ON NORMAL SESHADRI STRATIFICATIONS

ROCCO CHIRIVÌ, XIN FANG, AND PETER LITTELMANN

A Claudio, che ci mostra la via

Abstract. The existence of a Seshadri stratification on an embedded projective variety provides a flat degeneration of the variety to a union of projective toric varieties, called a semi-toric variety. Such a stratification is said to be normal when each irreducible component of the semi-toric variety is a normal toric variety. In this case, we show that a Gröbner basis of the defining ideal of the semi-toric variety can be lifted to define the embedded projective variety. Applications to Koszul and Gorenstein properties are discussed.

1. Introduction

Seshadri stratifications on an embedded projective variety $X \subseteq \mathbb{P}(V)$ have been introduced in [5] as a far reaching generalization of the construction in [15]. The aim is to provide a geometric framework of standard monomial theories such as Hodge algebras [12], LS-algebras [3], *etc*.

Such a stratification consists of certain projective subvarieties $X_p \subseteq X$ and homogeneous functions $f_p \in \mathrm{Sym}(V^*)$ indexed by a finite set A. The set A inherits a partially ordered set (poset) structure from the inclusion relation between the subvarieties X_p. These data: the collection of subvarieties X_p and of homogeneous functions f_p, $p \in A$, and the poset structure on A should satisfy the regularity and compatibility conditions in Definition 2.1.

Out of a Seshadri stratification we construct in [5] a quasi-valuation $\mathcal{V}$ on the homogeneous coordinate ring $R := \mathbb{K}[\hat{X}]$ taking values in the vector space $\mathbb{Q}^A$, where $\hat{X}$ is the affine cone of X. The quasi-valuation has one-dimensional leaves, hence its image in $\mathbb{Q}^A$, denoted by Γ, parametrizes a vector space basis of the homogeneous coordinate ring R. The set Γ, called a fan of monoids, carries fruitful structures: it is a finite union of finitely generated monoids in $\mathbb{Q}^A$, each monoid corresponds to a maximal chain in A. Geometrically, such a quasi-valuation provides a flat degeneration of X into a union of projective toric varieties[1] whose irreducible components arise from the monoids in Γ. Such a flat family is called a *semi-toric degeneration* of X. In general, the degeneration constructed in this way is different from the degeneration in Gröbner theory using a monomial order: the ideal defining the semi-toric variety is *radical*. Roughly speaking, it is the deepest degeneration without introducing any nilpotent elements.

We associate in [5] a Newton-Okounkov simplicial complex to a Seshadri stratification, and introduce an integral structure on it to establish a connection between the volume of the simplicial complex and the degree of X with respect to the embedding.

[1] In the article, toric varieties are reduced and irreducible, but not necessarily normal.

When all toric varieties appearing in the semi-toric degeneration are normal, or equivalently, all monoids in the fan of monoids Γ are saturated, such a Seshadri stratification is called *normal*. From such stratifications, we are able to derive a standard monomial theory in *loc.cit.*

As an application, the Lakshmibai-Seshadri path model [19, 20] for a Schubert variety is recovered from the Seshadri stratification consisting of Schubert subvarieties contained in it (see [7], [8] for details).

In this article, we study certain properties and applications of normal Seshadri stratifications.

First we will show (Theorem 3.2) that for such a stratification, the subduction algorithm lifts a reduced Gröbner basis of the defining ideal of the semi-toric variety to a reduced Gröbner basis of the defining ideal of X with respect to an embedding. The example of the flag variety SL_3/B in $\mathbb{P}(V(\rho))$, with the Seshadri stratification given by its Schubert varieties, is discussed in Section 5. As an application, we study how to determine the Koszul property of the homogeneous coordinate ring R from properties of the stratification. For this we introduce Seshadri stratifications of LS-type (Definition 2.6), and prove (Theorem 3.4): if the stratification is of LS-type and the functions f_p are linear, then the algebra R is Koszul. We also show that the Gorenstein property of the semi-toric variety can be lifted to R. As an application we show (Proposition 4.4) that the irreducible components of the semi-toric variety are not necessarily weighted projective spaces.

The Gröbner basis and the Koszul property have already been addressed for Schubert varieties in [18], and for LS-algebras in [2, 4]. Our approach in this article is different. For example, the Gröbner basis of the defining ideal of X is obtained in an algorithmic way by lifting the semi-toric relations; moreover, instead of being assumptions, weaker versions of quadratic straightening relations in the definition of LS-algebras become now consequences. In our paper [6] the relation between quasi-valuations and LS-algebras is studied in yet another way: starting from an LS-algebra and defining a quasi-valuation similar to the geometric one coming from Seshadri stratifications.

This article is organized as follows. In Section 2 we give a recollection on normal Seshadri stratifications and several constructions around them. Lifting Gröbner bases from the semi-toric varieties to the original variety is discussed in Section 3, which is then used to study the Koszul property. The Gorenstein property is discussed in Section 4; it is then applied to answer the question whether all irreducible components in the semi-toric variety are weighted projective spaces. Section 5 is devoted to an explicit example, when X is the flag variety SL_3/B, to illustrate the lifting procedure of Gröbner bases.

2. Seshadri stratifications

Throughout the paper we fix $\mathbb{K}$ to be an algebraically closed field and V to be a finite dimensional vector space over $\mathbb{K}$. The vanishing set of a homogeneous function $f \in \mathrm{Sym}(V^*)$ will be denoted by $\mathcal{H}_f := \{[v] \in \mathbb{P}(V) \mid f(v) = 0\}$. For a projective subvariety $X \subseteq \mathbb{P}(V)$, we let $\hat{X}$ denote its affine cone in V.

In this section we briefly recall the definition of a Seshadri stratification on an embedded projective variety. We quickly outline the construction of associated quasi-valuations and their associated fan of monoids.

Certain special classes, such as normal Seshadri stratifications and Seshadri stratifications of LS-type will be discussed. Details can be found in [5].

2.1. **Definition.** Let $X \subseteq \mathbb{P}(V)$ be an embedded projective variety, X_p, $p \in A$, be a finite collection of projective subvarieties of X and $f_p \in \mathrm{Sym}(V^*)$, $p \in A$, be homogeneous functions of positive degrees. The index set A inherits a poset structure by requiring: for $p, q \in A$, $p \geq q$ if $X_p \supseteq X_q$. We assume that there exists a unique maximal element $p_{\max} \in A$ with $X_{p_{\max}} = X$.

Definition 2.1 ([5])**.** The collection of subvarieties X_p and functions f_p for $p \in A$ is called a *Seshadri stratification* on X, if the following conditions are fulfilled:

(S1) the projective subvarieties X_p, $p \in A$, are smooth in codimension one; if $q < p$ is a covering relation in A, then X_q is a codimension one subvariety in X_p;
(S2) for $p, q \in A$ with $q \not\leq p$, the function f_q vanishes on X_p;
(S3) for $p \in A$, it holds set-theoretically

$$\mathcal{H}_{f_p} \cap X_p = \bigcup_{q \text{ covered by } p} X_q.$$

The functions f_p will be called *extremal functions*.

It is proved in [5, Lemma 2.2] that if X_p and f_p, $p \in A$, form a Seshadri stratification on X, then all maximal chains in A share the same length $\dim X$. This allows us to define the *length* $\ell(p)$ of $p \in A$ to be the length of a (hence any) maximal chain joining p with a minimal element in A. With this definition, $\ell(p) = \dim X_p$.

The set of all maximal chains in A will be denoted by $\mathcal{C}$.

To such a Seshadri stratification, we associate an edge-coloured directed graph $\mathcal{G}_A$: as a graph it is the Hasse diagram of the poset A; the edges, which correspond to covering relations in A, point to the larger element.

For a covering relation $p > q$ in A, the affine cone $\hat{X}_q$ is a prime divisor in $\hat{X}_p$. According to (S1), the local ring $\mathcal{O}_{\hat{X}_p, \hat{X}_q}$ is a discrete valuation ring (DVR). Let $\nu_{p,q} : \mathcal{O}_{\hat{X}_p, \hat{X}_q} \setminus \{0\} \to \mathbb{Z}$ be the associated discrete valuation. It extends to the field of rational functions $\mathbb{K}(\hat{X}_p) = \mathrm{Frac}(\mathcal{O}_{\hat{X}_p, \hat{X}_q})$, also denoted by $\nu_{p,q}$, by requiring

$$\nu_{p,q}\left(\frac{f}{g}\right) := \nu_{p,q}(f) - \nu_{p,q}(g), \quad \text{for } f, g \in \mathcal{O}_{\hat{X}_p, \hat{X}_q} \setminus \{0\}.$$

The edge $q \to p$ in the directed graph $\mathcal{G}_A$ is colored by the integer $b_{p,q} := \nu_{p,q}(f_p)$, called the *bond* between p and q. According to (S3), the bonds $b_{p,q} \geq 1$.

Since we will mainly work with the affine cones later in the article, it is helpful to extend the construction one step further. If $p \in A$ is a minimal element, the affine cone $\hat{X}_p$ is an affine line $\mathbb{A}^1$ hence $0 \in V$ is contained in $\hat{X}_p$. We set $\hat{A} := A \cup \{p_{-1}\}$ with $\hat{X}_{p_{-1}} := \{0\}$. The set $\hat{A}$ is endowed with the structure of a poset by requiring p_{-1} to be the unique minimal element. This partial order is compatible with the inclusion of affine cones $\hat{X}_p$ with $p \in \hat{A}$.

We associate to the extended poset $\hat{A}$ the directed graph $\mathcal{G}_{\hat{A}}$, an edge between a minimal element p in A and p_{-1} is colored by $b_{p,p_{-1}}$, the vanishing order of f_p at $\hat{X}_{p_{-1}} = \{0\}$: it is nothing but the degree of f_p.

2.2. **A family of higher rank valuations.** From now on we fix a Seshadri stratification on $X \subseteq \mathbb{P}(V)$. Let $R_p := \mathbb{K}[\hat{X}_p]$ denote the homogeneous coordinate ring of X_p and $\mathbb{K}(\hat{X}_p)$ the field of rational functions on X_p.

Let N be the least common multiple of all bonds appearing in $\mathcal{G}_{\hat{A}}$.

To a fixed maximal chain $\mathfrak{C} : p_{\max} = p_r > p_{r-1} > \ldots > p_1 > p_0$ in A, we associate a higher rank valuation $\mathcal{V}_{\mathfrak{C}} : \mathbb{K}[\hat{X}] \setminus \{0\} \to \mathbb{Q}^{\mathfrak{C}}$ as follows.

