Approximate Carathéodory bounds via Discrepancy Theory

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Abstract

The approximate Carathéodory problem in general form is as follows: Given two symmetric convex bodies $P, Q \subseteq \mathbb{R}^m$, a parameter $k \in \mathbb{N}$ and $z \in \text{conv}(X)$ with $X \subseteq P$, find $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in X$ so that $\|\boldsymbol{z} - \frac{1}{k} \sum_{i=1}^k \boldsymbol{v}_i\|_Q$ is minimized. Maurey showed that if both P and Q coincide with the $\|\cdot\|_p$ -ball, then an error of $O(\sqrt{p/k})$ is possible.

We prove a reduction to the vector balancing constant from discrepancy theory which for most cases can provide tight bounds for general *P* and *Q*. For the case where *P* and *Q* are both $\|\cdot\|_p$ -balls we prove an upper bound of $\sqrt{\frac{\min\{p,\log(\frac{2m}{k})\}}{k}}$. Interestingly, this bound cannot be obtained taking independent random samples; instead we use the Lovett-Meka random walk. We also prove an extension to the more general case where *P* and *Q* are $\|\cdot\|_p$ and $\|\cdot\|_q$ -balls with $2 \le p \le q \le \infty$.

1 Introduction

The (exact) Carathéodory Theorem is part of most introductory courses on the theory of linear programming: given any vector $z \in \text{conv}(X)$ where $X \subseteq \mathbb{R}^m$, there is a subset of points $X' \subseteq X$ with $|X'| \leq m + 1$ so that already $z \in \text{conv}(X')$. More recently, the *approximate* version gained interest, where only *k* vectors from *X* may be selected with uniform weights and the goal is to minimize the error in a given norm.

Barman [Bar15] used an approximate Carathéodory bound for algorithms to compute approximate Nash equilibria for bimatrix games as well as for finding *k*-densest subgraphs. The core argument of [Bar15] is as follows: if one has two players with *n* strategies each and some payoff matrix $A \in [-1,1]^{n \times n}$, then for any mixed strategy y of the column player (i.e. $y \in \mathbb{R}^n_{\geq 0}$ and $||y||_1 = 1$) one can apply the approximate Carathéodory Theorem for norm $|| \cdot ||_{\infty}$ (or rather equivalently for $|| \cdot ||_{\log(n)}$) and find $k := \Theta(\frac{\log(n)}{\varepsilon^2})$ columns a_1, \ldots, a_k of A so that $||Ay - \frac{1}{k}\sum_{i=1}^k a_i||_{\infty} \le \varepsilon$. In other words, any mixed strategy can be ε -approximated by the unweighted average of only $\Theta(\frac{\log(n)}{\varepsilon^2})$

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many pure strategies which then allows for an efficient enumeration. The approximate Carathéodory Theorem has also been useful in algebraic settings. For example Deligkas et al [DFMS22] use it to find approximate solutions to systems of polynomial (in)equalities and Bhargava, Saraf and Volkovich [BSV20] use approximate Carathéodory to prove that sparse polynomials have only sparse factors which then allows efficient deterministic factorization of sparse polynomials; both applications use the variant with respect to the $\|\cdot\|_{\infty}$ -norm.

To make the statements formal, for symmetric convex bodies $P, Q \subseteq \mathbb{R}^m$ and $k \in \mathbb{N}$, we denote

$$\operatorname{ac}_{k}(P,Q) := \sup_{\substack{X \subseteq P, \\ \boldsymbol{z} \in \operatorname{conv}(X)}} \inf_{\boldsymbol{v}_{1},\dots,\boldsymbol{v}_{k} \in X} \left\| \boldsymbol{z} - \frac{1}{k} \sum_{i=1}^{k} \boldsymbol{v}_{i} \right\|_{Q}$$

as the best error bound with respect to the $\|\cdot\|_Q$ -norm for approximating a point z in the convex hull of some points in P. We would like to point out that the vectors v_1, \ldots, v_k may be taken with repetition. Here, $\|\cdot\|_Q$ is the norm with $\|x\|_Q = \min\{s \ge 0 \mid x \in sQ\}$. A folklore result is that for the Euclidean norm one has $\operatorname{ac}_k(B_2^m, B_2^m) \le \frac{1}{\sqrt{k}}$ for any $k \ge 1$, which gives a *dimension free* bound. More generally, for $p \ge 1$, it is true that $\operatorname{ac}_k(B_p^m, B_p^m) \le O(\sqrt{\frac{p}{k}})$ where $B_p^m := \{x \in \mathbb{R}^m \mid \|x\|_p \le 1\}$ is the $\|\cdot\|_p$ -unit ball. This bound is derived from Maurey's Lemma from functional analysis (which was reported by Pisier [Pis81]; for an English version, see the appendix of Bourgain and Nelson [BN13]). Algorithmically, the result is simple: write $z = \sum_{i=1}^N \lambda_i u_i$ where $\lambda_1, \ldots, \lambda_N \ge 0$ and $\sum_{i=1}^N \lambda_i = 1$. Then sample $v_1, \ldots, v_k \in \{u_1, \ldots, u_N\}$ independently according to the probabilities λ_i (possibly with repetition).

Another approach in the literature by Mirrokni et al [MLVW17] is based on the desire to avoid the computation of $\boldsymbol{z} = \sum_{i=1}^{N} \lambda_i \boldsymbol{u}_i$. Instead they use the *Mirror Descent* algorithm from convex optimization to compute the sequence $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ directly. In fact they reprove the bound of $\operatorname{ac}_k(B_p^m, B_p^m) \leq O(\sqrt{\frac{p}{k}})$ using their framework. More recently, Combettes and Pokutta [CP21] show that the Frank-Wolfe algorithm can also be used to recover the same bounds. From the current state of the literature, there are two directions that appear natural to follow:

- Approximate Carathéodory for general pairs of norms. The existing bounds are for the case where *P* and *Q* are the same $\|\cdot\|_p$ -ball. Is there a convenient framework that can handle general symmetric convex bodies or at least $P = B_p^m$ and $Q = B_a^m$?
- *Tight bounds for approximate Carathéodory.* Generally, it is stated that for example the bound $\operatorname{ac}_k(B_p^m, B_p^m) \leq O(\sqrt{\frac{p}{k}})$ is tight (see e.g. [MLVW17]). But that is only true if one aims for a dimension independent bound. So for which regimes of *m* vs. *k* and *p* is it possible to improve the bound?

