On the size of matchings in 1-planar graph with high minimum degree*

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Abstract

A matching of a graph is a set of edges without common end vertex. A graph is called 1-planar if it admits a drawing in the plane such that each edge is crossed at most once. Recently, Biedl and Wittnebel proved that every 1-planar graph with minimum degree 3 and $n \geq 7$ vertices has a matching of size at least $\frac{n+12}{7}$, which is tight for some graphs. They also provided tight lower bounds for the sizes of matchings in 1-planar graphs with minimum degree 4 or 5. In this paper, we show that any 1-planar graph with minimum degree 6 and $n \geq 36$ vertices has a matching of size at least $\frac{3n+4}{7}$, and this lower bound is tight. Our result confirms a conjecture posed by Biedl and Wittnebel.

Keywords: matching, minimum degree, drawing, 1-planar graph.

1 Introduction

A *drawing* of a graph $G = (V, E)$ is a mapping D that assigns to each vertex in V a distinct point in the plane and to each edge uv in E a continuous arc connecting $D(u)$ and $D(v)$. We often make no distinction between a graph-theoretical object (such as a vertex, or an

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edge) and its drawing. All drawings considered here are such ones that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross. A graph is *planar* if it can be drawn in the plane without edge crossings. A drawing of a graph is 1-*planar* if each of its edges is crossed at most once. If a graph has a 1-planar drawing, then it is 1-*planar*. The notion of 1-planarity was introduced in 1965 by Ringel [19], and since then many properties of 1-planar graphs have been studied (e.g. see the survey paper [13]). For example, it is known that [2, 9, 18] any 1-planar graph with n (≥ 3) vertices has at most $4n-8$ edges, and thus has the minimum degree ≤ 7.

A *matching* of a graph is a set of edges without common end vertex. The study of matchings is one of the oldest and best-studied problems in graph theory, for example, see [15]. An earlier result, due to Nishizeki and Baybars [17], shows that every simple planar graph with $n \geq X$ vertices has a matching of size at least $Yn+Z$, where X, Y, Z depend on the minimum degree and the connectivity of the graph. In recent years, researchers have investigated the sizes of the matchings graphs that are "almost" planar. An interesting example is 1-planar graphs, which is a generalization of planar graphs, in some sense. Many papers studying the sizes of matchings in 1-planar graphs have been published, for example, see ([4, 5, 6, 7]). Recently, Biedl and Wittnebel [7] obtained the following results on the lower bounds of sizes of matchings in 1-planar graphs in terms of their minimum degrees.

Theorem 1 ([7])**.** *Any n-vertex simple 1-planar graph with minimum degree δ has a matching M of the following size:*

1. $|M| \geq \frac{n+12}{7}$ *if* $\delta = 3$ *and* $n \geq 7$*;*
2. $|M| \geq \frac{n+4}{3}$ *if* $\delta = 4$ *and* $n \geq 20$*; and*
3. $|M| \geq \frac{2n+3}{5}$ *if* $\delta = 5$ *and* $n \geq 21$*.*

The authors in [7] constructed 1-planar graphs which contain matchings of sizes equal to the lower bounds in Theorem 1. As for the minimum degree $\delta = 6$, they also constructed 1-planar graphs which contain matchings with a maximum size.

Theorem 2 ([7])**.** *For any positive integer N, there exists a simple 1-planar graph with minimum degree 6 and $n \geq N$ vertices in which each matching is of size at most $\frac{3}{7}n + \frac{4}{7}$.*

The authors in [7] suspected that this bound in Theorem 2 is tight and posed the following conjecture.

Conjecture 1 ([7])**.** *Any 1-planar graph with minimum degree 6 and $n \geq N$ vertices has a matching of size at least $\frac{3}{7}n + O(1)$.*

As for the minimum degree $\delta = 7$, both papers [4] and [7] constructively gave the following result.

Theorem 3 ([4, 7])**.** *For any N, there exists a simple 1-planar graph with minimum degree 7 and $n \geq N$ vertices for which any matching has size at most $\frac{11}{23}n + \frac{12}{23}$.*

Similarly, the authors in the papers [4] and [7] wondered whether this bound in Theorem 3 is tight, but this remains as an open problem.

In this paper we confirm Conjecture 1 above, and have the following result.

Theorem 4. *Any simple 1-planar graph with minimum degree 6 and $n \geq 36$ vertices has a matching of size at least $\frac{3}{7}n + \frac{4}{7}$, and this lower bound is tight.*

The paper is organized as follows. In Section 2 we explain some terminology and notations, and in Section 3 we provide some lemmas. The proof of Theorem 4 is given in Section 4. Some problems worthy of further study are presented in Section 5.

2 Terminology and notation

All graphs considered here are simple, and possibly disconnected. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph G, respectively.

Let G be a graph. The *degree* of a vertex v in G, denoted by $deg_G(v)$, is the number of edges of G incident to v (a loop edge is counted twice), and the *minimum degree* of G is denoted by $\delta(G)$. A *cycle* of G is a closed trail whose origin and internal vertices are distinct. For any subset $A \subseteq V(G)$, if $A \neq V(G)$, let $G \backslash A$ be the graph obtained from G by deleting all vertices in A together with their incident edges, and if $A \neq \emptyset$, the subgraph of G induced by A, denoted by $G[A]$, is the graph $G \backslash (V(G) \setminus A)$.

A component F in a graph is called a *k-vertex-component* if F has exactly k vertices, and moreover F is called an *odd component* (*even component*) if k is odd (even). For $S \subseteq V(G)$, let $odd(G \backslash S)$ denote the number of odd components of $G \backslash S$.

For any two disjoint vertex subsets A and B of a graph G, let $E_G(A, B)$ denote the set of edges in G which have one end vertex in A and the other in B, and for $v \in V(G)$, let $N_G(v, A)$ be the set of vertices in A which are adjacent to v. Write "$G \cong H$" when graphs G and H are isomorphic.

Let D be a drawing of G. An edge e of G is called *clean* under D if it does not cross with any other edge under the drawing D; a cycle C of G is called *clean* if each edge on C is clean under D.

Let H be a subgraph of G with a drawing D. The subdrawing $D|_H$ of H induced by D is called a *restricted drawing* of D.

For any drawing D, the *associated plane graph* $D^\times$ of D is the plane graph that is obtained from D by turning all crossings of D into new vertices of degree four. A cycle C of $D^\times$ partitions the plane into two open regions, the bounded one (i.e., the *interior* of C) and the unbounded one (i.e., the *exterior* of C). We denote by $int_{D^\times}(C)$ and $ext_{D^\times}(C)$ the interior and exterior of C, respectively.

For other terminology and notations not defined here we refer to [1].

3 Preliminary results

In this section we give some lemmas. The first one is the well-known Tutter-Berge Formula, which gives a formula on the size of a maximum matching of a graph.

