MARKED LENGTH SPECTRUM RIGIDITY FOR RELATIVELY HYPERBOLIC GROUPS

THANG NGUYEN AND SHI WANG

ABSTRACT. We consider a coarse version of the marked length spectrum rigidity: given a group with two left invariant metrics, if the marked length spectrum (the translation length function) under the two metrics are the same, then the two metrics are uniformly close. We prove the rigidity theorem for relatively hyperbolic groups. This generalizes a result of Fujiwara [Fuj15].

1. INTRODUCTION

Given a closed Riemannian manifold (M, g), if g has negative curvature, then each free homotopy class c associates a unique closed geodesic γ_c on M. Denote $\mathcal{C}$ the set of all free homotopy classes, or equivalently the set of conjugacy classes of $\pi_1(M)$. The function

$$\ell_g : \mathcal{C} \to \mathbb{R}^{\geq 0}$$

given by $\ell_g(c) := L(\gamma_c)$ is called the *marked length spectrum* of (M, g), where L denotes the length of a curve. The definition extends naturally to manifolds of nonpositive curvature, in which case the closed geodesic might not be unique, but its length is uniquely determined by the homotopy class.

The well-known marked length spectrum rigidity conjecture [BK85] states that if two closed negatively curved Riemannian manifolds have the same marked length spectrum, then they are isometric. This is known to be true for surfaces by the result of Otal [Ota90] and Croke [Cro90] independently. When one of the Riemannian manifolds is rank one locally symmetric, Hamenstädt [Ham99] proved the conjecture using the minimal entropy rigidity theorem of Besson-Courtois-Gallot [BCG95]. More recently, Guillarmou and Lefeuvre [GL19] showed that the conjecture holds if the two metrics are close enough. However, the conjecture remains open in general.

The main purpose of this draft is to consider a "coarse" version of the marked length spectrum rigidity problem. Let Γ be a finitely generated group acting isometrically on a metric space (X, d). A natural generalization of the marked length spectrum is given by the following.

Definition 1.1. *Let Γ be a group acting on a metric space (X, d) by isometries. For every $\gamma \in \Gamma$, the* translation length *of γ is defined as*

$$|\gamma|_\infty = \lim_{n \to +\infty} \frac{d(x, \gamma^n x)}{n},$$

for any (and thus all) $x \in X$. Similarly, we call the function

$$\ell_d : \Gamma \to [0, +\infty)$$

given by $\ell_d(\gamma) = |\gamma|_\infty$ the marked length spectrum *of the Γ-action on (X, d).*

We note that, when (X, d) is a CAT(0) space, the translation length defined above coincides with the usual definition of translation length of an isometry γ, which is defined as

$$\ell(\gamma) = \inf_{x \in X} d(x, \gamma x).$$

In Furman [Fur02], this notion is called *stable length* as it coincides with the usual stable length when X is a Cayley graph of Γ. Here we choose to use the terminology translation length as in Fujiwara's [Fuj15].

The translation length only depends on the conjugacy class of γ. In the case $\Gamma = \pi_1(M)$ where M is a nonpositively curved Riemannian manifold, the above defined marked length spectrum for the action of Γ on the Riemannian universal cover $\widetilde{M}$ coincides with the classical marked length spectrum for (M, g).

Unlike the Riemannian setting, the large scale geometry of the metric spaces is often considered in geometric group theory. Thus, we may naturally consider the following coarse marked length spectrum rigidity problem.

Question 1.2. *Let Γ be a finitely generated group, and let X and Y be metric spaces. We assume that Γ acts on X and Y cocompactly (or more generally coboundedly) by isometries. Suppose the two actions have the same marked length spectrum, is it true that X and Y must be roughly isometric, namely there exists $f : X \to Y$ and there exists $C > 0$ such that f is a $(1, C)$-quasi-isometry?*

We will say two metrics are *roughly equal* if they are $(1, C)$-quasi-isometric for some $C \geq 0$. This is also called "an almost isometry" in the literature and certain rigidity results have been studied in [KLS16, LSvL17]. When the metric spaces are proper (for example in the Riemannian context), a cobounded action is equivalent to a cocompact action. However, there are natural actions of certain groups on nonproper spaces. For example, when Γ is relatively hyperbolic, it naturally acts on a hyperbolic space which may not be proper. Another example is the mapping class group of a surface which acts on the (nonproper) curve complex.

We can further simplify our notions by pulling back the metric from the metric space to the group. Let Γ acts isometrically on a metric space (X, d). By choosing any base point $o \in X$, we may identify Γ with its orbit $\Gamma \cdot o \subset X$,

which is also a metric space viewed as a subspace of (X, d). Equivalently, it gives rise to a left invariant metric d_X on Γ, given by

$$d_X(\gamma_1, \gamma_2) = d(\gamma_1 \cdot o, \gamma_2 \cdot o) \quad \forall \gamma_1, \gamma_2 \in \Gamma.$$

Thus, via the left translation, Γ acts isometrically on (Γ, d_X), and it is clear that the marked length spectrum of this action coincides with that of the action on (X, d). Moreover, such a pull back metric d_X satisfies the following two additional properties:

(1) d_X is quasi-isometric to the word metric of Γ if the action is cocompact: this is due to Švarc-Milnor lemma. Also, the group Γ is finitely generated if the action is properly discontinuous.

