}^{m_2},\dots,\overbrace{\alpha_k,\dots,\alpha_k}^{m_k})$$

is called an $(m_1,\dots,m_k)$-tuple. Denote by $B=(v_1,\dots,v_m)$ an ordered homogeneous basis of V and equip it with an $(m_1,\dots,m_k)$-tuple $\Phi=(|v_1|,\dots,|v_m|)$ induced by B. Then the general linear Lie color algebra $\mathfrak{gl}(V,\Gamma)$ is isomorphic to its matrix-version $\mathfrak{gl}^\Phi(m_1,\dots,m_k)$ induced by tuple Φ, which has the underlying matrices space $\mathrm{M}(m)$ and a Γ-grading structure

$$\mathfrak{gl}^\Phi(m_1,\dots,m_k)=\oplus_{l=1}^k\mathfrak{gl}^\Phi_{\alpha_l}(m_1,\dots,m_k),$$

where

$$\mathfrak{gl}^{\Phi}_{\alpha_l}(m_1,\ldots,m_k)=\mathrm{Span}\{e_{ij}\mid \Phi_i-\Phi_j=\alpha_l\}.$$

Hereafter, Φ_n denotes the n-th entry of the tuple Φ. In particular, if

$$\Phi=(\overbrace{\alpha_1,\ldots,\alpha_1}^{m_1},\overbrace{\alpha_2,\ldots,\alpha_2}^{m_2},\ldots,\overbrace{\alpha_k,\ldots,\alpha_k}^{m_k}),$$

$\mathfrak{gl}^{\Phi}(m_1,\ldots,m_k)$ is denoted by $\mathfrak{gl}(m_1,\ldots,m_k)$ for short. For any $(m_1,\ldots,m_k)$-tuple Φ, write $\mathfrak{t}^{\Phi}(m_1,\ldots,m_k)$ or $\mathfrak{s}^{\Phi}(m_1,\ldots,m_k)$ for the subalgebras of $\mathfrak{gl}^{\Phi}(m_1,\ldots,m_k)$ consisting of upper triangular or strictly upper triangular matrices, respectively.

2. Maximal pre-nil or nil abelian subalgebras of $\mathfrak{gl}(V,\Gamma)$

In this section, we shall give a lower bound of the maximal dimensions for abelian subalgebras of the general linear Lie color algebra $\mathfrak{gl}(V,\Gamma)$. In the case $\Gamma=\mathbb{Z}_k$, the cyclic group of order k, we shall classify the pre-nil or nil maximal abelian subalgebras of $\mathfrak{gl}(V,\Gamma)$ in the sense of conjugation. In addition, we shall give a method to determine $\mu_{\text{pre-nil}}(L)$ and $\mu_{\text{nil}}(L)$ for any abelian Lie color algebra L.

2.1. Main lemmas. The following lemma realizes the triangulation of matrices for any pre-nil abelian subalgebra of $\mathfrak{gl}(m_1,\ldots,m_k)$, which is a color-version of [27, Lemma 3.3].

Lemma 2.1. Let $\mathfrak{a}$ be a pre-nil abelian subalgebra of $\mathfrak{gl}(m_1,\ldots,m_k)$. Then there exists an $(m_1,\ldots,m_k)$-tuple Φ such that $\mathfrak{a}$ is contained in $\mathfrak{t}^{\Phi}(m_1,\ldots,m_k)$. Furthermore, if $\mathfrak{a}$ is nil abelian, then there exists an $(m_1,\ldots,m_k)$-tuple Φ such that $\mathfrak{a}$ is contained in $\mathfrak{s}^{\Phi}(m_1,\ldots,m_k)$.

Proof. From Jacobson's Theorem on weakly closed sets, it is sufficient to prove that $\mathfrak{a}$ has common homogeneous eigenvectors in V. Since $\mathfrak{a}$ is pre-nil , for $A\in\mathfrak{gl}_{\gamma_l}(m_1,\ldots,m_k)$ with $\gamma_l\in\Gamma^*$, zero is the only eigenvalue of A, and hence

$$V_1:=\left\{x\in V\mid \mathfrak{gl}_{\gamma_l}(m_1,\ldots,m_k)x=0,\gamma_l\in\Gamma^*\right\}$$

is nonzero. It is clear that V_1 is $\mathfrak{gl}_0(m_1,\ldots,m_k)$-module. Then

$$V_2:=\{x\in V_1\mid \mathfrak{gl}_0(m_1,\ldots,m_k)x\in\mathbb{F}x\}$$

is also nonzero. Any nonzero homogeneous element in V_2 is a common eigenvector for $\mathfrak{a}$. □

The following lemma gives an HGMU decomposition (see below for a definition) of any abelian subalgebra for $\mathfrak{s}^{\Phi}(m_1,\ldots,m_k)$, which is a color-version of [24, Lemma 2.3].

Lemma 2.2. Let Φ be an $(m_1,\ldots,m_k)$-tuple and $m=\sum_{i=1}^{k}m_i$.

(1) If A is a homogeneous element in $\mathfrak{s}^{\Phi}(m_1,\ldots,m_k)$ with $\mathrm{ht}(A)=(i,j)$, then A is conjugate to a homogeneous generalized matrix unit $u_{i,j}$ in $\mathfrak{s}^{\Phi}(m_1,\ldots,m_k)$ and $|u_{i,j}|=|A|$.