A classical area within combinatorics that appears related to these questions is *discrepancy theory*. Let $S_1, ..., S_m \subseteq \{1, ..., n\}$ be a set system over *n* elements. Then the goal

is to find a bi-coloring $\mathbf{x} \in \{-1,1\}^n$ so that the worst imbalance $\max_{i=1,...,m} |\sum_{j \in S_i} x_j|$ is minimized. A seminal result of Spencer [Spe85] says that for $m \ge n$, the discrepancy is bounded by $O(\sqrt{n\log(\frac{2m}{n})})$. If no element is in more than t sets, then one can also prove a bound of 2t, see Beck and Fiala [BF81]. A convex geometry based method by Banaszczyk [Ban98] shows that for any $A \in \mathbb{R}^{m \times n}$ with column length $||A^j||_2 \le 1$ for all j = 1, ..., n and any symmetric convex body $K \subseteq \mathbb{R}^m$ with Gaussian measure at least 1/2 (for example $K = \Theta(\sqrt{\log(m)}) \cdot B_{\infty}^m$ or $K = \Theta(\sqrt{m}) \cdot B_2^m$ work), there is a coloring $\mathbf{x} \in \{-1, 1\}^n$ with $A\mathbf{x} \in 5K$. Interestingly, neither of these cited results of [Spe85, BF81, Ban98] can be obtained by merely taking a uniform random coloring \mathbf{x} . But for example, the result by Spencer allows for elegant algorithmic proofs. While the first such algorithm was due to Bansal [Ban10], we focus on the later work of Lovett and Meka [LM12] whose main claim can be paraphrased as follows:

Theorem 1 ([LM12]). Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{x}_0 \in [-1,1]^n$ and let $C_1, C_2 > 0$ be small enough constants. Then there is an efficiently computable distribution $\mathcal{D}(A, \mathbf{x}_0)$ with the following properties:

- (A) One has $|\{j \in [n] : x_j \in \{\pm 1\}\}| \ge \frac{n}{2}$ for all $\mathbf{x} \sim \mathcal{D}(\mathbf{A}, \mathbf{x}_0)$.
- (B) One has $\|\mathbf{A}(\mathbf{x} \mathbf{x}_0)\|_{\infty} \leq \Delta$ for all $\mathbf{x} \sim \mathcal{D}(\mathbf{A}, \mathbf{x}_0)$ where $\Delta \geq 0$ is any parameter satisfying $\sum_{i=1}^{m} \exp(-C_1 \frac{\Delta^2}{\|\mathbf{A}_i\|_2^2}) \leq C_2 n$.
- (C) One has $\mathbb{E}_{\boldsymbol{x}\sim\mathcal{D}(\boldsymbol{A},\boldsymbol{x}_0)}[\boldsymbol{x}] = \boldsymbol{x}_0$.
- (D) The random vector $\mathbf{x} \mathbf{x}_0$ is O(1)-subgaussian. In particular $\|\langle \mathbf{A}_i, \mathbf{x} \mathbf{x}_0 \rangle\|_{\psi_2} \lesssim \|\mathbf{A}_i\|_2$ for all i = 1, ..., m.

The running time to compute a sample is $T(m, n) \leq O(n^{1+\omega} + n^2 m)$.

Here we use the notation $A \leq B$ if there is a universal constant C > 0 so that $A \leq C \cdot B$. The statement differs in several aspects to the original statement of [LM12]. We discuss and justify the changes in Appendix A. Intuitively, $\mathcal{D}(A, \mathbf{x}_0)$ is simply the outcome of a *Brownian motion* starting at \mathbf{x}_0 that freezes coordinates as soon as they hit +1 or -1 and it freezes constraints if $|\langle A_i, \mathbf{x} - \mathbf{x}_0 \rangle| = \Delta$. For example, if $A \in \{0, 1\}^{m \times n}$ with $m \geq n$ is the incidence matrix in Spencer's setting, then one may choose $\Delta := \Theta(\sqrt{n\log(\frac{2m}{n})})$ and obtain a partial coloring of discrepancy Δ that colors at least half the coordinates².

¹Throughout this work, we will use T(m, n) as the best running time to generate a sample from the described distribution. Whenever we state a running time using $T(\cdot, \cdot)$ we implicitly assume that the function T is non-decreasing in m and n and that $T(m, n) \ge mn$ which corresponds to the input length.

²One can then obtain a full coloring by iterating the argument $O(\log n)$ times. In the *i*th such iteration (with $i \ge 0$) there are at most $n/2^i$ elements uncolored, so the suffered discrepancy decreases to $O(\sqrt{(n/2^i)\log\frac{2m}{n/2^i}})$. Summing over these terms gives a convergent sum of value $O(\Delta)$.

1.1 Our contribution

The *vector balancing constant* for two symmetric convex bodies $P, Q \subseteq \mathbb{R}^m$ is

$$\operatorname{vb}(P,Q) := \sup\left\{\min_{\boldsymbol{x}\in\{-1,1\}^n} \left\|\sum_{i=1}^n x_i \boldsymbol{\nu}_i\right\|_Q \mid n \in \mathbb{N}, \boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_n \in P\right\}$$

The connection between the approximate Carathéodory problem and vector balancing was already discovered by Dadush et al [DNTT18] who proved that for any symmetric convex bodies $P, Q \subseteq \mathbb{R}^m$ one has $ac_k(P,Q) \leq \frac{vb(P,Q)}{k}$. But for example for $P = Q = B_2^m$ one has $vb(B_2^m, B_2^m) = \Theta(\sqrt{m})$ and so the obtained bound is $O(\frac{\sqrt{m}}{k})$, which is suboptimal if $k \ll m$. Instead, we suggest a reduction to a slight variant of the vector balancing constant that allows for tight bounds. Let

$$\operatorname{vb}_{n}(P,Q) := \sup \left\{ \min_{\boldsymbol{x} \in \{-1,1\}^{n}} \left\| \sum_{i=1}^{n} x_{i} \boldsymbol{\nu}_{i} \right\|_{Q} \mid \boldsymbol{\nu}_{1}, \dots, \boldsymbol{\nu}_{n} \in P \right\}$$

be the vector balancing constant restricted to exactly n vectors³. We prove the following:

Theorem 2. For any symmetric convex bodies $P, Q \subseteq \mathbb{R}^m$ and any $k \in \mathbb{N}$ one has

$$ac_k(P,Q) \le 4\sum_{\ell \ge 1} \frac{1}{k2^\ell} \cdot vb_{k2^\ell}(P,Q)$$

The vectors $v_1, ..., v_k$ can be found in time $O(\log m)$ times the time to find a coloring behind $vb_t(P,Q)$ where $t \le m+1$, assuming we are given z as convex combination of at most m+1 vectors from X.

For most bodies the quantity $vb_k(P,Q)$ grows sublinear in k and the sum is dominated by the first term, in which case one has $ac_k(P,Q) \leq \frac{1}{k} \cdot vb_k(P,Q)$. For example if $P = Q = B_2^m$ one then has $vb_k(B_2^m, B_2^m) = \Theta(\sqrt{k})$ and so one recovers the $O(\frac{1}{\sqrt{k}})$ bound mentioned earlier⁴. Also note that by [LSV86], for any symmetric convex bodies $P, Q \subseteq \mathbb{R}^m$ and any $t \in \mathbb{N}$ one has $vb_t(P,Q) \leq 2vb_m(P,Q)$, meaning that the worst case is basically attained for m many vectors. Then the infinite sum in Theorem 2 is dominated by the first log(2m/k) terms if $k \leq m$ and the first term if $k \geq m$.

For balancing vectors in B_p^m into B_q^m we prove the following:

³We should note that this is the same as asking for *at most n* vectors as $\mathbf{0} \in P$.

⁴Though not all bodies allow for such a sublinear dependence. For example fix $1 \le k \le m/2$. Then one has $vb_k(B_1^m, B_1^m) = \Theta(k)$ and so Theorem 2 provides a suboptimal bound of $ac_k(B_1^m, B_1^m) \le O(\log \frac{m}{k})$. On the other hand, indeed it is true that $ac_k(B_1^m, B_1^m) \ge \Omega(1)$ meaning that the approximate Carathéodory bound does not even improve with k (at least as long as $k \le m/2$). To see this, consider $X := \{e_1, \dots, e_m\}$ and a target of $z := (\frac{1}{m}, \dots, \frac{1}{m})$. Then for any $v_1, \dots, v_k \in X$ one has $||z - \frac{1}{k}\sum_{i=1}^k v_i||_1 \ge \Omega(1)$ since any coordinate whose unit vector is not included will contribute $\frac{1}{m}$ to the norm.