(2) d_X is *roughly geodesic* (See the precise definition as follows): This is because (Γ, d_X) can be identified with $\Gamma \cdot o \subset X$, so any pair of points $\gamma_1 \cdot o, \gamma_2 \cdot o$ can be connected by a geodesic in X, and all points on the geodesic are always uniformly close to some orbit point since the action is cobounded.

We recall a definition given by Bonk and Schramm.

Definition 1.3 ([BS00])**.** *Let (X, d) be a metric space and let $\delta \geq 0$. We say the metric d is δ-roughly geodesic metric if for every $x, y \in X$ there is a $(1, \delta)$-quasigeodesic from x to y. A metric is said roughly geodesic if there is $\delta \geq 0$ such that it is δ-roughly geodesic.*

Inspired by the observations, our Question 1.2 can be reformulated (at least for proper metric spaces) as follows (cf. Burago and Margulis [Obe06, Problem session]):

Question 1.4. *Let Γ be a finitely generated group, and d_1, d_2 be two roughly geodesic left invariant metrics that are quasi-isometric to a word metric. If the actions under d_1, d_2 have the same marked length spectrum, then are d_1, d_2 roughly equal, that is, $|d_1 - d_2| \leq C$ for some constant C?*

The main result of this paper is to give an affirmative answer to the question when Γ is a relatively hyperbolic group. It is easy to see that the converse statement is also true: if two metrics are roughly equal, then their marked length spectra are the same.

Theorem 1.5. *Let Γ be a finitely generated relatively hyperbolic group with respect to a family of infinite finitely generated proper peripheral subgroups $\{H_1, ..., H_n\}$. Let d_1 and d_2 be two roughly geodesic left invariant metrics that are quasi-isometric to a proper word metric d. If the actions under d_1, d_2 have the same marked length spectrum, then d_1, d_2 are roughly equal.*

We would like to point out that the roughly geodesic property of the left-invariant metrics is necessary for the theorem. We consider the example where $\Gamma = F_2 = < a, b >$, the free group with two generators a and b. On Γ we let d_1 be the word metric defined by the generating set

$S = \{a, a^{-1}, b, b^{-1}\}$. We also consider another left invariant metric d_2 defined by $d_2(\gamma_1, \gamma_2) = d_1(\gamma_1, \gamma_2) + \sqrt{d_1(\gamma_1, \gamma_2)}$, for every $\gamma_1, \gamma_2 \in \Gamma$. We observe that d_2 is $(2, 1)$-quasi-isometric with d_1. The group Γ is nonelementarily hyperbolic and thus is relatively hyperbolic, for example, relatively to the cyclic subgroup $< a >$. The two metrics d_1 and d_2 have the same marked length spectrum, however they are not roughly equal. The result fails simply because d_2 is *not* a roughly geodesic metric.

Regarding the history, Question 1.4 was previously known to be true when

(1) Γ is abelian by Burago [Bur92],
(2) Γ is a 3-dimensional integer Heisenberg group by Krat [Kra99],
(3) Γ is finitely generated Gromov hyperbolic by Furman [Fur02],
(4) Γ is a reductive Lie groups and the metrics d_1, d_2 are word metrics, by Abels-Margulis [AM04],
(5) Γ is Gromov hyperbolic (a different proof from Furman's) or relatively hyperbolic groups with toral peripheral subgroups by Fujiwara [Fuj15].

Our result further generalizes Fujiwara's as we do not have additional assumptions on the peripheral subgroups. When Γ is non-elementary hyperbolic, we can treat Γ as a relatively hyperbolic group with respect to an infinite cyclic subgroup, thus our result also recovers the hyperbolic case. Moreover, our proof is different from both Fujiwara and Furman's arguments.

It is also worth pointing out that Question 1.4 is known to be *false* in general. Breuillard [Bre14, Section 8.3] gave a counterexample in the case $\Gamma = H^3 \times \mathbb{Z}$ where H^3 is the integer Heisenberg group of dimension 3.

Strategy of the proof. The main technical part of the proof is to establish the genericity of certain elements in Γ with a *good periodicity property*. These are elements g such that for every $n \in \mathbb{N}$, geodesics from 1 to g^n periodically passing through a uniform neigborhood of g^i for each $0 \leq i \leq n$, where the uniform constant depends on the metrics only and is independent of n. One key observation is that: if an element g has good periodicity property and it has the same translation lengths with respect to two metrics d_1 and d_2, then $d_1(1, g)$ and $d_2(1, g)$ must be uniformly close. So the rigidity result holds at least for elements with good periodicity property. Furthermore, we show that for a relatively hyperbolic group, elements with good periodicity property is generic in an appropriate sense, and can be obtained from perturbations of any given element. Thus by triangle inequality, the rigidity result holds for all elements, that is, d_1 and d_2 are roughly equal.

Compared to the previous works of [Bur92, Kra99, Fuj15], our notion of good periodicity property is new and is the main difference. In [Bur92, Kra99, Fuj15], they showed that for all/generic elements $g \in \Gamma$, and for all $n \in \mathbb{N}$, there exists a geodesic from 1 to g^n that are uniformly (does not depends on g and n) close to the union of consecutive geodesic segments from g^{i-1} to g^i, for every $1 \leq i \leq n$. However, this is a much stronger

property. For a general relatively hyperbolic group, we do not expect to have many such elements since the behaviors of geodesics along peripheral subgroups are mysterious. The good periodicity property we introduced is a good replacement and we are able to show for relatively hyperbolic groups, generic elements satisfy this property. The idea is to add a suitable prefix and suffix to any given element so that the resulting element initially travels transversely to all peripheral subsets and eventually stays in a particular peripheral subset. Then using the geometry of relatively hyperbolic group, we show this element has good periodicity property. This is done in Section 3.