(2) If $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{s}^{\Phi}(m_1,\dots,m_k)$ with $\mathrm{ht}(\mathfrak{a}) = (i,j)$, then there exist some homogeneous generalized matrix units $u_{i,k_1}, u_{i,k_2}, \dots, u_{i,k_r}$ in $\mathfrak{s}^{\Phi}(m_1,\dots,m_k)$, a homogeneous invertible matrix $T \in \mathfrak{gl}^{\Phi}(m_1,\dots,m_k)$ and an abelian subalgebra $\mathfrak{a}'$ of $\mathfrak{s}^{\Phi}(m_1,\dots,m_k)$ with $\mathrm{ht}(\mathfrak{a}') > (i,m)$, such that $\mathfrak{a}$ is conjugate to

$$T^{-1}\mathfrak{a}T = \mathbb{F}u_{i,k_1} \oplus \mathbb{F}u_{i,k_2} \oplus \cdots \oplus \mathbb{F}u_{i,k_r} \oplus \mathfrak{a}' \tag{2.1}$$

where $i < j = k_1 < k_2 < \cdots < k_r \le m$ and for each matrix X in $\mathfrak{a}'$:

- the k_jth row of X is zero for each $1 \le j \le r$.
- if the sth row (resp. column) of TXT^{-1} is zero, so is the sth row (resp. column) of X.

Proof. (1) On one hand, $\mathrm{h}_A(A)$ a generalized matrix unit $u_{i,j}$ since $\mathrm{ht}(A) = (i,j)$. On the other hand, since A is homogeneous and $\mathrm{ht}(A) = (i,j)$, $|A| = |e_{ij}| = \Phi_i - \Phi_j$. Then $\Phi_l = \Phi_j$ if a_{il} is nonzero. It follows that h_A is in $\mathrm{End}_0(\mathfrak{gl}^{\Phi}(m_1,\dots,m_k))$, which implies $|u_{i,j}| = |A|$.

(2) Let the homogeneous matrix $A_{ij} \in \mathfrak{a}$ satisfy $\mathrm{ht}\,(A_{ij}) = \mathrm{ht}(\mathfrak{a}) = (i,j)$. Then $\mathrm{h}_{A_{ij}}(A_{ij})$ is a generalized matrix unit $u_{i,j}$ by (1), and $\mathrm{h}_{A_{ij}}(\mathfrak{a})$ is a new abelian subalgebra contained in $\mathfrak{s}^{\Phi}(m_1,\dots,m_k)$. Let $\mathfrak{a}_{(1)}$ be the subspace of $\mathrm{h}_{A_{ij}}(\mathfrak{a})$ such that $\mathrm{ht}(\mathfrak{a}_{(1)}) > (i,k_1)$ and any matrix in $\mathrm{h}_{A_{ij}}(\mathfrak{a})$ is of the form $au_{i,k_1} + P$, where $a \in \mathbb{F}, k_1 = j$ and $P \in \mathfrak{a}_{(1)}$. Similarly, if $\mathrm{ht}(\mathfrak{a}_{(1)}) = (i,k_2)$, let the homogeneous matrix $A_{ik_2} \in \mathfrak{a}_{(1)}$ satisfy $\mathrm{ht}\,(A_{ik_2}) = (i,k_2)$. Consequently, $\mathrm{h}_{A_{ik_2}}(A_{ik_2})$ is also a homogenous generalized matrix unit, denoted by u_{i,k_2}, and $\mathrm{h}_{A_{ik_2}}\mathrm{h}_{A_{ik_1}}(\mathfrak{a}) \supset \mathrm{h}_{A_{ik_2}}(\mathfrak{a}_{(1)})$ are also abelian subalgebras of $\mathfrak{s}^{\Phi}(m_1,\dots,m_k)$. In particular, since $k_2 > k_1 > i$, $\mathrm{h}_{A_{ik_2}}(u_{i,k_1})$ is also a generalized matrix unit of (i,k_1)-form, denoted still by u_{i,k_1}. Let $\mathfrak{a}_{(2)}$ be the subspace of $\mathrm{h}_{A_{ik_2}}(\mathfrak{a}_{(1)})$ such that $\mathrm{ht}(\mathfrak{a}_{(2)}) > (i,k_2)$ and any a matrix in $\mathfrak{a}$ is of the form: $a_1u_{i,k_1} + a_2u_{i,k_2} + P$, where $a_1, a_2 \in \mathbb{F}$ and $P \in \mathfrak{a}_{(2)}$. By induction, there exists a positive integer r such that

$$\mathrm{ht}(\mathfrak{a}_{(r)}) > (i,m), \quad 1 \le r \le m - i$$

and

$$\mathrm{h}_{A_{ik_r}}\mathrm{h}_{A_{ik_{r-1}}}\cdots\mathrm{h}_{A_{ik_{t+1}}}\left(\mathfrak{a}_{(t)}\right) \subset \mathrm{h}_{A_{ik_r}}\mathrm{h}_{A_{ik_{r-1}}}\cdots\mathrm{h}_{A_{ik_t}}\left(\mathfrak{a}_{(t-1)}\right),$$

where $1 \le t \le r-1$ and $\mathfrak{a}_0 = \mathfrak{a}$. In addition, we also get r homogeneous generalized matrix units: $u_{i,k_1}, u_{i,k_2}, \dots, u_{i,k_r}$, where $i < k_1 < k_2 < \cdots < k_r \le m$. Write

$$\mathfrak{a}' = \mathfrak{a}_{(r)}, \quad T^{-1}\mathfrak{a}T = \mathrm{h}_{A_{ik_r}}\mathrm{h}_{A_{ik_{r-1}}}\cdots\mathrm{h}_{A_{ik_1}}(\mathfrak{a}).$$

Then

$$T^{-1}\mathfrak{a}T = \mathbb{F}u_{i,k_1} \oplus \cdots \oplus \mathbb{F}u_{i,k_r} \oplus \mathfrak{a}'.$$

For every $1 \le l \le r$ and any homogeneous matrix $X \in \mathfrak{a}'$, the ith row of $u_{i,k_l}X$ is the k_lth row of X and the ith row of Xu_{i,k_l} is 0 by $\mathrm{ht}\mathfrak{a}' > (i,m)$. Then since $T^{-1}\mathfrak{a}T$ is abelian and every u_{i,k_j} is homogeneous, the k_lth row of any homogeneous matrix in $\mathfrak{a}'$ is 0. Furthermore, the operators $\mathrm{h}_{A_{ik_t}}$ leave 0 rows but the k_lth rows of any a matrix invariant, where $1 \le t \le r$. Then (2) is true. □

(2.1) is called the *homogeneous generalized matrix unit decomposition* of $\mathfrak{a}$ (HGMU decomposition in short).

