GRASSMANNIANS IN THE LATTICE POINTS OF DILATIONS OF THE STANDARD SIMPLEX

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ABSTRACT. A remarkable connection between the cohomology ring $\mathrm{H}^*(\mathrm{Gr}(d, d + r), \mathbb{Z})$ of the Grasssmannian $\mathrm{Gr}(d, d + r)$ and the lattice points of the dilation $r\Delta_d$ of the standard d-simplex is investigated. The natural grading on the cohomology induces different gradings of the lattice points of $r\Delta_d$. This leads to different refinements of the Ehrhart polynomial $L_{\Delta_d}(r)$ of the standard d-simplex. We study two of these refinements which are defined by the weights $(1, 1, \ldots, 1)$ and $(1, 2, \ldots, d)$. One of the refinements interprets the Poincaré polynomial $\mathrm{P}(\mathrm{Gr}(d, d + r), z)$ as the counting of the lattice points which lie on the slicing hyperplanes of the dilation $r\Delta_d$. Therefore, on the combinatorial level the Poincaré polynomial of the Grassmannian $\mathrm{Gr}(d, d + r)$ is a refinement of the Ehrhart polynomial $L_{\Delta_d}(r)$ of the standard d-simplex Δ_d.

1. INTRODUCTION

Consider the diagonal sequence D_d of natural numbers realized from the Pascal triangle illustrated below

Pascal Triangle

One of the combinatorial interpretations of the terms of the sequence $D_d := \binom{r+d}{d}_{r=0}$, $d \in \mathbb{N}$, has to do with the counting of the lattice points associated with the dilations $r\Delta_d$ of the standard d-simplex Δ_d. By the standard d-simplex Δ_d we mean the convex hull of the set $\{\underline{0}, e_1, \ldots, e_d\}$ where $e_i's$, $1 \leq i \leq d$ are the standard vectors in $\mathbb{R}^d$ and $\underline{0}$ is the

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origin. That is,

$$(1.1) \qquad \Delta_d := \text{conv}(\underline{0}, e_1, \ldots, e_d) = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot e_i \geq 0, \ \sum_{i=1}^{d} \mathbf{x} \cdot e_i \leq 1\}$$

and the dilation $r\Delta_d$, is given by

$$(1.2) \qquad r\Delta_d = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot e_i \geq 0, \ \sum_{i=1}^{d} \mathbf{x} \cdot e_i \leq r, \ \ r \in \mathbb{N}\}$$

Lattice points are the points whose coordinates are integers. Asking for the lattice points on $r\Delta_d$ is tantamount to counting the integer solutions for the inequality

$$(1.3) \qquad \sum_{i=1}^{d} \mathbf{x} \cdot e_i \leq r$$

The number of lattice points on any given lattice polytope is well known. This is central theme of Ehrhart polynomials, [3], [6], [10], [11] and [16]. In fact the number of the lattice points on $r\Delta_d$ is given by

$$(1.4) \qquad |r\Delta_d \cap \mathbb{Z}_{\geq 0}^d| = \binom{r+d}{d}$$

and its generating function by

$$(1.5) \qquad \text{P}(r\Delta_d, z) = \sum_{r=0}^{\infty} A_r z^r = \frac{1}{(1-z)^{d+1}}, \ \text{where} \ \ A_r = \binom{r+d}{d}$$

On the other hand, Grassmannians are ubiquitous in nature and they constitute one of the best understood algebraic varieties. They admit algebraic, combinatorial and geometric structures which are very elegant. Their classical cohomology theory has taken the center stage in algebraic combinatorics in recent years, see [4], [5], [7] , [8], [9], [10] and [12]. It turns out that the lattice points on $r\Delta_d$ encode some vital information about the indexing partitions of the Schubert varieties contained in the Grassmannian $Gr(d, d + r)$. This sheds more light on the cohomology ring of the Grassmannian. It is well known that the multiplicative generators of the cohomology of the Grassmannian $\text{Gr}(d, d + r)$ are given by the special Schubert cycles σ_λ, see [3]. These cycles are indexed by one-row partitions $\lambda = (k), 1 \leq k \leq r$ and they constitute the total Chern class of the quotient bundle $\mathcal{Q}$, that is,

$$c(\mathcal{Q}) = 1 + \sigma_{\square} + \sigma_{\square\square} + \cdots + \sigma_{\underbrace{\square\square\cdots\square}_{1 \times r}}$$

We study the monomials identified with the semi standard tableaux of these one-row Young diagrams and realize a natural graded polynomial $T_r(t)$ called dilation polynomial. This is our first refinement of the Ehrhart polynomial $L_\Delta(r)$ of the standard d-simplex Δ_d. It comes with the natural weight $(1, 1, \ldots, 1)$. The second refinement is the the Poincaré polynomial $\mathbf{P}(Gr(d, d + r), z)$ of the Grassmannian $\text{Gr}(d, d + r)$ interpreted as the slicing of $r\Delta_d$ with hyperplanes with respect to the weight $(1, 2, \ldots, d)$. It is interesting to note that the natural grading on the cohomology of the Grassmannian $\text{Gr}(d, d + r)$ induces

different gradings of the lattice points of the dilation $r\Delta_d$ which give various refinements of the Ehrhart polynomial $L_\Delta(r)$. The paper is a generalisation of the previous studies in [1] and [2]. In section 2, we introduce a technique of counting lattice points by grading with respect to the weight $\mathbf{a} = (1, 1, \ldots, 1)$. This is just the slicing of the dilation $r\Delta_d$ into parallel regular $(d-1)$-simplices. The The polynomial

$$(1.6) \qquad T_r^{(1,\ldots,1)}(t) = \sum_{k=0}^{r} \binom{k+d-1}{d-1} t^k$$

refines the Ehrhart polynomial $L_\Delta(r)$. We give a generating function for such polynomials as r grows. This grading allows us to establish in Section 3, a bijection between the lattice points of the dilation $r\Delta_d$ and the semi standard tableaux of row Young diagrams indexing the special Schubert cycles of the Grassmanninan $\text{Gr}(d, d+r)$. By using another weight $\mathbf{h} = (1, 2, \ldots, d)$ which gives a different slicing of the simplex, we construct a polynomial

$$(1.7) \qquad P_{r\Delta_d}^{(1,2,\ldots,d)}(z) = \left[\binom{k+d}{d} \right]_z \quad \text{for } 0 \le k \le r$$

which is a z-binomial coefficient. This gives a bijection between the lattice points in $r\Delta_d$ and partitions fitting into an $r \times d$ rectangle, and establishes that the grading given here to a lattice point eventually identifies this polynomial with the Poincaré polynomial of the Grassmannian $\text{Gr}(d, d+r)$.