It is natural to further ask for which classes of groups our theorem still holds. Our proof, and also the argument of Fujiwara [Fuj15], use certain hyperbolic geometry of the groups, and we believe it is possible to extend to a broader context in which the groups share similar hyperbolic features. On the other hand, the cautionary example of Breuillard indicates that such feature of hyperbolicity may be necessary. We end this section by the following question:

Question 1.6. *Does our theorem hold for products of hyperbolic groups, mapping class groups, or more generally hierarchically hyperbolic groups?*

2. On relatively hyperbolic groups

Gromov in [Gro87] emphasized the parallelism between the notion of hyperbolicity in geometric group theory and manifolds of negative sectional curvature. The fundamental group of a closed negatively curved manifold has such hyperbolicity which is now known as Gromov hyperbolic (or δ-hyperbolic). He also suggested that there should be a suitable notion of *relative hyperbolicity* that serves as an analogous parallelism to include the fundamental group of finite volume negatively curved manifolds or more generally manifolds of non-positive sectional curvature.

Since the first attempt was given by Farb [Far98], there has been many analogous definitions and equivalent characterizations due to the work of Bowditch, Drutu–Sapir, Osin, Groves–Manning, and Sisto [Bow12, DS05, Osi06, GM08, Sis13]. In this paper, we use the notion of *relative hyperbolicity* in the sense of *strongly relative hyperbolicity* defined by Drutu and Sapir [DS05, Definition 8.4].

There are many well-known examples and constructions of relatively hyperbolic groups. Fujiwara, in his work [Fuj15], gave several examples of

relatively hyperbolic groups with toral peripheral subgroups. Since our the-
orem further generalizes to arbitrary relatively hyperbolic groups, we list
here a few more examples of relatively hyperbolic groups whose peripheral
subgroups are not toral.

(1) If G and H are two infinite groups then $G * H$ is a relatively hyper-
    bolic group with respect to the family $\{G, H\}$. If G and H are not
    hyperbolic and not toral then $G * H$ is an example of a relatively
    hyperbolic group with non toral peripheral subgroups.
(2) Let n be an integer that is at least 2. Non-uniform lattices in the fol-
    lowing rank one Lie groups $\mathrm{SU}(n, 1)$, $\mathrm{Sp}(n, 1)$, or F_4^{-20} are examples
    of relatively hyperbolic groups with nilpotent peripheral subgroups.
(3) One can construct closed smooth manifolds with relatively hyper-
    bolic fundamental group using the relative hyperbolization [Bel07]:
    For any compact manifold with boundary (e.g. a hyperbolic mani-
    fold with truncated cusps), after taking the relative hyperbolization,
    one obtains a new smooth manifold with boundary whose fundamen-
    tal group is relatively hyperbolic. Then by taking the double and
    glue along the boundary, one constructs a closed manifold whose
    fundamental group is still relatively hyperbolic.

3. Elements with good periodicity property

For this section, we consider a finitely generated group Γ, which is rel-
atively hyperbolic with respect to a finite family $\{H_1, ..., H_n\}$ of finitely
generated, infinite, proper peripheral subgroups. We note that we do not
lose any generality by assuming all peripheral subgroups are infinite as if any
peripheral subgroup is finite, we can remove it from the list of peripheral
subgroups while still keeping the relative hyperbolicity.

It is shown in [DS05, Theorem 9.1] that the Cayley graph X of Γ is asymp-
totically tree-graded with respect to the corresponding peripheral subsets
$\mathcal{P} = \{gH_i : g \in \Gamma, 1 \le i \le n\}$. Moreover, Sisto [Sis13, Theorem 2.14] proved
that $(X, \mathcal{P})$ is an almost projection system (see definition below). Since the
metric and the sets of peripheral subsets are (left) Γ-invariant, the system of
projections is clearly Γ-equivariant. In this draft, we will mainly use Sisto's
characterization. We recall here some of the definitions and results that we
will heavily use in the proofs later.

Denote d the Cayley metric on X, then (X, d) is a complete geodesic space
where Γ acts isometrically. For simplicity, we identify Γ as the vertex subset
in X, and denote $|g| = d(1, g)$, the norm measured in the word metric.

Definition 3.1. *([Sis13, Definition 1.12]) For every $x \in X$ and for every
peripheral subset $P \in \mathcal{P}$, the projection $\pi_P(x)$ is defined as the set of points
in P whose distances to x is less than $d(x, P) + 1$.*

We note that when X is a Cayley graph of Γ and $\mathcal{P}$ is the collection of
all left cosets of peripheral subgroups, then the projection $\pi_P(x)$ is precisely
the set of points in P whose distance equal $d(x, P)$.

