arXiv:2207.05941v2 [math.AT] 16 May 2024

CARTAN CALCULI ON THE FREE LOOP SPACES

KATSUHIKO KURIBAYASHI, TAKAHITO NAITO, SHUN WAKATSUKI, AND TOSHIHIRO YAMAGUCHI

Abstract. A typical example of a Cartan calculus consists of the Lie derivative and the contraction with vector fields of a manifold on the derivation ring of the de Rham complex. In this manuscript, a *second stage* of the Cartan calculus is investigated. In a general setting, the stage is formulated with operators obtained by the André–Quillen cohomology of a commutative differential graded algebra A on the Hochschild homology of A in terms of the homotopy Cartan calculus in the sense of Fiorenza and Kowalzig. Moreover, the Cartan calculus is interpreted geometrically with maps from the rational homotopy group of the monoid of self-homotopy equivalences on a space M to the derivation ring on the loop cohomology of M. We also give a geometric description to Sullivan's isomorphism, which relates the geometric Cartan calculus to the algebraic one, via the Γ_1 map due to Félix and Thomas.

Contents

2020 *Mathematics Subject Classification.* 55P50, 55P35, 55P62, 13D03, 19D55.

Key words and phrases. Cartan calculus, Hochschild homology, cyclic homology, André–Quillen cohomology, Brown–Szczarba model, free loop space, Sullivan model, derivation.

The first author was partially supported by a Grant-in-Aid for Scientific Research (B) 21H00982 from Japan Society for the Promotion of Science. The second author was supported by JSPS KAKENHI Grant Number JP18K13403. The third author was supported by JSPS KAKENHI Grant Number 20J00404. The fourth author was partially supported by JSPS KAKENHI Grant Number 20K03591.

1. Introduction

In the previous work [KNWY21], we consider a method to describe the string bracket [CS99] on the rational S^1-equivariant homology of the free loop space LM of a simply-connected manifold M in terms of the Gerstenhaber bracket on the loop homology of M, namely, the homology of LM. In particular, the reduction is possible if M is *BV exact*; that is, the reduced Batalin–Vilkovisky (BV) operator on the loop homology is exact; see [KNWY21, Definition 2.9, Theorem 2.15 and Corollary 2.16] for more details.

The result [KNWY21, Assertion 1.2] summarizes relationships between the BV exactness and other traditional homotopy invariants containing the formality of a space. Especially, we show that a simply-connected space is BV exact if the space admits *positive weights*; see [KNWY21, Theorem 2.21]. The key to proving the theorem is that two particular derivations on a Sullivan algebra associated with the space satisfy the *Cartan magic formula*; see Proposition 3.6 for the derivations that we use therein. The appearance of the formula has inspired us to consider algebraic and topological backgrounds for the derivations. In this article, we investigate such derivations in the framework of *homotopy Cartan calculi* introduced by Fiorenza and Kowalzig [FK20] and moreover give geometric descriptions to the *Lie derivative* and the *contraction operator*, which induce the two derivations mentioned above.

In order to describe our results in more detail, we first recall the classical Cartan calculus of the differential forms on a manifold M together with Connes' result on the Hochschild cohomology. The space of vector fields on M is considered a Lie algebra $\mathrm{Der}(C^\infty(M))$ of derivations on $C^\infty(M)$ the $\mathbb{R}$-algebra of smooth functions on M. The result [Con85, II Section 6. Example] due to Connes asserts that the *continuous* Hochschild cohomology $(HH^*_{conti}(C^\infty(M)), B)$ with Connes' B-operator B is isomorphic to the de Rham complex $(\Omega^*(M), d)$ as a complex provided M is compact. Thus the Lie derivative L_X and the contraction (interior product) ι_X for each vector field X are incorporated in the framework of a *Cartan calculus*

$$\mathrm{Der}(C^\infty(M)) \underset{\iota_{(\ )}}{\overset{L_{(\ )}}{\longrightarrow}} \big(\mathrm{Der}(\Omega^*(M)), d\big) \cong \big(\mathrm{Der}(HH^*_{conti}(C^\infty(M))), B\big) \tag{1.1}$$

in the sense that $L_{(\ )}$ is a Lie algebra representation and $\iota_{(\ )}$ is a linear map which satisfy, for any vector field X, Cartan's magic formula

$$L_X = [d, \iota_X].$$

The André–Quillen cohomology $H^{-*}_{AQ}(A)$ of a commutative differential graded algebra A is an important invariant for such differential objects; see, for example, [BL05] for its applications. Thus we may apply again cyclic theory, namely cyclic homology and Hochschild homology to the de Rham complex $(\Omega^*(M), d)$ involving the André–Quillen cohomology. Let $\mathrm{Der}\,(A)$ denote the derivation subalgebra of the endomorphism Lie algebra $\mathrm{End}\,(A)$ of a differential graded algebra A. While the assignment $\mathrm{Der}\,(\ )$ is not functorial, the André–Quillen cohomology is defined as a *derived version* of $\mathrm{Der}(A)$; see [BL05] and also Section 2 for the definition.

Let M be a simply-connected manifold and $\mathrm{aut}_1(M)$ the monoid of self-homotopy equivalences on M. Then, we obtain the isomorphism

$$\Phi : \pi_*(\mathrm{aut}_1(M)) \otimes \mathbb{R} \xrightarrow{\cong} H^{-*}_{AQ}(\Omega^*(M))$$

of Lie algebras due to Sullivan [Sul77]; see [FLS10, Theorems 3.6 and 4.3] and also Section 2 for the definition of Φ. Here, the homotopy group $\pi_*(\mathrm{aut}_1(M))$ is regarded as a Lie algebra endowed with the Samelson product. Main results (Propositions 4.3, 4.4 and Theorem 3.8) in this article enable us to obtain a Cartan calculus on the de Rham complex $\Omega^*(M)$ with values in the endomorphism ring of the Hochschild homology of $\Omega^*(M)$ and its geometric interpretation with the free loop space LM. More precisely, the assertions are summarized as follows.

Theorem 1.1. *Under the same notations and assumptions as above, there exists a commutative diagram*

$$\begin{tikzcd} H^{-*}_{AQ}(\Omega^*(M)) \arrow[r, "{}_aL", "e"'] & (\mathrm{End}(HH_*(\Omega^*(M))), B) \\ \pi_*(\mathrm{aut}_1(M)) \otimes \mathbb{R} \arrow[u, "\Phi", "\cong"'] \arrow[r, "L", "(-1)^* e"'] & (\mathrm{Der}(H^*(LM;\mathbb{R})), \Delta) \arrow[u, "\ell"'] \end{tikzcd}$$

for $* > 1$ *in which the upper row sequence is a Cartan calculus induced by a homotopy Cartan calculus in the sense of Fiorenza and Kowalzig* [FK20] *and the bottom row sequence is a Cartan calculus given geometrically by applying the loop construction to the adjoint of an element of homotopy group of* $\mathrm{aut}_1(M)$*; see (4.1) and (4.2). In particular, the calculi give the formulae*

$${}_aL_\eta = [B, e_\eta] \quad \textit{and} \quad L_\theta = [\Delta, \pm e_\theta]$$

for $\eta \in H^*_{AQ}(\Omega^*(M))$ *and* $\theta \in \pi_*(\mathrm{aut}_1(M)) \otimes \mathbb{R}$*. Here* B *and* Δ *denote Connes' B-operator on the Hochschild homology and the Batalin–Vilkovisky operator on the loop cohomology, respectively. Moreover, the right vertical map* ℓ *is a monomorphism induced by the isomorphism between the loop cohomology and the Hochschild homology in* [BV88, Theorem 2.4] *preserving operators* Δ *and* B.

Remark 1.2. The contraction operator e in the upper sequence in Theorem 1.1 is defined for $* \geq 1$. However, the operator e in the bottom sequence is defined for $* > 1$; see (4.2) below.

We observe that the square above for ${}_aL$ and L is commutative even if $* = 1$; see the proof of Theorem 1.1 in the end of Section 4.1. Moreover, it follows from Lemma 3.4 and Theorem 4.1 (1) that the maps ${}_aL$ and L are morphisms of Lie algebras, respectively.

We give more comments on Theorem 1.1 and its related results in this article. By the Cartan calculus in (1.1), we can regard the de Rham complex as appearing via the Hochschild homology theory for $C^\infty(M)$. The calculus in the upper sequence in the theorem is obtained by applying again the Hochschild homology theory to the de Rham complex. Therefore, it seems that the pair $({}_aL, e)$ of maps is in a *second stage* of Cartan calculi for the manifold M.

On the other hand, the dual of the calculus in the lower sequence in the theorem gives a Cartan calculus on the homology $H_*(LM;\mathbb{R})$. The second author reveals a relationship between the calculus and algebraic structures in string topology theory; see [Nai24, Theorem 1.1].

We stress that the pair in the upper sequence consists of the *Lie derivative* ${}_aL$ and the *contraction operator* e in the homotopy Cartan calculus [FK20, Definitions 3.1 and 3.7] associated with the Hochschild complex $\mathcal{H}$ and the Burghelea–Vigué-Poirrier complex $\mathcal{L}$ of $\Omega^*(M)$ in [BV88], respectively. As a consequence, we see that the two homotopy calculi coincide with each other on homology level; see Theorem 3.8. We remark that the Cartan magic formula holds in the complex $\mathcal{L}$ before taking homology, but not in $\mathcal{H}$ in general; see Propositions 3.6 and 3.7. Moreover, it is worth mentioning that the contraction e for the complex $\mathcal{H}$ is defined with the cap product between the Hochschild cochain and chain complexes of a commutative differential graded algebra; see, for example, [Men11] for the cap product.

Moreover, the contraction operator e is non-trivial in the following sense:

Theorem 1.3. *For a simply-connected closed manifold M, the contraction operator $e\colon \pi_*(\mathrm{aut}_1(M))\otimes\mathbb{R}\to \mathrm{Der}(H^*(LM;\mathbb{R}))$ is injective.*

This is an immediate consequence of Corollary 3.11 and Proposition 4.4. The former is proved by showing that the map invariably detects the fundamental class of a manifold M; see Theorem 3.10.

As for Sullivan's isomorphism Φ, we show that the isomorphism factors through the map Γ_1 from $\pi_*(\mathrm{aut}_1(M))\cong\pi_{*-1}(\Omega\mathrm{aut}_1(M))$ to the loop homology of simply-connected closed manifold M introduced by Félix and Thomas in [FT04]; see Theorem 4.7. Since the map Γ_1 is induced by the evaluation map of the space of sections of the evaluation fibration $\mathrm{ev}_0 : LM\to M$, it can be said that we give the isomorphism Φ a geometric interpretation. It is worth mentioning that the Brown–Szczarba model [BS97] for a function space plays a vital role in the argument on the geometric description of the isomorphism Φ.

We give comments on the André–Quillen cohomology. As mentioned above, taking the derivation algebra $\mathrm{Der}\,(A)$ for a commutative differential graded algebra A is not functorial. However, we see that a Sullivan model $\varphi\colon(\wedge V,d)\xrightarrow{\simeq}(A,d)$ induces a morphism $\widetilde{\varphi} : H^*(\mathrm{Der}(A))\to H^*(\mathrm{Der}(\wedge V))$ which is compatible with Cartan calculi for A and $\wedge V$. This is attained in Proposition 3.9. Such a map $\widetilde{\varphi}$ induced by φ is an isomorphism if the codomain A is also a Sullivan algebra; see Corollary 4.6. However, a quasi-isomorphism $\varphi\colon(\wedge V,d)\xrightarrow{\simeq}(A,d)$ does not necessarily induce a quasi-isomorphism between $\mathrm{Der}\,(A)$ and $\mathrm{Der}\,(\wedge V)$ in general; see Remark 5.3.

The rest of this manuscript is organized as follows. Section 2 recalls results in rational homotopy theory with which we develop our arguments. In Section 3, we recall the homotopy Cartan calculus mentioned above. Important examples of the calculi which come from a Sullivan algebra and the Hochschild complex of a DGA are given. The naturality of a Cartan calculus are discussed in Section 3.4. Section 4 is devoted to investigating geometric descriptions of the homotopy Cartan calculi considered in Section 3.1. In Section 4, after explaining geometric constructions of the operations L and e, we prove Theorem 1.1. In the rest of the section, we elaborate the proof of Theorem 4.7 mentioned above. Section 5 deals with computational examples of the Lie derivative L and the contraction operator e described in Theorem 1.1.

In Appendix A, we give a Sullivan representative for an adjoint map by using *twice* Brown–Szczarba models for function spaces. The result plays a crucial rule in giving the geometric description of Sullivan's isomorphism Φ. In Appendix B, we discuss an extension of the Lie derivative L to cyclic theory and its geometric counterpart with the cobar-type Eilenberg–Moore spectral sequence converging to the S^1-equivariant cohomology of the free loop space LM; see Theorem B.6 and Proposition B.13.

2. Preliminaries

We begin with the definitions of the Hochschild complex of a differential non-negatively graded algebra (DGA for short) over a field, the endomorphism ring of a DGA and Sullivan's isomorphism Φ which are used repeatedly in this manuscript. We assume that the underlying field is of characteristic zero unless otherwise stated.

Let $A = (A, d)$ be an augmented DGA, which is not necessarily graded commutative. We use the cohomological grading on A and then $\deg d = +1$. While the homological degree of a graded vector space W_* is also used, we freely apply the translation for homological and cohomological degrees with $W_* = W^{-*}$.

Let $C_*(A) = (A \otimes T(s\bar{A}), d = d_1 + d_2)$ be the Hochschild chain complex of (A, d). Here, $\bar{A}$ denotes the augmentation ideal of A and $s\bar{A}$ denotes the suspension of $\bar{A}$; that is, $(s\bar{A})^n = \bar{A}^{n+1}$. The differentials d_1 and d_2 are defined by $d_1 = \sum_i d_{1,i}$ and $d_2 = \sum_i d_{2,i}$ with

$$d_{1,i}(a_0[a_1|\cdots|a_n]) = \begin{cases} da_0[a_1|\cdots|a_n] & (i = 0), \\ (-1)^{\varepsilon_i+1}a_0[a_1|\cdots|da_i|\cdots|a_n] & (0 < i \leq n), \end{cases}$$

$$d_{2,i}(a_0[a_1|\cdots|a_n]) = \begin{cases} (-1)^{|a_0|}a_0a_1[a_2|\cdots|a_n] & (i = 0), \\ (-1)^{\varepsilon_{i+1}}a_0[a_1|\cdots|a_ia_{i+1}|\cdots|a_n] & (0 < i < n), \\ (-1)^{\varepsilon_n|sa_n|+1}a_ka_0[a_1|\cdots|a_{n-1}] & (i = n), \end{cases}$$

where $\varepsilon_i = |a_0| + \sum_{j<i} |sa_j|$.

Definition 2.1. (1) Let $(\mathcal{C}, d)$ be a cochain complex. A triple $(\mathcal{C}, d, B)$ is a *mixed complex* if $B \colon \mathcal{C} \to \mathcal{C}$ is a differential of degree -1 with $[d, B] := dB + Bd = 0$.

(2) A *mixed DGA* is a mixed complex (A, d, B) together with a graded algebra structure on A such that d and B are derivations with respect to it.

(3) A *mixed differential graded (dg) Lie algebra* is a mixed complex $(\mathfrak{h}, d, B)$ together with a graded Lie algebra structure $[\ ,\ ]$ on $\mathfrak{h}$ such that d and B are derivations with respect to $[\ ,\ ]$.

Let $(\mathcal{C}, d, B)$ be a mixed complex. We denote by $\mathrm{End}(\mathcal{C})$ the endomorphism ring $\mathrm{Hom}(\mathcal{C}, \mathcal{C})$ of linear maps (of any degree). The ring $\mathrm{End}(\mathcal{C})$ is considered the Lie algebra with the bracket $[\ ,\ ]$ defined by $[f, g] = fg - (-1)^{|f||g|}gf$ for f and $g \in \mathrm{End}(\mathcal{C})$. We observe that $\mathrm{End}(\mathcal{C})$ is endowed with a dg Lie algebra structure whose differential is defined by $[d, -]$ with the bracket and the differential d of $\mathcal{C}$. We see that a triple $(\mathrm{End}(\mathcal{C}), [d,\ ], [B,\ ])$ is a mixed dg Lie algebra. Moreover, for a DGA A, we define a differential graded Lie subalgebra $\mathrm{Der}(A)$ of $\mathrm{End}(A)$ consisting of derivations on A. If (A, d, B) is a mixed DGA, we observe that $\mathrm{Der}(A)$ is a mixed dg Lie subalgebra of $\mathrm{End}(A)$.

We recall a *derived version* of the non-positive derivations. Let A be a commutative differential graded algebra A (CDGA for short). Following Block and Lazarev [BL05], the André–Quillen cohomology $H^*_{AQ}(A)$ of A for $* \leq 0$ is defined by

$$H^*_{AQ}(A) := H^*(\mathrm{Der}(QA, A)) \cong H^*(\mathrm{Der}(QA, QA), [d_{QA}, -])$$

with a cofibrant replacement (QA, d_{QA}) of A in the category of CDGAs; see [BG76]. We regard $H^*_{AQ}(A)$ as a Lie algebra with the Lie bracket on $H^*(\mathrm{Der}(QA, QA))$. Here we may choose as QA a Sullivan model of A; see [FHT01, Section 12] for a general theory of Sullivan algebras.

Let $(\wedge V, d)$ be a Sullivan algebra for which $V^1 = 0$. Then we define a mixed DGA $(\wedge V \otimes \wedge \overline{V}, d, s)$, where s is the derivation of degree (-1) defined by $sv = \bar{v}$ and $s\bar{v} = 0$ for $v \in V$ and the differential d is the unique extension of $d\colon \wedge V \to \wedge V$ which satisfies the condition that $[d, s] = 0$. For simplicity of notation, we write $\mathcal{L} = \wedge V \otimes \wedge \overline{V}$ together with a decomposition $\mathcal{L} = \bigoplus_k \mathcal{L}_{(k)}$ of complexes, where $\mathcal{L}_{(k)} = \wedge V \otimes \wedge^k \overline{V}$. Observe that $H(\mathcal{L})$ is isomorphic to the Hochschild homology of $(\wedge V, d)$; see [BV88, Theorem 2.4(ii)].

Let X be a simply-connected space whose rational cohomology $H^*(X; \mathbb{Q})$ is of finite type; that is, $\dim H^i(X; \mathbb{Q}) < \infty$ for each $i \geq 0$. Let LX be the free loop space which is the space of maps from S^1 to X endowed with compact-open topology. Suppose that $(\wedge V, d)$ is a Sullivan model for X. Then the complex $\mathcal{L}$ mentioned above is a Sullivan model for LX; see [VPS76].

We recall Sullivan's isomorphism Φ described in Theorem 1.1. Consider a sequence of the homotopy sets

$$\pi_n(\mathrm{aut}_1(X)) \xrightarrow{k} [S^n \times X, X] \xrightarrow{\mu} [\mathcal{M}_X, \mathcal{M}_{S^n \times X}],$$

where $\mathcal{M}_Y$ denotes a minimal Sullivan model for a space Y and μ assigns a map f a Sullivan representative for f. We may replace $\mathcal{M}_{S^n \times X}$ with the DGA $H^*(S^n; \mathbb{Q}) \otimes \mathcal{M}_X$. Then we write

$$(\mu \circ k)(\theta) = 1 \otimes 1_{\mathcal{M}_X} + \iota \otimes \theta',$$

where ι is the generator of $H^n(S^n; \mathbb{Q})$. Then, Sullivan's isomorphism Φ of Lie algebras

$$\Phi : \pi_*(\mathrm{aut}_1(X)) \otimes \mathbb{Q} \to H^{-*}(\mathrm{Der}(\mathcal{M}_X), [d, -]) = H^{-*}_{AQ}(A^*_{PL}(X))$$

is defined by $\Phi(\theta) = \theta'$; see [Sul77] and also [FLS10, Theorems 3.6 and 4.3].

3. Algebraic Cartan calculi

The homotopy Cartan calculus due to Fiorenza and Kowalzig [FK20] provides a systematic way to endow the shifted homology of a mixed complex MC with the Batalin-Vilkovisky algebra structure and to give the Chas-Sullivan-Menichi [CS99, **?**] bracket to the negative cyclic homology of MC; see [FK20, Theorem D]. Thus, it is crucial to consider examples of such a homotopy calculus.

In this section, we recall the homotopy Cartan calculus with a slight generalization. Roughly speaking, the calculus consists of two operations (e and L) between complexes and two *homotopies* (S and T) between the two operations. We give examples of the calculi by using a Sullivan model of the free loop space of a simply-connected space and the Hochschild chain complex of a differential graded algebra (DGA). While the operations of the two homotopy calculi are identified on the homology level if a given DGA is a Sullivan algebra; see Theorem 3.8, the difference between the homotopy calculi appears in the homotopy between operations; see Proposition 3.6 and Proposition 3.7.

3.1. **Homotopy Cartan calculus with slight generalization.** Let $(\mathfrak{g}, \delta)$ be a chain complex and $(\mathfrak{h}, d, B)$ a mixed complex.

Definition 3.1 (cf. [FK20, Definition 3.1])**.** A tuple $(\mathfrak{g}, \mathfrak{h}, e, L, S)$ consisting of linear maps $e, L, S\colon \mathfrak{g} \to \mathfrak{h}$ of degrees 1, 0 and -1, respectively, is a *homotopy* *pre-Cartan calculus* if the equalities

$$\begin{aligned}
&L_\theta = B(e_\theta) + d(S_\theta) + S_{\delta\theta},\\
&d(e_\theta) + e_{\delta\theta} = 0 \quad \text{and}\\
&B(S_\theta) = 0
\end{aligned}$$

hold for any $\theta \in \mathfrak{g}$. The linear maps e and L are called a *contraction operator* (or *cap product*) and a *Lie derivative*, respectively.

The first two conditions imply that e and L are chain maps of degree 1 and 0, respectively.

Definition 3.2. [cf. [FK20, Definition 3.7]] Let $(\mathfrak{g}, \delta, [\ ,\ ])$ be a dg Lie algebra and $(\mathfrak{h}, d, B, [\ ,\ ])$ a mixed Lie algebra; see Definition 2.1. A *homotopy Cartan calculus* $(\mathfrak{g}, \mathfrak{h}, e, L, S, T)$ is a homotopy pre-Cartan calculus $(\mathfrak{g}, \mathfrak{h}, e, L, S)$ equipped with a linear map $T\colon \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{h}$ satisfying the following equalities for any $\theta, \rho \in \mathfrak{g}$

$$\begin{aligned}
&[e_\theta, L_\rho] - e_{[\theta,\rho]} = d(T_{\theta,\rho}) - T_{\delta\theta,\rho} - (-1)^{|\theta|} T_{\theta,\delta\rho},\\
&[S_\theta, L_\rho] - S_{[\theta,\rho]} = B(T_{\theta,\rho}).
\end{aligned}$$

Here T is called a *Gelfan'd-Daletskiĭ-Tsygan homotopy*.

