On Gauss factorials and their connection to the cyclotomic λ -invariants of imaginary quadratic fields.

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Abstract

In this paper we establish a connection between the Gauss factorials and Iwasawa's cyclotomic λ invariant for an imaginary quadratic field K. As a result, we will explain a correspondence between the 1-exceptional primes of Cosgrave and Dilcher [2], [3] for m = 3 and m = 4, and the primes for which the λ -invariants for $K = \mathbb{Q}(\sqrt{-3})$ and $K = \mathbb{Q}(i)$ are greater than one, respectively. We refer to the latter primes as "non-trivial" for their respective fields. We will also see that similar correspondences are true for $K = \mathbb{Q}(\sqrt{-d})$ when d = 2, 5 and 6. As a corollary we find that primes p of the form $p^2 = 3x^2 + 3x + 1$ are always non-trivial for $K = \mathbb{Q}(\sqrt{-3})$. Last, we show that the non-trivial primes p for $K = \mathbb{Q}(i)$ and $K = \mathbb{Q}(\sqrt{-3})$ are characterized by modulo p^2 congruences involving Euler and Glaisher numbers respectively.

1 Introduction and statement of main results

Let p be an odd prime, and d > 0 a square-free integer. Denote $K = \mathbb{Q}(\sqrt{-d})$ and $\lambda_p(K)$ to be Iwasawa's λ -invariant for the cyclotomic \mathbb{Z}_p -extension of K. In [4], Dummit, Ford, Kisilevsky and Sands compute $\lambda_p(K)$ for various primes and imaginary quadratic fields. They define the non-trivial primes of K to be those which satisfy $\lambda_p(K) > 1$ (non-trivial since $\lambda_p(K) > 0$ whenever p splits in K). For example, Table 1 gives the non-trivial primes for $K = \mathbb{Q}(\sqrt{-3})$ and $K = \mathbb{Q}(i)$ for primes $p < 10^7$ (see Table 1 in [4] for all other imaginary quadratic fields with discriminants up to 1,000).

Table 1: Non-trivial primes $p < 10^7$ of $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(i)$.

$K = \mathbb{Q}(\sqrt{-3})$	13	181	2521	76543	489061	6811741
$K = \mathbb{Q}(i)$	29789					

Authors such as Ellenberg, Jain, and Venkatesh [5], Horie [9], Ito [10], and Sands [17] have studied $\lambda_p(K)$ by fixing a prime p and varying the imaginary quadratic field K. Dummit, Ford, Kisilevsky, and Sands [4], and Gold [6] have studied the case when K is fixed and p varies (which is the point of view we take in this paper), but less seems to be known in this situation. Another point of view might be to fix both p and K and vary the \mathbb{Z}_p -extension of K. Interestingly, Sands [16] has shown that if p does not divide the class number of K, and the cyclotomic λ -invariant $\lambda_p(K) \leq 2$, then every other \mathbb{Z}_p -extension K_{∞}/K has $\lambda_p \leq 2$ and $\mu_p = 0$. Therefore, knowing the non-trivial primes p of K is important for our overall understanding of the other \mathbb{Z}_p -extensions of K.

On the other hand, for $m \in \mathbb{Z}^+$ we have the seemingly unrelated 1-exceptional primes p for m studied by Cosgrave and Dilcher, that is, primes $p \equiv 1 \pmod{m}$ such that $\left(\frac{p^2-1}{m}\right)_p^{p-1}! = \left(\prod_{\substack{p=1\\gcd(a,p)=1}}^{\frac{p^2-1}{m}} a\right)^{p-1} \equiv 1 \pmod{p^2}$. Surprisingly, the primes p in Table 1 are exactly the 1-exceptional primes for m = 3 and m = 4 respectively, with $p < 10^7$ (see the next section, or [2] and [3] to learn about 1-exceptional primes).