Definition 3.2. *([Sis13, Definition 2.1]) We say a family of maps $\Pi = \{\pi_P : X \to P\}$ is an almost projection system if there exists a constant $C > 0$ such that for any $P \in \mathcal{P}$, the following properities hold:*

(1) $\forall x \in X, \forall p \in P, d(x, p) \geq d(x, \pi_P(x)) + d(\pi_P(x), p) - C$,
(2) $\forall x \in X$ with $d(x, P) = d$, $\mathrm{diam}(\pi_P(B_d(x))) \leq C$,
(3) $\forall P \neq Q \in \mathcal{P}$, $\mathrm{diam}(\pi_P(Q)) \leq C$.

For our purpose, we will only use the property (3) in the above. As a special case, we have $\mathrm{diam}(\pi_P(x)) \leq C$ for any $x \in X$ and $P \in \mathcal{P}$.

Lemma 3.3. *Let $(X, \mathcal{P})$ be an almost projection system. Given $P \in \mathcal{P}$ and $x \in X$. If $z \in \pi_P(x)$ is any point and y is any point on a geodesic segment connecting x and z, then $z \in \pi_P(y)$.*

Proof. First we notice by triangle inequality that $d(x, P) \leq d(x, y) + d(y, P)$. Since $z \in \pi_P(x)$, we have $d(x, z) < d(x, P) + 1$. Since y is a point on a geodesic segment connecting x and z, we have $d(x, z) = d(x, y) + d(y, z)$. Now combining all the (in)equalities, we obtain $d(y, z) < d(y, P) + 1$. Thus by definition, $z \in \pi_P(y)$. $\qquad\square$

Besides, we will also need the following two geometric properties of this almost projection system.

Lemma 3.4. [Sis13, Lemma 2.3] *For every $x, y \in \Gamma$ and for every peripheral $P \in \mathcal{P}$,*
$$d(\pi_P(x), \pi_P(y)) < d(x, y) + 6C,$$
where C is the constant in Definition 3.2.

In what follows and later, we use $\mathrm{Nbhd}_R(A)$ to denote the R-neighborhood of a set A, that is the set of all elements whose distance to the set A is less than R.

Lemma 3.5. [Sis13, Lemma 1.15] *There exists L so that if $d(\pi_P(x), \pi_P(y)) \geq L$, then all (K_0, C_0)-quasi geodesics connecting x, y intersect $\mathrm{Nbhd}_R(\pi_P(x))$ and $\mathrm{Nbhd}_R(\pi_P(y))$ where $R = R(K_0, C_0)$.*

For the convenience, we introduce the following definitions.

Definition 3.6. *Let $T > 0$ be a constant and H be a peripheral subgroup. We say an element $g \in \Gamma$*

(1) has T-short head in H if $d(\pi_H(g), 1) < T$, and
(2) it has T-long tail in H if $d(\pi_{gH}(1), g) > T$.

Remark 3.7. *Note by the Γ-invariance of the metric, g^{-1} has T-short head in H if and only if $d(\pi_{gH}(1), g) < T$, in particular, g does not have T-long tail in H.*

From now on, we let $R = R(1, 0)$ be the constant from Lemma 3.5. We fix a constant R_0 such that

$$(3.1) \qquad\qquad R_0 > \max\{R + 10C, L + 3C\},$$

where C, R, L are defined as above. The following proposition shows that given an arbitrary element in Γ, we can always perturb by adding a prefix so that the new element has short head property.

Proposition 3.8. *For any $g \in \Gamma$ and any peripheral subgroup H, there exists $k \in \Gamma$ which satisfies*

(1) $|k| \leq 1$,
(2) kg has R_0-short head in H.

Proof. We divide into the following two cases.
Case 1: when g already has R_0-short hand in H. That is $d(\pi_H(g), 1) < R_0$ for the given peripheral subgroup H. Then we can simply choose $k = 1$, and the proposition holds automatically.
Case 2: when g does not have R_0-short hand in H. That is $d(\pi_H(g), 1) \geq R_0$ for the given peripheral subgroup H. We choose $k \in \Gamma \backslash H$ to be a generator of Γ so that $|k| = 1$. Such a k always exists since H is a proper subgroup of Γ. We claim that kg has R_0-short head in H. In other words, let $x \in \pi_H(kg)$ be an arbitrary element that achieves $d(1, x) = d(1, \pi_H(kg))$, we need to show $d(1, x) < R_0$.

Since $|k| = 1$, we have both $d(1, k) = d(k, H)$ and $d(1, k^{-1}) = d(k^{-1}, H)$, and it follows that $1 \in \pi_H(k)$ and $1 \in \pi_H(k^{-1})$. The latter containment implies $k \in \pi_{kH}(1) \subset \pi_{kH}(H)$ by the left invariance of the metric. Since $\text{diam}(\pi_{kH}(H)) < C$ according to Definition 3.2, we know $d(k, \pi_{kH}(x)) < C$. By triangle inequality,

$$
\begin{aligned}
&d(\pi_{kH}(x), \pi_{kH}(kg)) \\
&\geq d(k, \pi_{kH}(kg)) - d(k, \pi_{kH}(x)) - \text{diam}(\pi_{kH}(kg)) - \text{diam}(\pi_{kH}(x)) \\
&\geq d(k, \pi_{kH}(kg)) - d(k, \pi_{kH}(x)) - 2C \\
&\geq d(k, \pi_{kH}(kg)) - 3C \\
&= d(1, \pi_H(g)) - 3C \\
&\geq R_0 - 3C,
\end{aligned}
$$