Lemma 5 (Tutte-Berge [3]). *The size of a maximum matching M of a graph G with n vertices equals the minimum, over all $S \subseteq V(G)$, of $\frac{1}{2}(n - (odd(G\backslash S) - |S|))$.*

Lemma 6. *Let G be a 1-planar graph with minimum degree $\delta(G) \geq 6$, and S be a subset of $V(G)$ with $|S| \geq 2$. Denote by a_1 the number of 1-vertex-components of $G\backslash S$. Then $a_1 + 3 \leq |S|$.*

Proof. Let T be the set of isolated vertices in $G\backslash S$, namely, 1-vertex-components of $G\backslash S$, and let H be the subgraph of G with vertex set $S \cup T$ and edge set $E_G(S,T)$. As H is bipartite and 1-planar, by a result on the maximum size of a bipartite 1-planar graph due to Karpov [12], we have

$$|E(H)| \leq 3|V(H)| - 8 = 3(|S| + |T|) - 8 = 3|S| + 3a_1 - 8.$$

Since $\delta(G) \geq 6$,

$$6a_1 = 6|T| \leq \sum_{u \in T} deg_G(u) = |E(H)| \leq 3|S| + 3a_1 - 8,$$

implying that $a_1 + \frac{8}{3} \leq |S|$. As both a_1 and $|S|$ are integers, the result holds. □

The following result in [11] gives an upper bound of the size of a bipartite 1-planar graph $G(X,Y;E)$ in terms of $|X|$ and $|Y|$.

Lemma 7 ([11]). *Let G be a bipartite 1-planar graph (possibly disconnected) that has partite sets of sizes x and y with $2 \leq x \leq y$. Then we have that $|E(G)| \leq 2|V(G)| + 4x - 12$.*

We say that two 1-planar drawings of a graph are *isomorphic* if there is a homeomorphism of the sphere that maps one drawing to the other.

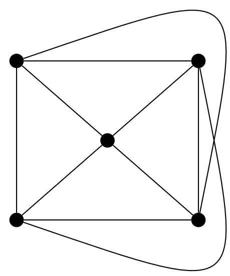

Figure 1: The unique 1-planar drawing of the complete graph K_5.

Lemma 8. *The complete graph K_5 has exactly one (up to isomorphism) 1-planar drawing as shown in Figure 1.*

Proof. See the proof of Lemma 7 in [14]. $\square$

Lemma 9. *Let G be a 1-planar graph with minimum degree $\delta(G) = 6$, $S \subseteq V(G)$, and F be a 5-vertex-component of $G \backslash S$. Then the following two statements hold:*

(a) $|E_G(V(F), S)| \geq 10$; and

(b) if $|E_G(V(F), S)| = 10$ or 11, then $F \cong K_5$.

Proof. (a). Since $|V(F)| = 5$, $|E(F)| \leq 10$. By the given condition,

$$30 \leq \sum_{u \in V(F)} deg_G(u) = 2|E(F)| + |E_G(V(F), S)|. \tag{1}$$

Thus, $|E_G(V(F), S)| \geq 30 - 2 \times 10 = 10$.

(b). If $|E_G(V(F), S)| = 10$ or 11, then (1) implies that

$$|E(F)| \geq \frac{1}{2}(30 - |E_G(V(F), S)|) \geq \frac{1}{2}(30 - 11) = 9.5,$$

implying that $|E(F)| \geq 10$ and so $F \cong K_5$. $\square$

Let D be a 1-planar drawing of a graph G. If $L = v_1 c_1 v_2 c_2 \cdots v_\ell c_\ell v_1$ is a cycle of the associated plane graph $D^\times$, which consists alternately of some vertices of G and crossing points of D, then we say that L is a *barrier loop* of D (see [16]). We can extend this concept "barrier loop" to any cycle L in $D^\times$ which contains some clean edges $e_1, e_2, \cdots, e_k$ such that after removing these edges, each of the remaining sections in L is either an isolated vertex or is a path in the form $v_i c_i v_{i+1} c_{i+1} \cdots v_{r+t-1} c_{r+t-1} v_{r+t}$ consisting alternately of some vertices $v_i, v_{i+1}, \cdots, v_{i+t}$ of G and crossing points $c_i, c_{i+1}, \cdots, c_{i+t-1}$ of D for some i and t, where the subindices are taken modulo ℓ.

The following proposition is obvious from the 1-planarity.

Lemma 10. *Let L be a barrier loop of a 1-planar drawing D of a graph G. For any $u, v \in V(G)$, if u and v locate in $int_{D^\times}(L)$ and $ext_{D^\times}(L)$ respectively, then u and v are not adjacent in G, and every common neighbor of u and v must be on L.*

Let D be a 1-planar drawing of G which has the minimum number of crossings and let Q be a restricted drawing of D. Let $\mathscr{F}(Q^\times)$ denote the set of faces of $Q^\times$. For any vertex u in Q, let $\mathscr{F}_u(Q^\times)$ be the set of faces F in $\mathscr{F}(Q^\times)$ such that either u is on the boundary of F, or u is on the boundary of some face $F_0 \in \mathscr{F}(Q^\times)$ whose boundary shares an edge e with the boundary of F, where e is neither clean under D and nor crossed in Q. Then, we can prove the following conclusion.

Lemma 11. *Let D be a 1-planar drawing of G and let Q be a restricted drawing of D. For any $u \in V(Q)$ and any $v \in N_G(u) \setminus V(Q)$, v must be within some face of $\mathscr{F}_u(Q^\times)$.*

Proof. As $v \notin V(Q)$, v is within some face F of $\mathscr{F}(Q^{\times})$.

Let Q' be the restricted drawing of D obtained from Q by adding a curve C representing edge uv. As D is a 1-planar drawing, C is crossed at most once. If C is not crossed in Q', then u must be on the boundary of F and so $F \in \mathscr{F}_u(Q^{\times})$.

If C is crossed once in Q' with some edge e' in Q, then e' is neither clean under D nor crossed in Q. In Q, e' is on the common boundary of face F and another face F_0 in $\mathscr{F}(Q^{\times})$. As e' is crossed once only, u must be on the boundary of F_0, implying that $F_0 \in \mathscr{F}_u(Q^{\times})$. Hence $F \in \mathscr{F}_u(Q^{\times})$ in this case. □

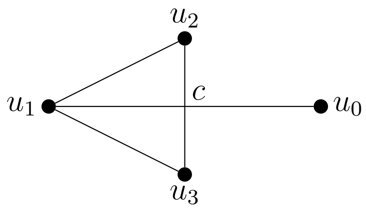

Figure 2: Both u_1u_2 and u_1u_3 are clean edges.