Remark 3.3. These definitions are equivalent to [FK20, Definitions 3.1 and 3.7] if $(\mathfrak{h}, d, B, [\ ,\ ])$ is the tuple $(\mathrm{End}(\mathcal{C}), [d',\ ], [B',\ ], [\ ,\ ])$ which is given by a mixed complex $(\mathcal{C}, d', B')$. In this case, we may call the calculus a homotopy Cartan calculus on the mixed complex $\mathcal{C}$.

The following is one of fundamental properties of a homotopy Cartan calculus.

Lemma 3.4 (cf. [FK20, Lemmas 3.4 and 3.10])**.** *Let $(\mathfrak{g}, \mathfrak{h}, e, L, S, T)$ be a homotopy Cartan calculus. Then the map $L\colon \mathfrak{g} \to \mathfrak{h}$ is a morphism of dg Lie algebras.*

In particular, we see that a homotopy Cartan calculus $(\mathfrak{g}, \mathfrak{h}, e, L, S, T)$ gives a $(H(\mathfrak{g}), [\ ,\ ])$-module structure to $H(\mathfrak{h})$ via the map $H(L) : H(\mathfrak{g}) \to H(\mathfrak{h})$. Moreover, it follows that $H(e) : H(\mathfrak{g}) \to H(\mathfrak{h})$ is a morphism of $(H(\mathfrak{g}), [\ ,\ ])$-modules.

If the linear map T in a homotopy Cartan calculus is trivial, then the map $e\colon \mathfrak{g} \to \mathfrak{h}$ is regarded as a morphism of $(\mathfrak{g}, \delta, [\ ,\ ])$-modules, where the $\mathfrak{g}$-module structure of $\mathfrak{h}$ is given by the morphism L of Lie algebras. We observe that our examples of homotopy Cartan calculi are in such a case; see Propositions 3.6 and 3.7 below.

3.2. **Homotopy Cartan calculus on the Sullivan model of free loop spaces.** In this section, we give a homotopy Cartan calculus induced by a Sullivan algebra. We recall the CDGA $\mathcal{L}$ described in Section 2.

Definition 3.5. For a derivation θ on $\wedge V$, we define derivations L_θ and e_θ on $\mathcal{L}$ by

$$\begin{aligned}
&L_\theta v = \theta v, \quad L_\theta \bar{v} = (-1)^{|\theta|} s\theta v,\\
&e_\theta v = 0, \quad e_\theta \bar{v} = (-1)^{|\theta|}\theta v
\end{aligned}$$

for $v \in V$. This defines linear maps $L\colon \mathrm{Der}(\wedge V) \to \mathrm{Der}(\mathcal{L})$ of degree 0 and $e\colon \mathrm{Der}(\wedge V) \to \mathrm{Der}(\mathcal{L})$ of degree (-1).

These derivations are introduced in [KNWY21, Proof of Theorem 2.21] by modifying constructions in [Vig94, Proposition 5].

Proposition 3.6. *The above maps give a homotopy Cartan calculus of the form* $(\mathrm{Der}(\wedge V), \mathrm{Der}(\mathcal{L}), e, L, S = 0, T = 0)$.

Proof. Since $S = T = 0$, we can reduce the equalities in Definition 3.1 and Definition 3.2 to

$$\begin{aligned} &L_\theta = [s, e_\theta], \\ &[d, e_\theta] + e_{\delta\theta} = 0, \\ &[e_\theta, L_\rho] - e_{[\theta,\rho]} = 0. \end{aligned}$$

A straightforward computation enables us to deduce that the equalities above hold on $V \oplus \overline{V}$. Since e_θ and L_θ are derivations for any $\theta \in \mathrm{Der}(\wedge V)$, we have the result. $\square$

3.3. **Homotopy Cartan calculus on the Hochschild chain complex.** In this section, we consider a homotopy Cartan calculus on the Hochschild chain complex of an augmented DGA. While the domain of our calculus is restricted to the derivation ring of a DGA, the calculus is regarded as a DGA version of a homotopy Cartan calculus on the Hochschild chain complex of an associative algebra described in [FK20, Example 3.13].

We recall the Hochschild complex $C_*(A)$ of a DGA (A, d) mentioned in Section 2. Then, Connes' B operator $B : C_*(A) \to C_*(A)$ is defined by

$$B_n := B|_{A\otimes T^n(s\bar{A})} = s \circ (1 + t_n + \cdots + t_n^n)$$

for $n \geq 0$. Here, $t_n : A\otimes T^n(s\bar{A}) \to A\otimes T^n(s\bar{A})$ and $s : A\otimes T^n(s\bar{A}) \to A\otimes T^{n+1}(s\bar{A})$ are morphisms given by $t_0 = 1$ and

$$\begin{aligned} t_n(a_0[a_1|\cdots|a_n]) &= (-1)^{|sa_n|(\varepsilon_n+1)} a_n[a_0|\cdots|a_{n-1}], \\ s(a_0[a_1|\cdots|a_n]) &= 1[a_0|a_1|\cdots|a_n], \end{aligned}$$

where ε_i is the notation described in Section 2. Then, it follows from [BV88, Example 1] that the triple $(C_*(A), d, B)$ is a mixed complex.

Let A' be an augmented DGA and $\varphi : A \to A'$ a morphism of DGAs. For a derivation $\theta \in \mathrm{Der}(A, A')$, we define $L_\theta : C_*(A) \to C_*(A')$ by $L_\theta = \sum_i L_{\theta,i}$ and

$$L_{\theta,i}(a_0[a_1|\cdots|a_n]) = \begin{cases} \theta(a_0)[\varphi(a_1)|\varphi(a_2)|\cdots|\varphi(a_n)] & (i = 0), \\ (-1)^{|\theta|(\varepsilon_i+1)}\varphi(a_0)[\varphi(a_1)|\cdots|\theta(a_i)|\cdots|\varphi(a_n)] & (1 \leq i \leq n). \end{cases}$$

We also define $e_\theta : C_*(A) \to C_*(A')$ by $e_\theta|_A = 0$ and

$$e_\theta(a_0[a_1|\cdots|a_n]) = (-1)^{|\theta||a_0|+|\theta|+|a_0|}\varphi(a_0)\theta(a_1)[\varphi(a_2)|\cdots|\varphi(a_n)].$$

Let e'_θ be the element in the Hochschild cochain complex $C^*(A; A')$ given by

$$e'_\theta(a_0[a_1|\cdots|a_n]a_{n+1}) = \begin{cases} (-1)^{|\theta||a_0|+|\theta|+|a_0|}\varphi(a_0)\theta(a_1)\varphi(a_2) & (n = 1), \\ 0 & (n \neq 1). \end{cases}$$

Then we see that $e_\theta = e'_\theta \cap -$, where the right-hand side is the cap product with e'_θ ; see [Men11, §3] for the cap product. Moreover, we define $S_\theta : C_*(A) \to C_*(A')$ by $S_\theta|_A = 0$ and, for $n \geq 1$,

$$S_\theta|_{A\otimes T^n(s\bar{A})} = \sum_{j=1}^{n} \left(\sum_{k=0}^{n-j} s \circ t_n^k \right) \circ L_{\theta,j}.$$

Proposition 3.7. *Let e, L and S be the morphisms described above.*

(1) *The tuple* $(\mathrm{Der}(A, A'), \mathrm{Hom}(C_*(A), C_*(A')), e, L, S)$ *is a homotopy pre-Cartan calculus.*
(2) *The tuple* $(\mathrm{Der}(A), \mathrm{End}(C_*(A)), e, L, S, T = 0)$ *is a homotopy Cartan calculus on the mixed complex* $(C_*(A), d, B)$.

Proof. In order to prove (1), it suffices to check the following equalities:

(i) $L_\theta = [B, e_\theta] + [d, S_\theta] + S_{\delta\theta}$,
(ii) $[d, e_\theta] + e_{\delta\theta} = 0$,
(iii) $[B, S_\theta] = 0$.

A straightforward computation allows us to deduce that

$$\begin{aligned}
&[d_1, S_\theta] = S_{\delta\theta},\\
&L_\theta = (-1)^{|\theta|} e_\theta \circ B_n + (d_{2,0} + d_{2,n+1}) \circ S_\theta \quad \text{and}\\
&B_{n-1} \circ e_\theta + \left(\sum_{i=1}^{n} d_{2,i}\right) \circ S_\theta + (-1)^{|\theta|} S_\theta \circ d_2 = 0.
\end{aligned}$$

By combining the equalities, we obtain the formula (i). The linear maps $d_{1,i}$, $d_{2,i}$ and e_θ satisfy the followings relations:

$$d_{1,i} \circ e_\theta = \begin{cases} (-1)^{|\theta|+1} e_\theta \circ (d_{1,0} + d_{1,1}) - e_{\delta\theta} & (i = 0),\\ (-1)^{|\theta|+1} e_\theta \circ d_{1,i+1} & (1 \leq i \leq n-1), \end{cases} \tag{3.1}$$
$$d_{2,i} \circ e_\theta = \begin{cases} (-1)^{|\theta|+1} e_\theta \circ (d_{2,0} + d_{2,1}) & (i = 0),\\ (-1)^{|\theta|+1} e_\theta \circ d_{2,i+1} & (1 \leq i \leq n-1). \end{cases}$$

Then, we have the formula (ii) by combining the equalities (3.1). Since $s \circ s = 0$, $t_n \circ s = 0$ and

$$L_{\theta,i} \circ s = (-1)^{|\theta|} s \circ L_{\theta,i-1} \tag{3.2}$$

for $i \geq 1$, it is immediate to verify the relation (iii). As a consequence, we have (1). We consider the case where $A = A'$. In order to prove the assertion (2), we show the following equalities

(iv) $[e_\theta, L_\rho] - e_{[\theta,\rho]} = 0$ and
(v) $[S_\theta, L_\rho] - S_{[\theta,\rho]} = 0$.

Observe that $e_\theta = d_{2,0} \circ L_{\theta,1}$. Moreover, we have

$$L_{\theta,i} \circ d_{2,0} = \begin{cases} (-1)^{|\theta|} d_{2,0} \circ (L_{\theta,0} + L_{\theta,1}) & (i = 0),\\ (-1)^{|\theta|} d_{2,0} \circ L_{\theta,i+1} & (i \geq 1) \end{cases} \quad \text{and} \tag{3.3}$$

$$L_{\theta,i} \circ L_{\rho,j} = \begin{cases} L_{\theta\rho,i} & (i = j),\\ (-1)^{|\theta||\rho|} L_{\rho,j} \circ L_{\theta,i} & (i \neq j). \end{cases} \tag{3.4}$$

The equalities (3.3) and (3.4) enable us to obtain the formula (iv). It is readily seen that

$$L_{\theta,i} \circ t_n = \begin{cases} t_n \circ L_{\theta,n} & (i = 0),\\ t_n \circ L_{\theta,i-1} & (1 \leq i \leq n). \end{cases} \tag{3.5}$$

Therefore, we have the formula (v) by combining (3.2), (3.4) and (3.5). $\square$

3.4. **Comparison among Cartan calculi.** In this section, we compare two Cartan calculi defined in Section 3.2 and Section 3.3; see Theorem 3.8. We also show that a Sullivan model induces morphisms of graded Lie algebras on the homology of Cartan calculi on the Hochschild complexes; see Proposition 3.9.

As mentioned in the beginning of Section 3, the homotopy Cartan calculi of a Sullivan algebra in Proposition 3.6 and the Hochschild complex in Proposition 3.7 coincide with each other on homology. To see this, we identify the Hochschild homology $HH_*(\wedge V)$ with the homology $H^*(\mathcal{L})$ by the quasi-isomorphism $\Theta : C_*(\wedge V) \to \mathcal{L}$ defined by $\Theta(a_0[a_1|\cdots|a_n]) = \frac{1}{n!}a_0 sa_1 \cdots sa_n$; see [BV88, Theorem 2.4]. Here s is the unique derivation on $\mathcal{L}$ stated in Section 2. We also recall the morphism $\Theta' : \mathcal{L} \to C_*(\wedge V)$ defined by $\Theta'(a_0 sa_1 \cdots sa_n) = a_0 * [a_1] * \cdots * [a_n]$, where $*$ denotes the shuffle product on the Hochschild complex; see [GJ90, Section 4]. Observe that $\Theta \circ \Theta' = 1$. Our main result in this section is described as follows.

Theorem 3.8. *With the same notation as above, one has a commutative diagram*

$$\begin{tikzcd} H(\mathrm{Der}(\wedge V)) \arrow[r, "L_{(\ )}", "(resp. e_{(\ )})"'] \arrow[dr, "L_{(\ )}", "(resp. e_{(\ )})"'] & \mathrm{End}(HH_*(\wedge V)) \\ & \mathrm{Der}(H^*(\mathcal{L})), \arrow[u, "i"'] \end{tikzcd}$$

where i is the monomorphism defined by the isomorphism $H(\Theta)$.

Proof. In order to prove the assertion, it suffices to show that the squares

$$\begin{tikzcd} C_*(\wedge V) \arrow[r, "\Theta"] \arrow[d, "L_\theta"'] & \mathcal{L} \arrow[d, "L_\theta"] \\ C_*(\wedge V) \arrow[r, "\Theta"'] & \mathcal{L}, \end{tikzcd} \qquad \begin{tikzcd} C_*(\wedge V) \arrow[d, "e_\theta"'] & \mathcal{L} \arrow[l, "\Theta'"'] \arrow[d, "e_\theta"] \\ C_*(\wedge V) \arrow[r, "\Theta"'] & \mathcal{L} \end{tikzcd}$$

are commutative for $\theta \in \mathrm{Der}(\wedge V)$. Observe that $[L_\theta, s] = 0$ in $\mathrm{End}(\mathcal{L})$ by Definition 3.5. Then, we get

$$\begin{aligned} &\Theta \circ L_\theta(a_0[a_1|a_2|\cdots|a_k]) \\ =&\frac{1}{k!}\left(\theta(a_0)sa_1sa_2\cdots sa_k + \sum_{i=1}^{k}(-1)^{|\theta|(\varepsilon_i+1)}a_0sa_1\cdots s\theta(a_i)\cdots sa_k\right) \\ =&\frac{1}{k!}\left(\theta(a_0)sa_1sa_2\cdots sa_k + \sum_{i=1}^{k}(-1)^{|\theta|\varepsilon_i}a_0sa_1\cdots L_\theta(sa_i)\cdots sa_k\right) \\ =&L_\theta \circ \Theta(a_0[a_1|a_2|\cdots|a_k]) \end{aligned}$$

which implies the commutativity of the left-hand side square. On the other hand, given $a\bar{v}_1\bar{v}_2\cdots\bar{v}_k \in \mathcal{L}$ for $a \in \wedge V$ and $v_i \in V$. The induction on k enables us to deduce that the shuffle product on $C_*(\wedge V)$ satisfies

$$\Theta'(a\bar{v}_1\bar{v}_2\cdots\bar{v}_k) = a * [v_1] * [v_2] * \cdots * [v_k] = \sum_{\sigma\in\mathfrak{S}}(-1)^{\varepsilon(\sigma)}a[v_{\sigma(1)}|v_{\sigma(2)}|\cdots|v_{\sigma(k)}],$$

where $\mathfrak{S}_k$ is the symmetric group of degree k and $(-1)^{\varepsilon(\sigma)}$ is the Koszul sign defined by the equality $(-1)^{\varepsilon(\sigma)}\bar{v}_{\sigma(1)}\bar{v}_{\sigma(2)}\cdots\bar{v}_{\sigma(k)} = \bar{v}_1\bar{v}_2\cdots\bar{v}_k$ in $\mathcal{L}$. Thus we have

$$\begin{aligned}
&\Theta\circ e_\theta\circ\Theta'(a\bar{v}_1\bar{v}_2\cdots\bar{v}_k)\\
=&\sum_{\sigma\in\mathfrak{S}_k}(-1)^{\varepsilon(\sigma)+|\theta||a|+|\theta|+|a|}\frac{1}{(k-1)!}a\theta(v_{\sigma(1)})\bar{v}_{\sigma(2)}\cdots\bar{v}_{\sigma(k)}\\
=&\sum_{i=1}^{k}(-1)^{|\theta||a|+|\theta|+|a|+|\bar{v}_i|(|\bar{v}_1|+\cdots+|\bar{v}_{i-1}|)}a\theta(v_i)\bar{v}_1\cdots\bar{v}_{i-1}\bar{v}_{i+1}\cdots\bar{v}_k\\
=&\sum_{i=1}^{k}(-1)^{|\theta||a|+|a|+(|\theta|+1)(|\bar{v}_1|+\cdots+|\bar{v}_{i-1}|)}a\bar{v}_1\cdots\bar{v}_{i-1}e_\theta(\bar{v}_i)\bar{v}_{i+1}\cdots\bar{v}_k\\
=&e_\theta(a\bar{v}_1\bar{v}_2\cdots\bar{v}_k).
\end{aligned}$$

This yields the commutativity of the right-hand side square. □

Let $\varphi\colon(\wedge V,d)\xrightarrow{\simeq}(A,d)$ be a (not necessarily connected) Sullivan model of an augmented CDGA (A,d). In the rest of this section, we show that the quasi-isomorphism φ induces a morphism of Lie algebras between the homology Lie algebras of derivations. Moreover, we relate two homotopy Cartan calculi

$$(\mathrm{Der}(A),\mathrm{End}(C_*(A)),e,L,S,0)\quad\text{and}\quad(\mathrm{Der}(\wedge V),\mathrm{End}(C_*(\wedge V)),e,L,S,0).$$

We refer the reader to Proposition 3.7 (2) for the calculi.

Proposition 3.9. *There exist a homomorphism $H^*(\mathrm{Der}(A))\to H^*(\mathrm{Der}(\wedge V))$ and an isomorphism $H^*(\mathrm{End}(C_*(A)))\xrightarrow{\cong}H^*(\mathrm{End}(C_*(\wedge V)))$ of graded Lie algebras such that the following diagrams commute:*

$$\begin{tikzcd}
H^*(\mathrm{Der}(\wedge V)) \arrow[r, "H(L)"] & H^*(\mathrm{End}(C_*(\wedge V))) \\
H^*(\mathrm{Der}(A)) \arrow[u] \arrow[r, "H(L)"] & H^*(\mathrm{End}(C_*(A))), \arrow[u, "\cong"']
\end{tikzcd}
\quad
\begin{tikzcd}
H^*(\mathrm{Der}(\wedge V)) \arrow[r, "H(e)"] & H^{*+1}(\mathrm{End}(C_*(\wedge V))) \\
H^*(\mathrm{Der}(A)) \arrow[u] \arrow[r, "H(e)"] & H^{*+1}(\mathrm{End}(C_*(A))). \arrow[u, "\cong"']
\end{tikzcd}$$

Proof. First we prove the proposition in the case where $\varphi\colon(\wedge V,d)\to(A,d)$ is a *surjective* quasi-isomorphism. The morphism φ of CDGAs gives rise to a commutative diagram

$$\begin{tikzcd}
H(\mathrm{Der}(\wedge V)) \arrow[r, "{H(L),H(e)}"] \arrow[d, "\varphi_*"', "\cong"] & H(\mathrm{End}(C_*(\wedge V))) \arrow[d, "\cong"', "\varphi_*"] \\
H(\mathrm{Der}(\wedge V,A)) \arrow[r, "{H(L),H(e)}"] & H(\mathrm{End}(C_*(\wedge V),C_*(A))) \\
H(\mathrm{Der}(A)) \arrow[u, "\varphi^*"] \arrow[r, "{H(L),H(e)}"] & H(\mathrm{End}(C_*(A))). \arrow[u, "\cong", "\varphi^*"']
\end{tikzcd}$$

Since the functor $C_*(-)$ preserves quasi-isomorphisms, it follows that the right vertical maps are isomorphisms. Corollary 4.6 (1) implies that the upper left map φ_* is an isomorphism. Now we need to prove that the two vertical composites are morphisms of Lie algebras. Since the right one can be proved similarly to (and easier than) the left one, we give only a proof for the left one.

It follows from Proposition 4.5 that the map $\varphi_*\colon\mathrm{Der}(\wedge V)\to\mathrm{Der}(\wedge V,A)$ is a surjective quasi-isomorphism. Then, for any elements $[f],[g]\in H(\mathrm{Der}(A))$, there are cocycles $f', g' \in \mathrm{Der}(\wedge V)$ such that $f\varphi = \varphi f'$ and $g\varphi = \varphi g'$. Therefore, we see that $(\varphi_*)^{-1} \circ \varphi^*[f] = [f']$ and $(\varphi_*)^{-1} \circ \varphi^*[g] = [g']$. By a straightforward computation, we have $[f, g]\varphi = \varphi[f', g']$ and this completes the proof for the particular case.

Next we deal with a general case. To this end, the "surjective trick" is applicable. In fact, the map $\varphi\colon (\wedge V, d) \to (A, d)$ factors as $(\wedge V, d) \underset{\simeq}{\xrightarrow{i}} (\wedge V, d) \otimes (E(A), \delta) \underset{\simeq}{\xrightarrow{\varphi'}} (A, d)$ with a contractible algebra $(E(A), \delta)$ and the canonical maps i and φ'; see [FHT01, Section 12 (b)] for details. Now we have a homotopy inverse $r\colon (\wedge W, d) \to (\wedge V, d)$ of i defined by sending A to zero, where $(\wedge W, d) = (\wedge V, d) \otimes (E(A), \delta)$. Observe that $W = V \oplus A \oplus \delta A$. Then r and φ' are *surjective* quasi-isomorphisms and hence the first half of the proof gives the following diagram:

$$\begin{tikzcd}
H(\mathrm{Der}(\wedge V)) \arrow[r, "{H(L),H(e)}"] \arrow[d, "r^*"', "\cong"] & H(\mathrm{End}(C_*(\wedge V))) \arrow[d, "\cong"', "r^*"] \\
H(\mathrm{Der}(\wedge W, \wedge V)) \arrow[r, "{H(L),H(e)}"] & H(\mathrm{End}(C_*(\wedge W), C_*(\wedge V))) \\
H(\mathrm{Der}(\wedge W)) \arrow[r, "{H(L),H(e)}"] \arrow[u, "r_*", "\cong"'] \arrow[d, "\varphi_*"', "\cong"] & H(\mathrm{End}(C_*(\wedge W))) \arrow[u, "\cong", "r_*"'] \arrow[d, "\cong"', "\varphi_*"] \\
H(\mathrm{Der}(\wedge W, A)) \arrow[r, "{H(L),H(e)}"] & H(\mathrm{End}(C_*(\wedge W), C_*(A))) \\
H(\mathrm{Der}(A)) \arrow[r, "{H(L),H(e)}"] \arrow[u, "\varphi^*"] & H(\mathrm{End}(C_*(A))). \arrow[u, "\cong", "\varphi^*"']
\end{tikzcd}$$

The upper left map r^* is an isomorphism with the inverse i^* by Corollary 4.6 (2). This completes the proof of the general case. □

3.5. **Injectivity of the contraction e on homology.** Let $(\wedge V, d)$ be a simply-connected Sullivan algebra whose homology satisfies the Poincaré duality. We assume that the fundamental class is in $H^m(\wedge V)$. In this section, we study properties of the derivation $H(e_\theta)\colon H(\mathcal{L}) \to H(\mathcal{L})$ and show injectivity of this map. We recall the differential graded module $\mathcal{L}_{(k)}$ defined in Section 3.2.