2. The Dilation Polynomial $T_{r\Delta_d}, r \ge 1$

We define the lexicographical order $<_{\text{lex}}$ on the set $r\Delta_d \cap \mathbb{Z}_{\ge 0}^d$ of lattice points on $r\Delta_d$ as follows: Let $\mathbf{a} = (a_1, \ldots, a_d)$ and $\mathbf{b} = (b_1, \ldots, b_d)$ be any two lattice points in $r\Delta_d \cap \mathbb{Z}_{\ge 0}^d$. We say $\mathbf{a} <_{\text{lex}} \mathbf{b}$ if, in the integer coordinate difference $\mathbf{a} - \mathbf{b} \in \mathbb{Z}^d$, the leftmost nonzero entry is negative. As noted earlier, the set $r\Delta_d \cap \mathbb{Z}_{\ge 0}^d$ of lattice points on $r\Delta_d$ is the integer solution set of the inequality (1.3). It turns out that the upper bound r in (1.3) defines a relation on the lattice points of the solution set which brings about the disjoint subdivisions of the integer solution set.

Proposition 2.1. *Let $\mathbf{a}$ and $\mathbf{b}$ be two lattice points in $r\Delta_d \cap \mathbb{Z}_{\ge 0}^d$ such that $\mathbf{a} <_{\text{lex}} \mathbf{b}$. The relation $\mathbf{a} \sim \mathbf{b}$ defined by $\sum_{i=1}^{d}(a_i - b_i) = 0$ is an equivalence relation.*

The relation partitions the set $r\Delta_d \cap \mathbb{Z}_{\ge 0}^d$ into disjoint equivalence classes. Notice that the integer solution set is complete with respect to the bound r in the sense that the sum of integer coordinates of the lattice points in $r\Delta_d \cap \mathbb{Z}_{\ge 0}^d$ takes all the values of the integers in the closed interval $[0, r]$. Completeness is one of the beautiful properties of the standard d-simplex not all the lattice polytopes enjoy this feature.

Corollary 2.2. *Any two lattice points in $r\Delta_d \cap \mathbb{Z}_{\ge 0}^d$ belong to the same class if and only if they share the same sum of their respective integer coordinates.*

Corollary 2.3. $|r\Delta_d \cap \mathbb{Z}_{\ge 0}^d / \sim| = r + 1$

Proof. This follows corollary 2 and the fact that $r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$ is complete

$$(2.1) \qquad r\Delta_d \cap \mathbb{Z}^d_{\geq 0}/ \sim := \{X_k : \sum_{i=1}^{d} x_i = k,\ 0 \leq k \leq r, \forall x = (x_1, \ldots, x_d) \in X_k\}$$

and hence, $|r\Delta_d \cap \mathbb{Z}^d_{\geq 0}/ \sim | = r + 1$

$\square$

Corollary 2.4. *The class of the origin $\underline{0} \in r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$ is a singleton set.*

Proof. The class of the origin denoted by X_0 is given by

$$(2.2) \qquad X_0 = \{\underline{x} = (x_1, \ldots, x_d) \in r\Delta_d \cap \mathbb{Z}^d_{\geq 0} : \sum_{i=1}^{d} x_i = 0\}$$

Suppose that there is a lattice point $\mathbf{a}$ which belongs to X_0 such that $\mathbf{a}$ is not the origin. Since the origin $\underline{0}$ is $<_{\text{lex}}$ than every lattice point $\mathbf{a} \in r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$, so, $\underline{0} \sim \mathbf{a}$ implies that $\sum_{i=1}^{d}(0 - a_i) < 0$, This integer value is not in $[0, r]$, therefore, there is no lattice point $r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$ which is equivalent to the origin apart from itself hence $|X_0| = 1$ $\square$

We now compute the size of each of the equivalence classes X_k such that $0 \leq k \leq r$.

Theorem 2.5. *Let $\mathcal{A} = r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$ denote the set of lattice points on $r\Delta_d$ and let $X_k \subset \mathcal{A}$ be the collection of lattice points whose sum of their integer coordinates is k such that $0 \leq k \leq r$. Then $|X_k| = \binom{k+d-1}{d-1}$.*

Proof. Notice that the chain of the following inclusions

$$\{(0, \ldots, 0)\} \subset \Delta_d \cap \mathbb{Z}^d_{\geq 0} \subset 2\Delta_d \cap \mathbb{Z}^d_{\geq 0} \cdots \subset r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$$

implies the following chain.

$$\Delta_d \cap \mathbb{Z}^d_{>0} \subset 2\Delta_d \cap \mathbb{Z}^d_{>0} \subset \cdots \subset r\Delta_d \cap \mathbb{Z}^d_{>0}$$

The subcollection X_k is given by

$$X_k = \{\underline{x} = (x_1, \ldots, x_d) \in \mathcal{A} : \sum_{i=1}^{d} x_i = k,\ \ 0 \leq k \leq r\}$$

$X_0 = \{(0, \ldots, 0)\}$, so $|X_k| = 1$. Observe that

$$X_k = k\Delta_d \cap \mathbb{Z}^d_{\geq 0}/(k-1)\Delta_d \cap \mathbb{Z}^d_{\geq 0},\ \ 2 \leq k \leq r$$