Lemma 12. *Let D be a 1-planar drawing of G which has the minimum number of crossings and let Q be a restricted drawing of D. Assume that $u_1u_2u_3u_1$ is a 3-cycle in Q. If u_2u_3 is crossed with an edge which is incident with u_1, as shown in Figure 2, then both u_1u_2 and u_1u_3 are clean edges with the drawing D.*

Proof. Suppose that u_1u_2 is not a clean edge. Then we redraw the edge u_1u_2 "most near" to one side of the sections u_1c and cu_2 so as to make no crossings, contradicting to the choice of D which has the minimum number of crossings. □

4 The Proof of Theorem 4

We only prove the former part of Theorem 4, because the tightness of the lower bound is direct from [7]. We first establish the following result for proving Theorem 4.

Proposition 13. *Theorem 4 is true if for every 1-planar graph G of order at least 36 and every $S \subseteq V(G)$ with $|S| \geq 2$, the following inequality holds:*

$$3a_1 + 2a_3 + a_5 + 4 \leq 4|S|, \tag{*}$$

where a_i is the number of i-vertex components of $G \backslash S$ for $i \in \{1, 3, 5\}$.

Proof. Let G be a 1-planar graph with $\delta(G) = 6$ and $n \geq 36$ vertices, and let $M(G)$ be a maximum matching of G. In order to prove Theorem 4, i.e., $|M(G)| \geq \frac{3}{7}n + \frac{4}{7}$, by Lemma 5, it suffices to prove that, for each subset $S \subseteq V(G)$,

$$odd(G\backslash S) - |S| \leq \frac{n-8}{7}. \tag{2}$$

If $|S| = 1$, because $\delta(G) = 6$, each odd component of $G\backslash S$ has at least 7 vertices. Therefore, $G\backslash S$ has at most $\frac{n-1}{7}$ odd components. So,

$$odd(G\backslash S) - |S| \le \frac{n-1}{7} - 1 = \frac{n-8}{7}.$$

Thus, (2) holds when $|S| = 1$.

If $|S| = 0$, because $\delta(G) = 6$ and the complete graph K_7 is not 1-planar (see [8], for example), each odd components of G ($= G\backslash S$) has at least 9 vertices. Hence G has at most $\frac{n}{9}$ odd components. Thus, for $n \ge 36$ we have

$$odd(G\backslash S) - |S| \le \frac{n}{9} \le \frac{n-8}{7}.$$

(2) holds as well when $|S| = 0$.

In the following we focus on the case that $|S| \ge 2$. For $i \ge 1$, let a_{2i-1} denote the number of components of $G\backslash S$ with $2i-1$ vertices, and let a_0 be the number of even components of $G\backslash S$. Noting that $n \ge |S| + 2a_0 + \sum_{i\ge1}(2i-1)a_{2i-1}$, in order to prove (2), we only need to prove

$$\sum_{i\ge1} a_{2i-1} - |S| \le \frac{1}{7}\Big(|S| + 2a_0 + \sum_{i\ge1}(2i-1)a_{2i-1} - 8\Big). \tag{3}$$

Because $a_{2i-1} \le \frac{2i-1}{7}a_{2i-1}$ for each $i \ge 4$, in order to prove (3), we only need to prove

$$(a_1 + a_3 + a_5) - |S| \le \frac{1}{7}(|S| + a_1 + 3a_3 + 5a_5 - 8), \tag{4}$$

namely $3a_1 + 2a_3 + a_5 + 4 \le 4|S|$.

Thus the result is proven. □

In the following, we always assume that G is a 1-planar graph of order at least 36 and S is a subset of $V(G)$ with $|S| \ge 2$. For each 5-vertex-component F of $G\backslash S$, $|E_G(V(F), S)| \ge 10$ by Lemma 9 (a). A 5-vertex component F of $G\backslash S$ is called *bad*, if $|E_G(V(F), S)| = 10$ or 11; otherwise, *good*.

The remainder of the proof of Theorem 4 consists of three subsections. In Subsection 4.1, we shall establish some properties on a bad 5-vertex component F of $G\backslash S$; in Subsection 4.2, a 1-planar bipartite graph G^* will be obtained from G by contracting or deleting some edges in G; and in the the last subsection, we will apply G^* to show that (*) holds and hence Theorem 4 follows.

4.1 Local properties of bad 5-vertex-components of $G\backslash S$

Let D be a 1-planar drawing of G such that D has the minimum number of crossings. In this subsection, we shall find some properties of the local structure of a bad 5-vertex-component of $G \backslash S$ under the drawing D. Let F be a bad 5-vertex-component of $G \backslash S$ with $V(F) = \{v_0, v_1, v_2, v_3, v_4\}$. For each vertex $v_i \in V(F)$, obviously, $\sum_{i=0}^{4} |N_G(v_i, S)| = |E_G(V(F), S)|$ by the simplicity of G. Again, since F is a bad 5-vertex-component of $G \backslash S$, it follows from Lemma 9 (b) that $F \cong K_5$.

Since D is a 1-planar drawing of G, the restricted drawing $D|_F$ is also a 1-planar drawing of F. Therefore, $D|_F$ is unique up to isomorphism by Lemma 8. Without loss of generality, in the following, we assume that the 1-planar drawing $D|_F$ of F is depicted in Figure 3, where v_1v_3 and v_2v_4 are two crossed edges with the crossing point c.

At this time we say that the 4-cycle $C = v_1v_2v_3v_4v_1$ is a *central* cycle of F, and v_0 is the *central* vertex of F.

Next we prove Propositions 14 and 15.

Proposition 14. *For a bad 5-vertex-component F of $G \backslash S$ with its central vertex v_0 and central cycle $C = v_1v_2v_3v_4v_1$, we have*

(a) $|N_G(v_i, S)| \geq 2$ for each $0 \leq i \leq 4$;

(b) C is a clean cycle under the drawing D;

(c) $\left|\bigcup_{i=1}^{4} N_G(v_i, S)\right| \geq 3$; and

(d) if $\left|\bigcup_{i=1}^{4} N_G(v_i, S)\right| = 3$, then some edge v_0v_i, where $1 \leq i \leq 4$, is clean under the drawing D.

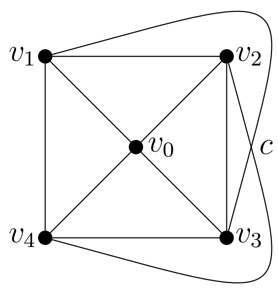

Figure 3: The unique 1-planar drawing of F of $G \backslash S$.

Proof. Let the restricted drawing $D|_F$ be as shown in Figure 3, where edges v_1v_3 and v_2v_4 cross each other at point c (see Figure 3).

(a) follows immediately from the facts that G is simple with $\delta(G) = 6$ and $F \cong K_5$.

