

Semisimple Field Theories Detect Stable Diffeomorphism

DAVID REUTTER AND CHRISTOPHER SCHOMMER-PRIES

ABSTRACT. Extending the work of the first author, we introduce a notion of semisimple topological field theory in arbitrary even dimension and show that such field theories necessarily lead to stable diffeomorphism invariants. The main result of this paper is a proof that this ‘upper bound’ is optimal: To this end, we introduce and study a class of semisimple topological field theories, generalizing the well known finite gauge theories constructed by Dijkgraaf-Witten, Freed and Quinn. We show that manifolds satisfying a certain finiteness condition — including 4-manifolds with finite fundamental group — are indistinguishable to these field theories if and only if they are stably diffeomorphic. Hence, such generalized Dijkgraaf-Witten theories provide the strongest semisimple TFT invariants possible. These results hold for a large class of ambient tangential structures.

We discuss a number of applications, including the constructions of unoriented 4-dimensional semisimple field theories which can distinguish unoriented smooth structure and oriented higher-dimensional semisimple field theories which can distinguish certain exotic spheres.

Along the way, we show that dimensional reductions of generalized Dijkgraaf-Witten theories are again generalized Dijkgraaf-Witten theories, we utilize ambidexterity in the rational setting, and we develop techniques related to the ∞ -categorical Möbius inversion principle of Gálvez-Carrillo-Kock-Tonks.

1. INTRODUCTION

Topological field theories (TFT) provide invariants of smooth manifolds. However what precisely these invariants measure and which manifolds can be distinguished remains largely unknown. In this paper, we focus on *semisimple field theories* in even dimensions (see Definition 5.13), a class of field theories which contains all currently known functorial topological field theories in more than two dimensions (see Example 5.14).

Extending a result of the first author [Reu20] to arbitrary even dimension, we show that such field theories only depend on the stable diffeomorphism class of a manifold¹, providing an ‘upper bound’ on the sensitivity of the induced manifold invariant.

Theorem A. *Stably diffeomorphic even-dimensional manifolds are indistinguishable by semisimple topological field theories.*

The precise version of this theorem, Theorem 5.21, allows for more general tangential structures; in this introduction we focus on the oriented and unoriented case. The main contribution of this paper is a proof that this ‘upper bound’ is optimal for manifolds fulfilling a certain finiteness condition.

Theorem B. *Connected closed $2q$ -manifolds with π -finite tangential $(q-1)$ -type² are stably diffeomorphic if and only if they are indistinguishable by semisimple field theories.*

¹Two $2q$ -dimensional manifolds M and N are stably diffeomorphic if there exists a natural number $n \geq 0$ and a diffeomorphism $M \#^n(S^q \times S^q) \cong N \#^n(S^q \times S^q)$.

²A $2q$ -manifold has π -finite tangential $(q-1)$ -type if $\pi_i F$ is finite for all $i \leq q-1$ and all basepoints, where F is the homotopy fiber of the classifying map $M \rightarrow \mathrm{BO}(2q)$ of the tangent bundle.

This theorem appears as Theorem 4.14. We will discuss a number of applications below, such as the existence of semisimple field theories which can distinguish smooth structure.

To prove Theorem B, we introduce and study a class of *super topological field theories* generalizing well known constructions of Dijkgraaf-Witten, Freed and Quinn [DW90, FQ93, Fre94, Qui95]. Associated to each such *generalized Dijkgraaf-Witten theory* is a certain numerical dimension, called its *type* and defined below at the end of Section 1.2. We show that generalized Dijkgraaf-Witten theories are semisimple (Theorem 5.25) and that type- $(q - 1)$ generalized Dijkgraaf-Witten theories distinguish *non* stably diffeomorphic $2q$ -manifolds, provided they have π -finite $(q - 1)$ -types (Theorem 5.25), proving Theorem B.

1.1. Manifold results. The four-dimensional oriented version of Theorem A appears as [Reu20, Theorem A], and generalizes earlier results on the sensitivity of positive TFTs [FKN⁺05]. In this dimension, Theorem B reduces to the following converse of Theorem A:

Theorem 1.1. *Let M and N be connected, closed 4-manifolds with finite fundamental groups. Then M and N are stably diffeomorphic if and only if they are indistinguishable by semisimple topological field theories if and only if they are indistinguishable by type-1 generalized Dijkgraaf-Witten theories.*

As explained in [Reu20], Theorem A implies that *oriented* semisimple 4-dimensional TFTs cannot detect smooth structure: It follows from a result of Gompf [Gom84] that smooth oriented 4-manifolds which are orientation preserving homeomorphic are stably diffeomorphic. However, Theorem 1.1 shows that semisimple oriented 4-dimensional TFTs can see more than the homotopy type of a manifold — in [Tei92, Ex. 5.2.4], Teichner constructed families of homotopy equivalent oriented 4-manifolds with finite fundamental group which are not stably diffeomorphic and hence can be distinguished by oriented semisimple field theories, answering Question 1.1 of [Reu20].

In contrast, Theorem 1.1 also shows that semisimple 4-dimensional TFTs can sometimes distinguish *unoriented* smooth structure: There are examples of unoriented smooth 4-manifolds which are homeomorphic but not stably diffeomorphic, such as $\mathbb{R}P^4$ and Cappell-Shaneson’s fake $\mathbb{R}P^4$. Provided they have finite fundamental groups, such manifolds can therefore be distinguished by semisimple 4-dimensional field theories, recovering and extending an example of A. Debray [Deb20] (a version of which also appears in [TKBB21]).

In higher dimensions, the shortcomings of semisimple field theories remain significant: For example, as observed by Kreck and Schafer [KS84], arbitrarily large families of $(2k - 1)$ -connected $4k$ -manifolds ($k \geq 2$) which are pairwise stably diffeomorphic, but pairwise not homotopy equivalent have been implicit in the literature since Wall’s classification of these manifolds up to the action of homotopy spheres [Wal62]. Many more examples have appeared more recently [CCPS21b, CCPS21a], including infinite families of pairwise stably diffeomorphic manifolds which are pairwise not homotopy equivalent. By Theorem A, all such manifolds are indistinguishable by semisimple TFT.

On the other hand, we may use Theorem B to construct semisimple field theories which can distinguish certain exotic smooth spheres (see Corollary 6.7).

Proposition 1.2. *In every dimension $d = 8k + 1$ or $8k + 2$ there exists pairs of distinct exotic spheres which are distinguished by certain oriented type-1 generalized Dijkgraaf-Witten theories.*

In fact, in dimensions > 4 , (exotic) spheres are diffeomorphic if and only if they are stably diffeomorphic [Rei]. Unfortunately, in the relevant dimensions $2q > 6$ homotopy spheres do

not have π -finite tangential $(q-1)$ -type, so that Theorem B does not apply. This raises the following question:

Question 1.3. Can the finiteness assumption in Theorem B be weakened? Do generalized Dijkgraaf-Witten theories, or more generally semisimple TFTs, distinguish all exotic spheres in dimensions > 4 ?

The finiteness assumption in Theorem B cannot be completely omitted: In Proposition 6.13 we provide an example of a 4-manifold with infinite fundamental group, but which nevertheless cannot be distinguished from S^4 by type-1 generalized Dijkgraaf-Witten theories.

In practice, we will use the following alternative characterization of $2q$ -manifolds having π -finite $(q-1)$ -type: For $4k \leq q$ let $p_{k,M}^{\mathbb{Q}} : \pi_{4k}M \rightarrow \mathbb{Q}$ denote the composite

$$\pi_{4k}M \xrightarrow{\pi_{4k}(T_M)} \pi_{4k}BO(2q) \rightarrow H_{4k}(BO(2q); \mathbb{Q}) \xrightarrow{p_{4k}^{\mathbb{Q}}} \mathbb{Q}$$

induced by the classifying map $T_M : M \rightarrow BO(2q)$ of the tangent bundle, the rational Hurewicz homomorphism, and the rational k^{th} Pontryagin class. Then, a connected $2q$ -manifold M has π -finite tangential $(q-1)$ -type if and only if for each $i \leq q-1$ with $i \neq 4k$ the group $\pi_i M$ is finite, for each $4k \leq q$ the homomorphism $p_{k,M}^{\mathbb{Q}}$ is non-zero, and for each $4k \leq q-1$ the kernel of $p_{k,M}^{\mathbb{Q}}$ is finite. The 6 and 8-dimensional analogues of Theorem 1.1 therefore become:

Corollary 1.4. *Connected closed 6-manifolds with finite π_1 and π_2 are stably diffeomorphic if and only if they are indistinguishable by semisimple topological field theories, if and only if they are indistinguishable by type-2 generalized Dijkgraaf-Witten theories.*

Connected closed 8-manifolds with finite π_1 , π_2 , and π_3 , and non-zero signature are stably diffeomorphic if and only if they are indistinguishable by semisimple topological field theories if and only if they are indistinguishable by type-3 generalized Dijkgraaf-Witten theories.

In dimension $2q$ with q even, stable diffeomorphism can fail to distinguish homotopically distinct manifolds. In contrast, when q is odd, simply connected manifolds which are stably diffeomorphic are themselves diffeomorphic [Kre99, Thm D]. This yields the following corollaries, the first of which partially answers a question raised in [KT08].

Corollary 1.5. *Simply connected 6-manifolds with finite π_2 are distinguished up to diffeomorphism by 6-dimensional semisimple TFTs; specifically by type-2 generalized Dijkgraaf-Witten theories.*

Corollary 1.6. *Simply connected 10-manifolds with finite π_2 and π_3 , $\pi_4 \otimes \mathbb{Q}$ rank one, and non-zero rational 1st Pontryagin class are distinguished up to diffeomorphism by 10-dimensional semisimple TFTs; specifically by type-4 generalized Dijkgraaf-Witten theories.*

Further examples and applications are discussed in Section 6.2.

1.2. Generalized Dijkgraaf-Witten theories. In [DW90], R. Dijkgraaf and E. Witten construct a gauge theory for a finite gauge group G , a construction which was later improved, streamlined and expressed as an oriented functorial TFT by D. Freed and F. Quinn in [Fre94, FQ93]. Such Dijkgraaf-Witten TFT were thereafter considered by many authors [Tro16, SW18, SW19, SW20, Har20].

Based on ideas of Kontsevich [Kon88], F. Quinn defined in [Qui95] oriented *twisted finite homotopy TQFTs*, a generalization of Dijkgraaf-Witten theories in which the classifying

space BG of the group G is replaced by any π -finite³ space Y and a cohomology class $\omega \in H^d(Y; \mathbb{C}^\times)$. In physical terms, Quinn’s version of Dijkgraaf-Witten theory may be understood as a ‘finite σ -model’. The ‘space of fields’ on a closed d -manifold M is the mapping space $\text{Map}(M, Y)$, and the partition function on M is computed via a ‘finite path integral’ involving $\omega \in H^d(Y; \mathbb{C}^\times)$ as a Lagrangian of the theory:

$$(1) \quad Z_{Y, \omega}(M) = \sum_{[f] \in \pi_0 \text{Map}(M, Y)} \#(\text{Map}(M, Y), f) \cdot \langle [M], f^* \omega \rangle.$$

Here, $\langle [M], - \rangle : H^d(M, \mathbb{C}^\times) \rightarrow \mathbb{C}^\times$ denotes the evaluation of a top-dimensional cohomology class against the fundamental class of M and $\#(Z, z)$ denotes the *homotopy cardinality of Z at z* (cf. [Qui95, Lecture 4]) defined for π -finite spaces as

$$(2) \quad \#(Z, z) := \prod_{i \geq 1} |\pi_i(Z, z)|^{(-1)^i} = |\pi_1(Z, z)|^{-1} |\pi_2(Z, z)| |\pi_3(Z, z)|^{-1} \cdots.$$

If $Y = BG$ is the classifying space of a finite group G , the space $\text{Map}(M, Y)$ is the space of principal G -bundles on M and (1) recovers the classical Dijkgraaf-Witten partition function.

Rather than summing over spaces of maps $\text{Map}(M, Y)$, we generalize this theory to allow for summations over more general tangential structures on M — for example, starting from a spin theory one may sum over spin structures to produce an oriented field theory. Such *generalized Dijkgraaf-Witten constructions* can be used to produce topological field theories of arbitrary dimension d and for arbitrary tangential structure, however for expository purposes we will focus on the oriented case in this introduction. In this case, one starts with a space X with a map $\xi : X \rightarrow BSO(d)$ whose homotopy fibers are π -finite. The space of fields will be the space of (X, ξ) -structures on an oriented manifold M , i.e. the space $\text{Map}_{BSO(d)}(M, X)$ of maps to X over $BSO(d)$ (in the appropriate homotopical sense), where M is viewed as a space over $BSO(d)$ via its tangent classifying map $T_M : M \rightarrow BSO(d)$. For example, if $X = Y \times BSO(d)$ the space of fields is $\text{Map}(M, Y)$ as in Quinn’s field theory, whereas for $X = BSpin(d)$ the space of fields is the space of spin structures on M which are compatible with the given orientation. Since X is π -finite, the space $\text{Map}_{BSO(d)}(M, X)$ will itself be π -finite.

The second input to our construction is a d -dimensional (X, ξ) -structured field theory \mathcal{W} . In Section 4.1, we produce from this data (X, ξ, \mathcal{W}) an oriented d -dimensional field theory with partition function on a closed oriented d -manifold M given by

$$DW_{\xi, \mathcal{W}}(M) = \sum_{[f] \in \pi_0 \text{Map}_{BSO(d)}(M, X)} \#(\text{Map}_{BSO(d)}(M, X), f) \cdot \mathcal{W}(M, f).$$

This field theory may be thought of as arising from *gauging* (or ‘integrating’) the (X, ξ) -structured field theory \mathcal{W} along $X \rightarrow BSO(d)$ to an oriented field theory.

For our purposes, we will mainly restrict attention to *invertible* super vector space valued (X, ξ) -theories \mathcal{W} . Such theories are classified by their partition function: For each tangential structure (X, ξ) , there is an associated finitely generated abelian group $\Omega_d^{T\xi}$ which may be thought of as an ‘unstable tangential’ version of more familiar bordism groups. In special cases, this group $\Omega_d^{T\xi}$ has been called the SKK-group [KKNO73] or the Reinhart vector field cobordism group [Rei63]. Closed d -manifolds with (X, ξ) -structures give elements in $\Omega_d^{T\xi}$,

³A space Y is π -finite if $\pi_0 Y$ is finite and if for each base point $\pi_n Y$ is finite for all n and non-zero for only finitely many n .

and invertible super field theories are uniquely determined by the induced group homomorphism (see Theorem 3.14)

$$\omega : \Omega_d^{T\xi} \rightarrow \mathbb{C}^\times.$$

The generalized Dijkgraaf-Witten construction obtained from (X, ξ) and an invertible theory determined by $\omega : \Omega_d^{T\xi} \rightarrow \mathbb{C}^\times$ results in an oriented super topological field theory $DW_{\xi, \omega}$ with partition function

$$(3) \quad DW_{\xi, \omega}(M) = \sum_{[f] \in \pi_0 \text{Map}_{BSO(d)}(M, X)} \#(\text{Map}_{BSO(d)}(M, X), f) \cdot \omega(M, f).$$

We say that a generalized Dijkgraaf-Witten theory $DW_{\xi, \omega}$ is of *type- n* if the homotopy fiber F of $\xi : X \rightarrow BSO(d)$ is an n -type, i.e. if $\pi_{\geq n+1}F = 0$.

A cohomology class $\omega \in H^d(Y; \mathbb{C}^\times)$ gives rise to an invertible theory for $X = Y \times BSO(d)$ and the resulting oriented TFT recovers Quinn's $Z_{Y, \omega}$ (see Example 4.6).

1.3. What is visible to generalized Dijkgraaf-Witten theories. The manifold invariants (3) arise from averaging (X, ξ) -structured bordism invariants over all (X, ξ) -structures on a given manifold M , weighted by homotopy cardinality. To study the ‘lossiness’ of this averaging procedure more systematically, we consider the following categorical generalization.

Let \mathcal{C} be an ∞ -category enriched in π -finite spaces and let $\Omega : \mathcal{C} \rightarrow \text{Ab}$ be a functor into the category of abelian groups. From this data, we obtain a linear pairing

$$(4) \quad \langle -, - \rangle_{\mathcal{C}, \Omega} : \bigoplus_{[c] \in \pi_0 \mathcal{C}} \mathbb{C}[\widehat{\Omega(c)}/\pi_0 \text{Aut}_{\mathcal{C}}(c)] \otimes \bigoplus_{[d] \in \pi_0 \mathcal{C}} \mathbb{C}[\Omega(d)/\pi_0 \text{Aut}_{\mathcal{C}}(d)] \rightarrow \mathbb{C}$$

where the direct sum is taken over the isomorphism classes of objects of \mathcal{C} , and where for an abelian group A with an action by a group G , $\widehat{A} := \text{Hom}(A, \mathbb{C}^\times)$ denotes the group of characters and A/G denotes the set of orbits of the G -action. The pairing is characterized by the following property: For each $c, d \in \mathcal{C}$, the induced pairing

$$\widehat{\Omega(c)} \times \Omega(d) \rightarrow \mathbb{C}$$

is given by

$$(\phi, v) \mapsto \sum_{f \in \pi_0 \mathcal{C}(d, c)} \#(\mathcal{C}(d, c), f) \phi(\Omega(f)(v)).$$

Note that this formula indeed only depends on the orbits of ϕ and v under the $\text{Aut}_{\mathcal{C}}(c)$ and $\text{Aut}_{\mathcal{C}}(d)$ action, respectively, and the isomorphism classes of c and d .

Example 1.7. Let \mathcal{C} be an ordinary 1-category with finite hom sets and let Ω be the constant functor with value the trivial group. Then, the pairing becomes a matrix indexed by the isomorphism classes of objects of \mathcal{C} with (x, y) -coefficient given by the cardinality $|\mathcal{C}(y, x)|$.

Example 1.8. Let $n, d \geq 0$ and consider the ∞ -category \mathcal{C} of spaces X equipped with maps $\xi : X \rightarrow BSO(d)$ whose homotopy fibers F are π -finite n -types (i.e. F is π -finite and $\pi_{\geq n+1}F = 0$). Let Ω be the functor $\Omega(X, \xi) = \Omega_d^{T\xi}$. If M is a closed d -manifold with π -finite tangential n -type, then the (oriented) generalized Dijkgraaf-Witten invariant associated to (X, ξ) and $\omega : \Omega_d^{X, \xi} \rightarrow \mathbb{C}^\times$ can be expressed in terms of the pairing $\langle -, - \rangle_{\mathcal{C}, \Omega}$:

$$DW_{\xi, \omega}(M) = \langle ((X, \xi), \omega), (\tau_{\leq n}M, [M]) \rangle_{\mathcal{C}, \Omega}.$$

Here, $\tau_{\leq n}M \rightarrow BSO(d)$ is the tangential n -type obtained from the oriented manifold M by factoring the classifying map of the tangent bundle

$$M \rightarrow \tau_{\leq n}M \rightarrow BSO(d)$$

into an n -connected map followed by an n -truncated map⁴ and

$$[M] \in \Omega_d^{\tau_{\leq n}(M)} / \pi_0 \text{Aut} \left(\tau_{\leq n}M \rightarrow BSO(d) \right)$$

is the induced orbit in the bordism group.

Our proof of Theorem B relies on a general criterion for when pairings $\langle -, - \rangle_{\mathcal{C}, \Omega}$ are non-degenerate, applied to Example 1.8. This question of non-degeneracy of $\langle -, - \rangle_{\mathcal{C}, \Omega}$ is closely related to the Möbius inversion principle⁵ for decomposition spaces (also known as 2-Segal spaces [DK19]) developed in [GCKT18a, GCKT18b, GCKT18c].

In the case where \mathcal{C} is a 1-category and Ω is the trivial functor, non-degeneracy of $\langle -, - \rangle_{\mathcal{C}, \Omega}$ is equivalent to the question of whether each object x is determined up to isomorphism by the sizes of the hom sets $\mathcal{C}(x, y)$ for all y , and vice versa. This question was studied by Lovasz [Lov67, Lov71] and answered positively for many categories of finite algebraic structure including various categories of finite graphs and the category of finite groups. An important consequence is for example that for any three finite groups (or finite graphs) G, H , and K if $G \times K \cong H \times K$, then $G \cong H$.

Lovasz's technique is, roughly, to use a factorization system on \mathcal{C} to write the pairing in simpler terms. For example, if \mathcal{C} is the category of finite sets, the set of isomorphism classes of objects of \mathcal{C} can be identified with \mathbb{N} and the pairing of finite sets n and m of cardinality n and m , respectively, is given by $\langle n, m \rangle = |\text{Hom}(m, n)| = n^m$. Factoring a function $m \rightarrow n$ as a surjection followed by an injection, it follows that

$$|\text{Hom}(m, n)| = \sum_{a \in \mathbb{N}} \frac{|\text{Surj}(m, a)| |\text{Inj}(a, n)|}{|\text{Aut}(a)|}.$$

As both $|\text{Surj}(-, -)|$ and $|\text{Inj}(-, -)|$ are given by (infinite) triangular matrices with non-zero diagonal entries, and hence lead to non-degenerate pairings, it follows that the pairing $|\text{Hom}(-, -)|$ itself is non-degenerate.

We generalize this method to categories \mathcal{C} enriched in π -finite spaces which are equipped with a nested sequence of factorizations systems, and prove the following theorem (see Theorem 2.38 for details).

Theorem 1.9. *Let \mathcal{C} be a ∞ -category enriched in π -finite spaces. For $k = 0, \dots, n$ let $(\mathcal{L}^{(k)}, \mathcal{R}^{(k)})$ be a collection of orthogonal factorization systems on \mathcal{C} , which are nested in the sense that $\mathcal{R}^{(k-1)} \subseteq \mathcal{R}^{(k)}$. Suppose that each of the subcategories*

$$\mathcal{R}^{(0)}, \quad \mathcal{R}^{(1)} \cap \mathcal{L}^{(0)}, \quad \mathcal{R}^{(2)} \cap \mathcal{L}^{(1)}, \quad \dots \quad \mathcal{R}^{(n)} \cap \mathcal{L}^{(n-1)}, \quad \mathcal{L}^{(n)}$$

satisfies the condition that all endomorphisms are equivalences. Then for every functor $\Omega : \mathcal{C} \rightarrow \text{Ab}$ the corresponding pairing $\langle -, - \rangle_{\mathcal{C}, \Omega}$ is non-degenerate.

⁴A map $f : X \rightarrow Y$ of spaces is n -connected if for all points $x \in X$, $\pi_i(f; x)$ is an isomorphism for $i < n + 1$ and surjective for $i = n + 1$, or equivalently if all homotopy fibers F of f are n -connected, i.e. $\pi_{\leq n}F = 0$. A map $f : X \rightarrow Y$ is n -truncated if $\pi_i(f; x)$ is injective for $i = n + 1$ and an isomorphism for $i > n + 1$, or equivalently if all homotopy fibers F are n -types, i.e. $\pi_{\geq (n+1)}F = 0$.

⁵Roughly speaking, the Möbius inversion principle states that the zeta function of any incidence algebra of a Möbius ∞ -category is invertible for the convolution product. Our sufficient condition in Theorem 1.9 does not require \mathcal{C} to be a Möbius category, and hence does neither imply nor is implied by the results in [GCKT18b].

The Postnikov factorization system into (k -connected/ k -truncated) maps for $-1 \leq k \leq n$ (see footnote 4) endows the ∞ -category \mathcal{C} from Example 1.8 with such a nested sequence of factorization systems (see Definition 2.21). Hence, applied to Example 1.8, Theorem 1.9 leads to the following precise characterization of the manifold topology visible to generalized Dijkgraaf-Witten theories.

Theorem 1.10. *Two d -manifolds M and N with π -finite tangential n -types $\tau_{\leq n}M$ and $\tau_{\leq n}N$ are indistinguishable by type- n generalized oriented Dijkgraaf-Witten theories if and only if they have equivalent tangential n -types $\tau_{\leq n}M \simeq (X, \xi) \simeq \tau_{\leq n}N$, and the bordism classes $[M]$ and $[N]$ lie in the same orbit in $\Omega_d^{T\xi}$ under the action of $\pi_0 \text{Aut}(X, \xi)$.*

A precise version of this theorem for general ambient tangential structure appears as Theorem 4.14. Theorem 1.10 should be compared to Kreck's theorem [Kre99] which implies (see Section 6.1) that the stable diffeomorphism class of a $2q$ -manifold M is precisely determined by the data appearing in Theorem 1.10: namely, its tangential $(q-1)$ -type (X, ξ) and the orbit of its bordism class $[M]$ in $\Omega_d^{T\xi}$ under the action of $\pi_0 \text{Aut}(X, \xi)$. Hence, combined with Kreck's result, Theorem 1.10 implies Theorem B. In other words, generalized Dijkgraaf-Witten theories are in a sense 'Pontryagin dual' to stable diffeomorphism classes.

Remark 1.11. An interesting consequence of Theorems 1.10, A, and B is that if two manifolds with π -finite tangential n -type and even dimension $d = 2q$ are distinguished by some semisimple field theories, then in fact they can be distinguished by generalized Dijkgraaf-Witten theories of type $(q-1)$, one less than the middle dimension. For example, using generalized Dijkgraaf-Witten theories of higher type does not result in stronger invariants.

On the other hand, non-degeneracy of our pairing also implies that in high enough dimensions d , generalized Dijkgraaf-Witten theories are distinguished by their partition functions.

Theorem 1.12. *Suppose that $n < \lfloor \frac{d}{2} \rfloor$. If $((Y_1, \xi_1), \omega_1)$ and $((Y_2, \xi_2), \omega_2)$ give rise to type- n generalized Dijkgraaf-Witten theories whose partition functions are identical on closed d -manifolds with π -finite tangential n -type, then $\xi_1 \simeq \xi_2 = \xi$, and ω_1 and ω_2 are in the same orbit of $\text{Hom}(\Omega_d^{T\xi}, \mathbb{C}^\times)$ under the action of $\pi_0 \text{Aut}(\xi)$. The resulting generalized Dijkgraaf-Witten theories are consequently isomorphic.*

This theorem appears for more general tangential structures as Theorem 4.16.

1.4. Organization of the paper. The structure of this paper can be roughly summarized as the reverse order of the introduction.

In Section 2, we start by reviewing homotopy cardinality and finite path integrals. This transitions to a discussion of spans of topological spaces equipped with local systems. Spans of π -finite spaces can be linearized/integrated to linear maps, as discussed in Section 2.2. Any ∞ -category naturally gives rise to a certain span, and in Section 2.3 we give criteria for when its linearization exists and is invertible. For ∞ -categories enriched in π -finite spaces but with infinitely many isomorphism classes of objects, this linearization does not exist, but we may still construct a certain pairing. This is discussed in Sections 2.4 and 2.5 along with conditions ensuring that this pairing is non-degenerate.

In Section 3, we discuss a variety of tangential structures, including structures on the stable normal bundle, structures on the stable tangent bundle, and on the unstable tangent bundle. Bordism groups for these are defined, as well as the bordism category. The classification of invertible topological field theories is discussed in Section 3.3. These are determined by characters on the homotopy groups of certain Madsen-Tillmann spectra.

Section 3.5 contains some computations that relate these bordism groups to more familiar classical bordism groups.

In Section 4, we construct generalized Dijkgraaf-Witten topological field theory. This utilizes the linearization of spans developed in Section 2. We show that generalized Dijkgraaf-Witten manifold invariants can be expressed in terms of the Pontryagin pairings constructed in Section 2. Our main Theorem 1.10 is proven in Section 4.4 which classifies the information that generalized Dijkgraaf-Witten theories can detect about manifolds. In Section 4.5, we also consider the process of dimensional reduction in the context of generalized Dijkgraaf-Witten theories. We show that the dimensional reduction of a generalized Dijkgraaf-Witten theory is again a generalized Dijkgraaf-Witten theory.

In Section 5, we first generalize the results of [Reu20] to higher dimensions, super field theories, and more general tangential structures. We define semisimple topological field theories and establish that they induce stable diffeomorphism invariants. Finally, we make use of the dimensional reduction results of the previous section to show that generalized Dijkgraaf-Witten theories are themselves semisimple topological field theories.

In Section 6, we recall Kreck’s classification of manifolds up to stable diffeomorphism [Kre99] and relate it to our main Theorem 1.10 concerning generalized Dijkgraaf-Witten theory (c.f. Section 4). From this we are able to deduce Theorem B and end with a discussion of various examples and applications.

Appendix A includes a technical discussion of finitely n -dominated spaces which is used to give conditions under which certain mapping spaces will be π -finite.

1.5. Acknowledgements. We are grateful to Peter Teichner and Noah Snyder for their continued interest in this work and many helpful discussions. We would like to thank Arun Debray for sparking our interest in the question of what generalized Dijkgraaf-Witten theories can detect. We also thank Dan Freed and Lukas Müller for helpful comments on an early draft of this manuscript. This material is based upon work supported by the National Science Foundation under Grant No. DMS-1440140, while the authors were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2020 semester. D.R. is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – 493608176 and is grateful for the hospitality and financial support of the Max-Planck Institute for Mathematics in Bonn. C.SP. is supported by the National Science Foundation under Grant No. DMS-2204297.

CONTENTS

| | |
|---|----|
| 1. Introduction | 1 |
| 1.1. Manifold results | 2 |
| 1.2. Generalized Dijkgraaf-Witten theories | 3 |
| 1.3. What is visible to generalized Dijkgraaf-Witten theories | 5 |
| 1.4. Organization of the paper | 7 |
| 1.5. Acknowledgements | 8 |
| 2. Spans and linearization | 9 |
| 2.1. Homotopy cardinality and finite path integrals | 9 |
| 2.2. Linearizing spans of spaces | 11 |
| 2.3. Linearizing π -finite categories | 14 |
| 2.4. Linearizing locally π -finite categories | 17 |
| 2.5. The Pontryagin pairing | 23 |
| 3. Bordism Groups and Invertible Topological Field Theories | 24 |

| | |
|--|----|
| 3.1. Stable structures and their normal and tangential bordism groups | 24 |
| 3.2. Unstable tangential structures, bordism categories and topological field theories | 25 |
| 3.3. Invertible topological field theories | 26 |
| 3.4. Computing unstable bordism groups | 29 |
| 3.5. Tangential n -types | 32 |
| 4. Dijkgraaf-Witten field theories and their topological sensitivity | 33 |
| 4.1. Dijkgraaf-Witten super topological field theories | 33 |
| 4.2. The Dijkgraaf-Witten invariant via Pontryagin pairings | 35 |
| 4.3. Dijkgraaf-Witten theories of type n | 35 |
| 4.4. The main theorem | 37 |
| 4.5. Dimensional reduction of Dijkgraaf-Witten theories | 38 |
| 5. Semisimple field theories and stable diffeomorphism | 40 |
| 5.1. Connected sum of B -manifolds and stable diffeomorphism | 41 |
| 5.2. Local operators control decompositions of topological field theories | 42 |
| 5.3. Semisimple topological field theories | 44 |
| 5.4. An eigenvalue equation in the bordism category | 45 |
| 5.5. Semisimple field theories lead to stable diffeomorphism invariants | 48 |
| 5.6. Dijkgraaf-Witten theories are semisimple | 50 |
| 6. Stable diffeomorphism and the Main Theorem | 53 |
| 6.1. The relation to stable diffeomorphism | 53 |
| 6.2. Examples and Applications | 55 |
| Appendix A. n -finitely dominated spaces | 57 |
| References | 60 |

2. SPANS AND LINEARIZATION

2.1. Homotopy cardinality and finite path integrals. Recall that a space X is π -finite if it has finitely many components, and for each choice of basepoint $x \in X$, $\pi_i(X, x)$ is finite and non-trivial for only finitely many i . For a π -finite space and a base point $x \in X$, define the *homotopy cardinality of X at x* to be (cf. [Qui95, Lecture 4])

$$(5) \quad \#(X, x) := \prod_{i \geq 1} |\pi_i(X, x)|^{(-1)^i} = |\pi_1(X, x)|^{-1} |\pi_2(X, x)| |\pi_3(X, x)|^{-1} \cdots \in \mathbb{Q}.$$

Evidently, this rational number only depends on the component $[x] \in \pi_0 X$ of x . Define the *total homotopy cardinality* of X to be $\#^{\text{tot}} X := \sum_{x \in \pi_0 X} \#(X, x)$. A key property of total homotopy cardinality is its compatibility with fiber sequences: If $X \rightarrow B$ is a map of spaces with connected π -finite base B and π -finite fiber F , it follows from the associated long exact sequence of homotopy groups that X is also π -finite and that

$$(6) \quad \#^{\text{tot}} X = (\#^{\text{tot}} B)(\#^{\text{tot}} F).$$

Homotopy cardinality can be used to develop a theory of ‘finite path integrals’.

Definition 2.1. If V is a rational vector space, X is a π -finite space and $\alpha : \pi_0 X \rightarrow V$ is a function, the *integral* of α over X is the vector

$$(7) \quad \int_X \alpha := \sum_{x \in \pi_0 X} \#(X, x) \alpha(x) \in V.$$

We will also utilize the notation $\int_{x \in X} \alpha(x)$, expressing α as a formula in $\pi_0 X$.

Most importantly, and most evidently, this integral is linear. Given a linear function $\phi : V \rightarrow W$ and a function $\alpha : \pi_0 X \rightarrow V$ where X is π -finite, the following elements of W agree:

$$(8) \quad \phi\left(\int_X \alpha\right) = \int_X \phi \circ \alpha$$

Equation (6) is an instance of the following ‘generalized Fubini theorem’.