Theorem 3. For $2 \le p \le q \le \infty$ and $n \le m$ one has

$$vb_n(B_p^m, B_q^m) \lesssim \frac{\sqrt{\min\{p, \log(\frac{2m}{n})\}}}{\frac{1}{2} - \frac{1}{p} + \frac{1}{q}} \cdot n^{1/2 - 1/p + 1/q}$$

The time to find the corresponding coloring is $O(\log n) \cdot T(m, n)$.

We should point out that the upper bound on $vb_n(B_p^m, B_q^m)$ itself is already proven in [RR20]. However, that argument goes via the Gaussian measure of the suitable partial colorings and the only known algorithms are via convex optimization resulting in large polynomial running times. In contrast, here we give a streamlined argument that shows that the more efficient Lovett-Meka algorithm can be used to obtain the same bound rather than relying on convex optimization.

Combining the results above then gives:

Theorem 4. Let $2 \le p \le q \le \infty$ and $k \in \mathbb{N}$. Then

$$ac_k(B_p^m, B_q^m) \lesssim \frac{1}{\frac{1}{2} - \frac{1}{p} + \frac{1}{q}} \cdot \frac{\sqrt{\min\{p, \log(\frac{2m}{k})\}}}{k^{1/2 + 1/p - 1/q}}$$

The vectors $v_1, ..., v_k$ can be found in time $O(\log^2 m) \cdot T(m, m+1) \le O(m^{1+\omega} \log^2(m))$ assuming we are given z as a convex combination of at most m+1 vectors in X.

To the best of our knowledge this is the first approximate Carathéodory bound for pairs of different $\|\cdot\|_p$ -norms. In particular, when p = q this improves upon the $O(\sqrt{\frac{p}{k}})$ bound in [MLVW17] whenever $p \ll \log(\frac{2m}{k})$:

Corollary 5. Let $2 \le p \le \infty$ and $k \in \mathbb{N}$. Then

$$ac_k(B_p^m, B_p^m) \lesssim \sqrt{\frac{\min\{p, \log(\frac{2m}{k})\}}{k}}$$

The vectors $v_1, ..., v_k$ can be found in time $O(\log^2 m) \cdot T(m, m+1) \le O(m^{1+\omega} \log^2(m))$ assuming we are given z as a convex combination of at most m + 1 vectors in X.

Finally, we show that the bound in Theorem 3 is tight up to a factor of $\frac{1}{2} - \frac{1}{p} + \frac{1}{q}$:

Theorem 6. Let $2 \le p \le q \le \infty$ and $n \le m \le 2^n$. Then

$$vb_n(B_p^m, B_q^m) \gtrsim \sqrt{\min\left\{p, \log\left(\frac{2m}{n}\right)\right\}} \cdot n^{1/2 - 1/p + 1/q}$$

2 Preliminaries

In this section we review a few facts that we later rely on. Let $S^{m-1} := \{ \mathbf{x} \in \mathbb{R}^m \mid ||\mathbf{x}||_2 = 1 \}$ be the sphere.

Convex functions. Recall the following well known fact:

Lemma 7 (Jensen Inequality for convave functions). Let *X* be any \mathbb{R} -valued random variable and let $F : \mathbb{R} \to \mathbb{R}$ be a concave function, then $F(\mathbb{E}[X]) \ge \mathbb{E}[F(X)]$.

Estimates on $\|\cdot\|_p$ **norms.** It will be useful to understand how the norm $\|z\|_p$ of a vector can change depending on $p \in [1,\infty]$.

Lemma 8. For any $z \in \mathbb{R}^m$ and $1 \le p \le q \le \infty$ one has $||z||_q \le ||z||_p \le m^{1/p-1/q} ||z||_q$. **Lemma 9.** For any $z \in \mathbb{R}^m$ and $1 \le p \le q \le \infty$, we have $||z||_q^q \le ||z||_p^p \cdot ||z||_{\infty}^{q-p}$.

The subgaussian norm. We introduce a concept from probability theory that is extremely useful and convenient when dealing with random variables that have Gaussian-type tails. For a random variable $X \in \mathbb{R}$ we define the *subgaussian norm* as

$$\|X\|_{\psi_2} := \inf\left\{s > 0 : \mathbb{E}\left[\exp\left(\frac{X^2}{s^2}\right)\right] \le 2\right\}$$

One may think of $||X||_{\psi_2}$ as the minimum number so that the tail of *X* is dominated by the Gaussian $N(0, ||X||_{\psi_2}^2)$. For example if $X \sim N(0, t^2)$ then $||X||_{\psi_2} = \Theta(t)$ and also if $X \sim \{-t, t\}$ uniformly, then $||X||_{\psi_2} = \Theta(t)$. It may not be obvious but indeed $|| \cdot ||_{\psi_2}$ is a norm on the space of jointly distributed random variables, i.e. $||tX||_{\psi_2} = |t| \cdot ||X||_{\psi_2}$ and $||X_1 + X_2||_{\psi_2} \le ||X_1||_{\psi_2} + ||X_2||_{\psi_2}$ for jointly distributed random variables X_1, X_2 (even if they are dependent). We will use in particular the following properties:

Lemma 10. The subgaussian norm satisfies the following:

- (A) For any real random variable X and any $p \ge 1$ one has $\mathbb{E}[|X|^p]^{1/p} \lesssim \sqrt{p} \cdot ||X||_{\psi_2}$.
- (B) If X_1, \ldots, X_N are independent real mean-zero random variables⁵, then

$$||X_1 + \dots + X_N||_{\psi_2} \lesssim \left(\sum_{i=1}^N ||X_i||_{\psi_2}^2\right)^{1/2}$$

- (C) For $\boldsymbol{a} \in \mathbb{R}^m$ and $\boldsymbol{x} \sim S^{m-1}$ uniformly one has $\|\langle \boldsymbol{a}, \boldsymbol{x} \rangle\|_{\psi_2} \lesssim \frac{1}{\sqrt{m}} \|\boldsymbol{a}\|_2$.
- (D) For any real random variable X with $\mathbb{E}[X] = 0$ and any $\lambda \ge 0$ one has

$$\Pr[|X| \ge \lambda ||X||_{\psi_2}] \le 2e^{-C\lambda^2},$$

where C > 0 is a universal constant.

⁵The argument also works in the Martingale setting. Suppose for any condititioning on X_1, \ldots, X_{i-1} one has $||X_i||_{\psi_2} \leq L_i$. Then $||X_1 + \cdots + X_N||_{\psi_2} \leq (\sum_{i=1}^N L_i^2)^{1/2}$.

We recommend the excellent textbook of Vershynin [Ver18] for details. For the lower bound we will use the following reverse Chernoff bound from [KY15]:

Lemma 11. Given independent random variables $x_1, \ldots, x_n \sim \{-1, 1\}$ and $\lambda \in [3, \sqrt{n}/2]$,

$$\Pr[x_1 + \dots + x_n \ge \lambda \sqrt{n}] \ge \exp(-9\lambda^2/2).$$

3 Reduction from Approximate Carathéodory to Vector Balancing

In this section, we prove the reduction of the approximate Carathéodory problem to vector balancing as stated in Theorem 2. The idea is to follow the classical approach of [LSV86]: begin with an arbitrary convex combination and round the coefficients bit-bybit. The same basic approach was also followed by Dadush et al [DNTT18]. We prove an auxiliary lemma that bounds the error when "doubling the fractionality".