First choose a non-zero rational function $g_r := g \in \mathbb{K}(\hat{X})$ and denote by a_r its vanishing order in the divisor $\hat{X}_{p_{r-1}} \subset \hat{X}_{p_r}$. We consider the following rational function

$$h := \frac{g_r^N}{f_{p_r}^{N\frac{a_r}{b_r}}} \in \mathbb{K}(\hat{X}_{p_r}),$$

where $b_r := b_{p_r,p_{r-1}}$ is the bond between p_r and p_{r-1}. By [5, Lemma 4.1], the restriction of h to $\hat{X}_{p_{r-1}}$ is a well-defined non-zero rational function on $\hat{X}_{p_{r-1}}$. Let g_{r-1} denote this rational function. This procedure can be iterated by restarting with the non-zero rational function g_{r-1} on $\hat{X}_{p_{r-1}}$. The output is a sequence of rational functions

$$g_{\mathfrak{C}} := (g_r, g_{r-1}, \ldots, g_1, g_0)$$

with $g_k \in \mathbb{K}(\hat{X}_{p_k}) \setminus \{0\}$.

Collecting the vanishing orders together, we define a map

$$\mathcal{V}_{\mathfrak{C}} : \mathbb{K}[\hat{X}] \setminus \{0\} \to \mathbb{Q}^{\mathfrak{C}},$$

$$g \mapsto \frac{\nu_r(g_r)}{b_r} e_{p_r} + \frac{1}{N}\frac{\nu_{r-1}(g_{r-1})}{b_{r-1}} e_{p_{r-1}} + \ldots + \frac{1}{N^r}\frac{\nu_0(g_0)}{b_0} e_{p_0},$$

where $\nu_k := \nu_{p_k,p_{k-1}}$ is the discrete valuation on the local ring $\mathcal{O}_{\hat{X}_{p_k},\hat{X}_{p_{k-1}}}$, extended to the fraction field, and e_{p_k} is the coordinate function in $\mathbb{Q}^{\mathfrak{C}}$ corresponding to $p_k \in \mathfrak{C}$. Such a map defines a valuation [5, Proposition 6.10] having at most one-dimensional leaves [5, Theorem 6.16].

2.3. **A higher rank quasi-valuation.** For a fixed maximal chain $\mathfrak{C} \in \mathcal{C}$, the image of the valuations $\mathcal{V}_{\mathfrak{C}}$ is not necessarily finitely generated. To overcome this problem we introduce a quasi-valuation by minimizing this family of valuations. We refer to [5, Section 3.1] for the definition and basic properties of quasi-valuations.

Definition 2.2. A linearization $>^t$ of the partial order on A is called *length preseving*, if for any $p, q \in A$ with $\ell(p) > \ell(q)$, $p >^t q$ holds.

We fix a length preserving linearization $>^t$ of A and enumerate elements in A as

$$q_M >^t q_{M-1} >^t \ldots >^t q_1 >^t q_0$$

to identify $\mathbb{Q}^A$ with $\mathbb{Q}^{M+1}$ by sending

$$\underline{a} = a_M e_{q_M} + a_{M-1} e_{q_{M-1}} + \ldots + a_0 e_{q_0} \in \mathbb{Q}^A$$

to $(a_M, a_{M-1}, \dots, a_1, a_0)$. We will consider the lexicographic ordering on $\mathbb{Q}^{M+1}$ defined by: for $\underline{a}, \underline{b} \in \mathbb{Q}^{M+1}$, $\underline{a} > \underline{b}$ if the first non-zero coordinate of $\underline{a} - \underline{b}$ is positive. We will write $\underline{a} \geq \underline{b}$ if either $\underline{a} = \underline{b}$ or $\underline{a} > \underline{b}$. The vector space $\mathbb{Q}^A$ is then endowed with a total order which is clearly compatible with vector addition.

We define a map

$$\mathcal{V} : \mathbb{K}[\hat{X}] \setminus \{0\} \to \mathbb{Q}^A, \quad g \mapsto \min\{\mathcal{V}_{\mathfrak{C}}(g) \mid \mathfrak{C} \in \mathcal{C}\},$$

where $\mathbb{Q}^{\mathfrak{C}}$ is naturally embedded into $\mathbb{Q}^A$ and the minimum is taken with respect to the total order defined above. By [5, Lemma 3.4], $\mathcal{V}$ is a quasi-valuation.

Let $\Gamma := \{\mathcal{V}(g) \mid g \in \mathbb{K}[\hat{X}] \setminus \{0\}\} \subseteq \mathbb{Q}^A$ be the image of the quasi-valuation. For a fixed maximal chain $\mathfrak{C} \in \mathcal{C}$, we define a subset $\Gamma_{\mathfrak{C}} := \{\underline{a} \in \Gamma \mid \operatorname{supp} \underline{a} \subseteq \mathfrak{C}\}$ of Γ where for $\underline{a} = \sum_{p \in A} a_p e_p \in \mathbb{Q}^A$, $\operatorname{supp} \underline{a} := \{p \in A \mid a_p \neq 0\}$.

Theorem 2.3 ([5, Proposition 8.6, Corollary 9.1, Lemma 9.6])**.** *The following hold:*

1. *The quasi-valuation $\mathcal{V}$ takes values in $\mathbb{Q}^A_{\geq 0}$.*
2. *The set Γ is a finite union of finitely generated monoids $\Gamma_{\mathfrak{C}}$.*

The set Γ will be called a *fan of monoids*.

For a homogeneous element $g \in R \setminus \{0\}$, we can recover its degree from its quasi-valuation [5, Corollary 7.5, Proposition 8.7]: we denote $\underline{a} := \mathcal{V}(g)$ with $\underline{a} = (a_p)_{p \in A}$, then $\deg(g) = \sum_{p \in A} \deg(f_p) a_p$ ([5, Corollary 7.5]). This suggests to define the degree of $\underline{a} = \sum_{p \in A} a_p e_p \in \mathbb{Q}^A$ to be

$$\deg(\underline{a}) := \sum_{p \in A} \deg(f_p) a_p. \tag{1}$$

2.4. **Fan of monoids, semi-toric degenerations.** We define a fan algebra $\mathbb{K}[\Gamma]$ as the quotient of the polynomial ring $\mathbb{K}[x_{\underline{a}} \mid \underline{a} \in \Gamma]$ by an ideal $I(\Gamma)$ generated by the following elements: (1) $x_{\underline{a}} x_{\underline{b}} - x_{\underline{a}+\underline{b}}$ if there exists a chain $C \subseteq A$ containing both $\operatorname{supp} \underline{a}$ and $\operatorname{supp} \underline{b}$; (2) $x_{\underline{a}} x_{\underline{b}}$ if there is no such a chain.

The quasi-valuation $\mathcal{V}$ defines a filtration on $R := \mathbb{K}[\hat{X}]$ as follows: for $\underline{a} \in \Gamma$ we define

$$R_{\geq \underline{a}} := \{g \in R \setminus \{0\} \mid \mathcal{V}(g) \geq \underline{a}\} \cup \{0\}$$

and similarly $R_{>\underline{a}}$ by replacing the inequality $\geq$ with $>$. By Theorem 2.3, $R_{\geq \underline{a}}$ and $R_{>\underline{a}}$ are ideals. The successive quotients $R_{\geq \underline{a}}/R_{>\underline{a}}$ is one-dimensional [5, Lemma 10.2], and the associated graded algebra

$$\operatorname{gr}_{\mathcal{V}} R := \bigoplus_{\underline{a} \in \Gamma} R_{\geq \underline{a}}/R_{>\underline{a}}$$

is isomorphic to the algebra $\mathbb{K}[\Gamma]$ [5, Theorem 11.1].

Geometrically, it means that there exists a flat family $\pi : \mathcal{X} \to \mathbb{A}^1$ with the generic fibre isomorphic to X and the special fibre $\operatorname{Proj}(\operatorname{gr}_{\mathcal{V}} R)$ a (reduced) union of toric varieties [5, Theorem 12.2]. The projective variety $\operatorname{Proj}(\operatorname{gr}_{\mathcal{V}} R)$ is called a semi-toric variety, and we say X admits a semi-toric degeneration to it.

2.5. **Normal Seshadri stratifications.** So far we have associated to a Seshadri stratification on $X \subseteq \mathbb{P}(V)$ a fan of monoids Γ, which is a finite union of finitely generated monoids $\Gamma_{\mathfrak{C}}$.

Definition 2.4. A Seshadri stratification is called *normal* if for any maximal chain $\mathfrak{C} \in \mathcal{C}$, the monoid $\Gamma_{\mathfrak{C}}$ is saturated, that is to say, $\mathcal{L}^{\mathfrak{C}} \cap \mathbb{Q}^{\mathfrak{C}}_{\geq 0} = \Gamma_{\mathfrak{C}}$, where $\mathcal{L}^{\mathfrak{C}}$ is the group generated by $\Gamma_{\mathfrak{C}}$.

When a Seshadri stratification is normal, we can characterize a nice generating set of the fan algebra $\mathbb{K}[\Gamma]$.

A non-zero element $\underline{a} \in \Gamma_{\mathfrak{C}}$ is called *indecomposable* if there does not exist non-zero elements $\underline{a}_1, \underline{a}_2 \in \Gamma_{\mathfrak{C}}$ with $\min \operatorname{supp} \underline{a}_1 \geq \max \operatorname{supp} \underline{a}_2$ such that $\underline{a} = \underline{a}_1 + \underline{a}_2$.

Every element $\underline{a} \in \Gamma_{\mathfrak{C}}$ admits [5, Proposition 15.3] a decomposition into a sum

$$\underline{a} = \underline{a}_1 + \ldots + \underline{a}_s$$

of indecomposable elements in $\Gamma_{\mathfrak{C}}$ satisfying $\min \operatorname{supp} \underline{a}_i \geq \max \operatorname{supp} \underline{a}_{i+1}$ for $i = 1, 2, \ldots, s-1$. Such a decomposition is unique if $\Gamma_{\mathfrak{C}}$ is saturated.

Let $\mathbb{G}$ be the set of indecomposable elements in $\Gamma \subseteq \mathbb{Q}^A$. If the Seshadri stratification is normal, then any $\underline{a} \in \Gamma$ admits a unique decomposition as above into a sum of elements in $\mathbb{G}$. The set $\mathbb{G}$ is not necessarily finite. In this article we will concentrate on the case when $\mathbb{G}$ is finite.

Definition 2.5. A normal Seshadri stratification is called *of finite type* if $\mathbb{G}$ is a finite set.

If this is the case, we let

$$S := \mathbb{K}[y_{\underline{u}_1}, \ldots, y_{\underline{u}_m}]$$

denote the polynomial ring indexed by $\mathbb{G}$. We sometimes write $y_i := y_{\underline{u}_i}$ for short.

In certain applications it is needed that the monoid $\Gamma_{\mathfrak{C}}$ is not only saturated, but also of some special form. For this we recall the LS-lattice and the LS-monoid associated to a maximal chain.