Proposition 2.2 (Fubini theorem). *Let $s : X \rightarrow A$ be a map of π -finite spaces and let $\alpha : \pi_0 X \rightarrow V$ be a function to a rational vector space V . Then, the fiber $\iota_a : X_a \rightarrow X$ of $X \rightarrow A$ at a point $a \in A$ is π -finite, and the following elements of V agree:*

$$\int_{x \in X} \alpha(x) = \int_{a \in A} \int_{f \in X_a} \alpha(\iota_a f)$$

Proof. It suffices to show that for every $[x] \in \pi_0 X$, the following equation holds in \mathbb{Q} :

$$(9) \quad \#(X, x) = \sum_{[a] \in \pi_0 A} \#(A, a) \sum_{[f] \in (\pi_0 \iota_a)^{-1}([x]) \subseteq \pi_0(X_a)} \#(X_a, f)$$

Proposition 2.2 follows from summing the product of this rational number (9) and the vector $\alpha([x]) \in V$ over $[x] \in \pi_0 X$. To show (9), note that for any $[a] \neq \pi_0 s([x])$ the preimage $(\pi_0 \iota_a)^{-1}([x])$ is empty, and hence the summand corresponding to such an $[a] \in \pi_0 A$ is zero. Therefore, equation (9) is equivalent to

$$(10) \quad \#(X, x) = \#(A, s([x])) \sum_{[f] \in (\pi_0 \iota_{s([x])})^{-1}([x])} \#(X_{s([x])}, f).$$

Let $X|_{[x]}$ denote the connected component of $[x] \in \pi_0 X$ and note that $s|_{[x]} : X|_{[x]} \rightarrow A|_{s([x])}$ has fiber $\sqcup_{[f] \in (\pi_0 \iota_{s([x])})^{-1}([x]) \subseteq \pi_0(X_{s([x])})} X_{s([x])}|_{[f]}$. Equation (10) follows from applying (6) to $s|_{[x]}$ and noting that the homotopy cardinality of a disjoint union of spaces is the sum of their respective homotopy cardinalities. \square

This generalized Fubini theorem indeed implies the more familiar statement that for a function $\alpha : \pi_0(X \times Y) \rightarrow V$ out of a product of π -finite spaces

$$\int_{(a,b) \in A \times B} \alpha(a, b) = \int_{a \in A} \int_{b \in B} \alpha(a, b).$$

This follows from applying Proposition 2.2 to the case where $s : X \rightarrow A$ is the projection $A \times B \rightarrow A$.

Another important property of this integral is its compatibility with group actions.

Lemma 2.3. *Let X be a π -finite space and let G be a (discrete) group acting on $\pi_0 X$ so that for all $[x] \in \pi_0 X$ and $g \in G$ the connected components $[x]$ and $g[x]$ of X are homotopy equivalent spaces. Then, for a rational vector space V with a G action and a G -equivariant function $\alpha : \pi_0 X \rightarrow V$, the integral $\int_X \alpha \in V$ is a G -fixed point in V .*

Proof. By assumption, $\#(X, [x]) = \#(X, g[x])$ for all components $[x] \in \pi_0 X$ and $g \in G$. The lemma then follows from reindexing the sum in Definition 2.1. \square

Of course, the main application of Lemma 2.3 is when there is a topological (or homotopy coherent) group \mathbb{G} with $\pi_0 \mathbb{G} = G$ and when the action of G on $\pi_0 X$ is induced by an action

of \mathbb{G} on the space X . In this case, the assumption that all components in a given G -orbit are homotopy equivalent is automatically satisfied.

In the next section, we unify these and further properties into Corollary 2.13.

2.2. Linearizing spans of spaces. For later applications to super field theories, we now generalize finite path integrals from rational vector spaces to objects in more general categories (such as the category of super vector spaces). For the remainder of this section, we therefore let \mathcal{V} be a 1-category which is enriched in rational vector spaces and which has small colimits. Most of the ideas and observations in this Section 2.2 have appeared before [Har20] and are only included here for the sake of completeness.

A (\mathcal{V} -valued) *local system* on a space A is a functor $\mathcal{L} : A \rightarrow \mathcal{V}$ and a map of local systems $\alpha : \mathcal{L} \rightarrow \mathcal{P}$ is a natural transformation, where A is viewed as an ∞ -groupoid. As \mathcal{V} is an ordinary 1-category, a local system is equivalent to a functor $\mathcal{L} : \pi_{\leq 1}A \rightarrow \mathcal{V}$ out of the fundamental 1-groupoid $\pi_{\leq 1}A$ of A . The *\mathcal{V} -object of sections* of \mathcal{L} is the limit of this functor $\mathcal{L} : \pi_{\leq 1}A \rightarrow \mathcal{V}$, the *\mathcal{V} -object of co-sections* is the colimit. Choosing a basepoint $a \in A$ in every component $[a] \in \pi_0A$, a local system therefore explicitly amounts to an object $\mathcal{L}(a) \in \mathcal{V}$ for every $[a] \in \pi_0A$ equipped with an action of $\pi_1(A, a)$. A map of local systems $\mathcal{L} \Rightarrow \mathcal{P}$ amounts to a choice of $\pi_1(A, a)$ intertwiner $\alpha_a : \mathcal{L}(a) \rightarrow \mathcal{P}(a)$ for every $[a] \in \pi_0A$. These choices of basepoints identify the \mathcal{V} -objects of sections of \mathcal{L} with the invariants $\lim \mathcal{L} \cong \prod_{[a] \in \pi_0A} \mathcal{L}_A(a)^{\pi_1(A, a)}$ and the \mathcal{V} -object of co-sections with the coinvariants $\operatorname{colim} \mathcal{L} \cong \coprod_{[a] \in \pi_0A} \mathcal{L}_A(a)_{\pi_1(A, a)}$.

Definition 2.4. Let (A, \mathcal{L}_A) and (B, \mathcal{L}_B) be spaces equipped with local systems. A *decorated span* (X, α) is a span of spaces $B \xleftarrow{t} X \xrightarrow{s} A$ equipped with a map of local systems $\alpha : s^* \mathcal{L}_A \rightarrow t^* \mathcal{L}_B$.

Two decorated spans (X, α) and (Y, β) between spaces equipped with local systems (A, \mathcal{L}_A) and (B, \mathcal{L}_B) are *isomorphic* if there is a homotopy equivalence $f : X \rightarrow Y$ and homotopies $s_Y \circ f \cong s_X$ and $t_Y \circ f \cong t_X$ which intertwine the natural transformations α and β .

We say that a decorated span (X, α) is *source finite* if $s : X \rightarrow A$ has π -finite (homotopy) fibers, *target finite* if $t : X \rightarrow B$ has π -finite fibers, and *π -finite* if all spaces A, B and X are π -finite.

Proposition 2.5. For a source finite decorated span $(B, \mathcal{L}_B) \leftarrow (X, \alpha) \rightarrow (A, \mathcal{L}_A)$ and a point $a \in A$ consider the following integral in the vector space $\operatorname{Hom}(\mathcal{L}_A(a), \operatorname{colim} \mathcal{L}_B)$:

$$(11) \quad (\Phi_{X, \alpha})_a := \int_{(x, \gamma) \in X_a} \left[\mathcal{L}_A(a) \xrightarrow{\mathcal{L}_A(\gamma)} \mathcal{L}_A(s(x)) \xrightarrow{\alpha_x} \mathcal{L}_B(t(x)) \xrightarrow{i_{t(x)}^B} \operatorname{colim} \mathcal{L}_B \right]$$

Here, X_a denotes the homotopy fiber of $s : X \rightarrow A$ at a , whose points are represented by pairs (x, γ) of a point $x \in X$ and a path $\gamma : a \rightsquigarrow s(x)$ in A , and $i_b^B : \mathcal{L}_B(b) \rightarrow \operatorname{colim} \mathcal{L}_B$ (for $b \in B$) denote the universal morphisms into the colimit.

These maps $(\Phi_{X, \alpha})_a : \mathcal{L}_A(a) \rightarrow \operatorname{colim} \mathcal{L}_B$ assemble into a map

$$\Phi_{X, \alpha} : \operatorname{colim} \mathcal{L}_A \rightarrow \operatorname{colim} \mathcal{L}_B$$

which only depends on the isomorphism class of the decorated span.

Proof. The morphism $(\Phi_{X, \alpha})_a \in \operatorname{Hom}(\mathcal{L}_A(a), \operatorname{colim} \mathcal{L}_B)$ is defined as the integral of the function $\pi_0 X_a \rightarrow \operatorname{Hom}(\mathcal{L}_A(a), \operatorname{colim} \mathcal{L}_B)$ given by $(x, \gamma) \mapsto i_{t(x)}^B \circ \alpha_x \circ \mathcal{L}_A(\gamma)$. This function intertwines the canonical $\pi_1(A, a)$ action on $\pi_0 X_a$ with the action on $\operatorname{Hom}(\mathcal{L}_A(a), \operatorname{colim} \mathcal{L}_B)$ induced by the local systems. Hence, it follows from Lemma 2.3 that this integral is a

$\pi_1(A, a)$ fixed point in $\text{Hom}(\mathcal{L}_A(a), \text{colim } \mathcal{L}_B)$, or equivalently that it defines an element in $\text{Hom}(\mathcal{L}_A(a)_{\pi_1(A, a)}, \text{colim } \mathcal{L}_B)$ and hence that the assignments (11) assemble into a map $\Phi_{X, \alpha} : \text{colim } \mathcal{L}_A \rightarrow \text{colim } \mathcal{L}_B$. \square

Henceforth, we refer to the map $\Phi_{X, \alpha} : \text{colim } \mathcal{L}_A \rightarrow \text{colim } \mathcal{L}_B$ associated to a decorated span as the *linearization* of this span.

Example 2.6. Let X be a π -finite space and consider the span $* \leftarrow X \rightarrow *$ decorated with the trivial local system in $\mathcal{V} = \text{Vec}_{\mathbb{Q}}$. Then, $\Phi_X : \mathbb{Q} \rightarrow \mathbb{Q}$ is given by the total homotopy cardinality $\#^{\text{tot}}(X)$.

Remark 2.7. If \mathcal{V} has limits and $(B, \mathcal{L}_B) \leftarrow (X, \alpha) \rightarrow (A, \mathcal{L}_A)$ is a target finite span (i.e. $t : X \rightarrow B$ has π -finite fibers), then a completely analogous construction leads to a morphism

$$\Phi^{X, \alpha} : \lim \mathcal{L}_A \rightarrow \lim \mathcal{L}_B$$

with component at $b \in B$ given by the following map $\lim \mathcal{L}_A \rightarrow \mathcal{L}_B(b)$:

$$\int_{(x, \gamma : t(x) \rightsquigarrow b) \in X_b} \left[\lim \mathcal{L}_A \rightarrow \mathcal{L}_A(s(x)) \xrightarrow{\alpha_x} \mathcal{L}_B(t(x)) \xrightarrow{\mathcal{L}_B(\gamma)} \mathcal{L}_B(b) \right]$$

Here, X_b denotes the fiber of $t : X \rightarrow B$ at $b \in B$. In fact, for any π -finite space X and rational vector space V , the integration function $\int_X : \text{Func}(\pi_0 X, V) \rightarrow V$ itself arises as the linearization Φ^X of the span $(*, \text{const}_V) \leftarrow (X, \text{id}) \rightarrow (X, \text{const}_V)$ where const_V denotes the constant local system at V with limit $\lim(\text{const}_V : X \rightarrow \text{Vec}_{\mathbb{Q}}) = \text{Func}(\pi_0 X, V)$.

Remark 2.8. For a π -finite space A , the *norm map* $\text{Nm}_A : \text{colim } \mathcal{L}_A \rightarrow \lim \mathcal{L}_A$ is defined as the map with coefficient $\mathcal{L}_A(a) \rightarrow \mathcal{L}_A(a')$ given by

$$\int_{\gamma \in \text{Path}_A(a, a')} \left[\mathcal{L}_A(a) \xrightarrow{\mathcal{L}_A(\gamma)} \mathcal{L}_A(a') \right].$$

In the category $\mathcal{V} = \text{Vec}_k$ of vector spaces over a field k of characteristic zero, and for connected A with a choice of basepoint $a \in A$, the norm map therefore explicitly unpacks to a map $\mathcal{L}_A(a)_{\pi_1(A, a)} \rightarrow \mathcal{L}_A(a)_{\pi_1(A, a)}$ which sends a coinvariant $[v]$ to the invariant

$$(12) \quad \#(A, a)^{-1} \frac{1}{|\pi_1(A, a)|} \sum_{\gamma \in \pi_1(A, a)} \mathcal{L}(\gamma)v.$$

Here, we used that for any loop $\gamma \in \Omega_a A$,

$$\#(\Omega_a A, \gamma) = \#(\Omega_a A, \text{id}) = |\pi_2(A, a)|^{-1} |\pi_3(A, a)| \cdots = \#(A, a)^{-1} |\pi_1(A, a)|^{-1}.$$

Since \mathcal{V} is $\text{Vec}_{\mathbb{Q}}$ -enriched, the norm map $\text{Nm}_A : \text{colim } \mathcal{L}_A \rightarrow \lim \mathcal{L}_A$ is invertible. If A , B and X are π -finite, the linearizations $\Phi_{X, \alpha} : \text{colim } \mathcal{L}_A \rightarrow \text{colim } \mathcal{L}_B$ and $\Phi^{X, \alpha} : \lim \mathcal{L}_A \rightarrow \lim \mathcal{L}_B$ are identified with one another under these norm isomorphisms Nm_A and Nm_B .

In fact, for this last π -finite case, it is sufficient to assume that the $\text{Vec}_{\mathbb{Q}}$ -enriched category \mathcal{V} is additive and idempotent complete, as all necessary limits and colimits can be computed in these terms.

Given a pair of decorated spans

$$(13) \quad \begin{array}{ccccc} & & (Y, \beta) & & (X, \alpha) \\ & t_Y \swarrow & & s_Y \swarrow & t_X \swarrow & & s_X \swarrow \\ (C, \mathcal{L}_C) & & & (B, \mathcal{L}_B) & & & (A, \mathcal{L}_A) \end{array}$$

their *pullback composite* is the span $C \xleftarrow{t_Y \circ p_Y} Y \times_B X \xrightarrow{s_X \circ p_X} A$, where $p_X : Y \times_B X \rightarrow X$ and $p_Y : Y \times_B X \rightarrow Y$ denote the projections, decorated with the local systems \mathcal{L}_C on C , \mathcal{L}_A on A and transformation $\beta \circ \alpha : p_X^* s_X^* \mathcal{L}_A \Rightarrow p_Y^* t_Y^* \mathcal{L}_C$ with component $(\beta \circ \alpha)_{(x,y,\gamma)}$ at a point $(y \in Y, x \in X, \gamma : t_X(x) \rightsquigarrow s_Y(y)) \in Y \times_B X$ given by the composite

$$\mathcal{L}_A(s_X(x)) \xrightarrow{\alpha_x} \mathcal{L}_B(t_X(x)) \xrightarrow{\mathcal{L}_B(\gamma)} \mathcal{L}_B(s_Y(y)) \xrightarrow{\beta_y} \mathcal{L}_C(t_Y(y)).$$

Theorem 2.9. *Given a pair of decorated spans as in (13) which are source finite (i.e. $s_Y : Y \rightarrow B$ and $s_X : X \rightarrow A$ have π -finite fibers). Then, their pullback composite $(C, \mathcal{L}_C) \leftarrow (Y \times_B X, \beta \circ \alpha) \rightarrow (A, \mathcal{L}_A)$ is again source finite and their linearizations compose:*

$$(14) \quad \Phi_{Y \times_B X, \beta \circ \alpha} = \Phi_{Y, \beta} \circ \Phi_{X, \alpha} : \text{colim } \mathcal{L}_A \rightarrow \text{colim } \mathcal{L}_C$$

Proof. The fiber of $Y \times_B X \rightarrow X$ at a point $x \in X$ is identified with the fiber of $Y \rightarrow B$ at the point $t_X(x) \in B$. Since $Y \rightarrow B$ has π -finite fibers, it therefore follows that $Y \times_B X \rightarrow X$ has π -finite fibers. The source map of the composite span is therefore a composite $Y \times_B X \rightarrow X \rightarrow A$ of maps with π -finite fibers, and hence has itself π -finite fibers.

A point in $Y \times_B X$ is represented by a triple (y, x, μ) consisting of a point $y \in Y$, a point $x \in X$ and a path $\mu : t_X(x) \rightsquigarrow s_Y(y)$ in B . Hence, a point in the fiber $(Y \times_B X)_a$ of $Y \times_B X \rightarrow X \rightarrow A$ at $a \in A$ is a quadruple (y, x, μ, γ) of a point $(y, x, \mu) \in Y \times_B X$ as above and a path $\gamma : a \rightarrow s_X(x)$. Using this notation, the map $(\Phi_{Y \times_B X, \beta \circ \alpha})_a \in \text{Hom}(\mathcal{L}_A(a), \text{colim } \mathcal{L}_C)$ is defined as the following integral (see (11)):

$$(15) \quad \int_{(y,x,\mu:t_X(x)\rightsquigarrow s_Y(y),\gamma:a\rightsquigarrow s_X(x)) \in (Y \times_B X)_a} \iota_{t_Y(y)}^C \circ \beta_y \circ \mathcal{L}_B(\mu) \circ \alpha_x \circ \mathcal{L}_A(\gamma)$$

$$= \int_{(x,\gamma:a\rightsquigarrow s_X(x)) \in X_a} \left(\int_{(y,\mu:t_X(x)\rightsquigarrow s_Y(y)) \in Y_{t_X(x)}} \iota_{t_Y(y)}^C \circ \beta_y \circ \mathcal{L}_B(\mu) \right) \circ \alpha_x \circ \mathcal{L}_A(\gamma)$$

$$(16) \quad = \int_{(x,\gamma:a\rightsquigarrow s_X(x)) \in X_a} (\Phi_{Y,\beta})_{t_X(x)} \circ \alpha_x \circ \mathcal{L}_A(\gamma)$$

$$(17) \quad = \Phi_{Y,\beta} \circ \left(\int_{(x,\gamma:a\rightsquigarrow s_X(x)) \in X_a} \iota_{t_X(x)}^B \circ \alpha_x \circ \mathcal{L}_A(\gamma) \right) = \Phi_{Y,\beta} \circ (\Phi_{X,\alpha})_a$$

In equation (15), we identified the fiber of $(Y \times_B X)_a \rightarrow X_a$ at (x, γ) with $Y_{t_X(x)}$ and used Proposition 2.2 to split the integral. Bilinearity of composition allows to interchange the integral and composition (see (8)). Equation (16) uses the definition (11) of $(\Phi_{Y,\beta})_{t_X(x)}$, and equation (17) uses the defining property $\Phi_{Y,\beta} \circ \iota_b^B = (\Phi_{Y,\beta})_b$ for $b \in B$ and again bilinearity of composition. \square

Example 2.10. Theorem 2.9 may be understood as a direct generalization of the compatibility of homotopy cardinality with fiber sequences. Indeed, let $X \rightarrow B$ be a map of spaces with π -finite fiber and connected pointed base B . Applying Theorem 2.9 to the sequence of spans $* \leftarrow * \rightarrow B \leftarrow X \rightarrow *$ with composite span $* \leftarrow F \rightarrow *$ (all decorated with the trivial local system in $\text{Vec}_{\mathbb{Q}}$) recovers (6).

Theorem 2.9 suggests to think of linearization Φ as a functor out of a category of decorated source finite spans.

Definition 2.11. We denote the 1-category of spaces and isomorphism classes of spans by $\text{Span}(\mathcal{S})$. Similarly, we denote the 1-category of spaces equipped with \mathcal{V} -valued local systems and isomorphism classes of decorated spans by $\text{Span}(\mathcal{S}, \mathcal{V})$. Any symmetric monoidal

structure on \mathcal{V} induces a symmetric monoidal structure on $\text{Span}(\mathcal{S}, \mathcal{V})$ given on objects by the cartesian product $A \times B$ equipped with the local system $\mathcal{L}_A \otimes \mathcal{L}_B : A \times B \rightarrow \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$.

We also denote the (symmetric monoidal) subcategory of source finite and target finite decorated spans (see Definition 2.4) by $\text{Span}^{s.f.}(\mathcal{S}, \mathcal{V})$ and $\text{Span}^{t.f.}(\mathcal{S}, \mathcal{V})$, respectively, and denote the further subcategory on π -finite spaces and π -finite decorated spans by $\text{Span}(\mathcal{S}^\pi, \mathcal{V})$.

Remark 2.12. 2-categories of spans in general categories were considered as early as [B67]. The importance of spans of spaces with local systems to topological field theories appears in [Lur09b, FHLT10]. Higher ∞ -categories of spans were constructed by [Bar13], and generalized to (∞, n) -categories of iterated spans with local systems in [Hau18]. Spans of groupoids with local systems were also considered in [Mor15].

Corollary 2.13. *If \mathcal{V} admits small colimits and is enriched in rational vector spaces, the linearizations $\Phi_{X, \mathcal{V}} : \text{colim } \mathcal{L}_A \rightarrow \text{colim } \mathcal{L}_B$ from Proposition 2.5 assemble into a functor*

$$\Phi : \text{Span}^{s.f.}(\mathcal{S}, \mathcal{V}) \rightarrow \mathcal{V}.$$

If \mathcal{V} has a symmetric monoidal structure which distributes over colimits, then this functor is symmetric monoidal.

Proof. Functoriality follows from Theorem 2.9 and the fact that the identity span on (A, \mathcal{L}_A) linearizes to the identity map on $\text{colim } \mathcal{L}_A$. Symmetric monoidality may ultimately be reduced to the following property of \int : If X and Y are π -finite spaces, and $\alpha : \pi_0 X \rightarrow V$ and $\beta : \pi_0 Y \rightarrow W$ are functions into rational vector spaces, then the following vectors in $V \otimes W$ agree:

$$\int_{X \times Y} \alpha \otimes \beta = \left(\int_X \alpha \right) \otimes \left(\int_Y \beta \right) \quad \square$$

Remark 2.14. The importance of the linearization functor to topological field theories was made clear in [FHLT10]. A construction of this linearization for spans of 1-groupoids with cocycles appears in [Mor15], and for spans of 1-groupoids with more general local systems in [Tro16, SW19]. Subsequently, the theory of *ambidexterity* has provided a far reaching generalization, see [HL13, HL14, CSY22] and especially [Har20].

Remark 2.15. As in Remark 2.7, there is a limit/target-finite variant of Corollary 2.13. If \mathcal{V} has limits (and a symmetric monoidal structure which distributes over limits), then the linearization $\Phi^{X, \alpha} : \lim \mathcal{L}_A \rightarrow \lim \mathcal{L}_B$ induces a (symmetric monoidal) functor $\text{Span}^{t.f.}(\mathcal{S}, \mathcal{V}) \rightarrow \mathcal{V}$. If one further restricts to π -finite spaces as in Remark 2.8, then the norm map isomorphisms induce a monoidal natural equivalence between the colimit and limit variant of the functor $\text{Span}(\mathcal{S}^\pi, \mathcal{V}) \rightarrow \mathcal{V}$.

2.3. Linearizing π -finite categories. For the remainder of this section, we restrict attention to the case $\mathcal{V} = \text{Vect}_k$ where k is a field of characteristic zero.

A rich source of decorated spans comes from π -finite categories. Let \mathcal{C} be an ∞ -category (which for our purposes can be taken to be a category enriched in spaces). We let $\mathcal{C}_0 = \text{Map}([0], \mathcal{C})$ and $\mathcal{C}_1 = \text{Map}([1], \mathcal{C})$ be the *moduli space of objects of \mathcal{C}* , respectively the *moduli space of arrows of \mathcal{C}* . Here $[0] = pt$ is the terminal category and $[1]$ is the ‘free-walking-arrow’.

The source and target map lead to a span of spaces associated to the ∞ -category \mathcal{C} :

$$\mathcal{C}_0 \xleftarrow{t} \mathcal{C}_1 \xrightarrow{s} \mathcal{C}_0.$$

Any functor $F : \mathcal{C} \rightarrow \text{Vect}_k$ restricts to a (Vect_k -valued) local system \mathcal{L}_F on \mathcal{C}_0 , and a map of local systems $\alpha_F : s^* \mathcal{L}_F \rightarrow t^* \mathcal{L}_F$ on \mathcal{C}_1 . Thus, any such functor induces a decorated span

$$(18) \quad (\mathcal{C}_0, \mathcal{L}_F) \xleftarrow{t} (\mathcal{C}_1, \alpha_F) \xrightarrow{s} (\mathcal{C}_0, \mathcal{L}_F).$$

We will also denote that span by (\mathcal{C}, F) or just \mathcal{C} when there is no confusion.

We will say that an ∞ -category \mathcal{C} is π -finite if it has finitely many isomorphism classes of objects and if for all objects $a, b \in \mathcal{C}$ the morphism spaces $\mathcal{C}(a, b)$ are π -finite. Equivalently, it is π -finite if both \mathcal{C}_0 and \mathcal{C}_1 are π -finite spaces.

Definition 2.16. Let \mathcal{C} be a π -finite ∞ -category and let $F : \mathcal{C} \rightarrow \text{Vec}_k$ be a functor into the category of vector spaces over a field k of characteristic zero. The *linearization* of (\mathcal{C}, F) is the linear map

$$\Phi_{\mathcal{C}, F} : \text{colim } \mathcal{L}_F \rightarrow \text{colim } \mathcal{L}_F$$

associated to the decorated span (\mathcal{C}, F) .

In the following, we say that an ∞ -category has the property that *endomorphisms are invertible* if any morphism $f : a \rightarrow b$ with isomorphic source and target is itself invertible.

Theorem 2.17 (Möbius inversion). *Let \mathcal{C} be a π -finite ∞ -category in which endomorphisms are invertible. Then, for any functor $F : \mathcal{C} \rightarrow \text{Vec}_k$, the linearization $\Phi_{\mathcal{C}, F}$ is invertible.*

Proof. Since \mathcal{C} is an ∞ -category (and in particular $\mathcal{C}_0, \mathcal{C}_1$ are the spaces of 0- and 1-simplices in a complete Segal space [Rez01]), the unit map $c_0 : \mathcal{C}_0 \rightarrow \mathcal{C}_1$ is an inclusion of components (i.e. an injection on π_0 and a bijection on higher homotopy groups) [Rez01, Sections 5 and 6]. Let $\mathcal{C}_1^<$ denote the complement of $c_0 : \mathcal{C}_0 \hookrightarrow \mathcal{C}_1$, so that $\mathcal{C}_1 \cong \mathcal{C}_0 \sqcup \mathcal{C}_1^<$. Explicitly $\mathcal{C}_1^<$ is the subspace of \mathcal{C}_1 of those morphisms $f : a \rightarrow b$ which are not invertible. More generally, let

$$(19) \quad \mathcal{C}_n^< := \mathcal{C}_1^< \times_{\mathcal{C}_0} \mathcal{C}_1^< \times_{\mathcal{C}_0} \cdots \times_{\mathcal{C}_0} \mathcal{C}_1^<$$

be the space of n -chains $a_0 \xrightarrow{f_{0,1}} a_1 \rightarrow \cdots \xrightarrow{f_{n-1,n}} a_n$ for which no $f_{i,i+1}$ is invertible. There are maps $t, s : \mathcal{C}_n^< \rightarrow \mathcal{C}_0$ which send a chain $a_0 \rightarrow a_1 \rightarrow \cdots \rightarrow a_n$ to a_n (target) and a_0 (source), yielding spans $\mathcal{C}_0 \xleftarrow{t} \mathcal{C}_n^< \xrightarrow{s} \mathcal{C}_0$. Abusing notation, we will henceforth simply denote these spans by $\mathcal{C}_n^<$.

As in (18), let $(\mathcal{C}_1, \alpha_F)$ denote the decoration of the span $\mathcal{C}_0 \xleftarrow{t} \mathcal{C}_1 \xrightarrow{s} \mathcal{C}_0$ induced by the functor $F : \mathcal{C} \rightarrow \text{Vec}_k$. Let $\alpha_F^<$ denote the restriction of α_F to the subspace $\mathcal{C}_1^< \hookrightarrow \mathcal{C}_1$. Unitality of F implies that the decomposition of spans $\mathcal{C}_1 \cong \mathcal{C}_0 \sqcup \mathcal{C}_1^<$ is compatible with decorations:

$$(\mathcal{C}_1, \alpha_F) \cong (\mathcal{C}_0, \text{id}) \sqcup (\mathcal{C}_1^<, \alpha_F^<)$$

Composition of morphisms defines a map of spans $c_n : \mathcal{C}_n^< \rightarrow \mathcal{C}_1$ (generalizing the inclusions $c_0 : \mathcal{C}_0 \hookrightarrow \mathcal{C}_1$ and $c_1 : \mathcal{C}_1^< \hookrightarrow \mathcal{C}_1$). Functoriality of F implies that the pulled-back decoration $c_n^* \alpha_F$ on $\mathcal{C}_n^<$ is compatible with the defining decomposition (19) of $\mathcal{C}_n^<$:

$$(20) \quad (\mathcal{C}_n^<, c_n^* \alpha_F) \cong (\mathcal{C}_1^<, \alpha_F^<) \times_{(\mathcal{C}_0, \mathcal{L}_F)} \cdots \times_{(\mathcal{C}_0, \mathcal{L}_F)} (\mathcal{C}_1^<, \alpha_F^<)$$

We can amalgamate the decorated spans $(\mathcal{C}_n^<, c_n^* \alpha_F)$ into a single decorated span $(\mathcal{C}_0, \mathcal{L}_F) \xleftarrow{t} (M, \mu) \xrightarrow{s} (\mathcal{C}_0, \mathcal{L}_F)$ with

$$(M, \mu) := \coprod_{n \geq 0} (\mathcal{C}_n^<, (-1)^n c_n^* \alpha_F) \cong (\mathcal{C}_0, \text{id}) \sqcup (\mathcal{C}_1^<, -\alpha_F) \sqcup (\mathcal{C}_2^<, \alpha_F \circ \alpha_F) \sqcup \cdots$$

Since endomorphisms in \mathcal{C} are invertible, for $[x], [y] \in \pi_0 \mathcal{C}_0$ the relation $[x] \leq [y]$ if there exists a morphism $x \rightarrow y$ in \mathcal{C} defines a poset structure on the set $\pi_0 \mathcal{C}_0$ of isomorphism classes of objects of \mathcal{C} . In particular, it follows that if a composite $g_1 \circ g_2 \circ \cdots \circ g_n$ of morphisms is invertible, then every single morphism g_i has isomorphic source and target and hence is also invertible. In particular, for any point in $\mathcal{C}_n^<$ represented by a chain $a_0 \xrightarrow{f_{0,1}} a_1 \rightarrow \cdots \xrightarrow{f_{n-1,n}} a_n$ of non-invertible morphisms, it follows that also the composite of any subchain

is non-invertible. Hence, if $n > |\pi_0 \mathcal{C}_0|$, the space $\mathcal{C}_n^<$ is empty. In particular, the space M is π -finite and the span (M, μ) can be linearized to a map $\Phi_{(M, \mu)} : \text{colim } \mathcal{L}_F \rightarrow \text{colim } \mathcal{L}_F$.

Using the decomposition of spans $(\mathcal{C}_1, \alpha_F) \cong (\mathcal{C}_0, \text{id}) \sqcup (\mathcal{C}_1^<, \alpha_F^<)$, it follows from (20) that

$$(\mathcal{C}_1, \alpha_F) \times_{(\mathcal{C}_0, \mathcal{L}_F)} (\mathcal{C}_n^<, c_n^* \alpha_F) \cong (\mathcal{C}_n^<, c_n^* \alpha_F) \sqcup (\mathcal{C}_{n+1}^<, c_{n+1}^* \alpha_F).$$

Since $\Phi_{(M, \mu)} = \sum_{n \geq 0} (-1)^n \Phi_{(\mathcal{C}_n^<, c_n^* \alpha_F)}$ (with only finitely many non-zero terms in this sum), the linearization $\Phi_{(M, \mu)}$ is inverse to the linearization $\Phi_{(\mathcal{C}_1, \alpha_F)}$. \square

Remark 2.18. Theorem 2.17 is closely related to the theory of decomposition spaces and Möbius inversion therein [GCKT18a, GCKT18b, GCKT18c]. Indeed, similar to [GCKT18b], the assumptions of Theorem 2.17 can be weakened: Instead of assuming that endomorphisms are invertible, it would have been sufficient to assume π -finiteness of \mathcal{C} and that every morphism $f : a \rightarrow b$ has *finite length*, in the sense that there is a finite upper bound on the length of a chain of non-invertible morphisms into which it can be decomposed. In the proof of Theorem 2.17, π -finiteness of \mathcal{C} together with this finite length condition is enough to ensure the eventual vanishing of $\mathcal{C}_n^<$. In fact, similar to [GCKT18b], we expect that it is not even necessary to assume that \mathcal{C} is an ∞ -category — a version of Theorem 2.17 should still apply to mere ‘decomposition spaces’ [GCKT18a, Definition 3.1] (also known as 2-Segal space [DK19]). However, the π -finite categories for which we show invertibility of $\Phi_{\mathcal{C}, F}$ in Corollary 2.24 do not in general fulfill this finite length condition and hence go beyond the categories considered in [GCKT18b]. See Remark 2.25.

Definition 2.19. [cf. [Lur09a, Prop. 5.2.8.17]] Let \mathcal{C} be an ∞ -category, and let $(\mathcal{L}, \mathcal{R})$ be a pair of subcategories, each of which contains all the isomorphisms of \mathcal{C} . The composition of morphism induces a natural map from the composite of the spans $\mathcal{C}_0 \xleftarrow{t} \mathcal{R}_1 \xrightarrow{s} \mathcal{C}_0$ and $\mathcal{C}_0 \xleftarrow{t} \mathcal{L}_1 \xrightarrow{s} \mathcal{C}_0$ to the span $\mathcal{C}_0 \xleftarrow{t} \mathcal{C}_1 \xrightarrow{s} \mathcal{C}_0$. The pair $(\mathcal{L}, \mathcal{R})$ is an *orthogonal factorization system* if this map is an equivalence of spans. We will often abuse notation and let \mathcal{L} and \mathcal{R} denote the corresponding spans, as well as the subcategories.

Remark 2.20. On homotopy fibers this condition becomes

$$\mathcal{C}(a, c) \simeq \coprod_{[b] \in \pi_0 \mathcal{C}_0} \mathcal{R}(b, c) \times_{\text{Aut}_{\mathcal{C}}(b)} \mathcal{L}(a, b).$$

In particular every morphism can be canonically factored into the composite of a morphism in \mathcal{L} followed by a morphism in \mathcal{R} .

