BANACH-MAZUR DISTANCE FROM ℓ_p^3 TO ℓ_∞^3

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ABSTRACT. The maximum of the Banach-Mazur distance $d_{BM}^M(X, \ell_\infty^n)$, where X ranges over the set of all n-dimensional real Banach spaces, is difficult to compute. In fact, it is already not easy to get the maximum of $d_{BM}^M(\ell_p^n, \ell_\infty^n)$ for all $p \in [1, \infty]$. We prove that $d_{BM}^M(\ell_p^3, \ell_\infty^3) \leq 9/5$, $\forall p \in [1, \infty]$. As an application, the following result related to Borsuk's partition problem in Banach spaces is obtained: any subset A of ℓ_p^3 having diameter 1 is the union of 8 subsets of A whose diameters are at most 0.9.

1. Introduction

The (multiplicative) *Banach-Mazur distance* between two isomorphic Banach spaces X and Y is defined as

$$d_{BM}^M(X, Y) = \inf \left\{ \|T\| \cdot \left\|T^{-1}\right\| \;\middle|\; T \text{ is an isomorphism from } X \text{ onto } Y \right\}.$$

It is well known that

$$d_{BM}^M(X, Y) \leq d_{BM}^M(X, Z) \cdot d_{BM}^M(Z, Y),$$

where X, Y, and Z are isomorphic Banach spaces, see, e.g., [10].

A compact convex subset of $\mathbb{R}^n$ having interior points is called a *convex body*. Let $\mathcal{K}^n$ be the set of all convex bodies in $\mathbb{R}^n$ and $\mathcal{C}^n$ be the set of convex bodies that are symmetric with respect to the *origin* o of $\mathbb{R}^n$. Let $\mathcal{A}^n$ be the set of all nonsingular affine transformations on $\mathbb{R}^n$. The Banach-Mazur distance between $K, L \in \mathcal{K}^n$ is defined by

$$d_{BM}^M(K, L) = \inf \left\{ \gamma \geq 1 \mid \exists T \in \mathcal{A}^n,\ x \in \mathbb{R}^n,\ \text{s.t. } T(L) \subseteq K \subseteq \gamma T(L) + x \right\}.$$

The infimum can be attained. When $K, L \in \mathcal{C}^n$, one can verify that

$$d_{BM}^M(K, L) = \inf \left\{ \gamma \geq 1 \mid \exists T \in \mathcal{T}^n,\ \text{s.t. } T(L) \subseteq K \subseteq \gamma T(L) \right\},$$

where $\mathcal{T}^n$ is the set of all nonsingular linear transformations on $\mathbb{R}^n$. Denote by B_X the unit ball of an n-dimensional Banach space $X = (\mathbb{R}^n, \|\cdot\|)$. We have $d_{BM}^M(X, Y) = d_{BM}^M(B_X, B_Y)$, which connects the Banach-Mazur distance between finite dimensional Banach spaces with the Banach-Mazur distance between two convex bodies (cf. e.g., [2, p. 15, p. 47]) and provides a link between Banach space theory and convex geometry. It is generally difficult to calculate the exact value of the Banach-Mazur distance between convex bodies (or isomorphic Banach spaces).

Denote by ℓ_p^n the space $(\mathbb{R}^n, \|\cdot\|_p)$, where the p-norm $\|\cdot\|_p$ is given by

$$\|(\alpha_1, \cdots, \alpha_n)\|_p = \left(\sum_{i \in [n]} |\alpha_i|^p \right)^{\frac{1}{p}},\ \forall p \in [1, \infty),$$

and

$$\|(\alpha_1, \cdots, \alpha_n)\|_\infty = \max_{i \in [n]} |\alpha_i|.$$

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Here we used the shorthand notation $[n] := \{i \in \mathbb{Z}^+ \mid 1 \le i \le n\}$. Denote by B_p^n the unit ball of ℓ_p^n. Clearly, $B_\infty^n = [-1,1]^n$. We have the following classical result:

Theorem 1 (cf. [10, Proposition 37.6])**.** *Let n be a positive integer and $1 \le p, q \le \infty$.*

- (i) *If $1 \le p \le q \le 2$ or $2 \le p \le q \le \infty$, then $d_{BM}^M(\ell_p^n, \ell_q^n) = n^{1/p-1/q}$.*
- (ii) *If $1 \le p < 2 < q \le \infty$, then $\gamma n^\alpha \le d_{BM}^M(\ell_p^n, \ell_q^n) \le \eta n^\alpha$, where $\alpha = \max\{1/p - 1/2\,, 1/2 - 1/q\}$, and γ, η are universal constants. If $n = 2^k$ ($k \in \mathbb{N}$), then $\eta = 1$.*