Lemma 12. Let $P, Q \subseteq \mathbb{R}^m$ be symmetric convex bodies and let $\delta > 0$. Let $\mathbf{z} = \sum_{i=1}^n \lambda_i \mathbf{v}_i$ where $\mathbf{v}_1, \dots, \mathbf{v}_n \in P$ and $\lambda \in \delta \mathbb{Z}_{\geq 0}^n$. Then there is a vector $\mathbf{z}' = \sum_{i=1}^n \lambda'_i \mathbf{v}_i$ where $\lambda' \in 2\delta \mathbb{Z}_{\geq 0}^n$ so that $\|\mathbf{z} - \mathbf{z}'\|_Q \leq \delta \cdot vb_n(P,Q)$ and $\sum_{i=1}^n \lambda'_i \leq \sum_{i=1}^n \lambda_i$.

Proof. Write $\lambda_i = 2\delta a_i + \delta b_i$ with $b_i \in \{0, 1\}$ and $a_i \in \mathbb{Z}_{\geq 0}$. Let $I := \{i \in [n] \mid b_i = 1\}$. Now, let $\mathbf{x} \in \{-1, 1\}^I$ be the coloring so that $\|\sum_{i \in I} x_i \mathbf{v}_i\|_Q \leq \operatorname{vb}_{|I|}(P,Q) \leq \operatorname{vb}_n(P,Q)$. We may assume that $\sum_{i \in I} x_i \leq 0$ — otherwise replace \mathbf{x} with $-\mathbf{x}$. We may extend the vector to $\mathbf{x} \in \{-1, 0, 1\}^m$ by setting $x_i := 0$ for $i \notin I$. We update $\lambda'_i := 2\delta a_i + \delta(1 + x_i)b_i \in 2\delta \mathbb{Z}_{\geq 0}$ for $i \in [n]$. Next, we define $\mathbf{z}' := \sum_{i=1}^n \lambda'_i \mathbf{v}_i$. Then

$$\|\boldsymbol{z} - \boldsymbol{z}'\|_{Q} = \delta \left\| \sum_{i \in I} x_{i} \boldsymbol{v}_{i} \right\|_{Q} \le \delta \cdot \operatorname{vb}_{n}(P, Q)$$

Note that $\sum_{i \in I} x_i \le 0$ implies that $\sum_{i=1}^n \lambda'_i \le \sum_{i=1}^n \lambda_i$. This gives the claim.

Next, we iteratively apply Lemma 12 to an initial convex combination until the convex coefficients are multiples of $\frac{1}{k}$. We almost obtain the desired claim, just that the number of vectors might be *less* than *k*.

Lemma 13. Let $P, Q \subseteq \mathbb{R}^m$ be symmetric convex bodies. Then for any $z \in conv(X)$ with $X \subseteq P$ and $k \in \mathbb{N}$ there are $s \in \{0, ..., k\}$ and $v_1, ..., v_s \in X$ so that

$$\left\|\boldsymbol{z} - \frac{1}{k}\sum_{i=1}^{s}\boldsymbol{v}_{i}\right\|_{Q} \leq \sum_{\ell \geq 1} \frac{2}{k2^{\ell}} \cdot vb_{k2^{\ell}}(\boldsymbol{P}, \boldsymbol{Q})$$

The vectors can be found in time $O(\log m)$ times the time to find the colorings in $vb_t(P,Q)$ where $t \le m + 1$.

Proof. Fix a point $z \in \text{conv}(X)$ where $X \subseteq P$. Then we can write $z = \sum_{i=1}^{n} \lambda_i v_i$ where $n \le m+1$, $v_1, \ldots, v_n \in X$, $\lambda_i \ge 0$ for all $i = 1, \ldots, m$ and $\sum_{i=1}^{m} \lambda_i = 1$. Without loss of generality we may assume that $\lambda \in \frac{2^{-L}}{k} \mathbb{Z}_{\geq 0}^n$ for some $L \in \mathbb{N}$. We abbreviate $z^{(L)} := z$. Now suppose for $\ell \in \{0, ..., L\}$ the current iterate is $\boldsymbol{z}^{(\ell)}$ so that $\boldsymbol{z}^{(\ell)} = \sum_{i=1}^{n} \lambda_i^{(\ell)} \boldsymbol{v}_i$ and $\boldsymbol{\lambda}^{(\ell)} \in \frac{2^{-\ell}}{k} \mathbb{Z}_{\geq 0}^n$. Then we apply Lemma 12 to obtain a vector $\boldsymbol{z}^{(\ell-1)} = \sum_{i=1}^{n} \lambda_i^{(\ell-1)} \boldsymbol{v}_i$ with $\boldsymbol{\lambda}^{(\ell-1)} \in \frac{2^{-(\ell-1)}}{k} \mathbb{Z}_{\geq 0}^n$.

and $\sum_{i=1}^{n} \lambda_i^{(\ell-1)} \leq 1$. Using that $|\operatorname{supp}(\boldsymbol{\lambda}^{(\ell)})| \leq k2^{\ell}$, the approximation error satisfies

$$\left\|\boldsymbol{z}^{(\ell)} - \boldsymbol{z}^{(\ell-1)}\right\|_{Q} \le \frac{1}{2^{\ell}k} \cdot \operatorname{vb}_{|\operatorname{supp}(\boldsymbol{\lambda}^{(\ell)})|}(\boldsymbol{P}, \boldsymbol{Q}) \le \frac{1}{2^{\ell}k} \cdot \operatorname{vb}_{k2^{\ell}}(\boldsymbol{P}, \boldsymbol{Q})$$

Note that the final iterate is of the form $\boldsymbol{z}^{(0)} = \sum_{i=1}^{n} \lambda_i^{(0)} \boldsymbol{v}_i$ with $\lambda_i^{(0)} \in \frac{\mathbb{Z}_{\geq 0}}{k}$ and $\sum_{i=1}^{n} \lambda_i^{(0)} \leq \frac{\mathbb{Z}_{\geq 0}}{k}$ 1. Then for $s := k \sum_{i=1}^{n} \lambda_i^{(0)} \in \{0, ..., k\}$, let $u_1, ..., u_s$ be a list of vectors that contains v_i exactly $k\lambda_i^{(0)} \in \mathbb{Z}_{\geq 0}$ times. Using the triangle inequality we obtain

$$\left\| \boldsymbol{z} - \frac{1}{k} \sum_{i=1}^{s} \boldsymbol{u}_{i} \right\|_{Q} \le \sum_{\ell=1}^{L} \| \boldsymbol{z}^{(\ell)} - \boldsymbol{z}^{(\ell-1)} \|_{Q} \le \sum_{\ell \ge 1} \frac{1}{k 2^{\ell}} \cdot \operatorname{vb}_{k 2^{\ell}}(\boldsymbol{P}, \boldsymbol{Q})$$

Now let us discuss the running time. First, we can choose $L \leq O(\log m)$ while possibly making a rounding error of max{ $||y||_{Q} : y \in P$ } $\leq vb_1(P,Q)$, which we absorb by paying an extra factor of 2. Then the running time is dominated by the time to find the colorings. Note that we call $vb_t(P,Q)$ only *L* times for parameters *t* with $t \le m+1$.