Theorem 3.10. *Let $(\mathrm{Der}(\wedge V), \mathrm{Der}(\mathcal{L}), e, L, 0, 0)$ be the homotopy Cartan calculus in Proposition 3.6. For any $[\theta] \neq 0 \in H^{-n-1}(\mathrm{Der}(\wedge V))$, there exists a cohomology class $[\alpha] \in H^{m+n}(\mathcal{L}_{(1)})$ such that $H(e_\theta)[\alpha]$ is the same as the fundamental class in $H^m(\wedge V) \subset H^m(\mathcal{L})$.*

This theorem immediately implies the following corollary.

Corollary 3.11. *Suppose that $H^*(\wedge V)$ satisfies the Poincaré duality. Then the map $H^*(\mathrm{Der}(\wedge V)) \to \mathrm{Der}^{*+1}(H(\mathcal{L}))$ induced by the contraction $e\colon \mathrm{Der}^*(\wedge V) \to \mathrm{Der}^{*+1}(\mathcal{L})$ is a monomorphism and hence so is the map $H^*(e)\colon H^*(\mathrm{Der}(\wedge V)) \to H^{*+1}(\mathrm{Der}(\mathcal{L}))$.*

Thanks to Corollary 3.11 and Proposition 4.4 below, we have Theorem 1.3. It is expected that the contraction operator e is injective for more general spaces.

The rest of this section is devoted to proving Theorem 3.10. Let $(\wedge V, d)_*$ be the linear dual of $(\wedge V, d)$ and $D\colon (\wedge V, d) \to (\wedge V, d)_*$ the duality map; that is, a quasi-isomorphism of $(\wedge V, d)$-modules of degree $(-m)$ given by the cap product with the representing cycle of the fundamental class. For a (non-negative) integer k, define a quasi-isomorphism pd by the composition

$$\mathrm{pd}\colon \mathrm{Hom}_{\wedge V}(\mathcal{L}_{(k)}, \wedge V) \xrightarrow[\simeq]{D_*} \mathrm{Hom}_{\wedge V}(\mathcal{L}_{(k)}, (\wedge V)_*) \xrightarrow[\cong]{\text{adjoint}} \mathrm{Hom}(\mathcal{L}_{(k)}, \mathbb{Q}) = (\mathcal{L}_{(k)})_*$$

and denote its adjoint by $\mathrm{ad(pd)}\colon \mathrm{Hom}^{-n}_{\wedge V}(\mathcal{L}_{(k)}, \wedge V) \otimes \mathcal{L}^{m+n}_{(k)} \to \mathbb{Q}$.

By a straightforward computation, we have

Lemma 3.12. *For any integer n and k, we have a commutative diagram*

$$\begin{tikzcd}
\mathrm{Hom}^{-n}_{\wedge V}(\mathcal{L}_{(k)}, \wedge V) \otimes \mathcal{L}^{m+n}_{(k)} \arrow[r, "\mathrm{ad(pd)}"] \arrow[d, "\mathrm{ev}"] & \mathbb{Q} \\
(\wedge V)^m \arrow[r, "D"] & ((\wedge V)_*)^0. \arrow[u, "\mathrm{ev}_1"', "\cong"]
\end{tikzcd}$$

Proposition 3.13. *Let n and k be integers.*

(1) The pairing

$$H(\mathrm{ev})\colon H^{-n}(\mathrm{Hom}_{\wedge V}(\mathcal{L}_{(k)}, \wedge V)) \otimes H^{m+n}(\mathcal{L}_{(k)}) \to H^m(\wedge V) \cong \mathbb{Q}$$

is non-degenerate.

(2) For any $[f] \neq 0 \in H^{-n}(\mathrm{Hom}_{\wedge V}(\mathcal{L}_{(k)}, \wedge V))$, there is a cohomology class $[\alpha] \in H^{m+n}(\mathcal{L}_{(k)})$ such that $[f(\alpha)] \in H^m(\wedge V)$ is the same as the fundamental class.

Proof. Since D induces an isomorphism on homology, Lemma 3.12 identifies $H(\mathrm{ev})$ and $H(\mathrm{ad(pd)})$ up to isomorphism. Hence the proposition follows from the fact that $H(\mathrm{pd})$ is an isomorphism. $\square$

In order to prove Theorem 3.10, we represent the contraction e as a composite of maps. Proposition 4.5 below asserts that the linear map $\lambda\colon \mathrm{Der}(\wedge V) \to \mathrm{Hom}_{\wedge V}(\mathcal{L}_{(1)}, \wedge V) \cong \mathrm{Hom}(\overline{V}, \wedge V)$ defined by $\lambda(\theta)(\bar{v}) = (-1)^{|\theta|}\theta(v)$ for $v \in V$ and $\theta \in \mathrm{Der}(\wedge V)$ is an isomorphism of complexes of degree 1. Moreover, we define a chain map $\tilde{e}\colon \mathrm{Hom}_{\wedge V}(\mathcal{L}_{(1)}, \wedge V) \to \mathrm{Der}(\mathcal{L})$ of degree 0 by $\tilde{e}(f)(v) = 0$ and $\tilde{e}(f)(\bar{v}) = f(\bar{v})$ for $v \in V$ and $f \in \mathrm{Hom}_{\wedge V}(\mathcal{L}_{(1)}, \wedge V)$. Then we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Der}(\wedge V) \arrow[r, "e"] \arrow[d, "\lambda", "\cong"'] & \mathrm{Der}(\mathcal{L}) \\
\mathrm{Hom}_{\wedge V}(\mathcal{L}_{(1)}, \wedge V). \arrow[ur, "\tilde{e}"] &
\end{tikzcd}$$

Proof of Theorem 3.10. Take an element $[\theta] \neq 0 \in H^{-n-1}(\mathrm{Der}(\wedge V))$. Since λ is an isomorphism, we have $H(\lambda)[\theta] \neq 0 \in H^{-n}(\mathrm{Hom}_{\wedge V}(\mathcal{L}_{(1)}, \wedge V))$. Hence, by Proposition 3.13 (2), there exists a cohomology class $[\alpha] \in H^{m+n}(\mathcal{L}_{(1)})$ such that $[\lambda(\theta)(\alpha)] \in H^m(\wedge V)$ is the same as the fundamental class. Then we see that $[(\tilde{e}\lambda(\theta))(\alpha)] \in H^m(\mathcal{L})$ is nothing but the fundamental class. Therefore, the theorem follows from the above commutative diagram. $\square$

4. Geometric counterparts of Cartan calculi

In this section, we assume that the underlying field is of arbitrary characteristic. Theorem 3.8 asserts that the operations L and e appeared in Proposition 3.6 coincide with the Lie representation and the contraction in Proposition 3.7 on homology, respectively. In this section, we consider geometric constructions of the operations L and e on homology. Moreover, as mentioned in Introduction, a geometric description of the isomorphism Φ of Lie algebras in Theorem 1.1 is given.

4.1. **Geometric descriptions of the operations L and e.** Given $\theta \in \pi_n(\mathrm{aut}_1(X))$ which is represented by $\theta : S^n \to \mathrm{aut}_1(X)$. Let $\mathrm{ad}(\theta) : S^n \times X \to X$ be the adjoint of θ and consider the map between the free loop spaces $L(\mathrm{ad}(\theta)) : LS^n \times LX \to LX$ defined by $(L(\mathrm{ad}(\theta)(l,\gamma))(t) = \mathrm{ad}(\theta)(l(t), \gamma(t))$ for $(l,\gamma) \in LS^n \times LX$ and $t \in S^1$. Let $\mathrm{ev}_0 : LS^n \to S^n$ be the evaluation map at 0. We define $L : \pi_n(\mathrm{aut}_1(X)) \to \mathrm{End}^{-n}(H^*(LX))$ by the composite

$$L_\theta : H^*(LX) \xrightarrow{L(\mathrm{ad}(\theta))^*} H^*(LS^n \times LX) \xrightarrow{\int_{[S^n]}} H^*(LX), \tag{4.1}$$

where $\int_{[S^n]}$ denotes the integration along the image of the fundamental class of S^n by the map $\mathrm{ev}_0^* : H^n(S^n) \to H^n(LS^n)$. The rotation on S^1 induces the action $\mu : S^1 \times LX \to LX$ on the free loop space. By definition, the BV-operator Δ on $H^*(LX)$ is the composite

$$\Delta : H^*(LX) \xrightarrow{\mu^*} H^*(S^1 \times LX) \xrightarrow{\int_{S^1}} H^{*-1}(LX),$$

where $\int_{S^1}$ is the integration along the fundamental class of S^1. Let $\overline{[S^n]}$ be the cohomology class in $H^{n-1}(LS^n)$ which is the image of the fundamental class of S^n by the composite

$$H^n(S^n) \xrightarrow{\mathrm{ev}_0^*} H^n(LS^n) \xrightarrow{\Delta} H^{n-1}(LS^n).$$

Then, we define a linear map $e : \pi_n(\mathrm{aut}_1(X)) \to \mathrm{End}^{-n+1}(H^*(LX))$ by the composite

$$e_\theta : H^*(LX) \xrightarrow{L(\mathrm{ad}(\theta))^*} H^*(LS^n \times LX) \xrightarrow{\int_{\overline{[S^n]}}} H^*(LX). \tag{4.2}$$

We first consider properties of the operation L.

Theorem 4.1. *(1) The map $L : \pi_*(\mathrm{aut}_1(X)) \to \mathrm{Der}^{-*}(H^*(LX))$ is a morphism of Lie algebras.*
(2) For each θ in $\pi_*(\mathrm{aut}_1(X))$, the derivation L_θ commutes with the BV operator $\Delta : H^*(LX) \to H^{*-1}(LX)$.

We postpone the proof to Appendix B; see the argument after Theorem B.2. The operation L in (4.1) is extended to that on the equivariant cohomology $H^*_{S^1}(LX)$. Theorem 4.1 is proved by considering a result on the equivariant version of the operator L.

Remark 4.2. One might expect that the same construction as that in (4.1) and (4.2) is applicable to *other* element in $H^*(LS^n)$, or more generally in $H^*(LS)$ for a simply-connected space S. In fact, it is possible. Moreover, we see that such operations incorporate e and L together with interesting properties, for instance, the Cartan formula such as that for Steenrod operations. The topic will be discussed in more detail in [NW22].

In the rest of this section, we assume that the underlying field and the coefficients of cohomology rings of spaces are rational unless otherwise specified.

We relate Sullivan's isomorphism Φ mentioned in Section 2 with the Lie derivative $L_{(\ )}$.

Proposition 4.3. *Let X be a simply-connected space of finite type and $\wedge V$ the minimal Sullivan model for X. Then there exists a commutative diagram*

$$\begin{tikzcd} \pi_*(\mathrm{aut}_1(X)) \otimes \mathbb{Q} \arrow[r, "L"] \arrow[d, "\Phi"', "\cong"] & \mathrm{Der}^{-*}(H^*(LX;\mathbb{Q})) \arrow[d, "\cong"] \\ H_*(\mathrm{Der}(\wedge V)) \arrow[r, "L"'] & \mathrm{Der}^{-*}(H^*(\mathcal{L})). \end{tikzcd}$$

We prove Proposition 4.3 in Appendix B together with its equivariant version; see Theorem B.6. Next we give a relationship between the operation e and the isomorphism Φ.

Proposition 4.4. *Under the same assumption as in Proposition 4.3, there exists a commutative diagram*

$$\begin{tikzcd} \pi_n(\mathrm{aut}_1(X)) \otimes \mathbb{Q} \arrow[r, "(-1)^n e"] \arrow[d, "\Phi"', "\cong"] & \mathrm{End}^{-n+1}(H^*(LX;\mathbb{Q})) \\ H_n(\mathrm{Der}(\wedge V)) \arrow[r, "e"] & \mathrm{Der}^{-n+1}(H^*(\mathcal{L})). \arrow[u, "a\ monomorphism"'] \end{tikzcd}$$

Proof. We first consider a rational model for e_θ described in (4.2). Let $\mathcal{M}_{S^n}$ be the Sullivan model for S^n which is of the form

$$\mathcal{M}_{S^n} = \begin{cases} (\wedge(u), 0) & (n : \mathrm{odd}), \\ (\wedge(u, u'), du' = u^2) & (n : \mathrm{even}), \end{cases}$$

where $|u| = n$, $|u'| = 2n - 1$. Let $\mathcal{L}_{S^n}$ be the Sullivan model for LS^n induced by $\mathcal{M}_{S^n}$ and $\varphi : \wedge V \to \mathcal{M}_{S^n} \otimes \wedge V$ a Sullivan representative for $\mathrm{ad}(\theta)$; see Section 2. It follows from Lemma 4.10 that a Sullivan representative $\mathcal{L}\varphi : \mathcal{L} \to \mathcal{L}_{S^n} \otimes \mathcal{L}$ for $L(\mathrm{ad}(\theta))$ is given by

$$\mathcal{L}\varphi(v) = \varphi(v) \quad \text{and} \quad \mathcal{L}\varphi(\bar{v}) = s\varphi(v) \tag{4.3}$$

for $v \in V$. We define a morphism $\int_{\bar{u}} : \mathcal{L}_{S^n} \to \mathbb{Q}$ of chain complexes of degree $-n+1$ by $\int_{\bar{u}}(\bar{u}) = 1$ and $\int_{\bar{u}}(w) = 0$ for bases w with $w \neq \bar{u}$. Since the cohomology class $\overline{[S^n]}$ is represented by $\bar{u}$, the definition of e_θ implies that the composite $(\int_{\bar{u}} \otimes 1) \circ \mathcal{L}\varphi$ is a rational model for e_θ. Now, we may write

$$\varphi(v) \equiv 1 \otimes v + u \otimes \theta'(v)$$

for $v \in V$ modulo $(\mathcal{M}_{S^n})^{>n} \otimes \wedge V$. By definition, we see that $\Phi(\theta)(v) = \theta'(v)$. Therefore, it follows from (4.3) that

$$\begin{aligned} \mathcal{L}\varphi(v) &\equiv 1 \otimes v + u \otimes \theta'(v), \\ \mathcal{L}\varphi(\bar{v}) &\equiv s(1 \otimes v + u \otimes \theta'(v)) = 1 \otimes \bar{v} + \bar{u} \otimes \theta'(v) + (-1)^n u \otimes s\theta'(v) \end{aligned}$$

modulo $(\mathcal{L}_{S^n})^{>n} \otimes \mathcal{L}$. We have $(\int_{\bar{u}} \otimes 1) \circ \mathcal{L}\varphi(v) = 0$ and $(\int_{\bar{u}} \otimes 1) \circ \mathcal{L}\varphi(\bar{v}) = (-1)^n e_\theta(\bar{v})$. This completes the proof. $\square$

Finally we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We observe that all results in this manuscript remain true even if the underlying field $\mathbb{Q}$ is replaced with $\mathbb{R}$. In fact, after taking the tensor product $-\otimes_{\mathbb{Q}}\mathbb{R}$, we have these results. Moreover, we recall the fact that there exists a sequence of quasi-isomorphisms connecting $\Omega^*(M)$ with $A_{PL}(M)\otimes_{\mathbb{Q}}\mathbb{R}$, where $A_{PL}(M)$ denotes the CDGA of polynomial differential forms on a manifold M; see [FHT01, Section 11 (d)] for the details. Therefore, a Sullivan minimal model $\wedge W \overset{\simeq}{\to} A_{PL}(M)$ gives rise to a minimal model $m : \wedge V \overset{\simeq}{\to} Q\Omega^*(M)$ in which $V = W\otimes_{\mathbb{Q}}\mathbb{R}$. Here $Q\Omega^*(M)$ denotes a cofibrant replacement of $\Omega^*(M)$ in the category $\mathcal{A}$ of CDGAs endowed with the model category structure described in [BG76].

Let M be a simply-connected manifold. To prove Theorem 1.1, we first construct an isomorphism between $H_*(\mathrm{Der}(Q\Omega^*(M)))$ and $H_*(\mathrm{Der}(\wedge V))$, by applying the proof of [BL05, Theorem 2.8]. With the notation above, the map m has a factorization

$$\begin{tikzcd} \wedge V \arrow[rr, "m", "\simeq"'] \arrow[dr, "\simeq", "i"'] & & Q\Omega^*(M) \\ & A \arrow[ur, "p"'] & \end{tikzcd}$$

in $\mathcal{A}$, where i is a trivial cofibration and p is a fibration, namely an epimorphism. Observe that p is also a quasi-isomorphism. Therefore, we have a right splitting g of p with $p\circ g = id_{Q\Omega^*(M)}$ and a map $h_1 : \mathrm{Der}(A) \to \mathrm{Der}(Q\Omega^*(M))$ defined by $h_1(\theta) = p\circ\theta\circ g$. Moreover, since each object in $\mathcal{A}$ is fibrant, it follows that i admits a left splitting $r : A \to \wedge V$ with $r\circ i = id_{\wedge V}$. Thus, a chain map $h_2 : \mathrm{Der}(A) \to \mathrm{Der}(\wedge V)$ is defined by $h_2(\theta) = r\circ\theta\circ i$. As a consequence, we have a diagram consisting of commutative squares

$$\begin{tikzcd} H_*(\mathrm{Der}(Q\Omega^*(M))) \arrow[r, "L", "e"'] & \mathrm{End}^{-*}(HH_*(\Omega^*(M))) \\ H_*(\mathrm{Der}(A)) \arrow[r, "L", "e"'] \arrow[u, "(h_1)_*", "\cong"'] \arrow[d, "(h_2)_*"', "\cong"] & \mathrm{End}^{-*}(HH_*(A)) \arrow[u, "\cong", "HH(p)\circ(\ )\circ HH(g)"'] \arrow[d, "\cong"', "HH(r)\circ(\ )\circ HH(i)"] \\ H_*(\mathrm{Der}(\wedge V)) \arrow[r, "L", "e"'] & \mathrm{End}^{-*}(HH_*(\wedge V)). \end{tikzcd}$$

We observe that left maps $(h_1)_*$ and $(h_2)_*$ are isomorphisms; see the proof of [BL05, Theorem 2.8 (1)].

Finally Theorem 3.8, Propositions 4.3 and 4.4 enable us to obtain the commutative diagram in Theorem 1.1. □

4.2. **The map Γ_1 due to Félix and Thomas.** Throughout this section, we assume that M is a simply-connected closed manifold of dimension m. Let $\wedge V = (\wedge V, d)$ be the minimal Sullivan model for M and $C_*(\wedge V)$ the Hochschild chain complex of $\wedge V$. Recall the direct sum decomposition $\mathcal{L} = \wedge V\otimes\wedge\overline{V} = \oplus_k\mathcal{L}_{(k)}$ of complexes from Section 2. Thus we have decompositions $H^*(\mathcal{L}) = \oplus_k H^*(\mathcal{L}_{(k)})$ and $H_*(LM;\mathbb{Q}) \cong \oplus_k H^{-*}(\mathrm{Hom}(\mathcal{L}_{(k)},\mathbb{Q}))$ which are called the Hodge decompositions. We put

$$H_*^{(k)}(LM) := H^{-*}(\mathrm{Hom}(\mathcal{L}_{(k)},\mathbb{Q})).$$

Since the Hochschild cohomology $HH^*(\wedge V, A)$ with coefficients in a $\wedge V$-module A is isomorphic to the homology of the complex $\mathrm{Hom}_{\wedge V}(\mathcal{L}, A)$, we also have the direct sum decomposition $HH^*(\wedge V, A) \cong \oplus_k HH^*_{(k)}(\wedge V, A)$, where

$$HH^*_{(k)}(\wedge V, A) = H^*(\mathrm{Hom}_{\wedge V}(\mathcal{L}_{(k)}, A)).$$

By a direct computation, we have

Proposition 4.5. *The map* $\lambda : \mathrm{Der}(\wedge V, A) \to \mathrm{Hom}_{\wedge V}(\mathcal{L}_{(1)}, A) \cong \mathrm{Hom}(\overline{V}, A)$ *of degree* 1 *defined by* $\lambda(\theta)(\bar{v}) = (-1)^{|\theta|}\theta(v)$ *for* $\theta \in \mathrm{Der}(\wedge V, A)$ *and* $\bar{v} \in \overline{V}$ *is an isomorphism of complexes of degree* 1*.*

Corollary 4.6. *Let* $\eta : \wedge V \to A$ *be a morphism of CDGAs, where* $\wedge V$ *is a Sullivan algebra.*

(1) *For any quasi-isomorphism* $\varphi : A \to B$ *of CDGAs, the map* $\varphi_* : H_*(\mathrm{Der}(\wedge V, A)) \to H_*(\mathrm{Der}(\wedge V, B))$ *induced by* φ *is an isomorphism, where* B *in the codomain is regarded as a* $\wedge V$*-module via the composite* $\varphi \circ \eta$*.*
(2) *For any quasi-isomorphism* $\psi : \wedge W \to \wedge V$ *of Sullivan algebras, the map* $\psi^* : H_*(\mathrm{Der}(\wedge V, A)) \to H_*(\mathrm{Der}(\wedge W, A))$ *induced by* ψ *is an isomorphism, where* A *in the codomain is regarded as a* $\wedge W$*-module via the composite* $\eta \circ \psi$*.*

Proof. The differential graded module $\mathcal{L}_{(1)}$ is a semifree $\wedge V$-module. Then the results follow from [FHT01, Theorem 6.10] and the naturality of λ in Proposition 4.5. □

Poincaré duality for manifolds gives rise to a duality between the direct summands $H_*^{(k)}(LM)$ and $HH^*_{(k)}(\wedge V, A)$. To see this, let A be an m-dimensional Poincaré duality model for a simply-connected manifold M introduced in [LS08] equipped with $\varphi : \wedge V \to A$ a quasi-isomorphism of CDGAs. Denote by $A_* := \mathrm{Hom}(A, \mathbb{Q})$ the linear dual of A with the differential defined by $\alpha \mapsto -(-1)^{|\alpha|}\alpha \circ d_A$ for $\alpha \in A_*$, where d_A denotes the differential of A; see Remark 4.12. Let $\{a_i\}_{i=1}^N$ be a homogeneous basis with $a_N = w_A$ a representative of the fundamental class of M. We denote by $\{a_i^*\}_{i=1}^N$ the dual basis. Let $D_A : A \to A_*$ be the duality map; that is, D_A is an isomorphism of A-modules defined by $D_A(a)(b) = \omega_A^*(ab)$. Observe that A is regarded as a $\wedge V$-module via φ.