From Theorem 1, it follows that $d_{BM}^M(\ell_p^n, \ell_\infty^n) = n^{1/p}$, $\forall p \in [2,\infty]$. In general, it is difficult to get the exact value of $d_{BM}^M(\ell_p^n, \ell_\infty^n)$ for $p \in [1,2)$. The case when $n = 2$ is an exception. Since ℓ_1^2 and ℓ_∞^2 are isometric,

$$d_{BM}^M(\ell_p^2, \ell_\infty^2) = d_{BM}^M(\ell_p^2, \ell_1^2) = 2^{1-1/p}, \ \forall p \in [1,2).$$

When $n = 2^k$ for some $k \in \mathbb{N}$, we have $d_{BM}^M(\ell_p^n, \ell_\infty^n) \le \sqrt{n}$, $\forall p \in [1,\infty]$. In particular, we have

$$d_{BM}^M(\ell_p^4, \ell_\infty^4) \le 2, \ \forall p \in [1,\infty]. \tag{1}$$

F. Xue [11] provided explicit upper bounds of $d_{BM}^M(\ell_1^n, \ell_\infty^n)$ for $n \in \{3,4,5,6,7,8\}$. and showed that

$$\alpha\sqrt{n} \le d_{BM}^M(\ell_1^n, \ell_\infty^n) \le (\sqrt{2}+1)\sqrt{n}, \ \forall n \in \mathbb{Z}^+,$$

where α is an absolute constant (cf. [11, Theorem 1.5]).

When $n = 3$, Y. Lian and S. Wu [6] proved that

$$d_{BM}^M(\ell_p^3, \ell_\infty^3) \le \frac{\sqrt{18 \cdot 19}}{10}, \ \forall p \in [1,2].$$

In this paper, we improve this result as follows:

Theorem 2. *We have*

$$d_{BM}^M(\ell_p^3, \ell_\infty^3) \le \frac{9}{5}, \ \forall p \in [1,\infty]. \tag{2}$$

Most likely, the estimations (1) and (2) are both tight. By Theorem 1, Theorem 2, and [6, Theorem 2], we have the following improvment of [6, Theorem 16]:

Corollary 3. *For each $p \in [1,\infty]$, any set A of ℓ_p^3 having diameter 1 is the union of 8 subsets of A whose diameters are at most 0.9.*

This result is closely related to Borsuk's partition problem in finite dimensional Banach spaces, see [6, 13] for more details. For Borsuk's problem in ℓ_2^3, Tolmachev et al. [9] proved that, if the diameter of $A \subseteq \ell_2^3$ is 1, then A can be partitioned into four subsets whose diameters are at most 0.966. Note that, a closed ball in ℓ_∞^3 cannot be split into 7 subsets having smaller diameters. Therefore we cannot replace 8 with a positive integer $m \le 7$ and obtain a result similar to Corollary 3.

2. Banach-Mazur distance to ℓ_∞^n

Denote by $\mathrm{GL}_n(\mathbb{R})$ the set of all nonsingular $n \times n$ matrices of real numbers. For $K, L \in \mathcal{C}^n$ and $A \in \mathrm{GL}_n(\mathbb{R})$, set

$$\gamma_1(K,L;A) = \inf\{\gamma \mid \gamma > 0 \text{ and } A(L) \subseteq \gamma K\},$$
$$\gamma_2(K,L;A) = \sup\{\gamma \mid \gamma > 0 \text{ and } \gamma K \subseteq A(L)\}.$$

Here we identify a member of $\mathrm{GL}_n(\mathbb{R})$ with the corresponding nonsingular linear transformation. Since both K and $A(L)$ contain the origin o in their interior,

$\gamma_1(K, L; A)$ and $\gamma_2(K, L; A)$ are well-defined and are positive. Moreover, since K and $A(L)$ are both compact, inf and sup in the definitions above can be replaced with min and max, respectively.

Inspired by the proof of [6, Lemma 14], we have Lemma 4 and Lemma 5.