Now we will use the same trick as Dadush et al [DNTT18] in order to obtain exactly k vectors, at the expense of a factor 2 in the approximation error:

Proof of Theorem 2. Let $z \in \text{conv}(X)$ where $X \subseteq P$. Fix a vector $u_0 \in X$ and write X' := $\{u - u_0 \mid u \in X\}$. Note that in particular $0 \in X'$ and $z - u_0 \in \text{conv}(X')$. We apply Lemma 13 and obtain vectors $v_1, \ldots, v_s \in X'$ with $s \le k$ so that

$$\left\| (\boldsymbol{z} - \boldsymbol{u}_0) - \frac{1}{k} \sum_{i=1}^{s} \boldsymbol{v}_i \right\|_Q \le \sum_{\ell \ge 1} \frac{2}{k 2^{\ell}} \cdot \operatorname{vb}_{k 2^{\ell}}(2P, Q)$$

using that $X' \subseteq 2P$. Since $\mathbf{0} \in X'$, we can extend this sequence to a list $\mathbf{v}_1, \ldots, \mathbf{v}_k$ of k vectors. Each $\boldsymbol{v}_i \in X'$ can be written as $\boldsymbol{v}_i = \boldsymbol{u}_i - \boldsymbol{u}_0$ with $\boldsymbol{u}_1, \dots, \boldsymbol{u}_k \in X$. Then

$$\left\| \boldsymbol{z} - \frac{1}{k} \sum_{i=1}^{k} \boldsymbol{u}_{k} \right\|_{Q} = \left\| (\boldsymbol{z} - \boldsymbol{u}_{0}) - \frac{1}{k} \sum_{i=1}^{k} (\boldsymbol{u}_{k} - \boldsymbol{u}_{0}) \right\|_{Q} \le 4 \sum_{\ell \ge 1} \frac{1}{k 2^{\ell}} \cdot \operatorname{vb}_{k 2^{\ell}}(\boldsymbol{P}, \boldsymbol{Q}).$$

Vector balancing from B_p^m to B_q^m 4

From now on, we focus on the case where $P = B_p^m$ and $Q = B_q^m$.

4.1 Balancing from B_p^m to B_p^m

Suppose we have a matrix $A \in \mathbb{R}^{m \times n}$ with $A^1, \ldots, A^n \in B_p^m$. Then for a uniform random $x \in \{-1, 1\}^n$ one has $\mathbb{E}[||Ax||_p] \leq \sqrt{pn}$. This result extends to the case where x is subgaussian rather than independent.

Lemma 14. Let $A^1, \ldots, A^n \in B_p^m$ for $p \ge 2$ and let $x_0 \in [-1, 1]^n$. Then

$$\mathbb{E}_{\boldsymbol{x}\sim\mathcal{D}(\boldsymbol{A},\boldsymbol{x}_0)}[\|\boldsymbol{A}(\boldsymbol{x}-\boldsymbol{x}_0)\|_p] \lesssim \sqrt{pn}$$

Proof. It will be convenient to abbreviate $\mathbf{B} \in \mathbb{R}^{m \times n}$ as the matrix with entries $B_{ij} := A_{ij}^2$. Then we have

$$\mathbb{E}_{\boldsymbol{x}\sim\mathcal{D}(\boldsymbol{A},\boldsymbol{x}_{0})} [\|\boldsymbol{A}(\boldsymbol{x}-\boldsymbol{x}_{0})\|_{p}] \stackrel{(*)}{\leq} \qquad \mathbb{E}_{\boldsymbol{x}\sim\mathcal{D}(\boldsymbol{A},\boldsymbol{x}_{0})} [\|\boldsymbol{A}(\boldsymbol{x}-\boldsymbol{x}_{0})\|_{p}^{p}]^{1/p} \\
= \left(\sum_{i=1}^{m} \mathbb{E}_{\boldsymbol{x}\sim\mathcal{D}(\boldsymbol{A},\boldsymbol{x}_{0})} [|\langle \boldsymbol{A}_{i},\boldsymbol{x}-\boldsymbol{x}_{0}\rangle|^{p}]\right)^{1/p} \\
\stackrel{\text{Thm 1.(D)+Lem 10.(A)}}{\leq} \left(\sum_{i=1}^{m} \left(\sqrt{p} \cdot \|\boldsymbol{A}_{i}\|_{2}\right)^{p}\right)^{1/p} \\
= \sqrt{p} \cdot \sum_{i=1}^{m} \left(\sum_{j=1}^{n} A_{ij}^{2}\right)^{p/2} \sum_{i=1}^{1/p} \left(\sum_{j=1}^{n} A_{ij}^{2}\right)^{p/2} \sum_{i=1}^{1/p} \sum_{j=1}^{n} A_{ij}^{2}\right)^{p/2} \sum_{i=1}^{1/p} \sum_{j=1}^{n} B_{ij} \sum_{j=1}^{n} \sum_{j=1}^{n} B_{ij} \sum_{j=1}^{n} \sum_{j=1}^{n} B_{ij} \sum_{j=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} B_{ij} \sum_{j=1}^{n} \sum_{j=1}^{n}$$

Here we are using Jensen's Inequality (Lemma 7) with the concavity of $f(z) = z^{1/p}$ in (*). Finally in (**) we have made use of

$$\|\boldsymbol{B}^{j}\|_{p/2} = \left(\sum_{i=1}^{m} (A_{ij}^{2})^{p/2}\right)^{2/p} = \left(\sum_{\substack{i=1\\\leq 1}}^{m} |A_{ij}|^{p}\right)^{2/p} \le 1.$$

This shows the claim.

4.2 A dimension-free upper bound on $\|\cdot\|_{\infty}$

Now suppose we have a matrix $A \in \mathbb{R}^{m \times n}$ with $A^1, \ldots, A^n \in B_p^m$ and we want to find a coloring x with small value of $||Ax||_{\infty}$. If we were to take a random coloring and the matrix

happens to contain a row of all-ones, then in expectation one would have $||Ax||_{\infty} \gtrsim \sqrt{n}$. It turns out that the Lovett-Meka distribution improves significantly over this bound. However, obtaining a bound solely dependent on *n* is slightly delicate since the Euclidean norm bound of $||A^j||_2 \le m^{1/2-1/p}$ on the columns might be tight.

Lemma 15. Let $p \ge 2$, $\mathbf{x}_0 \in [-1,1]^n$ and let $\mathbf{A}^{m \times n}$ be a matrix with $\|\mathbf{A}^j\|_p \le 1$ for all j = 1, ..., n. Then for any $\mathbf{x} \sim \mathcal{D}(\mathbf{A}, \mathbf{x}_0)$ one has $\|\mathbf{A}(\mathbf{x} - \mathbf{x}_0)\|_{\infty} \lesssim \sqrt{p} \cdot n^{1/2 - 1/p}$.