Definition 2.21. A *nested factorization system* is a collection $(\mathcal{L}^{(k)}, \mathcal{R}^{(k)})$ of orthogonal factorization systems for $1 \leq k \leq n$ such that $\mathcal{R}^{(k-1)} \subseteq \mathcal{R}^{(k)}$ for all k (and equivalently, $\mathcal{L}^{(k+1)} \subseteq \mathcal{L}^{(k)}$). Given a nested factorization system, we let $\mathcal{T}^{(\ell)}$ denote the following subcategories for $0 \leq \ell \leq n$:

$$\mathcal{T}^{(\ell)} = \begin{cases} \mathcal{R}^{(1)} & \text{if } \ell = 0 \\ \mathcal{R}^{(\ell+1)} \cap \mathcal{L}^{(\ell)} & \text{if } 1 \leq \ell \leq n-1 \\ \mathcal{L}^{(n)} & \text{if } \ell = n \end{cases}$$

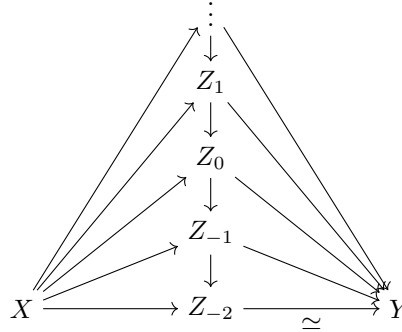
Remark 2.22. Each subcategory $\mathcal{T}^{(\ell)}$ gives a span $\mathcal{C}_0 \leftarrow \mathcal{T}^{(\ell)} \rightarrow \mathcal{C}_0$. For each $k = 1, \dots, n$ we have the following compositions of spans

$$\begin{aligned} \mathcal{R}^{(k)} &= \mathcal{T}^{(0)} \times_{\mathcal{C}_0} \mathcal{T}^{(1)} \times_{\mathcal{C}_0} \dots \times_{\mathcal{C}_0} \mathcal{T}^{(k-1)} \\ \mathcal{L}^{(k)} &= \mathcal{T}^{(k)} \times_{\mathcal{C}_0} \mathcal{T}^{(k+1)} \times_{\mathcal{C}_0} \dots \times_{\mathcal{C}_0} \mathcal{T}^{(n)} \\ \mathcal{C}_1 &= \mathcal{T}^{(0)} \times_{\mathcal{C}_0} \mathcal{T}^{(1)} \times_{\mathcal{C}_0} \dots \times_{\mathcal{C}_0} \mathcal{T}^{(n)}. \end{aligned}$$

The nested sequence of orthogonal factorization systems can be recovered from the subcategories $\mathcal{T}^{(\ell)}$, and so they provide an equivalent presentations.

Example 2.23. For $n \geq -2$, a map $f : X \rightarrow Y$ between topological spaces is called n -connected if for all points $x \in X$, $\pi_i(f; x)$ is an isomorphism for $i < n + 1$ and surjective for $i = n + 1$. A map $f : X \rightarrow Y$ is n -truncated if $\pi_i(f; x)$ is injective for $i = n + 1$ and an isomorphism for $i > n + 1$. Equivalently, f is n -truncated if the homotopy fiber at every basepoint in Y is n -truncated (i.e. is an n -type; all homotopy groups π_i for $i > n$ vanish) and it is n -connected if the homotopy fibers are n -connected (i.e. all homotopy groups π_i for $i \leq n$ vanish).

Any map $f : X \rightarrow Y$ of spaces may be factored, in an essentially unique way, into a n -connected map followed by a n -truncated map. Formally, this amounts to the assertion that the ∞ -category of spaces admits an orthogonal factorization system $(\mathcal{L}^{(n)}, \mathcal{R}^{(n)})$ in which the left class $\mathcal{L}^{(n)}$ consists of the $(n - 1)$ -connected maps and the right class $\mathcal{R}^{(n)}$ consists of the $(n - 1)$ -truncated maps, see [Lur09a, Example 5.2.8.16]. This system is nested in the sense that $\mathcal{R}^{(n-1)} \subseteq \mathcal{R}^{(n)}$ and $\mathcal{L}^{(n+1)} \subseteq \mathcal{L}^{(n)}$. Given a map $f : X \rightarrow Y$, for each integer $n \geq -2$ we may factor f as $X \rightarrow Z_n \rightarrow Y$ into an n -connected map followed by an n -truncated map. The fact that these orthogonal factorization systems are nested implies the the spaces $\{Z_n\}$ assemble into a tower:



This is the *Moore-Postnikov tower* of the map $f : X \rightarrow Y$.

Corollary 2.24. *Let \mathcal{C} be a π -finite ∞ -category which admits a nested factorization system $(\mathcal{L}^{(k)}, \mathcal{R}^{(k)})_{1 \leq k \leq n}$ such that for every $0 \leq \ell \leq n$, endomorphism in the category $\mathcal{T}^{(\ell)}$ are invertible. Then, for any functor $F : \mathcal{C} \rightarrow \text{Vec}_k$, the linearization $\Phi_{\mathcal{C}, F}$ is invertible.*

Proof. By definition, the decorated span $\mathcal{C}_0 \leftarrow \mathcal{C}_1 \rightarrow \mathcal{C}_0$ is the composite of decorated spans $\mathcal{T}^{(0)} \times_{\mathcal{C}_0} \dots \times_{\mathcal{C}_0} \mathcal{T}^{(n)}$ (with maps of local systems given by restricting the functor F to the subcategories $\mathcal{T}^{(\ell)}$). Therefore, the corollary follows from Theorems 2.17 and 2.9. \square

Remark 2.25. Let $\mathcal{C} = \text{FinSet}_{\leq N}$ be the category of finite sets with cardinality less than or equal to N . By limiting the cardinality, \mathcal{C} is a π -finite category. This category contains non-trivial idempotents, and thus does not satisfy the *Möbius condition* of [GCKT18b]. Nevertheless, epimorphisms and monomorphisms of finite sets which are endomorphisms are automatically bijections. Thus the single factorization system $(\text{Surj}, \text{Inj})$ satisfies the conditions of Corollary 2.24. Consequently for any functor $F : \text{FinSet}_{\leq N} \rightarrow \text{Vec}_k$, the linearization $\Phi_{\text{FinSet}_{\leq N}, F}$ is invertible.

2.4. Linearizing locally π -finite categories. In this section, we extend the results of Section 2.3 to ∞ -categories with possibly infinitely many isomorphism classes of objects.

An ∞ -category \mathcal{C} is *locally π -finite* if all its morphism spaces $\mathcal{C}(a, b)$ are π -finite for any $a, b \in \mathcal{C}$. For such categories, neither map in the span $\mathcal{C}_0 \leftarrow \mathcal{C}_1 \rightarrow \mathcal{C}_0$ necessarily has π -finite fibers. Hence, the linearization $\Phi_{\mathcal{C}, F}$ of Definition 2.16 is not defined. However, the span $* \leftarrow \mathcal{C}_1 \xrightarrow{(t, s)} \mathcal{C}_0 \times \mathcal{C}_0$ is still source finite (Definition 2.4) since the fiber of $(t, s) : \mathcal{C}_1 \rightarrow \mathcal{C}_0 \times \mathcal{C}_0$ at a point $(b, a) \in \mathcal{C}_0 \times \mathcal{C}_0$ is precisely the morphism space $\mathcal{C}(a, b)$.

Analogous to (18), any functor $F : \mathcal{C} \rightarrow \text{Vec}_k$ induces a decoration of this span

$$(21) \quad (*, k) \leftarrow (\mathcal{C}_1, p_F) \xrightarrow{(t, s)} (\mathcal{C}_0, \mathcal{L}_F^\vee) \times (\mathcal{C}_0, \mathcal{L}_F).$$

As before, $\mathcal{L}_F : \mathcal{C}_0 \rightarrow \text{Vec}_k$ denotes the restriction of F to \mathcal{C}_0 and $\mathcal{L}_F^\vee : \mathcal{C}_0 \rightarrow \text{Vec}_k$ denotes the local system with value at $d \in \mathcal{C}_0$ given by the dual vector space $F(d)^\vee := \text{Vec}_k(F(d), k)$. The transformation $p_F : t^* \mathcal{L}_F^\vee \otimes s^* \mathcal{L}_F \Rightarrow \text{const}_k$ of local systems on \mathcal{C}_1 is given at an $f : a \rightarrow b$ in \mathcal{C}_1 by

$$F(b)^\vee \otimes F(a) \xrightarrow{\text{id}_{F(b)^\vee} \otimes F(f)} F(b)^\vee \otimes F(b) \rightarrow k.$$

Definition 2.26. Let \mathcal{C} be a locally π -finite ∞ -category and let $F : \mathcal{C} \rightarrow \text{Vec}_k$ be a functor into the category of vector spaces over a field k of characteristic zero. The *linear pairing* associated to (\mathcal{C}, F) is the linear map

$$\langle -, - \rangle_{\mathcal{C}, F} : \text{colim } \mathcal{L}_F^\vee \otimes \text{colim } \mathcal{L}_F \rightarrow k$$

obtained from linearizing the above span $(*, k) \leftarrow (\mathcal{C}_1, p_F) \rightarrow (\mathcal{C}_0, \mathcal{L}_F^\vee) \times (\mathcal{C}_0, \mathcal{L}_F)$.

Remark 2.27. Expressing the colimit as a sum over coinvariants, the linear pairing associated to (\mathcal{C}, F) is a linear map

$$\langle -, - \rangle_{\mathcal{C}, F} : \left(\bigoplus_{[d] \in \pi_0 \mathcal{C}_0} (F(d)^\vee)_{\pi_0 \text{Aut}_{\mathcal{C}}(d)} \right) \otimes \left(\bigoplus_{[c] \in \pi_0 \mathcal{C}_0} F(c)_{\pi_0 \text{Aut}_{\mathcal{C}}(c)} \right) \rightarrow k.$$

Explicitly unpacking (11), given $(d \in \mathcal{C}_0, \phi \in F(d)^\vee)$ and $(c \in \mathcal{C}_0, v \in F(c))$, their pairing is given by

$$\langle (d, \phi), (c, v) \rangle_{\mathcal{C}, F} = \sum_{[f] \in \pi_0 \mathcal{C}(c, d)} \#(\mathcal{C}(c, d), f) \phi(F(f)(v)).$$

Note that this formula indeed only depends on the orbits of ϕ and v under the $\text{Aut}_{\mathcal{C}}(d)$ and $\text{Aut}_{\mathcal{C}}(c)$ action, respectively, and the isomorphism classes of c and d .

Example 2.28. As an important special case of Definition 2.26, suppose that \mathcal{C} is locally π -finite and that $F : \mathcal{C} \rightarrow \text{Vec}_k$ is the constant functor at the one-dimensional vector space k . In this case, the pairing $\langle -, - \rangle_{\mathcal{C}, \text{const}_k}$ (henceforth simply denoted by $\langle -, - \rangle_{\mathcal{C}}$) is a linear map

$$k[\pi_0 \mathcal{C}_0] \otimes k[\pi_0 \mathcal{C}_0] \rightarrow k$$

on the free k -vector spaces $k[\pi_0 \mathcal{C}_0]$ on the set $\pi_0 \mathcal{C}_0$ given at a $[c] \in \mathcal{C}_0$ and $[d] \in \mathcal{C}_0$ by

$$\langle [d], [c] \rangle_{\mathcal{C}} = \#^{\text{tot}} \mathcal{C}(c, d).$$

In particular, if \mathcal{C} is a 1-category, then $\langle [d], [c] \rangle_{\mathcal{C}} = |\mathcal{C}(c, d)|$ records the cardinalities of the hom sets. For example, for $\mathcal{C} = \text{FinSet}$ the category of finite sets and after identifying $\pi_0 \mathcal{C} \cong \mathbb{N}_{\geq 0}$, the pairing $\langle -, - \rangle_{\text{FinSet}}$ is represented by the $\mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 0}$ matrix with (n, m) -coefficient given by n^m .

Example 2.29. If \mathcal{C} is a locally π -finite ∞ -groupoid, or equivalently a locally π -finite space X (i.e. a possibly infinite disjoint union of π -finite spaces), equipped with a local system $\mathcal{L} : X \rightarrow \text{Vec}_k$, then it follows from the explicit formula in Remark 2.27 that the pairing

$$\langle -, - \rangle_{X, \mathcal{L}} : \left(\bigoplus_{[x] \in \pi_0 X} (\mathcal{L}(x)^\vee)_{\pi_1(X, x)} \right) \otimes \left(\bigoplus_{[y] \in \pi_0 X} \mathcal{L}(y)_{\pi_1(X, y)} \right) \rightarrow k$$

is *diagonal*, i.e.

$$\left\langle (x \in X, \phi \in \mathcal{L}(x)^\vee), (y \in X, v \in \mathcal{L}(y)) \right\rangle_{X, \mathcal{L}} = 0 \quad \text{if } [x] \neq [y] \in \pi_0 X,$$

and has diagonal entries

$$\left\langle (x, \phi \in \mathcal{L}(x)^\vee), (x, v \in \mathcal{L}(x)) \right\rangle_{X, \mathcal{L}} = \#(X, x)^{-1} \frac{1}{|\pi_1(X, x)|} \sum_{\gamma \in \pi_1(X, x)} \phi(\mathcal{L}(\gamma)(v)).$$

As in (12), we used that for any loop $\gamma \in \Omega_x X$, $\#(\Omega_x X, \gamma) = \#(X, x)^{-1} |\pi_1(X, x)|^{-1}$.

Remark 2.30. If the ∞ -category \mathcal{C} is not locally π -finite, the decorated span (21) is not source finite and hence cannot be linearized. However, its restriction to a pairing between certain full subcategories of \mathcal{C} can still be defined. Namely, if $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$ are full subcategories of \mathcal{C} such that for every $a \in \mathcal{A}$ and $b \in \mathcal{B}$, the mapping space $\mathcal{C}(a, b)$ is π -finite, then the decorated span $(*, k) \leftarrow (\mathcal{C}_1|_{\mathcal{A}, \mathcal{B}}, p_F|_{\mathcal{A}, \mathcal{B}}) \rightarrow (\mathcal{B}_0, \mathcal{L}_F^\vee|_{\mathcal{B}}) \times (\mathcal{A}_0, \mathcal{L}_F|_{\mathcal{A}})$ is source finite and can hence be linearized. Here, $\mathcal{C}_1|_{\mathcal{A}, \mathcal{B}}$ is the pullback

$$\begin{array}{ccc} \mathcal{C}_1|_{\mathcal{A}, \mathcal{B}} & \longrightarrow & \mathcal{C}_1 \\ \downarrow & \lrcorner & \downarrow (t, s) \\ \mathcal{B}_0 \times \mathcal{A}_0 & \longrightarrow & \mathcal{C}_0 \times \mathcal{C}_0 \end{array}$$

(i.e. the space of morphisms of \mathcal{C} whose source is in \mathcal{A} and whose target is in \mathcal{B}) and $\mathcal{L}_F|_{\mathcal{A}}$, $\mathcal{L}_F^\vee|_{\mathcal{B}}$, and $p_F|_{\mathcal{A}, \mathcal{B}}$ are the obvious restrictions of the local systems and map of local systems. As in Remark 2.27, this restricted pairing unpacks to a linear map

$$\langle -, - \rangle_{\mathcal{C}; F}^{\mathcal{A}, \mathcal{B}} : \left(\bigoplus_{[b] \in \pi_0 \mathcal{B}_0} (F(b)^\vee)_{\pi_0 \text{Aut}_{\mathcal{C}}(b)} \right) \otimes \left(\bigoplus_{[a] \in \pi_0 \mathcal{A}_0} F(a)_{\pi_0 \text{Aut}_{\mathcal{C}}(a)} \right) \rightarrow k.$$

explicitly given by

$$(22) \quad \left\langle (b \in \mathcal{B}_0, \phi \in F(b)^\vee), (a \in \mathcal{A}_0, v \in F(a)) \right\rangle_{\mathcal{C}, F}^{\mathcal{A}, \mathcal{B}} = \sum_{[f] \in \pi_0 \mathcal{C}(a, b)} \#(\mathcal{C}(a, b), f) \phi(F(f)(v)).$$

We say that a pairing $\langle -, - \rangle : V \otimes W \rightarrow k$ between k -vector spaces is *left non-degenerate* if the induced map $V \rightarrow \text{Hom}(W, k)$ is injective. We say it is *right non-degenerate* if the induced map $W \rightarrow \text{Hom}(V, k)$ is injective. We say the pairing is *non-degenerate* if it is both left and right non-degenerate. Note that a pairing between finite-dimensional vector spaces is non-degenerate if and only if it is *perfect*, i.e. if the induced map $V \rightarrow \text{Hom}(W, k)$ is an isomorphism. This is not true for infinite-dimensional vector spaces.

Proposition 2.31. *Let X be a locally π -finite space and let $\mathcal{L} : X \rightarrow \text{Vec}_k$ be a local system. Then, $\langle -, - \rangle_{X, \mathcal{L}}$ is non-degenerate.*

Proof. Following Example 2.29, $\langle -, - \rangle_{X, \mathcal{L}}$ is diagonal. Hence, it suffices to show that for every $x \in X$ the pairing $\langle -, - \rangle_x : \mathcal{L}(x)_{\pi_1(X, x)}^\vee \otimes \mathcal{L}(x)_{\pi_1(X, x)} \rightarrow k$ given by

$$(23) \quad \langle f \in \mathcal{L}(x)^\vee, v \in \mathcal{L}(x) \rangle_x := \#(X, x)^{-1} \frac{1}{|\pi_1(X, x)|} \sum_{\gamma \in \pi_1(X, x)} f(\mathcal{L}(\gamma)v)$$

is non-degenerate. Consider the pairings

$$\begin{aligned} (-, -)_x : (\mathcal{L}(x)^\vee)_{\pi_1(X, x)} \otimes \mathcal{L}(x)^{\pi_1(X, x)} &\rightarrow k & f, v &\mapsto f(v) \\ (-, -)^x : (\mathcal{L}(x)^\vee)^{\pi_1(X, x)} \otimes \mathcal{L}(x)_{\pi_1(X, x)} &\rightarrow k & f, v &\mapsto f(v). \end{aligned}$$

The maps $v \mapsto (-, v)_x$ and $f \mapsto (f, -)^x$ are injective. Let $\text{Nm}_{\mathcal{L}, X|_x} : \mathcal{L}(x)_{\pi_1(X, x)} \rightarrow \mathcal{L}(x)^{\pi_1(X, x)}$ denote the norm map (12) of the local system \mathcal{L} restricted to the connected component $X|_x$ of $x \in X$. Comparing the definition of the norm map with (23), it follows that $\langle -, - \rangle_x = (-, \text{Nm}_{\mathcal{L}, X|_x} -)_x$. Since $\text{Nm}_{\mathcal{L}, X|_x}$ is invertible, the map $v \mapsto \langle -, v \rangle_x$ is injective. On the other hand, let $\text{Nm}_{\mathcal{L}^\vee, X|_x} : (\mathcal{L}(x)^\vee)_{\pi_1(X, x)} \rightarrow (\mathcal{L}(x)^\vee)^{\pi_1(X, x)}$ denote the invertible norm map of the local system \mathcal{L}^\vee . It follows that $\langle -, - \rangle_x = (\text{Nm}_{\mathcal{L}^\vee, X|_x} -, -)^x$ and hence that also $f \mapsto \langle f, - \rangle_x$ is injective. \square

The linear pairing $\langle -, - \rangle_{\mathcal{C}, F}$ is closely related to the linearization $\Phi_{\mathcal{C}, F}$ of Definition 2.16.

Lemma 2.32. *Let \mathcal{C} be π -finite with a factorization system $(\mathcal{L}, \mathcal{R})$. Then, regarding \mathcal{L} and \mathcal{R} as categories, we have that for any $F : \mathcal{C} \rightarrow \text{Vec}_k$*

$$\langle -, - \rangle_{\mathcal{C}, F} = \langle -, \Phi_{\mathcal{L}, F}(-) \rangle_{\mathcal{R}, F} = \langle \Phi_{\mathcal{R}^{\text{op}}, F^\vee}(-), - \rangle_{\mathcal{L}, F}.$$

Proof. This follows from functoriality and monoidality of the linearization functor (Corollary 2.13) and the corresponding equation of spans in the category $\text{Span}(\mathcal{S}^\pi, \text{Vec}_k)$. \square

Corollary 2.33. *If \mathcal{C} is π -finite, then $\langle -, - \rangle_{\mathcal{C}, F} = \langle -, \Phi_{\mathcal{C}, F} - \rangle_{\mathcal{C}_0, \mathcal{L}_F}$.* \square

Proof. Apply Lemma 2.32 to the trivial factorization systems $(\mathcal{C}, \mathcal{C}_0)$ in which the left class are all maps in \mathcal{C} and the right class are just the invertible maps. \square

Combining Corollary 2.24 with Proposition 2.31 and Corollary 2.33 immediately leads to the following corollary.

Corollary 2.34. *Let \mathcal{C} be a π -finite ∞ -category equipped with a nested factorization system such that for every $0 \leq \ell \leq n$, endomorphism in the category $\mathcal{T}^{(\ell)}$ are invertible. Then, for any functor $F : \mathcal{C} \rightarrow \text{Vec}_k$, the linear pairing $\langle -, - \rangle_{\mathcal{C}, F}$ is non-degenerate.* \square

In the remainder of this subsection, we generalize Corollary 2.34 to ∞ -categories \mathcal{C} which are merely locally π -finite with possibly infinitely many isomorphism classes of objects. The main ingredient of this generalization will be the ability to restrict factorization systems to π -finite full subcategories.

Definition 2.35. Let \mathcal{C} be a locally π -finite ∞ -category with a factorization system $(\mathcal{L}, \mathcal{R})$. We call a full subcategory \mathcal{B} of \mathcal{C} *factorizable* if the factorization system restricts to \mathcal{B} . Equivalently \mathcal{B} is factorizable if for every morphism $f : a \rightarrow a'$ in \mathcal{B} , the intermediate object c in the $(\mathcal{L}, \mathcal{R})$ -factorization $a \rightarrow c \rightarrow a'$ of f remains in \mathcal{B} .

Lemma 2.36. *Suppose that \mathcal{C} is locally π -finite with a factorization system $(\mathcal{L}, \mathcal{R})$. Then any full π -finite subcategory $\mathcal{A} \subseteq \mathcal{C}$ is contained in a factorizable full π -finite subcategory $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$.*

Proof. We define \mathcal{B} to be the full subcategory of \mathcal{C} on those objects $c \in \mathcal{C}$ for which there exists objects $a, a' \in \mathcal{A}$ and morphisms $l : a \rightarrow c$ and $r : c \rightarrow a'$ with $l \in \mathcal{L}$ and $r \in \mathcal{R}$. Equivalently, \mathcal{B} is the full subcategory on all objects which occur as the middle object of an $(\mathcal{L}, \mathcal{R})$ -factorization of a map between objects of \mathcal{A} . From this second description we see that factorization with respect to $(\mathcal{L}, \mathcal{R})$ provides a map $\text{interm} : (\mathcal{A})_1 \rightarrow (\mathcal{B})_0$ sending a morphism to the intermediate object in its factorization. This is surjective on π_0 , showing that \mathcal{B} has finitely many isomorphism classes of objects.

In fact the category \mathcal{B} is factorizable, as we will now see. Let $b, b' \in \mathcal{B}$ be objects, and $f : b \rightarrow b'$ a map. Let

$$b \xrightarrow{l_1} c \xrightarrow{r_1} b'$$

be the $(\mathcal{L}, \mathcal{R})$ -factorization of f in \mathcal{C} . By assumption there exist objects $a, a' \in \mathcal{A}$, a map $l : a \rightarrow b$ in \mathcal{L} , and a map $r : b' \rightarrow a'$ in \mathcal{R} . Then $l_1 \circ l : a \rightarrow c$ and $r \circ r_1 : c \rightarrow a'$ are the $(\mathcal{L}, \mathcal{R})$ factorization of the map $r \circ f \circ l : a \rightarrow a'$. Thus the object c is contained in \mathcal{B} , showing that \mathcal{B} is factorizable. \square

Lemma 2.37. *Let \mathcal{C} be locally π -finite with a factorization system $(\mathcal{L}, \mathcal{R})$. Suppose that*

1. *the pairing $\langle -, - \rangle_{\mathcal{R}, F}$ is left non-degenerate;*
2. *in the left class \mathcal{L} , endomorphisms are invertible;*

then the pairing $\langle -, - \rangle_{\mathcal{C}, F}$ is left non-degenerate.

Suppose instead that

1. *the pairing $\langle -, - \rangle_{\mathcal{L}, F}$ is right non-degenerate;*
2. *in the right class \mathcal{R} , endomorphisms are invertible;*

then the pairing $\langle -, - \rangle_{\mathcal{C}, F}$ is right non-degenerate.

Proof. We will prove the first half of the lemma (regarding left non-degeneracy). The proof of the second half (right non-degeneracy) is completely analogous. Consider an arbitrary non-zero element

$$u \in \bigoplus_{[d] \in \pi_0 \mathcal{C}_0} (F(d)^\vee)_{\pi_0 \text{Aut}_{\mathcal{C}}(d)}.$$

We wish to show that there is a $v \in \bigoplus_{[c] \in \pi_0 \mathcal{C}_0} (F(c))_{\pi_0 \text{Aut}_{\mathcal{C}}(c)}$ such that $\langle u, v \rangle_{\mathcal{C}, F} \neq 0$.

By assumption the pairing $\langle -, - \rangle_{\mathcal{R}, F}$ is left non-degenerate, and so there exists a $w \in \bigoplus_{[c] \in \pi_0 \mathcal{C}_0} (F(c))_{\pi_0 \text{Aut}_{\mathcal{C}}(c)}$ such that

$$\langle u, w \rangle_{\mathcal{R}, F} \neq 0.$$

The vectors u and w are supported at a finite number of elements of $\pi_0 \mathcal{C}_0$. Since \mathcal{C} is locally π -finite, the full subcategory of \mathcal{C} on these objects is π -finite. By Lemma 2.36 this may be enlarged to a full π -finite factorizable subcategory $\mathcal{B} \subseteq \mathcal{C}$, containing both the supports of u and w . Theorem 2.17 applies to the π -finite category $\mathcal{B} \cap \mathcal{L}$ in which endomorphisms are invertible. Thus $\Phi_{\mathcal{B} \cap \mathcal{L}, F}$ is invertible. Set $v = \Phi_{\mathcal{B} \cap \mathcal{L}, F}^{-1}(w)$. Then, using Lemma 2.32, we have

$$\begin{aligned} \langle u, v \rangle_{\mathcal{C}, F} &= \langle u, v \rangle_{\mathcal{B}, F} \\ &= \langle u, \Phi_{\mathcal{B} \cap \mathcal{L}, F}^{-1}(w) \rangle_{\mathcal{B}, F} \\ &= \langle u, \Phi_{\mathcal{B} \cap \mathcal{L}, F} \Phi_{\mathcal{B} \cap \mathcal{L}, F}^{-1}(w) \rangle_{\mathcal{R} \cap \mathcal{B}, F} \\ &= \langle u, w \rangle_{\mathcal{R} \cap \mathcal{B}, F} \\ &= \langle u, w \rangle_{\mathcal{R}, F} \neq 0. \end{aligned}$$

This establishes that $\langle -, - \rangle_{\mathcal{C}, F}$ is left non-degenerate, as desired. \square

Theorem 2.38. *Let \mathcal{C} be a locally π -finite ∞ -category equipped with a nested factorization system such that for every $0 \leq \ell \leq n$, endomorphisms in the category $\mathcal{T}^{(\ell)}$ are invertible. Then, for any functor $F : \mathcal{C} \rightarrow \text{Vec}_k$, the linear pairing $\langle -, - \rangle_{\mathcal{C}, F}$ is non-degenerate.*

Proof. We will prove that $\langle -, - \rangle_{\mathcal{C}, F}$ is both left and right non-degenerate, separately. The proofs are similar, but dual to each other.

First, we will establish left non-degeneracy. We prove by induction that the pairing is left non-degenerate on the following sequence of subcategories:

$$\mathcal{C}_0, \mathcal{R}^{(1)}, \mathcal{R}^{(2)}, \dots, \mathcal{R}^{(n)}, \mathcal{C}.$$

For the base case we note that the restriction of F to \mathcal{C}_0 forms a local system on the locally π -finite space \mathcal{C}_0 . The non-degeneracy of $\langle -, - \rangle_{\mathcal{C}_0, F}$ was established in Prop. 2.31, and so in particular this pairing is left non-degenerate.

On the category $\mathcal{R}^{(1)}$ we have the factorization system $(\mathcal{R}^{(1)}, \mathcal{C}_0)$, that is the trivial factorization system in which the left class consists of all maps in $\mathcal{R}^{(1)}$ and the right class consists of just the isomorphisms. We have already shown the the pairing induced by the right class is left non-degenerate, and by assumption the left class $\mathcal{R}^{(1)} = \mathcal{T}^{(0)}$ satisfies that endomorphisms are invertible. Thus the conditions of Lemma 2.37 are satisfied, showing that $\langle -, - \rangle_{\mathcal{R}^{(1)}, F}$ is left non-degenerate.

Similarly, the category $\mathcal{R}^{(\ell)}$ admits a factorization system $(\mathcal{T}^{(\ell-1)}, \mathcal{R}^{(\ell-1)})$. The left class satisfies that endomorphisms are invertible by assumption, and by induction the pairing $\langle -, - \rangle_{\mathcal{R}^{(\ell-1)}, F}$ is left non-degenerate. Thus Lemma 2.37 shows that the pairing $\langle -, - \rangle_{\mathcal{R}^{(\ell)}, F}$ is left non-degenerate.

Finally, \mathcal{C} admits the factorization system $(\mathcal{L}^{(n)}, \mathcal{R}^{(n)})$. As we have seen, the pairing $\langle -, - \rangle_{\mathcal{R}^{(n)}, F}$ is left non-degenerate, and again by assumption the left class $\mathcal{L}^{(n)} = \mathcal{T}^{(n)}$ satisfies that endomorphisms are invertible. Thus once again Lemma 2.37 shows that the pairing $\langle -, - \rangle_{\mathcal{C}, F}$ is left non-degenerate, as desired.

To show that $\langle -, - \rangle_{\mathcal{C}, F}$ is right non-degenerate, we may use an analogous induction on the categories

$$\mathcal{C}_0, \mathcal{L}^{(n)}, \mathcal{L}^{(n-1)}, \dots, \mathcal{L}^{(1)}, \mathcal{C}.$$

The pairing on \mathcal{C}_0 is non-degenerate by Prop. 2.31, establishing the base case. For the induction step we consider the factorization systems corresponding to each category:

$$\begin{aligned} \mathcal{L}^{(n)} &: (\mathcal{C}_0, \mathcal{L}^{(n)}) \\ \mathcal{L}^{(\ell)} &: (\mathcal{L}^{(\ell+1)}, \mathcal{T}^{(\ell)}), \quad \ell < n \\ \mathcal{C} &: (\mathcal{L}^{(1)}, \mathcal{R}^{(1)}) \end{aligned}$$

In each case the right class satisfies that endomorphisms are invertible. By induction the pairing for the left class is right non-degenerate, and so Lemma 2.37 applies. This establishes the right non-degeneracy of the pairing for the subsequent left class. The final case establishes that the pairing $\langle -, - \rangle_{\mathcal{C}, F}$ is right non-degenerate, as claimed. \square

Example 2.39. The category FinSet of finite sets (or analogously, of finite graphs or finite groups etc.) is locally π -finite — its hom sets are finite — and admits a factorization system $\mathcal{L} = \mathcal{T}^{(1)} = \{\text{surjections}\}$ and $\mathcal{R} = \mathcal{T}^{(0)} = \{\text{injections}\}$. Since endomorphism injections and endomorphism surjections of finite sets are bijections, it follows that FinSet fulfills the conditions of Theorem 2.38. Hence, for any functor $F : \text{FinSet} \rightarrow \text{Vec}_k$, the associated pairing $\langle -, - \rangle_{\text{FinSet}, F}$ is non-degenerate. In particular, for $F = \text{const}_k$ the constant functor at k

the associated pairing $\langle n, m \rangle_{\text{FinSet}} = n^m$ is indeed non-degenerate (c.f. Remark 2.25). Categories like FinSet are more general than the Möbius categories considered by [GCKT18b].

2.5. The Pontryagin pairing. For our main application, we do not consider functors $F : \mathcal{C} \rightarrow \text{Vec}_k$ but rather functors $\Omega : \mathcal{C} \rightarrow \text{Ab}$ into the category of abelian groups. For such functors, we introduce a refinement of the pairing of Definition 2.26. Let k be an algebraically closed field of characteristic zero. For an abelian group A , let \widehat{A} denote the ‘dual group’ $\text{Hom}(A, k^\times)$ of group homomorphisms, and let $k[A]$ denote the k -vector space obtained from linearizing the underlying set of A (equivalently, the vector space of finitely supported k -valued functions on the underlying set of A). For a set A with action by a group G , we write A/G for its set of orbits.

Definition 2.40. Let \mathcal{C} be a locally π -finite ∞ -category, let $\Omega : \mathcal{C} \rightarrow \text{Ab}$ be a functor and let k be an algebraically closed field of characteristic zero. The *Pontryagin pairing* is the map of k -vector spaces

$$(24) \quad \langle -, - \rangle_{\mathcal{C}, \Omega} : \bigoplus_{[c] \in \pi_0 \mathcal{C}} k[\widehat{\Omega(c)}/\pi_0 \text{Aut}_{\mathcal{C}}(c)] \otimes \bigoplus_{[d] \in \pi_0 \mathcal{C}} k[\Omega(d)/\pi_0 \text{Aut}_{\mathcal{C}}(d)] \rightarrow k$$

with coefficient at $(c \in \mathcal{C}, \phi \in \widehat{\Omega(c)} = \text{Hom}(\Omega(c), k^\times))$ and $(d \in \mathcal{C}, x \in \Omega(d))$ given by

$$(25) \quad \sum_{[f] \in \pi_0 \mathcal{C}(d, c)} \#(\mathcal{C}(d, c), f) \phi(\Omega(f)(x)).$$

Since the sum is over $\pi_0 \mathcal{C}(c, d)$, the expression (25) only depends on the orbits of ϕ and x under the $\text{Aut}_{\mathcal{C}}(c)$ and $\text{Aut}_{\mathcal{C}}(d)$ action, respectively. Moreover, note that while $\widehat{\Omega(c)}$ and $\Omega(c)$ depend on a choice of basepoint c in $[c]$, the set of orbits $\widehat{\Omega(c)}/\pi_0 \text{Aut}_{\mathcal{C}}(c)$ is independent of that choice up to unique isomorphism, and hence the pairing (25) only depends on the isomorphism classes of the objects c and d .

Remark 2.41. As in Remark 2.30 and analogously to (22), if \mathcal{C} is not locally π -finite we may still define a Pontryagin pairing $\langle -, - \rangle_{\mathcal{C}, \Omega}^{\mathcal{A}, \mathcal{B}}$ between full subcategories $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$ for which for every $a \in \mathcal{A}$ and $b \in \mathcal{B}$ the mapping space $\mathcal{C}(a, b)$ is π -finite.