Lemma 4. *For $K, L \in \mathcal{C}^n$,*

$$d_{BM}^M(K, L) = \min\left\{ \frac{\gamma_1(K, L; A)}{\gamma_2(K, L; A)} \;\middle|\; A \in \mathrm{GL}_n(\mathbb{R}) \right\}.$$

Proof. Let B be an arbitrary element of $\mathrm{GL}_n(\mathbb{R})$. By the definitions of $\gamma_1(K, L; B)$ and $\gamma_2(K, L; B)$, we have $\gamma_2(K, L; B)K \subseteq B(L) \subseteq \gamma_1(K, L; B)K$, or equivalently,

$$\left(\frac{1}{\gamma_1(K, L; B)} B\right)(L) \subseteq K \subseteq \frac{\gamma_1(K, L; B)}{\gamma_2(K, L; B)} \left(\frac{1}{\gamma_1(K, L; B)} B\right)(L).$$

It follows that

$$d_{BM}^M(K, L) \leq \frac{\gamma_1(K, L; B)}{\gamma_2(K, L; B)}.$$

Hence

$$d_{BM}^M(K, L) \leq \inf\left\{ \frac{\gamma_1(K, L; A)}{\gamma_2(K, L; A)} \;\middle|\; A \in \mathrm{GL}_n(\mathbb{R}) \right\}.$$

Conversely, there exists $A_0 \in \mathrm{GL}_n(\mathbb{R})$ such that $A_0(L) \subseteq K \subseteq d_{BM}^M(K, L)A_0(L)$. Then $\gamma_1(K, L; A_0) \leq 1$ and $\gamma_2(K, L; A_0) \geq \left(d_{BM}^M(K, L)\right)^{-1}$. Hence

$$d_{BM}^M(K, L) \geq \frac{\gamma_1(K, L; A_0)}{\gamma_2(K, L; A_0)} \geq \inf\left\{ \frac{\gamma_1(K, L; A)}{\gamma_2(K, L; A)} \;\middle|\; A \in \mathrm{GL}_n(\mathbb{R}) \right\}.$$

This completes the proof. □

Let $(\mathbb{R}^n, \|\cdot\|)$ be a Banach space and let $(\mathbb{R}^n, \|\cdot\|_*)$ be its dual. Each $y \in \mathbb{R}^n$ defines a linear functional f on $(\mathbb{R}^n, \|\cdot\|)$ by $f(x) = y^T \cdot x$. When $y \neq o$, we have

$$\|f\|_* = \left(d(o, \{x \in \mathbb{R}^n \mid f(x) = 1\})\right)^{-1},$$

where $d(\cdot, \cdot)$ is the distance on $\mathbb{R}^n$ induced by $\|\cdot\|$.

Lemma 5. *Let $X = (\mathbb{R}^n, \|\cdot\|)$ be a Banach space and $A = (a_{ij})_{n\times n} \in \mathrm{GL}_n(\mathbb{R})$. Denote by A_{ij} the cofactor of a_{ij}, and set*

$$x_i = (a_{1i}, a_{2i}, \ldots, a_{ni})^\mathsf{T}, \quad y_j = (A_{1j}, A_{2j}, \ldots, A_{nj})^\mathsf{T}, \ \forall i, j \in [n].$$

Then

$$\gamma_1(B_X, B_\infty^n; A) = \max\left\{ \left\| \sum_{i\in[n]} \sigma_i x_i \right\| \;\middle|\; \sigma_i \in \{-1, 1\},\ \forall i \in [n] \right\}, \tag{3}$$

$$\gamma_2(B_X, B_\infty^n; A) = \min\left\{ \frac{|\det A|}{\|y_i\|_*} \;\middle|\; i \in [n] \right\}. \tag{4}$$

In particular,

$$d_{BM}^M(X, \ell_\infty^n)$$
$$= \min_{A\in\mathrm{GL}_n(\mathbb{R})} \max\left\{ (|\det A|)^{-1}\|y_i\|_* \left\| \sum_{j\in[n]} \sigma_j x_j \right\| \;\middle|\; i \in [n],\ \sigma_j \in \{-1, 1\},\ \forall j \in [n] \right\}.$$

Proof. Evidently, $A(B_\infty^n) = A([-1, 1]^n)$ is a convex polytope with

$$\left\{ \sum_{i\in[n]} \sigma_i x_i \;\middle|\; \sigma_i \in \{-1, 1\},\ \forall i \in [n] \right\}$$

as the set of vertices. Moreover, $A(B_\infty^n)$ is contained in γB_X if and only if every vertex of $A(B_\infty^n)$ is contained in γB_X. Thus (3) holds.