Put $\mathcal{L}^A := A \otimes_{\wedge V} \mathcal{L} \cong A \otimes \wedge\overline{V}$. We observe that $\mathcal{L}^A$ is also a rational model for LM. The direct sum decomposition of $\mathcal{L}$ mentioned above induces a decomposition $\mathcal{L}^A = \oplus_k \mathcal{L}^A_{(k)}$, where $\mathcal{L}^A_{(k)} := A \otimes_{\wedge V} \mathcal{L}_{(k)}$. Then, we have an isomorphism

$$\begin{tikzcd}[column sep=large] \mathrm{PD} : HH^n_{(k)}(\wedge V) \arrow[r, "\varphi_*", "\cong"'] & HH^n_{(k)}(\wedge V, A) \arrow[r, "D_{A*}", "\cong"'] & HH^{n-m}_{(k)}(\wedge V, A_*) \arrow[dl, "\text{adjoint}"', "\cong"] \\ & H^{n-m}(\mathrm{Hom}(\mathcal{L}^A_{(k)}, \mathbb{Q})) \arrow[r, "(\varphi\otimes 1)^*", "\cong"'] & H^{(k)}_{-n+m}(LM). \end{tikzcd} \tag{4.4}$$

Let $\Omega\mathrm{aut}_1(M)_0$ be the connected component of the based loop space $\Omega\mathrm{aut}_1(M)$ containing the constant loop at $id \in \mathrm{aut}_1(M)$. We here recall the morphism Γ_1 due to Félix and Thomas. Let $g : \Omega\mathrm{aut}_1(M)_0 \times M \to LM$ be a map defined by

$$g(\gamma, x)(t) = \gamma(t)(x)$$

for $\gamma \in \Omega\mathrm{aut}_1(M)_0$, $x \in M$ and $t \in S^1$. In [FT04], Félix and Thomas show that the morphism Γ_1 defined by the composite

$$\begin{array}{ccc} \pi_n(\Omega\mathrm{aut}_1(M)_0)\otimes\mathbb{Q} \xrightarrow{\mathrm{Hur}} H_n(\Omega\mathrm{aut}_1(M)_0;\mathbb{Q}) & & \\ \downarrow{\scriptstyle \times[M]} & & \\ H_{n+m}(\Omega\mathrm{aut}_1(M)_0\times M;\mathbb{Q}) & \xrightarrow{g_*} & H_{n+m}(LM;\mathbb{Q}) \end{array}$$

is injective for $n \geq 1$ and that the image of Γ_1 is isomorphic to $H^{(1)}_{n+m}(LM)$, where Hur denotes the Hurewicz map and $[M] \in H_m(M)$ is the fundamental class.

The main theorem in this section is as follows.

Theorem 4.7. *With the notation above, the diagram*

$$\begin{tikzcd} \pi_n(\mathrm{aut}_1(M))\otimes\mathbb{Q} \arrow[rr, "\Phi", "\cong"'] & & H_n(\mathrm{Der}(\wedge V)) \arrow[d, "\lambda", "\cong"'] \\ \pi_{n-1}(\Omega\mathrm{aut}_1(M)_0)\otimes\mathbb{Q} \arrow[u, "\partial", "\cong"'] \arrow[r, "\Gamma_1", "\cong"'] & H^{(1)}_{n+m-1}(LM) \arrow[r, "\mathrm{PD}^{-1}", "\cong"'] & HH^{-n+1}_{(1)}(\wedge V) \end{tikzcd}$$

is commutative, where ∂ is the adjoint map.

We note that Theorem 4.7 gives another proof of [FT04, Theorem 2]. In order to prove Theorem 4.7, we first observe rational models for $\Omega\mathrm{aut}_1(M)_0$ and the adjoint map ∂ by using the rational models for function spaces due to Brown and Szczarba [BS97]. Remark that the proof of the theorem due to Félix and Thomas uses a rational model for Γ_1 constructed by a Haefliger model [Hae82] for the space of sections of a fibration.

Let $(\wedge(V\otimes A_*), d)$ and $\wedge S_\varphi = \left(\wedge\left(\overline{(V\otimes A_*)}^1 \oplus (V\otimes A_*)^{\geq 2}\right), d\right)$ be the Brown-Szczarba models for $\mathrm{Map}(M,M)$ and $\mathrm{aut}_1(M)$, respectively. For the details of Brown-Szczarba models, see [BS97, BM06, HKO08] and also Appendix A.

Let $\mathcal{M}_{S^1} = (\wedge(u), 0)$ be the Sullivan model for S^1 with $|u| = 1$. Since $\mathrm{aut}_1(M)$ is connected, nilpotent space [HMR75] of finite type, it follows that the function space $L\mathrm{aut}_1(M)$ admits a Brown-Szczarba model of the form $(\wedge(S_\varphi\otimes\wedge(u)_*), d)$. A Sullivan representative for the constant loop in $L\mathrm{aut}_1(M)$ at $id \in \mathrm{aut}_1(M)$ is of the form $\wedge S_\varphi \to \wedge(u)$ defined by $w \mapsto 0$ for $w \in S_\varphi$. This induces an augmentation $\varepsilon : \wedge(S_\varphi\otimes\wedge(u)_*) \to \mathbb{Q}$ of the model for $L\mathrm{aut}_1(M)$. Therefore, by virtue of [BM06, Corollary 4.7], we have a model

$$(\wedge S_\varepsilon, d) = \left(\wedge\left(\overline{(S_\varphi\otimes\wedge(u)_*)}^1 \oplus (S_\varphi\otimes\wedge(u)_*)^{\geq 2}\right), d\right),$$

for the connected component $L\mathrm{aut}_1(M)_0$ of $L\mathrm{aut}_1(M)$ containing the constant loop. Moreover, it follows from [BM06, Proposition 4,2, Theorem 4.5] that the CDGA morphism

$$\wedge(S_\varphi\otimes\wedge(u)_*) \to \wedge S_\varepsilon, \quad w\otimes\beta \mapsto \begin{cases} \mathrm{pr}(w\otimes\beta) & (|w\otimes\beta| \geq 1) \\ 0 & (|w\otimes\beta| = 0) \end{cases}$$

for $w \in S_\varphi$ and $\beta \in \wedge(u)_*$ is a model for the inclusion $L\mathrm{aut}_1(M)_0 \hookrightarrow L\mathrm{aut}_1(M)$, where pr is the projection $(S_\varphi\otimes\wedge(u)_*)^{\geq 1} \to S_\varepsilon$. The result [BM06, Corollary 4.7] yields that a morphism

$$\omega_0 : \wedge S_\varphi \to \wedge S_\varepsilon$$

of CDGAs, which is defined by the projection onto $\overline{(S_\varphi \otimes \wedge(u)_*)}^1$ in S^1_φ and $\omega_0(w) = w \otimes 1^*$ for $w \in S^{\geq 2}_\varphi$, is a rational model of the evaluation map at the base point $\mathrm{ev}_0 : L\mathrm{aut}_1(M)_0 \to \mathrm{aut}_1(M)$.

Lemma 4.8. *The fiber of ω_0 at the canonical augmentation of $\wedge S_\varphi$ over $\mathbb{Q}$ is a Sullivan model for $\Omega\mathrm{aut}_1(M)_0$.*

We remark that the result [BM06, Corollary 4.8] is not applicable to the morphism ω_0 since $\mathrm{aut}_1(M)$ is not a simply-connected in general.

Proof of Lemma 4.8. For proving the assertion, it is enough to show that ω_0 is a KS-extension; that is, $\wedge S_\varepsilon$ is a relative Sullivan algebra with base $\wedge S_\varphi$ and ω_0 is the canonical inclusion; see, for example, [FHT01, Section 14] for relative Sullivan algebras. Observe that

$$(S_\varphi \otimes \wedge(u)_*)^0 = S^1_\varphi \otimes \mathbb{Q}u^* \quad \text{and} \quad (S_\varphi \otimes \wedge(u)_*)^1 = (S^1_\varphi \otimes \mathbb{Q}1^*) \oplus (S^2_\varphi \otimes \mathbb{Q}u^*).$$

It is readily seen that $\overline{(S_\varphi \otimes \wedge(u)_*)}^1$ coincides with the complement of the morphism

$$S^1_\varphi \otimes \mathbb{Q}u^* \xrightarrow{(0, d_0 \otimes 1)} (S^1_\varphi \otimes \mathbb{Q}1^*) \oplus (S^2_\varphi \otimes \mathbb{Q}u^*),$$

where d_0 is the linear part of the differential of $\wedge S_\varphi$. It follows that $\wedge S_\varepsilon$ is isomorphic to

$$\wedge\left((S_\varphi \otimes \mathbb{Q}1^*) \oplus \left(\overline{S^2_\varphi} \oplus S^{\geq 3}_\varphi \right) \otimes \mathbb{Q}u^* \right) \cong \wedge S_\varphi \otimes \wedge\left(\left(\overline{S^2_\varphi} \oplus S^{\geq 3}_\varphi \right) \otimes \mathbb{Q}u^* \right)$$

which is a relative Sullivan algebra with base $\wedge S_\varphi$. Here, $\overline{S^2_\varphi}$ is the quotient space $S^2_\varphi / d_0(S^1_\varphi)$. Therefore, the morphism ω_0 of CDGAs is a KS-extension with the fiber $\wedge\left(\left(\overline{S^2_\varphi} \oplus S^{\geq 3}_\varphi \right) \otimes \mathbb{Q}u^* \right)$. Since $\mathrm{aut}_1(M)$ is an H-space, we have a homotopy equivalence

$$\mathrm{aut}_1(M) \times \Omega\mathrm{aut}_1(M)_0 \simeq L\mathrm{aut}_1(M)_0$$

defined by $(x, \gamma) \mapsto x \cdot \gamma$. A homotopy, which defines the holonomy action of $\pi_1(\mathrm{aut}_1(M))$ on $H^*(\Omega\mathrm{aut}_1(M)_0)$, factors through the product. This implies that the $\pi_1(\mathrm{aut}_1(M))$-action is trivial and hence nilpotent. Therefore, by virtue of [Hal83, 20.3 Theorem], we see that the fiber $\wedge\left(\left(\overline{S^2_\varphi} \oplus S^{\geq 3}_\varphi \right) \otimes \mathbb{Q}u^* \right)$ is a Sullivan model for $\Omega\mathrm{aut}_1(M)_0$. □

Now, we recall facts on rational homotopy groups of nilpotent spaces. Let X be a connected nilpotent space of finite type and $\mathcal{M}_X = (\wedge W, d)$ a Sullivan model for X. It follows from [BG76, 11.3] that there is a natural isomorphism

$$\nu : \pi_n(X) \otimes \mathbb{Q} \longrightarrow \mathrm{Hom}(H^n(W, d_0), \mathbb{Q})$$

provided $\pi_n(X)$ is abelian, where d_0 is the linear part of the differential d. Let $f : S^n \to X$ be a map which represents an element in $\pi_n(X) \otimes \mathbb{Q}$. Then, the image $\nu(f)$ is defined by the linear part of $\mathcal{M}_f$ a Sullivan representative of the map f. We denote by $\mathcal{M}_{S^n}$ the Sullivan model for S^n described in [FHT01, §12 Example 1] and $\int_{S^n} : \mathcal{M}_{S^n} \to \mathbb{Q}$ the chain map which assigns 1 to a representative of the fundamental class of S^n. Since $\mathcal{M}_f(w)$ is indecomposable for any w in $(\wedge W)^n$, it follows that $\nu(f)$ coincides with the map induced by the chain map $\int_{S^n} \circ \mathcal{M}_f|_W$ on $H^n(W, d_0)$.

Let M be a simply-connected manifold. The monoid $\mathrm{aut}_1(M)$ and $\Omega\mathrm{aut}_1(M)_0$ are connected nilpotent H-spaces; see [HMR75]. Therefore, by the models mentioned above, we see that the dual spaces of $\pi_n(\mathrm{aut}_1(M))\otimes\mathbb{Q}$ and $\pi_n(\Omega_*\mathrm{aut}_1(M))\otimes\mathbb{Q}$ are isomorphic to the homology of S_φ and $\left(\overline{S^2_\varphi}\oplus S^{\geq 3}_\varphi\right)\otimes\mathbb{Q}u^*$ the linear parts of the Sullivan models for $\mathrm{aut}_1(M)$ and $\Omega\mathrm{aut}_1(M)_0$, respectively. Observe that the fundamental groups $\pi_1(\mathrm{aut}_1(M))$ and $\pi_1(\Omega\mathrm{aut}_1(M)_0)$ are abelian.

In what follows, we put

$$T_\varphi := \overline{S^2_\varphi}\oplus S^{\geq 3}_\varphi,$$

where the differential of T_φ is induced by the linear part (S_φ, d_0). Let $\iota : S_\varphi \to T_\varphi\otimes\mathbb{Q}u^*$ be the composite of the inclusion $T_\varphi \hookrightarrow T_\varphi\otimes\mathbb{Q}u^*$ defined by $w \mapsto (-1)^{|w|}w\otimes u^*$ and the projection $\mathrm{pr}' : (S_\varphi)^{\geq 2} \to T_\varphi$.

Proposition 4.9. *The morphism ι is a rational model for the adjoint map ∂; that is, the diagram*

$$\begin{tikzcd}
\pi_{n-1}(\Omega\mathrm{aut}_1(M)_0)\otimes\mathbb{Q} \arrow[r, "\nu", "\cong"'] \arrow[d, "\partial", "\cong"'] & \mathrm{Hom}_{\mathbb{Q}}\left(H^{n-1}\left(T_\varphi\otimes\mathbb{Q}u^*\right), \mathbb{Q}\right) \arrow[d, "H(\iota)^*"] \\
\pi_n(\mathrm{aut}_1(M))\otimes\mathbb{Q} \arrow[r, "\nu", "\cong"'] & \mathrm{Hom}_{\mathbb{Q}}(H^n(S_\varphi), \mathbb{Q}),
\end{tikzcd}$$

is commutative for $n\geq 2$. As a consequence, the morphism $H(\iota) : H^n(S_\varphi) \longrightarrow H^{n-1}(T_\varphi\otimes\mathbb{Q}u^*)$ induced by ι is an isomorphism for $n \geq 2$.

Proof. We may assume that M is a rational space without loss of generality. Let $f : S^{n-1} \to \Omega\mathrm{aut}_1(M)_0$ be a based map which represents an element in $\pi_{n-1}(\Omega\mathrm{aut}_1(M)_0) \cong \pi_{n-1}(\Omega\mathrm{aut}_1(M)_0)\otimes\mathbb{Q}$. Consider a commutative diagram

$$\begin{tikzcd}
S^n \arrow[d, equal] \arrow[rrd, bend left, "\partial(f)"] & & \\
S^{n-1}\wedge S^1 \arrow[r, "f\wedge 1"] & \Omega\mathrm{aut}_1(M)_0\wedge S^1 \arrow[r, "\mathrm{ev}"] & \mathrm{aut}_1(M) \arrow[d, equal] \\
S^{n-1}\times S^1 \arrow[u, two heads] \arrow[r, "f\times 1"] & \Omega\mathrm{aut}_1(M)_0\times S^1 \arrow[u, two heads] \arrow[r, "\mathrm{ev}"] & \mathrm{aut}_1(M),
\end{tikzcd} \tag{4.5}$$

where ev is the evaluation map. Let $\mathrm{inc}' : \Omega\mathrm{aut}_1(M)_0 \hookrightarrow L\mathrm{aut}_1(M)$ be the inclusion. By the construction of the model for $\Omega\mathrm{aut}_1(M)_0$ in the proof of Lemma 4.8, the CDGA morphism $\mathcal{M}_{\mathrm{inc}'} : \wedge(S_\varphi\otimes\wedge(u)_*) \to \wedge(T_\varphi\otimes\mathbb{Q}u^*)$ defined by $\mathcal{M}_{\mathrm{inc}'}(w\otimes 1^*) = 0$ and

$$\mathcal{M}_{\mathrm{inc}'}(w\otimes u^*) = \begin{cases} 0 & (|w|=1) \\ \mathrm{pr}'(w)\otimes u^* & (|w|\geq 2) \end{cases}$$

for $w\in S_\varphi$ is a rational model for inc'. Therefore, by combining the rational models of inc' and the evaluation map $L\mathrm{aut}_1(M)\times S^1 \to \mathrm{aut}_1(M)$ due to Buijs and Murillo [BM06], we see that ev admits a Sullivan representative

$$\mathcal{M}_{\mathrm{ev}} : \wedge S_\varphi \to \wedge(T_\varphi\otimes\mathbb{Q}u^*)\otimes\wedge(u)$$

defined by $\mathcal{M}_{\mathrm{ev}}(w) = -(\mathrm{pr}'(w)\otimes u^*)\otimes u$. Let $\mathcal{M}_f : \wedge(T_\varphi\otimes\mathbb{Q}u^*) \to \mathcal{M}_{S^{n-1}}$ and $\mathcal{M}_{\partial(f)} : \wedge S_\varphi \to \mathcal{M}_{S^n}$ be Sullivan representatives of f and $\partial(f)$, respectively. Then we have the following homotopy commutative diagram of CDGAs

$$
\begin{tikzcd}
 & & \mathcal{M}_{S^n} \arrow[r, "\int_{S^n}"] \arrow[d, "\pi"] & \mathbb{Q} \arrow[d, equal] \\
\wedge S_\varphi \arrow[r, "\mathcal{M}_{\mathrm{ev}}"] \arrow[dr, "\wedge\iota"'] \arrow[urr, bend left, "\mathcal{M}_{\partial(f)}"] & \wedge(T_\varphi \otimes \mathbb{Q}u^*) \otimes \wedge(u) \arrow[r, "\mathcal{M}_f \otimes 1"] \arrow[d, "1\otimes\int_{S^1}"] & \mathcal{M}_{S^{n-1}} \otimes \wedge(u) \arrow[r, "\int_{S^{n-1}} \otimes \int_{S^1}"] \arrow[d, "1\otimes\int_{S^1}"] & \mathbb{Q} \arrow[d, equal] \\
 & \wedge(T_\varphi \otimes \mathbb{Q}u^*) \arrow[r, "\mathcal{M}_f"] & \mathcal{M}_{S^{n-1}} \arrow[r, "\int_{S^{n-1}}"] & \mathbb{Q},
\end{tikzcd}
$$

where π is the canonical morphism which sends the fundamental class of S^n to the fundamental class of $S^{n-1} \times S^1$ on cohomology. The uniqueness up to homotopy of a Sullivan representative and commutativity of the diagram (4.5) enable us to conclude that the top and left-hand side diagram is homotopy commutative. Therefore, we have

$$
\begin{aligned}
\nu \circ \partial(f) = \left(\textstyle\int_{S^n} \circ \mathcal{M}_{\partial(f)}\right)|_{S_\varphi} &\simeq \left(\textstyle\int_{S^{n-1}} \circ \mathcal{M}_f \circ \wedge\iota\right)|_{S_\varphi} \\
&= \left(\textstyle\int_{S^{n-1}} \circ \mathcal{M}_f\right)|_{T_\varphi \otimes \mathbb{Q}u^*} \circ \iota = \nu(f) \circ \iota.
\end{aligned}
$$

This completes the proof. $\square$

We next consider a rational model for Γ_1. Let $\wedge(V \otimes \wedge(u)_*)$ be the Brown–Szczarba model for LM, where $\wedge V$ is a Sullivan model for M. Then we identify the model with the CDGA $\mathcal{L}$ mentioned in Section 2 by the isomorphism $\xi : \mathcal{L} \xrightarrow{\cong} \wedge(V \otimes \wedge(u)_*)$, defined by $v \mapsto v \otimes 1^*$ and $\bar{v} \mapsto (-1)^{|v|} v \otimes u^*$ for $v \in V$.

Lemma 4.10. *Let $\wedge V_i$ be a minimal Sullivan model of a simply-connected space X_i for each $i = 1$ and 2. Let $\mathcal{L}_{X_i}$ be the Sullivan model for LX_i and $\psi : \wedge V_2 \to \wedge V_1$ a Sullivan representative for $f : X_1 \to X_2$. Then, a CDGA morphism $\mathcal{L}\psi : \mathcal{L}_{X_2} \to \mathcal{L}_{X_1}$ defined by $\mathcal{L}\psi(v) = \varphi(v)$ and $\mathcal{L}\psi(\bar{v}) = s\psi(v)$ for $v \in V_2$ is a Sullivan representative for the map $Lf : LX_1 \to LX_2$.*

Proof. Let $\wedge(V_i \otimes \wedge(u)_*) \cong \wedge(\wedge V_i \otimes \wedge(u_*))/\mathcal{I}$ be the Brown–Szczarba model for LX_i; see Appendix A. A naturality of Brawn-Szczarba models implies that the induced morphism

$$
\wedge(\psi \otimes 1) : \wedge(\wedge V_2 \otimes \wedge(u_*))/\mathcal{I} \longrightarrow \wedge(\wedge V_2 \otimes \wedge(u_*))/\mathcal{I}
$$

is a rational model for Lf. It is readily seen that the square

$$
\begin{tikzcd}
\wedge(V_2 \otimes \wedge(u)_*) \arrow[d, "\rho"', "\cong"] & \mathcal{L}_{X_2} \arrow[l, "\xi"', "\cong"] \arrow[r, "\mathcal{L}\varphi"] & \mathcal{L}_{X_1} \arrow[r, "\xi", "\cong"'] & \wedge(V_1 \otimes \wedge(u)_*) \arrow[d, "\cong"', "\rho"] \\
\wedge(\wedge V_2 \otimes \wedge(u_*))/\mathcal{I} \arrow[rrr, "\wedge(\psi\otimes 1)"] & & & \wedge(\wedge V_1 \otimes \wedge(u_*))/\mathcal{I}
\end{tikzcd}
$$

is commutative, which proves the lemma. $\square$

By the restriction of the isomorphism ξ mentioned above to the direct summands of the Hodge decomposition, we see that $\mathcal{L}_{(k)}$ is isomorphic to $\wedge V \otimes \wedge^k(V \otimes \mathbb{Q}u^*)$ which is a direct summand of $\wedge(V \otimes \wedge(u)_*)$. Define the morphism $\mathcal{M}_{\Gamma_1}$ of CDGAs by the composite

$$
\begin{tikzcd}
\mathcal{L} \arrow[r, "\mathrm{proj}"] & \mathcal{L}_{(1)} \arrow[r, "\xi", "\cong"'] & \wedge V \otimes (V \otimes \mathbb{Q}u^*) \arrow[r, "\varphi\otimes 1"] & A \otimes (V \otimes \mathbb{Q}u^*) \arrow[dll, "D_A\otimes 1", "\cong"'] \\
& A_* \otimes (V \otimes \mathbb{Q}u^*) \arrow[r, "T", "\cong"'] & V \otimes A_* \otimes \mathbb{Q}u^* \arrow[r, "\mathrm{pr}''\otimes 1"] & T_\varphi \otimes \mathbb{Q}u^*.
\end{tikzcd}
\tag{4.6}
$$

Here, the map T is defined by $T(a\otimes v\otimes u^*) = (-1)^{|a||v|} v\otimes a\otimes u^*$ and $\mathrm{pr}'' : V\otimes A_* \to T_\varphi$ denotes the canonical projection.