The main result of this paper is Theorem 3.3, which is a criterion in terms of Gauss factorials that give $\lambda_p(K) > 1$ (this is a condition that works for every imaginary quadratic field K and any primes p that split in K). From this, we obtain an explanation for the apparent connection between the 1-exceptional primes for m = 3 and m = 4, and the non-trivial primes of $K = \mathbb{Q}(\sqrt{-3})$ and $K = \mathbb{Q}(i)$, as well as some similar results for $K = \mathbb{Q}(\sqrt{-d})$ with d = 2, 5 and 6:

Theorem 1.1. Let $K = \mathbb{Q}(\sqrt{-d})$ and D = 2d if $d \equiv 3 \pmod{4}$ and D = 4d otherwise. Let $r \in \mathbb{Z}^+$ such that $p^r \equiv 1 \pmod{D}$, and suppose that p does not divide the class number of K. Then for d = 1, 2, 3, 5 and 6 we have

$$\lambda_p(K) > 1 \iff \left(\frac{\left(\frac{p^{2r}-1}{D}\right)_p^2!}{\left(\frac{p^{2r}-1}{D/2}\right)_p!}\right)^{p-1} \equiv 1 \pmod{p^2}.$$

In particular, p is 1-exceptional for m = 3 if and only if $\lambda_p(\mathbb{Q}(\sqrt{-3}) > 1 \text{ and } p \text{ is 1-exceptional for } m = 4$ if and only if $\lambda_p(\mathbb{Q}(i)) > 1$.

The proof of Theorem 1.1 relies on the fact the fields $K = \mathbb{Q}(\sqrt{-d})$, where d = 1, 2, 3, 5 and 6, have so called "maximal class numbers" (see Definition 3.6). We will prove Theorem 3.9 which tells us that these are the only imaginary quadratic fields with such class numbers, under the assumption that the generalized Riemann hypothesis is true.

As a corollary of Theorem 1.1 we will see that primes p of the form $p^2 = 3x^2 + 3x + 1$ with $x \in \mathbb{Z}$ always give $\lambda_p(\sqrt{-3}) > 1$. However, the converse does not hold (see Remark 2.8). Theorem 1.1 also leads to

Corollary 1.2. For $K = \mathbb{Q}(\sqrt{-d})$ for d = 1, 2, 3, 5 and 6, we have

$$\lambda_p(K) > 1 \iff B_p(2/D) \equiv 2^p B_p(1/D) \pmod{p^3}$$

where $B_n(x)$ is the n-th Bernoulli polynomial.

In particular, we obtain some interesting conditions for the non-trivial primes of $K = \mathbb{Q}(i)$ and $K = \mathbb{Q}(\sqrt{-3})$ in terms of Glaisher and Euler numbers respectively. Recall the Euler numbers $\{E_n\}$ and Glaisher numbers $\{G_n\}$ are defined by

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \frac{2}{e^x + e^{-x}} \quad \text{and} \quad \sum_{n=0}^{\infty} G_n \frac{x^n}{n!} = \frac{3/2}{e^x + e^{-x} + 1}.$$

We will prove:

Corollary 1.3. Let $p \equiv 1 \pmod{4}$ be a prime and E_n denote the n-th Euler number. Then $\lambda_p(\mathbb{Q}(i)) > 1$ if and only if $E_{p-1} \equiv 0 \pmod{p^2}$.

Corollary 1.4. Let $p \equiv 1 \pmod{3}$ be a prime and G_n denote the n-th Glaisher number. Then $\lambda_p(\mathbb{Q}(\sqrt{-3})) > 1$ if and only if $G_{p-1} \equiv 0 \pmod{p^2}$.

Remark 1.5. The numbers $\{G_n\}$ were studied by Glaisher in [7] and [8] in which they are referred to as *I*-numbers.

An analogue of Theorem 3.3 for primes p giving $\lambda_p(\mathbb{Q}(\sqrt{-d})) > 2$ is proved in the author's PhD thesis, but uses a different technique involving p-adic L-functions.