Corollary 2.42. *Let \mathcal{C} be a locally π -finite ∞ -category with a nested factorization systems such that for every $0 \leq \ell \leq n$, endomorphisms in the category $\mathcal{T}^{(\ell)}$ are invertible. Then, for any functor $\Omega : \mathcal{C} \rightarrow \text{Ab}$ and any algebraically closed field k of characteristic zero, the Pontryagin pairing $\langle -, - \rangle_{\mathcal{C}, \Omega}$ is non-degenerate.*

Proof. For an abelian group A , let $\psi_A : k[\widehat{A}] \rightarrow k[A]^\vee$ denote the evident ‘linearization’ map from the linearization of the group of characters to the vector space dual $k[A]^\vee := \text{Hom}(k[A], k)$ of $k[A]$. It follows from linear independence of characters (see e.g. [The22, Section 9.13]) that for any abelian group A and any field k , this map ψ_A is injective. For any group G acting on the underlying set of A , the map ψ_A is G -equivariant for the evident induced actions of G on $k[\widehat{A}]$ and $k[A]^\vee$ and hence descends to a map on the coinvariants $(\psi_A)_G : k[\widehat{A}]_G \rightarrow (k[A]^\vee)_G$. For finite G , using the norm map and its inverse, this map $(\psi_A)_G$ may be expressed in terms of the injective map $(\psi_A)^G : k[\widehat{A}]^G \rightarrow (k[A]^\vee)^G$ on the invariants and hence is again injective.

Since for any set X with a group action, the linearization $k[X/G]$ is canonically isomorphic to the space of coinvariants $k[X]_G$, these maps assemble into an injective map

$$\Psi := \bigoplus_{[c] \in \pi_0 \mathcal{C}_0} (\psi_{\Omega(c)})_{\pi_0 \text{Aut}_{\mathcal{C}}(c)} : \bigoplus_{[c] \in \pi_0 \mathcal{C}_0} k[\widehat{\Omega(c)}/(\pi_0 \text{Aut}_{\mathcal{C}}(c))] \rightarrow \bigoplus_{[c] \in \pi_0 \mathcal{C}_0} (k[\Omega(c)]^\vee)_{\pi_0 \text{Aut}_{\mathcal{C}}(c)}$$

Let $F : \mathcal{C} \rightarrow \text{Vec}_k$ be the functor $k[\Omega(-)]$. Comparing (25) with the explicit expression for the linear pairing $\langle -, \rangle_{\mathcal{C}, F}$ from Remark 2.27, it follows that

$$\langle x, y \rangle_{\mathcal{C}, \Omega} = \langle \Psi x, y \rangle_{\mathcal{C}, F}.$$

Since Ψ is injective, it follows from Theorem 2.38 that $x \mapsto \langle x, - \rangle_{\mathcal{C}, \Omega}$ is injective.

To prove injectivity in the other slot, note that for any abelian group A and any algebraically closed field k , the canonical map $A \rightarrow \widehat{A}$ is injective.⁶

Let $\theta_A : k[A] \rightarrow k[\widehat{A}] \rightarrow \text{Hom}(k[\widehat{A}], k)$ denote the composite of this injective map with the injective map $\phi_{\widehat{A}}$. As before, we may assemble these maps into an injective map

$$\Theta := \bigoplus_{[d] \in \pi_0 \mathcal{C}_0} (\theta_{\Omega(d)})_{\pi_0 \text{Aut}_{\mathcal{C}}(d)} : \bigoplus_{[d] \in \pi_0 \mathcal{C}} k[\Omega(d)/\pi_0 \text{Aut}_{\mathcal{C}}(d)] \rightarrow \bigoplus_{[d] \in \pi_0 \mathcal{C}} (k[\widehat{\Omega(d)}])_{\pi_0 \text{Aut}_{\mathcal{C}}(d)}^{\vee}$$

Defining $G : \mathcal{C}^{\text{op}} \rightarrow \text{Vec}_k$ to be the functor $k[\widehat{\Omega(-)}]$, comparing (25) with Remark 2.27 it follows that

$$\langle x, y \rangle_{\mathcal{C}, \Omega} = \langle \Theta y, x \rangle_{\mathcal{C}^{\text{op}}, G}.$$

Since $\mathcal{C}^{\text{op}}, G$ also fulfills the requirements of Theorem 2.38, it follows from injectivity of Θ that $y \mapsto \langle -, y \rangle_{\mathcal{C}, \Omega}$ is also injective. \square

3. BORDISM GROUPS AND INVERTIBLE TOPOLOGICAL FIELD THEORIES

3.1. Stable structures and their normal and tangential bordism groups. Classical bordism theory, *à la* René Thom, considers structures on the *stable normal bundle*. Given a smooth manifold, it has a stable normal bundle which is classified by a map $\nu_M : M \rightarrow \text{BO}$. Define a *stable structure* to be a map $\beta : B \rightarrow \text{BO}$. A *(stable) normal (B, β) -structure* on M is a lift (in the homotopical sense)

$$\begin{array}{ccc} & & B \\ & \nearrow \nu_M & \downarrow \beta \\ M & \xrightarrow{\quad} & \text{BO} \end{array}$$

of ν_M through the map β .

If M_0 and M_1 are closed n -manifolds equipped with normal (B, β) -structures (f_0, θ_0) and (f_1, θ_1) , then a normal (B, β) -bordism from M_0 to M_1 is a d -dimensional bordism W from M_0 to M_1 together with a normal (B, β) -structure (f, θ) on W which *extends* the ones on M_0 and M_1 . Normal bordism is an equivalence relation and the equivalence classes form a group which is usually denoted Ω_n^B .

A similar theory can be developed using the *stable tangent bundle* in place of the stable normal bundle. Given a smooth manifold M of dimension n we fix a map $T_M : M \rightarrow \text{BO}(n)$ classifying the tangent bundle of M . The stable tangent bundle is the composite map $\tau_M : M \xrightarrow{T_M} \text{BO}(n) \xrightarrow{i_n} \text{BO}$. A *stable tangential (B, β) -structure* on M is a lift (in the

⁶We may see this as follows. Let $a \neq 1 \in A$. Pick a character $\chi : \langle a \rangle \rightarrow k^\times$ on the cyclic group generated by a , such that $\chi(a) \neq 1$ (this exists because k is algebraically closed). Then, since k^\times is an injective abelian group, any homomorphism $\langle a \rangle \rightarrow k^\times$ may be lifted along the injection $\langle a \rangle \rightarrow A$.

homotopical sense)

$$\begin{array}{ccc}
 & & B \\
 & \nearrow \tau_M & \downarrow \beta \\
 M & \xrightarrow{\tau_M} & BO
 \end{array}$$

of τ_M through the map β . Bordisms are defined similarly to the normal case, and so a priori we have two notions of stable bordism groups:

Definition 3.1. Let $\beta : B \rightarrow BO$ be a stable structure. The *stable tangential bordism group* $\Omega_d^{\tau\beta}$ is the bordism group of d -dimensional manifolds equipped with (B, β) -structures on their *stable tangent bundle* modulo the relation of cobordism with (B, β) -structures on their *stable tangent bundle*. The *stable normal bordism group* $\Omega_d^{\nu\beta}$ is the bordism group of d -dimensional manifolds equipped with (B, β) -structures on their *stable normal bundle* modulo the relation of cobordism with (B, β) -structures on their *stable normal bundle*.

However, stable tangential bordism does not actually provide a new notion. Given a stable structure $\beta : B \rightarrow BO$, we obtain a new stable structure $\bar{\beta} : B \rightarrow BO$ as the composite $B \rightarrow BO \xrightarrow{(-1)} BO$, where $(-1) : BO \simeq BO$ corresponds to inversion in the infinite loop space structure on BO (which may be thought of as taking the ‘orthogonal complement’ of a stable vector bundle). An $(B, \bar{\beta})$ -structure on the stable normal bundle of M is the same thing as an (B, β) -structure on the stable tangent bundle of M , and conversely.

$$\Omega_d^{\tau\beta} \cong \Omega_d^{\nu\bar{\beta}} \quad \Omega_d^{\nu\beta} \cong \Omega_d^{\tau\bar{\beta}}.$$

Remark 3.2. Sometimes it is the case that $\bar{\beta} \simeq \beta$. For example, this holds for orientations and spin structures - an orientation (or spin structure) on a manifold can equivalently be defined on the stable normal bundle or the (stable) tangent bundle. An example where $\bar{\beta} \not\simeq \beta$ is given by Pin^\pm -structures. Instead, the operation $(B, \beta) \mapsto (B, \bar{\beta})$ exchanges stable Pin^+ and Pin^- structures [KT90, Lem. 1.3].

Example 3.3. Let M be a smooth manifold, with stable normal map $\nu_M : M \rightarrow BO$, and fix a natural number n . The map ν_M factors uniquely up to homotopy $M \rightarrow B_M \rightarrow BO$ into a n -connected map followed by an n -truncated map (see Example 2.23). The space B_M is the (stable) normal n -type of M [Kre99]. The map from M to B_M realizes M as an element of $\Omega_n^{\nu B_M}$, the normal B_M -bordism group which appears in Kreck’s modified approach to surgery. See [Kre99] for details. Similarly, the *stable tangential n -type* of M is obtained as the n -connected/ n -truncated factorization of the stable tangent bundle $\tau_M : M \rightarrow BO$.

3.2. Unstable tangential structures, bordism categories and topological field theories. By an *unstable tangential structure* we mean a map $\beta : B \rightarrow BO(d)$, which we may equivalently view as a rank d vector bundle β over B . Given a smooth manifold M of dimensions $\dim(M) \leq d$, we fix a map $T_M : M \rightarrow BO(\dim(M))$ classifying the tangent bundle of M . A (B, β) -structure on M consists a lift (in the homotopical sense)

$$(26) \quad \begin{array}{ccc}
 & & B \\
 & \nearrow & \downarrow \beta \\
 M & \xrightarrow{T_M} BO(\dim(M)) \longrightarrow & BO(d)
 \end{array}$$

of the composite $M \rightarrow \mathrm{BO}(\dim(M)) \rightarrow \mathrm{BO}(d)$ through β . In particular, the mapping space $\mathrm{Map}_{\mathrm{BO}(d)}(M, B)$ in the $(\infty, 1)$ -category $\mathcal{S}_{/\mathrm{BO}(d)}$ of spaces over $\mathrm{BO}(d)$ is the *moduli space of (B, β) -structures on M* . (If $\beta : B \rightarrow \mathrm{BO}(d)$ is a fibration, this is the usual point-set mapping space.)

Remark 3.4. Alternatively, a (B, β) -structure on a smooth manifold M can be defined as a map $f : M \rightarrow B$ together with an isomorphism of vector bundles $\theta : T_M \oplus \varepsilon^{d-\dim(M)} \cong f^* \beta$ where ε is the trivial one-dimensional vector bundle.

For manifolds of dimension strictly less than d , there is a notion of (B, β) -bordism. However, in contrast to the stable case, the relation of (B, β) -bordism may fail to be reflexive and hence can fail to be an equivalence relation. Thus we do not define an unstable bordism group in the usual manner. However bordisms can be composed, and so we will obtain a bordism *category*. This can be thought of as a substitute in the unstable setting.

For $\beta : B \rightarrow \mathrm{BO}(d)$, we will describe (B, β) -bordisms between $(d-1)$ -manifolds as follows. If M_0 and M_1 are closed $(d-1)$ -manifolds equipped with (B, β) -structures (f_0, θ_0) and (f_1, θ_1) , then a (B, β) -bordism from M_0 to M_1 is a d -dimensional compact bordism W from M_0 to M_1 together with a (B, β) -structure (f, θ) on W and a choice of an *inward pointing normal vector* on the incoming boundary M_0 , and the *outward pointing normal* on the outgoing boundary M_1 . These latter structures are understood as appropriate isomorphisms $(T_{M_i} \oplus \varepsilon) \cong T_W|_{M_i}$. Furthermore the (B, β) -structure (f, θ) on W *extends* the ones on M_0 and M_1 . The meaning of ‘extends’ in this context is first that $f : W \rightarrow B$ restricts to f_0 and f_1 on M_0 and M_1 , and furthermore for $i = 1, 2$, the vector bundle isomorphism $\theta_i : T_{M_i} \oplus \varepsilon \cong f_i^* \beta$ agrees with

$$(T_{M_i} \oplus \varepsilon) \cong T_W|_{M_i} \stackrel{\theta|_{M_i}}{\cong} f_i^* \beta.$$

Bordisms may be composed giving rise to a symmetric monoidal bordism category of manifolds with (B, β) -structure $\mathrm{Bord}_d^{B, \beta}$. The objects of this category are closed $(d-1)$ -manifolds with (B, β) -structures, and the morphisms are equivalence classes of compact d -bordisms with (B, β) -structures. The equivalence relation on bordisms is generated by isotopy of (B, β) -structure together with diffeomorphism rel. boundary over B .

Definition 3.5. A (B, β) -structured topological field theory valued in a symmetric monoidal category \mathcal{C} is a symmetric monoidal functor $Z : \mathrm{Bord}_d^{B, \beta} \rightarrow \mathcal{C}$. A (B, β) -structured *topological super field theory* is a topological field theory valued in the symmetric monoidal category sVec of super vector spaces.

Remark 3.6. The category $\mathrm{Bord}_d^{B, \beta}$ is the homotopy 1-category of an $(\infty, 1)$ -category of (B, β) -bordisms (for example constructed as a topological category in [GRW10, GTMW09]) which also contains information about the moduli spaces of B -manifolds and B -bordisms. There are also (∞, n) -versions of the bordisms n -category considered in [Lur09b, CS19, SP17].

3.3. Invertible topological field theories. A topological super field theory is *invertible* if it factors through the groupoid sLine_κ whose objects are super lines (either odd or even) and whose morphisms are isomorphisms of super lines. In the following section, we briefly summarize the classification of such invertible super field theories. Variations of these results have for example appeared in [FH21, FH20]

A symmetric monoidal category in which all morphisms are invertible and in which all objects are \otimes -invertible is called a *Picard groupoid*. Every symmetric monoidal category

\mathcal{C} has a maximal Picard subgroupoid $\text{Pic}(\mathcal{C})$, and in particular any functor from a Picard groupoid into a symmetric monoidal category \mathcal{C} factors (essentially uniquely) through $\text{Pic}(\mathcal{C})$. Conversely, a functor from \mathcal{C} into a Picard groupoid factors through the *Picard completion* $\widehat{\mathcal{C}}$ of \mathcal{C} , which is obtained by freely inverting the morphisms, and then performing a group-completion on the objects. In other words, forming the Picard subgroupoid and the Picard completion are right and left adjoints to the inclusion of the 2-category of Picard groupoids into the 2-category of symmetric monoidal 1-categories. Equivalently, the Picard completion may be constructed by first observing that the geometric realization $\|\mathcal{C}\|$ of \mathcal{C} is an E_∞ -space, and then defining $\widehat{\mathcal{C}}$ to be the fundamental groupoid $\pi_{\leq 1}\Omega\text{B}\|\mathcal{C}\|$ of the group-completion of $\|\mathcal{C}\|$. If the category \mathcal{C} has duals, then $\|\mathcal{C}\|$ is already grouplike and $\Omega\text{B}\|\mathcal{C}\| \simeq \|\mathcal{C}\|$. For the bordism category the geometric realization is identified by the celebrated theorem of Galatius-Madsen-Tillmann-Weiss [GTMW09] (see also [GRW10]).

Theorem 3.7 ([GTMW09, GRW10]). *Let $\beta : B \rightarrow \text{BO}(d)$ be an unstable tangential structure. Then, $\|\text{Bord}_d^{B,\beta}\| \simeq \Omega^{\infty-1}MT\beta$, where the Madsen-Tillmann spectrum $MT\beta$ associated to $\beta : B \rightarrow \text{BO}(d)$ is the Thom spectrum $B^{-\beta}$ of the virtual vector bundle $-\beta$. \square*

Picard groupoids \mathbf{A} are classified by the following three *Postnikov invariants* [HS05, App. B] (see also [JO12]):

- An abelian group A_0 , corresponding to the isomorphism classes of objects of \mathbf{A} ;
- An abelian group A_1 , corresponding to the automorphisms of the unit object of \mathbf{A} ;
- A homomorphism $k_{\mathbf{A}} : A_0 \otimes \mathbb{Z}/2\mathbb{Z} \rightarrow A_1$, the *k-invariant* of \mathbf{A} .

The map $k_{\mathbf{A}}$ is defined as follows. For each object $x \in \mathbf{A}$ select a dual-inverse \bar{x} , equipped with unit $\eta : \mathbb{1} \rightarrow x \otimes \bar{x}$ and counit $\varepsilon : \bar{x} \otimes x \rightarrow \mathbb{1}$ isomorphisms, which satisfy the usual zig-zag equations

$$id_x = (id_x \otimes \varepsilon) \circ (\eta \otimes id_x), \quad id_{\bar{x}} = (\varepsilon \otimes id_{\bar{x}}) \circ (id_{\bar{x}} \otimes \eta).$$

Define $k_{\mathbf{A}}(x) : \mathbb{1} \rightarrow \mathbb{1}$ as

$$k_{\mathbf{A}}(x) = \varepsilon \circ \beta_{x,\bar{x}} \circ \eta$$

where $\beta_{x,\bar{x}} : x \otimes \bar{x} \rightarrow \bar{x} \otimes x$ is the braiding isomorphism of \mathbf{A} . The morphism $k_{\mathbf{A}}(x) : \mathbb{1} \rightarrow \mathbb{1}$ only depends on the isomorphism class of x . Since \mathbf{A} is symmetric monoidal, $k_{\mathbf{A}}(x) \in \text{End}_{\mathbf{A}}(\mathbb{1}) = A_1$ is an order 2 element and $k_{\mathbf{A}} : A_0 \otimes \mathbb{Z}/2\mathbb{Z} \rightarrow A_1$ defines a homomorphism.

Example 3.8. The Picard groupoid $\text{Pic}(\text{Vec}_k)$ of the category of vector spaces is the groupoid Line_k of one-dimensional vector spaces and isomorphisms between them. Its Postnikov invariants are $L_0 = 0$, $L_1 = k^\times$ and trivial k -invariant. The Picard groupoid $\text{Pic}(\text{sVec}_k)$ of the category of super vector spaces is the groupoid sLine_k of super lines (even and odd). Its Postnikov invariants are $sL_0 = \mathbb{Z}/2\mathbb{Z}$, $sL_1 = k^\times$, and the k -invariant is the homomorphism $k_{\text{sLine}_k} : \mathbb{Z}/2\mathbb{Z} \rightarrow k^\times$ which sends the non-trivial element to $-1 \in k^\times$. In particular, if the characteristic of k is not 2, k_{sLine_k} is injective.

Example 3.9 ([HS05, App. B]). Given a spectrum E and an integer i , we obtain a Picard groupoid $\pi_{\leq 1}\Omega^{\infty+i}E$, the fundamental groupoid of the shifted infinite loop space underlying E . It has Postnikov invariants:

- the isomorphism classes of objects are $\pi_i E$;
- the automorphisms of the unit object are $\pi_{i+1} E$;
- the k -invariant $k_E = (\eta \cdot -) : \pi_i E \rightarrow \pi_{i+1} E$ is given by multiplication by $\eta \in \pi_1 \mathbb{S}$ in the sphere spectrum.

All Picard groupoids arise in this way. (Indeed, the 2-groupoid of Picard groupoids is equivalent to the moduli space of spectra E with $\pi_i E = 0$ for $i \neq 0, 1$.)

Functors between Picard groupoids \mathbf{A} and \mathbf{B} can be described in terms of their Postnikov invariants [Sto]. Specifically, there is an exact sequence:

$$0 \rightarrow \text{Ext}(A_0, B_1) \rightarrow \pi_0 \text{Fun}(\mathbf{A}, \mathbf{B}) \rightarrow \text{Hom}(A_0, B_0) \times \text{Hom}(A_1, B_1) \rightarrow \text{Hom}(A_0 \otimes \mathbb{Z}/2\mathbb{Z}, B_1)$$

where the last map sends $(f, g) \in \text{Hom}(A_0, B_0) \times \text{Hom}(A_1, B_1)$ to $k_{\mathbf{B}} \circ f - g \circ k_{\mathbf{A}}$.

In particular, if B_1 is a divisible group and the map $B_0 \rightarrow B_0 \otimes \mathbb{Z}/2\mathbb{Z} \rightarrow B_1$ induced by the k -invariant of \mathbf{B} is injective, then $\pi_0 \text{Fun}(\mathbf{A}, \mathbf{B}) \cong \text{Hom}(A_1, B_1)$.

Corollary 3.10. *Let k be an algebraically closed field of characteristic $\neq 2$. Then, for all Picard groupoids \mathbf{A} :*

$$\pi_0 \text{Fun}(\mathbf{A}, \text{sLine}_k) \cong \text{Hom}(A_1, k^\times) \quad \square$$

Thus functors into the Picard groupoid of super lines (over an algebraically closed field, not of characteristic 2) are the same as characters on the automorphisms of the unit object, see [Fre19, Rmk. 6.91].

Remark 3.11. The Picard groupoid sLine_k also arises from a spectrum as in Example 3.9. Specifically let I_{k^\times} be the k^\times -based version of the Brown-Comenetz dual of the sphere spectrum [BC76]. Since k^\times is an injective abelian group, this spectrum has homotopy groups $\pi_i I_{k^\times} \cong \text{hom}(\pi_{-i} \mathbb{S}, k^\times)$. Thus if the characteristic of k is not 2, $\pi_{-1} I_{k^\times} \cong \mathbb{Z}/2\mathbb{Z} \cong sL_0$ and $\pi_0 I_{k^\times} \cong k^\times \cong sL_1$. The corresponding k -invariants coincide, and the universal property of sLine_k follows from the universal property of I_{k^\times} . See [BC76] and [HS05, App. B].

Definition 3.12. Given an unstable structure $\beta : B \rightarrow \text{BO}(d)$, we write $\Omega_k^{T\beta} := \Omega_k^{\tau(i_d \circ \beta)}$ where $i_d : \text{BO}(d) \rightarrow \text{BO}$ is the stabilization map. We refer to this as the *unstable bordism group*.

Explicitly, given an unstable structure $\beta : B \rightarrow \text{BO}(d)$ thought of as a d -dimensional vector bundle β on B , a stable $(B, i_d \circ \beta)$ -structure on a manifold M is a map $f : M \rightarrow B$ and an isomorphism

$$TM \oplus \varepsilon^N \cong f^* \beta \oplus \varepsilon^{N + \dim(M) - d}$$

for some $N \gg 0$. Note that any (unstable) (B, β) -structure on a closed manifold (with $\dim(M) \leq d$) as in (26) induces a stable tangential $(B, i_d \circ \beta)$ -structure and hence an element $[M] \in \Omega_n^{T\beta}$. Furthermore, it follows from the Pontryagin-Thom construction that the homotopy groups $MT\beta$ can be identified with these unstable bordism groups [RW].

Lemma 3.13. *Given an unstable structure $\beta : B \rightarrow \text{BO}(d)$, $\pi_{n-d} MT\beta \cong \Omega_n^{T\beta}$.* \square

Combining this computation, with the theorem of Galatius-Madsen-Tillmann-Weiss [GTMW09], we obtain the following proposition which asserts that invertible super topological field theories are uniquely characterized by their *partition functions*.

Theorem 3.14. *Let $\beta : B \rightarrow \text{BO}(d)$ be an unstable structure and let k be an algebraically closed field of characteristic $\neq 2$. Then, isomorphism classes of d -dimensional invertible (B, β) -structured super topological field theories $\mathcal{Z} : \text{Bord}_d^{B, \beta} \rightarrow \text{sVec}_k$ are in natural bijection with group homomorphisms $\omega : \Omega_d^{T\beta} \rightarrow k^\times$.*

Explicitly, the value of the TFT \mathcal{Z}_ω associated to a group homomorphism ω on a closed d -dimensional (B, β) -manifold W is given by

$$\mathcal{Z}_\omega(W) = \omega([W])$$

where $[W] \in \Omega_d^{T\beta}$ is the bordism class of W with stable tangential (B, β) -structure induced from its given unstable structure.

Proof. The bordism category $\text{Bord}_d^{(B, \beta)}$ has duals, and thus the Picard completion $\widehat{\text{Bord}_d^{(B, \beta)}}$ is the fundamental groupoid of the classifying space $\pi_{\leq 1} \|\text{Bord}_d^{(B, \beta)}\|$. In their famous work, Galatius, Madsen, Tillmann, and Weiss consider a category $\text{Cob}_d^{(B, \beta)}$ enriched in topological spaces. They identify the homotopy type of the geometric realization as $\|\text{Cob}_d^{(B, \beta)}\| \simeq \Omega^{\infty-1} MT\beta$ [GTMW09]. Our category $\text{Bord}_d^{(B, \beta)}$ is the homotopy category of $\text{Cob}_d^{(B, \beta)}$ and hence the canonical map $\text{Cob}_d^{(B, \beta)} \rightarrow \text{Bord}_d^{(B, \beta)}$ induces an equivalence on the fundamental 1-groupoid of classifying spaces

$$\pi_{\leq 1} \|\text{Cob}_d^{(B, \beta)}\| \cong \pi_{\leq 1} \|\text{Bord}_d^{(B, \beta)}\|.$$

Hence, it follows that $\pi_1(\widehat{\text{Bord}_d^{(B, \beta)}}) \cong \pi_0 MT\beta \cong \Omega_d^{T\beta}$. The proposition then follows from Corollary 3.10. \square

3.4. Computing unstable bordism groups. In this section, we explain how to compute unstable bordism groups from Definition 3.12 in terms of the more familiar stable tangential bordism groups from Definition 3.1.

Let $\beta_\infty : B_\infty \rightarrow BO$ be a stable tangential structure and for each $k \geq 0$, consider the unstable tangential structure $\beta_k : B_k \rightarrow BO(k)$ defined as the pullback:

$$\begin{array}{ccc} B_k & \longrightarrow & B_\infty \\ \beta_k \downarrow & \lrcorner & \downarrow \beta_\infty \\ BO(k) & \longrightarrow & BO \end{array}$$

Example 3.15. Consider $\beta_\infty : BSO \rightarrow BO$. Then, $\beta_k : BSO(k) \rightarrow BO(k)$, and similarly for $BSpin$ and all other structures in the Whitehead tower of BO .

Recall from Lemma 3.13 that for each $n \geq 0$ the unstable bordism groups $\Omega_n^{T\beta_k} = \Omega_n^{\tau_i \circ \beta_k}$ can be identified with homotopy groups of the Madsen-Tillmann spectrum $MT\beta_k$. We can compare these various bordism groups $\Omega_n^{T\beta_k}$ for varying n and k using the *Genauer fiber sequence*:

$$MT\beta_{k+1} \rightarrow \Sigma_+^\infty B_{k+1} \rightarrow MT\beta_k.$$

The existence of this fiber sequence of spectra is well-known to experts. Genauer constructed a (-1) -connective cover of this sequence using a bordism category of manifolds with boundary to model $\Sigma_+^\infty B_{k+1}$ [Gen12]. It is treated in detail in the unoriented case in [GTMW09, Prop. 3.1] and the case of general tangential structures is discussed in [GTMW09, Section 5]. It is also mentioned in [RS14, Section 6.1].

Lemma 3.16. *Let $\beta_\infty : B_\infty \rightarrow BO$ be a stable tangential structure and for each $k \geq 0$, let $\beta_k : B_k \rightarrow BO(k)$ denote the unstable tangential structure obtained by pulling back β_∞ to $BO(k)$.*

1. *Let $d < k$. Then $\Omega_d^{T\beta_k} \cong \Omega_d^{T\beta_{k+1}} \cong \Omega_d^{\tau\beta_\infty} \cong \Omega_d^{\nu\bar{\beta}_\infty}$.*
2. *There is an exact sequence*

$$\Omega_{k+1}^{T\beta_{k+1}} \rightarrow \pi_0 \Sigma_+^\infty B_{k+1} \rightarrow \Omega_k^{T\beta_k} \rightarrow \Omega_k^{T\beta_{k+1}} \rightarrow 0.$$

Suppose that B_k is connected.

3. If k is odd, then this last exact sequence reduces to a short exact sequence

$$0 \rightarrow \mathbb{Z}/\ell \rightarrow \Omega_k^{T\beta_k} \rightarrow \Omega_k^{\tau\beta_\infty} \rightarrow 0$$

for some $\ell \in \{0, 1, 2, \dots\}$.

4. If k is even, there is a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \Omega_k^{T\beta_k} & \longrightarrow & \Omega_k^{\tau\beta_\infty} & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \chi & & \downarrow \chi \text{ mod } 2 & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2\cdot} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \end{array}$$

where χ is the Euler characteristic function.

Proof. (1), (2), and (3) are direct consequences of the Genauer fiber sequence. We can also compare the Genauer fiber sequence for (B_k, β_k) to the sequence for $(BO(k), id)$. We obtain a map of long exact sequences:

$$\begin{array}{ccccccccc} \Omega_{k+1}^{T\beta_{k+1}} & \longrightarrow & \mathbb{Z} & \longrightarrow & \Omega_k^{T\beta_k} & \longrightarrow & \Omega_k^{\tau\beta_\infty} & \longrightarrow & 0 \\ \downarrow & & \downarrow \cong & & \downarrow & & \downarrow & & \\ \Omega_{k+1}^{TBO(k+1)} & \xrightarrow{\chi} & \mathbb{Z} & \longrightarrow & \Omega_k^{TBO(k)} & \longrightarrow & \mathcal{N}_k & \longrightarrow & 0 \end{array}$$

Here \mathcal{N}_k is the unoriented bordism group, and the first lower horizontal map χ is identified with the Euler characteristic [GTMW09, Eq. 3.6] (see also [BDS15]). Thus the ℓ in part (3) is the smallest common divisor of the possible Euler characteristics of $(k+1)$ -dimensional manifolds with (B_{k+1}, β_{k+1}) -structures. When k is even (so $k+1$ is odd), these Euler characteristics will be zero, and in that case we can extend to a diagram of short exact sequences as follows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \Omega_k^{T\beta_k} & \longrightarrow & \Omega_k^{\tau\beta_\infty} & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \Omega_k^{TBO(k)} & \longrightarrow & \mathcal{N}_k & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \chi & & \downarrow \chi \text{ mod } 2 & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2\cdot} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \end{array}$$

The bottom half of the diagram is identified in [BDS15]. Again χ is the Euler characteristic. \square

Example 3.17. Let $\beta_d : BSO(d) \rightarrow BO(d)$. This case has been widely studied. It was shown in [BS14] that the unstable oriented bordism group $\Omega_d^{T\beta} = \Omega_d^{TBSO(d)}$ is isomorphic to Reinhart's vector field cobordism group [Rei63]. This, in turn, was previously shown to agree with the SKK groups of manifolds [KKNO73], which were shown to fit into a short exact sequence

$$0 \rightarrow I_d \rightarrow SKK_d \rightarrow \Omega_d^{SO} \rightarrow 0$$

where Ω_d^{SO} denotes the classical bordism group of oriented manifolds and I_n is a cyclic summand - either zero, $\mathbb{Z}/2\mathbb{Z}$, or \mathbb{Z} depending on $d \bmod 4$. Moreover these extra summands

were known to be detected by the Euler characteristic or Kervaire's semi-characteristic (defined below). This was rediscovered in [BDS15]. We have:

$$\Omega_d^{TBSO(d)} \cong \begin{cases} \Omega_d^{SO} & d \equiv 3 \pmod{4} \\ \mathbb{Z} \oplus \Omega_d^{SO} & d \equiv 2 \pmod{4} \\ \mathbb{Z}/2\mathbb{Z} \oplus \Omega_d^{SO} & d \equiv 1 \pmod{4} \\ \mathbb{Z} \oplus \Omega_d^{SO} & d \equiv 0 \pmod{4} \end{cases}$$

where Ω_d^{SO} denotes the classical bordism group of oriented manifolds. If $\rho : \Omega_d^{TBSO(d)} \rightarrow \Omega_d^{SO}$ is the natural projection, then these isomorphisms are given, respectively, by $(\frac{1}{2}\chi, \rho)$ in dimensions $d = 4k + 2$, $(k_{\mathbb{R}}, \rho)$ in dimensions $d = 4k + 1$, and by $(\frac{1}{2}(\chi + \sigma), \rho)$ in dimensions $d = 4k$, where χ denotes the Euler characteristic, σ denotes the signature, and $k_{\mathbb{R}}$ is Kervaire's semi-characteristic (defined when $d = 4k + 1$):

$$k_{\mathbb{R}}(M) = \sum_{i=0}^{\frac{d-1}{2}} \dim H^{2i}(M; \mathbb{R}).$$

Pontryagin numbers and Stiefel-Whitney numbers can be used to give characters on the classical bordism group, and are well-known to completely determine Ω_d^{SO} [Wal60]. It follows then from these calculations and Theorem 3.14 that invertible oriented topological field theories determine precisely the following invariants:

- Pontryagin and Stiefel-Whitney numbers;
- The Euler characteristic;
- Kervaire's semi-characteristic.

Of these only Kervaire's semi-characteristic requires that the target category actually be *super* vector spaces.

Remark 3.18. Following Lemma 3.16, when $d = 2q$ is even we have an injection $(\chi, pr) : \Omega_{2q}^{T\beta_{2q}} \rightarrow \mathbb{Z} \oplus \Omega_{2q}^{\tau\beta_{\infty}}$. If B_{2q} is orientable (i.e. it factors as $B_{2q} \rightarrow BSO(2q) \rightarrow BO(2q)$), then the previous example allows us to be more precise. Specifically it follows that $(\frac{1}{2}\chi, pr) : \Omega_{2q}^{T\beta_{2q}} \cong \mathbb{Z} \oplus \Omega_{2q}^{\tau\beta_{\infty}}$ if q is odd, and $(\frac{1}{2}(\chi + \sigma), pr) : \Omega_{2q}^{T\beta_{2q}} \cong \mathbb{Z} \oplus \Omega_{2q}^{\tau\beta_{\infty}}$ if q is even, where χ is the Euler characteristic and σ is the signature.