For a maximal chain $\mathfrak{C} : p_r > p_{r-1} > \ldots > p_1 > p_0$ in A, we abbreviate $b_k := b_{p_k, p_{k-1}}$ to be the bond between p_k and p_{k-1}. The *LS-lattice* $\mathrm{LS}_{\mathfrak{C}}$ associated to $\mathfrak{C}$ is defined as follows

$$\mathrm{LS}_{\mathfrak{C}} := \left\{ \underline{u} = \begin{pmatrix} u_r \\ u_{r-1} \\ \vdots \\ u_0 \end{pmatrix} \in \mathbb{Q}^{\mathfrak{C}} \;\middle|\; \begin{array}{r} b_r u_r \in \mathbb{Z} \\ b_{r-1}(u_r + u_{r-1}) \in \mathbb{Z} \\ \ldots \\ b_1(u_r + u_{r-1} + \ldots + u_1) \in \mathbb{Z} \\ u_0 + u_1 + \ldots + u_r \in \mathbb{Z} \end{array} \right\}.$$

The *LS-monoid* is its intersection with the positive octant:

$$\mathrm{LS}^+_{\mathfrak{C}} := \mathrm{LS}_{\mathfrak{C}} \cap \mathbb{Q}^{\mathfrak{C}}_{\geq 0}.$$

Being an intersection of a lattice and an octant, the monoid $\mathrm{LS}^+_{\mathfrak{C}}$ is saturated.

Definition 2.6. A Seshadri stratification is called *of LS-type*, if for the extremal functions f_p, $p \in A$, are all of degree one, and for every maximal chain $\mathfrak{C} \in \mathcal{C}$, $\Gamma_{\mathfrak{C}} = \mathrm{LS}^+_{\mathfrak{C}}$.

Remark 2.7. A Seshadri stratification of LS-type is normal and of finite type (see Lemma 3.3).

For a fixed maximal chain $\mathfrak{C} : p_r > p_{r-1} > \ldots > p_0$ in $\mathcal{C}$ as above, a monomial basis of the algebra generated by the monoid $\mathrm{LS}_{\mathfrak{C}}^+$ can be described in the following way as in [4]. We set b_{r+1} and b_0 to be 1 and for $k = 0, 1, \ldots, r$, M_k to be the l.c.m of b_k and b_{k+1}. We consider the following map

$$\iota_{\mathfrak{C}} : \mathrm{LS}_{\mathfrak{C}}^+ \to \mathbb{K}[x_0, x_1, \ldots, x_r],$$

$$(u_r, u_{r-1}, \ldots, u_0) \mapsto x_0^{M_0 u_0} x_1^{M_1 u_1} \cdots x_r^{M_r u_r}.$$

We need to verify that for any $k = 0, 1, \ldots, r$, $M_k u_k \in \mathbb{N}$. Indeed, from $b_k(u_r + \ldots + u_k) \in \mathbb{N}$ it follows $M_k(u_r + \ldots + u_{k+1}) + M_k u_k \in \mathbb{N}$. Since b_{k+1} divides M_k, $M_k(u_r + \ldots + u_{k+1}) \in \mathbb{N}$ and hence $M_k u_k \in \mathbb{N}$.

It is then straightforward to show as in *loc.cit* that the map is injective and extends to an injective $\mathbb{K}$-algebra homomorphism $\iota_{\mathfrak{C}} : \mathbb{K}[\mathrm{LS}_{\mathfrak{C}}^+] \to \mathbb{K}[x_0, x_1, \ldots, x_r]$.

3. Gröbner bases and applications

3.1. **Lifting defining ideals.** We assume that the Seshadri stratification is normal and we keep the notation as in the previous sections. Let $\mathbb{G} = \{\underline{u}_i \mid i \in J\}$ be the set of indecomposable elements in $\Gamma \subseteq \mathbb{Q}^A$, indexed by the (possibly infinite) set J. For each $\underline{u}_i \in \mathbb{G}$ we fix a homogeneous element $g_{\underline{u}_i} \in R$ such that $\mathcal{V}(g_{\underline{u}_i}) = \underline{u}_i$. Again we use the abbreviation $g_i := g_{\underline{u}_i}$. According to [5, Proposition 15.6], $\{g_i \mid i \in J\}$ forms a generating set of the algebra R. Moreover, for $i \in J$ let $\overline{g}_i$ be the class of g_i in $\mathrm{gr}_{\mathcal{V}} R$. It is shown in *loc.cit* that $\{\overline{g}_i \mid i \in J\}$ generates $\mathrm{gr}_{\mathcal{V}} R$ as an algebra.

We consider the following commutative diagram of algebra homomorphisms:

$$\begin{array}{ccccc} S & \xrightarrow{\psi} & R & \qquad y_i & \longmapsto g_i \,. \\ & \searrow^{\varphi} & \downarrow & & \downarrow \\ & & \mathrm{gr}_{\mathcal{V}} R & & \overline{g}_i \end{array}$$

Let $I := \ker \psi$ and $I_{\mathcal{V}} := \ker \varphi$ be the defining ideals of R and $\mathrm{gr}_{\mathcal{V}} R$.

We recall the subduction algorithm from [5, Algorithm 15.15]. The input of the algorithm is a non-zero homogeneous element $f \in R$, and the output $\sum c_{\underline{a}_1,\ldots,\underline{a}_n} g_{\underline{a}_1} \cdots g_{\underline{a}_n}$ is a linear combination of standard monomials which coincides with f in R.

Algorithm:

(1). Compute $\underline{a} := \mathcal{V}(f)$.
(2). Decompose $\underline{a}$ into a sum of indecomposable elements $\underline{a} = \underline{a}_1 + \ldots + \underline{a}_s$ such that $\min \operatorname{supp} \underline{a}_i \geq \max \operatorname{supp} \underline{a}_{i+1}$.
(3). Compute $\overline{f}$ and $\overline{g}_{\underline{a}_1} \cdots \overline{g}_{\underline{a}_s}$ in $\mathrm{gr}_{\mathcal{V}} R$ to find $\lambda \in \mathbb{K}^*$ such that $\overline{f} = \lambda \overline{g}_{\underline{a}_1} \cdots \overline{g}_{\underline{a}_s}$.
(4). Print $\lambda g_{\underline{a}_1} \cdots g_{\underline{a}_s}$ and set $f_1 := f - \lambda g_{\underline{a}_1} \cdots g_{\underline{a}_s}$. When $f_1 \neq 0$ return to Step (1) with f replaced by f_1.
(5). Done.

We take $r \in I_{\mathcal{V}}$. To emphasize that it is a polynomial in y_i, we write it as $r(y_i)$. Let $g := r(g_i) \in R$ be its value at $y_i = g_i$ (i.e. its image under ψ). Applying the subduction algorithm to g returns the output $h \in R$, which is a linear combination of standard monomials in R. This allows us to write down the polynomial $h(y_i) \in S$ such that $h(g_i) = h$. We set

$$\widetilde{r}(y_i) := r(y_i) - h(y_i) \in S.$$

The element $\widetilde{r}(g_i) = g - h$ is contained in I. It has been shown in [5, Corollary 15.17] that the ideal I is generated by $\{\widetilde{r}(g_i) \mid r \in I_{\mathcal{V}}\}$.

3.2. **Lifting Gröbner bases.** In this paragraph we assume that the fixed normal Seshadri stratification is of finite type.

The ideal $I_{\mathcal{V}}$ is radical and generated by monomials and binomials. A Gröbner basis of such an ideal is not hard to describe. In this section we will lift a Gröbner basis of $I_{\mathcal{V}}$ to a Gröbner basis of the ideal I. Later in Section 5, we will work out as an example a Gröbner basis of the defining ideal of the complete flag varieties SL_3/B, embedded as a highest weight orbit.

We fix in this section a normal Seshadri stratification. Let $\mathbb{G} := \{\underline{u}_1, \ldots, \underline{u}_m\}$ be the set of indecomposable elements in Γ. Since the set $\mathbb{G}$, as a subset of Γ, is totally ordered by $>^t$, we assume without loss of generality that

$$\underline{u}_1 >^t \underline{u}_2 >^t \ldots >^t \underline{u}_m.$$

To be coherent with respect to the standard convention in Gröbner theory [10], we consider the following total order $\succ$ on monomials in $S := \mathbb{K}[y_{\underline{a}} \mid \underline{a} \in \mathbb{G}]$ defined by: for two monomials $y_{\underline{u}_1}^{k_1} \ldots y_{\underline{u}_m}^{k_m}$ and $y_{\underline{u}_1}^{\ell_1} \ldots y_{\underline{u}_m}^{\ell_m}$ with $k_1, \ldots, k_m, \ell_1, \ldots, \ell_m \geq 0$, we declare

$$y_{\underline{u}_1}^{k_1} \ldots y_{\underline{u}_m}^{k_m} \succ y_{\underline{u}_1}^{\ell_1} \ldots y_{\underline{u}_m}^{\ell_m}$$

if $\deg(y_{\underline{u}_1}^{k_1} \ldots y_{\underline{u}_m}^{k_m}) > \deg(y_{\underline{u}_1}^{\ell_1} \ldots y_{\underline{u}_m}^{\ell_m})$, or $\deg(y_{\underline{u}_1}^{k_1} \ldots y_{\underline{u}_m}^{k_m}) = \deg(y_{\underline{u}_1}^{\ell_1} \ldots y_{\underline{u}_m}^{\ell_m})$ and the first non-zero coordinate in the vector $(k_1 - \ell_1, \ldots, k_m - \ell_m)$ is negative. The total order $\succ$ is a monomial order.

Identifying the monomials in S with $\mathbb{N}^{\mathbb{G}}$, the above monomial order gives a monomial order on $\mathbb{N}^{\mathbb{G}}$. With this identification, the fan of monoids Γ can be embedded into both $\mathbb{Q}^A$ and $\mathbb{N}^{\mathbb{G}}$. Therefore Γ is endowed with two monomial orders $>^t$ and $\succ$.

Lemma 3.1. *For $\underline{a}, \underline{a}' \in \Gamma$ with $\deg \underline{a} = \deg \underline{a}'$, the following holds: if $\underline{a} >^t \underline{a}'$, then $\underline{a} \prec \underline{a}'$.*

Proof. Set $\operatorname{supp} \underline{a} = \{q_1, \ldots, q_s\}$ with $q_1 >^t \ldots >^t q_s$ and $\operatorname{supp} \underline{a}' = \{q_1', \ldots, q_{s'}'\}$ with $q_1' >^t \ldots >^t q_{s'}'$. This allows us to write

$$\underline{a} = \sum_{i=1}^{s} \lambda_i e_{q_i} \quad \text{and} \quad \underline{a}' = \sum_{i=1}^{s'} \lambda_i' e_{q_i'}$$

as elements in $\mathbb{Q}^A$ and

$$\underline{a} = \sum_{i=1}^{m} \mu_i e_{\underline{u}_i} \quad \text{and} \quad \underline{a}' = \sum_{i=1}^{m} \mu_i' e_{\underline{u}_i}$$

in $\mathbb{N}^{\mathbb{G}}$.