2 Gauss factorials and exceptional primes

In this section we define Gauss factorials and exceptional primes, as well as state some results that will be needed for the proof of Theorem 1.1 as well as Corollary 3.1. For $N, n \in \mathbb{Z}^+$ the Gauss factorial of N with respect to n is defined as

$$N_n! = \prod_{\substack{i=1\\ \gcd(i,n)=1}}^N i$$

In [3], Cosgrave and Dilcher investigate multiplicative orders modulo powers of p of the following Gauss factorials

$$\left(\frac{p^{\alpha}-1}{m}\right)_p!$$

where $m, \alpha \in \mathbb{Z}^+$, with m and α greater than 2, and $p \equiv 1 \pmod{m}$. If $\gamma_{\alpha+1}^m(p)$ is the multiplicative order of $\left(\frac{p^{\alpha+1}-1}{m}\right)_p!$ modulo $p^{\alpha+1}$, then Cosgrave and Dilcher define p to be α -exceptional for m if $\gamma_{\alpha+1}^m(p)$ and $\gamma_{\alpha}^m(p)$ are the same modulo a factor of $2^{\pm 1}$ (otherwise $\gamma_{\alpha+1}^m(p) = p\gamma_{\alpha}^m(p)$ or $\gamma_{\alpha+1}^m(p) = 2^{\pm 1}p\gamma_{\alpha}^m(p)$, see Theorem 1 and Definition 1 in [3]). Further, Theorem 3 in [3] shows that if p is α exceptional for m, then p is also $(\alpha - 1)$ -exceptional for m. For our purposes, we will not need this much precision on the multiplicative orders, and we will instead use the equivalent definition:

Definition 2.1. For $\alpha \in \mathbb{Z}^+$, we say that p is α -exceptional for m if and only if $\left(\frac{p^{\alpha+1}-1}{m}\right)_p^{p-1}! \equiv 1 \pmod{p^{\alpha+1}}$.

We contrast Definition 2.1 with the following definition of "non-trivial" primes. Theorem 1.1 will show that the primes in each of the two definitions are the same when m = 3 and $K = \mathbb{Q}(\sqrt{-3})$, and when m = 4and $K = \mathbb{Q}(i)$:

Definition 2.2. Given an imaginary quadratic field K, we say that p is non-trivial for K if $\lambda_p(K) > 1$.

Example 2.3. Let p = 13 and m = 3. Then $\gamma_1^3(13) = 12$, $\gamma_2^3(13) = 12$, $\gamma_3^3(13) = 12 \cdot 13$, $\gamma_4^3(13) = 12 \cdot 13^2$, $\gamma_5^3(13) = 12 \cdot 13^3$ and so on ("and so on" since Theorem 3 in [3] says that p is $(\alpha + 1)$ -exceptional for m implies p is also α -exceptional for m). The next few values of p such that $\gamma_1^3(p) = \gamma_2^3(p)$ are p = 181, 2521, 76543 and so on. On the other hand, if m = 4 and p = 29789, then $\gamma_2^4(p) = \frac{1}{2}\gamma_1^4(p)$, and is the only known such example for $p < 10^{11}$ (see also Table 1 in [2] for $\gamma_{\alpha}^4(p)$ with $1 \le \alpha \le 5$, $p \le 37$ and $p \equiv 1 \pmod{4}$).

The following results of Cosgrave and Dilcher will be important later on:

Theorem 2.4 (Cosgrave-Dilcher [3]). Let $p \equiv 1 \pmod{6}$ be a prime. Then p is 1-exceptional for m = 3 if and only if p is 1-exceptional for m = 6.

Theorem 2.5 (Cosgrave-Dilcher [3]). Let $p \equiv 1 \pmod{6}$ be a prime and $n \in \mathbb{Z}^+$. Then

$$\left(\left(\frac{p^n-1}{3}\right)_p!\right)^{24} \equiv \left(\left(\frac{p^n-1}{6}\right)_p!\right)^{12} \pmod{p^n}$$

Theorem 2.6 (Cosgrave-Dilcher [2]). Every prime $p \equiv 1 \pmod{6}$ that satisfies $p^2 = 3x^2 + 3x + 1$ for some $x \in \mathbb{Z}$ is 1-exceptional for m = 3. Equivalently, if $\gamma = 2 + \sqrt{3}$ and $q \in \mathbb{Z}^+$, then any prime of the form

$$p = \frac{\gamma^q + \gamma^{-q}}{4}$$

is 1-exceptional for m = 3.