2.2. Main results. Write

$$\begin{aligned} \mathfrak{E} &= \mathrm{Span}\{e_{ij} \in \mathfrak{gl}^{\Phi}(m_1,\dots,m_k) \mid 1 \le i \le \lceil m/2 \rceil, \lceil m/2 \rceil + 1 \le j \le m\}, \\ \mathfrak{F} &= \mathrm{Span}\{e_{ij} \in \mathfrak{gl}^{\Phi}(m_1,\dots,m_k) \mid 1 \le i \le \lfloor m/2 \rfloor, \lfloor m/2 \rfloor + 1 \le j \le m\}, \\ & \mathfrak{E}' = \mathfrak{E} \oplus \mathbb{F}\mathrm{I}_m, \quad \mathfrak{F}' = \mathfrak{F} \oplus \mathbb{F}\mathrm{I}_m, \end{aligned} \tag{2.2}$$

where $m = \sum_{i=1}^{k} m_i$.

Write $\mathbb{Z}_k = \{\bar{0}, \bar{1}, \dots, \overline{k-1}\}$ and fix the order $\bar{0}, \bar{1}, \dots, \overline{k-1}$ in the following. Now we are in the position to determine the pre-nil maximal abelian subalgebras of $\mathfrak{gl}(m_1,\dots,m_k)$, the idea of which mainly comes from Jacobson's paper [9].

Theorem 2.3. Let $\mathfrak{a}$ be a pre-nil maximal abelian subalgebra of $\mathfrak{gl}(m_1,\dots,m_k)$ and $m = \sum_{i=1}^{k} m_i$. Suppose $m > 3$. Then

(1) $\mathfrak{a}$ is conjugate to $\mathfrak{E}'$ or $\mathfrak{F}'$ for some $(m_1,\dots,m_k)$-tuple Φ.

(2) In case $\Gamma = \mathbb{Z}_k$, $\mathfrak{a}$ is of $\mathbb{Z}_k$-dimension

$$\left(\sum_{i=1}^{k} \dot{m}_{i+1}\ddot{m}_i, \sum_{i=1}^{k} \dot{m}_{i+2}\ddot{m}_i, \dots, \sum_{i=1}^{k} \dot{m}_{i+k-1}\ddot{m}_i, \left(\sum_{i=1}^{k} \dot{m}_{i+k}\ddot{m}_i \right) + 1 \right),$$

where $\dot{m}_i, \ddot{m}_i$ are nonnegative integers such that $\dot{m}_i + \ddot{m}_i = m_i$ and $\dot{m}_{k+i} = \dot{m}_i$ for $1 \le i \le k$. In particular, $\dim \mathfrak{a} = \lfloor m^2/4 \rfloor + 1$.

Proof. (1) Let V be the natural $\mathfrak{a}$-module. Similar to the case of Lie algebra, we get $V = \bigoplus_{\alpha \in \Delta} V_\alpha$, where

$$V_\alpha = \{v \in V \mid \text{for every } x \in \mathfrak{a}, \text{ there exists } l \in \mathbb{N} \text{ such that } (x - \alpha(x)\mathrm{id}_V)^l v = 0\}.$$

We use induction on $|\Delta|$.

(a) $|\boldsymbol{\Delta}| = \mathbf{1}$: From Lemma 2.1, there exists an $(m_1,\dots,m_k)$-tuple Φ such that $\mathfrak{a}$ is contained in $\mathfrak{t}^{\Phi}(m_1,\dots,m_k)$. Let $\mathfrak{b}$ be the abelian subalgebra in $\mathfrak{s}^{\Phi}(m_1,\dots,m_k)$ such that $\mathfrak{a} = \mathbb{F}\mathrm{I}_m \oplus \mathfrak{b}$. From Lemma 2.2, we get the following HGMU decompositions:

$$\begin{aligned} T_1^{-1}\mathfrak{b}T_1 &= \mathbb{F}u_{i_1,j_{11}} \oplus \cdots \oplus \mathbb{F}u_{i_1,j_{1r_1}} \oplus \mathfrak{b}_1 \\ T_2^{-1}\mathfrak{b}_1T_2 &= \mathbb{F}u_{i_2,j_{21}} \oplus \cdots \oplus \mathbb{F}u_{i_2,j_{2r_2}} \oplus \mathfrak{b}_2 \\ &\cdots \cdots \\ T_t^{-1}\mathfrak{b}_{t-1}T_t &= \mathbb{F}u_{i_t,j_{t1}} \oplus \cdots \oplus \mathbb{F}u_{i_t,j_{tr_t}} \oplus \mathfrak{b}_t \end{aligned} \tag{2.3}$$

where

- $(i_{l+1}, j_{l+1,1}) = \mathrm{ht}(\mathfrak{b}_l) > (i_l, m)$ for $0 \le l \le t-1$ with $\mathfrak{b}_0 = \mathfrak{b}, \mathfrak{b}_t = 0$, and then $\dim \mathfrak{b} = r_1 + r_2 + \cdots + r_t$;
- generalized matrix units in (2.3) are homogeneous, whose index-pairs satisfy

$$\text{no indexes in the first position appear in the second position.} \tag{2.4}$$