Proposition 4.11. *The morphism $\mathcal{M}_{\Gamma_1}$ is a rational model for the dual of Γ_1.*

Proof. We first consider the composite

$$
g' : \Omega\mathrm{aut}_1(M)_0 \xrightarrow{\mathrm{inc}'} L\mathrm{aut}_1(M) \xrightarrow{\mathrm{ad}} \mathrm{Map}(S^1 \times M, M) \underset{\cong}{\xleftarrow{\mathrm{ad}'}} \mathrm{Map}(M, LM),
$$

where ad and ad$'$ are the adjoint maps. By virtue of Lemma A.2, we see that the map ad is modeled by the morphism $\mathcal{M}_{\mathrm{ad}} : \wedge(V \otimes (\wedge(u)\otimes A)_*) \to \wedge(S_\varphi \otimes \wedge(u)_*)$ given by

$$
\mathcal{M}_{\mathrm{ad}}(v\otimes\alpha) = \rho^{-1}\left(\mathcal{M}_{\mathrm{inc}}(v\otimes\alpha_0)\otimes 1^* + (-1)^{|\alpha_1|}\mathcal{M}_{\mathrm{inc}}(v\otimes\alpha_1)\otimes u^*\right)
$$

for $\alpha \in (\wedge(u)\otimes A)_*$ and $\zeta(\alpha) = 1^*\otimes\alpha_0 + u^*\otimes\alpha_1$. Here, ρ and ζ are the isomorphisms described in Appendix A.

Since M is simply-connected, it follows that LM is connected. Thus, the same argument as in the proof of Lemma A.2 enables us to obtain a model for ad$'$ of the form $\mathcal{M}_{\mathrm{ad}'} : \wedge(V\otimes(\wedge(u)\otimes A)_*) \to \wedge((V\otimes\wedge(u)_*)\otimes A_*)$ which is induced by the isomorphisms $\zeta : (\wedge(u)\otimes A)_* \cong \wedge(u)_*\otimes A_*$. Therefore, the composite $\mathcal{M}_{\mathrm{inc}'}\circ\mathcal{M}_{\mathrm{ad}}\circ\mathcal{M}_{\mathrm{ad}'}^{-1}$ is a model for g', where $\mathcal{M}_{\mathrm{inc}'}$ is the model of inc$'$ described in the proof of Proposition 4.9. Let $\mathrm{ev}' : \mathrm{Map}(M, LM)\times M \to LM$ be the evaluation map. Then we have a model for ev$'$ of the form $\mathcal{M}_{\mathrm{ev}'} : \wedge(V\otimes\wedge(u)_*) \to \wedge((V\otimes\wedge(u)_*)\otimes A_*)\otimes A$ defined by

$$
\mathcal{M}_{\mathrm{ev}'}(v\otimes\beta) = \sum (-1)^{|a_i|}((v\otimes\beta)\otimes a_i^*)\otimes a_i.
$$

This follows from [BM06, Theorem 1.1] and [Kur06, Theorem 4.5]. We remark that the sign of the model $\mathcal{M}_{\mathrm{ev}'}$ is different from the original model due to Buijs and Murillo. For details of the sign, see Remark 4.12 after the proof.

Since the map g coincides with the composite $\mathrm{ev}'\circ(g'\times 1)$, it follows that

$$
\mathcal{M}_g = (\mathcal{M}_{\mathrm{inc}'}\circ\mathcal{M}_{\mathrm{ad}}\circ\mathcal{M}_{\mathrm{ad}'}^{-1}\otimes 1)\circ\mathcal{M}_{\mathrm{ev}'}
$$

is a rational model for g. Explicitly, we compute

$$
\begin{aligned}
\mathcal{M}_g(v) &= (\mathcal{M}_{\mathrm{inc}'} \circ \mathcal{M}_{\mathrm{ad}} \circ \mathcal{M}_{\mathrm{ad}'}^{-1} \otimes 1) \circ \mathcal{M}_{\mathrm{ev}'}(v \otimes 1^*) \\
&= (\mathcal{M}_{\mathrm{inc}'} \circ \mathcal{M}_{\mathrm{ad}} \otimes 1) \left(\sum_i (-1)^{|a_i|} (v \otimes (1 \otimes a_i)^*) \otimes a_i \right) \\
&= (\mathcal{M}_{\mathrm{inc}'} \otimes 1) \left(\sum_i (-1)^{|a_i|} \rho^{-1} \left(\mathcal{M}_{\mathrm{inc}}(v \otimes a_i^*) \otimes 1^*\right) \otimes a_i \right) \\
&= \sum_{|a_i|=|v|} a_i^*(\varphi(v)) \otimes a_i \quad \text{and} \\
\mathcal{M}_g(\bar{v}) &= (-1)^{|v|} (\mathcal{M}_{\mathrm{inc}'} \circ \mathcal{M}_{\mathrm{ad}} \circ \mathcal{M}_{\mathrm{ad}'}^{-1} \otimes 1) \circ \mathcal{M}_{\mathrm{ev}'}(v \otimes u^*) \\
&= (-1)^{|v|} (\mathcal{M}_{\mathrm{inc}'} \circ \mathcal{M}_{\mathrm{ad}} \otimes 1) \left(\sum_i (-1)^{|a_i|} ((v \otimes (u \otimes a_i)^*) \otimes a_i \right) \\
&= (-1)^{|v|} (\mathcal{M}_{\mathrm{inc}'} \otimes 1) \left(\sum_i \rho^{-1} \left(\mathcal{M}_{\mathrm{inc}}(v \otimes a_i^*) \otimes u^*\right) \otimes a_i \right) \\
&= (-1)^{|v|} \sum_{|a_i|<|v|} (\mathrm{pr}''(v \otimes a_i^*) \otimes u^*) \otimes a_i.
\end{aligned}
$$

Recall $\omega_A \in A^m$ the representative of the fundamental class $[M]$ described above, and define $\int_{\omega_A} : A \to \mathbb{Q}$ a linear map of degree $-m$ which maps ω_A to 1. It is immediate that $\int_{\omega_A}$ is a rational model of the dual of $\mathbb{Q} \to H_*(M;\mathbb{Q})$ defined by $1 \mapsto [M]$. Therefore, the definition of Γ_1 implies that the composite

$$
\mathcal{L} \xrightarrow[\cong]{\xi} \wedge(V \otimes \wedge(u)_*) \xrightarrow{\mathcal{M}_g} \wedge(T_\varphi \otimes \mathbb{Q}u^*) \otimes A \xrightarrow{1\otimes \int_{\omega_A}} \wedge(T_\varphi \otimes \mathbb{Q}u^*) \xrightarrow{\mathrm{proj}} T_\varphi \otimes \mathbb{Q}u^*
$$

induces the dual of Γ_1 on homology.

In order to complete the proof, it suffices to show that the composite above coincides with $\mathcal{M}_{\Gamma_1}$ in (4.6) on $\mathcal{L}_{(1)}$. We observe that $\varphi(v) = \sum_i a_i^*(\varphi(v))a_i$ for $v \in V$. Moreover, we may write $D_A(a_{i_1} \cdots a_{i_k}) = \sum_j \lambda_{(i_1,\ldots,i_k,j)} a_j^*$ for some $\lambda_{(i_1,\ldots,i_k,j)} \in \mathbb{Q}$. Thus it follows that $\int_{\omega_A}(a_{i_1} \cdots a_{i_k} a_i) = \omega_A^*(a_{i_1} \cdots a_{i_k} a_i) = D_A(a_{i_1} \cdots a_{i_k})(a_i) = \lambda_{(i_1,\ldots,i_k,i)}$. Then, the definition of $\mathcal{M}_{\Gamma_1}$ yields that

$$
\begin{aligned}
&\mathcal{M}_{\Gamma_1}(v_1 \cdots v_k \bar{v}) \\
=&(\mathrm{pr}'' \otimes 1) \circ T \circ (D_A \otimes 1) \\
&\qquad\qquad \Big((-1)^{|v|} \sum_{(i_1,\ldots,i_k)} a_{i_1}^*(\varphi(v_1)) \cdots a_{i_k}^*(\varphi(v_k)) a_{i_1} \cdots a_{i_k} \otimes v \otimes u^* \Big) \\
=&(\mathrm{pr}'' \otimes 1) \circ T \Big((-1)^{|v|} \sum_{(i_1,\ldots,i_k)} a_{i_1}^*(\varphi(v_1)) \cdots a_{i_k}^*(\varphi(v_k)) (\sum_i \lambda_{(i_1,\ldots,i_k,i)} a_i^*) \otimes v \otimes u^* \Big) \\
=&(-1)^{|v|} \sum_{(i_1,\ldots,i_k,i)} (-1)^{|a_i||v|} a_{i_1}^*(\varphi(v_1)) \cdots a_{i_k}^*(\varphi(v_k)) \mathrm{pr}''(v \otimes a_i^*) \otimes u^*) \lambda_{(i_1,\ldots,i_k,i)}.
\end{aligned}
$$

Moreover, we see that

$$
\begin{aligned}
&(\mathrm{proj}) \circ \left(1 \otimes \textstyle\int_{\omega_A}\right) \circ \mathcal{M}_g \circ \xi(v_1 \cdots v_k \bar{v}) \\
=&(-1)^{|v|}(\mathrm{proj}) \circ \left(1 \otimes \textstyle\int_{\omega_A}\right) \circ \mathcal{M}_g \left((v_1 \otimes 1^*) \cdots (v_k \otimes 1^*)(v \otimes u^*)\right) \\
=&(-1)^{|v|}(\mathrm{proj}) \circ \left(1 \otimes \textstyle\int_{\omega_A}\right) \\
&\left\{ \left(\sum_{(i_1,\dots,i_k)} a^*_{i_1}(\varphi(v_1)) \cdots a^*_{i_k}(\varphi(v_k)) a_{i_1} \cdots a_{i_k} \right) \left(\sum_i (\mathrm{pr}''(v \otimes a^*_i) \otimes u^*) \otimes a_i \right) \right\} \\
=&(-1)^{|v|} \sum_{(i_1,\dots,i_k,i)} (-1)^{(|a_i|-m)(|v|+|a_i|+1)+m(|v|+|a_i|+1)} \\
&\qquad\qquad a^*_{i_1}(\varphi(v_1)) \cdots a^*_{i_k}(\varphi(v_k)) \mathrm{pr}''(v \otimes a^*_i) \otimes u^*) \lambda_{(i_1,\dots,i_k,i)}.
\end{aligned}
$$

Observe that $\lambda_{(i_1,\dots,i_k,i)} = 0$ if $m \neq |a_{i_1} \cdots a_{i_k}| + |a_i|$. Thus, we have the result. $\square$

Remark 4.12. The sign of the rational model $\mathcal{M}_{\mathrm{ev}'}$ for the evaluation map in the proof of Proposition 4.11 is different from the model due to Buijs-Murillo [BM06] and Kuribayashi [Kur06]. It is caused by the difference of sings appeared in the differential of the dual space A_*. The differential d_* of A_* in[BS97, BM06, Kur06] is defined by $d_*(\alpha) = \alpha \circ d_A$ for $\alpha \in A_*$ with the differential d_A of A. In this paper, we adopt the differential d_* of A_* defined by $d_*(\alpha) = -(-1)^{|\alpha|}\alpha \circ d_A$ with the Koszul sign convention.

In [BM08, §2], Buijs and Murillo define a quasi-isomorphism

$$\Psi : \mathrm{Der}(\wedge V, A) \to \mathrm{Hom}(S_\varphi, \mathbb{Q})$$

by $\Psi(\theta)(v \otimes \alpha) := (-1)^{(|\theta|+|v|)|\alpha|}\alpha \circ \theta(v)$ for $\theta \in \mathrm{Der}(\wedge V, A)$ and $v \otimes \alpha \in S_\varphi$. Then, the isomorphism $H(\Psi)$ on homology is related to Sullivan's isomorphism Φ.

Lemma 4.13. *There exists a commutative diagram*

$$
\begin{tikzcd}
\pi_*(\mathrm{aut}_1(M)) \otimes \mathbb{Q} \arrow[rr, "\Phi", "\cong"'] \arrow[d, "\nu"', "\cong"] & & H^{-*}(\mathrm{Der}(\wedge V)) \arrow[d, "\cong"', "\varphi_*"] \\
\mathrm{Hom}(H^*(S_\varphi), \mathbb{Q}) & H^{-*}(\mathrm{Hom}(S_\varphi, \mathbb{Q})) \arrow[l, "\cong"'] & H^{-*}(\mathrm{Der}(\wedge V, A)), \arrow[l, "H(\Psi)"', "\cong"]
\end{tikzcd}
$$

where the unnamed arrow denotes the natural isomorphism.

Proof. Given $f \in \pi_n(\mathrm{aut}_1(M)) \otimes \mathbb{Q}$ which is represented by $f : S^n \to \mathrm{aut}_1(M)$. Here we assume that M is a rational space. Let $\mathrm{ad}(f) : S^n \times M \to M$ be the adjoint of f and $\mathcal{M}_{\mathrm{ad}(f)} : \wedge V \to \mathcal{M}_{S^n} \otimes \wedge V$ a Sullivan representative for $\mathrm{ad}(f)$. Note that $(1 \otimes \varphi) \circ \mathcal{M}_{\mathrm{ad}(f)}$ is also a Sullivan representative for $\mathrm{ad}(f)$. By the definition of Φ, we have

$$\varphi_* \circ \Phi(f) = \varphi_* \left\{ \left(\textstyle\int_{S^n} \otimes 1\right) \circ \mathcal{M}_{\mathrm{ad}(f)} \right\} = \left(\textstyle\int_{S^n} \otimes 1\right) \circ (1 \otimes \varphi) \circ \mathcal{M}_{\mathrm{ad}(f)}. \tag{4.7}$$

On the other hand, the adjoint $\mathrm{ad}(f)$ coincides with the composite

$$\mathrm{ad}(f) : S^n \times M \xrightarrow{f \times 1} \mathrm{aut}_1(M) \times M \xrightarrow{\mathrm{inc} \times 1} \mathrm{Map}(M, M) \times M \xrightarrow{\mathrm{ev}''} M, \tag{4.8}$$

where ev'' is the evaluation map. Let $\mathcal{M}_f : \wedge S_\varphi \to \mathcal{M}_{S^n}$ be a Sullivan representative for f and $\mathcal{M}_{\mathrm{ev}''} : \wedge V \to \wedge(V \otimes A_*) \otimes A$ the rational model for ev'' defined by

$$\mathcal{M}_{\mathrm{ev}''}(v) = \sum_j (-1)^{|a_j|}(v \otimes a_j^*) \otimes a_j$$

for $v \in V$. Let $\mathcal{M}_{\mathrm{inc}}$ be the rational model for inc described in Appendix A. It follows from (4.8) that the composite $(\mathcal{M}_f \circ \mathcal{M}_{\mathrm{inc}} \otimes 1) \circ \mathcal{M}_{\mathrm{ev}''}$ of morphisms of CDGAs is also a rational model for $\mathrm{ad}(f)$ and then it is homotopic to $(1 \otimes \varphi) \circ \mathcal{M}_{\mathrm{ad}(f)}$. Therefore, by (4.7), we have

$$\begin{aligned}
\Psi \circ \varphi_* \circ \Phi(f)(v \otimes a_i^*) &= (-1)^{(|f|+|v|)|a_i|} a_i^* \left\{ \left((\textstyle\int_{S^n} \circ \mathcal{M}_f \circ \mathcal{M}_{\mathrm{inc}} \otimes 1 \right) \circ \mathcal{M}_{\mathrm{ev}''}(v) \right\} \\
&= (-1)^{(|f|+|v|)|a_i|} a_i^* \left\{ \sum_j (-1)^{|a_j|} \left(\textstyle\int_{S^n} \circ \mathcal{M}_f (v \otimes a_j^*) \right) a_j \right\} \\
&= \textstyle\int_{S^n} \circ \mathcal{M}_f(v \otimes a_i^*) = \nu(f)(v \otimes a_i^*)
\end{aligned}$$

for $v \otimes a_i^* \in S_\varphi$. □

Proof of Theorem 4.7. By making use of isomorphisms $\overline{V} \cong V \otimes \mathbb{Q}u^*$ and $S_\varphi \otimes \mathbb{Q}u^* \cong S_\varphi$ defined by $\bar{v} \mapsto v \otimes u^*$ and $w \otimes u^* \mapsto (-1)^{|w|} w$, respectively, we obtain morphisms $\lambda' : \mathrm{Der}(\wedge V) \to \mathrm{Hom}(V \otimes \mathbb{Q}u^*, A)$ and $\Psi' : \mathrm{Hom}(V \otimes \mathbb{Q}u^*, A) \to \mathrm{Hom}(S_\varphi \otimes \mathbb{Q}u^*, \mathbb{Q})$ of chain complexes which fit in the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(\overline{V}, A) & \mathrm{Der}(\wedge V, A) \arrow[l, "\lambda"', "\cong"] \arrow[r, "\Psi"] \arrow[d, "\lambda'", "\cong"'] & \mathrm{Hom}(S_\varphi, \mathbb{Q}) \arrow[d, "\cong"] \\
 & \mathrm{Hom}(V \otimes \mathbb{Q}u^*, A) \arrow[ul, "\cong"] \arrow[r, "\Psi'"] & \mathrm{Hom}(S_\varphi \otimes \mathbb{Q}u^*, \mathbb{Q}).
\end{tikzcd}$$

Recall the morphisms PD in (4.4) and $\mathcal{M}_{\Gamma_1}$ in (4.6). Then, a straightforward computation shows that the following diagram

$$\begin{tikzcd}
\mathrm{Hom}(S_\varphi, \mathbb{Q}) \arrow[dr, "\cong"] & & \mathrm{Der}(\wedge V, A) \arrow[ll, "\Psi"'] \arrow[dl, bend left, "\lambda'"] \arrow[ddd, "\lambda"] \\
\mathrm{Hom}(T_\varphi \otimes \mathbb{Q}u^*, \mathbb{Q}) \arrow[u, "\iota"] \arrow[d, "(\mathrm{pr}''\otimes 1)^*"'] \arrow[dd, bend right=60, "\mathcal{M}_{\Gamma_1}^*"'] & \mathrm{Hom}(S_\varphi \otimes \mathbb{Q}u^*, \mathbb{Q}) \arrow[dl, "(\mathrm{pr}\otimes 1)^*"'] & \\
\mathrm{Hom}(A_* \otimes V \otimes \mathbb{Q}u^*, \mathbb{Q}) \arrow[d, "((D_A\circ\varphi\otimes 1)\circ\xi)^*"'] & \mathrm{Hom}(V \otimes \mathbb{Q}u^*, A) \arrow[l, "\mathrm{adj}"', "\cong"] \arrow[u, "\Psi'"] \arrow[dr, "\cong"] & \\
\mathrm{Hom}(\mathcal{L}_{(1)}, \mathbb{Q}) & & \mathrm{Hom}(\overline{V}, A). \arrow[ll, "\mathrm{PD}"']
\end{tikzcd}$$

is commutative. Therefore, by Proposition 4.9, 4.11 and Lemma 4.13, we have the commutativity of the diagram in the assertion. □

5. Examples

In this section, we describe explicitly the Lie representation L and the contraction e in Propositions 4.3 and 4.4 for manifolds and interesting spaces. For a Sullivan algebra $\wedge V$, we denote by (v, α) the derivation on $\wedge V$ that takes a generator v in V to an element α in $\wedge V$ and the other generators to 0.

Example 5.1. Let G be a simply-connected compact Lie group and $\alpha_1, \dots, \alpha_l$ the indecomposable elements of $H^*(G;\mathbb{Q})$ with maximal degree. The cohomology $H^*(LG;\mathbb{Q})$ is generated by $\alpha_1, \dots, \alpha_l$ as an algebra with the BV operator and derivations L_θ for suitable elements $\theta \in \pi_*(\mathrm{aut}_1(G)) \otimes \mathbb{Q}$. In this case, the Lie derivative $L : \pi_*(\mathrm{aut}_1(G)) \otimes \mathbb{Q} \to \mathrm{Der}_*(H^*(LG;\mathbb{Q}))$ is faithful.

Example 5.2. Let X be the complex projective plane $\mathbb{C}P^2$. Then a Sullivan minimal model $\mathcal{M}_X$ for X is given by $(\wedge(x,y),d)$ with $|x| = 2$, $|y| = 5$, $dx = 0$ and $dy = x^3$. Moreover, the free loop space LX admits the Sullivan minimal model $\mathcal{L} = (\wedge(x,y,\bar{x},\bar{y}),d)$ for which $|\bar{x}| = 1$, $|\bar{y}| = 4$, $dx = 0$, $dy = x^3$, $d\bar{x} = 0$ and $d\bar{y} = -3x^2\bar{x}$. Recall the result [KY97, Theorem 2.2(ii)] which asserts that

$$H^*(LX;\mathbb{Q}) \cong \frac{\mathbb{Q}[x] \otimes \wedge(\bar{x})}{(x^3, x^2\bar{x})} \oplus \Big((x,\bar{x})_A \otimes \mathbb{Q}^+[z]\Big) \ ; \qquad |z| = |\bar{y}| = 4$$

as an algebra, where $(x,\bar{x})_A$ is the ideal of $A := \mathbb{Q}[x] \otimes \wedge(\bar{x})/(x^3, x^2\bar{x})$. We choose a basis for $H^*(LX;\mathbb{Q})$ of the form

$$\{1,\ x\ ,x^2,\ \bar{x}\ (= \alpha_0)\ ,x\bar{x},\ \alpha_n,\ x\alpha_n,\ \beta_n, x\beta_n\}_{n\geq 1},$$

where $\alpha_n = \bar{x}\bar{y}^n$ and $\beta_n = x\bar{y}^n + 3n\bar{x}y\bar{y}^{n-1}$. Observe that

$$H_*(\mathrm{Der}(\mathcal{M}_X)) = \mathbb{Q}\{(y,1),(y,x)\}.$$

Let $e_1 := e_{(y,1)} = -(\bar{y},1)$, $e_2 := e_{(y,x)} = (\bar{y},x)$, $L_1 := L_{(y,1)} = (y,1)$ and $L_2 := L_{(y,x)} = (y,x) + (\bar{y},\bar{x})$. Then we see that $e_1(\alpha_n) = n\alpha_{n-1}$, $e_2(\alpha_n) = nx\alpha_{n-1}$, $e_1(\beta_1) = -x$, $e_1(\beta_n) = -n\beta_{n-1}$ $(n > 1)$, $e_2(\beta_1) = x^2$, $e_2(\beta_n) = nx\beta_{n-1}$ $(n > 1)$, $L_1(\alpha_n) = L_2(\alpha_n) = 0$, $L_1(\beta_n) = -3n\alpha_{n-1}$ and $L_2(\beta_n) = -2nx\alpha_{n-1}$. Thus L and e are injective.

Note that the calculations of the operations L and e yield that $e_1(x\otimes z^n) = -nx\otimes z^{n-1}$, $e_2(x\otimes z^n) = nx^2\otimes z^{n-1}$, $e_1(\bar{x}\otimes z^n) = n\bar{x}\otimes z^{n-1}$, $e_2(\bar{x}\otimes z^n) = nx\bar{x}\otimes z^{n-1}$, $L_1(x\otimes z^n) = -3n\bar{x}\otimes z^{n-1}$, $L_2(x\otimes z^n) = -2nx^2\otimes z^{n-1}$ and $L_1(\bar{x}\otimes z^n) = L_2(\bar{x}\otimes z^n) = 0$.