where the second inequality uses Definition 3.2. By the choice of R_0, we have that $R_0 - 3C > L$ where L is the constant from Lemma 3.5. It follows by this lemma that any geodesic from kg to x passes through a point y in R-neighborhood of $\pi_{kH}(x)$ (recall $R = R(1, 0)$). Since $d(k, \pi_{kH}(x)) < C$, the point y is contained in $(R + C)$-neighborhood of k, that is $d(y, k) \leq R + C$. On the other hand, since the geodesic from kg to $x \in \pi_H(kg)$ passes through y, by Lemma 3.3 the point x is in $\pi_H(y)$. Therefore, we have

$$
\begin{aligned}
d(\pi_H(kg), 1) = d(x, 1) &\leq d(\pi_H(y), 1) + C \\
&\leq d(\pi_H(y), \pi_H(k)) + d(\pi_H(k), 1) + 2C + C \\
&\leq (R + C) + 6C + 0 + 3C. \\
&\leq R + 10C,
\end{aligned}
$$

where the first inequality uses the fact that $x \in \pi_H(y)$ and also Definition 3.2, the second inequality follows from the triangle inequality and the third inequality uses Lemma 3.4. With the choice of R_0 such that $R_0 > R + 10C$, we have that $d(\pi_H(kg), 1) < R_0$. $\qquad \square$

In the next proposition, we show that the above element can be further perturbed by adding a suffix to obtain an element having the long tail property, while keeping the short head property.

Proposition 3.9. *Let $g \in G$ such that $d(1, g) > R_0 + 1$. Then for every $C_3 \geq \max\{1, C\}$, if there is a peripheral subgroup H such that g has R_0-short head, then there exists a perturbation $g' = gh$ such that*

(1) $|h| \leq 4C_3$,
(2) g' has C_3-long tail in H, and
(3) g' has $(R_0 + C)$-short head in H.

Proof. If $d(\pi_{gH}(1), g) > C_3$, we then just choose $h = 1$. Thus, we assume that $d(\pi_{gH}(1), g) \leq C_3$. Since H is an infinite subgroup, we can choose $h \in H$ such that $3C_3 < |h| \leq 4C_3$, and let $g' = gh$. By triangle inequality,

$$d(\pi_{g'H}(1), g') = d(\pi_{gH}(1), g') \geq d(g, g') - d(\pi_{gH}(1), g) - C$$
$$> 3C_3 - C_3 - C$$
$$\geq C_3.$$

It remains to check that the new element g' still has short head property. That is to check $d(\pi_H(g'), 1) < R_0 + C$. For this, we first note that $g \notin H$. Otherwise $g \in H$ implies that $\pi_H(g) = B_g(1)$, the ball radius 1 around g. Thus by triangle inequality $d(1, g) < d(1, \pi_H(g)) + 1$. Since $d(1, g) > R_0 + 1$, we have $d(1, \pi_H(g)) > R_0$, which contradicts with the assumption that g has R_0-short head in H.

Therefore $\text{diam}(\pi_H(g'H)) = \text{diam}(\pi_H(gH)) < C$ by Definition 3.2. It follows by triangle inequality that $d(\pi_H(g'), 1) \leq d(\pi_H(g), 1) + C < R_0 + C$. $\qquad \square$

Elements with short head and long tail in a peripheral subgroup will be shown later to satisfy *the good periodicity property*, and Proposition 3.8 and Proposition 3.9 simply descibes how such an element can be obtained from a perturbation.

Next we show that if an element g has short head and long tail in some peripheral subgroup H, then its power g^n also has short head and long tail in H, with possibly worse constants, but they are uniform on n.

Proposition 3.10. *For any $\delta > 0$, there exist constants $C_1, C_2, C_3 > 0$ explicitly given by*

$$C_1 = \delta + R + 9C$$
$$C_2 = C_1 + L + 3C$$

and

$$C_3 = C_2 + R + 10C + \delta + 2,$$

such that the following holds: for every $g \in G$, if there exists a peripheral subgroup H such that g has δ-short head and C_3-long tail in H, then for any $n \in \mathbb{N}$, we have

(1) *g^n has C_1-short head in H.*
(2) *g^n has C_2-long tail in H.*

Proof. We first note that the element g which satisfies the condition cannot belong to the peripheral subgroup H. Otherwise, $\pi_H(g)$ consists of elements in H whose distance to g are at most 1, so by triangle inequality $d(\pi_H(g), 1) \geq d(g, 1) - 1$. On the other hand, since g has C_3-long tail in H, we have $d(\pi_{gH}(1), g) > C_3$. It follows similarly that $d(\pi_{gH}(1), g) < d(1, g) + 1$. Thus, combining with the previous inequalities, we have

$$d(g, \pi_{gH}(1)) > C_3 - 2 > \delta,$$

which contradicts the δ-short head assumption of g.

For every $n \in \mathbb{N}$, we let $P(n)$ be the statement that (1) and (2) hold for n. We prove by induction that $P(n)$ holds for every $n \in \mathbb{N}$.

Base case $n = 1$: This follows immediately from the assumption of g and the choices that $C_1 > \delta$ and $C_3 > C_2$.

Induction hypothesis: Suppose $P(n)$ holds for all $n \leq k$ (where $k \in \mathbb{N}$).