In fact, $X'_k s$ define a partition of the set $r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$ of the lattice points on $r\Delta_d$, that is,

$$\bigcap_{k=0}^{r} X_k = \emptyset, \qquad \bigcup_{k=0}^{r} X_k = \mathcal{A}$$

From Ehrhart theory, using 1.4,

$$|\Delta_d \cap \mathbb{Z}^d_{\geq 0}| = \binom{1+d}{d} = |X_0 \cup X_1|.$$

This implies that $|X_1| = d$, Similarly,

$$|2\Delta_d \cap \mathbb{Z}_{\geq 0}^d| = \binom{2+d}{d} = |X_0 \cup X_1 \cup X_2|.$$

This gives

$$|X_2| = \binom{2+d}{d} - d - 1 = \binom{1+d}{d-1}$$

Continuing this way,

$$|X_k| = \binom{k+d}{d} - \sum_{j=1}^{k} \binom{k+d-j}{d} = \binom{k+d-1}{d-1}$$

$\square$

The disjoint union $\cup X_k$ of subcollections X_k, $0 \leq k \leq r$ of the set $r\Delta_d \cap \mathbb{Z}_{\geq 0}^d$ of lattice points on $r\Delta_d$ defines a polynomial $T_r(t)$ of degree r in variable t given by

$$(2.3) \qquad T_r(t) = \sum_{k=0}^{r} \binom{k+d-1}{d-1} t^k$$

We call $T_r(t)$ the dilation polynomial of degree r identified with the dilation $r\Delta_d$. This is precisely the slicing of $r\Delta_d$ with hyperplanes perpendicular to the direction $\mathbf{a} := (1, \ldots, 1)$ and enumerate all the lattice points in the different layers. That is,

$$(2.4) \qquad \binom{k+d-1}{d-1} = \#\{v \in r\Delta_d \cap \mathbb{Z}_{\geq 0}^d : v \cdot \mathbf{a} = k, 0 \leq k \leq r\}$$

The dilation polynomial $T_4(t)$ for the 4th dilation of the standard 3-simplex is illustrated in Figure 2.

Remark 2.6. Dilation polynomials identified with $r\Delta_2$ and $r\Delta_3$ are called triangular and tetrahedral polynomials respectively

Theorem 2.7. *Let $\mathcal{M} = \{T_r(t)\}_{r=0}$ be the sequence of dilation polynomials of lattice points counting on $r\Delta_d$ for $r \geq 0$. Then its generating series $G(t, z) = \sum_{r=0} T_r(t) z^r$ is given by*

$$G(t, z) = \frac{z}{(1-z)(1-tz)^d}.$$

Proof. Notice from the equation (2.3) that

$$T_r(t) = T_{r-1}(t) + \frac{(r+1)\cdots(r+d-1)}{(d-1)!} t^r \quad \text{and} \quad \sum_{r\geq 0} \frac{(r+1)\cdots(r+d-1)}{(d-1)!} z^r = \frac{1}{(1-z)^d}.$$

$$G(t, z) = \sum_{r\geq 0} T_r(t) z^r = \sum_{r\geq 0} \left[T_{r-1}(t) + \frac{(r+1)\cdots(r+d-2)}{(d-1)!} t^{r-1} \right] z^r.$$

$$G(t, z) = z G(t, z) + \sum_{r\geq 1} \left[\frac{(r+1)\cdots(r+d-1)}{(d-1)!} t^{r-1} \right] z^r, \quad \text{and so}$$

$$G(t, z) = \frac{z}{(1-z)(1-tz)^d}.$$

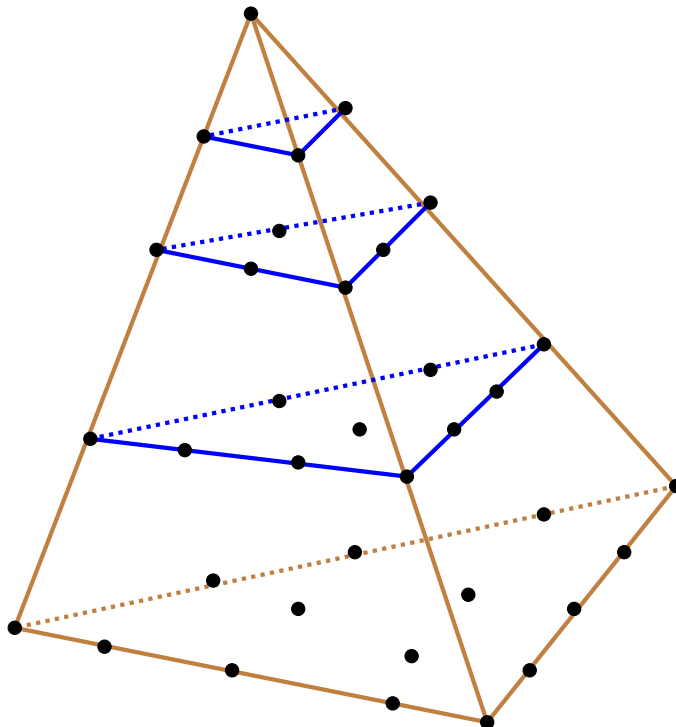$\square$

FIGURE 1. $T_4(t) = 1 + 3t + 6t^2 + 10t^3 + 15t^4$

3. The Cohomology ring of Grassmannian $Gr(d, d+r)$

Let V be an n-dimensional complex vector space. The set of all maximal chains of subspaces in V is called the flag variety $\mathcal{F}\ell_n(\mathbb{C})$ of dimension $\frac{n(n-1)}{2}$. That is,

$$\mathcal{F}\ell_n(\mathbb{C}) := \{V_\bullet := \{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_n = V \ \text{ such that } \ \dim V_i = i\}$$