Now prove (b). Assume to the contrary that there exists an edge of C, say v_1v_2, is a crossed edge under the drawing D. Then we redraw the edge v_1v_2 "most near" to one side of the sections v_1c and cv_2 so as to make no crossings, contradicting to the choice of D with the minimum number of crossings. This proves (b).

Then prove (c). Obviously, $\left|\bigcup_{i=1}^{4} N_G(v_i, S)\right| \geq 2$ by Proposition 14 (a). Assume to the contrary that $\left|\bigcup_{i=1}^{4} N_G(v_i, S)\right| \ngeq 3$. Then $\bigcup_{i=1}^{4} N_G(v_i, S) = \{x_1, x_2\}$ for two vertices x_1, x_2 in S. Observe that $G[\{v_0, v_1, v_2, v_3, v_4, x_1, x_2\}]$ contains a subgraph that is isomorphic to $K_7 \backslash K_3$ (the complete graph K_7 by deleting three edges of a 3-cycle). However, it is proved in [14] that $K_7 \backslash K_3$ is not 1-planar, a contradiction. This proves (c).

Finally prove (d). By the assumption, set $\bigcup_{i=1}^{4} N_G(v_i, S) = \{x_1, x_2, x_3\} \subseteq S$. For $1 \leq j \leq 3$, let $\ell(x_j) = |N_G(x_j, \{v_1, v_2, v_3, v_4\})|$. It follows from Proposition 14 (a) that

$$\sum_{j=1}^{3} \ell(x_j) = \sum_{i=1}^{4} |N_G(v_i, S)| \geq 4 \times 2 = 8.$$

So there exists a vertex in $\{x_1, x_2, x_3\}$, say x_1, such that $\ell(x_1) \geq 3$, implying that $N_G(x_1) \cap \{v_i, v_{i+1}\} \neq \emptyset$ for $i = 1, 2, 3, 4$, where v_5 represents the vertex v_1.

Note that $L_1 : v_3v_4cv_3$, $L_2 : v_1v_4cv_1$, $L_3 : v_1v_2cv_1$ and $L_4 : v_2v_3cv_2$ are barrier loops. As v_2, v_3 locate in $int_{D^\times}(L_2)$, by Lemma 10, x_1 must be in $int_{D^\times}(L_2)$. For $i = 1, 3, 4$, as v_i, v_{i+1} locate in $ext_{D^\times}(L_i)$, by Lemma 10, x_1 must be in $ext_{D^\times}(L_i)$, where v_5 represents the vertex v_1. Thus, x_1 must lie in one of the four regions bounded by the 3-cycles $v_1v_2v_0v_1$, $v_2v_3v_0v_2$, $v_3v_4v_0v_3$, and $v_4v_1v_0v_4$ (see Figure 3). Without loss of generality, let x_1 lie in the region bounded by the 3-cycle $v_4v_1v_0v_4$. As $\ell(x_1) \geq 3$, x_1 is adjacent to either v_2 or v_3, say v_2, implying that x_1v_2 crosses v_0v_1. By Lemma 12, edge v_0v_2 is a clean edge under the drawing D.

This proves (d). □

By Proposition 14 (c), $\left|\bigcup_{i=1}^{4} N_G(v_i, S)\right| \geq 3$. Now we continue to describe the local structure of F under the assumption that $\left|\bigcup_{i=1}^{4} N_G(v_i, S)\right| = 3$ and $N_G(v_0, S) \subseteq \bigcup_{i=1}^{4} N_G(v_i, S)$. Now we are going to prove the following conclusion.

Proposition 15. *For a bad 5-vertex component F of $G \backslash S$ with its central vertex v_0 and central cycle $C : v_1v_2v_3v_4v_1$, if $N_G(v_0, S) \subseteq \{x_1, x_2, x_3\} = \bigcup_{i=1}^{4} N_G(v_i, S)$, then there exist two non-adjacent edges $e = v_{j_1}v_{j_2}$ and $e' = v_{j_3}v_{j_4}$ on C such that*

$$|N_G(v_{j_1}) \cap N_G(v_{j_2}) \cap \{x_1, x_2, x_3\}| \leq 1 \quad \textit{and} \quad |N_G(v_{j_3}) \cap N_G(v_{j_4}) \cap \{x_1, x_2, x_3\}| \leq 1.$$

Proof. By assumption, $N_G(v_0, S) \subseteq \bigcup_{i=1}^{4} N_G(v_i, S) = \{x_1, x_2, x_3\} \subseteq S$. For $j = 1, 2, 3$, let $\ell(x_j) = |N_G(x_j, V(F))|$. By the simplicity of G, we have that $\ell(x_j) \leq 5$ for $1 \leq j \leq 3$, and

$$\sum_{i=0}^{4} |N_G(v_i, S)| = |E_G(V(F), S)| = \sum_{j=1}^{3} \ell(x_j).$$

It follows from Proposition 14 (a) that

$$10 = 5 \times 2 \leq \sum_{i=0}^{4} |N_G(v_i, S)| = \sum_{j=1}^{3} \ell(x_j) = |E_G(V(F), S)| \leq 11, \tag{5}$$

where the last inequality follows from the assumption that F is a bad 5-vertex component of $G \backslash S$.

Assume that $\ell(x_1) \geq \ell(x_i)$ for $i = 2, 3$. By (5), $\ell(x_1) \geq 4$. Note that $L_1 : v_3v_4cv_3$, $L_2 : v_1v_4cv_1$, $L_3 : v_1v_2cv_1$ and $L_4 : v_2v_3cv_2$ are barrier loops. As v_0, v_2, v_3 locate in $int_{D^\times}(L_2)$, by Lemma 10, x_1 must be in $int_{D^\times}(L_2)$. For $i = 1, 3, 4$, as v_0, v_i, v_{i+1} locate in $ext_{D^\times}(L_i)$, by Lemma 10, x_1 must be in $ext_{D^\times}(L_i)$, where v_5 represents the vertex v_1. Thus, x_1 must lie in one of the four regions bounded by the 3-cycles $v_1v_2v_0v_1$, $v_2v_3v_0v_2$, $v_3v_4v_0v_3$, and $v_4v_1v_0v_4$ (see Figure 3). Without loss of generality, in the following we always assume that x_1 lies in the region bounded by the 3-cycle $v_4v_1v_0v_4$ (for it is completely analogous for other cases). Then there are six possible subdrawings (B1)-(B6), as shown in Figure 4, where the first five subdrawings (B1)-(B5) correspond to that $\ell(x_1) = 4$, and the last subdrawing (B6) corresponds to that $\ell(x_1) = 5$.