Proof. The goal is to verify that the condition in Theorem 1.(B) applies for a value of Δ that is of the order $\sqrt{p} \cdot n^{1/2-1/p}$. First we convert the bound on the $\|\cdot\|_p$ -norm of the columns A^j into information about the $\|\cdot\|_2$ -norm of the rows A_i . In particular one has

$$\left(\frac{1}{n}\sum_{i=1}^{m}\|\boldsymbol{A}_{i}\|_{2}^{p}\right)^{1/p} \stackrel{\text{Lem 8}}{\leq} n^{1/2-1/p} \cdot \left(\frac{1}{n}\sum_{i=1}^{m}\|\boldsymbol{A}_{i}\|_{p}^{p}\right)^{1/p} = n^{1/2-1/p} \left(\frac{1}{n}\sum_{j=1}^{n}\|\boldsymbol{A}^{j}\|_{p}^{p}\right)^{1/p} \leq n^{1/2-1/p} \quad (***)$$

Then we use this to estimate that

$$\sum_{i=1}^{m} \exp\left(-C_1 \frac{\Delta^2}{\|A_i\|_2^2}\right) \leq \sum_{i=1}^{m} p^{p/2} \frac{C_1^{p/2} \|A_i\|_2^p}{\Delta^p} \leq n \cdot \left(\frac{\sqrt{C_1 p} n^{1/2 - 1/p}}{\Delta}\right)^p \leq C_2 n$$

if we set $\Delta := \frac{\sqrt{C_1}}{C_2^{1/p}} \sqrt{p} \cdot n^{1/2 - 1/p}$. Here we have used the following estimate:

Claim I. For $p \ge 1$ and y > 0 one has $\exp(-\frac{1}{y}) \le p^{p/2} y^{p/2}$. Indeed, for y > 1, one has $\exp(-\frac{1}{y}) < 1 < p^{p/2} y^{p/2}$. Since $\exp(\frac{1}{y}) = \sum_{k \in \mathbb{N}} \frac{1}{k! \cdot y^k} \ge \frac{1}{k! \cdot y^k}$ for any $k \in \mathbb{N}$, one has for $y \le 1$:

$$\exp\left(-\frac{1}{y}\right) \le \lceil p/2 \rceil! \cdot y^{\lceil p/2 \rceil} \le \lceil p/2 \rceil^{\lfloor p/2 \rfloor} y^{p/2} \le p^{p/2} y^{p/2}.$$

4.3 Balancing from B_p^m to B_a^m

Now we show how to find partial colorings for balancing vectors in the B_p^m -ball into scalars of the B_q^m -ball using the Lovett-Meka distribution:

Theorem 16. Let $2 \le p \le q \le \infty$, $x_0 \in [-1,1]^n$ and let $A \in \mathbb{R}^{m \times n}$ with $A^1, ..., A^n \in B_p^m$. Then $\mathbb{E} \left[\|A(\mathbf{r} - \mathbf{r}_0)\|_{2} \right] \le \sqrt{\min \{n \log(\frac{2m}{2})\}} \cdot n^{1/2 - 1/p + 1/q}$

$$\mathbb{E}_{\boldsymbol{x}\sim\mathcal{D}(\boldsymbol{A},\boldsymbol{x}_0)}\left[\|\boldsymbol{A}(\boldsymbol{x}-\boldsymbol{x}_0)\|_q\right] \lesssim \sqrt{\min\left\{p,\log(\frac{2m}{n})\right\}} \cdot n^{1/2-1/p+1/q}$$

Proof. We will make use of the inequality $\|\boldsymbol{z}\|_q \leq (\|\boldsymbol{z}\|_p^p \cdot \|\boldsymbol{z}\|_{\infty}^{q-p})^{1/q}$ for all $\boldsymbol{z} \in \mathbb{R}^m$, see

Lemma 9. Then combining the estimates from Lemma 14 and Lemma 15 we obtain

$$\mathbb{E}_{\mathbf{x}\sim\mathcal{D}(\mathbf{A},\mathbf{x}_{0})} [\|\mathbf{A}(\mathbf{x}-\mathbf{x}_{0})\|_{q}] \leq \mathbb{E}_{\mathbf{x}\sim\mathcal{D}(\mathbf{A},\mathbf{x}_{0})} [(\|\mathbf{A}(\mathbf{x}-\mathbf{x}_{0})\|_{p}^{p} \cdot \|\mathbf{A}(\mathbf{x}-\mathbf{x}_{0})\|_{\infty}^{q-p})^{1/q}] \quad (****)$$

$$\lim_{k \to \mathcal{D}(\mathbf{A},\mathbf{x}_{0})} \mathbb{E}_{\mathbf{x}\sim\mathcal{D}(\mathbf{A},\mathbf{x}_{0})} [\|\mathbf{A}(\mathbf{x}-\mathbf{x}_{0})\|_{p}^{p/q}] \cdot (\sqrt{p}n^{1/2-1/p})^{(q-p)/q}$$

$$\lim_{k \to \mathcal{D}(\mathbf{A},\mathbf{x}_{0})} \mathbb{E}_{\mathbf{x}\sim\mathcal{D}(\mathbf{A},\mathbf{x}_{0})} [\|\mathbf{A}(\mathbf{x}-\mathbf{x}_{0})\|_{p}]^{p/q} \cdot (\sqrt{p}n^{1/2-1/p})^{(q-p)/q}$$

$$\lim_{k \to \mathcal{D}(\mathbf{A},\mathbf{x}_{0})} \mathbb{E}_{\mathbf{x}\sim\mathcal{D}(\mathbf{A},\mathbf{x}_{0})} [\|\mathbf{A}(\mathbf{x}-\mathbf{x}_{0})\|_{p}]^{p/q} \cdot (\sqrt{p}n^{1/2-1/p})^{(q-p)/q}$$

$$\lim_{k \to \mathcal{D}(\mathbf{A},\mathbf{x}_{0})} \mathbb{E}_{\mathbf{x}\sim\mathcal{D}(\mathbf{A},\mathbf{x}_{0})} [\|\mathbf{A}(\mathbf{x}-\mathbf{x}_{0})\|_{p}]^{p/q} \cdot (\sqrt{p}n^{1/2-1/p})^{(q-p)/q}$$

Here we have used Jensen's inequality (see Lemma 7) with the concavity of the function $y \mapsto y^{p/q}$ for $0 < y < \infty$.

Note that this settles the claim in the parameter range $p \le \ln(e^2 \frac{m}{n})$. Now consider the case of $p > \ln(e^2 \frac{m}{n})$. We define $p_0 := \ln(e^2 \frac{m}{n})$. Note that $2 \le p_0 \le p$. Then applying the bound of (* * **) proven above, with parameters $2 \le p_0 \le q$ we obtain

$$\Pr_{\boldsymbol{x} \sim \mathcal{D}(\boldsymbol{A}, \boldsymbol{x}_{0})} \left[\|\boldsymbol{A}(\boldsymbol{x} - \boldsymbol{x}_{0})\|_{q} \right] \lesssim \sqrt{p_{0}} \cdot n^{1/2 - 1/p_{0} + 1/q} \cdot \max\{\|\boldsymbol{A}^{j}\|_{p_{0}} : j = 1, ..., n\}$$

$$\leq \sqrt{p_{0}} \cdot n^{1/2 - 1/p + 1/q} \cdot m^{1/p_{0} - 1/p} \cdot \max\{\|\boldsymbol{A}^{j}\|_{p} : j = 1, ..., n\}$$

$$= \sqrt{p_{0}} \cdot n^{1/2 - 1/p + 1/q} \cdot \left(\frac{m}{n}\right)^{-1/p} \cdot \left(\frac{m}{n}\right)^{1/p_{0}}$$

$$\leq \sqrt{\ln\left(e^{2}\frac{m}{n}\right)} \cdot n^{1/2 - 1/p + 1/q} \cdot \left(\frac{m}{n}\right)^{1/\ln(e^{2}\frac{m}{n})} \cdot e^{O(1)}$$

This completes the proof.

