CAUSAL BUBBLES IN GLOBALLY HYPERBOLIC SPACETIMES

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ABSTRACT. We give an example of a spacetime with a continuous metric which is globally hyperbolic and exhibits causal bubbling. The metric moreover splits orthogonally into a timelike and a spacelike part. We discuss our example in the context of energy conditions and the recently introduced synthetic timelike curvature-dimension (TCD) condition. In particular we observe that the TCD-condition does not, by itself, prevent causal bubbling.

1. INTRODUCTION

Spacetimes where the Lorentzian metric is merely continuous often appear in mathematical General Relativity as weak solutions of Einstein's Equations [4, 10, 19]. Chruściel and Grant [7] were the first to study their causal structure, discovering a phenomenon called *causal bubbling* (see below). Later, Sämann [18] studied globally hyperbolic spacetimes with continuous metrics in detail and showed that, just as in the smooth case, global hyperbolicity can be equivalently characterized by any of the following:

(i) Non-total imprisonment and compact causal diamonds,
(ii) Existence of a Cauchy hypersurface,
(iii) Existence of a Cauchy time function.

In the same paper (see Section 7) Sämann raises the question whether globally hyperbolic spacetimes can be causally bubbling. In this short note we give an example of a **globally hyperbolic spacetime with continuous metric** $-\mathrm{d}t^2 + \rho\,\mathrm{d}x^2$ **exhibiting causal bubbling**.

We define causal bubbling on a spacetime (M, g) with continuous metric g as follows. Set

$$\mathcal{B}^\pm(p) = J^\pm(p) \setminus \overline{I^\pm(p)}$$

The set $\mathcal{B}^\pm(p)$ is called a (future/past) *causal bubble* if it is non-empty.[1] Here $I^\pm(p)$ and $J^\pm(p)$ are the timelike and causal future/past cones defined using *Lipschitz* (or equivalently absolutely continuous) curves. We refer to [11] for a discussion on timelike and causal cones defined via smooth curves and their relationship with the notion used here. See also [9, 14] and [13, Section 5.1] for further analyses on causality with continuous metrics.

Our example stands out from previously known examples of causally bubbling spacetimes for two reasons.

(a) It is manifestly globally hyperbolic,
(b) The metric splits orthogonally into a timelike and a spacelike part.

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[1]This definition corresponds to *external* bubbling in [11] and is equivalent to the failure of the *push up* property, see [11, Theorem 2.12].

The second statement (b) would follow automatically from (a) if the metric were at least $\mathcal{C}^2$ [2], but not if it is merely continuous. In particular, while examples 3.1 and 3.2 in [11] (which exhibit *internal bubbling*) likely are globally hyperbolic, it is not clear whether the metric splits orthogonally as in (b). Whether Example 1.11 in [7] is globally hyperbolic is less clear, but it is known to be strongly causal (see [13, Section 5.1]).

2. The example

Consider the $(1 + 1)$-dimensional spacetime $\mathbb{R}^2$ equipped with the continuous Lorentzian metric

$$(2.1) \qquad g := -\mathrm{d}t^2 + \rho(t, x)\mathrm{d}x^2, \quad \rho(t, x) := 1 + \sqrt{(t - |x|)_+}.$$

With the natural choice of time orientation, t is a time function. Since the lightcones of the metric (2.1) are narrower than those of the Minkowski metric, t is a Cauchy time function, and hence the spacetime is globally hyperbolic.

To see that the causal future of the origin contains a causal bubble, we begin by considering the ODE for the null curves $\gamma(s) = (\alpha(s), \beta(s))$:

$$0 = -\alpha'(s)^2 + \left(1 + \sqrt{(\alpha(s) - |\beta(s)|)_+}\right)\beta'(s)^2.$$