Definition 2.7. We shall refer to the primes $p \equiv 1 \pmod{3}$ such that $p^2 = 3x^2 + 3x + 1$ for some $x \in \mathbb{Z}$ as Cosgrave-Dilcher primes.

Remark 2.8. In [2] and [3] Cosgrave and Dilcher rearranged the equation $p^2 = 3x^2 + 3x + 1$ into $(2p)^2 - 3(2x + 1)^2 = 1$, which can be viewed as the Pell equation $X^2 - 3Y^2 = 1$. It is from the theory of these equations that we obtain the primes $p = (\gamma^q + \gamma^{-q})/4$. Also, q is necessarily prime (see lemma 7 in [2]). It should be mentioned that the converse of Theorem 2.6 does not hold. For example, p = 76543 is 1-exceptional for 3 but is not a Cosgrave-Dilcher prime (p = 76543 is the only such example for $p < 10^{12}$). It is unknown whether or not there are infinitely many Cosgrave-Dilcher primes, and the question seems to be analogous to that of the infinitude of Fibonacci primes. In a moment we will list some new Cosgrave-Dilcher primes (see Example 3.2).

3 Proof of main Theorems

In this section we will prove Theorem 1.1 from which we immediately obtain as a Corollary:

Corollary 3.1. Let $p \equiv 1 \pmod{6}$ be a Cosgrave-Dilcher prime. Then $\lambda_p(\mathbb{Q}(\sqrt{-3})) > 1$.

Example 3.2. Using Corollary 3.1 we may add to the non-trivial primes of $\mathbb{Q}(\sqrt{-3})$ in Table 1 by searching for Cosgrave-Dilcher primes. The following table contains $p = (\gamma^q + \gamma^{-q})/4$ with $q \leq 79$:

q = 3	p = 13
q = 5	p = 181
q = 7	p = 2521
q = 11	p = 489061
q = 13	p = 6811741
q = 17	p = 1321442641
q = 19	p = 18405321661
q = 79	p = 381765135195632792959100810331957408101589361

One may further verify using any standard CAS that the primes $79 < q \leq 10,000$ giving 1-exceptional primes $p = (\gamma^q + \gamma^{-q})/4$ for m = 3 (and therefore non-trivial primes of $\mathbb{Q}(\sqrt{-3})$) are q = 151, 199, 233,251, 317, 863, 971, and q = 3049, 7451, and 7487 giving probable primes p (the non-trivial probable prime corresponding to q = 7487 is 4282 digits long).

Let d be a square-free integer, $K = \mathbb{Q}(\sqrt{-d})$, D = 2d if $d \equiv 3 \pmod{4}$ and D = 4d otherwise. Let p > 2be a prime such that $p \equiv 1 \pmod{D}$ with $p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$ and \mathcal{P} be a prime in $\mathbb{Q}(\zeta_D)$ above \mathfrak{p} , where ζ_D is a primitive D-th root of unity. Let $\overline{\mathcal{P}}$ be the complex conjugate of \mathcal{P} . Denote $G = \operatorname{Gal}(\mathbb{Q}(\zeta_D)/\mathbb{Q})$ and $\chi_K = \chi$ to be the imaginary quadratic character for K. We have for $x \in \mathbb{Q}(\zeta_D)$

$$N_{\mathbb{Q}(\zeta_D)/K}(x) = \prod_{\substack{i=1\\\chi(i)=1\\\gcd(i,D)=1}}^D \sigma_i(x) \in K$$

where $\sigma_i \in G$ acts by $\sigma_i(\zeta_D) = \zeta_D^i$. We will also denote $\mathcal{P}_i = \sigma_i(\mathcal{P})$ so that $N_{\mathbb{Q}(\zeta_D)/K}(\mathcal{P}_i) = \mathfrak{p}$. We will now work towards proving the following result from which Theorem 1.1 will follow.