Example 3.19. Fix a space Y and let $\beta_d : Y \times BSO(d) \rightarrow BO(d)$ be the composition of projection to the second factor followed by the canonical map. Then a β_d -structure on a d -manifold consists of an orientation together with a map to Y . There is a map of spectra

$$MT\beta_d \simeq MTSO(d) \wedge Y_+ \rightarrow \Sigma^{-d}H\mathbb{Z} \wedge Y_+$$

induced from the bottom Postnikov truncation $MTSO(d) \rightarrow \Sigma^{-d}H\mathbb{Z}$. Thus there is map

$$\Omega_d^{T\beta_d} \cong \pi_0(MTSO(d) \wedge Y_+) \rightarrow \pi_0(\Sigma^{-d}H\mathbb{Z} \wedge Y_+) \cong H_d(Y).$$

If k is an algebraically closed field (not of characteristic 2), then we get an induced map

$$H^d(Y; k^{\times}) \cong \text{Hom}(H_d(Y), k^{\times}) \rightarrow \text{Hom}(\Omega_d^{T\beta_d}, k^{\times}).$$

Thus, in light of Theorem 3.14, k^{\times} -valued cohomology classes on Y give rise to invertible β_d -structured topological field theories.

Fix $\alpha \in H^d(Y; k^{\times})$ and let (M, f) be a closed β_d -structured d -manifold, i.e. M is an oriented manifold and $f : M \rightarrow Y$ is a map. Then the *partition function* of the corresponding

invertible field theory is given by

$$\mathcal{Z}_{Y,\alpha}(M, f) = \langle [M], f^* \alpha \rangle$$

where $[M]$ is the fundamental class of M .

3.5. Tangential n -types. In Section 3.4, we showed that if an unstable tangential structure $\beta_k : B_k \rightarrow \mathbf{BO}(k)$ is pulled back from a stable structure $\beta_\infty : B_\infty \rightarrow \mathbf{BO}$, then the unstable bordism groups for β_k can be computed from the stable bordism groups for β_∞ . In this section, we study an important sufficient condition for an unstable structure β_k to arise from an β_∞ in this way.

Indeed, most tangential structures $\beta_d : B_d \rightarrow \mathbf{BO}(d)$ we will encounter in this paper will be n -truncated for some $n \geq 0$. Recall that this means that the homotopy fibers of β_d are n -types, equivalently for each basepoint the map $\pi_k B_d \rightarrow \pi_k \mathbf{BO}(d)$ is injective for $k = n + 1$ and an isomorphism for $k > n + 1$ (see Example 2.23).

Definition 3.20. A *tangential n -type* is an n -truncated tangential structure $\beta_d : B_d \rightarrow \mathbf{BO}(d)$.

For $k \leq d$, let $\beta_k : B_k \rightarrow \mathbf{BO}(k)$ denote the tangential structure obtained by pulling back β_d along $\mathbf{BO}(k) \rightarrow \mathbf{BO}(d)$. By the universal property of pullbacks, a B_k -structure on a k -manifold M ($k \leq d$) is the same as a B_d -structure on $T_M \oplus \varepsilon^{d-k}$, as defined in Section 3.2. If $\beta_d : B_d \rightarrow \mathbf{BO}(d)$ is n -truncated, then so is the pulled back map $\beta_k : B_k \rightarrow \mathbf{BO}(k)$.

If $\beta_d : B \rightarrow \mathbf{BO}(d)$ is n -truncated for some $n \leq d - 2$, it itself arises as a pullback of a $\beta_{d+1} : B \rightarrow \mathbf{BO}(d + 1)$ and hence ultimately of a stable tangential structure $\beta_\infty : B \rightarrow \mathbf{BO}$.

Proposition 3.21. *Assume that $\beta_d : B_d \rightarrow \mathbf{BO}(d)$ is n -truncated for some $n \leq d - 2$. Then, for all $k > d$ (including $k = \infty$) there are n -truncated maps $\beta_k : B_k \rightarrow \mathbf{BO}(k)$ and pullback squares:*

$$\begin{array}{ccccccc} B_d & \longrightarrow & B_{d+1} & \longrightarrow & \cdots & \longrightarrow & B_\infty \\ \downarrow \lrcorner & & \downarrow \lrcorner & & & \lrcorner & \downarrow \\ \mathbf{BO}(d) & \longrightarrow & \mathbf{BO}(d+1) & \longrightarrow & \cdots & \longrightarrow & \mathbf{BO} \end{array}$$

Proof. Factor the composite of β_d with the inclusion $\mathbf{BO}(d) \rightarrow \mathbf{BO}(d+1)$ into an n -connected map followed by an n -truncated map:

$$B_d \xrightarrow{n\text{-connected}} B_{d+1} \xrightarrow{n\text{-truncated}} \mathbf{BO}(d+1).$$

Since the map $\mathbf{BO}(d) \rightarrow \mathbf{BO}(d+1)$ is $(d-1)$ -connected and hence also $(n+1)$ -connected (as $n \leq d-2$), it follows that the maps in the following square have the indicated truncation/connectivity:

$$\begin{array}{ccc} B_d & \xrightarrow{n\text{-conn.}} & B_{d+1} \\ \downarrow n\text{-trunc.} & & \downarrow n\text{-trunc.} \\ \mathbf{BO}(d) & \xrightarrow{(n+1)\text{-conn.}} & \mathbf{BO}(d+1) \end{array}$$

Such squares are necessarily pullback squares. (In fact, it would have been sufficient to assume the right vertical map to only be $(n+1)$ -truncated.) Iterating, we obtain a sequence

of n -truncated $\beta_k : B_k \rightarrow \mathrm{BO}(k)$ for all k , including $\beta_\infty : B_\infty \rightarrow \mathrm{BO}$ (the $k = \infty$ case) such that (B_k, β_k) is obtained as a pull-back of the subsequent tangential structure. \square

4. DIJKGRAAF-WITTEN FIELD THEORIES AND THEIR TOPOLOGICAL SENSITIVITY

4.1. Dijkgraaf-Witten super topological field theories. As in Section 3, we fix a map $\beta : B \rightarrow \mathrm{BO}(d)$ thought of as an ambient tangential structure. Roughly speaking, to build generalized Dijkgraaf-Witten theories we start with a map $\xi : X \rightarrow B$ and a topological field theory $\mathcal{Z} : \mathrm{Bord}_d^{(X, \beta \circ \xi)} \rightarrow \mathrm{sVec}_k$ and ‘integrate out the fibers of ξ ’ to obtain a field theory $\mathrm{DW}_{X, \xi, \mathcal{Z}} : \mathrm{Bord}_d^{(B, \beta)} \rightarrow \mathrm{sVec}_k$.

Explicitly, to a closed $(d - 1)$ -dimensional (B, β) -manifold M , we associate the space $\mathrm{Map}_B(M, X)$ of maps $f : M \rightarrow X$ over B . We may think of this space as the moduli space of $(X, \beta \circ \xi)$ -structures on M which refine the given (B, β) -structure. Similarly, to every (B, β) -bordism W between (B, β) -manifolds M_0 and M_1 we associate the span of spaces

$$(27) \quad \mathrm{Map}_B(M_1, X) \xleftarrow{t} \mathrm{Map}_B(W, X) \xrightarrow{s} \mathrm{Map}_B(M_0, X)$$

which records the moduli space of X -structures on W together with the restrictions of such structures to the incoming and outgoing boundary M_0 and M_1 .

Let $\mathrm{Span}(\mathcal{S})$ denote the 1-category whose objects are spaces and whose morphisms are isomorphism classes of spans of spaces equipped with the symmetric monoidal structure given by the cartesian product of spaces (see Definition 2.11). Since the moduli space $\mathrm{Map}_B(W' \circ_{M_1} W)$ of X -structures on a glued bordism $W' \circ_M W$ is homotopy equivalent to the (homotopy) pullback of moduli spaces $\mathrm{Map}_B(W', X) \times_{\mathrm{Map}_B(M, X)} \mathrm{Map}_B(W, X)$, and since the moduli space of X -structures on a disjoint union of (B, β) -manifolds is a product of moduli spaces of the components, we immediately obtain the following lemma:

Lemma 4.1. *For every map $\xi : X \rightarrow B$, the assignments of the spaces $\mathrm{Map}_B(M, X)$ to closed $(d - 1)$ -dimensional (B, β) -manifolds M and spans (27) to d -dimensional (B, β) -bordisms W assemble into a symmetric monoidal functor $\mathcal{F}_{X, \xi} : \mathrm{Bord}_d^{B, \beta} \rightarrow \mathrm{Span}(\mathcal{S})$. \square*

For any (B, β) -manifold M , any point $f \in \mathrm{Map}_B(M, X)$ induces a $(X, \beta \circ \xi)$ -structure on M . Hence, given a topological field theory $\mathcal{Z} : \mathrm{Bord}_d^{(X, \beta \circ \xi)} \rightarrow \mathrm{sVec}_k$ and a point $f \in \mathrm{Map}_B(M, X)$ we obtain a vector space $\mathcal{Z}(M, f)$. Likewise, a path in $\mathrm{Map}_B(M, X)$ induces an isomorphism of $(X, \beta \circ \xi)$ -structures on M and hence the structure of an (invertible) $(X, \beta \circ \xi)$ -bordism on $M \times [0, 1]$ between its endpoints (M, f_0) and (M, f_1) . Applying the TFT \mathcal{Z} induces an isomorphism of vector spaces $\mathcal{Z}(M, f_0) \rightarrow \mathcal{Z}(M, f_1)$ which only depends on the homotopy class of the path. In this way, the topological field theory \mathcal{Z} induces an sVec_k -valued local system \mathcal{L}_M on the moduli space $\mathrm{Map}_B(M, X)$. Similarly, for a (B, β) -bordism W and a point $f \in \mathrm{Map}_B(W, X)$ with $s(f) = f_0 \in \mathrm{Map}_B(M_0, X)$ and $t(f) = f_1 \in \mathrm{Map}_B(M_1, X)$, the topological field theory induces a morphism in sVec_k :

$$\alpha_W(f) := \mathcal{Z}(W, f) : \mathcal{L}_{M_0}(f_0) = \mathcal{Z}(M_0, f_0) \rightarrow \mathcal{Z}(M_1, f_1) = \mathcal{L}_{M_1}(f_1).$$

Functoriality of the TFT \mathcal{Z} shows that this defines a map of local systems $\alpha_W : s^* \mathcal{L}_{M_0} \rightarrow t^* \mathcal{L}_{M_1}$.

As in Definition 2.11, let $\mathrm{Span}(\mathcal{S}, \mathrm{sVec}_k)$ denote the symmetric monoidal 1-category of spans of spaces equipped with sVec_k -valued local systems. The above assignments of local systems and maps of local systems are compatible with gluing of bordisms and disjoint unions and hence may be summarized in the following lemma.

Lemma 4.2. *Let $\xi : X \rightarrow B$ be a map of spaces and let $\mathcal{Z} : \text{Bord}_d^{(X, \beta \circ \xi)} \rightarrow \text{sVec}_k$ be a symmetric monoidal functor. Then, the above choices of local systems and maps of local systems lift the functor $\mathcal{F}_{X, \xi} : \text{Bord}_d^{(B, \beta)} \rightarrow \text{Span}(\mathcal{S})$ to a symmetric monoidal functor $\mathcal{F}_{X, \xi, \mathcal{Z}} : \text{Bord}_d^{(B, \beta)} \rightarrow \text{Span}(\mathcal{S}, \text{sVec}_k)$. \square*

Our generalized Dijkgraaf-Witten theory is built by composing this functor $\mathcal{F}_{X, \xi, \mathcal{Z}} : \text{Bord}_d^{(B, \beta)} \rightarrow \text{Span}(\mathcal{S}, \text{sVec}_k)$ with the linearization functor $\text{Span}(\mathcal{S}^\pi, \text{sVec}_k) \rightarrow \text{sVec}_k$ from Corollary 2.13. To do so, we need to ensure that the moduli spaces $\text{Map}_B(M, X)$ appearing in the definition of $\mathcal{F}_{X, \xi}$ are π -finite. The following is an easy exercise in obstruction theory.

Lemma 4.3. *Let $\xi : X \rightarrow B$ be a map with π -finite fibers. Then, for any map $Y \rightarrow B$ where Y has the homotopy type of a finite CW complex (such as $Y = M$ for a compact (B, β) -structured manifold M), the space $\text{Map}_B(Y, X)$ is π -finite. \square*

Definition 4.4. Fix a map $\beta : B \rightarrow \text{BO}(d)$ thought of as an ambient tangential structure. Let $\xi : X \rightarrow B$ be a map with π -finite fibers and let $\mathcal{Z} : \text{Bord}_d^{(X, \beta \circ \xi)} \rightarrow \text{sVec}_k$ be a symmetric monoidal functor. The associated *generalized Dijkgraaf-Witten topological field theory* $\text{DW}_{X, \xi, \mathcal{Z}} : \text{Bord}_d^{(B, \beta)} \rightarrow \text{sVec}_k$ is the composite

$$\text{Bord}_d^{(B, \xi)} \xrightarrow{\mathcal{F}_{X, \xi, \mathcal{Z}}} \text{Span}(\mathcal{S}^\pi, \text{sVec}_k) \xrightarrow{\Phi} \text{sVec}_k,$$

where $\mathcal{F}_{X, \xi, \mathcal{Z}}$ is the functor from Lemma 4.1 and Φ is the functor from Corollary 2.13. If \mathcal{Z} is an invertible field theory which by Theorem 3.14 is uniquely characterized by a character $\omega : \Omega_d^{T(\beta \circ \xi)} \rightarrow k^\times$, we denote the associated Dijkgraaf-Witten theory by $\text{DW}_{X, \xi, \omega}$.

Example 4.5. If W is a closed d -dimensional (B, β) -manifold, the value of $\text{DW}_{X, \xi, \mathcal{Z}}$ on W may be unpacked to the following formula:

$$\text{DW}_{X, \xi, \mathcal{Z}}(W) = \sum_{[f] \in \pi_0 \text{Map}_B(W, X)} \#(\text{Map}_B(W, X), f) \mathcal{Z}([W, f]),$$

where $\mathcal{Z}([W, f])$ denotes the value of \mathcal{Z} on W with its $(X, \beta \circ \xi)$ -structure induced by its (B, β) -structure and $f \in \text{Map}_B(W, X)$.

In particular, if \mathcal{Z} is invertible and hence determined by a character $\omega : \Omega_d^{T(\beta \circ \xi)} \rightarrow k^\times$, this becomes

$$(28) \quad \text{DW}_{X, \xi, \omega}(W) = \sum_{[f] \in \pi_0 \text{Map}_B(W, X)} \#(\text{Map}_B(W, X), f) \omega([W, f]),$$

where $[W, f] \in \Omega_d^{T(\beta \circ \xi)}$ is the stable tangential $(B, \beta \circ \xi)$ -structure on W induced by f and the (unstable) (B, β) -structure on W .

Example 4.6. Let Y be a π -finite space with a d -dimensional k^\times -valued cohomology class $\alpha \in H^d(Y, k^\times)$. As in Example 3.19 we obtain an invertible X -structured topological field theory where $X = Y \times \text{BSO}(d)$. An X -structured d -dimensional manifold consists of an oriented manifold M together with a map $f : M \rightarrow Y$. The partition function of this invertible topological field theory (i.e. the character $\omega : \Omega_d^{Y \times \text{BSO}} \rightarrow k^\times$) is given by pairing with the fundamental class $[M]$ of M :

$$M \mapsto \langle [M], f^* \alpha \rangle$$

Set $B = BSO(d)$ and let $\xi : X \rightarrow B$ be projection. Then Definition 4.4 produces an oriented topological field theory $DW_{X,\xi,\omega} : \text{Bord}_d^{\text{BSO}(d)} \rightarrow \text{sVec}$ out of the above invertible topological field theory. This field theory factors through Vec and recovers Quinn's twisted finite homotopy TQFT [Qui95] for (Y, α) as a special case of Definition 4.4.

In particular, if $Y = BG$ is the classifying space of a finite group G , this recovers classical twisted Dijkgraaf-Witten theory.

4.2. The Dijkgraaf-Witten invariant via Pontryagin pairings. Comparing Formula (28) with Definition 2.40, we see that $DW_{X,\xi,\omega}(W)$ should be understood as the Pontryagin pairing (as in Definition 2.40) of the ∞ -category $\mathcal{S}/_B$ of spaces over B (whose mapping spaces are precisely $\text{Map}_B(Y, X)$) and the functor $\Omega_d : \mathcal{S}/_B \rightarrow \text{Ab}$ which maps a $\xi : X \rightarrow B$ to the unstable bordism group $\Omega_d^{T(\beta \circ \xi)}$ (Definition 3.12). However, the category $\mathcal{S}/_B$ is not locally π -finite and its Pontryagin pairing is therefore not defined everywhere.

We therefore follow Remark 2.30 and 2.41 and consider the restricted Pontryagin pairing

$$\langle -, - \rangle_{\mathcal{S}/_B, \Omega_d}^{\mathcal{S}/_B^f, \mathcal{S}/_B^\pi} : \bigoplus_{[\xi] \in \pi_0 \mathcal{S}/_B^\pi} k[\widehat{\Omega_d^{T(\beta \circ \xi)}} / \pi_0 \text{Aut}_{\mathcal{S}/_B}(\xi)] \otimes \bigoplus_{[\psi] \in \pi_0 \mathcal{S}/_B^f} k[\Omega_d^{T(\beta \circ \psi)} / \pi_0 \text{Aut}_{\mathcal{S}/_B}(\psi)] \rightarrow k$$

between the full subcategory $\mathcal{S}/_B^f$ of $\mathcal{S}/_B$ on those $\psi : Y \rightarrow B$ for which Y is homotopy equivalent to a finite CW complex and the full subcategory $\mathcal{S}/_B^\pi$ of those $\phi : X \rightarrow B$ with π -finite fibers.

Lemma 4.7. *For any closed d -dimensional (B, β) -manifold W , the following holds:*

$$DW_{X,\xi,\omega}(W) = \left\langle (\xi : X \rightarrow B, \omega), (\theta : W \rightarrow B, [W]) \right\rangle_{\mathcal{S}/_B, \Omega_d}^{\mathcal{S}/_B^f, \mathcal{S}/_B^\pi}$$

Here, $\theta : W \rightarrow B$ is the map which comes as part of the (B, β) -structure on W , and $[W] \in \Omega_d^{T(\beta \circ \theta)}$ is the stable tangential $(W, \beta \circ \theta)$ -structured bordism class induced by W with its unstable $(W, \beta \circ \theta)$ -structure (where the lower triangle encodes the given (B, β) -structure on W):

$$\begin{array}{ccc} & & W \\ & \nearrow \text{id} & \downarrow \theta \\ & W & B \\ & \searrow \theta & \downarrow \beta \\ W & \xrightarrow{TW} & \text{BO}(d) \end{array}$$

Proof. This follows immediately from comparing the expression (28) with (24) and Remarks 2.41 and 2.30. \square

In particular, $DW_{X,\xi,\omega}(W) \in k$ only depends on the equivalence class of the map $\theta : W \rightarrow B$ in $\mathcal{S}/_B$ and the orbit of $[W] \in \Omega_d^{T(\beta \circ \theta)}$ under the $\text{Aut}_{\mathcal{S}/_B}(\theta)$ action. Likewise, it also only depends on ξ up to equivalence in $\mathcal{S}/_B$ and only on the orbit of ω under the $\text{Aut}_{\mathcal{S}/_B}(\xi)$ action on the group of characters $\widehat{\Omega_d^{T(\beta \circ \xi)}}$.

4.3. Dijkgraaf-Witten theories of type n . For a systematic study of the sensitivity of Dijkgraaf-Witten theories, we introduce the following filtration on the collection of Dijkgraaf-Witten theories.

Definition 4.8. Fix an ambient tangential structure $B \rightarrow \mathrm{BO}(d)$ and let $n \geq 0$ be an integer. We say that a Dijkgraaf-Witten theory $\mathrm{DW}_{X,\xi,\omega}$ associated to a map $\xi : X \rightarrow B$ and a character $\omega : \Omega^{T(\beta \circ \xi)} \rightarrow k^\times$ is of type n , if ξ is n -truncated (see Example 2.23).

We will also say that two closed B -structured d -manifolds M and N are *indistinguishable by type n Dijkgraaf-Witten theories* if for all n -truncated $\xi : X \rightarrow B$ and all characters $\omega : \Omega^{T(\beta \circ \xi)} \rightarrow k^\times$,

$$\mathrm{DW}_{X,\xi,\omega}(M) = \mathrm{DW}_{X,\xi,\omega}(N) \in k.$$

Note that if $\pi_0 B$ is finite, then any map $\xi : X \rightarrow B$ with π -finite fibers is n -truncated for some $n \geq 0$, and hence every generalized Dijkgraaf-Witten theory is of type n for some $n \geq 0$.

Let $\tau_{\leq n} \mathcal{S}_{/B}$ denote the full subcategory of $\mathcal{S}_{/B}$ on those $\xi : X \rightarrow B$ which are n -truncated.

Given a map $X \rightarrow B$, the n -connected/ n -truncated factorization system on the ∞ -category of spaces \mathcal{S} induces a factorization

$$X \rightarrow \tau_{\leq n}^B X \rightarrow B$$

into an n -connected map $X \rightarrow \tau_{\leq n}^B X$ followed by an n -truncated map $\tau_{\leq n}^B X \rightarrow B$. In this way, given an $X \in \mathcal{S}_{/B}$, we functorially obtain an $\tau_{\leq n}^B X \in \tau_{\leq n} \mathcal{S}_{/B}$ which we will henceforth refer to as the B - n -type of X .

Example 4.9. Let M be a smooth d -manifold with $T_M : M \rightarrow \mathrm{BO}(d)$ classifying the tangent bundle of M . The map T_M has a factorization (unique up to homotopy) $M \rightarrow \tau_{\leq n}^{\mathrm{BO}(d)} M \rightarrow \mathrm{BO}(d)$ into an n -connected map followed by an n -truncated map (see Example 2.23). $\tau_{\leq n}^{\mathrm{BO}(d)} M$ is the *tangential n -type* of M (c.f. the stable tangential and normal n -types of M (Example 3.3))

Definition 4.10. Given an $X \in \mathcal{S}_{/B}$, we will say that it has π -finite B - n -type if its B - n -type $\tau_{\leq n}^B X \rightarrow B$ has π -finite fibers. Equivalently, $X \in \mathcal{S}_{/B}$ has π -finite B - n -type if the first n homotopy groups $\pi_k, 0 \leq k \leq n$ of all fibers of $X \rightarrow B$ are finite. We let $\tau_{\leq n} \mathcal{S}_{/B}^\pi$ denote the full subcategory of $\mathcal{S}_{/B}$ on spaces with π -finite B - n -type.

To apply the non-degeneracy results of Section 2.5, we need to introduce a mild finiteness condition on the space B describing the ambient tangential structure. This finiteness condition is fulfilled by the classifying spaces $\mathrm{BO}(d)$ and their connective covers, and includes all examples of interest that we are aware of. This condition is discussed more thoroughly in Appendix A.

Assumption 4.11. For most of the following, we will assume that the space B is n -finitely dominated (Definition A.1). This includes the spaces BO , $\mathrm{BO}(d)$, and any k -connective covers of these spaces such as $\mathrm{BSO}(d)$, $\mathrm{BSpin}(d)$, etc. (Example A.7).

Under Assumption 4.11, the category $\tau_{\leq n} \mathcal{S}_{/B}^\pi$ is locally π -finite (Corollary A.9). Thus $\tau_{\leq n} \mathcal{S}_{/B}^\pi$ has an associated Pontryagin pairing. The generalized Dijkgraaf-Witten invariant can be expressed in terms of this pairing:

Lemma 4.12. *Let $\beta : B \rightarrow \mathrm{BO}(d)$ be an ambient tangential structure satisfying Assumption 4.11, let $\xi : X \rightarrow B$ be an n -truncated map with π -finite fibers, and let $\omega : \Omega_d^{T(\beta \circ \xi)} \rightarrow k^\times$ be a group homomorphism.*

Then, for any closed d -dimensional (B, β) -manifold W with π -finite B - n -type, the generalized Dijkgraaf-Witten invariant is given by

$$\mathrm{DW}_{X,\xi,\omega}(W) = \left\langle (\xi, \omega), (\tau_{\leq n}^B W, [W]) \right\rangle_{\tau_{\leq n} \mathcal{S}_{/B}^\pi, \Omega_d}$$

where $\langle -, - \rangle_{\tau_{\leq n} \mathcal{S}_{/B}^\pi, \Omega_d}$ denotes the Pontryagin pairing associated to the locally π -finite ∞ -category $\tau_{\leq n} \mathcal{S}_{/B}^\pi$ and the functor Ω_d (see Definition 2.40). Here, $[W] \in \Omega_d^{T(\beta \circ \tau_{\leq n}^B W)} / \pi_0 \text{Aut}_{\mathcal{S}_{/B}}(\tau_{\leq n}^B W \rightarrow B)$ is the bordism class of W with its canonical $\tau_{\leq n}^B W$ -structure.

Proof. The lemma immediately follows from (28) and the fact that the truncation functor $\mathcal{S}_{/B} \rightarrow \tau_{\leq n} \mathcal{S}_{/B}$ is left adjoint to the inclusion $\tau_{\leq n} \mathcal{S}_{/B} \rightarrow \mathcal{S}_{/B}$: for any $Y \rightarrow B$ and n -truncated $X \rightarrow B$ the canonical map $\text{Map}_B(\tau_{\leq n}^B Y, X) \rightarrow \text{Map}_B(Y, X)$ is an equivalence. \square

4.4. The main theorem. The k -connected/ k -truncated factorization system $(\mathcal{L}^{(k)}, \mathcal{R}^{(k)})$ on spaces induces one on $\mathcal{S}_{/B}$, the category of spaces over B . Explicitly, a map of spaces $f : X \rightarrow Y$ over B is k -connected or k -truncated if the underlying map of spaces $f : X \rightarrow Y$ is k -connected or k -truncated, respectively. This connected/truncated factorization system restricts to a factorization system on the subcategories $\tau_{\leq n} \mathcal{S}_{/B}$ of B - n -types and $\tau_{\leq n} \mathcal{S}_{/B}^\pi$ of π -finite B - n -types.

These define a nested sequence of factorization systems as in Definition 2.21, i.e. $\mathcal{R}^{(k-1)} \subseteq \mathcal{R}^{(k)}$ or equivalently $\mathcal{L}^{(k+1)} \subseteq \mathcal{L}^{(k)}$. In particular, we have subcategories $\mathcal{T}^{(\ell)} \subseteq \tau_{\leq n} \mathcal{S}_{/B}^\pi$ for $-2 \leq \ell \leq n$:

$$\mathcal{T}^{(\ell)} = \begin{cases} \mathcal{R}^{(-1)} & \text{if } \ell = -2 \\ \mathcal{R}^{(\ell+1)} \cap \mathcal{L}^{(\ell)} & \text{if } -1 \leq \ell \leq n-1 \\ \mathcal{L}^{(n)} & \text{if } \ell = n \end{cases}$$

The morphisms of \mathcal{T}^ℓ are those maps $f : X \rightarrow Y$ over B for which $\pi_\ell(f)$ is surjective, $\pi_{\ell+1}(f)$ is injective, and $\pi_i(f)$ is an isomorphism for all $i \neq \ell, \ell+1$.

Remark 4.13. In Definition 2.21 the indexing of the nested system was for $0 \leq \ell \leq n$, but here it is more convenient to index on $-2 \leq \ell \leq n$. This makes no substantive change and we hope it will not lead to confusion.

For π -finite B - n -types, endomorphisms in $\mathcal{T}^{(\ell)}$ are automatically weak equivalences since endomorphism of finite groups which are either injections or surjections are automatically isomorphisms. Combined with Theorem 2.38 and Corollary 2.42, we obtain the following theorem.

Theorem 4.14. *Let $\beta : B \rightarrow \text{BO}(d)$ be an ambient tangential structure and let $n \geq 0$. Let M and N be d -dimensional (B, β) -manifolds with π -finite B - n -types.*

Then M and N are indistinguishable by type- n generalized Dijkgraaf-Witten theories if and only if

1. *they have equivalent B - n -types $\tau_{\leq n}^B M \simeq (Y, \psi) \simeq \tau_{\leq n}^B N$;*
2. *the bordism classes $[M]$ and $[N]$ in $\Omega_d^{T(\beta \circ \psi)} / \pi_0 \text{Aut}_{\mathcal{S}_{/B}}(\psi)$ lie in the same orbit under the action of*

Proof. With Assumption 4.11, we may argue as follows. By Lemma 4.12, the value of $\text{DW}_{X, \xi, \omega}(M)$ for any n -truncated $X \rightarrow B$ and ω is given by the Pontryagin pairing, and hence only depends on the vector M induces in $\bigoplus_{\psi \in \pi_0 \tau_{\leq n} \mathcal{S}_{/B}^\pi} k[\Omega_d^{T(\beta \circ \psi)} / \pi_0 \text{Aut}_{\mathcal{S}_{/B}}(\psi)]$, and hence only on data (1) and (2).

Conversely, since the truncated/connected nested factorization system on $\tau_{\leq n} \mathcal{S}_{/B}^\pi$ has the property that endomorphisms in $\mathcal{T}^{(\ell)}$ are isomorphisms, it follows from Corollary 2.42 that the Pontryagin pairing is non-degenerate and hence that if M and N are indistinguishable by Dijkgraaf-Witten theories, then they induce the same data (1) and (2).

However Theorem 4.14 holds without Assumption 4.11. In that case in place of $\tau_{\leq n}\mathcal{S}_{/B}^\pi$ we use the category $\tau_{\leq n}\mathcal{S}_{/B}^{\pi, \text{f.d.}}$ described in Appendix A. Corollary A.9 states that it is locally π -finite, and thus, arguing as before, its Pontryagin pairing is non-degenerate. The tangential n -types of compact manifolds will be n -finitely dominated and hence live in $\tau_{\leq n}\mathcal{S}_{/B}^{\pi, \text{f.d.}}$. In this case it is in fact sufficient to use type- n generalized Dijkgraaf-Witten based on n -finitely dominated (Y, ψ) . \square

Lemma 4.12 shows that the partition function of generalized Dijkgraaf-Witten theories can be computed using the Pontryagin pairing for B -manifolds which have π -finite B - n -type. If every π -finite B - n -type ξ and element of $\Omega_d^T \xi$ were represented by such a manifold, then the non-degeneracy of the Pontryagin pairing on $\tau_{\leq n}\mathcal{S}_{/B}^\pi$ would immediately lead to a converse to Theorem 4.14 - that they are determined by their partition functions. However it is a subtle and interesting surgery question to determine which elements of $\Omega_d^T \xi$ can be so represented. The following result of Kreck will allow us to obtain a partial converse to Theorem 4.14.

Proposition 4.15 ([Kre99, Prop. 4]). *Suppose that $n < \lfloor \frac{d}{2} \rfloor$. Let $\xi : X \rightarrow B$ be a π -finite n - B -type (or more generally X need only be n -finitely dominated). Then every class $[M] \in \Omega^{TX_d}$ may be represented by a (X, ξ) -manifold N such that the map $N \rightarrow X$ identifies $\tau_{\leq n}^B N \simeq X$. \square*

Theorem 4.16. *Suppose that $n < \lfloor \frac{d}{2} \rfloor$ and that Assumption 4.11 holds. If $((X_1, \xi_1), \omega_1)$ and $((X_2, \xi_2), \omega_2)$ give rise to type- n generalized Dijkgraaf-Witten theories whose partition functions are identical on closed d -manifolds with π -finite tangential n -type, then $X_1 \simeq X_2 = X$ (over B), and ω_1 and ω_2 have the same restriction to the fixed points of $\Omega_d^{T\xi}$ under the action of $\pi_0 \text{Aut}(\xi)$. The resulting generalized Dijkgraaf-Witten theories are consequently isomorphic.*

Proof. Proposition 4.15 implies that every element in $\bigoplus_{\psi \in \pi_0 \tau_{\leq n} \mathcal{S}_{/B}^\pi} k[\Omega_d^{T(\beta \circ \psi)} / \pi_0 \text{Aut}_{\mathcal{S}_{/B}}(\psi)]$ is represented by a (linear combination) of B -manifolds with π -finite n - B -type, and Lemma 4.12 shows that the partition function of generalized Dijkgraaf-Witten theories on these manifolds can be computed using the Pontryagin pairing on $\tau_{\leq n}\mathcal{S}_{/B}^\pi$. The non-degeneracy of this pairing implies that $((X_1, \xi_1), \omega_1)$ and $((X_2, \xi_2), \omega_2)$ represent the same element in $\bigoplus_{[\xi] \in \pi_0 \mathcal{C}} k[\widehat{\Omega_d^T \xi} / \pi_0 \text{Aut}_{\mathcal{C}}(\xi)]$, which immediately implies the claimed result. \square

4.5. Dimensional reduction of Dijkgraaf-Witten theories. The *dimensional reduction* of a field theory $\mathcal{Z} : \text{Bord}_d \rightarrow \mathcal{C}$ along a closed $(d-k)$ -manifold M is defined to be the field theory $\mathcal{Z}(M \times -) : \text{Bord}_k \rightarrow \mathcal{C}$.

Dimensional reduction of tangentially structured field theories is considered in [SP18]. Given tangential structures $B_{d-k} \rightarrow \text{BO}(d-k)$, $B_k \rightarrow \text{BO}(k)$ and $B \rightarrow \text{BO}(d)$ fitting into a commutative diagram

$$(29) \quad \begin{array}{ccc} B_k \times B_{d-k} & \longrightarrow & B \\ \downarrow & & \downarrow \\ \text{BO}(k) \times \text{BO}(d-k) & \longrightarrow & \text{BO}(d) \end{array}$$

the product of a B_{d-k} -structured $(d-k)$ -manifold M and a B_k -structured k -manifold N induces a canonical B structure on $M \times N$.

Therefore, any closed B_{d-k} -structured $(d-k)$ -manifold M induces a symmetric monoidal functor $\text{red}_M = (M \times -) : \text{Bord}_k^{B_k} \rightarrow \text{Bord}_d^B$, and any B -structured d -dimensional field theory may be dimensionally reduced along M to a B_k -structured k -dimensional field theory.

In fact, for any (unstructured) $(d-k)$ -manifold M and any structure $B \rightarrow \text{BO}(d)$, there is a universal k -dimensional tangential structure $B_M \rightarrow \text{BO}(k)$ which is essentially defined so that the data of a B_M -structure on a k -manifold N is precisely the data of a B -structure on $M \times N$ and which hence gives rise to a *total dimensional reduction* functor $\text{red}_M^{\text{tot}} = (M \times -) : \text{Bord}_k^{B_M} \rightarrow \text{Bord}_d^B$ (see [SP18, Sec 9.2]).