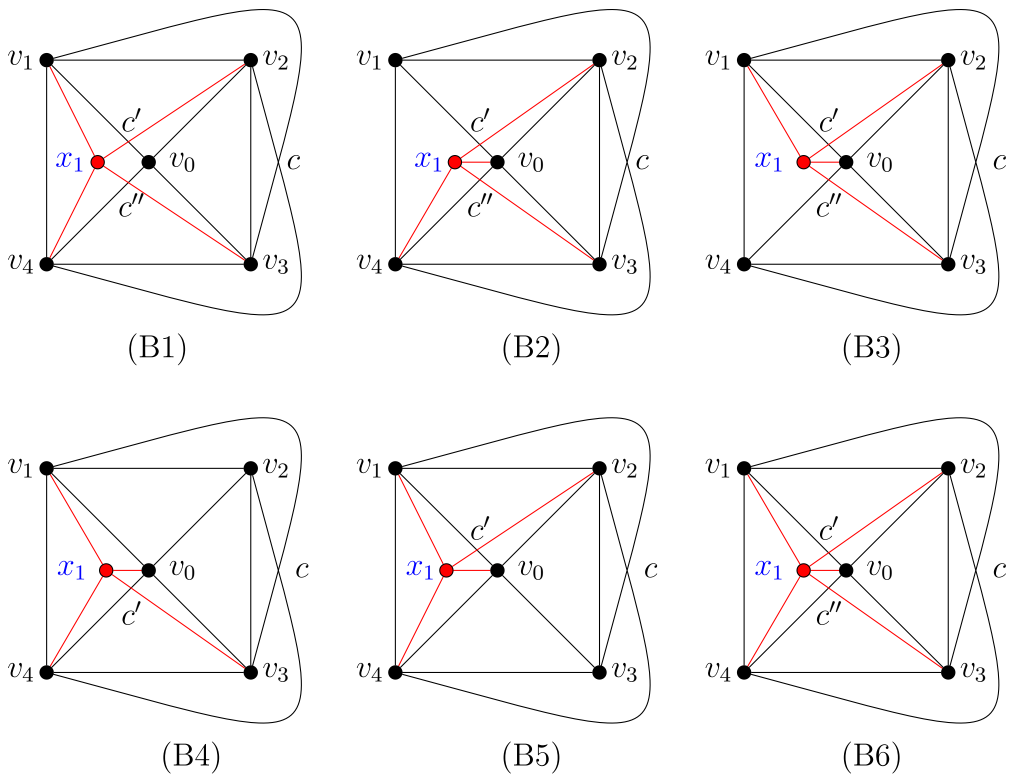

Figure 4: The possible subdrawings involving x_1

We shall prove the following claims to complete the proof.

Claim 1. *For $0 \le i \le 4$ and $1 \le t \le 3$, $N_G(v_i) \cap (\{x_1, x_2, x_3\} \setminus \{x_t\}) \neq \emptyset$, and if $x_t \notin N_G(v_i)$, then $\{x_1, x_2, x_3\} \setminus \{x_t\} \subseteq N_G(v_i)$.*

Proof. For $0 \le i \le 4$, by the given condition and Proposition 14 (a),

$$|N_G(v_i) \cap \{x_1, x_2, x_3\}| = |N_G(v_i) \cap S| \ge 2.$$

Thus, Claim 1 follows. □

Claim 2. *Subdrawings (B1), (B2) and (B3) cannot occur.*

Proof. Since (B2) and (B3) are symmetric, we consider (B1) and (B2) only.

Observe that in both (B1) and (B2), $C_1 = x_1 c' v_2 c v_3 c'' x_1$ is a barrier loop of D, and v_0 locates in $int_{D^\times}(C_1)$ while v_1 and v_4 locate in $ext_{D^\times}(C_1)$.

In (B1), as $x_1 v_0 \notin E(G)$, by Claim 1, $\{x_2, x_3\} \subseteq N_G(v_0)$. As v_0 is in $int_{D^\times}(C_1)$, by Lemma 10, both x_2 and x_3 are in $int_{D^\times}(C_1)$. Since v_1 locates in $ext_{D^\times}(C_1)$, by Lemma 10 again, we have $N_G(v_1) \cap \{x_1, x_2, x_3\} \subseteq \{x_1\}$, a contradiction to Proposition 14 (a).

In (B2), as $x_1 v_1 \notin E(G)$, by Claim 1, $\{x_2, x_3\} \subseteq N_G(v_1)$. As v_1 is in $ext_{D^\times}(C_1)$, by Lemma 10, both x_2 and x_3 are in $ext_{D^\times}(C_1)$. Since v_0 is in $int_{D^\times}(C_1)$, by Lemma 10 again, we have $N_G(v_0) \cap \{x_1, x_2, x_3\} = \{x_1\}$, a contradiction to Proposition 14 (a).

Hence Claim 2 holds. □

Claim 3. *Proposition 15 holds if subdrawing (B4) or (B5) occurs.*

Proof. We only consider (B4) because of the symmetry. Suppose (B4) happens.

Note that in (B4), $x_1 \notin N_G(v_2)$. By Claim 1, $N_G(v_2) \cap \{x_1, x_2, x_3\} = \{x_2, x_3\}$. By Claim 1 again, $\{x_2, x_3\} \cap N_G(v_4) \neq \emptyset$. Assume that $x_2 \in N_G(v_4)$. Then x_2 is adjacent to both v_2 and v_4. We shall show that

$$x_2 \notin N_G(v_3) \qquad \text{and} \qquad x_3 \notin N_G(v_1) \cup N_G(v_4). \tag{6}$$

As x_2 is adjacent to both v_2 and v_4, by Lemma 11, x_2 is within a common face F of $\mathscr{F}_{v_2}(Q^\times)$ and $\mathscr{F}_{v_4}(Q^\times)$, where Q is the restricted drawing of D shown in Figure 5 (B4).

It can be verified that $\mathscr{F}_{v_2}(Q^\times)$ and $\mathscr{F}_{v_4}(Q^\times)$ have exactly one common face, denoted by F_1, whose interior is $int_{D^\times}(C_2)$, where C_2 is the cycle $v_1 x_1 v_0 v_1$. Thus, x_2 must be within $int_{D^\times}(C_2)$, as shown in Figure 5 (B4-1), where c'' is the crossing point involving the two edges $x_2 v_2$ and $v_0 v_1$. As F_1 does not belong to $\mathscr{F}_{v_3}(Q^\times)$, $x_2 \notin N_G(v_3)$ by Lemma 11.

As $x_1 \notin N_G(v_2)$ and $x_2 \notin N_G(v_3)$, by Claim 1, $x_3 \in N_G(v_2) \cap N_G(v_3)$.

Note that $C_3 = v_0 c'' v_2 c v_4 c' v_0$ is a barrier loop, and v_3 is in $int_{D^\times}(C_3)$ while v_1 is in $ext_{D^\times}(C_3)$ (See Figure 5(B4-1)). As $x_3 \in N_G(v_3)$, by Lemma 10, we have $x_3 \notin N_G(v_1)$. Similarly, we know that $C_4 = v_0 c'' v_1 c v_3 c' v_0$ is a barrier loop, and v_2 is in $int_{D^\times}(C_4)$

while v_4 is in $ext_{D^\times}(C_4)$ (See Figure 5(B4-1)). As $x_3 \in N_G(v_2)$, by Lemma 10, we have $x_3 \notin N_G(v_4)$. Hence (6) holds.