Induction step: We need to show $P(k + 1)$ holds. We prove (1) and (2) separately, that is, $d(\pi_H(g^{k+1}), 1) < C_1$ and $d(\pi_{g^{k+1}H}(1), g^{k+1}) > C_2$ respectively.

Proof of (1): Let $x \in \pi_H(g^{k+1}) \subset H$ be an arbitrary element. We denote γ a geodesic segment from g^{k+1} to x. By induction hypothesis, we have

$$d(\pi_{gH}(g^{k+1}), g) = d(\pi_H(g^k), 1) < C_1,$$

and also $d(\pi_{gH}(1), g) > C_2$. Since $g \notin H$, by Definnition 3.2 we have $\text{diam}(\pi_{gH}(H)) < C$. So by the triangle inequality, we have

$$d(\pi_{gH}(x), g) \geq d(\pi_{gH}(1), g) - \text{diam}(\pi_{gH}(H)) > C_2 - C.$$

Thus,

$$\begin{aligned}
&d(\pi_{gH}(x), \pi_{gH}(g^{k+1})) \\
&\geq d(\pi_{gH}(x), g) - d(\pi_{gH}(g^{k+1}), g) - \text{diam}(\pi_{gH}(x)) - \text{diam}(\pi_{gH}(g^{k+1})) \\
&\geq C_2 - C - C_1 - C - C \\
&\geq L,
\end{aligned}$$

by the choice of C_1, C_2. Apply Lemma 3.5 on γ, we see that γ intersects with $\text{Nbhd}_R(\pi_{gH}(x))$, and if we let $z \in \gamma \cap \text{Nbhd}_R(\pi_{gH}(x))$ and let $z' \in \pi_{gH}(x)$ such that $d(z, z') < R$, then by Lemma 3.4, we have

$$\begin{aligned}
d(\pi_H(z), \pi_H(z')) &< d(z, z') + 6C \\
&\leq R + 6C.
\end{aligned}$$

Since γ is a geodesic connecting g^{k+1} with x and $x \in \pi_H(g^{k+1})$, we have $x \in \pi_H(z)$ by Lemma 3.3. On the other hand, since both z' and g belongs to

the peripheral subset gH, we see that $\pi_H(z')$ and $\pi_H(g)$ are both contained in the set $\pi_H(gH)$ whose diameter is bounded by C. Thus we have

$$d(1, \pi_H(z')) \leq d(1, \pi_H(g)) + C < \delta + C,$$

where the last inequality uses the assumption that g has δ-short head in H.

Finally, we can estimate

$$\begin{aligned}
d(1, x) &\leq d(1, \pi_H(z')) + d(\pi_H(z'), \pi_H(z)) + d(\pi_H(z), x) + 2C \\
&< (\delta + C) + (R + 6C) + 0 + 2C \\
&= C_1,
\end{aligned}$$

by the choice of C_1. Thus, (1) of $P(k+1)$ holds.

Proof of (2): Let $y \in \pi_{g^{k+1}H}(1)$ be arbitrary. We denote γ' a geodesic path from 1 to y. By induction hypothesis, we have

$$d(\pi_{g^k H}(g^{k+1}), g^k) = d(\pi_H(g), 1) < \delta,$$

and also $d(\pi_{g^k H}(1), g^k) > C_2$. Note that

$$\mathrm{diam}(\pi_{g^k H}(g^{k+1}H)) = \mathrm{diam}(\pi_H(gH)) < C$$

since $g \notin H$. So $d(\pi_{g^k H}(y), \pi_{g^k H}(g^{k+1})) \leq \mathrm{diam}(\pi_{g^k H}(g^{k+1}H) \leq C$. Thus by the triangle inequality, we have

$$\begin{aligned}
d(\pi_{g^k H}(y), \pi_{g^k H}(1)) &\geq d(\pi_{g^k H}(1), g^k) - d(\pi_{g^k H}(g^{k+1}), g^k) - \mathrm{diam}(\pi_{g^k H}(g^{k+1})) \\
&> C_2 - \delta - C \\
&> L.
\end{aligned}$$

By Lemma 3.5, we have $\gamma' \cap \mathrm{Nbhd}_R(\pi_{g^k H}(y)) \neq \emptyset$. We choose $w \in \pi_{g^k H}(y)$ and $w' \in \gamma' \cap \mathrm{Nbhd}_R(\pi_{g^k H}(y))$ such that $d(w, w') < R$. Then by triangle inequality, we have

$$\begin{aligned}
d(w', g^k) &\leq d(w', \pi_{g^k H}(y)) + \mathrm{diam}(\pi_{g^k H}(g^{k+1}H)) + d(\pi_{g^k H}(g^{k+1}), g^k) \\
&< d(w', w) + C + d(\pi_H(g), 1) \\
&< R + C + \delta,
\end{aligned}$$

where the second inequality follows from the fact that $w \in \pi_{g^k H}(y)$, by Lemma 3.3. On the other hand, since γ' is a geodesic and $y \in \pi_{g^{k+1}H}(1)$, by Lemma 3.3 we have $y \in \pi_{g^{k+1}H}(w')$. It follows by Lemma 3.4 that

$$\begin{aligned}
d(y, \pi_{g^{k+1}H}(g^k)) &\leq d(y, \pi_{g^{k+1}H}(w')) + d(\pi_{g^{k+1}H}(w'), \pi_{g^{k+1}H}(g^k)) \\
&\quad + \mathrm{diam}(\pi_{g^{k+1}H}(w')) + \mathrm{diam}(\pi_{g^{k+1}H}(g^k)) \\
&< 0 + d(w', g^k) + 6C + 2C \\
&< R + C + \delta + 8C \\
&= R + 9C + \delta.
\end{aligned}$$