The Grassmannian $Gr(d, n)$ is the spacial case of the flag variety being the set of all d-dimensional subspaces in V. Its dimension is $d(n-d)$. There is a projection

$$\pi : \mathcal{F}\ell_n(\mathbb{C}) \longrightarrow Gr(d, n)$$

from the full flag variety $\mathcal{F}\ell_n(\mathbb{C})$ to the Grassmannian $Gr(d, n)$ with $\pi^{-1}(X_\lambda(F_\bullet)) = X_{w(\lambda)}(F_\bullet)$, where $X_\lambda(F_\bullet)$ is a Schubert variety in the Grassmannian $Gr(d, n)$ defined as the closure of the Schubert cell $C_\lambda(F_\bullet)$ given by

$$C_\lambda(F_\bullet) = \{V_d \in Gr(d, n) : \dim V_d \cap F_{n+i-\lambda_i} = i, \ 1 \leq i \leq d\},$$

with respect to the fixed flag $F_\bullet$:

$$F_\bullet := \{0\} \subset F_1 \subset F_2 \subset \cdots \subset F_n = V \ \text{ such that } \ \dim F_i = i$$

The partition λ is called fitted in the sense that it has at most length d and each part cannot exceed $n - d$. The permutation $w(\lambda)$ identified with the partition $\lambda = (\lambda_1, \ldots, \lambda_d)$ is given by

(3.1) $$w_i = i + \lambda_{d+1-i}, \ 1 \leq i \leq d \ \text{and} \ w_j < w_{j+1}, \ d+1 \leq j \leq n.$$
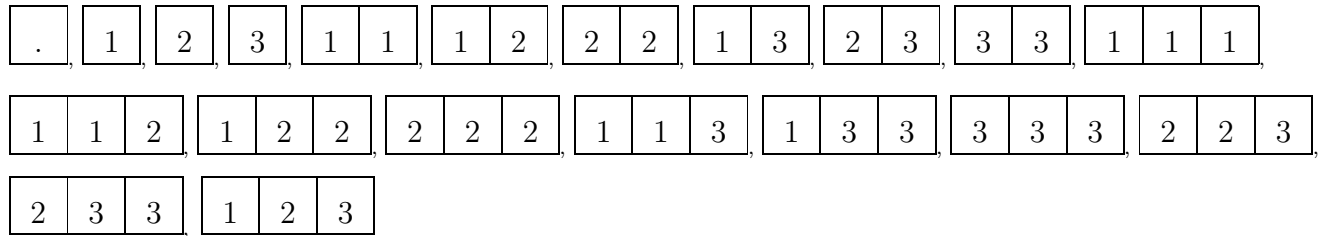
This permutation is called Grassmannian in that it has a unique descent by definition. Every such permutation has the code $c(w(\lambda))$ of the form $(w_1-1, w_2-2, \ldots, w_d-d, 0, \ldots, 0)$ which can be represented by $(m_1, m_2, \ldots, m_d)$ by disregarding the string of zeros at the right hand. It turns out that the partition λ indexing the Schubert variety X_λ can be recovered from this code as $\lambda = (m_{i_1}, m_{i_2}, \ldots, m_{i_d})$ where $m_{i_1} \geq m_{i_2} \geq \cdots \geq m_{i_d}$ and $m_{i_p} \neq 0$, $1 \leq i_p \leq d$. Recall that for any permutation w in the symmetric group S_n, the code $c(w)$ of w is the sequence $(c_1(w), \ldots, c_n(w))$ where $c_i(w) = |\{j : 1 \leq i < j \leq n \text{ and } w(i) > w(j)\}|$. For instance the code $c(w)$ of the permutation $w = 315426 \in S_6$ is $(2, 0, 2, 1, 0, 0)$. The string of zeros at the right hand may be discarded. Notice that $c_i(w) \leq n - i$. The length $\ell(w)$ of w is $\#\{(i,j) : w(i) > w(j), 1 \leq i < j \leq n\}$, the number of inversions in w, that is, the sum of integer coordinates of the code of w. It is well known that the cohomology ring of the Grassmannian $Gr(d, n)$ is generated by the Schubert cycles σ_λ. These are Poincaré dual of the fundamental classes in the homology of Schubert varieties. The Grassmannian $\mathrm{Gr}(d, n)$ admits many important vector bundles, most importantly there is a universal short exact sequence: $0 \longrightarrow \mathcal{S} \longrightarrow \mathbb{C}^n \times \mathrm{Gr}(d, n) \longrightarrow \mathcal{Q} \longrightarrow 0$ of bundles on $Gr(d, n)$ which makes it easy to compute the Chern class $c(\mathcal{Q})$ of the quotient bundle $\mathcal{Q}$ on the Grassmannian $Gr(d, n)$. Recall that $\mathcal{Q}$ is a globally generated vector bundle of rank $r := n - d$ and all its global sections are from the trivial bundle $\mathbb{C}^{d+r} \times \mathrm{Gr}(d, d+r)$. The total Chern class is the sum over all the one-row partitions inside the rectangle $\square_{r \times d}$. That is,

$$(3.2) \qquad c(\mathcal{Q}) = 1 + \sigma_{\square} + \sigma_{\square\square} + \cdots + \sigma_{\square\square \cdot \square_{1 \times r}}$$

It turns out that the set of all one-row Young diagrams indexing the multiplicative generators of the cohomology of the Grassmannian $\mathrm{Gr}(d, d+r)$ is deeply connected with the lattice points of $r\Delta_d$. Let $\mathcal{C}_{d,r}$ be the set of row Young diagrams with at most r boxes and adjoin the empty set ϕ. That is,

$$\mathcal{C}_{d,r} = \{\square_{1 \times k} : 1 \leq k \leq r\} \cup \emptyset.$$