Theorem 3.3. Let $K = \mathbb{Q}(\sqrt{-d})$ be any imaginary quadratic field, and D be as above. Let p be a prime and $r \in \mathbb{Z}^+$ such that $p^r \equiv 1 \pmod{D}$, $p \nmid h_K$, and $p \neq 3$ whenever $\chi_K(2) = -1$ and $K \neq \mathbb{Q}(\sqrt{-3})$. Then,

$$\lambda_p(K) > 1 \iff \left(\prod_{\substack{i=D/2\\\chi(i)=1\\\gcd(i,D)=1}}^{D} \frac{\left((D-i)\frac{p^{2r}-1}{D}\right)_p^2!}{\left((D-i)\frac{p^{2r}-1}{D/2}\right)_p!} \prod_{\substack{i=1\\\chi(i)=1\\\gcd(i,D)=1}}^{D/2} \frac{\left(i\frac{p^{2r}-1}{D/2}\right)_p!}{\left(i\frac{p^{2r}-1}{D}\right)_p^2!}\right)^{p-1} \equiv 1 \pmod{p^2}.$$

The first step of the proof is to write $\bar{\mathfrak{p}}$ in terms of Jacobi sums. Consider the multiplicative character $\psi: \mathcal{O}_{\mathbb{Q}(\zeta_D)}/\mathcal{P} \to \mathbb{C}^{\times}$ of order D modulo \mathcal{P} . We denote

$$J(\psi) = \sum_{a \in \mathbb{F}_p} \psi(a)\psi(1-a)$$

to be the Jacobi sum for ψ . Denote $0 \le L(j) < D$ to be reduction of j modulo D, and for $1 \le i < D/2$ we define

$$S_i(D) = \{j : 0 < j < D; \gcd(j, D) = 1; L(ji) < D/2\}$$

Then from Theorem 2.1.14 in [1] we have

$$J(\psi^i)\mathcal{O}_{\mathbb{Q}(\zeta_D)} = \prod_{j \in S_i(D)} \mathcal{P}_{j^{-1}}.$$

Proposition 3.4. Denote $h_K = h$ to be the class number for $K = \mathbb{Q}(\sqrt{-d})$. With the notation fixed above, we have

$$\bar{\mathfrak{p}}^{t} = \left(\left. \prod_{\substack{i=D/2\\\chi(i)=1\\\gcd(i,D)=1}}^{D} J(\psi^{i}) \middle/ \prod_{\substack{i=1\\\chi(i)=1\\\gcd(i,D)=1}}^{D/2} J(\psi^{-i}) \right) \mathcal{O}_{\mathbb{Q}(\zeta_{D})}$$

where $t = \pm h(2 - \chi(2))$ if $d \neq 1$ or 3, else $t = \pm 1$. The sign of t depends on the number of quadratic residues modulo D between 1 and D/2.

Proof. Denote

$$a^{+} = \#\{0 < j < D/2 \ \operatorname{gcd}(j, D) = 1, \ \chi(j) = 1\}$$
$$a^{-} = \#\{0 < j < D/2 \ \operatorname{gcd}(j, D) = 1, \ \chi(j) = -1\}.$$

It is well known that $\pm h = (a^+ - a^-)/(2 - \chi(2))$ when d is not 1 or 3 (it is easy to see what happens in those cases, so we will assume d > 3). If N is the norm from $\mathbb{Q}(\zeta_D)$ to K then

$$N(J(\psi^{-1}))\mathcal{O}_{\mathbb{Q}(\zeta_D)} = \prod_{\substack{j=1\\\gcd(j,D)=1}}^{D/2} N(\bar{\mathcal{P}}_{j^{-1}}) = \mathfrak{p}^{a^-}\bar{\mathfrak{p}}^{a^+}$$