A null curve γ starting at the origin, parametrized as $(\alpha(s), s)$ thus satisfies $\alpha'(s)^2 = 1 + \sqrt{(\alpha(s) - s)_+}$ or, by denoting $y(s) = \alpha(s) - s \geq 0$,

$$(2.2) \qquad y'(s) + 1 = \sqrt{1 + \sqrt{y(s)}}, \quad y(0) = 0.$$

In addition to the trivial solution $y \equiv 0$, the initial value problem (2.2) also admits another solution, expressed in implicit form as

$$s = \frac{4}{3}\left[\left(1 + \sqrt{y(s)}\right)^{3/2} - 1\right] + 2\sqrt{y(s)}.$$
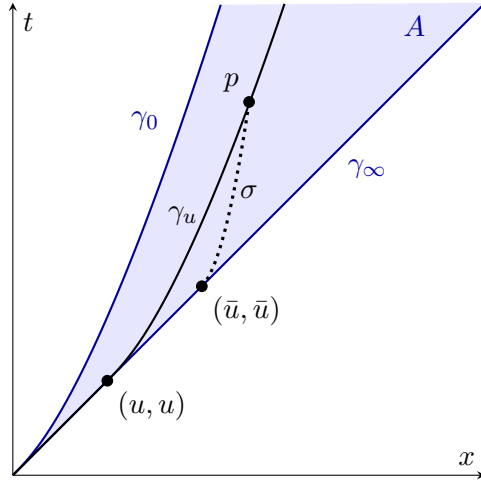
Indeed, differentiating both sides yields

$$1 = 2\frac{\mathrm{d}}{\mathrm{d}s}\sqrt{y(s)}(1 + \sqrt{y(s)})^{1/2} + 2\frac{\mathrm{d}}{\mathrm{d}s}\sqrt{y(s)}$$

$$= y'(s)\frac{\sqrt{1 + \sqrt{y(s)}} + 1}{\sqrt{y(s)}} = \frac{y'(s)}{\sqrt{1 + \sqrt{y(s)}} - 1}.$$

Denoting

$$f(y) := \frac{4}{3}\left[(1 + \sqrt{y})^{3/2} - 1\right] + 2\sqrt{y},$$

we conclude that $\gamma = (s + f^{-1}(s), s)$ is a null curve as well as the straight line given by $\gamma = (s, s)$. In fact there is a 1-parameter family of null curves starting at 0, given by

$$(2.3) \qquad \gamma_u(s) := \begin{cases} (s, s) & \text{for } 0 \leq s < u, \\ (s + f^{-1}(s - u), s) & \text{for } s \geq u, \end{cases}$$

FIGURE 1. The causal bubble A as in Proposition 2.1.

where $u \in [0, \infty]$. One can check that the curves γ_u are smooth for all parameter values except $s = u$, where they are only $\mathcal{C}^{1,1}$ regular. Note also that on smooth, two-dimensional spacetimes, every null curve is a null geodesic. Hence, for $u < \infty$, $\gamma_u|_{(u,\infty)}$ is a null geodesic, as it is contained in the region where the metric is smooth. At $s = u$, however, γ_u is not locally length maximizing.

Proposition 2.1. The non-empty open set $A := \{(t, x) : x > 0, \ 0 < t - x < f^{-1}(x)\} \subset \mathbb{R}^2$ consists of points in $J^+(0) \setminus \overline{I^+(0)}$.

Consequently $\mathbb{R}^2$ equipped with the globally hyperbolic metric (2.1) contains causal bubbles.

Proof. The inclusion $A \subset J^+(0)$ is clear. Let $p \in A$. We first show that $p \notin I^+(0)$. Since the set A is foliated by the null curves (2.3), there is a unique $u \in (0, \infty)$ such that γ_u passes through p. Because the metric is smooth outside of $t = \pm x$, γ_u is a null geodesic generating the boundary of $I^-(p)$, at least from (u, u) until p. This means that any past-directed timelike curve σ starting at p must intersect the diagonal at some point $(\bar{u}, \bar{u})$ with $\bar{u} \geq u > 0$ (see Figure 1). It follows that σ cannot reach 0; indeed, following the diagonal would introduce a null piece, while leaving the diagonal violates the causality of the curve (the metric (2.1) has narrower lightcones than those of the Minkowski metric). We conclude that $0 \notin I^-(p)$, (equivalently, $p \notin I^+(0)$) and hence $A \cap I^+(0) = \emptyset$. But since A is open, it cannot contain any boundary points of $I^+(0)$ either, hence also $A \cap \overline{I^+(0)} = \emptyset$, concluding the proof. $\square$