For $1 \le l \le t$, the fact $i_l > i_{l-1} > \cdots > i_1$ and (2.4) imply $r_l \le m - i_l - (t-l)$, and then

$$\begin{aligned} \dim \mathfrak{b} &= r_1 + \cdots + r_t \\ &\le tm - (i_1 + \cdots + i_t) - (1 + \cdots + t - 1) \\ &\le tm - (1 + \cdots + t) - (1 + \cdots + t - 1) \\ &= t(m-t) \\ &\le \lfloor m^2/4 \rfloor. \end{aligned} \tag{2.5}$$

Since $\dim \mathfrak{a} \geq \dim \mathfrak{E}' = \lfloor m^2/4 \rfloor + 1$, $\dim \mathfrak{b} \geq \lfloor m^2/4 \rfloor$. This forces (2.5) is a equation, that is,

$$i_l = l, \quad r_l = m - t, \quad t = \lceil m/2 \rceil \text{ or } \lfloor m/2 \rfloor, \tag{2.6}$$

for $1 \leq l \leq t$.

If $m = 2n$ is even, then

$$i_l = l, \quad t = n, \quad r_l = m - t = n \text{ for } 1 \leq l \leq n$$

by (2.6). Then by (2.4), generalized matrix units in (2.3) are $u_{i,j}$ for $1 \leq i \leq n, n+1 \leq j \leq 2n$. It follows that the last n rows of any matrix in $\mathfrak{b}_1 \cup \mathfrak{b}_2 \cdots \cup \mathfrak{b}_{n-1}$ is zero rows from Lemma 2.2(2). In particular, $u_{n,j}$ in (2.3) is e_{nj}, where $n+1 \leq j \leq 2n$. Furthermore, $u_{n-1,j}$ in (2.3) may be viewed as e_{n-1j}, $n+1 \leq j \leq 2n$. By induction, $u_{l,j}$ in (2.3) may be viewed as e_{lj}, where $n+1 \leq j \leq 2n$ and $2 \leq l \leq n$. Then $\mathfrak{b}_1$ is spanned by $\{e_{ij} \mid 2 \leq i \leq n, n+1 \leq j \leq 2n\}$. Since

$$0 = [u_{l,h}, e_{ij}] = u_{l,h} e_{ij} - \varepsilon(|u_{l,h}|, |e_{ij}|) e_{ij} u_{l,h} = -\varepsilon(|u_{l,h}|, |e_{ij}|) e_{ij} u_{l,h}$$

for $1 \leq i \leq n, n+1 \leq j, h \leq 2n$, the last n rows of each $u_{l,h}$ are zero rows. Consequently, $\mathfrak{b}$ is conjugate to $\mathfrak{E}$ or $\mathfrak{F}$, and $\mathfrak{a} = \mathbb{F}\mathrm{I}_m \oplus \mathfrak{b}$ is conjugate to $\mathfrak{E}'$ or $\mathfrak{F}'$.

The remaining case $s = 2n+1$ can be analogously treated.

(b) $|\boldsymbol{\Delta}| > \mathbf{1}$: Let $\Delta = \{\beta_1, \beta_2, \ldots, \beta_l\}$ where $l > 1$. We may suppose that $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$, where $\mathfrak{a}_i$ is an abelian subalgebra of $\mathfrak{gl}(V_i)$ with $V_1 = V_{\beta_1}$ and $V_2 = \bigoplus_{i=2}^{l} V_{\beta_i}$. For $i = 1, 2$, denote by Δ_i the weight set of V_i with respect to $\mathfrak{a}_i$. Note that $\Delta_1 = \{\beta_1\}$ and $\Delta_2 = \{\beta_j \mid 2 \leq j \leq l\}$ by abuse of language. Since $|\Delta_i| < |\Delta|$ for $i = 1, 2$, by inductive hypothesis we may assume that the theorem holds for $\mathfrak{gl}(V_1)$ and $\mathfrak{gl}(V_2)$. Let $(\dot{m}_1, \ldots, \dot{m}_k)$ or $(\ddot{m}_1, \ldots, \ddot{m}_k)$ be Γ-dimension of V_1 or V_2, respectively. Write $\dot{m} = \sum_{i=1}^{k} \dot{m}_i$ and $\ddot{m} = \sum_{i=1}^{k} \ddot{m}_i$. Then $m = \dot{m} + \ddot{m}$ and $m_i = \dot{m}_i + \ddot{m}_i$, $1 \leq i \leq k$.

(I) $m = 2t-1, \dot{m} = 2t_1 - 1, \ddot{m} = 2t_2$: Here $t = t_1 + t_2$ and

$$\dim \mathfrak{a} \leq t_1(t_1 - 1) + 1 + t_2^2 + 1 \leq t(t-1) + 1.$$

Equality holds between the last terms only when $m = 3$.

(II) $m = 2t, \dot{m} = 2t_1 - 1, \ddot{m} = 2t_2 - 1$: Here $t = t_1 + t_2 - 1$ and

$$\dim \mathfrak{a} \leq t_1(t_1 - 1) + 1 + t_2(t_2 - 1) + 1 \leq t^2 + 1.$$

Equality holds between the last terms only when $m = 2$.

(III) $m = 2t, \dot{m} = 2t_1, \ddot{m} = 2t_2$: Here $t = t_1 + t_2$ and

$$\dim \mathfrak{a} \leq t_1^2 + 1 + t_2^2 + 1 < t^2 + 1.$$

Hence (1) holds.