For each $i \in [n]$, let H_i be the hyperplane passing through x_i and parallel to the hyperplane spanned by $\{x_j \mid j \in [n] \setminus \{i\}\}$. We easily verify that, $\pm H_1, \dots, \pm H_n$ are the bounding hyperplanes of $A(B_\infty^n)$. For each $i \in [n]$, the null space of the linear functional f_i defined by $f_i(x) = y_i^\intercal \cdot x$ is precisely $\operatorname{span}\{x_j \mid j \in [n] \setminus \{i\}\}$. Thus

$$H_i = \{x \in \mathbb{R}^n \mid y_i^\intercal \cdot x = y_i^\intercal \cdot x_i\} = \left\{ x \in \mathbb{R}^n \;\middle|\; \frac{y_i^\intercal}{\det A} \cdot x = 1 \right\}.$$

It follows that

$$d(o, H_i) = \left(\left\| \frac{y_i}{\det A} \right\|_* \right)^{-1} = \frac{|\det A|}{\|y_i\|_*}.$$

Assume that $\gamma > 0$. Then $\gamma B_X \subseteq A(B_\infty^n)$ if and only if $\gamma \leq \min\{d(o, H_i) \mid i \in [n]\}$. Hence the equality (4) follows. □

Remark 6. Clearly, $\gamma_2(B_X, B_\infty^n; A)$ is the reciprocal of the operator norm $\|A^{-1}\|$ of A^{-1}. We can also deduce (4) using the fact that $\|A^{-1}\|$ equals the operator norm of its adjoint (cf. e.g., [1, Lemma 9.1]).

Remark 7. Set

$$R_\infty^n = \max\left\{ d_{BM}^M(X, \ell_\infty^n) \;\middle|\; X \text{ is an } n\text{-dimensional Banach space} \right\}.$$

It is shown in [4] that there exists a universal constant $c > 0$ such that $R_\infty^n \leq c n^{5/6}$. S. Taschuk [8] proved that, for $n \geq 3$,

$$R_\infty^n \leq \sqrt{n^2 - 2n + 2 + \frac{2}{\sqrt{n+2} - 1}}. \tag{5}$$

P. Youssef [12] showed that $R_\infty^n \leq (2n)^{5/6}$, which is better than the estimation in (5) when $n \geq 22$. Lemma 5 provides a way for estimating R_∞^n when n is small.

3. Banach-Mazur distance from ℓ_p^3 to ℓ_∞^3

Assume that $A = (a_{ij})_{3\times 3} \in \mathrm{GL}_3(\mathbb{R})$ and A_{ij} is the cofactor of a_{ij}, $\forall i, j \in [3]$. Let x_1, x_2, x_3 be the column vectors of A and set $y_i = (A_{1i}, A_{2i}, A_{3i})^\intercal$, $\forall i \in [3]$. For $p \in [1, \infty]$, put

$$g_p(A) = \frac{1}{|\det A|} \max\left\{ \|y_i\|_q \|x_1 + \sigma_2 x_2 + \sigma_3 x_3\|_p \;\middle|\; i \in [3],\ \sigma_1, \sigma_2 \in \{-1, 1\} \right\}, \tag{6}$$

where q is the conjugate of p. Set $d(p) = d_{BM}^M(\ell_p^3, \ell_\infty^3)$, $\forall p \in [1, \infty]$. By Lemma 5, $d(p)$ is the optimal value of the optimization problem

$$\min_{A \in \mathrm{GL}_3(\mathbb{R})} \quad g_p(A). \tag{7}$$

By (5), $d(p) \leq \sqrt{2\left(\sqrt{5} + 11\right)} \Big/ 2 \approx 2.572553$. Put

$$\mathcal{J} = \left\{ A \in \mathrm{GL}_3(\mathbb{R}) \;\middle|\; g_p(A) \leq \frac{\sqrt{2\left(\sqrt{5}+11\right)}}{2} \right\}.$$

Then (7) is equivalent to the optimization problem

$$\min_{A \in \mathcal{J}} \quad g_p(A).$$

We use the Nelder-Mead simplex algorithm (cf. [5, 7]) to find a local minimum of $g_p(A)$ starting from some $A \in \mathcal{J}$, and apply a particle swarm algorithm (cf. [3]) to process a global search. Numerical experiments yield estimations for upper bounds

p	1	1.2	1.4	1.6	1.8	2
upper bound of $d(p)$	1.8000	1.71533	1.67744	1.67601	1.69732	1.73205

TABLE 1. Several estimations of $d(p)$

of $d(p)$, see Table 1. When $p = 2$, the estimation in Table 1 is very close to $\sqrt{3}$, which is the exact value of $d(2)$.