4.4 From partial colorings to full colorings

The result from Theorem 16 allows us to find a partial coloring — the next step is to iterate the argument in order to find a full coloring:

Proof of Theorem 3. Consider *n* vectors from B_p^m that we conveniently write as column vectors $A^1, \ldots, A^n \in B_p^m$. We set $\mathbf{x}_0 := \mathbf{0} \in \mathbb{R}^n$ and $n_0 := n$. In iteration $t = 0, 1, \ldots$ we have maintained a vector $\mathbf{x}_t \in [-1, 1]^n$ where $n_t := |\{i \in [n] \mid -1 < x_t(i) < 1\}|$ denotes the number of uncolored elements. We draw $\mathbf{x}_{t+1} \sim \mathcal{D}(\mathbf{A}, \mathbf{x}_t)$ and repeat until the bound of the expectation provided by Theorem 16 is attained (say up to a factor of 2). In particular

$$\left\|\boldsymbol{A}(\boldsymbol{x}_{t+1}-\boldsymbol{x}_t)\right\|_q \lesssim \sqrt{\min\left\{p,\log\left(\frac{2m}{n_t}\right)\right\}} \cdot n_t^{1/2-1/p+1/q}$$

and $n_{t+1} \le n_t/2$. Then $n_{t+1} \le \frac{n}{2^t}$ and so $\mathbf{x}^* := \mathbf{x}_{\log_2(n)+1}$ will be in $\{-1, 1\}^n$ and by the triangle inequality, the discrepancy is at most

$$\|\mathbf{A}\mathbf{x}^*\|_q \lesssim \sum_{t \ge 0} \sqrt{\min\{p, \log(\frac{2m}{n/2^t})\}} \cdot (n/2^t)^{1/2 - 1/p + 1/q} \lesssim \frac{\sqrt{\min\{p, \log(\frac{2m}{n})\}}}{1/2 - 1/p + 1/q} n^{1/2 - 1/p + 1/q}$$

To see the ultimate inequality, consider the exponent $\alpha := 1/2 - 1/p + 1/q > 0$. Then it takes $1/\alpha$ iterations until the quantity n_t^{α} has decreased by a factor of 2 (while the term $\log(2m/n_t)$ has a miniscule growth). Then the cumulated discrepancy is dominated by the first $\frac{1}{\alpha}$ terms. The running time is bounded by $O(\log n) \cdot T(m, n)$.

5 Approximate Carathéodory bounds for $\|\cdot\|_p$ norms

Next, we prove the bound on $ac_k(B_p^m, B_q^m)$ claimed in Theorem 4:

Proof of Theorem 4. Let $2 \le p \le q \le \infty$. We apply the reduction to the vector balancing constant from Theorem 2 and combine this with the bound from Theorem 3:

$$\begin{aligned} \operatorname{ac}_{k}(B_{p}^{m}, B_{q}^{m}) &\stackrel{\operatorname{Thm 2}}{\leq} & \sum_{\ell \geq 1} \frac{1}{k2^{\ell}} \cdot \operatorname{vb}_{k2^{\ell}}(B_{p}^{m}, B_{q}^{m}) \\ &\stackrel{\operatorname{Thm 3}}{\lesssim} & \frac{1}{\frac{1}{2} - \frac{1}{p} + \frac{1}{q}} \sum_{\ell \geq 1} \frac{\sqrt{\min\{p, \log(\frac{2m}{k2^{\ell}})\}}}{(k2^{\ell})^{1/2 + 1/p - 1/q}} \\ &\lesssim & \frac{1}{\frac{1}{2} - \frac{1}{p} + \frac{1}{q}} \cdot \frac{\sqrt{\min\{p, \log(\frac{2m}{k})\}}}{k^{1/2 + 1/p - 1/q}}. \end{aligned}$$

Note that the exponent $\alpha := 1/2 + 1/p - 1/q$ satisfies $\alpha \ge 1/2$ and so the sum is already dominated by the very first term. The running time to find the vectors v_1, \ldots, v_k is dominated by $O(\log m)$ calls to find the coloring behind $vb_t(B_p^m, B_q^m)$ where $t \le m + 1$ which results in a total running time of $O(\log^2 m) \cdot T(m, m + 1) \le O(m^{1+\omega} \log^2(m))$ as $\omega \ge 2$. \Box

Remark 1. In the case where p = 2 and $q = \infty$, we can apply Theorem 4 with $q' := \log_2 m$ to obtain $\operatorname{ac}_k(B_2^m, B_\infty^m) \leq \frac{\log m}{k}$ by noting that Lemma 8 implies $\|\boldsymbol{x}\|_{\infty} \leq \|\boldsymbol{x}\|_{\log_2 m} \leq 2\|\boldsymbol{x}\|_{\infty}$ for any $\boldsymbol{x} \in \mathbb{R}^m$. But for $P = B_2^m$ and $Q = B_\infty^m$ one has $\operatorname{vb}_k(B_2^m, B_\infty^m) \leq \sqrt{\log m}$ by the result of Banaszczyk [Ban98], so that Theorem 2 yields the improved bound

$$\operatorname{ac}_k(B_2^m, B_\infty^m) \lesssim \frac{\sqrt{\log m}}{k}.$$

It remains an interesting open question whether this may be improved to $O(\frac{1}{k})$.

6 Lower bounds for vector balancing

In this section we show that the vector balancing bounds in Theorem 3 are tight up to the factor of $\frac{1}{2} - \frac{1}{p} + \frac{1}{q}$:

Proof of Theorem 6. First, let us focus on the case when $m \le 2^p n$ so that $\log(2m/n) = \min\left\{p, \log\left(\frac{2m}{n}\right)\right\}$. We will also assume that $m \ge 8n$, since otherwise we can use instead $n' := \lfloor n/8 \rfloor$ and add n - n' columns of zeros. Let **B** denote an $m \times n$ random matrix with i.i.d ± 1 entries. For any fixed $\mathbf{x} \in \{-1, 1\}^n$, let $N_{\mathbf{x}}$ denote the number of rows $i \in [m]$ with $\langle \mathbf{B}_i, \mathbf{x} \rangle \ge \lambda \sqrt{n}$ for $\lambda := \sqrt{\frac{2}{9} \log\left(\frac{m}{2n}\right)}$. Since $\lambda \le \sqrt{n}/2$, by Lemma 11 we have

$$\Pr\left[\langle \boldsymbol{B}_i, \boldsymbol{x} \rangle \geq \lambda \sqrt{n}\right] \geq \frac{2n}{m},$$

so that $\mathbb{E}[N_x] \ge 2n$. The standard Chernoff bound then gives $\Pr[N_x \le (1-0.9) \cdot 2n] < 2^{-n}$, so that by the union bound there exists a matrix $\mathbf{B} \in \{-1, 1\}^{m \times n}$ for which $N_x \ge n/5$ for all $\mathbf{x} \in \{-1, 1\}^n$. Thus for any such \mathbf{x} ,

$$\|\boldsymbol{B}\boldsymbol{x}\|_{q} \ge (|N_{\boldsymbol{x}}| \cdot (\lambda \sqrt{n})^{q})^{1/q} \gtrsim n^{1/2+1/q} \log\left(\frac{m}{2n}\right)^{1/2} \gtrsim n^{1/2+1/q} \log\left(\frac{2m}{n}\right)^{1/2}$$

where in the last step we used $m \ge 8n$. The matrix $A := m^{-1/p} B$ has columns in B_p^m and

$$\|\mathbf{A}\mathbf{x}\|_q \gtrsim \sqrt{\log\left(\frac{m}{2n}\right)} \cdot \frac{n^{1/2+1/q}}{m^{1/p}} \gtrsim \sqrt{\log\left(\frac{m}{2n}\right)} \cdot n^{1/2-1/p+1/q}$$

for all $x \in \{-1, 1\}^n$, as claimed.