The filiing of the boxes of the row Young diagrams in $\mathcal{C}_{d,r}$ using the numbers in $[d] := \{1, \ldots, d\}$ is semi standard, that is, the numbers weakly increase from the left to the right. We denote the collection of all such fillings by $\mathcal{C}^d_{d,r}$ and call it the d-filling set of the dilation $r\Delta_d$. For instance, the 3-filling set $\mathcal{C}^3_{3,3}$ associated the second dilation $3\Delta_3$ of the standard 3-simplex is the following collection

$$\boxed{.} \;,\; \boxed{1} \;,\; \boxed{2} \;,\; \boxed{3} \;,\; \boxed{1\,1} \;,\; \boxed{1\,2} \;,\; \boxed{2\,2} \;,\; \boxed{1\,3} \;,\; \boxed{2\,3} \;,\; \boxed{3\,3} \;,\; \boxed{1\,1\,1} \;,$$

$$\boxed{1\,1\,2} \;,\; \boxed{1\,2\,2} \;,\; \boxed{2\,2\,2} \;,\; \boxed{1\,1\,3} \;,\; \boxed{1\,3\,3} \;,\; \boxed{3\,3\,3} \;,\; \boxed{2\,2\,3} \;,$$

$$\boxed{2\,3\,3} \;,\; \boxed{1\,2\,3}$$

These 20 semi standard Young tableaux can be organized in terms of their defining Young diagrams. It turns out that this arrangement can be expressed as a polynomial, given by $T_3(t) = 1 + 3t + 6t^2 + 10t^3$. This is the graded semi-standard polynomial of degree 3 illustrated in Figure 2.
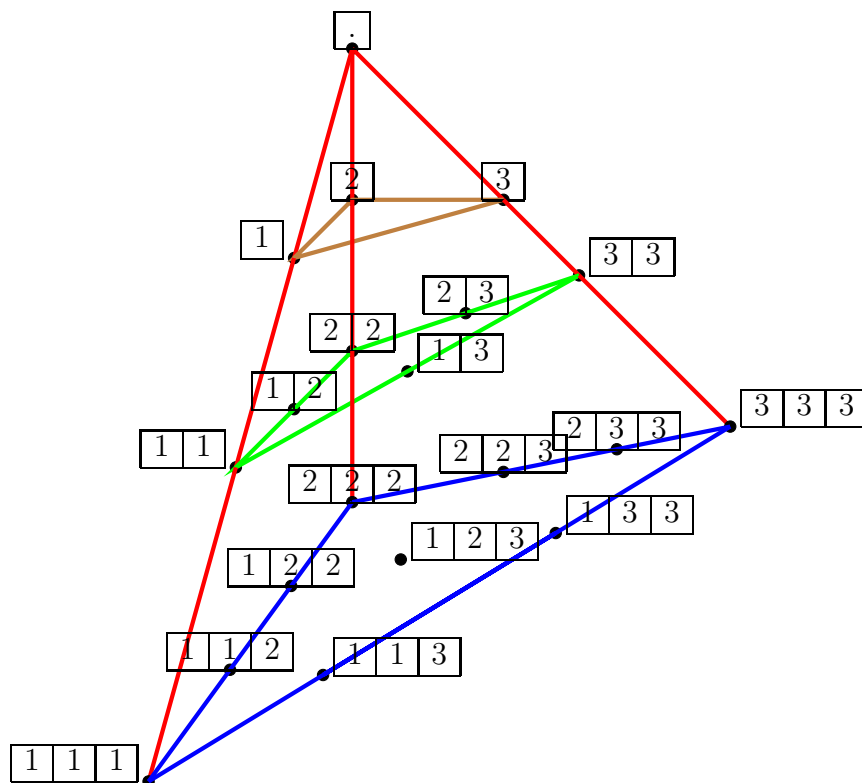
FIGURE 2. $T_3(t) = P_3(t) = 1 + 3t + 6t^2 + 10t^3$

Theorem 3.1.   (i.) *The size $L^d(r)$ of the d-filling set $\mathcal{C}^d_{d,r}$ is $\binom{r+d}{d}$ and the sequence $(L^d(r))^\infty_{r=0}$ of cardinalities as r grows is recorded by the generating function*

$$P(\mathcal{C}^d_{(d,r)}, z) = \frac{1}{(1-z)^{d+1}}$$

(ii.) *More is true, there is a graded counting polynomial of the semi standard tableaux in $\mathcal{C}^d_{d,r}$ given by*

$$P_r(t) = \sum_{k=0}^{r} \binom{k+d-1}{d-1} t^k$$

that is, a k-box row diagram gives $\binom{k+d-1}{d-1}$ semi standard Young tableaux. This has a generating function

$$G(t,z) = \frac{z}{(1-z)(1-tz)^d}.$$

Theorem 3.2. *There is a bijection $T \mapsto v(T)$ between the set $\mathcal{C}^d_{d,r}$ and the set $r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$ of the lattice points of the dilation $r\Delta_d$. Furthermore, the semi-standard polynomial $P_r(t)$ is precisely the dilation polynomial $T_r(t)$ identified with $r\Delta_d$.*

Proof. To each semi standard tableau $T \in \mathcal{C}^d_{(d,r)}$ there exists a unique exponent vector $v(T) := (v(T)_1, \ldots, v(T)_d)$ in which the coordinate $v(T)_j$ is the number of appearances of j in T, $1 \leq j \leq d$. Thi is a bijection.