Lemma 8. *For $p \in [1, 1.7]$, $d(p) \le 9/5$.*

Proof. By the proof of [6, Lemma 14], we have

$$d(1) \le \frac{\|(1,4,1)\|_1 \|(3,1,3)\|_\infty}{10} = \frac{9}{5},$$
$$d(p) \le \frac{1}{10}(4^p + 2)^{\frac{1}{p}} \cdot (2 \cdot 3^{\frac{p}{p-1}} + 1)^{\frac{p-1}{p}}, \forall p \in (1, 2]. \tag{8}$$

Thus we only need to consider the case when $p \in (1, 1.7]$. Set

$$f(p) = \ln(2 + 4^p) + (p-1) \cdot \ln\left(2 \cdot 3^{\frac{p}{p-1}} + 1\right), \ \forall p \in (1, 2].$$

Then $(4^p + 2)^{\frac{1}{p}} \cdot (2 \cdot 3^{\frac{p}{p-1}} + 1)^{\frac{p-1}{p}} = e^{\frac{f(p)}{p}}$. For $p \in (1, 2]$, put $r(p) = f(p)/p$ and $w(p) = pf'(p) - f(p)$. We have $r'(p) = w(p)/p^2$ and $w'(p) = pf''(p)$, where

$$f'(p) = \frac{4^p}{2 + 4^p} \cdot \ln 4 + \ln\left(2 + 3^{-\frac{p}{p-1}}\right) + \ln 3 + \frac{\ln 3}{(p-1)(2 \cdot 3^{\frac{p}{p-1}} + 1)},$$
$$f''(p) = \frac{2 \cdot \ln^2 4 \cdot 4^p}{(2 + 4^p)^2} + \frac{\ln 3}{(p-1)^3} \cdot \frac{2 \cdot \ln 3 \cdot 3^{\frac{p}{p-1}}}{(2 \cdot 3^{\frac{p}{p-1}} + 1)^2}.$$

Obviously, $\lim_{p \to 1^+} f(p) = \ln 18$ and $\lim_{p \to 1^+} f'(p) = \frac{2}{3} \cdot \ln 4 + \ln 6$. Therefore,

$$\lim_{p \to 1^+} w(p) = \frac{2}{3} \cdot \ln 4 - \ln 3 < 0. \tag{9}$$

Moreover,

$$w(2) = \frac{16}{9} \ln 4 + 2 \ln 19 - \frac{36}{19} \ln 3 - \ln(18 \cdot 19) > 0. \tag{10}$$

Since $f''(p)$ is positive on $(1, 2]$, $w(p)$ is strictly increasing on $(1, 2]$. By (9) and (10), there exists a unique $p_0 \in (1, 2)$ satisfying $w(p_0) = 0$. Therefore, $r'(p) \le 0$ for $p \in (1, p_0]$ and $r'(p) > 0$ for $p \in (p_0, 2]$. Hence $r(p)$ decreases on $(1, p_0]$ and increases on $(p_0, 2]$. Since

$$2.8904 \approx \ln 18 = \lim_{p \to 1^+} r(p) > r(1.7) \approx 2.8864,$$

we have $r(p) \le \lim_{p \to 1^+} r(p) = \ln 18, \ \forall p \in (1, 1.7]$. By (8),

$$d(p) \le \frac{e^{r(p)}}{10} \le \frac{e^{\ln 18}}{10} = \frac{9}{5}, \ \forall p \in (1, 1.7]. \qquad \square$$

Now we are ready to prove Theorem 2.