There are two cases to consider:

(1) $q_1 = q_1'$, $\lambda_1 = \lambda_1'$, ..., $q_{k-1} = q_{k-1}'$, $\lambda_{k-1} = \lambda_{k-1}'$ but $q_k > q_k'$. Let $1 \le t \le m$ (resp. $1 \le t' \le m$) be minimum such that $q_k \in \operatorname{supp} \underline{u}_t$ and $\mu_t \neq 0$ (resp. $q_k' \in \operatorname{supp} \underline{u}_{t'}$ and $\mu_{t'} \neq 0$). From $q_k >^t q_k'$ it follows $t \le t'$. When $t = t'$, q_k will appear in $\operatorname{supp} \underline{a}'$, which is not possible because this would imply that $q_k = q_k'$. Therefore $t < t'$ and hence $\underline{a} \prec \underline{a}'$.

(2) $q_1 = q_1'$, $\lambda_1 = \lambda_1'$, ..., $q_{k-1} = q_{k-1}'$, $\lambda_{k-1} = \lambda_{k-1}'$, $q_k = q_k'$ but $\lambda_k > \lambda_k'$. Let $1 \le t \le m$ be minimum such that $q_k \in \operatorname{supp} \underline{u}_t$ and $\mu_t \neq \mu_t'$. Such a minimum exists since $\lambda_1 = \lambda_1', \ldots, \lambda_{k-1} = \lambda_{k-1}'$ and $\lambda_k > \lambda_k'$.
    There are three possibilities:
    (a) $\operatorname{supp} \underline{u}_t = \{q_k\}$: By the proof of [5, Proposition 15.3], $\lambda_k > \lambda_k'$ and $\mu_t \neq \mu_t'$ imply $\mu_t > \mu_t'$, hence $\underline{a} \prec \underline{a}'$.
    (b) $q_k = \max \operatorname{supp} \underline{u}_t \neq \min \operatorname{supp} \underline{u}_t$: In this case, for any $t < s \le m$ with $\mu_s \neq 0$ or $\mu_s' \neq 0$, $q_k \notin \operatorname{supp} \underline{u}_s$, hence $\mu_t > \mu_t'$ and $\underline{a} \prec \underline{a}'$.
    (c) $q_k = \min \operatorname{supp} \underline{u}_t \neq \max \operatorname{supp} \underline{u}_t$: If $\mu_t > \mu_t'$ then we are done. Assume that $\mu_t < \mu_t'$. First notice that $\max \operatorname{supp} \underline{u}_t = q_{k-1}$, otherwise $\mu_t < \mu_t'$ would imply $\lambda_{k-1} < \lambda_{k-1}'$, contradicts to the assumption. Consider $\underline{b} := \sum_{i=1}^{t-1} \mu_i e_{\underline{u}_i}$ and $\underline{b}' := \sum_{i=1}^{t-1} \mu_i' e_{\underline{u}_i}$. From $\mu_t < \mu_t'$ it follows $\operatorname{supp} \underline{b}' \subseteq \operatorname{supp} \underline{b} = \{q_1, \ldots, q_{k-1}\}$. Writing $\underline{b} := \sum_{j=1}^{k-1} \delta_j e_{q_j}$ and $\underline{b}' := \sum_{j=1}^{k-1} \delta_j' e_{q_j}$, we have $\delta_{k-1} > \delta_{k-1}'$. Proceeding by induction on k, we will eventually fall into one of the above two cases (a) and (b), or even the case (1) when $\delta_{k-1}' = 0$. From the definition of $\succ$, $\underline{b} \prec \underline{b}'$ implies $\underline{a} \prec \underline{a}'$.

•

For a polynomial $f \in S$ (resp. an ideal $J \subseteq S$), let $\mathrm{in}_\succ(f)$ (resp. $\mathrm{in}_\succ(J)$) be the initial term of f (resp. initial ideal of J). Let $\mathcal{G}_{\mathrm{red}}(I_\mathcal{V}, \succ)$ denote the reduced Gröbner basis of $I_\mathcal{V}$ with respect to $\succ$.

Theorem 3.2. *The set $\{\widetilde{r} \mid r \in \mathcal{G}_{\mathrm{red}}(I_\mathcal{V}, \succ)\}$ forms a reduced Gröbner basis of I with respect to $\succ$.*

Proof. In the proof we will slightly abuse the notation: for $f \in S$, we will write $\mathcal{V}(f)$ for $\mathcal{V}(\psi(f))$, the quasi-valuation of the value of f at g_i.

We first show that the set $\{\widetilde{r} \mid r \in \mathcal{G}_{\mathrm{red}}(I_\mathcal{V}, \succ)\}$ forms a Gröbner basis. Let $\mathcal{G}_{\mathrm{red}}(I_\mathcal{V}, \succ) = \{r_1, \cdots, r_p\}$. According to [5, Theorem 11.1], $\mathrm{gr}_\mathcal{V} R$ is isomorphic to $\mathbb{K}[\Gamma]$ as $\mathbb{K}$-algebra, hence the ideal $I_\mathcal{V}$ is generated by homogeneous binomials and monomials. By Buchberger algorithm [10, Chapter 2, Section 7], for each $1 \le i \le p$, if r_i is not a monomial, then it has the form $\mathrm{in}_\succ(r_i) - s_i$, where $s_i \notin \mathrm{in}_\succ(I_\mathcal{V})$ is a monomial in S. In this case we have $\mathcal{V}(\mathrm{in}_\succ(r_i)) = \mathcal{V}(s_i)$, hence $\mathcal{V}(r_i) \ge^t \mathcal{V}(\mathrm{in}_\succ(r_i))$.

We claim that $\widetilde{r}_i = \mathrm{in}_\succ(r_i) + t_i$ where $1 \le i \le p$ and t_i is a linear combination of monomials which are strictly smaller than $\mathrm{in}_\succ(r_i)$ with respect to $\succ$. Indeed, the monomials appearing in t_i are either s_i, or, according to the subduction algorithm, those strictly larger than $\mathcal{V}(r_i)$ with respect to $>^t$, hence they are strictly larger than $\mathcal{V}(\mathrm{in}_\succ(r_i))$ with respect to $>^t$. Since the homogeneity is preserved in the subduction algorithm, by Lemma 3.1, all monomials appearing in t_i are strictly smaller than $\mathrm{in}_\succ(r_i)$ with respect to $\succ$.

Since $\{r_1,\ldots,r_p\}$ is a Gröbner basis of $I_{\mathcal{V}}$ with respect to $\succ$, we have:

$$\begin{aligned}\mathrm{in}_\succ(I_{\mathcal{V}}) &= (\mathrm{in}_\succ(r_1),\ldots,\mathrm{in}_\succ(r_p))\\ &= (\mathrm{in}_\succ(\widetilde{r}_1),\ldots,\mathrm{in}_\succ(\widetilde{r}_p))\\ &\subseteq \mathrm{in}_\succ((\widetilde{r}_1,\ldots,\widetilde{r}_p))\\ &\subseteq \mathrm{in}_\succ(I).\end{aligned}$$

As $\mathrm{gr}_{\mathcal{V}}R$ is the associated graded algebra of R, the above inclusion implies $\mathrm{in}_\succ(I_{\mathcal{V}}) = \mathrm{in}_\succ(I)$. This shows that $\{\widetilde{r}_1,\ldots,\widetilde{r}_p\}$ is a Gröbner basis of I with respect to the monomial order $\succ$.

For the reducedness, it suffices to notice that monomials appearing in t_i are not contained in the initial ideal $\mathrm{in}_\succ(I)$. •

3.3. **Koszul property.** We apply Theorem 3.2 to study the Koszul property of the homogeneous coordinate ring R. In the case of Schubert varieties, the Koszul property is sketched in [18, Remark 7.6] from a standard monomial theoretic point of view. For LS-algebras such a property is proved in [2, 4].

In this paragraph we fix a Seshadri stratification of LS-type on X. We keep the notation introduced in the in previous sections.

Recall that for an indecomposable element $\underline{u}_i \in \mathbb{G}$, we have fixed a homogeneous element $g_{\underline{u}_i} \in R$ with $\mathcal{V}(g_{\underline{u}_i}) = \underline{u}_i$.

Lemma 3.3. *Assume that the Seshadri stratification is of LS-type. The degree of any indecomposable element $\underline{u} \in \mathbb{G}$ is one, hence $\deg(g_{\underline{u}}) = 1$. In particular, the Seshadri stratification is of finite type.*

Proof. Let $\underline{u} \in \Gamma$ be an indecomposable element and let $\mathfrak{C} : p_r > p_{r-1} > \ldots > p_0$ be a maximal chain in A such that $\mathrm{supp}\,\underline{u} \subseteq \mathfrak{C}$. We will look at $\underline{u}$ as an element in $\mathbb{Q}^{\mathfrak{C}}$ and abbreviate its coordinate u_{p_k} to be u_k for $0 \le k \le r$. Assume that $\deg(\underline{u}) > 1$ (the degree is defined in (1)). There exists a maximal index j such that

$$u_r + u_{r-1} + \ldots + u_j \ge 1.$$

We consider $\underline{u}' \in \mathbb{Q}^A$ with $\mathrm{supp}\,\underline{u}' \subseteq \mathfrak{C}$ defined by:

$$u'_k := \begin{cases} u_k, & \text{if } k > j;\\ 1-(u_r+\ldots+u_{j+1}), & \text{if } k = j;\\ 0, & \text{if } k < j;\end{cases}$$

where we wrote $u'_k := u'_{p_k}$ for short.

We show that $\underline{u}' \in \Gamma_{\mathfrak{C}}$. By the assumption $\Gamma_{\mathfrak{C}} = \mathrm{LS}^+_{\mathfrak{C}}$, it suffices to show that for any $1 \le k \le r$, $b_k(u'_r + \ldots + u'_k) \in \mathbb{N}$. When $k > j$, it follows from the corresponding property of $\underline{u}$; when $k \le j$, it suffices to notice that $u'_r + \ldots + u'_k = 1$ and $b_k \in \mathbb{N}$.