By (6) and the fact that $x_1 \notin N_G(v_2)$, we have

$$N_G(v_1) \cap N_G(v_2) \cap \{x_1, x_3\} = \emptyset \quad \text{and} \quad N_G(v_3) \cap N_G(v_4) \cap \{x_2, x_3\} = \emptyset,$$

implying that Proposition 15 holds with $e = v_1v_2$ and $e' = v_3v_4$ and Claim 3 is proven. $\square$

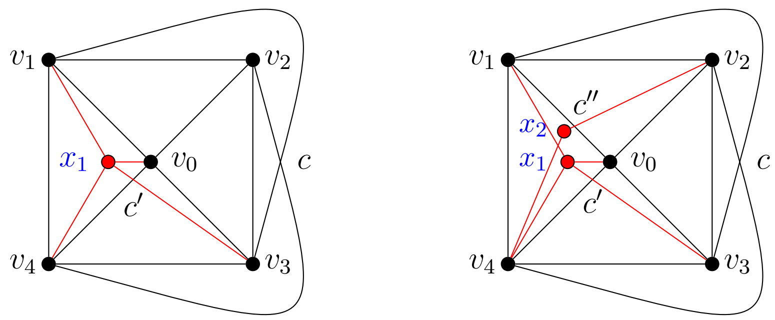

Figure 5: Restricted drawing Q and the one Q' obtained after adding a vertex x_2 and edges joining x_2 to both v_2 and v_4

Claim 4. *Proposition 15 holds if subdrawing (B6) occurs.*

Proof. Assume that (B6) happens. We first show that for both $i = 2, 3$,

$$\{v_1, v_3\} \not\subseteq N_G(x_i) \quad \text{and} \quad \{v_2, v_4\} \not\subseteq N_G(x_i). \tag{7}$$

Let C_5 be the barrier loop $v_0c'v_2cv_4c''v_0$. Observe that v_1 is in $ext_{D^\times}(C_5)$ and v_3 is in $int_{D^\times}(C_5)$. By Lemma 10, $\{v_1, v_3\} \not\subseteq N_G(x_i)$ for both $i = 2, 3$. Similarly, it can be proved that $\{v_2, v_4\} \not\subseteq N_G(x_i)$ for both $i = 2, 3$. Thus, (7) holds. It implies that each x_i is adjacent to at most two consecutive vertices on C for both $i = 2, 3$. On the other hand, by Claim 1,

$$\{v_1, v_2, v_3, v_4\} \subseteq N_G(x_2) \cup N_G(x_3), \tag{8}$$

Hence, for $i = 2, 3$, x_i is adjacent to exactly two consecutive vertices on C and x_2, x_3 together are adjacent to all four vertives on C. Equivalently, one of the two cases below happens for some $t \in \{2, 3\}$:

(i) $N_G(x_t) \cap \{v_1, v_2, v_3, v_4\} = \{v_1, v_2\}$ and $N_G(x_{5-t}) \cap \{v_1, v_2, v_3, v_4\} = \{v_3, v_4\}$, or

(ii) $N_G(x_t) \cap \{v_1, v_2, v_3, v_4\} = \{v_1, v_4\}$ and $N_G(x_{5-t}) \cap \{v_1, v_2, v_3, v_4\} = \{v_2, v_3\}$.

But we can show that Case (i) above cannot happen. Note that $C_6 : x_1c'v_2cv_3c''x_1$ is a barrier loop, and v_0 is in $int_D(C_6)$ while both v_1 and v_4 are in $ext_D(C_6)$. If Case (i) happens, then $N_G(x_t) \cap \{v_1, v_4\} \neq \emptyset$ for both $t = 2, 3$. By Lemma 10, $x_t \notin N_G(v_0)$ for both $t = 2, 3$, a contradiction to Claim 1.

Thus, Case (ii) above happens, and

$$N_G(v_1) \cap N_G(v_2) \cap \{x_2, x_3\} = \emptyset = N_G(v_3) \cap N_G(v_4) \cap \{x_2, x_3\},$$

implying that Proposition 15 holds with $e = v_1v_2$ and $e' = v_3v_4$.

Hence Claim 4 holds. □

By Claims 2, 3 and 4, Proposition 15 is proven. □

4.2 To form a bipartite 1-planar graph G^* from G

In this section, associated with the given 1-planar graph G and 1-planar drawing D of G with the minimum number of crossings, we perform several operations on G so as to obtain a desired bipartite 1-planar (simple) graph G^*, in which one bipartite set is S and the other consists of some original vertices in $V(G)\backslash S$ and some new vertices.

Let F be a bad 5-vertex-component of $G\backslash S$ with $V(F) = \{v_0, v_1, v_2, v_3, v_4\}$, its central cycle $C = v_1v_2v_3v_4v_1$ and its central vertex v_0. Then we know that C is clean under the drawing D by Proposition 14 (a). Then $|\bigcup_{i=1}^{4} N_G(v_i, S)| \geq 3$ by Proposition 14(c).

In the following, we define three operations on F according to the value of $|\bigcup_{i=1}^{4} N_G(v_i, S)|$ and whether $N_G(v_0, S)$ is a subset of $\bigcup_{i=1}^{4} N_G(v_i, S)$.

Operation A (When $|\bigcup_{i=1}^{4} N_G(v_i, S)| \geq 4$) *Contract all the edges of C such that the four vertices of C are merged into a new vertex v^*, and delete all loops and all possible parallel edges but one for each pair of distinct vertices which appear after edge-contraction.*

Since C is clean under the drawing D, the edge contraction in Operation A is performable and does not affect the 1-planarity. After performing Operation A, F is transformed into a 2-vertex graph F'' with $V(F'') = \{v^*, v_0\}$. Moreover, we easily know that v^* is still adjacent to each vertex in $\bigcup_{i=1}^{4} N_G(v_i, S)$. Let $\omega(F'', S)$ denote the number of edges with one end in $V(F'')$ and the other end in S after performing Operation A. Because $|N_G(v_0, S)| \geq 2$ by Proposition 14 (a), we get that

$$\omega(F'', S) = \left|\bigcup_{i=1}^{4} N_G(v_i, S)\right| + |N_G(v_0, S)| \geq 6. \tag{9}$$

If $|\bigcup_{i=1}^{4} N_G(v_i, S)| = 3$, it then follows from Proposition 14(d) that there exists a clean edge e under the drawing D which joins v_0 to some vertex in C. Without loss of generality, let $e = v_0v_1$. We shall distinguish the two cases below: $N_G(v_0, S) \nsubseteq \bigcup_{i=1}^{4} N_G(v_i, S)$ and $N_G(v_0, S) \subseteq \bigcup_{i=1}^{4} N_G(v_i, S)$.

