How to project onto the intersection of a closed affine subspace and a hyperplane

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Abstract

Let A be a closed affine subspace and let B be a hyperplane in a Hilbert space. Suppose we are given their associated nearest point mappings P_A and P_B, respectively. We present a formula for the projection onto their intersection $A \cap B$. As a special case, we derive a formula for the projection onto the intersection of two hyperplanes. These formulas provides useful information even if $A \cap B$ is empty. Examples and numerical experiments are also provided.

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1 Introduction

Throughout, we assume that

$$X \text{ is a real Hilbert space with inner product } \langle\cdot,\cdot\rangle : X \times X \to \mathbb{R}, \tag{1}$$

and induced norm $\|\cdot\|$. We also assume that

$$A \text{ is a closed affine subspace of } X \text{ with parallel space } U := A - A. \tag{2}$$

Denote the nearest point mapping associated with A by P_A, and set $a_0 := P_A(0)$. Then $a_0 \in U^\perp$ and $P_A(x) = a_0 + P_U(x)$. This formula allows us to move back and forth between P_A and P_U as needed. Furthermore, we assume that B is a hyperplane given by

$$B := \big\{x \in X \;\big|\; \langle x, v\rangle = \beta\big\}, \text{ where } v \in X \text{ and } \|v\| = 1, \tag{3}$$

which in turn yields $B = \beta v + \{v\}^\perp$ and $P_B \colon x \mapsto x - (\langle x, v\rangle - \beta)v$.

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The aim of this note is to present and prove a formula for $P_{A\cap B}$ that relies on P_A and P_B.

Indeed, we have:

Theorem 1.1 (main result). *Suppose that $A \cap B \neq \varnothing$, and let $x \in X$. Then exactly one of the following holds:*

(i) *If $P_U(v) = 0$, then $P_{A\cap B}(x) = P_A(x)$.*

(ii) *If $P_U(v) \neq 0$, then $P_{A\cap B}(x) = P_A(x) + \dfrac{\beta - \langle P_A(x), v\rangle}{\|P_U(v)\|^2} P_U(v)$.*

The analysis we carry out will reveal useful information *even if $A \cap B = \varnothing$.*

Organization of the paper. In Section 2, we analyze the case when $P_U(v) = 0$. In Section 3, we turn to the situation when $P_U(v) \neq 0$. A utility version of our main result as well as the special case of two hyperplanes is discussed in Section 4. In Section 5, we outline a numerical application.

2 The case when $P_U(v) = 0$

Throughout this section, we assume that

$$P_U(v) = 0. \tag{4}$$

Note that this assumption allows the following additional characterizatons:

$$v \in U^\perp \Leftrightarrow \mathbb{R}v \subseteq U^\perp \Leftrightarrow U \subseteq \{v\}^\perp. \tag{5}$$

We shall not impose that $A \cap B \neq \varnothing$. In order to handle this case, we well define

$$g := P_{B-A}(0) \;\text{ and }\; E := A \cap (B - g) \neq \varnothing. \tag{6}$$

Note that $B - A = (U + \{v\}^\perp) + (\beta v - a_0)$. Because $\{v\}^\perp$ is codimension 1, the Minkowski sum $U + \{v\}^\perp$ is closed by combining [3, Theorem 9.35 and Corollary 9.37]. Hence $B - A$ is a closed affine subspace which makes the vector g not only well-defined but it also yields $E \neq \varnothing$. Moreover, [2, Example 2.2] yields

$$g = P_{U^\perp \cap \mathbb{R}v}(b - a) \in U^\perp \cap \mathbb{R}v \tag{7}$$

no matter how $(a, b) \in A \times B$ is chosen. Note that

$$\text{if } A \cap B \neq \varnothing, \text{ then } g = 0 \text{ and so } E = A \cap B; \tag{8}$$

consequently, E can be thought of as the *generalized* intersection of A and B. Finally, by [1, Lemma 2.2.(i)], the generalized intersection E has also a description that does not involve the gap vector g:

$$E = \operatorname{Fix}(P_A P_B) := \big\{x \in X \;\big|\; x = P_A(P_B(x))\big\}. \tag{9}$$