The difference $\underline{u} - \underline{u}'$ lies in the lattice $\mathrm{LS}_{\mathfrak{C}}$, and by construction its coordinates are non-negative. Since the LS-monoid is saturated,

$$\underline{u} - \underline{u}' \in \mathrm{LS}_{\mathfrak{C}} \cap \mathbb{Q}^A_{\ge 0} = \mathrm{LS}^+_{\mathfrak{C}}.$$

By comparing the degree, $\underline{u} - \underline{u}' \ne 0$, contradicts to the assumption that $\underline{u}$ is indecomposable. The other statement $\deg(g_{\underline{u}}) = 1$ follows from [5, Corollary 7.5]. •

As an application of the lifting of Gröbner basis, we prove the following

Theorem 3.4. *The homogeneous coordinate ring* $R := \mathbb{K}[\hat{X}]$ *is a Koszul algebra.*

Proof. The algebra R is generated by $\{g_{\underline{u}} \mid \underline{u} \in \mathbb{G}\}$. We prove that R admits a quadratic Gröbner basis, hence by [1, Page 654], R is Koszul. According to Theorem 3.2 and the fact that the lifting preserves the degree, it suffices to show the following lemma:

Lemma 3.5. *The fan algebra* $\mathbb{K}[\Gamma]$ *is generated by degree* 2 *elements.*

Proof. We first define an ideal $J \subseteq \mathbb{K}[y_{\underline{u}_1}, \ldots, y_{\underline{u}_m}]$ generated by $J(\underline{u}_i, \underline{u}_j)$ for $\underline{u}_i, \underline{u}_j \in \mathbb{G}$ with $1 \leq i, j \leq m$. These elements $J(\underline{u}_i, \underline{u}_j)$ are defined as follows:

(1) If $\operatorname{supp} \underline{u}_i \cup \operatorname{supp} \underline{u}_j$ is not contained in a maximal chain in A, then
$$J(\underline{u}_i, \underline{u}_j) := y_{\underline{u}_i} y_{\underline{u}_j}.$$

(2) Otherwise $\underline{u}_i + \underline{u}_j \in \Gamma$ is well-defined. If $\min \operatorname{supp} \underline{u}_i \not\geq \max \operatorname{supp} \underline{u}_j$ and $\min \operatorname{supp} \underline{u}_j \not\geq \max \operatorname{supp} \underline{u}_i$, then by [5, Proposition 15.3], we can write
$$\underline{u}_i + \underline{u}_j = \underline{u}_{\ell_1} + \ldots + \underline{u}_{\ell_s}.$$
Comparing the degree using Lemma 3.3, we have $s = 2$. By assumption we have $\min \operatorname{supp} \underline{u}_{\ell_1} \geq \max \operatorname{supp} \underline{u}_{\ell_2}$, then define
$$J(\underline{u}_i, \underline{u}_j) := y_{\underline{u}_i} y_{\underline{u}_j} - y_{\underline{u}_{\ell_1}} y_{\underline{u}_{\ell_2}}.$$

(3) For the remaining cases we set $J(\underline{u}_i, \underline{u}_j) := 0$.

We single out a property which will be used later in the proof: in the case (2), if $\underline{u}_i >^t \underline{u}_j$ then from the proof of Lemma 3.3, $\underline{u}_{\ell_1} >^t \underline{u}_i$.

We consider an algebra homomorphism
$$\varphi : \mathbb{K}[y_{\underline{u}_1}, \ldots, y_{\underline{u}_m}] \to \mathbb{K}[\Gamma], \quad y_{\underline{u}_i} \mapsto x_{\underline{u}_i}.$$
Recall that for $\underline{a}_1, \ldots, \underline{a}_k \in \mathbb{G}$, the monomial $x_{\underline{a}_1} \cdots x_{\underline{a}_k}$ is called *standard* if for any $i = 1, \ldots, k-1$, $\min \operatorname{supp} \underline{a}_i \geq \max \operatorname{supp} \underline{a}_{i+1}$. This notion is similarly defined for monomials in $\mathbb{K}[y_{\underline{u}_1}, \ldots, y_{\underline{u}_m}]$. The standard monomials form a linear basis of $\mathbb{K}[\Gamma]$. This implies that the map φ is surjective.

From the definition of the defining ideal $I(\Gamma)$ of $\mathbb{K}[\Gamma]$, φ sends the ideal J to zero. The map φ induces a surjective algebra homomorphism $\overline{\varphi} : \mathbb{K}[y_{\underline{u}_1}, \ldots, y_{\underline{u}_m}]/J \to \mathbb{K}[\Gamma]$. We show that modulo the ideal J, we can write any non-zero monomial in $y_{\underline{u}_1}, \ldots, y_{\underline{u}_m}$ as a standard monomial, hence standard monomials generate $\mathbb{K}[y_{\underline{u}_1}, \ldots, y_{\underline{u}_m}]/J$, implying that $\overline{\varphi}$ is an isomorphism.

Indeed, we consider a non-zero monomial $y_{\underline{a}_1} \cdots y_{\underline{a}_s}$ where $\underline{a}_1, \ldots, \underline{a}_s \in \mathbb{G}$ and proceed by induction on s. We assume that this monomial is not standard because otherwise there is nothing to prove. When $s = 2$, we can use $J(\underline{a}_1, \underline{a}_2) \in J$ to write it as a standard monomial. For general $s > 2$, without loss of generality we can assume that
$$\underline{a}_1 \geq^t \underline{a}_2 \geq^t \ldots \geq^t \underline{a}_s$$
with respect to the total order $>^t$ on $\mathbb{Q}^A$, and their supports are contained in a maximal chain $\mathfrak{C}$ in A. There are two cases to consider:

(Case 1). If $y_{\underline{a}_1}y_{\underline{a}_2}$ is standard, then apply induction hypothesis to write $y_{\underline{a}_2}\cdots y_{\underline{a}_s}$ into a standard monomial $y_{\underline{b}_2}\cdots y_{\underline{b}_s}$ with $\underline{b}_2,\dots,\underline{b}_s\in\mathbb{G}$. Since $\underline{a}_2$ is the largest element among $\underline{a}_2,\dots,\underline{a}_s$ with respect to $>^t$, we have $\max\operatorname{supp}\underline{b}_2=\max\operatorname{supp}\underline{a}_2$, hence $\min\operatorname{supp}\underline{a}_1\geq\max\operatorname{supp}\underline{b}_2$ and the monomial $y_{\underline{a}_1}y_{\underline{b}_2}\cdots y_{\underline{b}_s}$ is standard.

(Case 2). If $y_{\underline{a}_1}y_{\underline{a}_2}$ is not standard, we use the $s=2$ case to write it into a standard monomial $y_{\underline{a}_{\ell_1}}y_{\underline{a}_{\ell_2}}$: we have furthermore $\underline{a}_{\ell_1}>^t\underline{a}_1$. If the monomial $y_{\underline{a}_{\ell_2}}y_{\underline{a}_3}\cdots y_{\underline{a}_s}$ is standard, then we are done. Otherwise we apply the induction hypothesis to write it into a standard monomial $y_{\underline{b}_2}\cdots y_{\underline{b}_s}$. Denote $\underline{b}_1:=\underline{a}_{\ell_1}$, we obtain a monomial $y_{\underline{b}_1}\cdots y_{\underline{b}_s}$ with $y_{\underline{b}_1}>^t y_{\underline{a}_1}$. If $y_{\underline{b}_1}y_{\underline{b}_2}$ is standard then we are done, otherwise repeat the above procedure. Such a process will eventually terminate because there are only finitely many elements in $\mathbb{G}$.

The lemma is proved. •

The proof of the theorem is then complete. •

Remark 3.6. One may also argue as in [2, 4]: By [5, Theorem 12.1], there exists a flat family over $\mathbb{A}^1$ with special fibre $\operatorname{Spec}(\operatorname{gr}_{\mathcal{V}}R)$ and generic fibre $\operatorname{Spec}(R)$. By [17, Theorem 1], if $\operatorname{gr}_{\mathcal{V}}R$ is Koszul, so is R. Then one uses [16] and Lemma 3.5.

Example 3.7. Let $X(\tau)\subseteq\mathbb{P}(V(\lambda))$ be a Schubert variety in a partial flag variety G/Q where G is a semi-simple simply connected algebraic group, Q is a parabolic subgroup in G and $V(\lambda)$ is the irreducible representation of G with a regular highest weight λ with respect to Q. We consider the Seshadri stratification on $X(\tau)$ defined in [7] consisting of all Schubert subvarieties in $X(\tau)$ and the extremal weight functions (see also Section 5). In *loc.cit.* we have proved that this Seshadri stratification is of LS-type. Theorem 3.4 implies that the homogeneous coordinate ring $\mathbb{K}[\hat{X}(\tau)]$ is a Koszul algebra (see also [18]).

3.4. **Relations to LS-algebras.** We briefly discuss in this paragraph relations between Seshadri stratifications and LS-algebras [3, 6]. In [6] we have proved that given an LS algebra with some additional assumptions, then one can construct a quasi-valuation on the algebra having as values exactly the LS-paths. Here we see how from a Seshadri stratification of LS-type one can recover a "partial" LS algebra structure. For the definition of an LS-algebra we refer to the version in [4, 6]. Note that the conditions (LS1), (LS2) and (LS3) in [6] are labeled (LSA1), (LSA2) and (LSA3) in [4].

Assume that the Seshadri stratification on $X\subseteq\mathbb{P}(V)$ is of LS-type. We examine which conditions for being an LS-algebra hold on the homogeneous coordinate ring $R:=\mathbb{K}[\hat{X}]$.

Elements in the fan of monoids

$$\Gamma=\bigcup_{\mathfrak{C}\in\mathcal{C}}\mathrm{LS}_{\mathfrak{C}}^{+}$$

are called LS-paths. By Lemma 3.3, the set of indecomposable elements $\mathbb{G}$ coincides with the set of all degree one elements in Γ. For each $\underline{u}\in\mathbb{G}$ we fix homogeneous element $g_{\underline{u}}\in R$ of degree one satisfying $\mathcal{V}(g_{\underline{u}})=\underline{u}$.

The condition (LS1) is fulfilled because by [5, Proposition 15.6], standard monomials form a linear basis of R.

The condition (LS3) also holds by [5, Theorem 11.1, Proposition 15.6]. Indeed, since the associated graded algebra $\mathrm{gr}_{\mathcal{V}}R$ is isomorphic to the fan algebra $\mathbb{K}[\Gamma]$, the homogeneous generators $g_{\underline{u}}$, $\underline{u} \in \mathbb{G}$, of R can be chosen to meet the coefficient 1 condition in (LS3).

The condition (LS2) is almost satisfied by R. First, the degree two straightening relations in (LS2) are guaranteed by Theorem 3.2 and Lemma 3.5. But in the definition of an LS-algebra it is required that the standard monomials appearing in a straightening relation should be larger than the non-standard monomial with respect to a stronger relation $\trianglelefteq$.

Summary: *If the Seshadri stratification on $X \subseteq \mathbb{P}(V)$ is of LS-type, then the homogeneous coordinate ring R of X admits a structure of a "partial" LS-algebra, where "partial" means that in the definition of $\trianglelefteq$ one replaces "for any total order refining the partial order" by "for the fixed total order $\geq^t$". Note that R is an LS-algebra when the Seshadri stratification is balanced with respect to all linearizations of the order on A; see [8] for details.*

4. Gorenstein property

Following [2, 4], we study the Gorenstein property of R from the viewpoint of Seshadri stratifications. As an application, we will show that the irreducible components of the semi-toric variety $\mathrm{Proj}(\mathrm{gr}_{\mathcal{V}}R)$ are not necessarily weighted projective spaces.

We assume that the collection X_p and f_p, $p \in A$ defines a Seshadri stratification on the embedded projective variety $X \subseteq \mathbb{P}(V)$, and denote by $R := \mathbb{K}[\hat{X}]$ the homogeneous coordinate ring.