(2) From (1), it is sufficient to consider the cases $\mathfrak{a} = \mathfrak{E}'$ and $\mathfrak{F}'$. Let $\dot{m}_i$ be the cardinality of the set

$$\left\{ j \mid \Phi_j = \bar{i}, 1 \leq j \leq \lfloor \tfrac{m}{2} \rfloor \right\} \quad \text{if} \quad \mathfrak{a} = \mathfrak{F}'$$

or

$$\left\{ j \mid \Phi_j = \bar{i}, 1 \leq j \leq \lceil \tfrac{m}{2} \rceil \right\} \quad \text{if} \quad \mathfrak{a} = \mathfrak{E}',$$

and write $\ddot{m}_i = m_i - \dot{m}_i$, where $1 \leq i \leq k$. For $1 \leq l \leq k$, denote by $\mathfrak{a}_{\bar{l}}$ the homogeneous subspace of degree $\bar{l}$ for $\mathfrak{a}$. Since the matrix unit e_{st} is of the degree $\Phi_s - \Phi_t$, we get

$$\mathfrak{a}_{\bar{l}} = \mathrm{Span}\left\{e_{st} \mid \Phi_s = \overline{i+l}, \Phi_t = \bar{i}, 1 \leq i \leq k\right\} + \delta_{\bar{l},0} I_m.$$

Hereafter $\delta_{i,j}$ is 1 if $i = j$ and 0 otherwise. It follows that

$$\dim \mathfrak{a}_{\bar{l}} = \sum_{i=1}^{k} \dot{m}_{i+l}\ddot{m}_i + \delta_{\bar{l},0} 1.$$

□

By a direct computation, we get the following theorem, which complements the Theorem 2.3.

Theorem 2.4. Let $\mathfrak{a}$ be a pre-nil maximal abelian subalgebra of $\mathfrak{gl}(m_1, \ldots, m_k)$ and $m = \sum_{i=1}^{k} m_i$.

(1) If $m = 3$, $\mathfrak{a}$ is conjugate to one of the following

$$\mathrm{Span}\{\mathrm{I}_3, e_{12}, e_{13}\}, \quad \mathrm{Span}\{\mathrm{I}_3, e_{23}, e_{13}\},$$

$$\mathrm{Span}\{e_{11} + e_{22}, e_{12}, e_{33}\}, \quad \mathrm{Span}\{e_{11}, e_{22}, e_{33}\}.$$

(2) If $m = 2$, $\mathfrak{a}$ is conjugate to $\mathrm{Span}\{\mathrm{I}_2, e_{12}\}$ or $\mathrm{Span}\{e_{11}, e_{22}\}$.

Since any abelian subalgebra of the general linear Lie superalgebra is a pre-nil one, Theorems 2.3 and 2.4 cover Schur's work and its super-version. As a by-product, we may determine all nil maximal abelian subalgebras of $\mathfrak{gl}(m_1, \ldots, m_k)$ as follows.

Corollary 2.5. Let $\mathfrak{b}$ be a nil maximal abelian subalgebra of $\mathfrak{gl}(m_1, \ldots, m_k)$ and $m = \sum_{i=1}^{k} m_i$.

(1) Suppose $m > 3$. Then

- $\mathfrak{b}$ is conjugate to $\mathfrak{E}$ or $\mathfrak{F}$ for some $(m_1, \ldots, m_k)$-tuple Φ.
- If $\Gamma = \mathbb{Z}_k$, then $\mathfrak{b}$ is of Γ-dimension

$$\left(\sum_{i=1}^{k} \dot{m}_{i+1}\ddot{m}_i, \sum_{i=1}^{k} \dot{m}_{i+2}\ddot{m}_i, \ldots, \sum_{i=1}^{k} \dot{m}_{i+k-1}\ddot{m}_i, \sum_{i=1}^{k} \dot{m}_{i+k}\ddot{m}_i\right),$$

where $\dot{m}_i, \ddot{m}_i$ are nonnegative numbers such that $\dot{m}_i + \ddot{m}_i = m_i$, and $\dot{m}_{k+i} = \dot{m}_i$ for $1 \leq i \leq k$. In particular, $\dim \mathfrak{b} = \lfloor m^2/4 \rfloor$.

(2) If $m = 3$, then $\mathfrak{b}$ is conjugate to one of the following

$$\mathrm{Span}\{e_{12}, e_{13}\}, \quad \mathrm{Span}\{e_{23}, e_{13}\}, \quad \mathrm{Span}\{e_{12}, e_{33}\}.$$

(3) If $m = 2$, then $\mathfrak{b}$ is conjugate to $\mathrm{Span}\{e_{12}\}$.

From Theorems 2.4 and 2.5, we have the following remark. If we focus only on the maximal dimension of pre-nil or nil abelian subalgebras for $\mathfrak{gl}(m_1, \dots, m_k)$, we may give a direct proof of the following remark by virtue of Mirzakhani's idea in [13], for which readers may see the Appendix.

Remark 2.6. (1) Any nil maximal abelian subalgebra of $\mathfrak{gl}(m_1, \dots, m_k)$ is of dimension $\lfloor \frac{(\sum_{i=1}^{k} m_i)^2}{4} \rfloor$.

(2) Any pre-nil maximal abelian subalgebra of $\mathfrak{gl}(m_1, \dots, m_k)$ is of dimension $\lfloor \frac{(\sum_{i=1}^{k} m_i)^2}{4} \rfloor + 1$.

2.3. Applications. For any $\mathbb{Z}_k$-graded abelian Lie color algebra L, the following theorem gives a method of determining $\mu_{\text{pre-nil}}(L)$ and $\mu_{\text{nil}}(L)$.