We will later need a version of this construction relative to given tangential structures in a commutative square (29): Suppose $X \rightarrow B$ is a space over B . Then, given any B_{d-k} -structured closed $(d-k)$ -manifold $M = (M, \theta)$, we define a map $X_M \rightarrow B_k$ as the following pullback:

$$(30) \quad \begin{array}{ccc} X_M & \xrightarrow{\quad\quad\quad} & \text{Map}(M, X) \\ \downarrow \lrcorner & & \downarrow \\ B_k \times \{\theta\} & \longrightarrow & B_k \times \text{Map}(M, B_{d-k}) \longrightarrow \text{Map}(M, B) \end{array}$$

As in [SP18, Sec 9.2], the structure $X_M \rightarrow B_k$ is universal in the sense that for a B_k -structured k -manifold N , a lift of the canonical B -structure on $M \times N$ along $X \rightarrow B$ to an X structure precisely amounts to a lift of the B_k -structure on N along $X_M \rightarrow B_k$ to a X_M structure. In particular, M induces a *total dimensional reduction* symmetric monoidal functor $\text{red}_M^{\text{tot}} = (M \times -) : \text{Bord}_k^{X_M} \rightarrow \text{Bord}_d^X$.

Remark 4.17. The dimensional reduction and total dimensional reductions of invertible TFTs are again invertible.

4.5.1. Dimensional reduction of Dijkgraaf-Witten theories.

Theorem 4.18. *Consider a B -structured d -dimensional generalized Dijkgraaf-Witten theory $\text{DW}_{X, \mathcal{W}} : \text{Bord}_d^B \rightarrow \text{sVect}$ constructed as in Def. 4.4 from a map $X \rightarrow B$ with π -finite fibers and a field theory $\mathcal{W} : \text{Bord}_d^X \rightarrow \text{sVect}$. Given tangential structures as in the commutative square (29) and a closed B_{d-k} -structured $(d-k)$ -manifold M , the dimensionally reduced theory $\text{DW}_{(X, \mathcal{W})} \circ \text{red}_M : \text{Bord}_k^{B_k} \rightarrow \text{sVect}$ is equivalent to the generalized Dijkgraaf-Witten theory $\text{DW}_{(X_M, \mathcal{W} \circ \text{red}_M^{\text{tot}})}$ built from the map $X_M \rightarrow B_k$ defined in (30) and the total dimensional reduction of \mathcal{W} to a field theory $\mathcal{W} \circ \text{red}_M^{\text{tot}} = \mathcal{W}(M \times -) : \text{Bord}_k^{X_M} \rightarrow \text{sVect}$.*

Proof. First we establish that $X_M \rightarrow B_k$ has π -finite homotopy fibers, so that the dimensional reduction $\text{DW}_{(X_M, \mathcal{W} \circ \text{red}_M^{\text{tot}})}$ is meaningful. Since (30) is a pullback square, the fiber of $X_M \rightarrow B_k$ at a point $b_k \in B_k$ agrees with the fiber of $\text{Map}(M, X) \rightarrow \text{Map}(M, B)$ at the map $M \rightarrow B_{d-k} \xrightarrow{B_{d-k} \times b_k} B_{d-k} \times B_k \rightarrow B$. Since the fibers of $X \rightarrow M$ are π -finite and M is a compact manifold, this fiber is also π -finite.

Furthermore, by construction for each B_k -manifold N we have a natural equivalence of spaces:

$$\text{Map}_{B_k}(N, X_M) \simeq \text{Map}_B(M \times N, X),$$

and a commuting diagram:

$$\begin{array}{ccc}
\text{Map}_B(M \times N, X) & \xrightarrow{\mathcal{W}} & \text{sVect} \\
\text{red}_M^{\text{tot}} \uparrow \simeq & & \\
\text{Map}_{B_k}(N, X_M) & \xrightarrow{\mathcal{W} \circ \text{red}_M^{\text{tot}}} &
\end{array}$$

Thus we have a natural equivalence of (super) vector spaces:

$$\begin{aligned}
\text{DW}_{(X, \mathcal{W})} \circ \text{red}_M(N) &= \bigoplus_{[\psi] \in \pi_0 \text{Map}_B(M \times N, X)} (\mathcal{W}(M \times N, \psi))_{\pi_1(\text{Map}_B(M \times N, X), \psi)} \\
&\cong \bigoplus_{[\psi] \in \pi_0 \text{Map}_{B_k}(N, X_M)} (\mathcal{W} \circ \text{red}_M^{\text{tot}}(N))_{\pi_1(\text{Map}_{B_k}(N, X_M), \psi)} \\
&= \text{DW}_{(X_M, \mathcal{W} \circ \text{red}_M^{\text{tot}})}(N).
\end{aligned}$$

Similarly, for a k -dimensional B_k -bordism W from N_0 to N_1 , the value of $\text{DW}_{(X, \mathcal{W})} \circ \text{red}_M(W)$ is the linearization of the following decorated span

$$(\text{Map}_B(M \times N_1, X), \mathcal{L}_{\mathcal{W}}) \xleftarrow{t} (\text{Map}(M \times W, X), \alpha^{\mathcal{W}}) \xrightarrow{s} (\text{Map}_B(M \times N_0, X), \mathcal{L}_{\mathcal{W}}).$$

The value of $\text{DW}_{(X_M, \mathcal{W} \circ \text{red}_M^{\text{tot}})}(W)$ is the linearization of the following equivalent decorated span

$$(\text{Map}_{B_k}(N_1, X_M), \mathcal{L}_{\mathcal{W} \circ \text{red}_M^{\text{tot}}}) \xleftarrow{t} (\text{Map}_{B_k}(W, X_M), \alpha^{\mathcal{W} \circ \text{red}_M^{\text{tot}}}) \xrightarrow{s} (\text{Map}_{B_k}(N_0, X_M), \mathcal{L}_{\mathcal{W} \circ \text{red}_M^{\text{tot}}})$$

and thus the two TFTs are isomorphic. \square

Example 4.19. Consider a Dijkgraaf-Witten theory in the classical sense, where $B = BSO(d)$, $X = BSO(d) \times BG \rightarrow BSO(d)$ for G a finite group, and $\mathcal{W} : \text{Bord}_d^{or, G} \rightarrow \text{sVec}$ is a G -equivariant field theory, which leads to an oriented field theory $\text{DW}_{\mathcal{W}, G} : \text{Bord}_k^{or} \rightarrow \text{sVec}$. Let M be an oriented $(d-r)$ -manifold. Then, Theorem 4.18 asserts that the dimensionally reduced⁷ theory $\text{DW}_{\mathcal{W}, G}(M \times -) : \text{Bord}_k^{or} \rightarrow \text{sVec}$ is the k -dimensional oriented Dijkgraaf-Witten theory built from the finite 1-groupoid

$$\text{Map}(M, BG) \cong \bigsqcup_{\substack{\text{isomorphism classes } [P] \\ \text{of } G\text{-bundles over } M}} \text{BAut}(P)$$

and $\mathcal{W}(M \times -)$ considered as a $\text{Map}(M, BG)$ -equivariant field theory. This recovers (the unextended version of) a result of Müller and Woike [MW21].

5. SEMISIMPLE FIELD THEORIES AND STABLE DIFFEOMORPHISM

The goal of this section is two-fold. In [Reu20], the first author showed that *semisimple* oriented four-dimensional field theories valued in the category of vector spaces lead to stable diffeomorphism invariants. Our first aim is to generalize this to higher dimensions, more general tangential structures, and to TFTs valued in super vector spaces. These results give an upper bound on what this class of topological field theories can detect about smooth manifolds. Our second aim is to show that generalized Dijkgraaf-Witten theories satisfy this semisimplicity condition.

⁷In the notation of Theorem 4.18, we set $B_k = BSO(k)$ and $B_{d-k} = BSO(d-k)$ and consider the evident maps in the square (29).

As in [Reu20], an oriented four-dimensional field theory is defined to be semisimple, if the algebras $Z(S^3)$ and $Z(S^2 \times S^1)$, with multiplication and unit given as follows, respectively, are semisimple:

$$(31) \quad Z(D^4 \setminus (D^4 \sqcup D^4)) \quad Z(D^4)$$

$$(32) \quad Z((D^3 \setminus D^3 \sqcup D^3) \times S^1) \quad Z(D^3 \times S^1)$$

A physical interpretation of these algebra objects is given in [Reu20, Rem 2.2], similar interpretations apply to the algebra objects in the rest of this section.

In order to generalize these results to B -structured field theories, the spheres and bordisms inducing the unital multiplication must all admit B -structures. Likewise, in order to define stable diffeomorphism of B -manifolds, one needs to have a well-defined notion of connected sum, and for the product of middle dimensional spheres to admit a canonical B -structure. These two issues are closely related, and, as we will see shortly, will require us to make mild assumptions on the ambient tangential structure B .

5.1. Connected sum of B -manifolds and stable diffeomorphism. Let $\beta : B \rightarrow \text{BO}(d)$ be a tangential structure. The tangent bundle of the disk D^d is trivial, and thus it is always possible to give the disk a B -structure. In fact if we assume that B is connected, up to isotopy B -structures on the disk correspond to elements of $\pi_0 F$, where F is the homotopy fiber of β , and this can either be one element or two depending on whether $\pi_1 B \rightarrow \pi_1 \text{BO}(d)$ is surjective or not. Equivalently it depends on whether $\beta^* w_1$ is non-trivial or trivial. In the latter case the two possible isotopy classes of lifts result in diffeomorphic B -manifolds which are exchanged by an orientation reversing diffeomorphism.

The $(d-1)$ -sphere S^{d-1} is the boundary of D^d , but to get an induced B -structure on S^{d-1} , we must choose either an inward pointing or outward pointing of the normal bundle of S^{d-1} in D^d . These correspond to viewing the disk either as a bordism from the empty manifold to the sphere, or as a bordism from the sphere to the empty manifold. There are thus potentially two canonical diffeomorphism classes of B -structures on the $(d-1)$ -sphere — the *bounding* and the *cobounding* B -structures. Without further restrictions on B it is possible for these to be distinct. For example, when the tangential structure β is tangential framing $* \rightarrow \text{BO}(d)$, then the bounding and cobounding framings on S^{d-1} only agree for $d = 0, 1, 3, 7$, i.e. precisely when S^d admits a tangential framing.

The operation of connected sum requires removing disks from the manifolds in question and then replacing the result with a cylinder. In the presence of tangential B -structures this requires that the disks, cylinders, and their boundary spheres all have compatible B -structures. In particular, the bounding and cobounding B -structures on S^{d-1} need to coincide. For this we need to place some restrictions on the ambient tangential structure B .

Lemma 5.1. *Let $\beta : B \rightarrow \text{BO}(d)$ be a tangential n -type. Assume that B is connected and that $n \leq d - 2$, then S^{d-1} admits a unique B -structure up to orientation reversal diffeomorphism. This B -structure is both bounding and cobounding; it extends both over the incoming and outgoing disk D^d . The gluing of these B -disks defines the unique B -structure on S^d up to orientation reversal diffeomorphism.*

Proof. Obstruction theory shows that S^{d-1} admit a B -structure and that there can be at most two depending on whether $\beta^* w_1$ is non-trivial or trivial. On the other hand, D^d admits precisely two B -structures which are exchanged by an orientation reversing diffeomorphism. Hence, by uniqueness, the two B -structures on S^{d-1} must bound these B -structures on D^d and are exchanged by the restriction of the orientation reversing diffeomorphism of D^d to

the boundary. Similarly, both B -structures on S^{d-1} are cobounding. By obstruction theory, S^d has a unique B -structure (up to orientation-reversal) which hence must be given by the gluing of the unique (up to orientation-reversal) B -structures on D^d . \square

In particular, if B fulfills the assumptions of Lemma 5.1, the *connected sum* $M\#N$ of two connected B -manifolds may be defined by excising a disk from both M and N and gluing the resulting boundary spheres (possibly along an orientation reversal diffeomorphism). Lemma 5.1 guarantees that this procedure results in a well-defined B -manifold.

We can also obtain canonical B -structures on products of spheres. As in Section 3.5, we can compose $\beta : B \rightarrow \mathrm{BO}(d)$ with the inclusion $\mathrm{BO}(d) \rightarrow \mathrm{BO}(d+2)$, and then factor the resulting map into an n -connected map followed by an n -truncated map:

$$B \xrightarrow{n\text{-connected}} B_{d+2} \xrightarrow{n\text{-truncated}} \mathrm{BO}(d+2).$$

By Proposition 3.21, if $n \leq d-2$, the resulting diagram

$$\begin{array}{ccc} B & \longrightarrow & B_{d+2} \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{BO}(d) & \longrightarrow & \mathrm{BO}(d+2) \end{array}$$

is a homotopy pull-back diagram. Let $a+b=d$, and consider the disk $D^{a+1} \times D^{b+1}$. The disk admits a B_{d+2} -structure (which is unique up to orientation reversal). This restricts to a B_{d+2} -structure on $S^a \times S^b$, which is equivalent to a B -structure on $S^a \times S^b$ by virtue of the above square being a pullback.

Definition 5.2. The B -structure on $S^a \times S^b$ so obtained will be called the *canonical B -structure* and we will denote it by $\bar{\theta}$.

Unlike the case of Lemma 5.1, this B -structure is in general not unique — there can be many other B -structures on $S^a \times S^b$.

Definition 5.3. If $d=2q$, and $\beta : B \rightarrow \mathrm{BO}(d)$ satisfies the assumptions of Lemma 5.1, then two B -structured connected manifolds (M_1, ψ_1) and (M_2, ψ_2) are *B -stably diffeomorphic* if there exists a natural number N and a diffeomorphism of B -manifolds

$$(M_1, \psi_1) \# (\#^N(S^q \times S^q, \bar{\theta})) \cong (M_2, \psi_2) \# (\#^N(S^q \times S^q, \bar{\theta})).$$

where $\bar{\theta}$ is the canonical B -structure on $S^q \times S^q$.

Assuming sufficient additional truncatedness of β , the canonical B -structure on $S^q \times S^q$ used in Definition 5.3 is indeed unique and may be omitted from the notation.

Lemma 5.4. *If $d=2q$, and $\beta : B \rightarrow \mathrm{BO}(d)$ is a tangential $(q-1)$ -type with B connected, then the canonical B -structure on $S^q \times S^q$ is unique (up to orientation reversal). \square*

5.2. Local operators control decompositions of topological field theories. The *algebra of local operators* of a d -dimensional oriented TFT is the Frobenius algebra $Z(S^{d-1})$ with unit $Z(D^d)$ and multiplication $Z(D^d \setminus (D^d \sqcup D^d))$. This algebra may be generalized to other B -structures as follows:

Lemma 5.5. *Let $\beta : B \rightarrow \mathrm{BO}(d)$ satisfy the assumptions of Lemma 5.1. Then for any topological field theory $Z : \mathrm{Bord}_d^B \rightarrow \mathrm{sVec}$, the value of the $(d-1)$ -sphere $Z(S^{d-1})$ (with its*

unique B -structure) is a super (commutative for $d > 1$) Frobenius algebra with multiplication and unit given as follows:

$$Z(D^d \setminus (D^d \sqcup D^d)) \quad Z(D^d).$$

Proof. Assuming that B is connected and $\beta : B \rightarrow \mathrm{BO}(d)$ is a tangential $(d - 2)$ -type, consider the unique (up to orientation-reversal) B -structure on S^d . Removing a number of incoming and outgoing disks results in a bordism from $\sqcup_m S^{d-1}$ to $\sqcup_\ell S^{d-1}$ where each of these $(d - 1)$ -spheres is given the unique (up to orientation-reversal) B -structure which is simultaneously bounding and cobounding. Analogous to the oriented case, these B -bordisms assemble into the structure of a commutative Frobenius algebra. \square

Following a theorem of Sawin [Saw95], we will now show that this algebra $Z(S^{d-1})$ controls direct sum decomposition of the field theory. Given a finite family of field theories $Z_i : \mathrm{Bord}_d^B \rightarrow \mathrm{sVec}$, $i \in I$, we follow [DJ94, Saw95] and define their *direct sum* $\bigoplus_{i \in I} Z_i$ as the following field theory: To a non-empty connected $(n - 1)$ -manifold M , it assigns the super vector space $\bigoplus_{i \in I} Z_i(M)$ and the tensor product of these to non-connected manifolds. Similarly, to a non-empty connected compact bordism W , the field theory $\bigoplus_{i \in I} Z_i$ assigns the direct sum $\bigoplus_{i \in I} Z_i(W)$, interpreted appropriately as a map between the appropriate tensor products of direct sums.

Direct sums decompositions of a field theory are determined by direct sum decompositions of the algebra $Z(S^{d-1})$.

Lemma 5.6. *Let $\beta : B \rightarrow \mathrm{BO}(d)$ satisfy the assumptions of Lemma 5.1 and let $Z : \mathrm{Bord}_d^B \rightarrow \mathrm{sVec}$ be a topological field theory. Suppose that there is a direct sum decomposition $Z(S^{d-1}) \cong \bigoplus_{i \in I} A_i$ of super-algebras. Then, Z admits a direct sum decomposition $Z \cong \bigoplus_{i \in I} Z_i$ into field theories with algebra isomorphisms $Z_i(S^{n-1}) \cong A_i$.*

Proof. In the oriented case this is [Saw95, Thm. 1]. The proof in the presence of general B -structure (satisfying the assumptions of Lemma 5.1) is identical. \square

Remark 5.7. Both the definition of the direct sum of field theories, and Lemma 5.6 hold more generally for arbitrary semi-additive and idempotent complete target categories replacing the category of super vector spaces.

Lemma 5.8. *A finite-dimensional super-commutative semisimple superalgebra is isomorphic to a direct sum of trivial algebras k concentrated in purely even degree.*

Proof. By super-commutativity, any odd element is nilpotent. However, by semisimplicity there cannot be any non-zero nilpotent elements. Hence, the algebra is even and hence is of the claimed form by Artin-Wedderburn. \square

Definition 5.9. Let $\beta : B \rightarrow \mathrm{BO}(d)$ satisfy the assumptions of Lemma 5.1. We will say that a field theory is *simple* if the super vector space $Z(S^{d-1})$ is isomorphic to $k^{1|0}$, i.e. is one-dimensional and in even degree. We will say that a field theory is *indecomposable* if it cannot be decomposed into a direct sum of non-zero field theories.

Using the classification of semisimple super-algebras, indecomposability and simplicity of a sufficiently semisimple super field theory are equivalent:

Corollary 5.10. *Let $d \geq 2$, $\beta : B \rightarrow \mathrm{BO}(d)$ satisfy the conditions of Lemma 5.1 and let $Z : \mathrm{Bord}_d^B \rightarrow \mathrm{sVec}$ be a d -dimensional topological field theory for which $Z(S^{d-1})$ is a semisimple algebra.*

Then, Z is indecomposable if and only if it is simple. Moreover, any d -dimensional topological field theory with semisimple $Z(S^{d-1})$ decomposes into a finite direct sum of simple field theories.

Proof. Immediate by Lemmas 5.8 and 5.6 and the observation that a summand of a semisimple algebra is again a semisimple algebra. \square

We recall from [Reu20, Prop 3.2] that simple field theories are multiplicative under connected sums.

Proposition 5.11. *Let $\beta : B \rightarrow \text{BO}(d)$ satisfy the conditions of Lemma 5.1 and let $Z : \text{Bord}_d^B \rightarrow \mathcal{C}$ be a simple topological field theory. Then, $Z(S^d)$ is invertible and Z is multiplicative under connected sums: For a connected closed B -structured d -manifold M and a connected d -dimensional B -bordism $W : A \rightarrow B$, the following holds, where the connected sum is taken in the interior of W :*

$$Z(M\#W) = Z(S^n)^{-1}Z(M)Z(W)$$

Proof. The proof is the same as [Reu20, Prop 3.2]. \square

5.3. Semisimple topological field theories. In Section 5.2, we used the unique B -structure on S^d to obtain a commutative Frobenius algebra structure on S^{d-1} . In a similar way, for $a + b = d$, we can use the canonical B -structure $\bar{\theta}$ on $S^a \times S^b$ from Definition 5.2 to obtain a Frobenius algebra structure on the value of $S^{a-1} \times S^b$. Specifically, we may decompose S^a as a union of an incoming disk followed by an outgoing disk, and then take the product with S^b . This gives rise to a B -structure on $S^{a-1} \times S^b$ which we call the *bounding-cobounding* B -structure, which we will also denote $\bar{\theta}$. Removing disks from S^a , and crossing with S^b give B -structured bordisms which are easily seen to satisfy the axioms of a Frobenius algebra object which is commutative for $a \geq 2$.

Lemma 5.12. *Let $\beta : B \rightarrow \text{BO}(d)$ satisfy the assumptions of Lemma 5.1. Then for any topological field theory $Z : \text{Bord}_d^B \rightarrow \text{sVec}$, and any $a + b = d$, the value of $Z(S^{a-1} \times S^b, \bar{\theta})$ is a (commutative for $a \geq 2$) super Frobenius algebra with multiplication and unit given as*

$$(33) \quad Z((D^a \setminus D^a \sqcup D^a) \times S^b, \bar{\theta}) \quad Z(D^a \times S^b, \bar{\theta}).$$

We may now define *semisimple* even-dimensional topological field theories.

Definition 5.13. Let $d = 2q$ and let $\beta : B \rightarrow \text{BO}(d)$ be a tangential $(q-1)$ -type for which B is connected. Then a topological field theory $Z : \text{Bord}_d^B \rightarrow \text{sVec}$, is called *semisimple* if the algebras $Z(S^{2q-1}, \bar{\theta})$ and $Z(S^q \times S^{q-1}, \bar{\theta})$, with multiplication and unit given as in Lemmas 5.5 and 5.12, are semisimple.

We will show in Section 5.5 that semisimple topological field theories lead to stable diffeomorphism invariants.

Example 5.14. The examples in [Reu20] of semisimple TFTs still apply. Specifically ([Reu20, Thm. 2.9]) any unitary topological field theory is semisimple, and ([Reu20, Thm. 2.10]) any once-extended topological field theory valued in the 2-category of Cauchy complete k -linear categories and k -linear functors, or the 2-category of k -algebras and k -bimodules is semisimple.

Remark 5.15. As stated, Definition 5.13 would make sense for arbitrary tangential structure $\beta : B \rightarrow \text{BO}(d)$. However, in the case where β is not $(q-1)$ -truncated, we do not expect this to lead to a good notion of semisimplicity. Indeed, one essential reason that Definition 5.13

imposes a considerable constraint on our field theory is that the middle-dimensional sphere S^q admits a unique B -structure which we access via the canonical B -structure $\bar{\theta}$ on $S^q \times S^{q-1}$. For arbitrary tangential structures, we expect a better notion of semisimplicity to impose conditions on all B -structured middle-dimensional spheres S^q .

For our current comparison with the notion of stable diffeomorphism, we only need to consider tangential $(q-1)$ -types (B, β) for which Definition 5.13 suffices.

5.4. An eigenvalue equation in the bordism category. In the next section, we will generalize the arguments of [Reu20] and show that the manifold invariant induced by any even-dimensional semisimple field theory only depends on the stable diffeomorphism class of the manifold. Following [Reu20, Prop. 3.4], the main step of this argument involves a diffeomorphism between certain B -bordisms which we now construct.

For a Frobenius algebra the composition of the comultiplication followed by the multiplication is called the window map. Under the assumptions of Lemma 5.1, for every $a+b=d$, $S^{a-1} \times S^b$ becomes a Frobenius algebra object. The corresponding *window bordism* is given by

$$W_{S^{a-1} \times S^b} = W_{S^{a-1}} \times S^b = ((S^{a-1} \times S^1) \setminus (D^a \sqcup D^a)) \times S^b$$

viewed as a bordism from $S^{a-1} \times S^b$ to $S^{a-1} \times S^b$.

The first goal of this section is to prove the following proposition:

Proposition 5.16. *Under the assumptions of Lemma 5.1, we have a diffeomorphism of B -structured bordisms $\emptyset \Rightarrow (S^{a-1} \times S^b, \bar{\theta})$:*

$$(W_{S^{a-1}} \times S^b, \bar{\theta}) \circ (S^{a-1} \times D^{b+1}, \bar{\theta}) \cong (S^{a-1} \times D^{b+1}, \bar{\theta}) \# (S^1 \times S^{d-1}, \theta_1) \# (S^{a-1} \times S^{b+1}, \theta_2)$$

where $\#$ denotes connected sum on the interior, each $\bar{\theta}$ denotes a canonical B -structure from Section 5.1, and θ_1 and θ_2 are some B -structures on $S^1 \times S^{d-1}$ and $S^{a-1} \times S^{b+1}$, respectively.

As in [Reu20], interpreting the connected sum as a ‘scalar multiplication’ operation, the diffeomorphism of Proposition 5.16 may be understood as an ‘eigenvalue equation’ in the bordism category; asserting that the endomorphism $W_{S^{a-1} \times S^b}$ has ‘eigenvector’ $S^{a-1} \times D^{b+1}$ with eigenvalue $(S^1 \times S^{d-1}) \# (S^{a-1} \times S^{b+1})$.

Lemma 5.17. *Let M be a smooth d -dimensional manifold, $a+b=d$, and $i : D^a \times S^b \hookrightarrow M$ an embedding. Suppose that i may be changed by isotopy so that the image lies in an embedded d -disk and the corresponding S^b is unknotted inside this disk. Then there is a diffeomorphism:*

$$(M \setminus D^a \times S^b) \cong (S^{a-1} \times D^{b+1}) \# M.$$

Here $\#$ denotes connected sum on the interior.

Proof. By assumption we may change i by an isotopy so that its image lies inside a disk. Thus

$$(M \setminus D^a \times S^b) \cong (S^d \setminus i(D^a \times S^b)) \# M.$$

Since the image of the core S^b is assumed to be unknotted, we have $(S^d \setminus i(D^a \times S^b)) \cong (S^{a-1} \times D^{b+1})$, which proves the result. \square

Corollary 5.18. *There exists a diffeomorphism of smooth manifolds:*

$$\begin{aligned} & (W_{S^{a-1}} \times S^b) \circ (S^{a-1} \times D^{b+1}) \\ & \cong (S^{a-1} \times D^{b+1}) \# [(S^{a-1} \times S^1 \times S^b \setminus D^a \times S^b) \cup_{S^{a-1} \times S^b} S^{a-1} \times D^{b+1}]. \end{aligned}$$

Proof. The manifold $W_{S^{a-1}} \times S^b$ is $S^{a-1} \times S^1 \times S^b$ with two parallel copies of $D^a \times S^b$ removed. In $(W_{S^{a-1}} \times S^b) \circ (S^{a-1} \times D^{b+1})$, a copy of $S^{a-1} \times D^{b+1}$ has been attached to the boundary of one of these $D^a \times S^b$ (i.e. we performed surgery on one of these $D^a \times S^b$). In the resulting manifold, the core $\{0\} \times S^b$ of the remaining $D^a \times S^b$ now bounds an embedded $(b+1)$ -disk. Thus it can be isotoped into a d -disk in which it is unknotted. Now Lemma 5.17 applies and gives the desired result. \square

Note that the second summand on the right hand side of Corollary 5.18 is the manifold obtained from $S^{a-1} \times S^1 \times S^b$ by surgery along the embedded $\{0\} \times \{0\} \times S^b$.

Lemma 5.19. *The manifold $(S^{a-1} \times S^1 \times S^b \setminus D^a \times S^b) \cup_{S^{a-1} \times S^b} S^{a-1} \times D^{b+1}$ obtained from surgery along $\{0\} \times \{0\} \times S^b \hookrightarrow S^{a-1} \times S^1 \times S^b$ is diffeomorphic to*

$$S^1 \times S^{d-1} \# S^{a-1} \times S^{b+1}.$$

Proof. We will begin by writing $S^{a-1} = D_+^{a-1} \cup_{S^{a-2}} D_-^{a-1}$ and $S^1 = D_+^1 \cup_{S^0} D_-^1$ as unions of upper and lower hemispheres. This gives the decomposition depicted in Figure 1 of $S^{a-1} \times S^1 \times S^b$ as a gluing of manifolds with corners. The surgered manifold is obtained

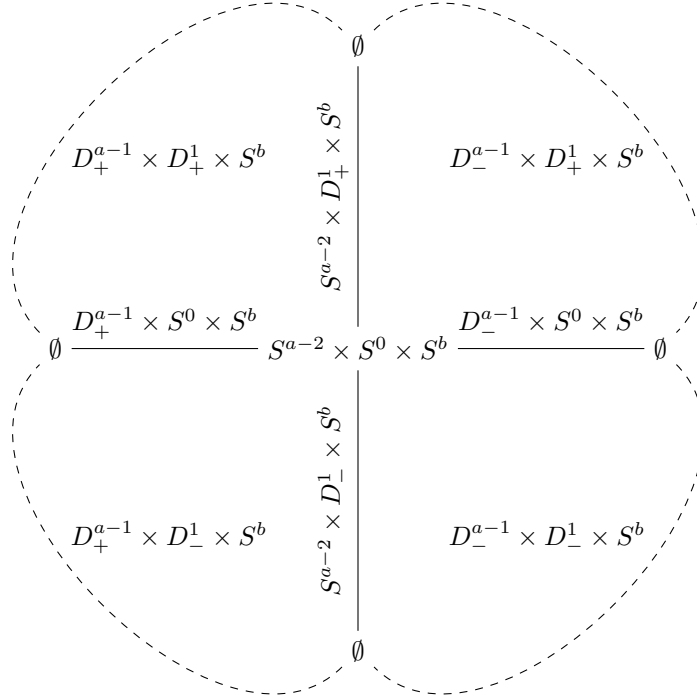


FIGURE 1. A decomposition of $S^{a-1} \times S^1 \times S^b$.

from $S^{a-1} \times S^1 \times S^b$ by removing a copy of $D^a \times S^b$ and replacing it with $S^{a-1} \times D^{b+1}$ (i.e. doing surgery). The copy of $D^a \times S^b$ that we shall use is $D_-^{a-1} \times D_-^1 \times S^b$, denoted in the

lower right-hand corner of the above decomposition. We also have

$$\begin{aligned} S^{a-1} \times D^{b+1} &\cong (S^{a-2} \times D_-^1 \cup_{S^{a-2} \times S^0} D_-^{a-1} \times S^0) \times D^{b+1} \\ &\cong (S^{a-2} \times D_-^1 \times D^{b+1}) \cup_{S^{a-2} \times S^0 \times D^{b+1}} (D_-^{a-1} \times S^0 \times D^{b+1}) \end{aligned}$$

and so our surgered manifold is given as the decomposition depicted in Figure 2 as a gluing of manifolds with corners.

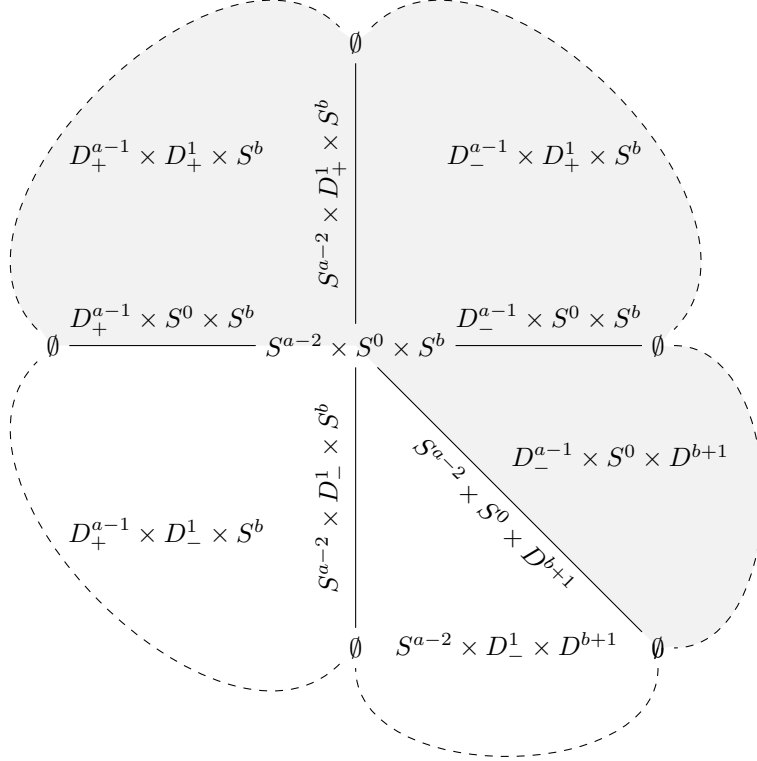


FIGURE 2. A decomposition of the manifold L .

Let X be the manifold corresponding to the shaded portion of the decomposition in Figure 2. The manifold corresponding to the unshaded portion is diffeomorphic to the cylinder $S^{d-1} \times D_-^1$. Since the surgered manifold is orientable, the two disks along which the cylinder is attached to X are isotopic. Thus to prove the lemma it is sufficient to show that X (corresponding to the shaded region) is diffeomorphic to $(S^{a-1} \times S^{b+1}) \setminus (D^d \times S^0)$. This in turn follows if $X \cup_{S^{a-1} \times S^0} (D^d \times S^0)$ is diffeomorphic to $S^{a-1} \times S^{b+1}$, which is what we will show.

The manifold $X \cup_{S^{a-1} \times S^0} (D^d \times S^0)$ is obtained by taking the manifold from the shaded region of the decomposition in Figure 2 and attaching $D^d \times S^0 \cong D_+^{a-1} \times S^0 \times D^{b+1}$ in place of the unshaded region. This gives the decomposition of $X \cup_{S^{a-1} \times S^0} (D^d \times S^0)$ as a gluing of manifolds with corners depicted in Figure 3.