The number of semi standard fillings of each of the row diagram with shape $\lambda = (k)$, $0 \leq k \leq r$ using the elements of the set $\{1, \ldots, d\}$ has a well known closed formula. Notice that for a fixed point $\mathbf{a} = (1, \ldots, 1)$ the following identity holds

$$\prod_{1 \leq i < j \leq d} \frac{\lambda_i - \lambda_j + j - i}{j - i} = \binom{k + d - 1}{d - 1} = \#\{v \in r\Delta_d \cap \mathbb{Z}^d_{\geq 0} : v \cdot \mathbf{a} = k, 0 \leq k \leq r\}$$

Therefore, the semi-standard polynomial $P_r(t)$ can be viewed as the dilation polynomial $T_r(t)$. The bijection is a polynomial preserving map, see figure 2 □

4. GRASSMANNIAN MONOMIALS

It is clear from the Theorem 3.1 that every standard tableau $T \in \mathcal{C}^d_{(d,r)}$ defines a monomial $\mathbf{t}^{v(T)}$ where $v(T) := (v(T)_1, \ldots, v(T)_d)$, that is,

$$(4.1) \qquad \mathbf{t}^{v(T)} := \prod_{j=1}^{d} t_j^{\# \text{ times j appears in T}}, \quad \text{where} \quad v(T) \in r\Delta_d \cap \mathbb{Z}^d_{\geq 0}$$

For instance, the monomial defined by $T = $ | 1 | 1 | 2 | 3 | 3 | $\in \mathcal{C}^4_{(4,5)}$ is given by $\mathbf{t}^{\mathbf{a}} = t_1^2 t_2 t_3^2$ where $\mathbf{a} = (2, 1, 2, 0)$. We call such monomials in $\mathcal{C}^d_{(d,r)}$ Grassmannian because they encode the data of indexing partitions of Schubert varieties in the Grassmannian $\mathrm{Gr}(d, d + r)$. We denote these monomials by W^r_d, that is,

$$W^r_d := \{t_1^{a_1} \cdots t_d^{a_d} : \sum_{i=1}^{d} a_i \leq r, \ 0 \leq a_i \leq r\}$$

Proposition 4.1. *Let W^r_d and $W^{r'}_d$ be two Grassmannian monomial sets such that $r \leq r'$. Then $W^r_d \subseteq W^{r'}_d$.*

Proposition 4.2. *Every monomial $\mathbf{t}^{\mathbf{a}} \in \mathbb{Z}[t_1, \ldots, t_d]$ is Grassmannian.*

Proof. It suffices to produce a Grassmannian set W^r_d containing $\mathbf{t}^{\mathbf{a}}$. By (4.1) there is a semi-standard tableau T which encodes the exponent vector $\mathbf{a}$ and this implies that there exists $r \in \mathbb{N}$ such that T is an element of the d- filling set $\mathcal{C}^d_{(d,r)}$, so $\mathbf{t}^{\mathbf{a}}$ belongs to the Grassmannian monomial set W^r_d. □

Corollary 4.3. *If $r = \sum_{i=1}^{d} a_i$, where a_i is an integer coordinate of $\mathbf{a}$ then the Grassmannian set W^r_d is the smallest set containing the monomial $\mathbf{t}^{\mathbf{a}}$.*

It is important to quickly point out that the sum $P_r(t_1, \ldots, t_d)$ of all the monomials in W^r_d, that is,

$$(4.2) \qquad P_r(t_1, \ldots, t_d) = \sum_{T \in \mathcal{C}^d_{(d,r)}} \prod_{j=1}^{d} t_j^{\# \text{ times j appears in T}}$$

is deeply connected with a polynomial representation (V, ρ) of the general linear group $GL_d(\mathbb{C})$ where $V := \bigoplus_{k=0}^{r} \mathrm{Sym}^k(\mathbb{C}^d)$ is the space of the direct sum of homogeneous symmetric polynomials of dgree k in d variables. Let $\mathbb{C}[X] := \mathbb{C}[x_{11}, x_{12} \ldots, x_{dd}]$ be the ring of polynomial functions on $d \times d$ matrices. There is an action of $G = GL_d(\mathbb{C})$ on $\mathbb{C}[X]$ by conjugation. The character of the polynomial representation (V, ρ) is the polynomial $\chi_\rho \in \mathbb{C}[X]$ given by the trace of the matrix $\rho(X)$. Recall that the character χ_ρ of every polynomial representation (V, ρ) lies in the invariant ring $\mathbb{C}[X]^G$. Interested reader can consult [15] and [17].

Theorem 4.4. *The character χ_V of $V := \bigoplus_{k=0}^{r} \mathrm{Sym}^k(\mathbb{C}^d)$ as a polynomial representation ρ of the general linear group $GL_d(\mathbb{C})$ is $P_r(t_1, \ldots, t_d)$, that is,*

$$\chi_V = \sum_{T \in \mathcal{C}_{(d,r)}^d} \prod_{j=1}^{d} t_j^{\#\ \text{times j appears in T}}$$

The sum ranging over all the semi standard fillings of the row diagrams with at most r boxes.

Proof. Let $t_1, \ldots, t_d$ be eigenvalues of a generic $d \times d$ matrix X. The map $\mathbb{C}[X]^G \longrightarrow \mathbb{C}[t_1, t_2, \ldots, t_d]^{\mathcal{S}_n}$ defined by $f \mapsto f(\mathrm{diag}(t_1, \ldots, t_d))$ is an isomorphism. Set $\lambda = (k)$ since $k's$ define the rows diagrams with at most r boxes, so the image of the character $f_\rho(X)$ is

$$\sum_{k=0}^{r} \frac{\det(t_i^{\lambda_i + d - j})_{1 \leq i,j \leq d}}{\det(t_i^{d-j})_{1 \leq i,j \leq d}}.$$

$\square$

Corollary 4.5. *The dimension of the vector space $V := \bigoplus_{k=0}^{r} \mathrm{Sym}^k(\mathbb{C}^d)$ is $\chi_V(1, 1, \ldots, 1) := |r\Delta_d \cap \mathbb{Z}_{\geq 0}|$, the number of lattice points of the dilation $r\Delta_d$.*

Proof. The Grassmannian set W_d^r spans the vector space $V := \bigoplus_{k=0}^{r} \mathrm{Sym}^k(\mathbb{C}^d)$. $\square$