Remark 5.3. We see that $H_*(\mathrm{Der}(H^*(\mathbb{C}P^2;\mathbb{Q}))) = \mathrm{Der}_*(H^*(\mathbb{C}P^2;\mathbb{Q})) = 0$. In fact, with the same notation as in Example 5.2, every derivation assigning an element in $H^0(\mathbb{C}P^2;\mathbb{Q}) = \mathbb{Q}$ to the generator x should be trivial. As mentioned above, the homology Lie algebra $H_*(\mathrm{Der}(\mathcal{M}_{\mathbb{C}P^2}))$ is non trivial. Observe that $\mathbb{C}P^2$ is *formal*; see [FHT01, §12(c)]. Thus, a quasi-isomorphism does not induce an isomorphism between the homology Lie algebras of derivations in general.

Example 5.4. Let X be a non-formal space whose minimal model $\mathcal{M}_X$ has the form $(\wedge(x,y,z),d)$, where $dx = dy = 0$, $dz = xy$, $|x| = |y| = 3$ and $|z| = 5$. We observe that $\mathcal{M}_X$ is realized by a manifold of dimension 11; see [Sul77, Theorem 13.2]. Then $H_*(\mathrm{Der}(\mathcal{M}_X)) = \mathbb{Q}\{(z,1)\}$ and $H^*(X;\mathbb{Q}) = \wedge(x,y) \otimes \mathbb{Q}[w,u]/(xy, xw, yu, xu + yw, w^2, wu, u^2)$ for $w = [xz]$ and $u = [yz]$. Thus the natural map $\psi : H_*(\mathrm{Der}(\mathcal{M}_X)) \to \mathrm{Der}_*(H^*(X;\mathbb{Q}))$ is faithful since $\psi(z,1)(w) = [x]$. Moreover e and L are injective since $e_{(z,1)}([xyz\bar{z}]) = [xyz]$ and $L_{(z,1)}([xyz\bar{z}]) = [xy\bar{z}]$.

In the following examples, we rely on the software *Kohomology* [Wak] for determining bases for $H^*(X;\mathbb{Q})$ and $H^*(LX;\mathbb{Q})$ and computing actions of L_θ and e_θ on $H^*(LX;\mathbb{Q})$ with data of a Sullivan model for a given space X.

Example 5.5. Let X be a non-formal manifold of dimension 14 whose minimal model $\mathcal{M}_X$ is of the form $(\wedge(a,x,y,b,v,w),d)$, where $|a|=2$, $|x|=|y|=3$, $|b|=4$, $|v|=5$, $|w|=7$, $da=dx=0$, $dy=a^2$, $db=ax$, $dv=ab+xy$, $dw=2xv+b^2$; see [FHT01, p.439] and [Sul77, Theorem 13.2]. Note that $H^*(X;\mathbb{Q})$ is generated by

$$\{a, x, xb, av-yb, a^2w-abv+xyv, 3axw+b^3\}$$

as an algebra. Then $H_*(\mathrm{Der}(\mathcal{M}_X)) = \mathbb{Q}\{(w,1),(w,a)\}$ and the natural map $\psi : H_*(\mathrm{Der}(\mathcal{M}_X)) \to \mathrm{Der}_*(H^*(X;\mathbb{Q}))$ is zero though $\mathrm{Der}_*(H^*(X;\mathbb{Q})) \neq 0$; see [Yam05]. However, the Lie representation $L : H_*(\mathrm{Der}(\mathcal{M}_X)) \to \mathrm{Der}_*(H^*(LX;\mathbb{Q}))$ is non-trivial. Indeed, we see that $L_{(w,1)} \neq 0$ and $L_{(w,a)} \neq 0$ since

$$L_{(w,1)}(xbv\bar{w} - axw\bar{w}) = 2xv\bar{b} + ax\bar{w} \quad \text{and}$$
$$L_{(w,a)}(-axw\bar{w} + xbv\bar{w} + xvw\bar{b}) = 3(\frac{1}{2}a^2x\bar{w} + axv\bar{b}) + (axb\bar{v} + axw\bar{a} + xyb\bar{b})$$

as non-zero cohomology classes [Wak]. Thus it follows that the representation $L : H_*(\mathrm{Der}(\mathcal{M}_X)) \to \mathrm{End}_*(H^*(\mathcal{L}_{(1)}))$ is faithful. Here $\mathcal{L}_{(k)} = \wedge V \otimes \wedge^k \overline{V}$ for $\mathcal{M}_X = (\wedge V, d)$ in Section 2.

The contraction e is injective as seen in Corollary 3.11. In this case, we can check the faithfulness with explicit calculations. Indeed, we see in [Wak] that

$$e_{(w,1)}(a^2xw\bar{w} - axbv\bar{w} - 2axvw\bar{b}) = a^2xw - xbva \quad \text{and}$$
$$e_{(w,a)}(axw\bar{w} - xbv\bar{w} - 2xvw\bar{b}) = a^2xw - xbva,$$

where $[a^2xw - xbva] \in H^{14}(X;\mathbb{Q})(= H^{14}(\mathcal{L}_{(0)}))$ is the fundamental class of X.

Example 5.6. When X does not have positive weights, $L : H_*(\mathrm{Der}(\mathcal{M}_X)) \to \mathrm{End}_*(H^*(\mathcal{L}_{(1)}))$ may not be faithful. Let X be an elliptic manifold of dimension 228 with

$$\mathcal{M}_X = (\wedge(x_1, x_2, y_1, y_2, y_3, z), d)$$

given in [AL00, Example 5.2], where $|x_1| = 10$, $|x_2| = 12$, $|y_1| = 41$, $|y_2| = 43$, $|y_3| = 45$, $|z| = 119$ and the differential is defined by

$$\begin{aligned} &dx_1 = 0 && dy_1 = x_1^3x_2 && dz = x_2(y_1x_2 - x_1y_2)(y_2x_2 - x_1y_3) + x_1^{12} + x_2^{10}. \\ &dx_2 = 0 && dy_2 = x_1^2x_2^2 && \\ & && dy_3 = x_1x_2^3 && \end{aligned}$$

Then we see that $L_\theta : H_*(\mathrm{Der}(\mathcal{M}_X)) \to \mathrm{End}_*(H^*(\mathcal{L}_{(1)}))$ is not zero except $\theta = (z, x_2^9)$, $(z, x_1^2x_2)$ and $(z, x_1x_2^2)$ [Wak]. Observe that $L_\theta : H_*(\mathrm{Der}(\mathcal{M}_X)) \to \mathrm{End}_*(H^*(\mathcal{L}_{(4)}))$ for $\theta = (z, x_1^2x_2)$ and $(z, x_1x_2^2)$ is non trivial for elements of degree 221 and 219 of $H^*(\mathcal{L}_{(4)})$, respectively. Unfortunately, a calculation of $L_{(z,x_2^9)} : H_*(\mathrm{Der}(\mathcal{M}_X)) \to \mathrm{Der}_*(H^*(LX;\mathbb{Q}))$ with Kohomology [Wak] shows that the representation is trivial for degrees less than or equal to 355.

We do not know whether the operator L is a faithful representation in general. In Example 5.6, it is expected that L_θ is not zero for some derivation θ with higher degree.

Problem 5.7. Is $L : H_*(\mathrm{Der}(\mathcal{M}_X)) \to \mathrm{Der}_*(H^*(LX;\mathbb{Q}))$ faithful when X is a closed manifold ?

Acknowledgements

The authors thank the referees for their careful reading and valuable comments.

Appendix A. A Sullivan representative for an adjoint map

We begin by recalling the rational models due to Brown and Szczarba [BS97]. Let $\wedge V$ be a minimal Sullivan algebra, A a finite dimensional CDGA and $A^q = 0$ for $q < 0$. We denote by $A_* = \mathrm{Hom}(A, \mathbb{Q})$ the dual of A with the coproduct Δ_A of A_* induced by the multiplication of A; see Remark 4.12. We consider the CDGA $\wedge(\wedge V \otimes A_*)$ and the differential ideal $\mathcal{I}$ of $\wedge(\wedge V \otimes A_*)$ generated by $1 \otimes 1^* - 1$ and

$$w_1 w_2 \otimes \alpha - \sum (-1)^{|w_2||\alpha_i'|}(w_1 \otimes \alpha_i')(w_2 \otimes \alpha_i''),$$

where $w_i \in \wedge V$, $\alpha \in A_*$ and $\Delta_A(\alpha) = \sum \alpha_i' \otimes \alpha_i''$. Then, it follows from [BS97, Theorem 3.5] that the composite

$$\rho : \wedge(V \otimes A_*) \xrightarrow{\text{incl}} \wedge(\wedge V \otimes A_*) \xrightarrow{\text{proj}} \wedge(\wedge V \otimes A_*)/\mathcal{I}$$

is an isomorphism of graded algebras. We define the differential d_{BS} of $\wedge(V \otimes A_*)$ by $\rho^{-1} d \rho$, where d is the differential of $\wedge(\wedge V \otimes A_*)/\mathcal{I}$.

Assume that $\wedge V$ is a minimal Sullivan model for a connected nilpotent space Y of finite type and A is a finite dimensional commutative model for a finite CW complex X. Then, we see that $(\wedge(V \otimes A_*), d_{BS})$ is a rational model of $\mathrm{Map}(X, Y)$; see [BS97, Theorem 1.3].

Let $\varphi : \wedge V \to A$ be a Sullivan representative for a continuous map $f : X \to Y$. The morphism of CDGAs induces the augmentation $\varphi : \wedge(V \otimes A_*) \to \mathbb{Q}$ which is denoted by the same notation. It follows from [BM06, Proposition 4.2, Theorem 4.5] and [HKO08, Remark 3.4] that the connected component $\mathrm{Map}_f(X, Y)$ of $\mathrm{Map}(X, Y)$ containing f has a Sullivan model of the form

$$(\wedge S_\varphi, d) = \left(\wedge \left(\overline{(V \otimes A_*)}^1 \oplus (V \otimes A_*)^{\geq 2} \right), d \right),$$

where $\overline{(V \otimes A_*)}^1$ is the complement of the image of the composite

$$(V \otimes A_*)^0 \xrightarrow{d} (\wedge(V \otimes A_*))^1 \twoheadrightarrow (\wedge(V \otimes A_*)/K_\varphi)^1 \xleftarrow[\cong]{\text{proj}} (V \otimes A_*)^1$$

in which K_φ is the differential ideal of $\wedge(V \otimes A_*)$ generated by $(V \otimes A_*)^{<0}$ and $\{w - \varphi(w) \mid w \in (V \otimes A_*)^0\}$. We observe that the differential d is induced by the differential d_{BS} of $\wedge(V \otimes A_*)$; see [BM06, The proof of Proposition 4.2]. Moreover, the morphism $\mathcal{M}_{\mathrm{inc}} : \wedge(V \otimes A_*) \to \wedge S_\varphi$ of CDGAs defined by

$$\mathcal{M}_{\mathrm{inc}}(w) = \begin{cases} \mathrm{pr}(w) & (|w| > 0), \\ \varphi(w) & (|w| = 0), \\ 0 & (|w| < 0) \end{cases}$$

is a rational model for the inclusion $\mathrm{inc} : \mathrm{Map}_f(X, Y) \hookrightarrow \mathrm{Map}(X, Y)$, where $\mathrm{pr} : (V \otimes A_*)^{\geq 1} \to S_\varphi$ is the canonical projection.

In what follows, let X_i and Y_i be connected nilpotent spaces of finite type for $i = 1, 2$. We further assume that X_i is a finite CW complex. Moreover, let A_i be a finite dimensional commutative model for X_i and $\wedge V_i$ a minimal Sullivan model for Y_i. We first construct a rational model for the map

$$- \times g : \mathrm{Map}(X_1, Y_1) \to \mathrm{Map}(X_1 \times X_2, Y_1 \times Y_2)$$

defined by $f \mapsto f \times g$ with a continuous map $g : X_2 \to Y_2$. Let $\iota_i : A_i \hookrightarrow A_1 \otimes A_2$ denote the inclusion which is a rational model for the projection $\mathrm{pr}_i : X_1 \times X_2 \to X_i$. Remark that $\wedge V_1 \otimes \wedge V_2 \cong \wedge(V_1 \oplus V_2)$ is a Sullivan model for $Y_1 \times Y_2$; that is, the Brown-Szczarba model for $\mathrm{Map}(X_1 \times X_2, Y_1 \times Y_2)$ is of the form $\wedge((V_1 \oplus V_2) \otimes (A_1 \otimes A_2)_*)$. Let $\psi : \wedge V_2 \to A_2$ be a Sullivan representative for g. Then, we define the CDGA morphism $\widehat{\psi}$ as the following composite;

$$\begin{tikzcd}
\wedge((V_1 \oplus V_2) \otimes (A_1 \otimes A_2)_*) \arrow[r, "\rho", "\cong"'] & \wedge(\wedge V_1 \otimes \wedge V_2 \otimes (A_1 \otimes A_2)_*)/\mathcal{I} \arrow[d, "\wedge(1 \otimes \psi \otimes 1)"'] \\
& \wedge(\wedge V_1 \otimes A_2 \otimes (A_1 \otimes A_2)_*)/\mathcal{I} \arrow[d, "\widetilde{\eta}"'] \\
\wedge(V_1 \otimes A_{1*}) & \wedge(\wedge V_1 \otimes A_{1*})/\mathcal{I}, \arrow[l, "\rho^{-1}"', "\cong"]
\end{tikzcd}$$

where $\widetilde{\eta}$ is the CDGA morphism which is induced by the natural isomorphism $\zeta : (A_1 \otimes A_2)_* \xrightarrow{\cong} A_{1*} \otimes A_{2*}$ and the paring $\eta : A_2 \otimes A_{2*} \to \mathbb{Q}$.

Lemma A.1. *The morphism $\widehat{\psi}$ is a rational model for $- \times g$.*

Proof. First, we see that the map $- \times g$ coincides with the composite

$$\begin{tikzcd}
\mathrm{Map}(X_1, Y_1) \arrow[r, "(1, c_g)"] & \mathrm{Map}(X_1, Y_1) \times \mathrm{Map}(X_2, Y_2) \arrow[dl, "\mathrm{pr}_1^* \times \mathrm{pr}_2^*"'] \\
\mathrm{Map}(X_1 \times X_2, Y_1) \times \mathrm{Map}(X_1 \times X_2, Y_2) \arrow[r, "\cong"] & \mathrm{Map}(X_1 \times X_2, Y_1 \times Y_2),
\end{tikzcd} \tag{A.1}$$

where $c_g : \mathrm{Map}(X_1, Y_1) \to \mathrm{Map}(X_2, Y_2)$ is the constant map at g. Let $\mathrm{Map}_g(X_2, Y_2)$ be the connected component of $\mathrm{Map}(X_2, Y_2)$ containing g. We also see that c_g is regarded as the composite

$$\mathrm{Map}(X_1, Y_1) \longrightarrow \mathrm{pt} \xrightarrow{c_g} \mathrm{Map}_g(X_2, Y_2) \xrightarrow{\mathrm{inc}} \mathrm{Map}(X_2, Y_2).$$

It follows from the rational model for the inclusion $\mathrm{Map}_g(X_2, Y_2) \hookrightarrow \mathrm{Map}(X_2, Y_2)$ described in [BM06, Proposition 4.2, Theorem 4.5] and [HKO08, Remark 3.4] that the morphism

$$\mathcal{M}_{c_g} : \wedge(V_2 \otimes A_{2*}) \to \wedge(V_1 \otimes A_{1*})$$

defined by $v_2 \otimes \alpha_2 \mapsto (-1)^{|v_2||\alpha_2|}\alpha_2(\psi(v_2))$ for $v_2 \otimes \alpha_2 \in V_2 \otimes A_{2*}$ is a rational model of c_g. We choose the inclusion ι_i into the ith factor as a model for the projection pr_i in the ith factor. Then the description (A.1) and the naturality of the Brown-Szczarba models shows that the composite

$$\begin{tikzcd}
\wedge((V_1 \oplus V_2) \otimes (A_1 \otimes A_2)_*) \arrow[r, "\cong"] & \wedge(V_1 \otimes (A_1 \otimes A_2)_*) \otimes \wedge(V_2 \otimes (A_1 \otimes A_2)_*) \arrow[dl, "\iota_1^* \otimes \iota_2^*"'] \\
\wedge(V_1 \otimes A_{1*}) \otimes \wedge(V_2 \otimes A_{2*}) \arrow[r, "1 \cdot \mathcal{M}_{c_g}"] & \wedge(V_1 \otimes A_{1*})
\end{tikzcd} \tag{A.2}$$

is a model for the map $- \times g$. For any element $\alpha \in (A_1 \otimes A_2)_*$, we may write

$$\zeta(\alpha) = \alpha \circ \iota_1 \otimes 1^* + 1^* \otimes \alpha \circ \iota_2 + \widetilde{\alpha}$$

with $\widetilde{\alpha} \in (A_1^+)_* \otimes (A_2^+)_*$. Observe that $1 \otimes \widetilde{\alpha}_1$ is zero in $\wedge(\wedge V \otimes A_{1*})/\mathcal{I}$ for any $\widetilde{\alpha}_1 \in (A_1^+)_*$. Therefore, we can check that the composite (A.2) coincides with $\widehat{\psi}$ and the proof is complete. ∎

We are ready to construct a rational model for the adjoint map

$$\mathrm{ad} : \mathrm{Map}(X_1, \mathrm{Map}_f(X_2, Y)) \to \mathrm{Map}(X_1 \times X_2, Y) \tag{A.3}$$

defined by $\mathrm{ad}(g)(x_1, x_2) = g(x_1)(x_2)$ for $f : X_2 \to Y$, $g : X_1 \to \mathrm{Map}_f(X_2, Y)$ and $x_i \in X_i$. In order to apply the rational models due to Brown and Szczarba to our objects, we need to consider the connected component $\mathrm{Map}_f(X_2, Y)$ containing f.

Let φ be a Sullivan representative for f, $\varphi : \wedge(V \otimes A_{2*}) \to \mathbb{Q}$ the augmentation induced by φ and $\wedge S_\varphi$ the Brown-Szczarba model for $\mathrm{Map}_f(X_2, Y)$ mentioned above. Since $\mathrm{Map}_f(X_2, Y)$ is a connected and nilpotent space of finite type; see [HMR75], by applying the construction of the Brown-Szczarba model to X_1 and $\mathrm{Map}_f(X_2, Y)$, we have a model for $\mathrm{Map}(X_1, \mathrm{Map}_f(X_2, Y))$ of the form $\wedge(S_\varphi \otimes A_{1*})$.

Proposition A.2. *The morphism $\mathcal{M}_{\mathrm{ad}} : \wedge(V \otimes (A_1 \otimes A_2)_*) \to \wedge(S_\varphi \otimes A_{1*})$ of CDGAs defined by*

$$\mathcal{M}_{\mathrm{ad}}(v \otimes \alpha) = \sum (-1)^{|\alpha_1||\alpha_2|} \rho^{-1} \left(\mathcal{M}_{\mathrm{inc}}(v \otimes \alpha_2) \otimes \alpha_1 \right)$$

is a rational model for the adjoint map ad *in* (A.3)*, where $v \in V$, $\alpha \in (A_1 \otimes A_2)_*$ and $\zeta(\alpha) = \sum \alpha_1 \otimes \alpha_2$; see the paragraph before Lemma A.1 for the maps ρ and ζ.*

Proof. The map ad fits in the commutative diagram

$$\begin{tikzcd}
\mathrm{Map}(X_1, \mathrm{Map}_f(X_2, Y)) \arrow[rr, "\mathrm{ad}"] \arrow[dr, "- \times id"'] & & \mathrm{Map}(X_1 \times X_2, Y) \\
& \mathrm{Map}(X_1 \times X_2, \mathrm{Map}_f(X_2, Y) \times X_2), \arrow[ur, "\mathrm{ev}_*"'] &
\end{tikzcd}$$

where $\mathrm{ev} : \mathrm{Map}_f(X_2, Y) \times X_2 \to Y$ is the evaluation map. The result [BM06, Theorem 1.1] enables us to obtain a Sullivan model

$$\mathcal{M}_{\mathrm{ev}} : \wedge V \to \wedge S_\varphi \otimes A_2$$

for ev defined by $\mathcal{M}_{\mathrm{ev}}(v) = \sum_i (-1)^{|a_i|} \mathcal{M}_{\mathrm{inc}}(v \otimes a_i^*) \otimes a_i$, where $\{a_i\}$ is a basis of A_2 and $\{a_i^*\}$ is the dual basis of A_{2*}. By the surjective trick [FHT01, p.148], there exist a Sullivan model $\wedge W$ for X_2 and a surjective Sullivan representative $\sigma : \wedge W \to A_2$ for the identity on X_2. The lifting lemma [FHT01, Lemma 12.4] shows that there exists a morphism $\mathcal{M}'_{\mathrm{ev}} : \wedge V \to \wedge S_\varphi \otimes \wedge W$ such that $(1 \otimes \sigma) \circ \mathcal{M}'_{\mathrm{ev}} = \mathcal{M}_{\mathrm{ev}}$. Lemma A.1 is applicable to the map $- \times id$. Thus we see that the composite

$$\begin{tikzcd}
& \wedge(V \otimes (A_1 \otimes A_2)_*) \arrow[r, "\rho", "\cong"'] & \wedge(\wedge V \otimes (A_1 \otimes A_2)_*)/\mathcal{I} \arrow[d, "\wedge(\mathcal{M}'_{\mathrm{ev}} \otimes 1)"] \\
& & \wedge(\wedge S_\varphi \otimes \wedge W \otimes (A_1 \otimes A_2)_*)/\mathcal{I} \arrow[d, "\rho^{-1}", "\cong"'] \\
& \wedge(S_\varphi \otimes A_{1*}) & \wedge((S_\varphi \oplus \wedge W) \otimes (A_1 \otimes A_2)_*) \arrow[l, "\widehat{\psi}"']
\end{tikzcd} \tag{A.4}$$

is a model for ad, where $\widehat{\psi}$ is the morphism of CDGAs defined in the paragraph before Lemma A.1. Explicitly, we compute

$$\begin{aligned}
&\widehat{\psi}\circ\rho^{-1}\circ\wedge(\mathcal{M}'_{\mathrm{ev}}\otimes 1)\circ\rho(v\otimes\alpha)\\
=&\rho^{-1}\circ\widetilde{\eta}\circ\wedge(\mathcal{M}_{\mathrm{ev}}\otimes 1)(v\otimes\alpha)\\
=&\rho^{-1}\circ\widetilde{\eta}\left(\sum_i(-1)^{|a_i|}\mathcal{M}_{\mathrm{inc}}(v\otimes a_i^*)\otimes a_i\otimes\alpha\right)\\
=&\rho^{-1}\left(\sum_i\sum(-1)^{|a_i|+|\alpha_2|(|a_i|+|\alpha_1|)}\mathcal{M}_{\mathrm{inc}}(v\otimes a_i^*)\cdot\alpha_2(a_i)\otimes\alpha_1\right)\\
=&\sum(-1)^{|\alpha_1||\alpha_2|}\rho^{-1}\left(\mathcal{M}_{\mathrm{inc}}(v\otimes\alpha_2)\otimes\alpha_1\right).
\end{aligned}$$

Therefore, the model (A.4) for the adjoint map ad is nothing but the morphism $\mathcal{M}_{\mathrm{ad}}$ mentioned in the assertion. □

Appendix B. An equivariant version of the Lie derivative L

B.1. **A geometric construction of a Lie derivative.** In this section, we assume that the underlying field is of arbitrary characteristic.