4.1. Gorenstein property.

We start from the following

Proposition 4.1. *If the fan algebra $\mathbb{K}[\Gamma]$ is Gorenstein, then R is Gorenstein.*

Proof. By [5, Theorem 11.1], $\mathbb{K}[\Gamma]$ is isomorphic to $\mathrm{gr}_{\mathcal{V}}R$ as an algebra. The latter is the special fibre in a flat family [5, Theorem 12.1], the proposition follows from the fact that being Gorenstein is an open property. •

Remark 4.2. For LS-algebras, under certain assumptions, the above proposition is proved in [2, 4].

When the poset A is linearly ordered, the Gorenstein property of R can be determined effectively.

Let the poset $A = \{p_0, \ldots, p_r\}$ in the Seshadri stratification be linearly ordered with $p_r > p_{r-1} > \ldots > p_0$. The bond between p_k and p_{k-1} will be denoted by b_k. Let M_k be the l.c.m of b_k and b_{k+1} where b_0 and b_{r+1} are set to be 1. Assume furthermore that the Seshadri stratification is of LS-type (Definition 2.6).

Theorem 4.3 ([4, Theorem 7.3])**.** *Under the above assumptions, the algebra R is Gorenstein if and only if for any $k = 0, 1, \ldots, r$,*

$$b_k \left(\frac{1}{M_r} + \frac{1}{M_{r-1}} \ldots + \frac{1}{M_k} \right) \in \mathbb{N}.$$

The proof of the theorem realizes R as an invariant algebra of a finite abelian group acting on a polynomial ring. Such a group can be chosen to contain no pseudo-reflections, then the Gorenstein criterion in [22] can be applied. In the proof, to show that R is indeed the invariant algebra, one makes use of the homomorphism $\iota_{\mathfrak{C}}$ after Definition 2.6: this is the reason why the Seshadri stratification is assumed to be of LS-type.

4.2. **Weighted projective spaces.** If all bonds appearing in the extended graph $\mathcal{G}_{\hat{A}}$ are 1, such a Seshadri stratification is called of Hodge type [5, Section 16.1]. In this case the irreducible components appearing in the semi-toric variety are all projective spaces. It is natural to ask whether in general the irreducible components are weighted projective spaces. In this section we give a Seshadri stratification of LS-type on a toric variety, which is not a weighted projective space, such that the semi-toric variety associated to the stratification is the toric variety itself.

We consider the graded $\mathbb{C}$-algebra

$$R := \mathbb{C}[x_1, x_2, \ldots, x_6]/(x_2^2 - x_1x_3, x_5^2 - x_4x_6).$$

Let $X := \mathrm{Proj}(R) \subseteq \mathbb{P}^5$ be the associated projective variety where the embedding comes from the canonical surjection $\mathbb{C}[x_1, x_2, \ldots, x_6] \to R$.

We consider the following subvarieties in X: $X_{p_3} := X$,

$$X_{p_2} := X_{p_3} \cap \{[0:0:a:b:c:d] \in \mathbb{P}^5 \mid a, b, c, d \in \mathbb{C}\},$$

$$X_{p_1} := X_{p_2} \cap \{[0:0:0:b:c:d] \in \mathbb{P}^5 \mid b, c, d \in \mathbb{C}\},$$

$$X_{p_0} := X_{p_1} \cap \{[0:0:0:0:0:d] \in \mathbb{P}^5 \mid d \in \mathbb{C}\};$$

they are projective subvarieties by taking the reduced structure. Let

$$f_{p_3} := x_1, \quad f_{p_2} := x_3, \quad f_{p_1} := x_4, \quad f_{p_0} := x_6.$$

We leave it to the reader to verify that these data define indeed a Seshadri stratification on X with the following colored Hasse graph

$$p_3 \xleftarrow{2} p_2 \xleftarrow{1} p_1 \xleftarrow{2} p_0.$$

The index set A is a linear poset.

This Seshadri stratification is of LS-type. Indeed, we need to show that

$$\Gamma = \left\{ \underline{u} = \begin{pmatrix} u_3 \\ u_2 \\ u_1 \\ u_0 \end{pmatrix} \in \mathbb{Q}^4 \;\middle|\; \begin{array}{r} 2u_3 \in \mathbb{N} \\ u_3 + u_2 \in \mathbb{N} \\ 2(u_3 + u_2 + u_1) \in \mathbb{N} \\ u_3 + u_2 + u_1 + u_0 \in \mathbb{N} \end{array} \right\}.$$

Since there exists only one maximal chain, the quasi-valuation $\mathcal{V}$ is in fact a valuation. It is straightforward to verify that for a monomial $x_1^{a_1} \cdots x_6^{a_6}$,

$$\mathcal{V}(x_1^{a_1} \cdots x_6^{a_6}) = \begin{pmatrix} a_1 \\ a_3 \\ a_4 \\ a_6 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} a_2 \\ a_2 \\ a_5 \\ a_5 \end{pmatrix}.$$

The monomials

$$\{x_1^{a_1}\cdots x_6^{a_6} \mid a_1, a_5 \in \{0,1\}, a_2, a_3, a_4, a_6 \in \mathbb{N}\}$$

generate the ring R, and they have different valuations. As a consequence, Γ is contained in the LS-monoid LS_A^+. To show the other inclusion, for $\underline{u} := (u_3, u_2, u_1, u_0) \in \mathrm{LS}_A^+$, the monomial with exponent

$$([u_3], 2(u_3 - [u_3]), u_2 - (u_3 - [u_3]), [u_1], 2(u_1 - [u_1]), u_0 - (u_1 - [u_1]))$$

has $\underline{u}$ as valuation, where $[u]$ is the integral part of u.

The associated graded algebra $\mathrm{gr}_{\mathcal{V}} R$ is isomorphic to R, and the flat family over $\mathbb{A}^1$ is trivial. So X itself appears as the irreducible component in the degenerate variety.

Proposition 4.4. *The projective variety X is not isomorphic to a weighted projective space.*

Proof. Since $\dim X = 3$, we consider the weighted projective spaces $\mathbb{P}(\mathbf{a})$ with $\mathbf{a} = (a_0, a_1, a_2, a_3)$ where $a_0 \leq a_1 \leq a_2 \leq a_3$. Without loss of generality, we assume that the weights $\mathbf{a}$ are normalized, that is to say,

$$\text{g.c.d}(a_1, a_2, a_3) = \text{g.c.d}(a_0, a_2, a_3) = \text{g.c.d}(a_0, a_1, a_3) = \text{g.c.d}(a_0, a_1, a_2) = 1.$$

By Theorem 4.3, the algebra R is Gorenstein. It suffices to consider those weighted projective spaces which are Gorenstein. For weighted projective spaces with normalized weights, being Gorenstein and being Fano are equivalent, hence by [11, Example 8.3.3, Exercise 8.3.2], $\mathbb{P}(\mathbf{a})$ is Gorenstein if and only if

$$a_i \mid a_0 + a_1 + a_2 + a_3 \quad \text{for} \quad i = 0, 1, 2, 3.$$

It is not hard to see that there are only 14 of them (see also [13, Table 1]) with

$$\mathbf{a} = (1,1,1,1),\ (1,1,1,3),\ (1,1,2,2), (1,1,2,4),\ (1,1,4,6),\ (1,2,2,5),\ (1,2,3,6),$$
$$(1,2,6,9),\ (1,3,4,4),\ (1,3,8,12),\ (1,4,5,10),\ (1,6,14,21),\ (2,3,3,4),\ (2,3,10,15).$$

We compare the singular locus of X to $\mathbb{P}(\mathbf{a})$ with the weights $\mathbf{a}$ from the above list. The singular locus of X is a disjoint union of two $\mathbb{P}^1$. To determine the singular locus of $\mathbb{P}(\mathbf{a})$, we use the criterion from [14, Section 1]. For a prime number p, denote

$$\mathbb{P}_p(\mathbf{a}) := \{\underline{x} \in \mathbb{P}(\mathbf{a}) \mid p \mid a_i \text{ for those } i \text{ with } x_i \neq 0\}.$$

Then the singular locus of $\mathbb{P}(\mathbf{a})$ is given by the union of all $\mathbb{P}_p(\mathbf{a})$.

From this description, it is clear that only $\mathbb{P}(2,3,3,4)$ has as singular locus a disjoint union of two copies of $\mathbb{P}^1$.

It remains to show that X is not isomorphic to $\mathbb{P}(2,3,3,4)$. The variety X (resp. $\mathbb{P}(2,3,3,4)$) is a toric variety with torus $T = (\mathbb{C}^*)^4$ (resp. $T' = (\mathbb{C}^*)^4$). Since $\mathbb{P}(2,3,3,4)$ is a complete simplicial toric variety, if they were isomorphic as abstract varieties, such an isomorphism would be in the automorphism group of $\mathbb{P}(2,3,3,4)$. By [9, Corollary 4.7], such an automorphism group is a linear algebraic group with maximal torus T'. Since the maximal tori are conjugate, the isomorphism between X and $\mathbb{P}(2,3,3,4)$ could be chosen to be toric. The toric variety X has 4 two-dimensional torus orbit closures which are all isomorphic. However, $\mathbb{P}(2,3,3,4)$ has $\mathbb{P}(2,3,4) \cong \mathbb{P}(1,2,3)$ and $\mathbb{P}(2,3,3) \cong \mathbb{P}(1,1,2)$ as two-dimensional torus orbit closures; they are not isomorphic by looking at the singularities. This contradiction completes the proof. •

Remark 4.5. If X admits a Seshadri stratification of LS-type, the proof of [4, Theorem 6.1] can be applied *verbatim* to show that the irreducible components appearing in the semi-toric variety (see Section 2.4) are quotients of a projective space by a finite abelian subgroup in a general linear group. Moreover, such a subgroup can be chosen to contain no pseudo-reflections.

5. Example

In this last section, we illustrate the lifting procedure in Theorem 3.2 in an example related to flag varieties. To avoid technical assumptions we fix in this section $\mathbb{C}$ as the base field.

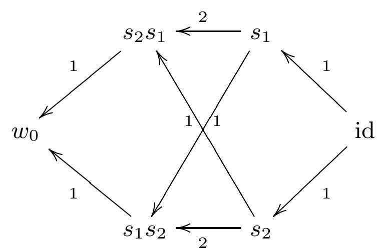
5.1. **Setup.** Let G be a simple simply connected algebraic group, $B \subseteq G$ be a fixed Borel subgroup and T be the maximal torus contained in B. Let $\alpha_1, \ldots, \alpha_n$ be the simple roots with respect to the above choice, Δ_+ be the set of positive roots and $\varpi_1, \ldots, \varpi_n$ be the fundamental weights generating the weight lattice Λ. For a positive root $\beta \in \Delta_+$, $U_{\pm\beta}$ are the root subgroups in G associated to $\pm\beta$. Let $W := N_G(T)/T$ be the Weyl group of G. For $\tau \in W$, we denote $\Delta_\tau := \{\gamma \in \Delta_+ \mid \tau^{-1}(\gamma) \notin \Delta_+\}$ and $U_\tau := \prod_{\gamma\in\Delta_\tau} U_\gamma$ for any chosen ordering of elements in Δ_+.