Operation B $\big($When $|\bigcup_{i=1}^{4} N_G(v_i, S)| = 3$ and $N_G(v_0, S) \nsubseteq \bigcup_{i=1}^{4} N_G(v_i, S)\big)$. *First contract all the edges of C such that the four vertices of C are merged into a new vertex v^*; then contract the edge $e = v_0v_1$ $(= v_0v^*)$ such that the two end vertices of e are merged into a new vertex v^{**}; finally delete all loops and all parallel edges but one for each pair of distinct vertices which appear after edge-contraction.*

Similarly, the edge contraction in Operation B is performable and does not affect the 1-planarity. After performing Operation B, F is transformed into a 1-vertex graph F' with $V(F') = \{v^{**}\}$. Similarly, we also easily see that v^{**} is still adjacent to each vertex in $\bigcup_{i=1}^{4} N_G(v_i, S) \cup N_G(v_0, S)$ after performing Operation B.

Let $\omega(F', S)$ denote the number of edges with one end in $V(F')$ $(= \{v^{**}\})$ and the other end in S. Because of the assumption that $\big|\bigcup_{i=1}^{4} N_G(v_i, S)\big| = 3$ and $N_G(v_0, S) \nsubseteq \bigcup_{i=1}^{4} N_G(v_i, S)$, we therefore get that

$$\omega(F', S) = \left|\bigcup_{i=0}^{4} N_G(v_i, S)\right| \geq \left|\bigcup_{i=1}^{4} N_G(v_i, S)\right| + 1 = 4. \tag{10}$$

If $\big|\bigcup_{i=1}^{4} N_G(v_i, S)\big| = 3$ and $N_G(v_0, S) \subseteq \bigcup_{i=1}^{4} N_G(v_i, S)$, let $\bigcup_{i=1}^{4} N_G(v_i, S) = \{x_1, x_2, x_3\}$, where $x_1, x_2, x_3 \in S$. Then, by Proposition 15, there are two non-adjacent edges e and e' on C such that the two end vertices of e (resp. e') have at most one common neighbor x (resp. x') in $\{x_1, x_2, x_3\}$.

At this time we define the following operation.

Operation C $\big($When $|\bigcup_{i=1}^{4} N_G(v_i, S)| = 3$ and $N_G(v_0, S) \subseteq \bigcup_{i=1}^{4} N_G(v_i, S)\big)$. *Contract the edge e (resp. e') such that its two end vertices are merged into a new vertex v' (resp. v''), and then delete all parallel edges but one for each pair of distinct vertices which appear after edge-contraction.*

Similarly, the edge contraction in Operation C can be implemented and does not change the 1-planarity. After finishing Operation C, F is transformed into a 3-vertex graph F'''' with $V(F''') = \{v', v^{''}, v_0\}$. By the reasons as just stated before Operation C, we know that performing Operation C yields at most two multiple edges with one end vertex is in $\{v', v''\}$ and the other end vertex in S.

Let $\omega(F^{'''}, S)$ denote the number of edges with one end in $V(F'')$ and the other end in S after performing Operation C. Therefore, by Proposition 14 (a), we get that

$$\omega(F''', S) \geq |N_G(v_0, S)| + \Big(\sum_{i=1}^{4} |N_G(v_i, S)| - 2\Big) \geq 8. \tag{11}$$

Now we are going to construct the desired graph G^* from G by going through all the following steps.

Step 0. (deleting components). Delete from $G\backslash S$ all even components and all i-vertex odd components with $i \geq 7$ (deleting a vertex must also delete its all incident edges).

Step 1. Perform Operation A at every bad 5-vertex-component F of $G\backslash S$ if $|\bigcup_{i=1}^{4} N_G(v_i, S)| \geq 4$. By the definition of Operation A, each F is changed into a 2-vertex graph F''.

Step 2. Perform Operation B at every bad 5-vertex-component F of $G\backslash S$ if $|\bigcup_{i=1}^{4} N_G(v_i, S)| = 3$ and $N_G(v_0, S) \nsubseteq \bigcup_{i=1}^{4} N_G(v_i, S)$. By the definition of Operation B, F is changed into a 1-vertex graph F'.

Step 3. Perform Operation C at every bad 5-vertex-components F of $G\backslash S$ if $|\bigcup_{i=1}^{4} N_G(v_i, S)| = 3$ and $N_G(v_0, S) \subseteq \bigcup_{i=1}^{4} N_G(v_i, S)$. By the definition of Operation C, F is changed into a 3-vertex graph F'''.

Step 4. (deleting edges) Delete the edges in all 3-vertex-components and all good 5-vertex-components of $G\backslash S$; delete the edge in every 2-vertex graph F'' obtained in Step 1, and all edges in each 3-vertex graph F''' obtained by Step 3; and delete all edges which joint two vertices in S.

After doing all the steps above, we see that the resulting graph G^* is a bipartite 1-planar (simple) graph with bipartite sets $|S|$ and $|T|$, where $T = V(G^*)\backslash S$.

Remark: *The edge contraction in Steps 1-3 may cause two adjacent edges to cross with each other in the resulting 1-planar drawing of G^*. If this phenomenon appears, we can modify the drawing so that this two adjacent edges are no longer crossed.*

4.3 Applying G^* to prove Theorem 4

From the previous subsection, we know that G^* is a 1-planar bipartite graph with bipartite set S and T. Now we compute the sizes of $|T|$ and $|E(G^*)|$. Form the process of obtaining G^* from G, we have the following facts on T and $E_{G^*}(S, T)$.

Fact (1). Each 1-vertex-component F of $G\backslash S$ exactly contributes one vertex to T, and at least 6 edges to G^*.

Fact (2). Each 3-vertex-component F of $G\backslash S$ exactly contributes 3 vertices to T and at least 12 edges to G^*, because $|E_G(V(F), S)| \geq 3 \times (\delta(G) - 2) \geq 3 \times 4 = 12$.

Fact (3). Each good 5-vertex-component F of $G\backslash S$ exactly contributes 5 vertices to T and at least 12 edges to G^*, because $|E_G(V(F), S)| \geq 12$ by the definition of good 5-vertex-components of $G\backslash S$.

Fact (4). Each bad 5-vertex-component F of $G\backslash S$ with $\big|\bigcup\limits_{i=1}^{4} N_G(v_i, S)\big| \geq 4$ exactly contributes 2 vertices to T and at least 6 edges to G^* by (9).