Note that $d(\pi_{g^{k+1}H}(g^k), g^{k+1}) = d(\pi_{gH}(1), g) > C_3$ by the assumption. Thus,

$$
\begin{aligned}
d(y, g^{k+1}) &\geq d(\pi_{g^{k+1}H}(g^k), g^{k+1}) - d(y, \pi_{g^{k+1}H}(g^k)) - \mathrm{diam}(\pi_{g^{k+1}H}(g^k)) \\
&> C_3 - (R + 9C + \delta) - C \\
&> C_2.
\end{aligned}
$$

Hence (2) of $P(k+1)$ holds.

Therefore, by induction, the proposition holds.   □

We now fix all constants so that they depend only on the projection system, or equivalently a finitely generated word metric on Γ. We first fix R_0 by choosing

$$(3.2) \qquad\qquad R_0 = R + L + 10C,$$

where we recall that $R = R(1,0)$ and L are constants from Lemma 3.5. In particular, it satisfies inequality (3.1). Besides, we also fix $\delta = R_0 + C$ as in the following proposition. We emphasize that by doing so, all constants including R, L, C, C_1, C_2, and C_3 are now determined and they only depend on the projection system. We summarize the above discussions into the following proposition, which will be the main technical preparation of our theorem.

Proposition 3.11. *There exist C_1, C_2, C_3 explicitly given by*

$$C_1 = \delta + R + 9C$$

$$C_2 = C_1 + L + 3C$$

$$C_3 = C_2 + R + 10C + \delta + 2,$$

where

$$\delta = R_0 + C = R + L + 11C,$$

with the following properties: for every element $g_0 \in \Gamma$ with $|g_0| > R_0 + 2$, there exist an element $g = kg_0h \in \Gamma$ and a peripheral subgroup H such that

(1) $|k| \leq 1$.
(2) $|h| \leq 4C_3$.
(3) for every $n \in \mathbb{N}$, g^n has C_1-short head and C_2-long tail in H.

Proof. Given any element $g_0 \in \Gamma$, by Proposition 3.8, there exist $k \in \Gamma$ and a peripheral subgroup H such that $|k| \leq 1$ and kg_0 has R_0-short head in H. Since $|g_0| > R_0 + 2$, by triangle inequality we have $|kg_0| \geq |g_0| - |k| > R_0 + 1$. By applying Proposition 3.9 on the element kg_0, there exists $h \in \Gamma$ such that $|h| < 4C_3$ and kg_0h has $(R_0 + C)$-short head, C_3-long tail in H. Apply Proposition 3.10 to the element $g = kg_0h$ and on the scale $\delta = R_0 + C$, we conclude that g^n has C_1-short head and C_2-long tail in H for any $n \in \mathbb{N}$.   □

One significance of obtaining such an element is that all its iterated powers belong to a uniform neighborhood of a geodesic. This is what we were referring to as *the good periodicity property* in the introduction, and we will

show in the next proposition. Thus, together with the above proposition, we show that up to appropriate perturbations, essentially all elements in a relatively hyperbolic group will have good periodicity property.

Proposition 3.12. *Let g be an element such that for every $n \in \mathbb{N}$, g^n has C_1-short head and C_2-long tail in a peripheral subgroup H, where C_1 and C_2 are constants from Proposition 3.11. For every $\tau \geq 1$ and $\eta \geq 0$, there is a constant $\kappa > 0$, depending only on τ, η and Γ, such that for every $n \in \mathbb{N}$ and for every (τ, η)-quasigeodesic γ from 1 to g^n, g^i is in κ-neighborhood of γ for each $0 \leq i \leq n$.*

Proof. For every $0 < i < n$, since g^i has C_2-long tail in H, we know $d(\pi_{g^i H}(1), g^i) > C_2$. Since g^{n-i} has C_1-short head in H, we have

$$d(g^i, \pi_{g^i H}(g^n)) = d(1, \pi_H(g^{n-i})) < C_1.$$

Thus, by triangle inequality, we have

$$d(\pi_{g^i H}(1), \pi_{g^i H}(g^n))$$
$$\geq d(\pi_{g^i H}(1), g^i) - d(g^i, \pi_{g^i H}(g^n)) - \text{diam}(\pi_{g^i H}(g^n)) - \text{diam}(\pi_{g^i H}(1))$$
$$> C_2 - C_1 - 2C$$
$$\geq L$$

by the choice of C_1, C_2. By Lemma 3.5, there exists a constant $R(\tau, \eta)$ such that γ intersects the neighborhood $\text{Nbhd}_{R(\tau,\eta)}(\pi_{g^i H}(g^n))$. On the other hand, since $d(\pi_{g^i H}(g^n), g^i) = d(\pi_H(g^{n-i}), 1) < C_1$, it follows by the triangle inequality that g^i is in $(R(\tau, \eta) + C_1 + C)$-neighborhood of γ.