Proof of Theorem 2. By Theorem 1 and Lemma 8, we only need to consider the case when $p \in [1.7, 2]$. Set

$$A_1 = \begin{pmatrix} 13 & -24 & 24 \\ -24 & 13 & 24 \\ 24 & 24 & 13 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 9 & -17 & 17 \\ -17 & 9 & 17 \\ 17 & 17 & 9 \end{pmatrix}.$$

Using (6), we get $d(1.7) \leq g_{1.7}(A_1) \leq 1.6967$ and $d(1.8) \leq g_{1.8}(A_2) \leq 1.7033$. By Theorem 1,

$$d_{BM}^{M}\left(\ell_{1.7}^3, \ell_p^3\right) = 3^{1/1.7-1/p} \leq 3^{1/1.7-1/1.8} \leq 1.0366,\ \forall p \in [1.7, 1.8],$$
$$d_{BM}^{M}\left(\ell_{1.8}^3, \ell_p^3\right) = 3^{1/1.8-1/p} \leq 3^{1/1.8-1/1.9} \leq 1.0327,\ \forall p \in [1.8, 1.9],$$
$$d_{BM}^{M}\left(\ell_{2}^3, \ell_p^3\right) = 3^{1/p-1/2} \leq 3^{1/1.9-1/2} \leq 1.0294,\ \forall p \in [1.9, 2].$$

It follows that

$$d_{BM}^{M}(\ell_p^3, \ell_\infty^3) \leq d_{BM}^{M}(\ell_{1.7}^3, \ell_\infty^3) \cdot d_{BM}^{M}\left(\ell_{1.7}^3, \ell_p^3\right) < \frac{9}{5},\ \forall p \in [1.7, 1.8],$$
$$d_{BM}^{M}(\ell_p^3, \ell_\infty^3) \leq d_{BM}^{M}(\ell_{1.8}^3, \ell_\infty^3) \cdot d_{BM}^{M}\left(\ell_{1.8}^3, \ell_p^3\right) < \frac{9}{5},\ \forall p \in [1.8, 1.9],$$
$$d_{BM}^{M}(\ell_p^3, \ell_\infty^3) \leq d_{BM}^{M}(\ell_{2}^3, \ell_\infty^3) \cdot d_{BM}^{M}\left(\ell_{2}^3, \ell_p^3\right) < \frac{9}{5},\ \forall p \in [1.9, 2].$$

Thus $d(p) \leq 9/5$, $\forall p \in [1.7, 2]$. This completes the proof. □

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References

[1] C. Clason, *Introduction to functional analysis*, Compact Textbooks in Mathematics, Birkhäuser/Springer, Cham, 2020. MR 4182425

[2] M. Fabian, P. Habala, P. Hájek, V. Montesinos, and V. Zizler, *Banach Space Theory: The Basis for Linear and Nonlinear Analysis*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, New York, 2011. MR 2766381 (2012h:46001)

[3] V. Gazi and K.M. Passino, *Swarm stability and optimization*, Springer, New York, 2011. MR 3235758

[4] A.A. Giannopoulos, *A note on the Banach-Mazur distance to the cube*, Geometric aspects of functional analysis (Israel, 1992–1994), Oper. Theory Adv. Appl., vol. 77, Birkhäuser, Basel, 1995, pp. 67–73. MR 1353450

[5] J.C. Lagarias, J.A. Reeds, M.H. Wright, and P.E. Wright, *Convergence properties of the Nelder-Mead simplex method in low dimensions*, SIAM J. Optim. **9** (1999), no. 1, 112–147. MR 1662563

[6] Yanlu Lian and Senlin Wu, *Partition bounded sets into sets having smaller diameters*, Results Math. **76** (2021), no. 3, Paper No. 116, 15. MR 4261748

[7] L.M. Rios and N.V. Sahinidis, *Derivative-free optimization: a review of algorithms and comparison of software implementations*, J. Global Optim. **56** (2013), no. 3, 1247–1293. MR 3070154

[8] S. Taschuk, *The Banach-Mazur distance to the cube in low dimensions*, Discrete Comput. Geom. **46** (2011), no. 1, 175–183. MR 2794363

[9] A.D. Tolmachev, D.S. Protasov, and V.A. Voronov, *Coverings of planar and three-dimensional sets with subsets of smaller diameter*, Discrete Appl. Math. **320** (2022), 270–281. MR 4441253

[10] N. Tomczak-Jaegermann, *Banach-Mazur Distances and Finite-dimensional Operator Ideals*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 38, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989. MR 993774

[11] Fei Xue, *On the Banach-Mazur distance between the cube and the crosspolytope*, Math. Inequal. Appl. **21** (2018), no. 4, 931–943. MR 3868093

[12] P. Youssef, *Restricted invertibility and the Banach-Mazur distance to the cube*, Mathematika **60** (2014), no. 1, 201–218. MR 3164527

[13] Chuanming Zong, *Borsuk's partition conjecture*, Jpn. J. Math. **16** (2021), no. 2, 185–201. MR 4338243

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