3. DISCUSSION

3.1. **Strong energy condition.** As in previously known examples [7, 9, 11, 14] causal bubbling arises from the branching of null geodesics. Chruściel and Grant noted [7, Rem. 1.19] that, in the Riemannian case, branching is associated with curvature being unbounded from below. Indeed, in

our example (in $\{t \neq |x|\}$) the Ricci scalar, given by

$$R = \begin{cases} -\frac{\rho + (\rho-1)^2}{4\rho^2 (\rho-1)^3} & \text{if } t > |x| \\ 0 & \text{if } t < |x|, \end{cases}$$

diverges to $-\infty$ as $t \searrow |x|$. Note however that in dimension $1 + 1$ the Ricci *tensor* is given by $\mathrm{Ric} = \frac{1}{2} R g$ and thus

(3.1) $$\mathrm{Ric}(v, v) \geq 0 \quad \text{for all causal vectors } v \text{ in } \{t \neq |x|\}.$$

In other words the strong energy condition is satisfied. Physically speaking causal bubbling appears to be a consequence of the presence of infinite (but positive) effective energy density (see [12] for an in-depth discussion of energy conditions).

3.2. **Synthetic curvature bounds.** Recently, a *synthetic* notion of *timelike curvature dimension (TCD) bounds* on (non-smooth) *Lorentzian pre-length spaces* has been put forth using optimal transport [5, 17], in analogy with the very successful metric theory [1, 15, 21, 22]. The *entropic convexity* condition defining $\mathrm{TCD}(K, N)$-spaces asks that the *Rényi entropy*

$$\mathrm{Ent}(f \, \mathrm{vol}) := \int f \log f \, \mathrm{d} \, \mathrm{vol}$$

is (K, N)-convex along timelike geodesics in a space of probability measures. We refer to [13, Definition 2.8] and [3, Definition 2.17] (see also [5, 6]) for the definitions and properties of Lorentzian pre-length spaces and (K, N)-convexity, respectively. On smooth spacetimes, such convexity properties characterize the strong energy condition, cf. [17].

Despite satisfying (3.1) the Rényi entropy associated to the volume measure of the metric (2.1) is not (K, N)-convex along Lorentz-Wasserstein geodesics, i.e. **the metric** (2.1) **does not satisfy the entropic convexity condition for any** (K, N).[2] This follows from the fact that $t \mapsto -\log \rho(t, x)$ is not (K, N)-convex for any $x \in \mathbb{R}$, cf. [20].

However, restricted to a suitable subset, our example demonstrates that the TCD-condition does not prevent causal bubbling. Indeed, the closed set $Y := \{t \geq |x|\} \subset \mathbb{R}^2$ with the restriction of the metric (2.1) satisfies the (weak) entropic $(0, 2)$-convexity condition but contains causal bubbles. Notice that (Y, g) is obtained as the uniform pointwise limit of the sequence of smooth metrics

(3.2) $$g_j := -\mathrm{d}t^2 + \rho_j(t, x)\mathrm{d}x^2, \quad \rho_j(t, x) = \rho(t + 1/j, x)$$

on Y, which all satisfy the strong energy condition (3.1) everywhere on Y and are thus $\mathrm{TCD}(0, 2)$-spaces.[3]

In closing we point out that, in the metric setting, branching of geodesics is excluded by the *Riemannian curvature-dimension* (RCD) condition but not by the (weak) CD-condition, see [8] and [16], respectively.

Remark 3.1. Note that all the conclusions in this note, including those about curvature, are valid in higher dimensions as seen by considering metrics $g = -\mathrm{d}t^2 + \rho(t, |\boldsymbol{x}|)\mathrm{d}\boldsymbol{x}^2$ in $\mathbb{R}^{n+1}$.

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[2]While Lorentzian pre-length spaces are required (by definition) to have the push up property the entropic convexity condition makes sense regardless of the validity of push up.