The manifold corresponding to the shaded half of the decomposition in Figure 3 is $D_-^{a-1} \times S^{b+1}$. The manifold corresponding to the unshaded half is $D_+^{a-1} \times S^{b+1}$ and they are

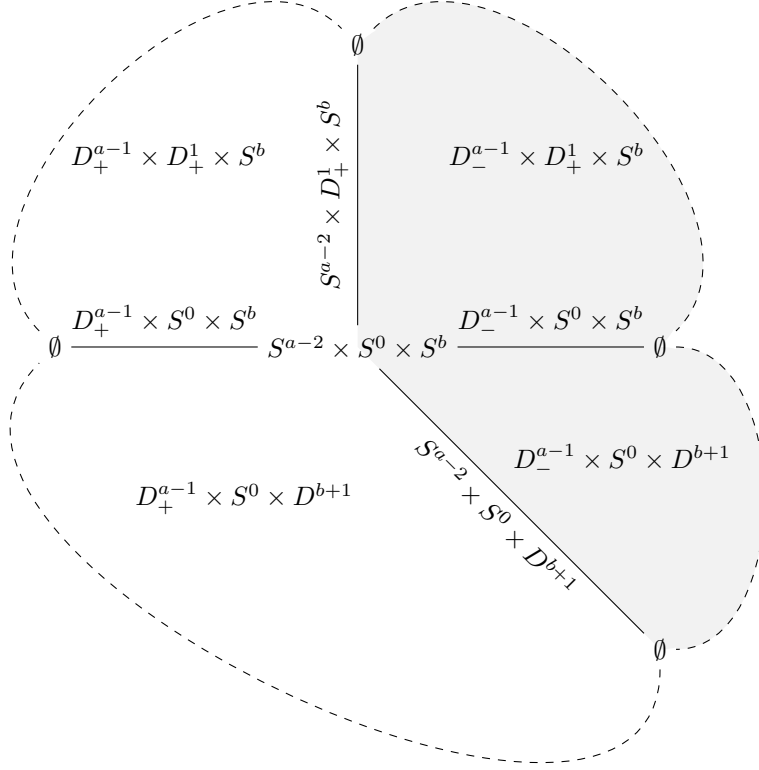


FIGURE 3. The manifold X with two disks attached.

glued together along $S^{a-2} \times S^{b+1}$. Whence the manifold in question is $S^{a-1} \times S^{b+1}$, as claimed. \square

Proof of Prop. 5.16. It follows immediately from Lemma 5.19 and Corollary 5.18 that there exists a diffeomorphism of bordisms

$$(W_{S^{a-1} \times S^b}) \circ (S^{a-1} \times D^{b+1}) \cong (S^{a-1} \times D^{b+1}) \# S^1 \times S^{d-1} \# S^{a-1} \times S^{b+1}.$$

The bordisms on the left-hand side can be equipped with canonical B -structures, and these induce B -structures on the right-hand side. The fact that these B -structures come from the connected sum of ones on $S^{a-1} \times D^{b+1}$, $S^1 \times S^{d-1}$, and $S^{a-1} \times S^{b+1}$ separately follows from the uniqueness of B -structures on D^d and S^{d-1} (implicit in the assumptions of Lemma 5.1). The proof of Lemma 5.17 shows that the B -structure on $S^{a-1} \times D^{b+1}$ is induced by one on S^d , which must be the canonical B -structure, and hence is the canonical B -structure on $S^{a-1} \times D^{b+1}$. \square

5.5. Semisimple field theories lead to stable diffeomorphism invariants. We now show that semisimple topological field theories Z lead to stable diffeomorphism invariants. We first consider the case where Z is furthermore simple, i.e. $Z(S^{d-1}) \cong k$. For such theories, we may use the 'eigenvalue equation' of Proposition 5.16, to identify $Z(S^q \times S^q)$ as

a (multiplicative factor of an) eigenvalue of an invertible endomorphism and hence conclude that it is non-zero.

Lemma 5.20. *Let $d = 2q$, let $\beta : B \rightarrow \mathbf{B}(d)$ satisfy the conditions of Definition 5.13, and let $Z : \mathbf{Bord}_d^B \rightarrow \mathbf{sVec}$ be a semisimple topological field theory with $Z(S^{d-1}) \cong k$. Then, $Z(S^q \times S^q, \bar{\theta})$ is non-zero.*

Proof. By definition, the Frobenius algebra $Z(S^q \times S^{q-1}, \bar{\theta})$ is semisimple. By [Reu20, Prop. 3.3] the window endomorphism $m \circ \Delta : A \rightarrow A$ is invertible for any semisimple Frobenius algebra A . Whence,

$$Z(W_{S^q} \times S^{q-1}, \bar{\theta}) : Z(S^q \times S^{q-1}, \bar{\theta}) \rightarrow Z(S^q \times S^{q-1}, \bar{\theta})$$

is invertible. Applying Z to the diffeomorphism of Proposition 5.16 and using multiplicativity under connected sums (Proposition 5.11), we find:

$$Z(W_{S^q} \times S^{q-1}, \bar{\theta}) \circ Z(S^q \times D^q, \bar{\theta}) = Z(S^d, \bar{\theta})^{-2} Z(S^{2q-1} \times S^1, \theta_1) Z(S^q \times S^q, \theta_2) \cdot Z(S^q \times D^q, \bar{\theta})$$

for the respective canonical B -structures $\bar{\theta}$ and some B -structures θ_1, θ_2 . By Lemma 5.4, the canonical B -structure on $S^q \times S^q$ is in fact the unique one up to orientation reversing diffeomorphism, and hence $(S^q \times S^q, \theta_2) \cong (S^q \times S^q, \bar{\theta})$.

The composite morphism

$$Z(\emptyset) \xrightarrow{Z(S^q \times D^q, \bar{\theta})} Z(S^q \times S^{q-1}, \theta) \xrightarrow{Z(D^{q+1} \times S^{q-1}, \bar{\theta})} Z(\emptyset)$$

is $Z(S^d, \bar{\theta})$, which is non-zero (Proposition 5.11). Thus

$$k = Z(\emptyset) \xrightarrow{Z(S^q \times D^q, \bar{\theta})} Z(S^q \times S^{q-1}, \theta)$$

is a non-zero vector in $Z(S^q \times S^{q-1}, \theta)$. It follows that $Z(S^d, \bar{\theta})^{-2} Z(S^{2q-1} \times S^1, \theta_1) Z(S^q \times S^q, \bar{\theta})$ is an eigenvalue of the invertible endomorphism $Z(W_{S^q} \times S^{q-1}, \bar{\theta})$, and is therefore invertible itself. \square

Decomposing any semisimple field theory into simple summands, the main theorem of this section follows.

Theorem 5.21. *Let $\beta : B \rightarrow \mathbf{BO}(d)$ satisfy the conditions of Definition 5.13, and let $Z : \mathbf{Bord}_d^B \rightarrow \mathbf{sVec}$ be any semisimple topological field theory (Definition 5.13). If B -structured manifolds (M_1, ψ_1) and (M_2, ψ_1) are B -stably diffeomorphic, then $Z(M_1, \psi_1) = Z(M_2, \psi_2)$.*

Proof. Any semisimple topological field theory decomposes as a direct sum of simple topological field theories (Cor. 5.10), and hence it is sufficient to assume Z is simple. By assumption there exists a natural number N and a diffeomorphism of B -manifolds

$$(M_1, \psi_1) \# (\#^N(S^q \times S^q, \bar{\theta})) \cong (M_2, \psi_2) \# (\#^N(S^q \times S^q, \bar{\theta})).$$

where $\bar{\theta}$ is the canonical B -structure on $S^q \times S^q$. By the multiplicativity of simple topological field theories (Proposition 5.11) we have

$$Z(S^d)^{-N} Z(S^q \times S^q, \bar{\theta})^N Z(M_1, \psi_1) = Z(S^d)^{-N} Z(S^q \times S^q, \bar{\theta})^N Z(M_2, \psi_2).$$

However $Z(S^d)$ is invertible (Proposition 5.11) and $Z(S^q \times S^q, \bar{\theta})$ is invertible (Lemma 5.20), and thus $Z(M_1, \psi_1) = Z(M_2, \psi_2)$. \square

5.6. Dijkgraaf-Witten theories are semisimple. Our goal in this section is to show that generalized Dijkgraaf-Witten theories induced from invertible field theories are semisimple.

Our main proof will rely on the fact that any generalized Dijkgraaf-Witten theory as in Definition 4.4 can be *once extended* to a symmetric monoidal 2-functor $\text{Bord}_{d,d-1,d-2}^B \rightarrow 2\text{sVec}_k$. Here, $\text{Bord}_{d,d-1,d-2}^B$ denotes the symmetric monoidal bicategory of closed $(d-2)$ - B -manifolds, compact compatibly B -structured $(d-1)$ -bordisms and compact compatibly B -structured d -dimensional bordisms with corners between them (see e.g. [SP11] for details on the construction as a bicategory), and 2sVec_k denotes the symmetric monoidal bicategory of k -linear finite semisimple sVec_k -module categories, module functors and module natural transformations. (This bicategory 2sVec_k is equivalent to the symmetric monoidal bicategory sAlg_k of semisimple super algebra (i.e. semisimple algebra objects in sVec_k), super bimodules and grading-preserving bimodule maps.)

In Remark 5.26 below, we will sketch another more elementary proof which does not use any bicategorical machinery and instead proceeds by explicitly studying the algebras associated to products of spheres.

We first show that invertible sVec_k -valued field theories can be uniquely extended to 2sVec_k . (The analogous statement about extending Vec_k -valued theories to 2Vec_k fails spectacularly, see [SP17].)

Lemma 5.22. *Let $W : \text{Bord}_d^X \rightarrow \text{sVec}_k$ be any invertible topological field theory. Then, there exists a unique 2-categorical invertible field theory $\widetilde{W} : \text{Bord}_{d,d-1,d-2}^B \rightarrow 2\text{sVec}_k$ which extends W .*

Before proving the lemma, we make a brief detour on Picard 2-groupoids. A Picard 2-groupoid is a symmetric monoidal bicategory, in which all objects, 1-morphisms and 2-morphisms are invertible. Equivalently [GJO19], a Picard 2-groupoid is a spectrum whose only non-trivial homotopy groups are π_0, π_1, π_2 . Out of any Picard 2-groupoid \mathcal{A} we may extract two Picard 1-groupoids $\mathcal{A}_{[12]} := \tau_{\geq 1}\mathcal{A}$ and $\mathcal{A}_{[01]} := \tau_{\leq 1}\mathcal{A}$. The former may be identified with the groupoid of automorphisms of the unit object of \mathcal{A} , and the later is the groupoid whose objects are those of \mathcal{A} and whose morphisms are isomorphism classes of 1-morphisms in \mathcal{A} . As in Section 3.3, these Picard 1-groupoids are determined by their *Postnikov invariants* [HS05, App. B] (see also [JO12]), which take the form $(\pi_1\mathcal{A}, \pi_2\mathcal{A}, k_{\mathcal{A}_{[12]}} : \pi_1\mathcal{A} \otimes \mathbb{Z}/2\mathbb{Z} \rightarrow \pi_2\mathcal{A})$ and $(\pi_0\mathcal{A}, \pi_1\mathcal{A}, k_{\mathcal{A}_{[01]}} : \pi_0\mathcal{A} \otimes \mathbb{Z}/2\mathbb{Z} \rightarrow \pi_1\mathcal{A})$. We will call these the *elementary Postnikov invariants* of \mathcal{A} .

Let 2sLine_k denote the underlying Picard 2-groupoid of the symmetric monoidal bicategory 2sVect_k . By definition, 2sVect_k is a delooping of sVect_k and hence we have $(2\text{sLine}_k)_{[12]} \simeq \text{sLine}_k$, considered in Example 3.8. On the other hand, $\pi_0 2\text{sLine}_k \cong \pi_0(2\text{sLine}_k)_{[01]} \cong \mathbb{Z}/2\mathbb{Z}$ corresponding to the super Morita equivalence classes of Clifford algebras, also known as the Brauer-Wall group or the super Brauer group [Wal64]. A simple calculation shows that the k -invariant

$$k_{(2\text{sLine}_k)_{[01]}} : \pi_0 2\text{sLine}_k \cong \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \cong \pi_1 2\text{sLine}_k = \pi_0 \text{sLine}_k$$

is an isomorphism: The $Cl_1 \otimes Cl_1$ - $Cl_1 \otimes Cl_1$ super bimodule corresponding to the graded twist automorphism turns into the odd line after employing the super Morita equivalence $Cl_1 \otimes Cl_1 \cong Cl_2 \sim k$.

In general, the elementary Postnikov invariants do not determine a Picard 2-groupoid \mathcal{A} . However we have the following special case.

Lemma 5.23. *Let k be an algebraically closed field, not of characteristic 2. Let \mathcal{A} be a Picard 2-groupoid with $\pi_0\mathcal{A} \cong \mathbb{Z}/2\mathbb{Z}$, $\pi_1\mathcal{A} \cong \mathbb{Z}/2\mathbb{Z}$, $\pi_2\mathcal{A} \cong k^\times$, and with non-trivial*

elementary k -invariants

$$\begin{aligned} k_{\mathcal{A}_{[01]}} : \pi_0 \mathcal{A} &\cong \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}/2\mathbb{Z} \cong \pi_1 \mathcal{A} \\ k_{\mathcal{A}_{[12]}} : \pi_1 \mathcal{A} &\cong \mathbb{Z}/2\mathbb{Z} \hookrightarrow k^\times \cong \pi_2 \mathcal{A} \end{aligned}$$

Then the underlying E_∞ -2-type of \mathcal{A} is equivalent to $\tau_{\geq 0} \Sigma^2 I_{k^\times}$, the connective cover of the two-fold suspension of the k^\times -Brown-Comenetz dual of the sphere.

Proof. We have $\pi_0(\tau_{\geq 0} \Sigma^2 I_{k^\times}) \cong \mathbb{Z}/2\mathbb{Z}$, $\pi_1(\tau_{\geq 0} \Sigma^2 I_{k^\times}) \cong \mathbb{Z}/2\mathbb{Z}$, and $\pi_2(\tau_{\geq 0} \Sigma^2 I_{k^\times}) \cong k^\times$. The universal property of the k^\times -Brown-Comenetz dual of the sphere implies that

$$\pi_0 2\mathrm{Fun}(\mathcal{A}, \tau_{\geq 0} \Sigma^2 I_{k^\times}) \cong \mathrm{Hom}(\pi_2 \mathcal{A}, k^\times),$$

and thus there is a map $f : \mathcal{A} \rightarrow \tau_{\geq 0} \Sigma^2 I_{k^\times}$ which induces an isomorphism on π_2 . Because the elementary k -invariant $k_{\mathcal{A}_{[12]}}$ is injective, it follows that the elementary k -invariant $k_{(\tau_{\geq 0} \Sigma^2 I_{k^\times})_{[12]}}$ is injective and that f induces an isomorphism on π_1 . Likewise, because the elementary k -invariant $k_{\mathcal{A}_{[01]}}$ is an isomorphism, it follows that the elementary k -invariant $k_{(\tau_{\geq 0} \Sigma^2 I_{k^\times})_{[01]}}$ is an isomorphism and that f induces an isomorphism on π_0 . Since f is an isomorphism on all homotopy groups, it is an equivalence of E_∞ -2-types. \square

It follows from Lemma 5.23 that the Picard 2-groupoid $2\mathrm{sLine}_k$ of $2\mathrm{sVec}_k$ has the underlying E_∞ -2-type of $\tau_{\geq 0} \Sigma^2 I_{k^\times}$. See [BLM22] for another proof of this and related results, as well as [Fre12, Fre14, FH20, FH21]. In other words, the Picard 2-groupoid $2\mathrm{sLine}_k$ can be characterized by the following universal property, analogous to the characterization of sLine_k in Corollary 3.10.

Corollary 5.24. *Let k be an algebraically closed field of characteristic $\neq 2$. Then, $2\mathrm{sLine}_k$ is the unique Picard 2-groupoid with the property that for any Picard 2-groupoid \mathcal{A} , the restriction map*

$$(34) \quad \pi_0 2\mathrm{Fun}(\mathcal{A}, 2\mathrm{sLine}_k) \rightarrow \mathrm{Hom}(\pi_2 \mathcal{A}, k^\times)$$

is an isomorphism.

This immediately leads to a proof of Lemma 5.22.

Proof of Lemma 5.22. Let \mathcal{B} be the Picard completion of the 2-category $\mathrm{Bord}_{d,d-1,d-2}^X$. It follows from [SP17] that $\mathcal{B}_{[12]}$ is the Picard completion of $\mathrm{Bord}_{d,d-1}^X$. Thus the restriction map

$$\pi_0 2\mathrm{Fun}(\mathrm{Bord}_{d,d-1,d-2}^X, 2\mathrm{sLine}_k) \rightarrow \pi_0 \mathrm{Fun}(\mathrm{Bord}_{d,d-1}^X, \mathrm{sLine}_k)$$

is an isomorphism, since they are both isomorphic to $\mathrm{Hom}(\Omega_d^{TX}, k^\times)$ by Corollaries 3.10 and 5.24. \square

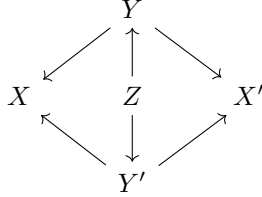
Theorem 5.25. *If $d = 2q$ and $\beta : B \rightarrow \mathrm{BO}(d)$ is a tangential $(q-1)$ -type with B connected (i.e. the assumptions for Definitions 5.13 and 5.3), then $\mathrm{DW}_{\xi,\omega} : \mathrm{Bord}_d^B \rightarrow \mathrm{sVec}_k$ is semisimple for any choice of $\xi : X \rightarrow B$ and ω , and hence leads to stable B -diffeomorphism invariants.*

Proof sketch. Recall that the character $\omega : \Omega^{T(\beta \circ \xi)} \rightarrow k^\times$ classifies an invertible field theory $\mathcal{W} : \mathrm{Bord}_d^X \rightarrow \mathrm{sLine}_k$. By Lemma 5.22, there is therefore a unique invertible extension $\widetilde{\mathcal{W}} : \mathrm{Bord}_{d,d-1,d-2}^B \rightarrow 2\mathrm{sLine}_k$.

Essentially, Theorem 5.25 then follows since any Dijkgraaf-Witten theory $\mathrm{DW}_{X,\mathcal{W}} : \mathrm{Bord}_d^B \rightarrow \mathrm{sVec}_k$ built from an invertible field theory $\mathcal{W} : \mathrm{Bord}_d^X \rightarrow \mathrm{sVec}_k$ can be once extended to a 2-categorical Dijkgraaf-Witten field theory $\widetilde{\mathrm{DW}}_{X,\widetilde{\mathcal{W}}} : \mathrm{Bord}_{d,d-1,d-2}^X \rightarrow 2\mathrm{sVec}_k$

constructed from this unique extension $\widetilde{\mathcal{W}} : \text{Bord}_{d,d-1,d-2}^X \rightarrow 2\text{sVec}_k$. The construction of $\widetilde{DW}_{X,\widetilde{\mathcal{W}}}$ may be outlined as follows:

Let $\text{Span}(\mathcal{S}^\pi, 2\text{sVec})$ denote the symmetric monoidal bicategory whose objects are π -finite spaces equipped with 2-functors $\mathcal{L} : \tau_{\leq 2}X \rightarrow 2\text{sVec}$, whose 1-morphisms are spans $X \xleftarrow{f} Y \xrightarrow{f'} X'$ of π -finite spaces equipped with natural transformations $f^*\mathcal{L} \Rightarrow (f')^*\mathcal{L}'$ of 2-functors $\tau_{\leq 2}Y \rightarrow 2\text{sVec}_k$ and whose 2-morphisms are appropriate equivalence classes of spans of spans



equipped with modifications between the pulled back natural transformations. Composition of 1-morphisms and 2-morphisms is given by homotopy pullback of spans with appropriate composition of the associated natural transformations and modifications. In the case where all spaces are 1-groupoids (equivalently, 1-types), a detailed construction of this symmetric monoidal bicategory can be found in [SW18], the general case is entirely analogous. (This symmetric monoidal bicategory $\text{Span}(\mathcal{S}^\pi, 2\text{sVec}_k)$ is the homotopy bicategory of the symmetric monoidal $(\infty, 2)$ -category of iterated spans with local systems constructed in [Hau18].)

Analogously to Definition 4.4, our extended Dijkgraaf-Witten theory $\widetilde{DW}_{X,\widetilde{\mathcal{W}}}$ is defined as a composite $\text{Bord}_{d,d-1,d-2}^B \rightarrow \text{Span}(\mathcal{S}^\pi, 2\text{sVec}_k) \rightarrow 2\text{sVec}_k$. Here, the first functor maps a closed $(d-2)$ - B -manifold W to the space of X -structures on W refining the given B structure, equipped with the 2sVec_k -valued local system determined by $\widetilde{\mathcal{W}} : \text{Bord}_{d,d-1,d-2}^B \rightarrow 2\text{sVec}_k$ and analogous assignments to higher-dimensional bordisms. The second functor $\text{Span}(\mathcal{S}^\pi, 2\text{sVec}_k) \rightarrow 2\text{sVec}_k$ is a 2-categorical linearization functor extending the 1-categorical linearization functor from Corollary 2.13. In the case where all π -finite spaces considered are 1-groupoids, and the target is 2Vec instead of 2sVec , this linearization 2-functor is constructed in detail by Schweigert and Woike in [SW18]. It is straight-forward to generalize their construction to arbitrary π -finite spaces, essentially by replacing their ‘integral with respect to groupoid cardinality’ by our finite path integral from Definition 2.1 (i.e. by inserting various homotopy cardinality factors to account for higher homotopies), and similarly to generalize from 2Vec_k -local systems to 2sVec_k -valued local systems. (Note that Schweigert and Woike’s construction [SW18] more evidently extends the ‘limit variant’ of our linearization functor from Remark 2.15. However, as explained in Remark 2.15, since all spaces considered here are π -finite, the limit and colimit variants of the linearization functors are equivalent.)

By construction, this new bicategorical Dijkgraaf-Witten theory extends our given field theory $DW_{\xi,\omega}$. Analogously to [Reu20, Thm. 2.10], any once-extendable field theory is automatically semisimple, proving the theorem. \square

Remark 5.26. We now outline an alternative proof of Theorem 5.25 which avoids bicategories and proceeds by explicitly computing the algebras associated by $DW_{\xi,\omega}$ to products of spheres.

First, we claim that for any d , any $(d-2)$ -truncated map $\beta : B \rightarrow \mathrm{BO}(d)$ with connected B , any $\xi : X \rightarrow B$ with π -finite fibers, and any character $\omega : \Omega^{T\xi} \rightarrow k^\times$ (equivalently, invertible field theory $W : \mathrm{Bord}_d^X \rightarrow \mathrm{sLine}_k$), the sphere algebra $\mathrm{DW}_{\xi,\omega}(S^{d-1})$ (with its unique B -structure, cf. Lemma 5.5) of the associated generalized Dijkgraaf-Witten theory $\mathrm{DW}_{\xi,\omega} : \mathrm{Bord}_d^B \rightarrow \mathrm{sVec}_k$ is semisimple. This essentially follows from an explicit computation: Let F denote the fiber of $\xi : X \rightarrow B$ (at some choice of basepoint) and assume without loss of generality that F is connected (else, $\mathrm{DW}_{\xi,\omega}(S^{d-1})$ decomposes into a direct sum of algebras, one for each component of F). Then, it can be shown that $\mathrm{DW}_{\xi,\omega}(S^{d-1})$ is either zero or may be identified with the fixed point subalgebra $(k^\alpha[\pi_{d-1}F])^{\pi_1(F)}$ of the $\pi_1(F)$ action on the twisted group algebra $k^\alpha[\pi_{d-1}(F)]$, twisted by a certain 2-cocycle $\alpha \in \mathrm{H}^2(\pi_{d-1}F, k^\times)$ which is determined by the invertible field theory W . In characteristic zero, twisted group algebras and their fixed point subalgebras are semisimple [Lev35, Thm. 9].

It remains to show that for Dijkgraaf-Witten theories fulfilling the assumptions of Theorem 5.25, the algebra associated to the product of middle dimensional spheres is semisimple. Indeed, for such a theory (i.e. one for which $d = 2q$ and for which $\beta : B \rightarrow \mathrm{BO}(2q)$ is $(q-1)$ -truncated and B is connected), the algebra $\mathrm{DW}_{\xi,\omega}(S^q \times S^{q-1})$ is the algebra associated to S^q by the dimensionally reduced theory $\mathrm{DW}_{X,\omega}(S^{q-1} \times -) : \mathrm{Bord}_{q+1}^{B_{q+1}} \rightarrow \mathrm{sVec}_k$ where $B_{q+1} \rightarrow \mathrm{BO}(q+1)$ is the pull back of our original ambient structure $B \rightarrow \mathrm{BO}(2q)$. By Theorem 4.18, this reduced theory is again a generalized Dijkgraaf-Witten theory. Since we assumed $B \rightarrow \mathrm{BO}(2q)$ to be $(q-1)$ -truncated, so is $B_{q+1} \rightarrow \mathrm{BO}(q+1)$ and the preceding paragraph implies that the algebra associated by $\mathrm{DW}_{X,\omega}(S^{q-1} \times -)$ to S^q is semisimple.

6. STABLE DIFFEOMORPHISM AND THE MAIN THEOREM

6.1. The relation to stable diffeomorphism. In this section we summarize Kreck's classification of manifolds up to stable diffeomorphism [Kre99], and explain the connection to generalized Dijkgraaf-Witten topological field theories.

We consider manifolds of dimension $d = 2q$, and we fix an ambient tangential structure $\beta : B \rightarrow \mathrm{BO}(2q)$, which is required to be a connected $(q-1)$ -tangential type (e.g. $\mathrm{BSO}(2q)$, $\mathrm{BSpin}(2q)$, etc.). By Proposition 3.21, if $q \geq 1$, there is a corresponding stable tangential structure $\beta_\infty : B_\infty \rightarrow \mathrm{BO}$ and a pullback diagram

$$\begin{array}{ccc} B & \longrightarrow & B_\infty \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{BO}(2q) & \longrightarrow & \mathrm{BO} \end{array}$$

Similarly, there is an associated stable normal structure $(\overline{B}_\infty, \overline{\beta})$ defined as the composite $B_\infty \xrightarrow{\beta} \mathrm{BO} \xrightarrow{-1} \mathrm{BO}$. Hence, for a $2q$ -manifold M there are natural equivalences between the space of B -structures on the tangent bundle T_M , the space of B_∞ -structures on the stable tangent bundle τ_M and the space of \overline{B}_∞ -structures on the stable normal bundle of M . We will intentionally conflate these three types of structure and simply refer manifolds with any of these structures as B -manifolds.

Definition 6.1. A $2q$ -dimensional B -manifold M has:

- a *tangential $(q-1)$ -type* $M \rightarrow \tau_{\leq n}^B M \rightarrow B$, defined by factoring the map to B into a $(q-1)$ -connected map followed by a $(q-1)$ -truncated map;

- a *stable tangential* $(q-1)$ -type $M \rightarrow \tau_{\leq n}^{B_\infty} M \rightarrow B_\infty$, defined by factoring the map to B_∞ into a $(q-1)$ -connected map followed by a $(q-1)$ -truncated map;
- a *stable normal* $(q-1)$ -type $M \rightarrow \tau_{\leq n}^{\overline{B}_\infty} M \rightarrow \overline{B}_\infty$, defined by factoring the map to \overline{B}_∞ into a $(q-1)$ -connected map followed by a $(q-1)$ -truncated map;

Definition 6.2. Let M be a B -manifold and $X \rightarrow B$ a tangential $(q-1)$ -type. An X -structure on M is a *tangential* $(q-1)$ -smoothing if it induces an equivalence $X \simeq \tau_{\leq q-1}^B M$ of spaces over B .

There is a corresponding stable tangential $(q-1)$ -type $X_\infty \rightarrow B_\infty$ and stable normal $(q-1)$ -type $\overline{X}_\infty \rightarrow \overline{B}_\infty$. If an X -structure on M is a tangential $(q-1)$ -smoothing it also induces equivalences $X_\infty \simeq \tau_{\leq n}^{B_\infty} M$ and $\overline{X}_\infty \simeq \tau_{\leq n}^{\overline{B}_\infty} M$. Thus we could equally have called the X -structure a *normal* $(q-1)$ -smoothing, as is done in [Kre99].

A B -manifold M with a choice of tangential $(q-1)$ -smoothing $M \rightarrow X \rightarrow B$, gives rise to an element in the unstable bordism group Ω_{2q}^{TX} (see Definition 3.12), and thus also in the stable bordism group $\Omega_{2q}^{\tau X_\infty} \cong \Omega_{2q}^{\nu \overline{X}_\infty}$ (Section 3.5). By Lemma 3.16 (see Remark 3.18) the map

$$(\chi, p) : \Omega_{2q}^{TX} \rightarrow \mathbb{Z} \oplus \Omega_{2q}^{\tau X_\infty} \cong \mathbb{Z} \oplus \Omega_{2q}^{\nu \overline{X}_\infty}$$

is injective, where $p : \Omega_{2q}^{TX} \rightarrow \Omega_{2q}^{\tau X_\infty} \cong \Omega_{2q}^{\nu \overline{X}_\infty}$ is the natural projection (and in particular surjective) and χ is the Euler characteristic. Thus, a class in the unstable tangential X -bordism group can be thought of as the normal X -bordism class together with information about the Euler characteristic.

There are natural equivalences $\text{Aut}_B(X) \simeq \text{Aut}_{B_\infty}(X_\infty) \simeq \text{Aut}_{\overline{B}_\infty}(\overline{X}_\infty)$, and the group $\pi_0 \text{Aut}_B(X)$ of (homotopy classes) of automorphisms of X over B acts effectively and transitively on the equivalence classes of tangential $(q-1)$ -smoothings on M . This action extends to the respective actions on the bordism groups Ω_{2q}^{TX} , and $\Omega_{2q}^{\tau X_\infty} \cong \Omega_{2q}^{\nu \overline{X}_\infty}$.

Our assumption that B is a connected tangential $(q-1)$ -type, implies (assuming $q \geq 1$) that it satisfies the assumptions from Section 5.1, and in particular that there is a well defined notion of connected sum of B -manifolds. Moreover there exists a canonical (unique up to orientation reversal) B -structure on $S^q \times S^q$. By construction this agrees with the canonical (normal) \overline{B}_∞ -structure constructed by [Kre99].

Recall (Definition 5.3) that two compact connected closed $2q$ -dimensional B -manifolds, M and N , are B -stably diffeomorphic if there exists a natural number r and a B -structured diffeomorphism between $M \#^r S^q \times S^q$ and $N \#^r S^q \times S^q$. Note well that the notion of stable diffeomorphism used here takes connected sum with the same number of copies of $S^q \times S^q$ on both M and N .

Theorem 6.3 ([Kre99]). *Let M and N be compact connected closed $2q$ -dimensional B -manifolds with the same stable normal $(q-1)$ -type $\overline{X}_\infty \rightarrow \overline{B}_\infty$. Then the following are equivalent*

1. M and N are B -stably diffeomorphic
2. M and N have the same Euler characteristic and represent elements in the stable normal bordism group $\Omega_{2q}^{\nu \overline{X}_\infty}$ which lie in the same orbit under the action of $\pi_0 \text{Aut}_B(X) \cong \pi_0 \text{Aut}_{\overline{B}_\infty}(\overline{X}_\infty)$. \square

In what follows, we will use a tangential unstable version of Kreck's theorem.

Corollary 6.4. *Let M and N be compact connected closed $2q$ -dimensional B -manifolds. Then, the following are equivalent:*

1. M and N are B -stably diffeomorphic
2. M and N have the same tangential $(q-1)$ -type $X \rightarrow B$ and represent elements in the unstable tangential bordism group Ω_{2q}^{TX} which lie in the same orbit under the action of $\pi_0 \text{Aut}_B(X)$.

Proof. As we have observed having equivalent tangential $(q-1)$ -type is equivalent to having equivalent stable normal $(q-1)$ -type. By Lemma 3.16 (see Remark 3.18) the map

$$(\chi, p) : \Omega_{2q}^{TX} \rightarrow \mathbb{Z} \oplus \Omega_{2q}^{\tau X_\infty} \cong \mathbb{Z} \oplus \Omega_{2q}^{\nu \bar{X}_\infty}$$

is injective, where $p : \Omega_{2q}^{TX} \rightarrow \Omega_{2q}^{\tau X_\infty} \cong \Omega_{2q}^{\nu \bar{X}_\infty}$ is the natural projection and χ is the Euler characteristic. The action of $\pi_0 \text{Aut}_B(X)$ on Ω_{2q}^{TX} leaves the Euler characteristic invariant. Hence, the Corollary is a direct consequence of Kreck's stable diffeomorphism classification Theorem 6.3. \square

Using this corollary, our main Theorem 4.14 may be re-interpreted as follows.

Corollary 6.5. *Let M and N be two $2q$ -dimensional B -manifolds. Suppose that M and N have π -finite B -tangential $(q-1)$ -types. Then M and N are indistinguishable by type- $(q-1)$ generalized Dijkgraaf-Witten theories if and only if they are B -stably diffeomorphic. \square*

6.2. Examples and Applications.

6.2.1. *Hitchin Exotic Spheres.* If $d > 1$, the tangential 1-type of a homotopy sphere is given by $B\text{Spin}(d)$, and each homotopy sphere admits a unique spin structure, yielding a homomorphism $\Theta_d \rightarrow \Omega_d^{T\text{Spin}(d)}$ from the group Θ_d of h-cobordism classes of smooth homotopy d -spheres (equivalently, for $d > 4$, the group of diffeomorphism classes of homotopy spheres). Since all homotopy spheres of the same dimension have the same Euler characteristic, the image injects into Ω_d^{Spin} , and we may as well consider the induced homomorphism $\eta : \Theta_d \rightarrow \Omega_d^{Spin}$. Each class $[\Sigma] = \eta(\Sigma)$ represented by a homotopy sphere is invariant under the action of $\text{Aut}_B(B\text{Spin}(d))$ where $B = BSO(d)$ or $B = BO(d)$. It follows directly from Theorem 4.14 that:

Corollary 6.6. *Two homotopy spheres Σ and Σ' are indistinguishable by type-1 generalized Dijkgraaf-Witten theories if and only if $\eta(\Sigma) = \eta(\Sigma')$, i.e. if and only if they represent the same element in Spin bordism. \square*

The α -invariant is a map $\alpha : \Omega_d^{Spin} \rightarrow KO_d(pt)$ from Spin bordism to the real K-theory of a point. Hitchin [Hit74] (based on results in [Lic63]) proved that the α invariant of a closed spin manifold is an obstruction to the existence of a metric of positive scalar curvature on it. On the other hand, Stolz [Sto90] proved that a simply connected spin manifold of dimension $d \geq 5$ admits a metric of positive scalar curvature if its α -invariant vanishes. The proof uses surgery results obtained (independently) in [GL80] and [SY79], as well as involved calculations within stable homotopy. Hence, a d -dimensional homotopy sphere Σ with $d \geq 5$ admits a metric of positive scalar curvature if and only if $\alpha(\Sigma)$ is trivial. Homotopy spheres for which $\alpha(\Sigma)$ is non-trivial, or which equivalently do not admit a metric of positive scalar curvature, are called *Hitchin spheres*. In [ABP66], Anderson-Brown-Peterson showed that Hitchin spheres exist precisely in dimension $d = 8k + 1$ and $8k + 2$ and in fact that for any homotopy sphere Σ , the eta invariant $\eta(\Sigma) \neq 0$ if and only if $\alpha \circ \eta(\Sigma) \neq 0$. Hence, the following is an immediate corollary of Corollary 6.6.