Theorem 2.7. Let L be an abelian Lie color algebra.

(1) $\mu_{\text{pre-nil}}(L) \geq \lceil 2\sqrt{\dim L - 1} \rceil, \quad \mu_{\text{nil}}(L) \geq \lceil 2\sqrt{\dim L} \rceil.$

(2) In case $\Gamma = \mathbb{Z}_k$, L possesses a pre-nil faithful representation of dimension m if and only if m admits a $2k$-partition $(\dot{m}_1, \ddot{m}_1, \dots, \dot{m}_k, \ddot{m}_k)$ such that $\sum_{i=1}^{k} \dot{m}_{i+l}\ddot{m}_i + \delta_{\bar{l},0}1 \geq \dim L_{\bar{l}}$, where $\dot{m}_{k+i} = \dot{m}_i$ and $1 \leq l \leq k-1$.

(3) In case $\Gamma = \mathbb{Z}_k$, L possesses a nil faithful representation of dimension m if and only if m admits a $2k$-partition $(\dot{m}_1, \ddot{m}_1, \dots, \dot{m}_k, \ddot{m}_k)$ such that $\sum_{i=1}^{k} \dot{m}_{i+l}\ddot{m}_i \geq \dim L_{\bar{l}}$, where $\dot{m}_{k+i} = \dot{m}_i$ and $1 \leq l \leq k-1$.

Proof. (1) Let $\iota : L \longrightarrow \mathfrak{gl}(m_1, \dots, m_k)$ be a pre-nil faithful representation of L. Then $\iota(L)$ is a pre-nil abelian subalgebra of $\mathfrak{gl}(m_1, \dots, m_k)$. From Theorem 2.3(1),

$$\dim L = \dim \iota(L) \leq \lfloor (\sum_{i=1}^{k} m_i)^2/4 \rfloor + 1.$$

Consequently, $\sum_{i=1}^{k} m_i \geq \lceil 2\sqrt{\dim L - 1} \rceil$.

Furthermore, if ι is nil, then each element in $\iota(L)$ is nilpotent. From Corollary 2.5,

$$\dim L = \dim \iota(L) \leq \lfloor (\sum_{i=1}^{k} m_i)^2/4 \rfloor.$$

Consequently, $\sum_{i=1}^{k} m_i \geq \lceil 2\sqrt{\dim L} \rceil$.

(2) and (3) are true by virtue of Theorem 2.5(2) and Corollary 2.5(2), respectively. □

3. Appendix: The maximal dimension of pre-nil or nil abelian subalgebras for $\mathfrak{gl}(m_1, \dots, m_k)$

In this section, we give a direct proof of Remark 2.6. The main idea comes from Mirzakhani's work [13].

Let $\mathfrak{a}$ be any maximal pre-nil abelian subalgebra of $\mathfrak{gl}(m_1, \dots, m_k)$. We may assume that $\mathfrak{a}$ is contained in $\mathfrak{t}^{\Phi}(m_1, \dots, m_k)$ from Lemma 2.1. Let us use induction on m to show that $\dim\mathfrak{a} \leq \lfloor m^2/4\rfloor + 1$. When $m = 1$, the conclusion holds since $\mathfrak{gl}^{\Phi}(m_1, \ldots, m_k)$ is of dimension 1. Assume that the conclusion holds for $m - 1$. Let us consider the case m. Suppose the contrary, that is, $\dim\mathfrak{a} > \lfloor m^2/4\rfloor + 1$. Then $\mathfrak{a}$ contains an abelian subalgebra, say $\mathfrak{n}$, of dimension

$$\nu(m) := \lfloor m^2/4\rfloor + 2. \tag{3.1}$$

Fix a homogeneous basis of $\mathfrak{n}$: $\{A_i \mid 1 \leq i \leq \nu(m)\}$. For $1 \leq i \leq \nu(m)$, let $\bar{A}_i$ and $\tilde{A}_i$ be $(m-1)\times(m-1)$ matrices such that

$$A_i = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ & & \\ \vdots & \bar{A}_i & \\ a_{m1} & & \end{bmatrix} = \begin{bmatrix} & & a_{1m} \\ & \tilde{A}_i & \vdots \\ & & \\ a_{m1} & \cdots & a_{mm} \end{bmatrix}.$$

Write

$$\bar{\Phi} = (\hat{\Phi}_1, \Phi_2, \ldots, \Phi_m) \text{ and } \tilde{\Phi} = (\Phi_1, \Phi_2, \ldots, \hat{\Phi}_m),$$

where the sign $\hat{}$ means that the element under it is omitted. It is obvious that $\bar{\Phi}$ is an $(m_1, \ldots, m_g - 1, \ldots, m_k)$-tuple if $\Phi_1 = \alpha_g$, $\tilde{\Phi}$ is an $(m_1, \ldots, m_h - 1, \ldots, m_k)$-tuple if $\Phi_m = \alpha_h$. Then $\bar{A}_i$ and $\tilde{A}_i$ are in $\mathfrak{gl}^{\bar{\Phi}}(m_1, \ldots, m_g - 1, \ldots, m_k)$ and $\mathfrak{gl}^{\tilde{\Phi}}(m_1, \ldots, m_h - 1, \ldots, m_k)$ respectively, and $|\bar{A}_i| = |A_i| = |\tilde{A}_i|$. By $[A_i, A_j] = 0$, we have $A_iA_j = \varepsilon(|A_i|, |A_j|)\, A_jA_i$. Then

$$\begin{bmatrix} * & \cdots & * \\ 0 & & \\ \vdots & \bar{A}_j\bar{A}_j & \\ 0 & & \end{bmatrix} = \varepsilon(|A_i|, |A_j|) \begin{bmatrix} * & \cdots & * \\ 0 & & \\ \vdots & \bar{A}_j\bar{A}_i & \\ 0 & & \end{bmatrix},$$

$$\begin{bmatrix} & & & * \\ & \tilde{A}_i\tilde{A}_j & & \vdots \\ & & & \\ 0 & \cdots & 0 & * \end{bmatrix} = \varepsilon(|A_i|, |A_j|) \begin{bmatrix} & & & * \\ & \tilde{A}_j\tilde{A}_i & & \vdots \\ & & & \\ 0 & \cdots & 0 & * \end{bmatrix}.$$