We now prove

Theorem 2.1. *The assumption that $P_U(v) = 0$ implies $A \subseteq B - g$ and so $E = A$.*

Proof. Take $e \in E = A \cap (B - g)$. Then $e = P_A(e) = a_0 + P_U(e)$ and $e = \beta v + v^\perp - g$ for some $v^\perp \in \{v\}^\perp$. Using the linearity of $P_{U^\perp}$, the fact that $a_0 \in U^\perp$, (5), and (7), we obtain $a_0 = P_{U^\perp}(e) = \beta P_{U^\perp}(v) + P_{U^\perp}(v^\perp) - P_{U^\perp}(g) = \beta v + v^\perp - P_U(v^\perp) - g$. Therefore, using (5) again,

$$\begin{aligned} A = a_0 + U &= \beta v + v^\perp - P_U(v^\perp) - g + U && \text{(10a)} \\ &= \beta v + v^\perp + (U - P_U(v^\perp)) - g && \text{(10b)} \\ &= \beta v + v^\perp + U - g && \text{(10c)} \\ &\subseteq \beta v + \{v\}^\perp + \{v\}^\perp - g && \text{(10d)} \\ &= (\beta v + \{v\}^\perp) - g && \text{(10e)} \\ &= B - g, && \text{(10f)} \end{aligned}$$

as claimed. Because $E = A \cap (B - g)$, it now follows that $E = A$. ∎

Corollary 2.2. *The assumption that $P_U(v) = 0$ yields $P_E = P_A$. Let $x \in X$. Then exactly one of the following holds:*

(i) $P_A(x) \in B$, $g = 0$, $E = A \cap B \neq \varnothing$, *and* $P_{A\cap B}(x) = P_A(x)$.

(ii) $P_A(x) \notin B$, $g \neq 0$, $E \neq A \cap B = \varnothing$, *and* $P_E(x) = P_A(x)$.

Proof. This is a direct consequence of Theorem 2.1. ∎

Remark 2.3. *Note that Corollary 2.2(i) yields Theorem 1.1(i).*

3 The case when $P_U(v) \neq 0$

Throughout this section, we assume that

$$P_U(v) \neq 0. \tag{11}$$

Then

$$0 < \|P_U(v)\|^2 = \langle P_U(v), v\rangle . \tag{12}$$

Now set

$$Q\colon X \to X\colon x \mapsto P_A(x) + \frac{\beta - \langle P_A(x), v\rangle}{\|P_U(v)\|^2} P_U(v) \in A + U = A. \tag{13}$$

Proposition 3.1. *The assumption that $P_U(v) \neq 0$ implies* $\operatorname{ran} Q \subseteq A \cap B$*; hence, $A \cap B \neq \varnothing$.*

Proof. Let $x \in X$. Using (13) and (12), we have $Q(x) \in A$ and

$$\begin{aligned} \langle Q(x), v\rangle &= \langle P_A(x), v\rangle + \frac{\beta - \langle P_A(x), v\rangle}{\|P_U(v)\|^2} \langle P_U(v), v\rangle && \text{(14a)} \\ &= \langle P_A(x), v\rangle + \frac{\beta - \langle P_A(x), v\rangle}{\|P_U(v)\|^2} \|P_U(v)\|^2 && \text{(14b)} \\ &= \beta. && \text{(14c)} \end{aligned}$$

Hence $Q(x) \in B$ and we are done. ∎

Proposition 3.2. *The assumption that $P_U(v) \neq 0$ implies*

$$(A \cap B) - c = U \cap \{v\}^{\perp}, \tag{15}$$

$$\big((A \cap B) - c\big)^{\perp} = U^{\perp} + \mathbb{R}v. \tag{16}$$

for every $c \in A \cap B$.