Corollary 6.7. *There exist oriented and unoriented semisimple topological field theories which distinguish Hitchin spheres from non-Hitchin homotopy spheres. \square*

Remark 6.8. In dimensions $d = 8k + 1$ or $8k + 2$ the alpha invariant $\alpha : \Omega_d^{Spin} \rightarrow \mathbb{Z}/2\mathbb{Z}$ takes values in $\mathbb{Z}/2\mathbb{Z}$ and can be viewed as a character by identifying $\mathbb{Z}/2\mathbb{Z} \cong \{\pm 1\} \subseteq \mathbb{C}^*$. Thus this gives rise to an invertible spin topological field theory. The generalized Dijkgraaf-Witten construction applied to this invertible theory produces an oriented theory which takes value $-\frac{1}{2}$ on Hitchin spheres and value $\frac{1}{2}$ on non-Hitchin spheres.

6.2.2. *Examples in dimension 4.* In dimension $2q = 4$, Theorem 6.5 says that two 4-manifolds with π -finite tangential 1-types are stably diffeomorphic if and only if they are indistinguishable by type-1 generalized Dijkgraaf-Witten theories. A connected manifold has π -finite tangential 1-type precisely if it has a finite fundamental group. In [Tei92] Teichner constructed the following examples of homotopy equivalent smooth manifolds

Theorem 6.9 ([Tei92]). *Let π be a finite group with generalized quaternion 2-Sylow subgroup of order larger than eight. Then there exist connected oriented smooth 4-manifolds M_0 and M_1 whose fundamental groups $\pi_1 M_0 \cong \pi_1 M_1 \cong \pi$, such that M_0 is (orientation preserving) homotopy equivalent to M_1 , but M_0 and M_1 are not stably diffeomorphic. These manifolds have the following properties*

1. M_0 and M_1 have the same tangential 1-type $\xi : X \rightarrow BO(4)$.
2. M_i is oriented, and the universal cover of M_i is spin.
3. The manifolds M_i are not spin; the second Stiefel Whitney class is a particular specified class induced from a particular class in the cohomology of the generalized quaternion 2-Sylow subgroup.
4. M_0 and M_1 are distinguished by a certain ‘tertiary’ bordism invariant $ter : \Omega_4^{TX} \rightarrow \mathbb{Z}/2\mathbb{Z}$.

This leads to the following corollary of Theorem 6.5.

Corollary 6.10. *In each instance of Theorem 6.9, there are oriented and unoriented type-1 generalized Dijkgraaf-Witten theories which distinguish the manifolds M_0 and M_1 . \square*

In fact an explicit generalized Dijkgraaf-Witten theory can be constructed using the bordism invariant ter , much as in Remark 6.8. We also recall that Gompf [Gom84] has shown that smooth manifolds which are orientation preserving homeomorphic are stably diffeomorphic. So while the manifolds M_0 and M_1 are homotopy equivalent, they are not homeomorphic.

In the case of non-orientable 4-manifolds, there are examples of smooth manifolds which are homeomorphic but are not stably diffeomorphic. For example, the tangential 1-type of $\mathbb{R}P^4$ is $BPin^+(4) \rightarrow BO(4)$ (the stable normal 1-type is $BPin^-(4) \rightarrow BO$). The action of $\pi_0 \text{Aut}_{BO}(BPin^+) \cong \mathbb{Z}/2\mathbb{Z}$ on $\Omega_4^{\tau BPin^+}$ is given by inversion. The smooth $\mathbb{R}P^4$, with its two Pin^+ -structures, represents the classes $\{\pm 1\} \subseteq \mathbb{Z}/16\mathbb{Z}$, while the Cappell-Shaneson exotic $\mathbb{R}P_{C,S}^4$ [CS76], with its two Pin^+ -structures, represents the classes $\{\pm 9\} \subseteq \mathbb{Z}/16\mathbb{Z}$, see [Sto88]. This gives an example of homeomorphic manifolds which have the same tangential 1-type but lie in distinct orbits in $\Omega_4^{\tau BPin^+}$, and which are thus not stably diffeomorphic. More generally, we can take a (possibly mixed) S^1 -connected sum (an operation which preserves tangential 1-type, see [HKT94]) of $\mathbb{R}P^4$ and Cappell-Shaneson exotic $\mathbb{R}P_{C,S}^4$ which remain homeomorphic. Hence, we obtain the following corollary of Theorem 6.5 generalizing Debray’s result [Deb20] mentioned in the introduction.

Corollary 6.11. *When a (possibly mixed) S^1 -connected sum [HKT94] of $\mathbb{R}P^4$ and Cappell-Shaneson exotic $\mathbb{R}P_{C,S}^4$ yield homeomorphic, but not stably diffeomorphic 4-manifolds, they are distinguished by type-1 unoriented generalized Dijkgraaf-Witten theories.*

This happens for example with the S^1 -connected sums $\#_{S^1}^r \mathbb{R}P^4$ and $\#_{S^1}^r \mathbb{R}P_{C.S}^4$ for $r \not\equiv 0 \pmod{4}$. \square

When $r = 4$, these $\#_{S^1}^4 \mathbb{R}P^4$ and $\#_{S^1}^4 \mathbb{R}P_{C.S}^4$ lie in the same orbit in $\Omega_4^{TBSpin^+}$, and thus are stably diffeomorphic. This raises the following question:

Question 6.12. Is the S^1 -connected sum of four copies of real projective space $\mathbb{R}P^4$ diffeomorphic to the S^1 -connected sum of four copies of Cappell-Shaneson's exotic $\mathbb{R}P^4$?

Similar examples can be obtained for 4-manifolds whose fundamental group is a finite of order $2 \pmod{4}$, see [Deb21].

6.2.3. *Manifolds whose fundamental group is the Thompson group.* The next example shows that we cannot completely ignore the π -finite condition in Theorem 6.5.

Let π be a finitely presented group. Let K_π be a connected finite two complex with $\pi_1 K \cong \pi$. We can embed K into \mathbb{R}^5 and take a regular neighborhood $W \subset \mathbb{R}^5$, which deformation retracts onto K . The boundary of W is a smooth 4-manifold $M = \partial W$ with $\pi_1 M \cong \pi$. By construction M bounds a framed 5-manifold (the regular neighborhood of K_π). Whence the signature of M is zero, the tangent bundle of M_π is stably trivial, M is oriented, and spin. The corresponding tangential 1-type is $BSpin(4) \times B\pi \rightarrow BO(4)$.

Suppose now that π is also infinite and simple. Then for any π -finite tangential 1-type (X, ξ) , the space of (X, ξ) -structures on M may be computed as

$$\text{Map}_{BO(4)}(M, X) \simeq \text{Map}_{BO(4)}(BSpin(4) \times B\pi, X) \simeq \text{Map}_{BO(4)}(BSpin(4), X)$$

where the last equivalence is induced from the projection map $BSpin(4) \times B\pi \rightarrow BSpin(4)$. Thus the type-1 generalized Dijkgraaf-Witten invariants of M are completely determined by the corresponding tangential spin bordism class $[M] \in \Omega_4^{TBSpin(4)} \cong \mathbb{Z} \oplus \mathbb{Z}$. By Remark 3.18, this bordism class is determined by the Euler characteristic of M (and the signature which we have already seen is zero).

If further π is acyclic, for example if $\pi = V$ is the large Thompson group, then Lefschetz and Alexander duality imply that $\chi(M) = 2 = \chi(S^4)$. It follows that $[M] = [S^4] \in \Omega_4^{TBSpin(4)}$. Thus we obtain:

Proposition 6.13. *When $\pi = V$ is the large Thompson group, or any other infinite finitely presented acyclic group, then the 4-manifold manifold M constructed above and the 4-sphere S^4 are not stably diffeomorphic, yet are indistinguishable by type-1 generalized Dijkgraaf-Witten theories.* \square

We note the obvious fact that the 4-sphere S^4 and M are not even homotopy equivalent. Thus in some cases, with infinite fundamental group, type-1 generalized Dijkgraaf-Witten theories can fail to detect homotopy type of smooth 4-manifold.

APPENDIX A. n -FINITELY DOMINATED SPACES

Definition A.1. A space X is *n -finitely dominated* if there exists an n -dimensional finite CW complex K and an $(n-1)$ -connected map $K \rightarrow X$.

Recall that a map being $(n-1)$ -connected means that the induced map $\pi_n(K) \rightarrow \pi_n(X)$ is surjective and $\pi_k(K) \cong \pi_k(X)$ for all $k < n$. This definition is relevant to us because of the following lemma whose proof is a standard exercise in obstruction theory.

Lemma A.2. *If $\psi : Y \rightarrow B$ is n -truncated, then $\text{Map}_B(X, Y)$ is an n -type for all $\xi : X \rightarrow B$. If $\psi : Y \rightarrow B$ moreover has π -finite fibers and X is n -finitely dominated, then $\text{Map}_B(X, Y)$ is π -finite.*

The following lemma collects sufficient conditions for a space to be n -finitely dominated.

Lemma A.3. *A space X satisfying any of the following conditions is n -finitely dominated:*

1. X is homotopy equivalent to a CW complex with finite n -skeleton.
2. X is a homotopy pushout of n -finitely dominated spaces.
3. There is a n -finitely dominated space B and a map $\xi : X \rightarrow B$ all of whose homotopy fibers are n -finitely dominated.
4. X is simply connected and $H_i(X)$ is finitely generated for each $i \leq n$.
5. X is simply connected and the based loop space ΩX is n -finitely dominated.
6. The n -type of X is π -finite (i.e. X has finitely many components and the first n homotopy groups of every component are finite).

Proof. For (1), the inclusion $\text{sk}_n X \rightarrow X$ of the n -skeleton is $(n-1)$ -connected and hence witnesses X being n -finitely dominated. To prove (2), given a diagram $U_1 \leftarrow U_3 \rightarrow U_2$ of n -finitely dominated spaces, let K_1, K_2, K_3 be finite n -dimensional CW complexes with $(n-1)$ -connected maps $K_i \rightarrow U_i$. Since K_3 is n -dimensional and the maps $K_i \rightarrow U_i$ for $i = 1, 2$ are $(n-1)$ -connected, it follows from obstruction theory that there are maps, as indicated by the dashed arrows, making the squares commute up to homotopy:

$$\begin{array}{ccccc}
 K_1 & \xleftarrow{\text{-----}} & K_3 & \xrightarrow{\text{-----}} & K_2 \\
 \downarrow & & \downarrow & & \downarrow \\
 U_1 & \xleftarrow{\text{-----}} & U_1 \cap U_2 & \xrightarrow{\text{-----}} & U_2
 \end{array}$$

Taking homotopy pushouts, this defines a map from the homotopy pushout $K = K_1 \cup_{K_3} K_2$ to the homotopy pushout $X = U_1 \cup_{U_3} U_2$. Since the pushout of finite n -dimensional CW complexes is again finite and n -dimensional, and since the pushout of $(n-1)$ -connected maps is again $(n-1)$ -connected [tD08, Theorem 6.7.9], it follows that $K \rightarrow X$ witnesses X being n -finitely dominated.

For (3), let $K_0 \rightarrow B$ be an $(n-1)$ -connected map from a finite n -dimensional CW complex. Then the map $X \times_B K_0 \rightarrow X$ pulled back from $K_0 \rightarrow B$ is also $(n-1)$ -connected. Thus, it suffices to show that $X \times_B K_0$ is n -finitely dominated — in other words, without loss of generality, we can assume that $B = K$ is a finite CW complex. Now we can induct on the cell structure of B . When B is a finite union of 0-cells, then X is a finite disjoint union of n -finitely dominated spaces and hence is itself n -finitely dominated. Thus we may assume that $B = B_0 \cup_{\partial D^k} D^k$ and that $X|_{B_0}$ is n -finitely dominated. Consider the commuting cube

$$\begin{array}{ccccc}
 & & Z & \xrightarrow{\quad} & F \\
 & \swarrow & \downarrow & & \swarrow \\
 X|_{B_0} & \xrightarrow{\quad} & X & & \\
 \downarrow & & \downarrow & & \downarrow \\
 & & \partial D^k & \xrightarrow{\quad} & D^k \\
 \downarrow & \swarrow & \downarrow & & \swarrow \\
 B_0 & \xrightarrow{\quad} & B & &
 \end{array}$$

where F is the homotopy pullback of $X \rightarrow B \leftarrow D^k$ (and hence homotopy equivalent to the homotopy fiber of $X \rightarrow B$ at any point in the image of $D^k \rightarrow B$) and Z is the homotopy

pullback of $\partial D^k \rightarrow D^k \leftarrow F$ (and hence homotopy equivalent to the product $F \times \partial D^k$). By definition, all four side squares are homotopy pullback squares and the lower square is a homotopy pushout. Hence, it follows that the upper square is a homotopy pushout square. Notice that since $\partial D^k = S^{k-1}$ is a compact manifold, it is n -finitely dominated, and since by assumption F and $X|_{B_0}$ are n -finitely dominated, so is also the product $Z \simeq F \times \partial D^k$ and hence it follows from (2) that the pushout X of $X|_{B_0} \leftarrow Z \rightarrow F$ is also n -finitely dominated.

Assertion (4) is a direct corollary of Wall [Wal65, Thm. B]. Assertion (5) follows from (4) and a Serre spectral sequence argument we learned from [Sad14]. Consider the based path-loop fibration $\Omega X \rightarrow P_* X \rightarrow X$. Since ΩX is n -finitely the homology groups $H_k(\Omega X)$ are finitely generated for $k \leq n$. If i be the first dimension in which ΩX has non-zero reduced homology, then

$$H_{i+1}(X) \cong \pi_{i+1}(X) \cong \pi_i(\Omega X) \cong H_i(\Omega X).$$

Thus $H_{i+1}(X)$ is finitely generated and all lower reduced homology groups are trivial. Now suppose that $H_k(X)$ is finitely generated for $k \leq \ell$, and consider the E^2 -term of the Serre Spectral Sequence for the path-loop fibration. We want to show that $H_{\ell+1}(X)$ is finitely generated. The relevant E^∞ groups are zero, and since $E_{n+1,0}^\infty = E_{n+1,0}^{n+2}$, we have $d_{n+1,0}^{n+1}$ is an isomorphism. The target is a quotient of $H_n(\Omega X)$, hence finitely generated. So $E_{n+1,0}^{n+2}$ is finitely generated. Now if we assume by induction that $E_{n+1,0}^k$ is finitely generated for $2 < k \leq n+2$, we examine the differential

$$E_{n+1,0}^{k-1} \rightarrow E_{n+2-k,k-2}^{k-1}$$

$E_{n+1,0}^k$ is the kernel of that map and the image is finitely generated as it is a subgroup of a finitely generated group. So we have a short exact sequence

$$0 \rightarrow E_{n+1,0}^k \rightarrow E_{n+1,0}^{k-1} \rightarrow \text{im}(d_{n+1,0}^{k-1}) \rightarrow 0$$

Since the two groups on the end are finitely generated, so is the group in the middle. We conclude that $E_{n+1,0}^2 = H_{n+1}(H)$ is finitely generated.

To prove (6), first observe that for finite π the space $K(\pi, 1)$ is homotopy equivalent to a CW complex with finite n -skeleton (for any n) and hence is n -finitely dominated. This can for example be seen by presenting $K(\pi, 1)$ as the realization of a simplicial bar complex built from the finite group π . Then, it follows by iteratively applying (5) and observing that $\Omega K(\pi, k) \simeq K(\pi, k-1)$ that $K(\pi, k)$ is n -finitely dominated. Hence, using (3) and inducting on the homotopy groups, it follows that every π -finite space is n -finitely dominated. Lastly, applying (3) to the map $X \rightarrow \tau_{\leq n} X$ to the n -type of X , it follows that it suffices that the n -type of X is π -finite. \square

Remark A.4. By work of Wall [Wal65, Thm. A], the first condition of Lemma A.3 is equivalent to the seemingly weaker condition that X is a homotopy retract of a CW complex with a finite n -skeleton.

Example A.5. Every compact manifold (of arbitrary dimension) is presented by a finite CW complex and hence is n -finitely dominated for every $n \geq 0$.

Example A.6. The circle $S^1 \simeq K(\mathbb{Z}, 1)$ is presented by a finite CW complex and hence is n -finitely dominated for every $n \geq 0$. It then follows that $K(\mathbb{Z}, k)$ is n -finitely dominated by iteratively applying Lemma A.3(5) and observing that $\Omega K(\mathbb{Z}, k) \simeq K(\mathbb{Z}, k-1)$.

Example A.7. The space BO and for any $d \geq 0$ the spaces $BO(d)$ are n -finitely dominated for every $n \geq 0$. Moreover, any connective cover $BO(d)\langle k \rangle \rightarrow BO(d)$ is n -finitely dominated.

This can be seen as follows. The space $BO(d)$ is the Grassmannian of d -planes in infinite dimensional space. The map from the finite Grassmannian $Gr_d(n+d) \rightarrow BO(d)$ is $n-1$ -connected, and since $Gr_d(n+d)$ is a compact manifold, it follows that $BO(d)$ is n -finitely dominated for all n . Likewise the map $BO(n) \rightarrow BO$ is $n-1$ -connected, and thus BO is n -finitely dominated for all n . The connective cover $BO(d)\langle k \rangle \rightarrow BO(d)$ sits in a fibration sequence

$$K(\pi_k(BO(d)), k-1) \rightarrow BO(d)\langle k \rangle \rightarrow BO(d)\langle k-1 \rangle.$$

Thus iteratively applying Lemma A.3(3) and Example A.6 shows that the connective covers too are n -finitely dominated.

For our purposes, a central corollary of Lemma A.3 is the following:

Corollary A.8. *Let $n \geq 0$ and assume that B is n -finitely dominated. If $\xi : X \rightarrow B$ is a map of spaces with π -finite fibers, then X is n -finitely dominated.*

Proof. By Lemma A.3 (4), the fibers of ξ are n -finitely dominated, and hence X is n -finitely dominated by Lemma A.3(3). \square

Let $\tau_{\leq n}\mathcal{S}_{/B}^\pi$ denote the full subcategory of $\tau_{\leq n}\mathcal{S}_{/B}$ on those n -truncated $\xi : X \rightarrow B$ whose fibers are π -finite, and let $\tau_{\leq n}\mathcal{S}_{/B}^{\pi, f.d.}$ denote the further full subcategory on those n -truncated $\xi : X \rightarrow B$ with π -finite fibers for which furthermore X is n -finitely dominated.

Corollary A.9. *The ∞ -category $\tau_{\leq n}\mathcal{S}_{/B}^{\pi, f.d.}$ is locally π -finite. Moreover, if B is n -finitely dominated, then $\tau_{\leq n}\mathcal{S}_{/B}^{\pi, f.d.} = \tau_{\leq n}\mathcal{S}_{/B}^\pi$.*

Proof. This is a direct consequence of Lemma A.2 and Corollary A.8. \square

REFERENCES

- [ABP66] D. W. Anderson, E. H. Brown, Jr., and F. P. Peterson. SU-cobordism, KO-characteristic numbers, and the Kervaire invariant. *Ann. of Math. (2)*, 83:54–67, 1966.
- [B67] J. Bénabou. Introduction to bicategories. In *Reports of the Midwest Category Seminar*, pp. 1–77. Springer, Berlin, 1967.
- [Bar13] C. Barwick. On the \mathbb{Q} construction for exact quasicategories. 2013. [arXiv:1301.4725](https://arxiv.org/abs/1301.4725).
- [BC76] E. H. Brown, Jr. and M. Comenetz. Pontrjagin duality for generalized homology and cohomology theories. *Amer. J. Math.*, 98(1):1–27, 1976.
- [BDS15] M. Bökstedt, J. Dupont, and A. M. Svane. Cobordism obstructions to independent vector fields. *Q. J. Math.*, 66(1):13–61, 2015.
- [BLM22] J. Beardsley, K. Luecke, and J. Morave. Brauer-wall groups and truncated picard spectra of k -theory. preprint, October 2022.
- [BS14] M. Bökstedt and A. M. Svane. A geometric interpretation of the homotopy groups of the cobordism category. *Algebr. Geom. Topol.*, 14(3):1649–1676, 2014.
- [CCPS21a] A. Conway, D. Crowley, M. Powell, and J. Sixt. Simply-connected manifolds with large homotopy stable classes. 2021. [arXiv:2109.00654](https://arxiv.org/abs/2109.00654).
- [CCPS21b] A. Conway, D. Crowley, M. Powell, and J. Sixt. Stably diffeomorphic manifolds and modified surgery obstructions. 2021. <https://arxiv.org/pdf/2109.05632.pdf>.
- [CS76] S. E. Cappell and J. L. Shaneson. Some new four-manifolds. *Ann. of Math. (2)*, 104(1):61–72, 1976.
- [CS19] D. Calaque and C. Scheimbauer. A note on the (∞, n) -category of cobordisms. *Algebr. Geom. Topol.*, 19(2):533–655, 2019.
- [CSY22] S. Carmeli, T. M. Schlank, and L. Yanovski. Ambidexterity in chromatic homotopy theory. *Invent. Math.*, 228(3):1145–1254, 2022.
- [Deb20] A. Debray. Can triangulations (or some related combinatorial structure) distinguish smooth structures on $\mathbb{R}P^4$? *MathOverflow*, 2020. <https://mathoverflow.net/q/362517> (version: 2020-06-08).

- [Deb21] A. Debray. Stable diffeomorphism classification of some unorientable 4-manifolds. 2021. [arXiv: 2102.03965](https://arxiv.org/abs/2102.03965).
- [DJ94] B. Durhuus and T. Jónsson. Classification and construction of unitary topological field theories in two dimensions. vol. 35, pp. 5306–5313. 1994. *Topology and physics*.
- [DK19] T. Dyckerhoff and M. Kapranov. *Higher Segal spaces*, vol. 2244 of *Lecture Notes in Mathematics*. Springer, Cham, 2019.
- [DW90] R. Dijkgraaf and E. Witten. Topological gauge theories and group cohomology. *Comm. Math. Phys.*, 129(2):393–429, 1990.
- [FH20] D. S. Freed and M. J. Hopkins. Invertible phases of matter with spatial symmetry. *Adv. Theor. Math. Phys.*, 24(7):1773–1788, 2020.
- [FH21] D. S. Freed and M. J. Hopkins. Reflection positivity and invertible topological phases. *Geom. Topol.*, 25(3):1165–1330, 2021.
- [FHLT10] D. S. Freed, M. J. Hopkins, J. Lurie, and C. Teleman. Topological quantum field theories from compact Lie groups. In *A celebration of the mathematical legacy of Raoul Bott*, vol. 50 of *CRM Proc. Lecture Notes*, pp. 367–403. Amer. Math. Soc., Providence, RI, 2010.
- [FKN⁺05] M. H. Freedman, A. Kitaev, C. Nayak, J. K. Slingerland, K. Walker, and Z. Wang. Universal manifold pairings and positivity. *Geom. Topol.*, 9:2303–2317, 2005.
- [FQ93] D. S. Freed and F. Quinn. Chern-Simons theory with finite gauge group. *Comm. Math. Phys.*, 156(3):435–472, 1993.
- [Fre94] D. S. Freed. Higher algebraic structures and quantization. *Comm. Math. Phys.*, 159(2):343–398, 1994.
- [Fre12] D. Freed. Lectures on twisted k-theory and orientifolds. Available at <https://web.ma.utexas.edu/users/dafr/vienna.pdf>, 2012.
- [Fre14] D. S. Freed. Short-range entanglement and invertible field theories. 06 2014.
- [Fre19] D. S. Freed. *Lectures on field theory and topology*, vol. 133 of *CBMS Regional Conference Series in Mathematics*. American Mathematical Society, Providence, RI, 2019. Published for the Conference Board of the Mathematical Sciences.
- [GCKT18a] I. Gálvez-Carrillo, J. Kock, and A. Tonks. Decomposition spaces, incidence algebras and Möbius inversion I: Basic theory. *Adv. Math.*, 331:952–1015, 2018.
- [GCKT18b] I. Gálvez-Carrillo, J. Kock, and A. Tonks. Decomposition spaces, incidence algebras and Möbius inversion II: Completeness, length filtration, and finiteness. *Adv. Math.*, 333:1242–1292, 2018.
- [GCKT18c] I. Gálvez-Carrillo, J. Kock, and A. Tonks. Decomposition spaces, incidence algebras and Möbius inversion III: The decomposition space of Möbius intervals. *Adv. Math.*, 334:544–584, 2018.
- [Gen12] J. Genauer. Cobordism categories of manifolds with corners. *Trans. Amer. Math. Soc.*, 364(1):519–550, 2012.
- [GJO19] N. Gurski, N. Johnson, and A. M. Osorno. The 2-dimensional stable homotopy hypothesis. *J. Pure Appl. Algebra*, 223(10):4348–4383, 2019.
- [GL80] M. Gromov and H. B. Lawson, Jr. The classification of simply connected manifolds of positive scalar curvature. *Ann. of Math. (2)*, 111(3):423–434, 1980.
- [Gom84] R. E. Gompf. Stable diffeomorphism of compact 4-manifolds. *Topology Appl.*, 18(2-3):115–120, 1984.
- [GRW10] S. Galatius and O. Randal-Williams. Monoids of moduli spaces of manifolds. *Geom. Topol.*, 14(3):1243–1302, 2010.
- [GTMW09] S. Galatius, U. Tillmann, I. Madsen, and M. Weiss. The homotopy type of the cobordism category. *Acta Math.*, 202(2):195–239, 2009.
- [Har20] Y. Harpaz. Ambidexterity and the universality of finite spans. *Proc. Lond. Math. Soc. (3)*, 121(5):1121–1170, 2020.
- [Hau18] R. Haugseng. Iterated spans and classical topological field theories. *Math. Z.*, 289(3-4):1427–1488, 2018.
- [Hit74] N. Hitchin. Harmonic spinors. *Advances in Math.*, 14:1–55, 1974.
- [HKT94] I. Hambleton, M. Kreck, and P. Teichner. Nonorientable 4-manifolds with fundamental group of order 2. *Trans. Amer. Math. Soc.*, 344(2):649–665, 1994.
- [HL13] M. Hopkins and J. Lurie. Ambidexterity in $K(n)$ -local stable homotopy theory. December 2013. available: <https://www.math.ias.edu/~lurie/>.
- [HL14] G. Heuts and J. Lurie. Ambidexterity. In *Topology and field theories*, vol. 613 of *Contemp. Math.*, pp. 79–110. Amer. Math. Soc., Providence, RI, 2014.

- [HS05] M. J. Hopkins and I. M. Singer. Quadratic functions in geometry, topology, and M-theory. *J. Differential Geom.*, 70(3):329–452, 2005.
- [JO12] N. Johnson and A. M. Osorno. Modeling stable one-types. *Theory Appl. Categ.*, 26:No. 20, 520–537, 2012.
- [KKNO73] U. Karras, M. Kreck, W. D. Neumann, and E. Ossa. *Cutting and pasting of manifolds; SK-groups*. Mathematics Lecture Series, No. 1. Publish or Perish, Inc., Boston, Mass., 1973.
- [Kon88] M. Kontsevich. Rational conformal field theory and invariants of 3-dimensional manifolds. 1988. preprint.
- [Kre99] M. Kreck. Surgery and duality. *Ann. of Math. (2)*, 149(3):707–754, 1999.
- [KS84] M. Kreck and J. A. Schafer. Classification and stable classification of manifolds: some examples. *Comment. Math. Helv.*, 59(1):12–38, 1984.
- [KT90] R. C. Kirby and L. R. Taylor. Pin structures on low-dimensional manifolds. In *Geometry of low-dimensional manifolds, 2 (Durham, 1989)*, vol. 151 of *London Math. Soc. Lecture Note Ser.*, pp. 177–242. Cambridge Univ. Press, Cambridge, 1990.
- [KT08] M. Kreck and P. Teichner. Positivity of topological field theories in dimension at least 5. *J. Topol.*, 1(3):663–670, 2008.
- [Lev35] J. Levitzki. On automorphisms of certain rings. *Ann. of Math. (2)*, 36(4):984–992, 1935.
- [Lic63] A. Lichnerowicz. Spineurs harmoniques. *C. R. Acad. Sci. Paris*, 257:7–9, 1963.
- [Lov67] L. Lovász. Operations with structures. *Acta Math. Acad. Sci. Hungar.*, 18:321–328, 1967.
- [Lov71] L. Lovász. On the cancellation law among finite relational structures. *Period. Math. Hungar.*, 1(2):145–156, 1971.
- [Lur09a] J. Lurie. *Higher topos theory*, vol. 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.
- [Lur09b] J. Lurie. On the classification of topological field theories. In *Current developments in mathematics, 2008*, pp. 129–280. Int. Press, Somerville, MA, 2009.
- [Mor15] J. C. Morton. Cohomological twisting of 2-linearization and extended TQFT. *J. Homotopy Relat. Struct.*, 10(2):127–187, 2015.
- [MW21] L. Müller and L. Woike. Dimensional reduction, extended topological field theories and orbifoldization. *Bull. Lond. Math. Soc.*, 53(2):392–403, 2021.
- [Qui95] F. Quinn. Lectures on axiomatic topological quantum field theory. In *Geometry and quantum field theory (Park City, UT, 1991)*, vol. 1 of *IAS/Park City Math. Ser.*, pp. 323–453. Amer. Math. Soc., Providence, RI, 1995.
- [Rei] J. Reinhold. What is known about exotic spheres up to stable diffeomorphism? MathOverflow. [https://mathoverflow.net/q/385015\(version:2021-02-26\)](https://mathoverflow.net/q/385015(version:2021-02-26)).
- [Rei63] B. L. Reinhart. Cobordism and the Euler number. *Topology*, 2:173–177, 1963.
- [Reu20] D. Reutter. Semisimple 4-dimensional topological field theories cannot detect exotic smooth structure. 2020. [arXiv:2001.02288](https://arxiv.org/abs/2001.02288).
- [Rez01] C. Rezk. A model for the homotopy theory of homotopy theory. *Trans. Amer. Math. Soc.*, 353(3):973–1007, 2001.
- [RS14] G. Raptis and W. Steimle. On the map of Bökstedt-Madsen from the cobordism category to A -theory. *Algebr. Geom. Topol.*, 14(1):299–347, 2014.
- [RW] O. Randal-Williams. Homology theory represented by Madsen-Tillmann spectra. MathOverflow. [https://mathoverflow.net/q/179327\(version:2014-08-25\)](https://mathoverflow.net/q/179327(version:2014-08-25)).
- [Sad14] H. Sadofsky. Spectral sequence notes: Finiteness of homotopy groups. <https://pages.uoregon.edu/sadofsky/691/sseq-finite-homotopy.pdf>, 2014.
- [Saw95] S. Sawin. Direct sum decompositions and indecomposable TQFTs. *J. Math. Phys.*, 36(12):6673–6680, 1995.
- [SP11] C. Schommer-Pries. *The Classification of Two-Dimensional Extended Topological Field Theories*. PhD thesis, 2011. [arXiv:1112.1000](https://arxiv.org/abs/1112.1000).
- [SP17] C. Schommer-Pries. Invertible topological field theories. 2017. [arXiv:1712.08029](https://arxiv.org/abs/1712.08029).
- [SP18] C. J. Schommer-Pries. Tori detect invertibility of topological field theories. *Geom. Topol.*, 22(5):2713–2756, 2018.
- [Sto] S. Stolz. Classification of picard groupoids and functors between them. Appendix to unpublished preprint.
- [Sto88] S. Stolz. Exotic structures on 4-manifolds detected by spectral invariants. *Invent. Math.*, 94(1):147–162, 1988.

- [Sto90] S. Stolz. Simply connected manifolds of positive scalar curvature. *Bull. Amer. Math. Soc. (N.S.)*, 23(2):427–432, 1990.
- [SW18] C. Schweigert and L. Woike. A parallel section functor for 2-vector bundles. *Theory Appl. Categ.*, 33:Paper No. 23, 644–690, 2018.
- [SW19] C. Schweigert and L. Woike. Orbifold construction for topological field theories. *J. Pure Appl. Algebra*, 223(3):1167–1192, 2019.
- [SW20] C. Schweigert and L. Woike. Extended homotopy quantum field theories and their orbifoldization. *J. Pure Appl. Algebra*, 224(4):106213, 42, 2020.
- [SY79] R. Schoen and S. T. Yau. On the structure of manifolds with positive scalar curvature. *Manuscripta Math.*, 28(1-3):159–183, 1979.
- [tD08] T. tom Dieck. *Algebraic topology*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008.
- [Tei92] P. Teichner. *Topological 4-manifolds with finite fundamental group*. PhD thesis, University of Mainz, Germany, 1992. Shaker Verlag 1992, ISBN 3-86111-182-9. Available at <http://people.mpim-bonn.mpg.de/teichner/Math/ewExternalFiles/phd.pdf>.
- [The22] The Stacks project authors. The stacks project. <https://stacks.math.columbia.edu>, 2022.
- [TKBB21] S. Tata, R. Kobayashi, D. Bulmash, and M. Barkeshli. Anomalies in (2+1)d fermionic topological phases and (3+1)d path integral state sums for fermionic SPTs. 2021. [arXiv:2104.14567](https://arxiv.org/abs/2104.14567).
- [Tro16] F. Trova. Nakayama categories and groupoid quantization. 02 2016. [arXiv:1602.01019](https://arxiv.org/abs/1602.01019).
- [Wal60] C. T. C. Wall. Determination of the cobordism ring. *Ann. of Math. (2)*, 72:292–311, 1960.
- [Wal62] C. T. C. Wall. Classification of $(n - 1)$ -connected $2n$ -manifolds. *Ann. of Math. (2)*, 75:163–189, 1962.
- [Wal65] C. T. C. Wall. Finiteness conditions for CW-complexes. *Ann. of Math. (2)*, 81:56–69, 1965.
- [Wal64] C. T. C. Wall. Graded Brauer groups. *J. Reine Angew. Math.*, 213:187–199, 1963/64.

UNIVERSITÄT HAMBURG

Email address: david.reutter@uni-hamburg.de

URL: <https://www.davidreutter.com>

UNIVERSITY OF NOTRE DAME

Email address: cschomme@nd.edu

URL: <https://sites.nd.edu/chris-schommer-pries/>