[3]Note that, while the spaces (Y, g_j) are Lorentzian *pre*-length spaces, every point on $\{t = |x|\} \subset Y$ has empty chronological past and thus they fail to be Lorentzian length spaces, cf. [13, Definition 3.22].

References

[1] L. Ambrosio, N. Gigli, and G. Savaré. Metric measure spaces with Riemannian Ricci curvature bounded from below. *Duke Math. J.*, 163(7):1405–1490, 2014.

[2] A. N. Bernal and M. Sánchez. Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes. *Comm. Math. Phys.*, 257(1):43–50, 2005.

[3] M. Braun. Good geodesics satisfying the timelike curvature-dimension condition. Preprint (arXiv:2205.06950), 2022.

[4] A. Y. Burtscher and P. G. LeFloch. The formation of trapped surfaces in spherically-symmetric Einstein–Euler spacetimes with bounded variation. *Journal de Mathématiques Pures et Appliquées*, 102(6):1164–1217, 2014.

[5] F. Cavalletti and A. Mondino. Optimal transport in lorentzian synthetic spaces, synthetic timelike ricci curvature lower bounds and applications. Preprint (arXiv:2004.08934), 2020.

[6] F. Cavalletti and A. Mondino. A review of lorentzian synthetic theory of timelike ricci curvature bounds. Preprint (arXiv:2204.13330), 2022.

[7] P. T. Chruściel and J. D. E. Grant. On Lorentzian causality with continuous metrics. *Classical Quantum Gravity*, 29(14):145001, 32, 2012.

[8] Q. Deng. *Holder Continuity of Tangent Cones and Non-Branching in RCD(K,N) Spaces*. ProQuest LLC, Ann Arbor, MI, 2021. Thesis (Ph.D.)–University of Toronto (Canada).

[9] L. García-Heveling. Causality theory of spacetimes with continuous Lorentzian metrics revisited. *Classical Quantum Gravity*, 38(14):145028, 13, 2021.

[10] R. P. Geroch and J. H. Traschen. Strings and other distributional sources in general relativity. *Phys. Rev. D*, 36:1017–1031, 1987.

[11] J. D. E. Grant, M. Kunzinger, C. Sämann, and R. Steinbauer. The future is not always open. *Lett. Math. Phys.*, 110(1):83–103, 2020.

[12] E.-A. Kontou and K. Sanders. Energy conditions in general relativity and quantum field theory. *Classical Quantum Gravity*, 37(19):193001, 63, 2020.

[13] M. Kunzinger and C. Sämann. Lorentzian length spaces. *Ann. Global Anal. Geom.*, 54(3):399–447, 2018.

[14] E. Ling. Aspects of C^0 causal theory. *Gen. Relativity Gravitation*, 52(6):Paper No. 57, 40, 2020.

[15] J. Lott and C. Villani. Ricci curvature for metric-measure spaces via optimal transport. *Ann. of Math. (2)*, 169(3):903–991, 2009.

[16] M. Magnabosco. Example of a highly branching CD space. *J. Geom. Anal.*, 32(6):Paper No. 173, 32, 2022.

[17] R. J. McCann. Displacement convexity of Boltzmann's entropy characterizes the strong energy condition from general relativity. *Camb. J. Math.*, 8(3):609–681, 2020.

[18] C. Sämann. Global hyperbolicity for spacetimes with continuous metrics. *Ann. Henri Poincaré*, 17(6):1429–1455, 2016.

[19] J. Sbierski. The C^0-inextendibility of the Schwarzschild spacetime and the spacelike diameter in Lorentzian geometry. *J. Diff. Geom.*, 108(2):319–378, 2018.

[20] E. Soultanis. Globally hyperbolic lorentzian length spaces with orthogonally split metric. In preparation, 2022.

[21] K.-T. Sturm. On the geometry of metric measure spaces. I. *Acta Math.*, 196(1):65–131, 2006.

[22] K.-T. Sturm. On the geometry of metric measure spaces. II. *Acta Math.*, 196(1):133–177, 2006.

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