For the case when $m \ge 2^p n$ we may use the same construction for $m' := \lfloor 2^p n \rfloor$ with m - m' additional rows of zeros.

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A The Lovett-Meka random walk

In this section we want to justify Theorem 1. First, the original main technical statement of Lovett and Meka [LM12] is as follows:

Theorem 17 ([LM12]). Let $A_1, ..., A_m \in \mathbb{R}^n$, $x_0 \in [-1,1]^n$, $\delta > 0$ small enough and let $\lambda_1, ..., \lambda_m \ge 0$ so that $\sum_{i=1}^m \exp(-\lambda_i^2/16) \le \frac{n}{16}$. Then there exists an efficient randomized algorithm which with probability at least $\frac{1}{10}$ finds a point $x \in [-1,1]^n$ so that

- (i) $|\langle \mathbf{A}_i, \mathbf{x} \mathbf{x}_0 \rangle| \leq \lambda_i ||\mathbf{A}_i||_2$ for all $i = 1, \dots, m$.
- (ii) $|x_i| \ge 1 \delta$ for at least n/2 many indices.

Moreover, the algorithm runs in time $O((m+n)^3 \cdot \delta^{-2} \log(mn/\delta))$.

First, note that if Δ is a parameter with $\sum_{i=1}^{m} \exp(-\frac{\Delta^2}{16\|A_i\|_2^2}) \leq \frac{n}{16}$, then $\lambda_i := \frac{\Delta}{\|A_i\|_2}$ is a feasible choice. The algorithm behind Theorem 17 works as follows: let $\gamma > 0$ be a parameter that is logarithmically smaller than δ . We compute a random sequence $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ of points. In iteration t we have already computed \mathbf{x}_t ; next we define a subspace $V_t \subseteq \mathbb{R}^n$ that is orthogonal to coordinates \mathbf{e}_i with $|x_i| \geq 1 - \delta$ and to directions A_i with $|\langle A_i, \mathbf{x} - \mathbf{x}_0 \rangle| \geq \lambda_i - \delta$. Then update $\mathbf{x}_{t+1} = \mathbf{x}_t + \gamma \mathbf{u}_t$ where $\mathbf{u}_t \sim N(\mathbf{0}, \mathbf{I}_{V_t})$ is a random Gaussian from that subspace V_t .

What is slightly unsatisfactory is that we will not actually have coordinates equal to ± 1 . But there is an easy way to remedy this: consider the polyhedron $K := \{x \in [-1,1]^n \mid |\langle A_i, x - x_0 \rangle| \le \lambda_i ||A_i||_2 \quad \forall i = 1,...,m\}$. We say that a coordinate *j* is *tight* for a point *x* if $|\langle A_i, x - x_0 \rangle| = \lambda_i ||A_i||_2$. Then consider the following random walk:

- (1) FOR t = 1, 2, ..., 8n DO
 - (2) Let $V_t := \{ \mathbf{y} \in \mathbb{R}^n \mid \langle \mathbf{y}, \mathbf{x}_t \rangle = 0; y_j = 0 \text{ if } j \in [n] \text{ is tight for } \mathbf{x}_t; \langle \mathbf{A}_i, \mathbf{y} \rangle = 0 \text{ if } i \text{ is tight for } \mathbf{x}_t \}.$ If dim $(V_t) \le \frac{n}{8}$ then return FAIL
 - (3) Let $\boldsymbol{u}_t \sim S^{n-1} \cap V_t$.
 - (4) Let $r_t := \max\{r \ge 0 \mid x_t + r u_t \in K \text{ and } x_t r u_t \in K\}$
 - (5) Let $\delta_t := \min\{1, r_t\}$
 - (6) Update $\mathbf{x}_{t+1} := \mathbf{x}_t + \sigma_t \delta_t \mathbf{u}_t$ where $\sigma_t \sim \{-1, 1\}$
 - (7) IF x_{t+1} has at least n/2 coordinates tight THEN return x_{t+1} .
- (8) RETURN FAIL

The key modification compared to [LM12] is that the step length is not uniform but it is capped so that we do not exit *K*. As we walk orthogonal to the current point, one has $\|\mathbf{x}_{t+1}\|_2^2 = \|\mathbf{x}_t\|_2^2 + \delta_t^2$ for all *t* and so $\sum_t \delta_t^2 \le 4n$. Let us call a step *t long* if $\delta_t = 1$ and *short* otherwise. So we cannot have more than 4n many long steps. But in each short step (i.e. $\delta_r = r_t$) we have a probability of at least 1/2 that some coordinate or constraint becomes tight and this cannot happen more than $\frac{7}{8}n$ times before exiting at (2). So the probability of reaching (8) is less than $(1/2)^n$.

Now let \mathbf{x} be the random vector when the algorithm stops (either in (2), (7) or (8)). Fix a constraint i; the next step is to analyze the concentration behaviour of $\langle \mathbf{A}_i, \mathbf{x} - \mathbf{x}_0 \rangle$. In each iteration t one has $\|\langle \mathbf{A}_i, \sigma_t \delta_t \mathbf{u}_t \rangle\|_{\psi_2} \leq \|\langle \mathbf{A}_i, \mathbf{u}_t \rangle\|_{\psi_2} \leq \frac{\|\mathbf{A}_i\|_2}{\sqrt{n}}$ by Lemma 10 as \mathbf{u}_t is a unit vector from a subspace of dimension at least n/8. As there are at most 8n iterations, we have $\|\langle \mathbf{A}_i, \mathbf{x} - \mathbf{x}_0 \rangle\|_{\psi_2} \leq 8n \cdot (\frac{\|\mathbf{A}_i\|_2}{\sqrt{n}})^2 \leq \|\mathbf{A}_i\|_2$. Then $\Pr[|\langle \mathbf{A}_i, \mathbf{x}_0 - \mathbf{x} \rangle| \geq \lambda_i \|\mathbf{A}_i\|_2] \leq \exp(-C_1\lambda_i^2)$ by Lemma 10.(D) for some constant $C_1 > 0$. In other words, the probability that the *i*th constraint becomes tight at any point in the algorithm is upper bounded by $\exp(-C_1\lambda_i^2)$ and so with good probability the number of tight constraints remains below, say, $\frac{n}{4}$. This gives an upper bound on the probability to exit in (2). Finally, we define $\mathcal{D}(\mathbf{A}, \mathbf{x}_0)$ as the vector \mathbf{x}_{t+1} returned in (7), that means conditioned on the run being successful.

We briefly discuss the running time. In every of the at most 8*n* iterations we compute a basis of V_t which using fast matrix multiplication can be done in time $O(n^{\omega})$. Then to compute r_t (and to determine which constraints determine V_t) can be done in time O(nm). This results in the claimed bound of $T(m, n) \le O(n^{1+\omega} + n^2m)$.