Let $\mathfrak{g}$ be the Lie algebra of G with the Cartan decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ such that $\mathfrak{n}_+ \oplus \mathfrak{h}$ is the Lie algebra of B. For a positive root $\beta \in \Delta_+$ we fix root vectors $X_{\pm\beta} \in \mathfrak{n}_\pm$ of weights $\pm\beta$. When $\beta = \alpha_i$ is a simple root, we abbreviate $X_{\pm i} := X_{\pm\alpha_i}$ for $1 \leq i \leq n$. For $k \in \mathbb{N}$, the k-th divided power of $X_{\pm i}$ is denoted by $X_{\pm i}^{(k)}$. For a dominant integral weight $\lambda \in \Lambda$, we denote $V(\lambda)$ the finite dimensional irreducible representation of $\mathfrak{g}$: it is a highest weight representation and we choose a highest weight vector $v_\lambda \in V(\lambda)$. We associate to a fixed element $\tau \in W$ with reduced decomposition $\underline{\tau} = s_{i_1} \cdots s_{i_\ell}$ an extremal weight vector $v_\tau := X_{-i_1}^{(m_1)} \cdots X_{-i_\ell}^{(m_\ell)} v_\lambda \in V(\lambda)$ with m_k the maximal natural number such that $X_{-i_k}^{(m_k)} \cdots X_{-i_\ell}^{(m_\ell)} v_\lambda \neq 0$. By Verma relations, v_τ is independent of the choice of the reduced decomposition of τ. The dual vector of v_τ is denoted by $p_\tau \in V(\lambda)^*$.

In the following example we will take $G = \mathrm{SL}_3$. Let $X := \mathrm{SL}_3/B$ be the complete flag variety embedded into $\mathbb{P}(V(\rho))$, where $\rho = \varpi_1 + \varpi_2$, as the highest weight orbit $\mathrm{SL}_3 \cdot [v_\rho]$ through the chosen highest weight line $[v_\rho] \in \mathbb{P}(V(\rho))$. The homogeneous coordinate ring will be denoted by $R := \mathbb{C}[\hat{X}]$, where the degree k component is $V(k\rho)^*$. In this case the Weyl group W is the symmetric group $\mathfrak{S}_3$. The longest element in W will be denoted by w_0.

We consider the Seshadri stratification on SL_3/B as in [7] given by the Schubert varieties $X(\tau)$ and the extremal weight functions p_τ for $\tau \in W$.

The Hasse diagram with bonds associated to this Seshadri stratification is depicted as follows:



We choose $N := 2$ to be the l.c.m of all bonds appearing in the above diagram.

By [7, Theorem 7.1, Theorem 7.3], for any maximal chain $\mathfrak{C} \in \mathcal{C}$, the monoid $\Gamma_{\mathfrak{C}}$ coincides with the LS-monoid $\mathrm{LS}_{\mathfrak{C}}^{+}$, the Seshadri stratification is therefore normal. Moreover, the fan of monoid Γ is independent of the choice of the linearlization of the partial order on A. Without loss of generality, we choose the following length preserving linear extension of the Bruhat order on W:

$$w_0 >^t s_1s_2 >^t s_2s_1 >^t s_1 >^t s_2 >^t \mathrm{id}.$$

With this total order one identifies $\mathbb{Q}^W$ with $\mathbb{Q}^6$.

The indecomposable elements in $\mathbb{G}$ are $e_1, \ldots, e_6$ in $\mathbb{Q}^6$ and the following two extra elements:

$$\pi_1 := \left(0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0\right), \quad \pi_2 := \left(0, \frac{1}{2}, 0, 0, \frac{1}{2}, 0\right).$$

For each element $\underline{a} \in \mathbb{G}$, in [21] and [7, Lemma 13.3] with $\ell = 2$ we have introduced the path vector associated to $\underline{a}$ and ℓ, denoted by $p_{\underline{a},\ell}$. More precisely, for $\tau \in W$, the path vector associated to the coordinate function $e_\tau \in \mathbb{Q}^W$ is the extremal functions p_τ. For $\pi_1, \pi_2 \in \mathbb{G}$, we denote the associated path vector by p_{π_1} and p_{π_2}. By [7, Theorem 7.1], for $\tau \in W$, $\mathcal{V}(p_\tau) = e_\tau$; $\mathcal{V}(p_{\pi_1}) = \pi_1$ and $\mathcal{V}(p_{\pi_2}) = \pi_2$.

On the polynomial ring

$$S := [y_{w_0}, y_{s_1s_2}, y_{\pi_2}, y_{s_2s_1}, y_{\pi_1}, y_{s_1}, y_{s_2}, y_{\mathrm{id}}].$$

we consider the following monomial order $\succ$. The generators of S are enumerated with respect to $>^t$:

$$y_{w_0} >^t y_{s_1s_2} >^t y_{\pi_2} >^t y_{s_2s_1} >^t y_{\pi_1} >^t y_{s_1} >^t y_{s_2} >^t y_{\mathrm{id}},$$

then the monomial order $\succ$ is the one defined in Section 3.2

The associated graded algebra $\mathrm{gr}_{\mathcal{V}}R$ is generated by

$$\overline{p}_{\mathrm{id}}, \quad \overline{p}_{s_1}, \quad \overline{p}_{s_2}, \quad \overline{p}_{s_1s_2}, \quad \overline{p}_{s_2s_1}, \quad \overline{p}_{w_0}, \quad \overline{p}_{\pi_1}, \quad \overline{p}_{\pi_2}$$

subject to the following relations:

$$\begin{aligned} &\overline{p}_{s_2s_1}\overline{p}_{s_1s_2} = 0, \quad \overline{p}_{s_2}\overline{p}_{s_1} = 0, \quad \overline{p}_{\pi_1}\overline{p}_{s_1s_2} = 0, \quad \overline{p}_{\pi_1}\overline{p}_{s_2} = 0, \quad \overline{p}_{\pi_2}\overline{p}_{s_2s_1} = 0, \\ &\overline{p}_{\pi_2}\overline{p}_{s_1} = 0, \quad \overline{p}_{\pi_1}^2 - \overline{p}_{s_2s_1}\overline{p}_{s_1} = 0, \quad \overline{p}_{\pi_2}^2 - \overline{p}_{s_1s_2}\overline{p}_{s_2} = 0, \quad \overline{p}_{\pi_2}\overline{p}_{\pi_1} = 0. \end{aligned} \tag{2}$$

They form a reduced Gröbner basis of the defining ideal of $\mathrm{gr}_{\mathcal{V}}R$ in S with respect to the monomial order $\succ$.

5.2. **Birational charts.** There are four maximal chains in W:

$$\mathfrak{C}_1 : w_0 > s_2s_1 > s_1 > \mathrm{id}, \quad \mathfrak{C}_2 : w_0 > s_1s_2 > s_2 > \mathrm{id},$$

$$\mathfrak{C}_3 : w_0 > s_1s_2 > s_1 > \mathrm{id}, \quad \mathfrak{C}_4 : w_0 > s_2s_1 > s_2 > \mathrm{id}.$$

For $i = 1, 2, 3, 4$, we let $\mathcal{V}_{\mathfrak{C}_i}$ denote the valuation associated to the maximal chain $\mathfrak{C}_i$ in Section 2.2. We will introduce birational charts of SL_3/B and its Schubert varieties to calculate these valuations.

We will work out $\mathcal{V}_{\mathfrak{C}_1}(p_{\pi_2})$ and the method can be applied in a straightforward way to determine other valuations. We will freely use the notations in [7, Section 12, 13].

First consider the following birational chart of SL_3/B introduced in [7, Lemma 3.2]: we write $\beta = \alpha_1 + \alpha_2$,

$$U_\beta \times U_{\alpha_2} \times U_{-\alpha_1} \to \mathrm{SL}_3/B \to \mathbb{P}(V(\rho)) \tag{3}$$

$$(\exp(t_1X_\beta), \exp(t_2X_{\alpha_2}), \exp(yX_{-\alpha_1})) \mapsto \exp(t_1X_\beta)\exp(t_2X_{\alpha_2})\exp(yX_{-\alpha_1}) \cdot [v_{s_2s_1}].$$

The vanishing order of the path vector $g := p_{\pi_2} \in V(\rho)^*$ along the Schubert variety $X(s_2s_1)$ is the lowest degree of y in the polynomial

$$p_{\pi_2}\left(\exp(t_1X_\beta)\exp(t_2X_{\alpha_2})\exp(yX_{-\alpha_1}) \cdot v_{s_2s_1}\right) \in \mathbb{C}[t_1, t_2][y]. \tag{4}$$

To compute this polynomial we work in the tensor product of Weyl module $M(\rho)\otimes M(\rho)$ as in [7, Section 12.4, Lemma 13.3], where the embedding of $V(\rho)$ into $M(\rho) \otimes M(\rho)$ is uniquely determined by $v_\rho \mapsto m_\rho \otimes m_\rho$. The path vector p_{π_2} is defined as the restriction of $x_{s_1s_2} \otimes x_{s_2} \in M(\rho)^* \otimes M(\rho)^*$ to $V(\rho)^*$ (notation in [7, Section 13]). Direct computation shows that the polynomial in (4) equals to $-t_1y$, hence the vanishing order of p_{π_2} along $X(s_2s_1)$ is 1.

The rational function $\frac{p_{\pi_2}^2}{p_{w_0}^2}$ coincides with $\frac{p_{s_2}^2}{p_{s_2s_1}^2}$ on the birational chart (3) because both of them evaluate to the polynomial t_1^2 on the above chart. The function $g_2 := \frac{p_{s_2}^2}{p_{s_2s_1}^2}$ is a rational function on $X(s_2s_1)$.

In order to determine the vanishing order of g_2 along the Schubert variety $X(s_1)$, we make use of the following birational chart

$$U_{\alpha_1} \times U_{-\alpha_2} \to X(s_2s_1) \to \mathbb{P}(V(\rho)_{s_2s_1}),$$

$$(\exp(t_1X_{\alpha_1}), \exp(yX_{-\alpha_2})) \mapsto \exp(t_1X_{\alpha_1})\exp(yX_{-\alpha_2}) \cdot [v_{s_1}]$$

where $V(\rho)_{s_2s_1}$ is the Demazure module associated to $s_2s_1 \in W$. From similar computation as above, $p_{s_2}^2$ (resp. $p_{s_2s_1}^2$) evaluates to the polynomial $t_1^2y^2$ (resp. y^4), hence g_2 has a pole of order 2 along $X(s_1)$.