We discuss an equivariant cohomology (cyclic homology) version of the Lie derivative L that we consider in Section 4.1. We begin by recalling a morphism of Lie algebras related to the Hochschild homology and the cyclic homology of an algebra. Let A be an unital algebra over a commutative ring k. For a derivation D on A, we define a map L_D on the Hochschild complex $C_*(A)$ by

$$L_D(a_0,...,a_n)=\sum_{i\geq 0}(a_0,..,a_{i-1},Da_i,a_{i+1},...,a_n).$$

We also recall the Hochschild cohomology $HH^*(A,A)$ of A. In particular, the first cohomology $HH^1(A,A)$ is isomorphic to $\mathrm{Der}(A)/\{\text{inner derivations}\}$ as a k-module. Then, we have

Proposition B.1. ([Lod98, 4.1.6 Corollary]) *There are well-defined homomorphisms of Lie algebras $[D]\mapsto L_D$:*

$$HH^1(A,A)\to\mathrm{End}_k(HH_n(A))\quad\textit{and}\quad HH^1(A,A)\to\mathrm{End}_k(HC_n(A)).$$

In the body of this manuscript, we discuss a geometric description of the Lie derivative on the endomorphism algebra of the Hochschild homology of a DGA. The above result motivates us to consider its cyclic version. In this section, we deal with the topics. As a consequence, our main theorem, Theorem B.6 below is obtained.

We work on the category of compactly generated spaces [Ste67] or the category NG of numerically generated spaces, which is obtained by adjoint functors between the category of topological spaces and that of diffeological spaces; see [SYH18]. Thus, we can consider a space in such a Cartesian closed category without changing the weak homotopy type. Observe that the category NG is also complete and cocomplete.

Let X be a simply-connected space of finite type and $\mathrm{aut}_1(X)$ the monoid of self-homotopy equivalences on X. We recall that the homotopy group $\pi_*(\mathrm{aut}_1(X))$ is a

Lie algebra with the Samelson product; see [Whi78, Chapter III]. For an element θ in the homotopy group $\pi_n(\mathrm{aut}_1(X))$ for $n > 1$, we define a map u_θ by the composite

$$u_\theta := L(\ ) \circ inc \circ \theta : S^n \longrightarrow \mathrm{aut}_1(X) \longrightarrow \mathrm{Map}(X, X) \xrightarrow{L} \mathrm{Map}(LX, LX),$$

where inc denotes the inclusion and L is the map which assigns $Lf : LX \to LX$ defined by $Lf(l) = f \circ l$ to a map $f : X \to X$. Then, the adjoint map $ad(u_\theta) : S^n \times LX \to LX$ gives rise to the derivation

$$L'_\theta : H^*(LX) \xrightarrow{(ad(u_\theta))^*} H^*(S^n) \otimes H^*(LX) \xrightarrow{\int_{S^n}} H^{*-n}(LX)$$

on the cohomology $H^*(LX)$, where $\int_{S^n}$ denotes the integration along the fiber. The map L_θ in (4.1) is regarded as the composite $\int_{S^n} \circ (s \times 1)^* \circ L(ad(\theta))^*$, where $s : S^n \to LS^n$ is the section of the evaluation map ev_0 defined by $s(x)(t) = x$ for $x \in S^n$ and $t \in S^1$. Since $ad(u_\theta) = L(ad(\theta)) \circ (s \times 1)$, it follows that L'_θ coincides with L_θ in (4.1). In what follows, we may write L_θ for L'_θ.

Observe that the adjoint map $ad(u_\theta) : S^n \times LX \to LX$ is an S^1-equivariant map, where the S^1-action on S^n is defined to be trivial. Thus, we have a map $\overline{ad(u_\theta) \times_{S^1} 1} : (S^n \times LX) \times_{S^1} ES^1 \to LX \times_{S^1} ES^1$ between the Borel constructions. Therefore, the same construction as that of L_θ with the integration enables us to obtain a derivation

$$\overline{L}_\theta : H^*_{S^1}(LX) \longrightarrow H^{*-n}_{S^1}(LX)$$

of degree $-n$.

The assertion below describes geometric counterparts of the morphisms of Lie algebras described in Proposition B.1.

Theorem B.2. *The map $\overline{L}_{(\ )} : \pi_*(\mathrm{aut}_1(X)) \to \mathrm{Der}_*(H^*_{S^1}(LX))$ is a morphism of Lie algebras.*

Proofs of Theorems 4.1 and B.2. As mentioned above, the map $ad(u_\theta)$ is an S^1-equivariant map. Then, the operation L'_θ commutes with the BV operator. We have Theorem 4.1 (2).

In order to prove Theorem 4.1 (1) and B.2, we first recall that

$$L_\theta = \int_{S^n} \circ H^*(ad(L_*(\theta))) \ \text{ and } \ \overline{L}_\theta = \int_{S^n} \circ H^*(ad(L_*(\theta)) \times_{S^1} 1_{ES^1})$$

for $\theta \in \pi_n(\mathrm{aut}_1(X))$. We may write $ad(L_*(\theta))^\sigma$ for $ad(L_*(\theta)) \times_{S^1} 1_{ES^1}$. The map L mentioned above induces a homomorphism $L_* : \pi_*(\mathrm{aut}_1(X)) \to \pi_*(\mathrm{aut}_1(LX))$. Let θ_1 and θ_2 be homotopic maps which represent an element in $\pi_n(\mathrm{aut}_1(X))$. Then, we see that the maps $ad(L_*(\theta_1))$ and $ad(L_*(\theta_2))$ from $S^n \times LX$ to LX are homotopic with an S^1-equivariant homotopy. This implies that L_θ and $\overline{L}_\theta$ are well defined. In what follows, we prove that $\overline{L}_{(\ )}$ is a morphism of Lie algebras. The same argument as that for $\overline{L}_{(\ )}$ is applicable to showing the result on $L_{(\ )}$. As a consequence, we have Theorem 4.1 (1).

We apply the same strategy as that for [FLS10, Lemma 4.1 and Theorems 3.6, 4.2 and 4.3]. In order to prove that $\overline{L}_{(\ )}$ is a homomorphism, we consider a diagram

$$\begin{tikzcd}
(S^n \vee S^n) \times LX \times_{S^1} ES^1 \arrow[r, "(ad(L_*(\theta))^\sigma | ad(L_*(\theta'))^\sigma)"] & LX \times_{S^1} ES^1 \\
S^n \times LX \times_{S^1} ES^1 \arrow[u, "\tau \times 1"] \arrow[ur, swap, "ad(L_*(\theta+\theta'))^\sigma = ad(L_*(\theta)+L_*(\theta'))^\sigma"] &
\end{tikzcd}$$

in which $(ad(L_*(\theta))^\sigma \mid ad(L_*(\theta'))^\sigma)\circ i_1 = ad(L_*(\theta))^\sigma$ and $(ad(L_*(\theta))^\sigma \mid ad(L_*(\theta'))^\sigma)\circ i_2 = ad(L_*(\theta'))^\sigma$, where τ is the pinch map and i_j is the map induced by the inclusion $S^n \to S^n \vee S^n$ in the j factor. Then, it follows that the horizontal arrow assigns $\chi + u\overline{L}_\theta(\chi) + v\overline{L}_{\theta'}(\chi)$ to an element $\chi \in H^*(LX \times_{S^1} ES^1)$, where $(u,0)$ and $(0,v)$ denotes the generators of $H^n(S^n \vee S^n)$. The definition of the summation in $\pi_*(\mathrm{aut}_1(LX))$ implies that the diagram above is commutative. Moreover, by definition, the slant arrow induces $\overline{L}_{\theta+\theta'}$. This yields that $\overline{L}_\theta + \overline{L}_{\theta'} = \overline{L}_{\theta+\theta'}$.

Let $\overline{\theta}$ be the inverse of θ in $\pi_*(\mathrm{aut}_1(X))$ with respect to the multiplication of the monoid $\mathrm{aut}_1(X)$. Since $\overline{L}_{(\ )}$ is a homomorphism, it follows that $\overline{L}_{\overline{\theta}} = -\overline{L}_\theta$.

We recall the Samelson product $\langle\ ,\ \rangle$ on the homotopy group $\pi_*(\mathrm{aut}_1(X))$. For elements $\theta : \pi_p(\mathrm{aut}_1(X))$ and $\theta' \in \pi_q(\mathrm{aut}_1(X))$, the product is induced by the map $\gamma : S^p \times S^q \to \mathrm{aut}_1(X)$ defined by $\gamma(x,y) = \theta(x)\circ\theta'(y)\circ\overline{\theta}(x)\circ\overline{\theta'}(y)$. Then, we have

$$(L\circ\gamma)(x,y) = L_*(\theta)(x)\circ L_*(\theta')(y)\circ L_*(\overline{\theta})(x)\circ L_*(\overline{\theta'})(y).$$

Observe that L is a morphism of monoids. Therefore, the adjoint Γ to $L\circ\gamma$ fits in the commutative diagram

$$\begin{tikzcd}
S^p\times S^q\times LX\times_{S^1} ES^1 \arrow[r, "\mathrm{Diag}\times\mathrm{Diag}\times_{S^1}1"] \arrow[ddr, "\Gamma\times_{S^1}1"'] & S^p\times S^p\times S^q\times S^q\times LX\times_{S^1} ES^1 \arrow[d, "1_{S^p}\times T\times 1_{S^q}\times 1_{LX\times_{S^1}1_{ES^1}}"] \\
 & S^p\times S^q\times S^p\times S^q\times LX\times_{S^1} ES^1 \arrow[d, "{[F,G]\times_{S^1}1_{ES^1}}"] \\
 & LX\times_{S^1} ES^1
\end{tikzcd}$$

where Diag is the diagonal map, T denotes the transposition and $[F,G]$ is defined by the composite

$$ad(L_*(\theta))\circ(1_{S^p}\times ad(L_*(\theta')))\circ(1_{S^p\times S^q}\times ad(L_*(\overline{\theta})))\circ(1_{S^p\times S^q\times S^p}\times ad(L_*(\overline{\theta'}))).$$

The commutativity follows from the same consideration as in the proof of [FLS10, Theorem 4.3]. Moreover, the same computation as in [FLS10, page 394] works well on homology. It turns out that $\overline{L}_{\langle\theta,\theta'\rangle} = \overline{L}_\theta\overline{L}_{\theta'} - (-1)^{pq}\overline{L}_{\theta'}\overline{L}_\theta$. This completes the proof of Theorem B.2. □

B.2. **An algebraic construction of $\overline{L}$.** In what follows, we assume that the underling field is rational. The assertion below shows that the geometric derivations $L_{(\ )}$ and $\overline{L}_{(\ )}$ are related to the Loday's derivations ${}_aL_{(\ )}$ and ${}_a\overline{L}_{(\ )}$ in Proposition B.1, respectively. Observe that ${}_aL_\theta$ is the derivation L_θ in Definition 3.5.

Proof of Proposition 4.3. The standard algebraic model for the evaluation map $\mathrm{ev} : LX\times S^1 \to X$ plays an important role in our proof; see [VPS76] for the model for ev. We consider the following commutative diagram consisting of continuous maps

(B.1)
$$\begin{tikzcd}
\mathrm{Map}(S^n,\mathrm{Map}(X,X)) \arrow[r, "L_*"] \arrow[d, "ad"', "\cong"] & \mathrm{Map}(S^n,\mathrm{Map}(LX,LX)) \arrow[d, "\cong"', "ad_1"] \\
\mathrm{Map}(S^n\times X,X) \arrow[r, "ad(L_*)"] \arrow[dr, "\psi:=ad_2\circ ad(L_*)"'] & \mathrm{Map}(S^n\times LX,LX) \arrow[d, "\cong"', "ad_2"] \\
 & \mathrm{Map}(S^n\times LX\times S^1,X).
\end{tikzcd}$$

It follows that $ad_1(u_\theta) = ad(L_*)(ad(\theta))$ for $\theta \in \mathrm{Map}(S^n, \mathrm{Map}(X,X))$ and $\psi(\phi) = \phi \circ (1 \times \mathrm{ev})$ for $\phi \in \mathrm{Map}(S^n \times X, X)$. In what follows, we may assume that X is a rational space. Then, we have the following diagram for the homotopy sets

$$\begin{tikzcd}
\pi_n(\mathrm{aut}_1(X)) \arrow[r, "k"] & {[S^n \times X, X]} \arrow[r, "ad(L_*)"] \arrow[d, "\mu"', "\cong"] & {[S^n \times LX, LX]} \arrow[r, "ad_2", "\cong"'] \arrow[d, "\mu"', "\cong"] & {[S^n \times LX \times S^1, X]} \arrow[d, "\mu"', "\cong"] \\
 & {[\mathcal{M}_X, \mathcal{M}_{S^n \times X}]} & {[\mathcal{M}_{LX}, \mathcal{M}_{S^n \times LX}]} & {[\mathcal{M}_X, \mathcal{M}_{S^n \times LX \times S^1}]}
\end{tikzcd}$$

which are given by the sets of continuous maps mentioned above. Here k is induced by the adjoint ad mentioned above and $\mathcal{M}_Y$ denotes a minimal Sullivan model for a space Y and μ is the Sullivan–de Rham correspondence between rational spaces and CDGAs; see, for example, [BG76]. We use the same notation for a map as that for its homotopy class.

We may replace $\mathcal{M}_{S^n \times X}$ with the CDGA $H^*(S^n;\mathbb{Q}) \otimes \mathcal{M}_X$; see [FHT01, Proposition 12.9]. Then we write $(\mu \circ k)(\theta') = 1 \otimes 1_{\mathcal{M}_X} + \iota \otimes \theta$, where ι is the generator of $H^n(S^n;\mathbb{Q})$. Observe that, by definition, $\Phi(\theta') = \theta$ for the map Φ in Proposition 4.3. In order to prove Proposition 4.3, it suffices to show

Lemma B.3. *For $\theta' \in \pi_n(\mathrm{aut}_1(X))$, one has $(\mu \circ ad(L_*) \circ k)(\theta') = 1 \otimes 1_{\mathcal{M}_{LX}} + \iota_a L_\theta$.*

In fact, applying the integration $\int_{S^n}$ to the equality in Lemma B.3 on the cohomology yields the commutativity of the diagram in Proposition 4.3. $\square$

Proof of Lemma B.3. We consider the adjoint map ad_2. The uniqueness of the adjoint correspondence shows that if we have a morphism $L(\theta')$ of CDGAs which makes the following triangle

$$\begin{tikzcd}
H^*(S^n;\mathbb{Q}) \otimes \mathcal{M}_{LX} \otimes \wedge(t) & & \mathcal{M}_{LX} \otimes \wedge(t) \arrow[ll, "L(\theta') \otimes 1"'] \\
 & \mathcal{M}_X = (\wedge V, d) \arrow[ul, "\mu(1\times \mathrm{ev})\circ\mu(k(\theta'))=\mu(k(\theta')\circ(1\times \mathrm{ev}))"] \arrow[ur, "\mu(\mathrm{ev})"'] &
\end{tikzcd}$$

commutative up to homotopy, the map $L(\theta')$ is nothing but the map $(\mu \circ ad(L_*) \circ k)(\theta')$. In fact, the map ad_2 assigns the realization $|L(\theta')|$ of $L(\theta')$ to the realization $|\mu(k(\theta') \circ (1 \times \mathrm{ev}))|$, which is homotopic to $ad_2 \circ ad(L_*)(\theta') = k(\theta') \circ (1 \times \mathrm{ev}))$. Observe that the equality follows from the commutativity of the diagram (B.1). Then we have $ad_2(|L(\theta')|) \simeq |\mu(k(\theta') \circ (1 \times \mathrm{ev}))| \simeq ad_2(ad(L_*)(k(\theta')))$. The injectivity of the map ad_2 yields that $|L(\theta')| \simeq ad(L_*)(k(\theta'))$. This implies that the map $L(\theta')$ is a model for $ad(L_*)(k(\theta'))$; that is, $L(\theta') = (\mu \circ ad(L_*) \circ k)(\theta')$.

We recall the Sullivan model $\mathcal{M}_{LX} = \mathcal{L}$ described in Section 2. Moreover, we may choose a model $\mu(\mathrm{ev})$ for the evaluation map so that $\mu(\mathrm{ev})(\omega) = \omega \otimes 1 + (-1)^{|\omega|-1} s\omega \otimes t$ for $\omega \in V$; see [VPS76]. Since $(\mu \circ k)(\theta') = 1 \otimes 1_{\mathcal{M}_X} + \iota \otimes \theta$, it follows from the commutativity for the triangle that

$$\begin{aligned}
& L(\theta')\omega \otimes 1 + (-1)^{|\omega|-1} L(\theta')(s\omega) \otimes t \\
= \quad & 1 \otimes (\omega \otimes 1 + (-1)^{|\omega|-1} s\omega \otimes t) + \iota \otimes (\theta(\omega) \otimes 1 + (-1)^{|\theta(\omega)|-1} s\theta(\omega) \otimes t)
\end{aligned}$$

for $\omega \in V$. Therefore, we see that $L(\theta')\omega = 1 \otimes \omega + \iota\theta(\omega)$ and $L(\theta')(s\omega) = 1 \otimes s\omega + (-1)^{|\theta|} s\theta(\omega)$. The definition of ${}_aL_\theta$ shows that $L(\theta') = 1 \otimes 1_{\mathcal{M}_{LX}} + \iota_a L_\theta$. This completes the proof. $\square$

Next we review the relationship with cyclic homology. Let $\mathcal{C} = (\mathcal{C}, d, B)$ be a non-negatively graded mixed complex. We introduce a variable u of degree 2 and consider the graded module $\mathcal{C}[u] = \mathcal{C} \otimes \mathbb{Q}[u]$.

Definition B.4. The *cyclic complex* $CC(\mathcal{C})$ of $\mathcal{C}$ is the complex $(\mathcal{C}[u], d_u)$, where d_u is the $\mathbb{Q}[u]$-linear map defined by $d_u = d + uB$. Its cohomology will be called the *cyclic cohomology* of $\mathcal{C}$ and denoted by $HC(\mathcal{C})$.

We recall the mixed DGA $(\mathcal{L}, d, s)$ mentioned in Section 2. With the model, the minimal Sullivan model $\mathcal{E}$ of the orbit space $ES^1 \times_{S^1} LX$ is defined by $\mathcal{E} := (\mathcal{L}[u], d + us)$; see [VPB85, Theorem A]. Thus we have an isomorphism $H(\mathcal{E}) \cong H^*(ES^1 \times_{S^1} LX; \mathbb{Q})$. Observe that $CC(\mathcal{L})$ is nothing but the complex $\mathcal{E}$ defined above and then $H(CC(\mathcal{L}))$ is isomorphic to the cyclic homology [BV88] of $(\wedge V, d)$.

Let $(\mathfrak{g}, e, L, S, T)$ be a homotopy Cartan calculus on $\mathcal{C}$. For $\theta \in \mathfrak{g}$, define ${}_a\overline{L}_\theta \in \mathrm{End}(CC(\mathcal{C}))$ by extending L_θ to a $\mathbb{Q}[u]$-linear map. This gives a linear map ${}_a\overline{L} \colon \mathfrak{g} \to \mathrm{End}(CC(\mathcal{C}))$.

Lemma B.5 ([FK20, Lemmas 3.4 and 3.10])**.** *The map* ${}_a\overline{L} \colon \mathfrak{g} \to \mathrm{End}(CC(\mathcal{C}))$ *is a morphism of dg Lie algebras.*

For the homotopy Cartan calculus in Proposition 3.6, we obtain the morphism ${}_a\overline{L}_{(\ )} : \mathrm{Der}(\wedge V) \to \mathrm{Der}(\mathcal{E})$ defined by ${}_a\overline{L}_\theta = {}_aL_\theta \otimes 1_{\mathbb{Q}[u]}$ on $\mathcal{E}$ for $\theta \in \mathrm{Der}(\wedge V)$. We recall the cobar-type Eilenberg-Moore spectral sequence (EMSS) in [KNWY21, Theorem 7.5] converging to the string cohomology $H^*_{S^1}(LX; \mathbb{Q})$ with

$$E_2^{*,*} \cong \mathrm{Cotor}^{*,*}_{H^*(S^1;\mathbb{Q})}(H^*(LX; \mathbb{Q}), \mathbb{Q}).$$

Let $\{F^p\}_{\geq 0}$ be the decreasing filtration of $H^*_{S^1}(LX; \mathbb{Q})$ associated with the EMSS.