Proof. By Proposition 3.1, $A \cap B \neq \varnothing$. Let $c \in A \cap B$. Then

$$(A \cap B) - c = (A - c) \cap (B - c) = U \cap \{v\}^{\perp}, \tag{17}$$

which is (15). Hence, using also [3, Theorem 9.35 and Corollary 9.37], we see that

$$\big((A \cap B) - c\big)^{\perp} = \big(U \cap \{v\}^{\perp}\big)^{\perp} = \overline{U^{\perp} + \mathbb{R}v} = U^{\perp} + \mathbb{R}v. \tag{18}$$

Therefore, (16) is verified and we are done. ■

Theorem 3.3. *The assumption that $P_U(v) \neq 0$ implies $Q = P_{A \cap B}$.*

Proof. Let $x \in X$. By Proposition 3.1,

$$Q(x) \in A \cap B. \tag{19}$$

Using (13), we have

$$\begin{aligned}
x - Q(x) &= x - P_A(x) - \frac{\beta - \langle P_A(x), v\rangle}{\|P_U(v)\|^2} P_U(v) && \text{(20a)}\\
&= \big(P_U(x) + P_{U^{\perp}}(x)\big) - \big(a_0 + P_U(x)\big) - \frac{\beta - \langle P_A(x), v\rangle}{\|P_U(v)\|^2}(v - P_{U^{\perp}}v) && \text{(20b)}\\
&= \left(P_{U^{\perp}}(x) - a_0 + \frac{\beta - \langle P_A(x), v\rangle}{\|P_U(v)\|^2} P_{U^{\perp}}(v)\right) - \frac{\beta - \langle P_A(x), v\rangle}{\|P_U(v)\|^2} v && \text{(20c)}\\
&\in U^{\perp} + \mathbb{R}v. && \text{(20d)}
\end{aligned}$$

Now (19), (16), and (20) yield

$$x - Q(x) \in \big((A \cap B) - Q(x)\big)^{\perp}. \tag{21}$$

Combining (19) and (21), we conclude that $P_{A \cap B}(x) = Q(x)$. ■

Remark 3.4. *Note that Theorem 3.3 and (13) imply Theorem 1.1(ii).*

4 A utility version of the main result and the case of two hyperplanes

The analysis in the previous section was simplified because of our assumption that $\|v\| = 1$. It is worthwhile to record our results when we drop this normalization requirement.

Theorem 4.1 (trichotomy). *Let A be as in (2) and let H be a hyperplane given by*

$$H := \big\{x \in X \mid \langle x, c\rangle = \gamma\big\}, \tag{22}$$

where $c \in X \smallsetminus \{0\}$ and $\gamma \in \mathbb{R}$. Let $x \in X$. Then exactly one of the following holds:

(i) *$P_U(c) = 0$, $\langle P_A(x), c\rangle = \gamma$, $A \cap H \neq \varnothing$, and $P_{A \cap H}(x) = P_A(x)$.*

(ii) $P_U(c) = 0$, $\langle P_A(x), c\rangle \neq \gamma$, $A \cap H = \varnothing$, *and* $P_{\operatorname{Fix}(P_A P_H)}(x) = P_A(x)$.

(iii) $P_U(c) \neq 0$, $A \cap H \neq \varnothing$, *and* $P_{A\cap H}(x) = P_A(x) + \dfrac{\gamma - \langle P_A(x), c\rangle}{\|P_U(c)\|^2} P_U(c)$.

Proof. Suppose that

$$v = \frac{c}{\|c\|} \quad \text{and} \quad \beta = \frac{\gamma}{\|c\|}. \tag{23}$$

Then $H = B$ (see (3)). Note that

$$P_U(v) = \frac{P_U(c)}{\|c\|}, \tag{24}$$

which shows that $P_U(v) = 0 \Leftrightarrow P_U(c) = 0$.