Fact (5). Each bad 5-vertex-component F of $G\backslash S$ with $\big|\bigcup\limits_{i=1}^{4} N_G(v_i, S)\big| = 3$ and $N_G(v_0, S) \nsubseteq \bigcup\limits_{i=1}^{4} N_G(v_i, S)$ exactly contributes one vertex to T and at least 4 edges to G^* by (10).

Fact (6). Each bad 5-vertex-component F of $G\backslash S$ with $\big|\bigcup\limits_{i=1}^{4} N_G(v_i, S)\big| = 3$ and $N_G(v_0, S) \subseteq \bigcup\limits_{i=1}^{4} N_G(v_i, S)$ exactly contributes 3 vertices to T and at least 8 edges to G^* by (11).

Now we are ready to prove Theorem 4.

Proof of Theorem 4: Denote by $a_5^{'}$ the number of bad 5-vertex-components F of $G\backslash S$ with $\big|\bigcup\limits_{i=1}^{4} N_G(v_i, S)\big| = 3$ and $N_G(v_0, S) \nsubseteq \bigcup\limits_{i=1}^{4} N_G(v_i, S)$, $a_5^{''}$ the number of bad 5-vertex-components F of $G\backslash S$ with $\big|\bigcup\limits_{i=1}^{4} N_G(v_i, S)\big| \geq 4$, and $a_5^{'''}$ the number of bad 5-vertex-components F of $G\backslash S$ with $\big|\bigcup\limits_{i=1}^{4} N_G(v_i, S)\big| = 3$ and $N_G(v_0, S) \subseteq \bigcup\limits_{i=1}^{4} N_G(v_i, S)$.

Therefore, based on Facts (1)-(6) above it follows that

$$\begin{aligned} |T| &= a_1 + 3a_3 + 5(a_5 - a_5^{'} - a_5^{''} - a_5^{'''}) + a_5^{'} + 2a_5^{''} + 3a_5^{'''} \\ &= a_1 + 3a_3 + 5a_5 - 4a_5^{'} - 3a_5^{''} - 2a_5^{'''}, \end{aligned} \tag{12}$$

$$\begin{aligned} |E(G^*)| &\geq 6a_1 + 12a_3 + 12(a_5 - a_5^{'} - a_5^{''} - a_5^{'''}) + 4a_5^{'} + 6a_5^{''} + 8a_5^{'''} \\ &= 6a_1 + 12a_3 + 12a_5 - 8a_5^{'} - 6a_5^{''} - 4a_5^{'''}. \end{aligned} \tag{13}$$

Case 1. $|S| \geq |T| + 1$.

Noting $a_5 \geq a_5^{'} + a_5^{''} + a_5^{'''}$, we have

$$|S| \geq (a_1 + 3a_3 + 5a_5 - 4a_5^{'} - 3a_5^{''} - 2a_5^{'''}) + 1 \geq a_1 + 3a_3 + a_5 + 1.$$

Therefore,

$$4|S| \geq 4(a_1 + 3a_3 + a_5 + 1) \geq 3a_1 + 2a_3 + a_5 + 4. \tag{14}$$

Theorem 4 follows directly from (14) and Proposition 13.

Case 2. $|S| \leq |T|$.

Because G^* is a bipartite 1-planar (simple) graph with bipartite sets $|S|$ and $|T|$, where $|S| \geq 2$, from Lemma 7, we have

$$\begin{aligned}
|E(G^*)| &\leq 2(|T| + |S|) + 4|S| - 12 \\
&= 2(a_1 + 3a_3 + 5a_5 - 4a_5^{'} - 3a_5^{''} - 2a_5^{'''} + |S|) + 4|S| - 12 \\
&= 6|S| + 2a_1 + 6a_3 + 10a_5 - 8a_5^{'} - 6a_5^{''} - 4a_5^{'''} - 12.
\end{aligned} \tag{15}$$

By (13) and (15), we have $3|S| \geq 2a_1 + 3a_3 + a_5 + 6$. On the other hand, $|S| \geq a_1 + 3$ by Lemma 6. Therefore,

$$4|S| \geq (2a_1 + 3a_3 + a_5 + 6) + (a_1 + 3) = 3a_1 + 3a_3 + a_5 + 9 > 3a_1 + 2a_3 + a_5 + 4. \tag{16}$$

Theorem 4 follows directly from (16) and Proposition 13. □

5 Further study

For each 1-planar graph G, as $|E(G)| \leq 4|V(G)| - 8$, we have $\delta(G) \leq 7$. For any positive integer n and δ, where $3 \leq \delta \leq 7$, let $\nu_{n,\delta}$ be the maximum number such that every 1-planar graph G of order n and minimum degree δ has a matching of size $\nu_{n,\delta}$.

Theorem 1 shows that $\nu_{n,3} \geq \frac{n+12}{7}$ when $n \geq 7$, $\nu_{n,4} \geq \frac{n+4}{3}$ when $n \geq 20$, and $\nu_{n,5} \geq \frac{2n+3}{5}$ when $n \geq 21$, while Theorem 4 shows that $\nu_{n,6} \geq \frac{3n+4}{7}$ when $n \geq 36$.

Regarding $\nu_{n,7}$, the following conjecture posed by Biedl and Wittnebel [7] is still open.

Conjecture 2. *There exists an integer N such that $\nu_{n,7} \geq \frac{11n+12}{23}$ when $n \geq N$.*

Biedl [4] studied the maximum size of matchings in 4-connected (resp. 5-connected) 1-planar graphs, and obtained the following results.

Theorem 16 ([4])**.** *(a). For any integer N, there exists a 4-connected 1-planar graph G with $n \geq N$ vertices in which every matching has its size at most $\frac{n+4}{3}$.*

(b). For any integer N, there exists a 5-connected 1-planar graph G with $n \geq N$ vertices in which every matching has its size at most $\frac{n-2}{2}$.

Biedl [4] conjectured that every 5-connected 1-planar graph with n vertices has a matching of size $\frac{n}{2} - O(1)$. Recently this conjecture was disapproved by Huang [10] who shows that for any integer N, there exists a 5-connected 1-planar graph G with $n \geq N$ vertices such that every matching in G has its size at most $\frac{n}{2} - \frac{3}{8}\sqrt{n}$.

We end this article with the following problem on the study of maximum matchings in t-connected 1-planar graphs, where $t \leq 7$.

Problem 1. *Let G be a t-connected 1-planar graph with n vertices, where $t \le 7$, and let $M(G)$ be a maximum matching of G. Study the lower bound of $|M(G)|$. In particular,*

(a). for $t = 5$, is there a constant b such that $|M(G)| \ge \frac{n}{2} - b\sqrt{n}$ when $n \ge N$ for some integer N?

(b). for $t = 6$ or 7, does G have a near-perfect matching, namely, $|M(G)| = \lfloor \frac{n}{2} \rfloor$ when $n \ge N$ for some integer N?

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