Now to every monomial $\mathbf{t}^{\mathbf{a}} \in \mathbb{Z}[t_1, \ldots, t_d]$ we associate a weight $w_{\mathbf{a}}$ defined by

$$(4.3) \qquad\qquad\qquad\qquad w_{\mathbf{a}} = \sum_{k=1}^{d} k a_k$$

It turns out that $w_{\mathbf{a}}$ admits two important partitions $\lambda, \lambda^* \vdash w_{\mathbf{a}}$ which can be identified with the monomial $\mathbf{t}^{\mathbf{a}}$. These partitions, λ and λ^* are called α-partition and β-partition respectively. A partition $\lambda \vdash w_{\mathbf{a}}$ is said to be the α-partition of the monomial $t_1^{a_1} \cdots t_d^{a_d}$ if the number of parts of size i in λ is a_i, $1 \leq i \leq d$. The length $\ell(\lambda)$ of α-partition is $a_1 + \cdots + a_d$. The β partition $\lambda^* = (\lambda_1^*, \ldots, \lambda_d^*)$ of $w_{\mathbf{a}}$ is such that $\lambda_k^* = \sum_{i \geq k}^{d} a_i$, $1 \leq k \leq d$ and its length is d. For instance, the α-partition associated with the monomial $t_1^3 t_2^2 t_3^3 t_4^2 \in \mathbb{Z}[t_1, t_2, t_3, t_4]$ is $(4,4,3,3,3,2,2,1,1,1)$ while its β partition i λ^* is $(10, 7, 5, 2)$. In fact, α and β partitions identified with the monomial $\mathbf{t}^{\mathbf{a}}$ can be realized in terms of the sum of the entries of the $d \times d$ upper triangular matrix $\mathrm{M}_{\mathbf{a}}$ associated with the exponent vector $\mathrm{a} = (a_1, \ldots, a_d)$ of the monomial, that is,

$$(4.4) \qquad M_{\mathbf{a}} = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_d \\ & a_2 & a_3 & \cdots & a_d \\ & & a_3 & \cdots & a_d \\ & & & & \vdots \\ & & & & a_d \end{bmatrix}$$

The sum of the entries in the column k divided by k is the number of parts of size k in the α- partition λ of $w_{\mathbf{a}}$. The β partition $\lambda^* = (\lambda_1^* \ldots \lambda_d^*)$ of $w_{\mathbf{a}}$ is such that λ_k^* is the sum of the entries in the row k where $1 \leq k \leq d$. For instance, the 4×4 matrix $M_{\mathbf{a}}$ corresponding to the monomial $t_1^3 t_2^2 t_3^3 t_4^2 \in \mathbb{Z}[t_1, t_2, t_3, t_4]$ is

$$M_{\mathbf{a}} = \begin{bmatrix} 3 & 2 & 3 & 2 \\ & 2 & 3 & 2 \\ & & 3 & 2 \\ & & & 2 \end{bmatrix}$$

so the α-partition λ and the β-partition λ^* identified with the matrix $M_{\mathbf{a}}$ are $1^3 2^2 3^3 4^2$ and $(10, 7, 5, 2)$ respectively.

Proposition 4.6. *Let λ be the α-partition of $w_{\mathbf{a}}$ associated with the monomial $\mathbf{t^a} = t_1^{a_1} \cdots t_d^{a_d} \in \mathbb{Z}[t_1, \ldots, t_d]$. Then its corresponding β-partition λ^* is the transpose of λ and vice versa.*

Proof. Let $\lambda = (\lambda_1, \ldots, \lambda_{a_1 + \cdots + a_d})$ and $\lambda^* = (\lambda_1^*, \ldots, \lambda_d^*)$. It is obvious that these partitions satisfy the following identity

$$\sum_{k=1}^{a_1 + \cdots + a_d} (2k - 1)\lambda_k = \sum_{k=1}^{d} \lambda_k^{*2}.$$

$\square$

It would be interesting to characterize and study all the monomials for which α-partition and β-partition coincide. This amounts to the characterization of all self conjugate partitions. Recall that for all $n \in \mathbb{N}$ such that $n > 2$ there is a bijection between the set of self conjugate partitions of n and the set of all distinct odd parts partitions of n. For instance, a square free monomial of the form $t_1 t_2 \cdots t_d$ admits the stair case partition $(d, d-1, \ldots, 1)$, this is deeply connected with the distribution of triangular numbers in the set $\mathbb{N}$ of natural numbers. We give a few other examples of such monomials.

Example 4.7. Some monomials for which α and β-partitions coincide:

(i.) All monomials of the form $t_{\frac{d}{2}}^{\frac{d}{2}} t_d^{\frac{d}{2}} \in \mathbb{Z}[t_1, t_2, \ldots, t_d]$ for even d.

(ii) All monomials of the form $t_1 t^{d-2} t_d \in \mathbb{Z}[t_1, t_2, \ldots, t_d]$.

(iii.) All monomials of the form $t_1^{d-1} t_d \in \mathbb{Z}[t_1, t_2, \ldots, t_d]$.

(iv.) All monomials of the form $t_{d-2}t_{d-1}t_d^{d-2} \in \mathbb{Z}[t_1, t_2, \ldots, t_d]$.

Lemma 4.8. *Let $\lambda^* = (\lambda_1, \ldots, \lambda_d)$ be the β-partition identified with the monomial $t_1^{a_1} \cdots t_d^{a_d} \in \mathbb{Z}[t_1, t_2, \ldots, t_d]$. Then the exponent vector $(a_1, \ldots, a_d)$ is equivalent to $(\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{d-1} - \lambda_d, \lambda_d)$.*

Proof. It follows from the construction of the β partition λ^* from the exponent vector $(a_1, \ldots, a_d)$. $\qquad\square$

Theorem 4.9. *Let $\mathbf{t^a} \in W_d^r$ be a Grassmannian monomial associated with exponent vector $\mathbf{a} \in r\Delta_d \cap \mathbb{Z}_{\geq 0}^d$. If a partition λ^* is the β-partition identified with $\mathbf{t^a}$ then the length $\ell(w(\lambda^*))$ of the Grassmannian permutation $w(\lambda^*)$ is the weight $w_\mathbf{a}$.*