Continue this computation, we obtain

$$g_{\mathfrak{C}_1} = \left(p_{\pi_2}, \frac{p_{s_2}^2}{p_{s_2s_1}^2}, \frac{p_{\mathrm{id}}^4}{p_{s_1}^2}, p_{\mathrm{id}}^8\right),$$

and the valuation is hence

$$\mathcal{V}_{\mathfrak{C}_1}(p_{\pi_2}) = \left(1, -\frac{1}{2}, -\frac{1}{2}, 1\right).$$

5.3. **Lift semi-toric relations.** As an example, we lift the relation $\overline{p}_{s_2s_1}\overline{p}_{s_1s_2} = 0$ to R. Other relations can be dealt with similarly.

In order to determine $\mathcal{V}(p_{s_2s_1}p_{s_1s_2})$, we need to work out the above four valuations on $p_{s_2s_1}$ and $p_{s_1s_2}$. By [5, Example 6.8], the valuation $\mathcal{V}_{\mathfrak{C}_1}(p_{s_2s_1}) = (0,1,0,0)$.

Similar computation as in the previous paragraph, one has:

$$\mathcal{V}_{\mathfrak{C}_1}(p_{s_1s_2}) = \left(1, -\frac{1}{2}, \frac{1}{2}, 0\right).$$

Summing them up we obtain:

$$\mathcal{V}_{\mathfrak{C}_1}(p_{s_2s_1}p_{s_1s_2}) = \left(1, \frac{1}{2}, \frac{1}{2}, 0\right).$$

In the same way we have:

$$\mathcal{V}_{\mathfrak{C}_2}(p_{s_2s_1}p_{s_1s_2}) = \left(1, \frac{1}{2}, \frac{1}{2}, 0\right), \quad \mathcal{V}_{\mathfrak{C}_3}(p_{s_2s_1}p_{s_1s_2}) = \left(1, 1, -\frac{1}{2}, \frac{1}{2}\right).$$

$$\mathcal{V}_{\mathfrak{C}_4}(p_{s_2s_1}p_{s_1s_2}) = (1,1,-1,1).$$

Taking the minimum with respect to the total order defined above, we obtain

$$\mathcal{V}(p_{s_2s_1}p_{s_1s_2}) = \left(1, 0, \frac{1}{2}, \frac{1}{2}, 0, 0\right).$$

It decomposes into indecomposable elements in $\mathbb{G}$ as follows:

$$\left(1, 0, \frac{1}{2}, \frac{1}{2}, 0, 0\right) = (1,0,0,0,0,0) + \left(0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0\right).$$

The standard monomial having this quasi-valuation is hence $p_{w_0}p_{\pi_1}$.

In the next step we consider the function $p_{s_2s_1}p_{s_1s_2} - p_{w_0}p_{\pi_1}$. The coefficient -1 is uniquely determined by the property

$$\mathcal{V}(p_{s_2s_1}p_{s_1s_2}) < \mathcal{V}(p_{s_2s_1}p_{s_1s_2} + \lambda p_{w_0}p_{\pi_1})$$

for $\lambda \in \mathbb{C}$, where both sides can be computed using the birational chart (3).

Along the maximal chains $\mathfrak{C}_2$ and $\mathfrak{C}_3$, the valuations of $p_{s_2s_1}p_{s_1s_2}$ and $p_{w_0}p_{\pi_1}$ are different. It follows:

$$\mathcal{V}_{\mathfrak{C}_2}(p_{s_2s_1}p_{s_1s_2} - p_{w_0}p_{\pi_1}) = \left(1, \frac{1}{2}, \frac{1}{2}, 0\right),$$

$$\mathcal{V}_{\mathfrak{C}_3}(p_{s_2s_1}p_{s_1s_2} - p_{w_0}p_{\pi_1}) = \left(1, 1, -\frac{1}{2}, \frac{1}{2}\right).$$

Along both maximal chains $\mathfrak{C}_1$ and $\mathfrak{C}_4$, both valuations $\mathcal{V}_{\mathfrak{C}_1}$ and $\mathcal{V}_{\mathfrak{C}_4}$ on $p_{s_2s_1}p_{s_1s_2} - p_{w_0}p_{\pi_1}$ have the first coordinate 2. Since this element is homogeneous of degree 2, from [5, Corollary 7.5], in both of the valuations there exist at least one negative coordinate. According to the non-negativity of the quasi-valuation [5, Proposition 8.6], neither of them can be the minimum.

As a summary, we have shown that

$$\mathcal{V}(p_{s_2s_1}p_{s_1s_2} - p_{w_0}p_{\pi_1}) = \left(1, \frac{1}{2}, 0, 0, \frac{1}{2}, 0\right).$$

Again decompose it into indecomposable elements

$$\left(1, \frac{1}{2}, 0, 0, \frac{1}{2}, 0\right) = (1, 0, 0, 0, 0, 0) + \left(0, \frac{1}{2}, 0, 0, \frac{1}{2}, 0\right),$$

we obtain the next standard monomial $p_{w_0} p_{\pi_2}$.

On the birational chart (3) we have used before, the function $p_{s_2 s_1} p_{s_1 s_2} - p_{w_0} p_{\pi_2} - p_{w_0} p_{\pi_1}$ is zero, giving out the lifted relation

$$p_{s_2 s_1} p_{s_1 s_2} - p_{w_0} p_{\pi_2} - p_{w_0} p_{\pi_1} = 0.$$

By lifting all relations in (2), the reduced Gröbner basis of the defining ideal of SL_3/B in $\mathbb{P}(V(\rho))$ with respect to $\succ$ is given by:

$$p_{s_1} p_{s_2} = p_{\mathrm{id}} p_{\pi_1} + p_{\mathrm{id}} p_{\pi_2}, \quad p_{s_1} p_{\pi_2} = p_{s_1 s_2} p_{\mathrm{id}},$$
$$p_{\pi_1}^2 = p_{s_2 s_1} p_{s_1} - p_{\mathrm{id}} p_{w_0}, \quad p_{\pi_1} p_{s_2} = p_{s_2 s_1} p_{\mathrm{id}},$$
$$p_{\pi_1} p_{\pi_2} = p_{w_0} p_{\mathrm{id}}, \quad p_{\pi_1} p_{s_1 s_2} = p_{w_0} p_{s_1}, \quad p_{s_2 s_1} p_{\pi_2} = p_{w_0} p_{s_2},$$
$$p_{s_2 s_1} p_{s_1 s_2} = p_{w_0} p_{\pi_1} + p_{w_0} p_{\pi_2}, \quad p_{\pi_2}^2 = p_{s_1 s_2} p_{s_2} - p_{\mathrm{id}} p_{w_0}.$$

These relations coincide with those given in [2], although the bases are defined in a different way.

Remark 5.1. The Seshadri stratification on $\mathrm{SL}_3/B \subseteq \mathbb{P}(V(\rho))$ consisting of Schubert varieties is normal and balanced (see [7, Theorem 7.3] for details on the balanced condition). This property can be used to determine a Gröbner basis of the defining ideal of a Schubert variety in SL_3/B.

References

[1] D. Anick, *On the Homology of Associative Algebras*, Transactions of the American Mathematical Society, Aug., 1986, Vol. 296, No. 2, 641–659.

[2] R. Chirivì, *LS algebras and Schubert varieties*, Ph.D thesis, Scuola Normale Superiore, 2000.

[3] R. Chirivì, *LS-algebras and application to Schubert varieties*, Transformation Groups, Vol. 5, No. 3, 2000, pp. 245–264.

[4] R. Chirivì, *On some properties of LS algebras*, Communications in Contemporary Mathematics, Vol. 22, No. 02, 1850085 (2020).

[5] R. Chirivì, X. Fang, P. Littelmann, *Seshadri stratifications and standard monomial theory*, arXiv:2112.03776, preprint 2021.

[6] R. Chirivì, X. Fang, P. Littelmann, *LS algebras, valuations and Schubert varieties*, to appear in Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.

[7] R. Chirivì, X. Fang, P. Littelmann, *Seshadri stratification for Schubert varieties and Standard Monomial Theory*, arXiv:2203.13569, preprint 2022.

[8] R. Chirivì, X. Fang, P. Littelmann, *Standard monomial theory for Schubert varieties: a new approach*, preprint 2022.

[9] D. Cox, *The homogeneous coordinate ring of a toric variety*, J. Alg. Geometry 4 (1995), 17–50.

[10] D. Cox, J. Little, D. O'Shea, *Ideals, varieties, and algorithms. An introduction to computational algebraic geometry and commutative algebra*. Fourth edition. Undergraduate Texts in Mathematics. Springer, Cham, 2015. xvi+646 pp.

[11] D. Cox, J. Little, H. Schenck, *Toric Varieties*. Graduate Studies in Mathematics. vol. 124. American Mathematical Society, Providence (2011).

[12] C. De Concini, D. Eisenbud and C. Procesi, *Hodge algebras*, Astérisque, 91, Société Mathématique de France, Paris 1982.

[13] T. Dedieu, E. Sernesi, *Deformations and extensions of Gorenstein weighted projective spaces*, to appear in a Springer volume on the occasion of Ciro Ciliberto's 70th birthday.
[14] A. Dimca, *Singularities and coverings of weighted complete intersections*, J. Reine Angew. Math. 366 (1986), 184–193.
[15] X. Fang, P. Littelmann, *From standard monomial theory to semi-toric degenerations via Newton-Okounkov bodies*, Trans. Moscow Math. Soc. 2017, 275–297.
[16] R. Fröberg, *Determination of a class of Poincaré series*, Math. Scand., 37(1), 29–39, 1975.
[17] G. Kempf, *Some wonderful rings in algebraic geometry*, J. Algebra, 134(1), 222–224, 1990.
[18] V. Lakshmibai, P. Littelmann, P. Magyar, *Standard Monomial Theory and applications*, "Representation Theories and Algebraic Geometry" (A. Broer, ed.), Kluwer Academic Publishers (1998), pp 319–364.
[19] P. Littelmann, *A Littlewood-Richardson formula for symmetrizabIe Kac-Moody algebras*, Invent. Math. 116 (1994), 329–346.
[20] P. Littelmann, *Paths and root operators in representation theory*, Ann. Math. 142 (1995), 299–525.
[21] P. Littelmann, *Contracting modules and standard monomial theory for symmetrizable Kac-Moody algebras*, J. Amer. Math. Soc. 11 (1998), no. 3, 551–567.
[22] R. Stanley, *Hilbert functions of graded algebras*, Adv. Math, Volume 28, Issue 1, April 1978, 57–83.

Dipartimento di Matematica e Fisica "Ennio De Giorgi", Università del Salento, Lecce, Italy
Email address: rocco.chirivi@unisalento.it

Department Mathematik/Informatik, Universität zu Köln, 50931, Cologne, Germany
Email address: xinfang.math@gmail.com

Department Mathematik/Informatik, Universität zu Köln, 50931, Cologne, Germany
Email address: peter.littelmann@math.uni-koeln.de