(i): This is clear from Corollary 2.2(i). (ii): Combine Corollary 2.2(ii) with (9). (iii): Combining Theorem 3.3, (13), and (23) yields

$$\begin{aligned} P_{A\cap B}(x) &= P_A(x) + \frac{\beta - \langle P_A(x), v\rangle}{\|P_U(v)\|^2} P_U(v) && \text{(25a)}\\ &= P_A(x) + \frac{\gamma/\|c\| - \langle P_A(x), c/\|c\|\rangle}{\|P_U(c/\|c\|)\|^2} P_U(c/\|c\|) && \text{(25b)}\\ &= P_A(x) + \frac{\gamma - \langle P_A(x), c\rangle}{\|P_U(c)\|^2} P_U(c), && \text{(25c)} \end{aligned}$$

as claimed. ■

Corollary 4.2 (two hyperplanes). *Suppose that*

$$H_1 := \big\{x \in X \mid \langle x, c_1\rangle = \gamma_1\big\} \ \text{ and } \ H_2 := \big\{x \in X \mid \langle x, c_2\rangle = \gamma_2\big\}, \tag{26}$$

where c_1, c_2 *lie in* $X \smallsetminus \{0\}$, *and* γ_1, γ_2 *belong to* $\mathbb{R}$. *Let* $x \in X$. *Then the following hold:*

(i) *If* $\langle c_1, c_2\rangle^2 = \|c_1\|^2\|c_2\|^2$ *and* $\|c_1\|^2(\langle x, c_2\rangle - \gamma) = \langle c_1, c_2\rangle\,(\langle x, c_1\rangle - \gamma_1)$, *then* $H_1 = H_2$ *and*

$$P_{H_1\cap H_2}(x) = P_{H_1}(x) = x - \frac{\langle x, c_1\rangle - \gamma_1}{\|c_1\|^2} c_1. \tag{27}$$

(ii) *If* $\langle c_1, c_2\rangle^2 = \|c_1\|^2\|c_2\|^2$ *and* $\|c_1\|^2(\langle x, c_2\rangle - \gamma) \neq \langle c_1, c_2\rangle\,(\langle x, c_1\rangle - \gamma_1)$, *then* H_1 *and* H_2 *are parallel but distinct* ($H_1 \cap H_2 = \varnothing$), *and*

$$P_{\operatorname{Fix}(P_{H_1}P_{H_2})}(x) = P_{H_1}(x) = x - \frac{\langle x, c_1\rangle - \gamma_1}{\|c_1\|^2} c_1. \tag{28}$$

(iii) *If* $\langle c_1, c_2\rangle^2 \neq \|c_1\|^2\|c_2\|^2$, *then* $H_1 \cap H_2 \neq \varnothing$ *and*

$$\begin{aligned} P_{H_1\cap H_2}(x) = x &+ \frac{\|c_2\|^2\big(\gamma_1 - \langle x, c_1\rangle\big) + \langle c_1, c_2\rangle\,\big(\langle x, c_2\rangle - \gamma_2\big)}{\|c_1\|^2\|c_2\|^2 - \langle c_1, c_2\rangle^2} c_1 && \text{(29a)}\\ &+ \frac{\|c_1\|^2\big(\gamma_2 - \langle x, c_2\rangle\big) + \langle c_1, c_2\rangle\,\big(\langle x, c_1\rangle - \gamma_1\big)}{\|c_1\|^2\|c_2\|^2 - \langle c_1, c_2\rangle^2} c_2. && \text{(29b)} \end{aligned}$$

Proof. We apply Theorem 4.1 with $A = H_1$, $H = H_2$, $c = c_2$, and $\gamma = \gamma_2$. We have

$$P_A(x) = P_{H_1}(x) = x - \frac{\langle x, c_1\rangle - \gamma_1}{\|c_1\|^2} c_1, \tag{30}$$

$U = \{c_1\}^\perp$, and so

$$P_U(c) = P_{\{c_1\}^\perp}(c_2) = c_2 - P_{\mathbb{R}c_1}(c_2) = c_2 - \frac{\langle c_2, c_1 \rangle}{\|c_1\|^2} c_1. \tag{31}$$