That is, $[\bar{A}_i, \bar{A}_j] = 0 = [\tilde{A}_i, \tilde{A}_j]$. Let $\bar{W}$ and $\tilde{W}$ be the Γ-graded vector spaces spanned by $\{\bar{A}_i \mid 1 \leq i \leq \nu(m)\}$ and $\{\tilde{A}_i \mid 1 \leq i \leq \nu(m)\}$, respectively. Then $\bar{W}$ and $\tilde{W}$ are also abelian subalgebras of $\mathfrak{t}^{\bar{\Phi}}(m_1, \ldots, m_g - 1, \ldots, m_k)$ and $\mathfrak{t}^{\tilde{\Phi}}(m_1, \ldots, m_h - 1, \ldots, m_k)$ for some g and h, respectively. Write $r = \dim\bar{W}$ and $t = \dim\tilde{W}$. By inductive hypothesis, we have

$$\begin{aligned} r &\leq \lfloor (m-1)^2/4\rfloor + 1, \\ t &\leq \lfloor (m-1)^2/4\rfloor + 1. \end{aligned} \tag{3.2}$$

Without loss of generality, we may assume that $\{\bar{A}_i \mid 1 \leq i \leq r\}$ are linearly independent, so are $\{\tilde{A}_i \mid 1 \leq i \leq t\}$. Let

$$\bar{A}_i = \textstyle\sum_{k=1}^{r} \bar{m}_{ik}\bar{A}_k, \text{ where } \bar{m}_{ik} \in \mathbb{F} \text{ and } r + 1 \leq i \leq \nu(m),$$

$$\tilde{A}_j = \textstyle\sum_{k=1}^{t} \tilde{m}_{jk}\tilde{A}_k, \text{ where } \tilde{m}_{jk} \in \mathbb{F} \text{ and } t + 1 \leq j \leq \nu(m).$$

Note that $|\bar{A}_i| = |\bar{A}_j|$ if $\bar{m}_{ij} \neq 0$, and $|\tilde{A}_i| = |\tilde{A}_j|$ if $\tilde{m}_{ij} \neq 0$. Then

$$|A_i| = |A_j| \text{ if } \bar{m}_{ij} \neq 0 \text{ or } \tilde{m}_{ij} \neq 0.$$

For $r+1 \leq i \leq \nu(m)$ and $t+1 \leq j \leq \nu(m)$, write

$$\bar{B}_i = A_i - \sum_{k=1}^{r} \bar{m}_{ik} A_k, \quad \tilde{B}_j = A_j - \sum_{k=1}^{t} \tilde{m}_{jk} A_k.$$

Thus each $\bar{B}_i$ (resp. $\tilde{B}_j$) is homogeneous and of the form $[\bar{b}_i,\ O]^{\mathrm{t}}$ (resp. $\left[O, \tilde{b}_j\right]$), where $\bar{b}_i^{\mathrm{t}} = \bar{a}_i - \sum_{k=1}^{r} \bar{m}_{ik}\bar{a}_k$ is a $1 \times m$ matrix and $\bar{a}_q$ is the first row of A_q for $1 \leq q \leq \nu(m)$ (resp. $\tilde{b}_j = \tilde{a}_j - \sum_{k=1}^{l} \tilde{m}_{jk}\tilde{a}_k$ is an $m \times 1$ matrix and $\tilde{a}_q$ is the last column of A_q, $1 \leq q \leq \nu(m)$). Hereafter X^{t} denotes the transpose of a matrix X. Clearly,

$$\left\{\bar{B}_i \mid r+1 \leq i \leq \nu(m)\right\} \text{ (resp. } \left\{\tilde{B}_i \mid t+1 \leq i \leq \nu(m)\right\})$$

are linearly independent, and so are

$$\left\{\bar{b}_i^{\mathrm{t}} \mid r+1 \leq i \leq \nu(m)\right\} \text{ (resp. } \left\{\tilde{b}_i \mid t+1 \leq i \leq \nu(m)\right\}).$$

Let $M = \left[\bar{b}_{r+1}, \bar{b}_{r+2}, \ldots, \bar{b}_{\nu(m)}\right]^{\mathrm{t}}$. Clearly,

$$\operatorname{rank} M = \nu(m) - r. \tag{3.3}$$

Denote by W the set $\{X \in \mathbb{F}^m \mid MX = 0\}$. Then

$$\dim W = m - \operatorname{rank} M. \tag{3.4}$$

For $r+1 \leq i \leq \nu(m)$ and $t+1 \leq j \leq \nu(m)$, $\bar{B}_i, \tilde{B}_j$ are homogenous matrices in $\mathfrak{n}$. Note that $\tilde{B}_j\bar{B}_i = 0$, then $\bar{B}_i\tilde{B}_j = 0$ by $[\bar{B}_i, \tilde{B}_j] = 0$. Consequently,

$$\bar{b}_i^{\mathrm{t}}\tilde{b}_j = 0 \text{ for } r+1 \leq i \leq \nu(m) \text{ and } t+1 \leq j \leq \nu(m),$$

that is, the set $\left\{\tilde{b}_j \mid t+1 \leq j \leq \nu(m)\right\}$ is contained in W, which are linearly independent. Consequently,

$$\dim W \geq \nu(m) - t. \tag{3.5}$$

Therefore

$$\begin{aligned} m &\overset{(3.4)}{=} \operatorname{rank} M + \dim W \\ &\overset{(3.3)(3.5)}{\geq} \nu(m) - r + \nu(m) - t \\ &\overset{(3.1)}{=} 2(\lfloor m^2/4 \rfloor + 2) - r - t \\ &\overset{(3.2)}{\geq} 2(\lfloor m^2/4 \rfloor - \lfloor (m-1)^2/4 \rfloor + 1). \end{aligned}$$