Theorem B.6. *There exists a commutative diagram*

$$\begin{tikzcd}
\pi_*(\mathrm{aut}_1(X)) \otimes \mathbb{Q} \arrow[r, "\overline{L}_{(\ )}"] \arrow[d, "\Phi", "\cong"'] & \mathrm{Der}_*(H^*_{S^1}(LX; \mathbb{Q})) \arrow[d, "\cong"] \\
H_*(\mathrm{Der}(\wedge V)) \arrow[r, "{}_a\overline{L}_{(\ )}"'] & \mathrm{Der}_*(H^*(\mathcal{E}))
\end{tikzcd}$$

modulo the filtration of the EMSS in the sense that $({}_a\overline{L}_\theta - \overline{L}_\theta)(F^p) \subset F^{p+1}$ *for* θ *in* $\pi_*(\mathrm{aut}_1(X)) \otimes \mathbb{Q}$ *and* $p \geq 0$*, where* $\{F^p\}$ *is the filtration of* $H^*_{S^1}(LX; \mathbb{Q})$ *associated with the EMSS mentioned above.*

Proof. The key to proving the result is that the *projection* of a model for the derivation $\overline{L}_\theta$ on $H^*_{S^1}(LX; \mathbb{Q})$ is the model ${}_aL_\theta$ the derivation on $\mathcal{L}$ considered in the proof of Proposition 4.3. Let u_θ be the map stated in section B.1. Consider the commutative diagram

$$\begin{tikzcd}
& (S^n \times LX) \times_{S^1} ES^1 \arrow[dl] \arrow[dd, "ad_1(u_\theta) \times 1_{ES^1} = \overline{v_\theta}"] & S^n \times LX \arrow[l] \arrow[dd, "ad_1(u_\theta) = v_\theta"] \\
BS^1 & & \\
& LX \times_{S^1} ES^1 \arrow[ul] & LX \arrow[l]
\end{tikzcd}$$

whose row sequences are the fibrations associated with the universal S^1-bundle $S^1 \to ES^1 \to BS^1$. For simplicity, we put $v_\theta := ad_1(u_\theta)$ and $\overline{v_\theta} := ad_1(u_\theta) \times 1_{ES^1}$. Observe that v_θ is an S^1-equivariant map. We moreover consider the relative

Sullivan models for the fibrations and the morphism between them described in [FHT01, (15.9) pages 204–205]. Then we have a commutative diagram

$$\begin{tikzcd}
 & H^*(S^n;\mathbb{Q})\otimes\mathcal{E} \arrow[r] & H^*(S^n;\mathbb{Q})\otimes\mathcal{L} \\
\mathbb{Q}[u] \arrow[ur] \arrow[dr] & & \\
 & \mathcal{E} \arrow[uu, "\mathcal{M}(\overline{v_\theta})"'] \arrow[r] & \mathcal{L} \arrow[uu, "\overline{\mathcal{M}(\overline{v_\theta})}"']
\end{tikzcd}$$

in which $\mathcal{M}(\overline{v_\theta})$ and $\overline{\mathcal{M}(\overline{v_\theta})}$ are algebraic models for $\overline{v_\theta}$ and v_θ, respectively. Then Lemma B.3 implies that $\int_{S^n}\circ\overline{\mathcal{M}(\overline{v_\theta})}$ is chain homotopic to ${}_aL_\theta$; that is, there exists a homotopy h'_θ of degree -1 with $\int_{S^n}\circ\overline{\mathcal{M}(\overline{v_\theta})}-{}_aL_\theta = dh'_\theta + h'_\theta d$ in $\mathcal{L}$. Observe that $\mathcal{M}(\overline{v_\theta})$ is a morphism of $\mathbb{Q}[u]$-modules. Thus we see that for $x\in\widetilde{F}^p := \mathcal{E}\cdot\mathbb{Q}^{\geq p}[u]$,

$$\left(\int_{S^n}\circ\mathcal{M}(\overline{v_\theta}) - {}_aL_\theta\otimes 1_{\mathbb{Q}[u]}\right)x = (dh_\theta + h_\theta d)x + \alpha_{\theta,x} \tag{B.2}$$

with $h_\theta = 1_{H^*(S^n)}\otimes h'_\theta$ and for some $\alpha_{\theta,x}$ in $\widetilde{F}^{p+1}$. By construction, the filtration $\{\widetilde{F}^p\}_{p\geq 0}$ gives rise to the EMSS that we deal with. Moreover, the filtration $\{F_p\}_{p\geq 0}$ associated with the EMSS is induced by $\{\widetilde{F}^p\}_{p\geq 0}$.

Suppose that x is a cocycle with respect to the differential $D := d + us$ of $\mathcal{E}$. We may write $x = (x^0, x^1, ...)$. By applying D to the both sides of the equality (B.2), we have $0 = D(dh_\theta + h_\theta d)x^0 + D\alpha'_{\theta,x}$, where $\alpha'_{\theta,x} := (dh_\theta + h_\theta d)x^{\geq 1} + \alpha_{\theta,x}$. Observe $\int_{S^n}\circ\mathcal{M}(\overline{v_\theta})$ and ${}_aL_\theta\otimes 1_{\mathbb{Q}[u]}$ are cochain maps. Since $dx^0 = 0$, it follows that $0 = us(dh_\theta x^0) + D\alpha'_{\theta,x}$. Thus we see that the element $-ush_\theta x^0 + \alpha'_{\theta,x}$ is a cocycle in $\widetilde{F}^{p+1}$. It turns out that

$$\left(\int_{S^n}\circ\mathcal{M}(\overline{v_\theta}) - {}_aL_\theta\otimes 1_{\mathbb{Q}[u]}\right)x = Dh'_\theta x^0 + (-ush_\theta x^0 + \alpha'_{\theta,x}).$$

By definition, we have $\int_{S^n}\circ\mathcal{M}(\overline{v_\theta}) = \overline{L}_\theta$ and ${}_aL_\theta\otimes 1_{\mathbb{Q}[u]} = {}_a\overline{L}_\theta$ on the homology. This fact and the equality above yield the result. □

In a particular case, the square in Theorem B.6 is commutative. To see this, we first recall the BV-exactness of a space, which is a new homotopy invariant introduced in [KNWY21].

Definition B.7. ([KNWY21, Definition 2.9]) A simply-connected space X is *BV exact* if $\operatorname{Im}\widetilde{\Delta} = \operatorname{Ker}\widetilde{\Delta}$ for the reduced BV operator $\widetilde{\Delta} : \widetilde{H}_*(LX)\to\widetilde{H}_{*+1}(LX)$.

We observe that a formal space and a space which admits positive weights are BV exact; see [KNWY21, Assertion 1.2].

Corollary B.8. *Let X be a BV exact space. Then the diagram in Theorem B.6 is indeed commutative.*

Proof. By assumption, the space X is BV exact. Then it follows from [KNWY21, Corollary 7.4] that the EMSS collapses at the E_2-term. Moreover, the result [KNWY21, Lemma 7.5] implies that $F^p = 0$ for $p\geq 1$. This completes the proof. □

Remark B.9. Let M be a BV-exact manifold. Then the results [KNWY21, Theorem 2.15 and Corollary 2.16] assert that the string bracket $[\ ,\ ]$ on $H_*^{S^1}(LM)$ is a restriction of the loop bracket $\{\ ,\ \}$ on the loop homology $\mathbb{H}_*(LM)$. More precisely, we have a commutative diagram

$$\begin{tikzcd}
H^{S^1}_*(LM;\mathbb{K})^{\otimes 2} \arrow[r, "\Phi\otimes\Phi", "\cong"'] \arrow[d, "{[\ ,\ ]}"'] & (\mathrm{Ker}\,\widetilde{\Delta}\oplus\mathbb{K}[u])^{\otimes 2} \arrow[r, "(inc.\oplus 0)^{\otimes 2}"] \arrow[d, "\pm\{\ ,\ \}"'] & H_*(LM;\mathbb{K})^{\otimes 2} \arrow[d, "\text{the loop product}"', "\bullet"] \\
H^{S^1}_*(LM;\mathbb{K}) \arrow[r, "\Phi", "\cong"'] & (\mathrm{Ker}\,\widetilde{\Delta}\oplus\mathbb{K}[u]) & H_*(LM;\mathbb{K}), \arrow[l, "\Delta"]
\end{tikzcd}$$

where $\pm\{a,b\} := (-1)^{|a|}\{a,b\}$ for $a,b\in\mathrm{Ker}\,\widetilde{\Delta}$, $|u|=2$, Δ is the BV operator, *inc.* denotes the inclusion and Φ is the isomorphism described in [KNWY21, Theorem 2.15].

It seems that the representation $\overline{L}$ has a different property from that for L.

Example B.10. (cf. Example 5.2) We determine explicitly the Lie representation $\overline{L} : \pi_*(\mathrm{aut}_1(X))\otimes\mathbb{Q} \cong H_*(\mathrm{Der}(\mathcal{M}_X)) \to \mathrm{Der}_*(H^*_{S^1}(LX;\mathbb{Q}))$ for a simply-connected space X whose rational cohomology is isomorphic to $\mathbb{Q}[x]/(x^{n+1})$ as an algebra, where $n\geq 1$. Let $\mathcal{M}_X$ be the minimal model for X. We see that $\mathcal{M}_X\cong(\wedge(x,y),d)$ in which $dx=0$ and $dy=x^{n+1}$. Then the results [KY97, Theorem 2.2] and [KY00, Theorem 0.2] yield that

$$H^*_{S^1}(LX;\mathbb{Q})\cong\oplus_{k\geq 0,1\leq j\leq n}\mathbb{Q}\{\alpha(j,k)\}\oplus\mathbb{Q}[u]$$

as an algebra, where $\alpha(j,k)=[x^{j-1}\bar{x}\bar{y}^k]$. Moreover, we see that

$$H_*(\mathrm{Der}(\mathcal{M}_X))=\mathbb{Q}\{(y,1),(y,x),\cdots,(y,x^{n-1})\}.$$

Since ${}_a\overline{L}_{(y,x^i)}(x^{j-1}\bar{x}\bar{y}^k)=kix^{j-1}\bar{x}x^{i-1}\bar{x}\bar{y}^{k-1}=0$ for $0<i<n$ and ${}_a\overline{L}_{(y,x^i)}(x^{j-1}\bar{x})=0$, it follows that ${}_a\overline{L} : H_*(\mathrm{Der}(\mathcal{M}_X))\to\mathrm{Der}_*(H^*_{S^1}(LX;\mathbb{Q}))$ is trivial. The space X is formal and especially BV-exact. Thus Corollary B.8 yields that $\overline{L}=0$.

We conclude this appendix with a brief discussion on the Lie representation $\overline{L}$ for a more general simply-connected space X, which is not necessarily BV exact. We consider a behavior of the operator $\overline{L}_\theta$ in the EMSS for each element $\theta\in\pi_*(\mathrm{aut}_1(X))\otimes\mathbb{Q}$. Let $\{E_r^{*,*},d_r\}$ be the EMSS mentioned above and $\{F^p\}_{p\geq 0}$ the filtration of the target $H^*_{S^1}(LX;\mathbb{Q})$ associated with the EMSS. Then, we have a decomposition

$$H^*_{S^1}(LX;\mathbb{Q})=(H^*_{S^1}(LX;\mathbb{Q})/F^1)\oplus F^1.$$

Moreover, it follows that the map $i^* : H^*_{S^1}(LX;\mathbb{Q})\to H^*(LX;\mathbb{Q})$ defined by the inclusion of the fibration $LX\xrightarrow{i}LX\times_{S^1}ES^1\to BS^1$ induces a monomorphism

$$i^* : H^*_{S^1}(LX;\mathbb{Q})/F^1\to H^*(LX;\mathbb{Q}).$$

We also recall the decomposition of the EMSS

$$\{E_r^{*,*},d_r\}=\bigoplus_{N\in\mathbb{Z}}\{{}_{(N)}E_r^{*,*},d_r\}\oplus\{\mathbb{Q}[u],0\}$$

introduced in [KNWY21, Section 7].

The derivation $\overline{L}_\theta$ is well-behaved in the vertical edge of the EMSS while it acts trivially apart from the edge. As seen in the proof of the proposition below, the Cartan calculus, in particular, the contraction e plays a crucial role in describing the property of $\overline{L}$ in the EMSS.

Proposition B.11. *For each θ in $\pi_*(\mathrm{aut}_1(X)) \otimes \mathbb{Q}$ and $p \geq 1$, $\overline{L}_\theta(F^p) \subset F^{p+1}$.*

In order to prove Proposition B.11, we recall the Cartan calculus on a Sullivan algebra in Section 3.2. The proof of Proposition 3.6 allows us to obtain

Lemma B.12. *For each $\theta \in \pi_*(\mathrm{aut}_1(X)) \otimes \mathbb{Q}$, one has $[e_\theta, d] = 0$, $[e_\theta, B] = {}_aL_\theta$.*

Proof of Proposition B.11. By Lemma B.12, we see that $[e_\theta u^{-1}, d + uB] = {}_aL_\theta$ in $\mathbb{Q}^+[u] \cdot \mathcal{E}$. This implies that $({}_a\overline{L}_\theta)(F^p) = 0$ for θ in $\pi_*(\mathrm{aut}_1(X)) \otimes \mathbb{Q}$ and $p > 0$. By virtue of Theorem B.6, we have the result. □

Proposition B.13. *(1) For $r \geq 2$ and $n > 1$, there exist morphisms of Lie algebras*

$$\overline{L}_{(\ )} : \pi_n(\mathrm{aut}_1(X)) \otimes \mathbb{Q} \to \mathrm{Der}_{n,0}(E_r^{*,*}) \quad \textit{and}$$
$$\overline{L}_{(\ )} : \pi_n(\mathrm{aut}_1(X)) \otimes \mathbb{Q} \to \mathrm{End}_{n,0}({}_{(N)}E_r^{*,*})$$

for which $\overline{L}_\theta$ is compatible with the differential d_r and respects to the derivation $\overline{L}_\theta$ on $H^*_{S^1}(LX;\mathbb{Q})$ in the E_∞-term for each $\theta \in \pi_n(\mathrm{aut}_1(X)) \otimes \mathbb{Q}$. Moreover, up to isomorphism, the morphism $\overline{L}_{(\ )}$ of Lie algebras coincides with the map ${}_a\overline{L}_{(\ )}$.

(2) The map $\overline{L}_{(\ )}$ acts trivially on $E_r^{*,q}$ for $q > 1$. As a consequence, for $\theta \in \pi_n(\mathrm{aut}_1(X)) \otimes \mathbb{Q}$, one has a commutative diagram

$$\begin{tikzcd}
H^*_{S^1}(LX;\mathbb{Q})/F^1 \arrow[r, "i^*"] \arrow[d, "\overline{L_\theta}"'] & H^*(LX;\mathbb{Q}) \arrow[d, "L_\theta"] \\
H^{*-n}_{S^1}(LX;\mathbb{Q})/F^1 \arrow[r, "i^*"] & H^{*-n}(LX;\mathbb{Q}).
\end{tikzcd}$$

Proof. We first observe that the multiplication on $E_r^{*,*}$ induces the map ${}_{(N)}E_r^{p,q} \otimes {}_{(N')}E_r^{p',q'} \to {}_{(N+N')}E_r^{p+p',q+q'}$. The definition of the decomposition gives the result; see the discussion after [KNWY21, Remark 7.1].

We use a rational model $\mathcal{E}$ for the Borel construction $LX \times_{S^1} ES^1$ described above. Then, for an element $\theta \in \pi_n(\mathrm{aut}_1(X))$, the morphism $\mathcal{M}(\overline{v_\theta}) : \mathcal{E} \to H^*(S^n;\mathbb{Q}) \otimes \mathcal{E}$ of CDGAs in the proof of Theorem B.6 gives rise to a linear map $\overline{L}_\theta : E_r^{p,q} \to E_r^{p-n,q}$ for $p, q \geq 0$. In fact, the map $\mathcal{M}(\overline{v_\theta})$ preserves the filtration which constructs the EMSS. Therefore, we also see that $\overline{L}_\theta$ is compatible with the differential of each term of the EMSS. The equality (B.2) enables us to deduce that the map $\overline{L}_\theta$ on $E_1^{*,*}$ coincides with the derivation ${}_aL_\theta$. The map ${}_aL_{(\ )} : \pi_*(\mathrm{aut}_1(X)) \otimes \mathbb{Q} \to \mathrm{Der}^{-*,0}(E_r^{*',*'})$ is a morphism of Lie algebras and the so is $\overline{L}_{(\ )}$. This completes the proof of (1).

(2) Let θ be a representative of an element in $\pi_n(\mathrm{aut}_1(X)) \otimes \mathbb{Q}$. A mentioned in the proof of Proposition B.11, it follows that ${}_aL_\theta(x) = 0$ for $x \in E_1^{*,q}$ with $q > 1$. In fact, such element x is represented by one in the ideal $\mathbb{Q}^+[u] \cdot \mathcal{E}$. Thus, the first half of the assertion of (2) follows from the result (1).

As for the latter half of the assertion, we have a commutative diagram

$$\begin{tikzcd}
H^*_{S^1}(LX;\mathbb{Q})/F^1 \arrow[r, "\cong"'] \arrow[d, "\overline{L_\theta}"'] \arrow[rr, bend left, "i^*"] & E_\infty^{*,0} \arrow[r, tail] \arrow[d, "{}_aL_\theta"] & H^*(LX;\mathbb{Q}) \arrow[d, "L_\theta"] \\
H^{*-n}_{S^1}(LX;\mathbb{Q})/F^1 \arrow[r, "\cong"] \arrow[rr, bend right, "i^*"'] & E_\infty^{*-n,0} \arrow[r, tail] & H^{*-n}(LX;\mathbb{Q}).
\end{tikzcd}$$

The commutativity of the diagrams containing i^* follows from a property of the EMSS. By the definition of ${}_aL_\theta$ and Proposition 4.3, we see that the right-hand side diagram is commutative. It follows from (1) that the left-hand side diagram is commutative. We have the result. □

Corollary B.14. *Let x be an element in the image of the derivation $\overline{L}_\theta : E_r^{0,*} \to E_r^{0,*}$ for some θ. Then $d_r(x) = 0$.*

Proof. The operation $\overline{L}_\theta$ is compatible with the differential d_r. Proposition B.13 (2) implies the result. □

References

[AL00] M. Arkowitz and G. Lupton. Rational obstruction theory and rational homotopy sets. *Math. Z.*, 235:525–539, 2000.

[BL05] J. Block and Andrey Lazarev. André–Quillen cohomology and rational homotopy of function spaces. *Adv. Math.*, 193(1):18–39, 2005.

[BG76] A. K. Bousfield and V. K. A. M. Gugenheim. *On PL De Rham theory and rational homotopy type*, volume 179. Providence, RI: American Mathematical Society (AMS), 1976.

[BS97] Edgar H. jun. Brown and Robert H. Szczarba. On the rational homotopy type of function spaces. *Trans. Am. Math. Soc.*, 349(12):4931–4951, 1997.

[BM06] Urtzi Buijs and Aniceto Murillo. Basic constructions in rational homotopy theory of function spaces. *Ann. Inst. Fourier*, 56(3):815–838, 2006.

[BM08] Urtzi Buijs and Aniceto Murillo. The rational homotopy Lie algebra of function spaces. *Comment. Math. Helv.*, 83(4):723–739, 2008.

[BV88] Dan Burghelea and Micheline Vigué Poirrier. Cyclic homology of commutative algebras. I. Algebraic topology, rational homotopy, Proc. Conf., Louvain-la-Neuve/Belg. 1986, Lect. Notes Math. 1318, 51-72 (1988)., 1988.

[CS99] Moira Chas and Dennis Sullivan. String topology. preprint, arXiv:math/9911159v1, 1999.

[Con85] Alain Connes. Non-commutative differential geometry. *Publ. Math., Inst. Hautes Étud. Sci.*, 62:41–144, 1985.

[FHT01] Yves Félix, Stephen Halperin, and Jean-Claude Thomas. *Rational homotopy theory*, volume 205. New York, NY: Springer, 2001.

[FLS10] Yves Félix, Gregory Lupton, and Samuel B. Smith. The rational homotopy type of the space of self-equivalences of a fibration. *Homology Homotopy Appl.*, 12(2):371–400, 2010.

[FT04] Yves Félix and Jean-Claude Thomas. Monoid of self-equivalences and free loop spaces. *Proc. Am. Math. Soc.*, 132(1):305–312, 2004.

[FK20] Domenico Fiorenza and Niels Kowalzig. Higher brackets on cyclic and negative cyclic (co)homology. *Int. Math. Res. Not.*, 2020(23):9148–9209, 2020.

[GJ90] Ezra Getzler and John D. S. Jones. A_∞-algebras and the cyclic bar complex. *Ill. J. Math.*, 34(2):256–283, 1990.

[Hae82] Andre Haefliger. Rational homotopy of the space of sections of a nilpotent bundle. *Trans. Am. Math. Soc.*, 273:609–620, 1982.

[Hal83] S. Halperin. Lectures on minimal models. *Mém. Soc. Math. Fr., Nouv. Sér.*, 9-10:261, 1983.

[HMR75] Peter Hilton, Guido Mislin, and Joe Roitberg. *Localization of nilpotent groups and spaces*, volume 15 of *North-Holland Math. Stud.* Elsevier, Amsterdam, 1975.

[HKO08] Yoshihiro Hirato, Katsuhiko Kuribayashi, and Nobuyuki Oda. A function space model approach to the rational evaluation subgroups. *Math. Z.*, 258(3):521–555, 2008.

[Kur06] Katsuhiko Kuribayashi. A rational model for the evaluation map. *Georgian Math. J.*, 13(1):127–141, 2006.

[KNWY21] Katsuhiko Kuribayashi, Takahito Naito, Shun Wakatsuki, and Toshihiro Yamaguchi. A reduction of the string bracket to the loop product. *Algebraic Geom. Topol.* (2021), in press, arXiv:2109.10536v1.

[KY97] Katsuhiko Kuribayashi and Toshihiro Yamaguchi. The cohomology algebra of certain free loop spaces. *Fundamenta Mathematicae*, 154:57–73, 1997.

[KY00] Katsuhiko Kuribayashi and Toshihiro Yamaguchi. On additive K-theory with the Loday–Quillen *-product. *Math. Scand.*, 87:5–21, 2000.

[LS08] Pascal Lambrechts and Don Stanley. Poincaré duality and commutative differential graded algebras. *Ann. Sci. Éc. Norm. Supér. (4)*, 41(4):497–511, 2008.

[Lod98] Jean-Louis Loday. *Cyclic homology.*, volume 301. Berlin: Springer, 1998.

[Men11] Luc Menichi. Van Den Bergh isomorphisms in string topology. *J. Noncommut. Geom.*, 5(1):69–105, 2011.

[Nai24] Takahito Naito. Cartan calculus in string topology. *Proc. Am. Math. Soc.*, 152(4):1789–1801, 2024.

[NW22] Takahito Naito and Shun Wakatsuki. 'On loop-slant operations' in preparation, 2022.

[SYH18] Kazuhisa Shimakawa, Kohei Yoshida, and Tadayuki Haraguchi. Homology and cohomology via enriched bifunctors. *Kyushu J. Math.*, 72(2):239–252, 2018.

[Ste67] N. E. Steenrod. A convenient category of topological spaces. *Mich. Math. J.*, 14:133–152, 1967.

[Sul77] Dennis Sullivan. Infinitesimal computations in topology. *Inst. Hautes Études Sci. Publ. Math.*, (47):269–331 (1978), 1977.

[Vig94] Micheline Vigué-Poirrier. Homologie cyclique des espaces formels. *J. Pure Appl. Algebra*, 91(1-3):347–354, 1994.

[VPB85] Micheline Vigué-Poirrier and Dan Burghelea. A model for cyclic homology and algebraic K-theory of 1-connected topological spaces. *J. Differ. Geom.*, 22:243–253, 1985.

[VPS76] Micheline Vigue-Poirrier and Dennis Sullivan. The homology theory of the closed geodesic problem. *J. Differ. Geom.*, 11:633–644, 1976.

[Wak] Shun Wakatsuki. Kohomology. `https://github.com/shwaka/kohomology`.

[Whi78] George W. Whitehead. *Elements of homotopy theory*, volume 61. Springer, New York, NY, 1978.

[Yam05] Toshihiro Yamaguchi. An example of fiber in fibrations whose Serre spectral sequences collapse. *Czechoslovak Math. J.*, 55(130):997–1001, 2005.

Department of Mathematical Sciences, Faculty of Science, Shinshu University, Matsumoto, Nagano 390-8621, Japan
Email address: `kuri@math.shinshu-u.ac.jp`

Nippon Institute of Technology, Gakuendai, Miyashiro-machi, Minamisaitama-gun, Saitama 345-8501, Japan
Email address: `naito.takahito@nit.ac.jp`

Graduate School of Mathematics, Nagoya University, Furo-cho, Chikusa-ku, Nagoya, Aichi 464-8601, Japan
Email address: `shun.wakatsuki@math.nagoya-u.ac.jp`

Faculty of Education, Kochi University, Akebono-cho, Kochi 780-8520, Japan
Email address: `tyamag@kochi-u.ac.jp`