Therefore, (31) implies

$$\|P_U(c)\|^2 = \left\| c_2 - \frac{\langle c_1, c_2 \rangle}{\|c_1\|^2} c_1 \right\|^2 \tag{32a}$$

$$= \|c_2\|^2 - \frac{2\langle c_1, c_2 \rangle^2}{\|c_1\|^2} + \frac{\langle c_1, c_2 \rangle^2}{\|c_1\|^4} \|c_1\|^2 \tag{32b}$$

$$= \|c_2\|^2 - \frac{\langle c_1, c_2 \rangle^2}{\|c_1\|^2}. \tag{32c}$$

Hence

$$P_U(c) = 0 \quad \Leftrightarrow \quad \|P_U(c)\|^2 = 0 \quad \Leftrightarrow \quad \langle c_1, c_2 \rangle^2 = \|c_1\|^2 \|c_2\|^2. \tag{33}$$

Next, (30) implies

$$\langle P_A(x), c \rangle = \langle P_{H_1}(x), c_2 \rangle = \langle x, c_2 \rangle - \frac{\langle x, c_1 \rangle - \gamma_1}{\|c_1\|^2} \langle c_1, c_2 \rangle. \tag{34}$$

Thus

$$\langle P_A(x), c \rangle = \gamma \quad \Leftrightarrow \quad \|c_1\|^2 \big(\langle x, c_2 \rangle - \gamma_2 \big) = \langle c_1, c_2 \rangle \big(\langle x, c_1 \rangle - \gamma_1 \big). \tag{35}$$

(i): The hypothesis in this case corresponds to $P_U(c) = 0$ and $\langle P_A(x), c \rangle = \gamma$. By Theorem 4.1(i), $A \cap H \neq \varnothing$ which means the gap vector $g = 0$ by Corollary 2.2(i). By Theorem 2.1, $H_1 \subseteq H_2$. Because both H_1 and H_2 are hyperplanes, we have $H_1 = H_2 = H_1 \cap H_2$.

(ii): The hypothesis in this case corresponds to $P_U(c) = 0$ and $\langle P_A(x), c \rangle \neq \gamma$. The conclusion now follows from Theorem 4.1(ii).

(iii): Using (33), Theorem 4.1(iii), (30), (34), (32), and (31), we have $H_1 \cap H_2 = A \cap H \neq \varnothing$ and

$$P_{H_1 \cap H_2}(x) = x - \frac{\langle x, c_1 \rangle - \gamma_1}{\|c_1\|^2} c_1 + \frac{\gamma_2 - \Big(\langle x, c_2 \rangle - \frac{\langle x, c_1 \rangle - \gamma_1}{\|c_1\|^2} \langle c_1, c_2 \rangle \Big)}{\|c_2\|^2 - \frac{\langle c_1, c_2 \rangle^2}{\|c_1\|^2}} \Big(c_2 - \frac{\langle c_2, c_1 \rangle}{\|c_1\|^2} c_1 \Big) \tag{36a}$$

$$= x - \frac{\langle x, c_1 \rangle - \gamma_1}{\|c_1\|^2} c_1 \tag{36b}$$

$$+ \frac{\|c_1\|^2 \big(\gamma_2 - \langle x, c_2 \rangle \big) + \langle c_1, c_2 \rangle \big(\langle x, c_1 \rangle - \gamma_1 \big)}{\|c_1\|^2 \|c_2\|^2 - \langle c_1, c_2 \rangle^2} \Big(c_2 - \frac{\langle c_2, c_1 \rangle}{\|c_1\|^2} c_1 \Big) \tag{36c}$$

$$= x + \frac{\|c_1\|^2 \big(\gamma_2 - \langle x, c_2 \rangle \big) + \langle c_1, c_2 \rangle \big(\langle x, c_1 \rangle - \gamma_1 \big)}{\|c_1\|^2 \|c_2\|^2 - \langle c_1, c_2 \rangle^2} c_2 \tag{36d}$$

$$- \Big(\frac{\langle x, c_1 \rangle - \gamma_1}{\|c_1\|^2} + \frac{\|c_1\|^2 \big(\gamma_2 - \langle x, c_2 \rangle \big) + \langle c_1, c_2 \rangle \big(\langle x, c_1 \rangle - \gamma_1 \big)}{\|c_1\|^2 \|c_2\|^2 - \langle c_1, c_2 \rangle^2} \frac{\langle c_2, c_1 \rangle}{\|c_1\|^2} \Big) c_1 \tag{36e}$$