Thus, if $m = 2q$ is even, then $2q \geq 2(q+1)$, a contradiction; if $m = 2q+1$ is odd, then $2q+1 \geq 2(q+1)$, also a contradiction. Hence $\dim \mathfrak{a} \leq \lfloor m^2/4 \rfloor + 1$. By (2.2), we have $\dim \mathfrak{a} \geq= \dim \mathfrak{E}' = \lfloor m^2/4 \rfloor + 1$.

Furthermore, if each element in $\mathfrak{a}$ is nilpotemt, we may assume that $\mathfrak{a}$ is contained in $\mathfrak{s}^{\Phi}(m_1, \ldots, m_k)$ from Lemma 2.1. Thus

$$\lfloor m^2/4 \rfloor = \dim \mathfrak{E} \leq \dim \mathfrak{a} < \lfloor m^2/4 \rfloor + 1.$$

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References

[1] A. Martín and J. Delgado, On the structure of split Lie color algebras, Linear Algebra Appl., 436 (2012): 307–315.

[2] Y. Benoist, Une Nilvariete Non Affine, J. Diff. Geom., 41 (1995): 21–52.

[3] G. Birkhoff, Representability of Lie algebras and Lie groups by matrices, Ann. Math., 38(2) (1937): 526–532.

[4] D. Burde, Affine structures on nilmanifolds, Internat. J. Math., 7(5) (1996): 599–616.

[5] D. Burde and F. Grunewald, Modules for certain Lie algebras of maximal class, J. Pure and Appl. Algebra, 99 (1995): 239–254.

[6] D. Burde and W. Moens, Minimal Faithful Representations of Reductive Lie Algebras, Archiv der Mathematik, 89 (2007): 513–523.

[7] D. Burde, On a refinement of Ado's Theorem, Arch. Math., 70 (1998): 118–127.

[8] L. Cagliero and N. Rojas, Faithful representations of minimal dimension of current Heisenberg Lie algebras, Internat. J. Math., 20(11) (2009): 1347–1362.

[9] N. Jacobson, Schur's theorems on commutative matrices, Bull. Amer. Math. Soc., 50 (1944): 431–436.

[10] W. Liu and M. Chen, The minimal dimensions of faithful representations for Heisenberg Lie superalgebras, J. Geom. Phys., 89 (2015): 17–24.

[11] W. Liu and S. Wang, Minimal faithful representations of abelian Jordan algebras and Lie superalgebras, Linear Algebra Appl., 437 (2012): 1293–1299.

[12] J. Milnor, On fundamental groups of complete affinely flat manifolds, Adv. Math., 25 (1977): 178–187.

[13] M. Mirzakhani, A simple proof of a theorem of Schur, Amer. Math. Monthly, 105 (1998): 260–262.

[14] R. Ree, Generalized Lie elements, Canad. J. Math., 12 (1960): 439–502.

[15] B. Reed, Representations of solvable Lie algebras, Michigan Math. J., 16 (1969): 227–233.

[16] N. Rojas, Minimal faithful representation of the Heisenberg Lie algebra with abelian factor, J. Lie Theory, 23 (2013): 1105–1114.

[17] M. Scheunert, Generalized Lie algebras, J. Math. Phys., 20 (1979): 712–720.

[18] M. Scheunert and R. Zhang, Cohomology of Lie superalgebras and their generalizations, J. Math. Phys., 39(9) (1998), 5024–5061.

[19] I. Schur, Zur Theorie vertauschbaren Matrizen, J. Reine Angew. Math., 130 (1905): 66–76.

[20] Y. Su, K. Zhao and L. Zhu, Simple Lie color algebras of Weyl type, Israel J. Math., 137 (2003): 109–123.

[21] Y. Su, K. Zhao and L. Zhu, Classification of derivation-simple color algebras related to locally finite derivations, J. Math. Phys., 45(1) (2004): 525–536.

[22] Q. Wang, H.J. Chen and W.D. Liu, On representations of the filiform Lie superalgebra $L_{m,n}$, J. Geom. Phys., 97 (2015): 93–104.

[23] S. Wang and W. Liu, Minimal dimensions of faithful representations for abelian Lie superalgebras, Linear Multilinear Algebra, 64(7) (2016): 1354–1361.

[24] S. Wang and W. Liu, The abelian subalgebras of maximal dimensions for general linear Lie superalgebras, Linear Multilinear Algebra, 64(10) (2016): 2081–2089.

[25] S. Wang and W. Liu, On the minimal dimension of faithful representations for current Heisenberg Lie superalgebras, Colloq. Math., 165(1) (2021): 45–52

[26] Y. Wei and T. Wettig, Bosonic color-flavor transformation for the special unitary group, J. Math. Phys., 46(7) (2005): 072306(25pp).

[27] Y. Yao, Non-restricted representations of simple Lie superalgebras of special type and Hamiltonian type, Sci. China Math., 56 (2013): 239–252.

[28] C. Zhang, Representations of the restricted Lie color algebras, Algebr. Represent. Theory, 16 (2013): 445–470.