For $i = 0$ or $i = n$, g^i is in the η-neighborhood of γ. Hence if we choose $\kappa = R(\tau, \eta) + C_1 + C + \eta$, the proposition follows. $\square$

4. RIGIDITY

In this section, we assume that d_1 and d_2 are two left invariant roughly geodesic metrics on a finitely generated relatively hyperbolic group Γ. Furthermore, assume that d_1 and d_2 have the same marked length spectrum. We let $\Delta = |d_1 - d_2|$. Suppose that both metrics are (L_0, C_0)-quasi-isometric to a word metric d defined by a finite generating set.

Fujiwara proved the following almost additivity for the difference between two metrics.

Lemma 4.1. [Fuj15, Lemma 2.3] *For $i = 1, 2$, let γ_i be a $(1, \delta)$-roughly geodesic from x to y with respect to d_i metric. For every $\kappa > 0$ there exists $\overline{\kappa} > 0$ such that the following holds: if $z \in \gamma_1$ is a point such that there exists $z' \in \gamma_2$ with $d_2(z, z') \leq \kappa$, then $|\Delta(x, z) + \Delta(z, y) - \Delta(x, y)| < \overline{\kappa}$.*

We are now ready to give a proof of our main theorem.

Proof of Theorem 1.5. Suppose $\tau \geq 1$ and $\eta \geq 0$ are constants such that d_1 and d_2 are (τ, η)-quasi-isometric to a word metric d. The marked length

spectrum assumption implies that for every $g \in \Gamma$, we have the following sub-linear growth $\Delta(1, g^n) = o(n)$.

Suppose, for the sake of contradiction, that d_1 and d_2 are not roughly equal. Thus for every $\mathcal{E} > 0$, there is an element (hence infinitely many elements) $g' \in \Gamma$ such that $\Delta(1, g') > \mathcal{E}$. By possibly avoiding finitely many elements, we may assume $|g'| > R_0 + 2$ where we recall $|\cdot|$ is the norm measured in word metric d and R_0 is the constant chosen in (3.2). By Proposition 3.11, there is a peripheral subgroup H, elements $k, h \in \Gamma$, and constants C_1, C_2, C_3 such that

(1) $|d(1, k)| \leq 1$,
(2) $|d(1, h)| \leq 4C_3$,
(3) For every n, the element g^n has C_1-short head and C_2-long tail in H, where $g = kg'h$.

Since d_1 and d_2 are roughly geodesic, there is δ_0 such that for any $x, y \in \Gamma$ are $(1, \delta_0)$ quasi-geodesics from x to y with respect to d_1 and d_2. As d_1 and d are (τ, η) quasi-isometric, any $(1, \delta_0)$-quasi-geodesic in d_1 metric is a $(\tau, \tau\delta_0 + \eta)$-quasi-geodesic in the word metric d.

By Proposition 3.12, there exists κ for every $n \in \mathbb{N}$, any $(1, \delta_0)$-quasi-geodesic γ_n in d_1 metric from 1 to g^n passes through κ-neighborhoods of g^i for every $0 \leq i \leq n$. Similarly, for every $(1, \delta_0)$-quasi-geodesic β_n from 1 to g^n in d_2 metric, β_n passes through κ-neighborhoods of g^i for every $0 \leq i \leq n$. By Lemma 4.1, there exists $\overline{\kappa} \geq 0$ such that

$$|\Delta(1, g) + \Delta(g, g^2) + \cdots + \Delta(g^{n-1}, g^n) - \Delta(1, g^n)| < n\overline{\kappa}.$$

By left-invariance of d_1, d_2, and thus of Δ, this is equivalent with

$$|n\Delta(1, g) - \Delta(1, g^n)| < n\overline{\kappa}.$$

Therefore, $\Delta(1, g^n) > n(\Delta(1, g) - \overline{\kappa})$. We note that C_1, C_2, C_3 and thus κ, $\overline{\kappa}$ depend only on δ_0, τ, η, and the metric d, but they are independent of the constant $\mathcal{E}$. Since $g = kg'h$ with $|d(1, k)| \leq 1$ and $|d(1, h)| \leq 4C_3$, using triangle inequalities we have that

$$\Delta(1, g) > \Delta(1, g') - 2(\tau + \eta + \tau \cdot 4C_3 + \eta).$$

Hence if we choose $\mathcal{E} > 2(\tau + \eta + \tau \cdot 4C_3 + \eta) + \overline{\kappa} + 1$ then $\Delta(1, g^n) > n$ for every $n \in \mathbb{N}$. This contradicts with the sub-linear growth $\Delta(1, g^n) = o(n)$.

Therefore d_1 and d_2 are roughly equal. $\square$

As an immediate corollary of our Theorem 1.5, we have the following:

Corollary 4.2. *Let M be a closed smooth manifold whose fundamental group is relatively hyperbolic. If g_1 and g_2 are two non-positively curved Riemannian metrics with the same marked length spectrum, then there exists a constant $C \geq 0$ and a Γ-equivariant $(1, C)$-quasi-isometry between their universal covers $(\widetilde{M}, \widetilde{g}_1)$ and $(\widetilde{M}, \widetilde{g}_2)$.*

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Florida State University, Tallahassee, Florida, USA
Email address: thang.q.nguyen7@gmail.com

ShanghaiTech University, Shanghai, China
Email address: shiwang.math@gmail.com