Proof. The code $c(w(\lambda^*))$ of the Grassmannian permutation $w(\lambda^*)$ is of the form $(m_1, m_2, \ldots, m_d, 0, 0, \ldots, 0)$. The rearrangement of $m_1, m_2, \ldots, m_d$ in weakly decreasing order yields the fitted partition $\lambda^* = (\lambda_1^*, \ldots, \lambda_d^*)$. The sum of the entries of the code $c(w) = (c_1(w), c_2(w), \ldots, c_n(w))$ of any permutation w is the length $\ell(w)$ of the partition, since each entry $c_i(w)$ is the number of inversions associated to the value w_i in the position i. Hence the length $\ell(w(\lambda^*))$ of $w(\lambda^*)$ is the size $|\lambda^*|$ of λ^*. Next we show that the weight $w_\mathbf{a}$ of the exponent vector $\mathbf{a} = (a_1, \ldots, a_d)$ of the Grassmannian monomial $\mathbf{t^a} = t_1^{a_1} \cdots t_d^{a_d}$ is $|\lambda^*|$. From Lemma 3.12 $a_i = \lambda_i^* - \lambda_{i+1}^*$, $1 \leq i \leq d-1$, $a_d = \lambda_d^*$. Therefore, the weight $w_\mathbf{a} = \sum_{i=1}^{d-1} i(\lambda_i^* - \lambda_{i+1}^*) + d\lambda_d^* = |\lambda^*|$. $\qquad\square$

Corollary 4.10. *Every β-partition λ^* identified with each of the monomials $\mathbf{t^a} \in W_d^r$ fits into the $r \times d$ rectangle $\square_{r \times d}$.*

Proof. It is sufficient to establish that the parts of λ^* cannot exceed r and the length $\ell(\lambda^*)$ of λ^* is d. Notice that the exponent vector $\mathbf{a}$ is a lattice point of $r\Delta_d$ and by definition $a_1 + \cdots + a_d \leq r$. Therefore each part λ_k^* of λ^* is at most r and length $\ell(\lambda^*)$ is d by the definition of λ^*. $\qquad\square$

Corollary 4.11. *The set of β-partitions λ^* identified with monomials in W_d^r index the Schubert varieties in the Grassmannian $Gr(d, d + r)$, giving a bijection between lattice points in $r\Delta_d$ and partitions fitting into an $r \times d$ rectangle.*

The weight $w_\mathbf{a}$ defined in the equation (3.2) gives another refinement $P_{r\Delta_d}^h(z)$ of the Ehrhart polynomial of $r\Delta_d$ with respect to a fixed point $h = (1, 2, \ldots d)$.

$$(4.5) \qquad\qquad P_{r\Delta_d}^h(z) = \sum_{m=0}^{dr} A_m z^m.$$

where $A_m = \#\{\mathbf{a} \in r\Delta_d \cap \mathbb{Z}_{\geq 0}^d : \mathbf{a} \cdot h = m, \ 0 \leq m \leq dr\}$, that is, the number of exponent vectors $\mathbf{a}$ which share the weight m. We call $P_{r\Delta_d}^h(z)$ the weighted polynomial associated with the dilation $r\Delta_d$.

Lemma 4.12. *The polynomial $P_{r\Delta_d}^h(z) = \sum_{m=0}^{dr} A_m z^m$ specializes at $z = 1$ to the Ehrhart polynomial $L_{\Delta_d}(r)$.*

Remark 4.13. Notice that A_m is precisely the number of lattice points in the intersection of the dilation $r\Delta_d$ with the hyperplane H_m perpendicular to the direction $\mathbf{h} := (1, 2, \ldots, d)$. It is also interesting to note that the grading given here to a lattice point eventually identifies the weighted polynomial $P^h_{r\Delta_d}(z)$ with the Poincaré polynomial of the Grassmannian $\mathrm{Gr}(d, d+r)$.

Theorem 4.14. *Let $P^h_{r\Delta_2}(z)$ be the weighted polynomial of the lattice points of the dilation $r\Delta_d$. Then the Poincaré polynomial $P(Gr(d, d+r), t)$ of the Grassmannian $Gr(d, d+r)$ coincides with the weighted polynomial $P^h_{r\Delta_d}(z)$.*

Proof. It is well known from the Borel presentation of the cohomology ring $H^*(Gr(d, d+r), \mathbb{Z})$ of the Grassmannian $\mathrm{Gr}(d, d+r)$ that the Poincaré polynomial $\mathrm{P}(\mathrm{Gr}(d, d+r), t)$ is given by the following Gaussian polynomial

$$\frac{(1-t)(1-t^2)\cdots(1-t^{d+r})}{(1-t)\cdots(1-t^d)(1-t)\cdots(1-t^r)}$$

This is combinatorially simplified as

$$\sum_{\lambda \subseteq \square_{d \times r}} t^{|\lambda|}$$

where $|\lambda|$ is the number of boxes in the Young diagram of shape λ. The size $|\lambda|$ coincides with the length $\ell(w(\lambda))$ (the number of inversions) of the Grassmannian permutation $w(\lambda)$ identified with λ in the equation (3.1). Notice that $|\lambda| \leq dr$, therefore, It follows from the Theorem 4.9 that $|\lambda|$ is the weight $w_{\mathbf{a}}$ of the monomial $t^{\mathbf{a}} \in W_d^r$, $a \in r\Delta \cap \mathbb{Z}_{\geq 0}^d$, therefore, $\sum_{\lambda \subseteq \square_{2 \times r}} t^{|\lambda|}$ is precisely the polynomial $\sum_{m=0}^{dr} A_m z^m$. $\square$

Question: Does the set $r\Delta_d \cap \mathbb{Z}_{\geq 0}^d$ encode some data about the degree and the Hilbert polynomial of $\mathrm{Gr}(d, d+r)$?

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