$$= x + \frac{\|c_1\|^2 \big(\gamma_2 - \langle x, c_2 \rangle \big) + \langle c_1, c_2 \rangle \big(\langle x, c_1 \rangle - \gamma_1 \big)}{\|c_1\|^2 \|c_2\|^2 - \langle c_1, c_2 \rangle^2} c_2 \tag{36f}$$

$$+ \frac{\|c_2\|^2 \big(\gamma_1 - \langle x, c_1 \rangle \big) + \langle c_1, c_2 \rangle \big(\langle x, c_2 \rangle - \gamma_2 \big)}{\|c_1\|^2 \|c_2\|^2 - \langle c_1, c_2 \rangle^2} c_1, \tag{36g}$$

as announced. ■

Remark 4.3. *Having just found a formula for the projection onto the intersection of two hyperplanes, it is in principle possible to present a formula for the intersection of three (or more) hyperplanes; however, the result would of course be significantly more complicated than the formulas presented in Corollary 4.2.*

We conclude this section with the following limiting example which shows that there does not appear to exist a straightforward extension of the main result in Theorem 1.1. Indeed, Example 4.4 below verifies that Theorem 1.1 does not generalize when we replace A by a cone K. Observe that in this case $a_0 = 0$, hence U is replaced by K as well.

Example 4.4. *Suppose that $X = \mathbb{R}^2$, and that $K = \mathbb{R}^2_+ := \{(\xi_1, \xi_2) \in \mathbb{R}^2 \mid \xi_1 \geq 0, \xi_2 \geq 0\}$. Let $v_1 = \frac{1}{\sqrt{2}}(-1,-1)$, let $v_2 = \frac{1}{\sqrt{2}}(1,1)$, let $\beta_1 = 0$ and let $\beta_2 = \frac{1}{\sqrt{2}}$. Set $(\forall i \in \{1,2\})$ $B_i := \{x \in \mathbb{R}^2 \mid \langle x, v_i \rangle = \beta_i\}$. Then the following hold:*

(i) *Let $x \in K \smallsetminus \{(0,0)\}$. Then:*

  (a) $K \cap B_1 = \{(0,0)\}$.

  (b) $P_K(v_1) = (0,0)$.

  (c) $P_{K\cap B_1} \equiv (0,0)$.

  (d) $(0,0) = P_{K\cap B_1}(x) \neq P_K(x) = x$.

(ii) *Set $S := \{(\xi, 0) \in \mathbb{R}^2 \mid \xi > \sqrt{2}\}$ and set $\widetilde{Q} := P_K + \dfrac{\beta_2 - \langle P_K(\cdot), v_2\rangle}{\|P_K(v_2)\|^2} P_K(v_2)$. Then:*

  (a) $K \cap B_2 = \operatorname{conv}\{(1,0),(0,1)\}$.

  (b) $P_K(v_2) = v_2 \neq (0,0)$.

  (c) $\widetilde{Q} = P_K + \big(\frac{1}{\sqrt{2}} - \langle P_K(\cdot), v_2\rangle\big) v_2$.

  (d) $(\forall(\xi,0) \in S)$ $P_{K\cap B_2}(\xi,0)) = (1,0)$.

  (e) $(\forall(\xi,0) \in S)$ $\widetilde{Q}((\xi,0)) = \frac{1}{2}(\sqrt{2}+\xi, \sqrt{2}-\xi) \notin K$.

5 A numerical experiment

In this section we provide a numerical experiment to evaluate the performance of the formula developed in Corollary 4.2 when employed to find the projection onto the intersection of finitely many hyperplanes.

We randomly generate 100 matrices M each of size 10×50. For each matrix M, we randomly generate $\overline{x} \in \mathbb{R}^{50}$ and set $b = M\overline{x}$. This guarantees that the underdetermined system of equations $Mx = b$ is consistent, i.e., it has a solution. For each random instance of the matrix M we randomly generate 100 starting points. Because the i^{th} row in each of the randomly generated systems of equation $Mx = b$ defines a hyperplane, namely $H_i := \{x \in \mathbb{R}^{50} \mid \langle m_i, x\rangle = b_i\}$, we set

$$P := P_{H_{10}} P_{H_9} \cdots P_{H_2} P_{H_1} \tag{37}$$

and

$$Q := P_{H_{10}\cap H_9} P_{H_8\cap H_7} \cdots P_{H_2\cap H_1}. \tag{38}$$

For each of the randomly generated problems with data (M, b), and for a randomly generated starting point x_0, let $x^* = P_C(x_0)$, where $C = M^{-1}(b)$. We generate then two sequences via

$$(\forall n \in \mathbb{N}) \quad p_n := P^n x_0 \;\text{ and }\; q_n := Q^n x_0. \tag{39}$$

Both $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ are incarnations of the method of *cyclic projections*, and thus both sequences converge to x^*, by e.g., [3, Theorem 9.27] or [4, Chapter 3]. At each iteration index n, we measure the decibel (dB) value of the proximity function which we choose to be the relative distance of the iterate to the solution x^*:

$$20\log_{10}\frac{\|p_n - x^*\|}{\|p_0 - x^*\|} \;\text{ and }\; 20\log_{10}\frac{\|q_n - x^*\|}{\|q_0 - x^*\|}. \tag{40}$$

Figure 1 reports the progress of the proximity function of both sequences as a function of the iteration index where the median is calculated over all 100 instances of the matrix M and then over 100 randomly generated starting points, resulting in 10,000 numerical scenarios. We observe a notable improvement in the speed of convergence using Corollary 4.2 — this suggests that experimenting with this result may improve performance of projection algorithms involving hyperplanes.

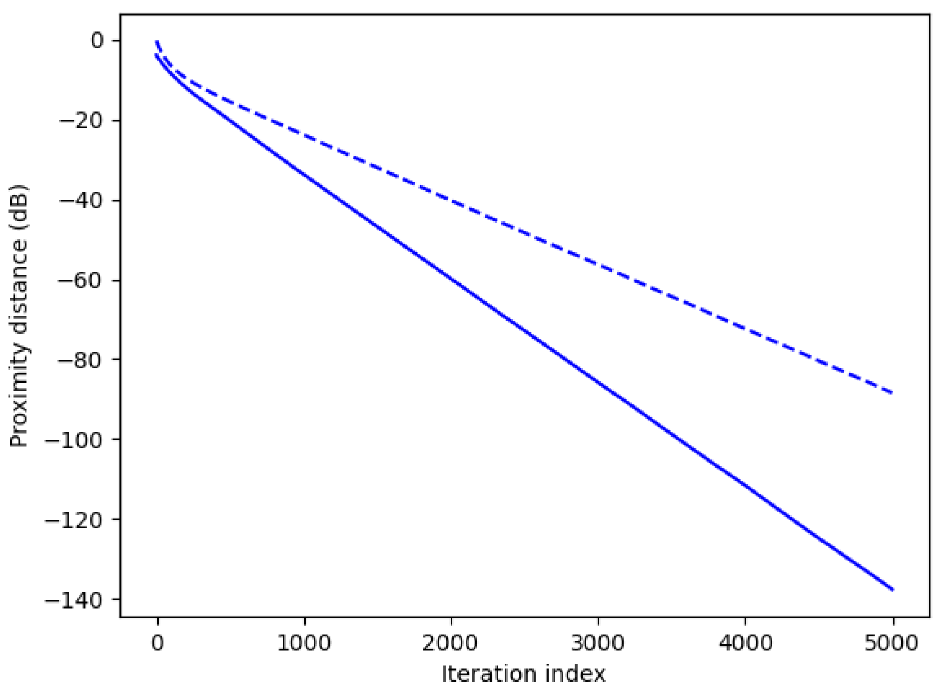

Figure 1: Plot of the decibel (dB) value of the median of the proximity function for the sequences $(p_n)_{n\in\mathbb{N}}$ (the dashed curve) and $(q_n)_{n\in\mathbb{N}}$ (the solid curve).

Declaration of competing interest

The authors declare that they have no competing interest.

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