and also $J(\psi^i)J(\psi^{-i}) = p$. Then the ideal

$$\left(\prod_{\substack{i=D/2\\\chi(i)=1\\\gcd(i,D)=1}}^{D} J(\psi^{i}) \middle/ \prod_{\substack{i=1\\\chi(i)=1\\\gcd(i,D)=1}}^{D} J(\psi^{-i}) \right) = \left(\prod_{\substack{i=D/2\\\chi(i)=1\\\gcd(i,D)=1}}^{D} J(\psi^{i}) \prod_{\substack{i=D/2\\\chi(i)=1\\\gcd(i,D)=1}}^{D} J(\psi^{i}) \prod_{\substack{i=D/2\\\chi(i)=1\\\gcd(i,D)=1}}^{D} J(\psi^{i}) \prod_{\substack{i=D/2\\\chi(i)=1\\\gcd(i,D)=1}}^{D} J(\psi^{-i}) \right) \\ = \left(\prod_{\substack{i=D/2\\\chi(i)=1\\\gcd(i,D)=1}}^{D} p \middle/ N(J(\psi^{-1})) \right) = \frac{(\mathfrak{p}\bar{\mathfrak{p}})^{a^{-}}}{\mathfrak{p}^{a^{+}}} = \bar{\mathfrak{p}}^{\pm h(2-\chi(2))}.$$

Theorem 3.3 will now follow from Gold's criterion:

Theorem 3.5 (Gold's criterion (Theorem 4 in [6])). Let K be an imaginary quadratic field, and p > 2 be a prime such that p does not divide the class number h_K of K.

- i. If p splits in K then $\lambda_p(K) > 0$.
- ii. Suppose $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$ and write $\mathfrak{p}^{h_K} = (\alpha)$. Then $\lambda_p(K) > 1$ if and only if $\alpha^{p-1} \equiv 1 \pmod{\bar{\mathfrak{p}}^2}$.

Proof of Theorem 3.3. Let $r \in \mathbb{Z}^+$ such that $p^r \equiv 1 \pmod{D}$. Working inside the localization $K_{\mathfrak{p}} \cong \mathbb{Q}_p$, we have $J(\psi^{-i}) \equiv \frac{\left(i\frac{p^{2r}-1}{D}\right)_p!}{\left(i\frac{p^{2r}-1}{D}\right)_p^2!} \pmod{p^2\mathbb{Z}_p}$ from (9.3.6) in [1] (which is essentially the Gross-Koblitz formula). The result now follows from Proposition 3.4 and Gold's criterion 3.5.

We will see that the condition in Theorem 3.3 becomes more compact for a certain family of imaginary quadratic fields.

Definition 3.6. Let $\chi_K = \chi$ be the imaginary quadratic character for K and D be as above. We say that K has maximal class number if $\chi_K(i) = 1$ for each i co-prime to D and $1 \le i \le D/2$.

If h_K is the class number for $K \neq \mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$, we have that $(2 - \chi(2))h_K = \left|\sum_{i=1}^{D/2} \chi(i)\right|$. Then $\chi(i) = 1$ for each *i* co-prime to *D* and $1 \leq i \leq D/2$ if and only if $h_K = \varphi(D)/2(2 - \chi(2))$.

Theorem 3.7. Suppose $r \in \mathbb{Z}^+$ such that $p^r \equiv 1 \pmod{D}$ and all other notation is as above. If K has maximal class number, and $p \nmid h_K$, then

$$\lambda_p(K) > 1 \iff \left(\frac{\left(\frac{p^{2r}-1}{D}\right)_p^2!}{\left(\frac{p^{2r}-1}{D/2}\right)_p!}\right)^{p-1} \equiv 1 \pmod{p^2}.$$

Proof. Here we will view $K \subseteq K_{\mathfrak{p}} \cong \mathbb{Q}_p$. If K has maximal class number, then $S_1(D)$ accounts for all of the quadratic residues between 1 and D/2, and so $J(\psi^{-1}) \in N(\bar{\mathcal{P}}) = \bar{\mathfrak{p}}$. Therefore, if $\bar{\mathfrak{p}}^{h_K} = (\alpha)$ for some $\alpha \in K$, we have $J(\psi^{-1})^{h_K} \equiv \alpha u \pmod{p^2 \mathbb{Z}_p}$ where $u \in \mathcal{O}_K^{\times}$. Now, since $p \nmid h_K$ we have $J(\psi^{-1})^{h_K(p-1)} \equiv 1 \pmod{p^2 \mathbb{Z}_p}$ if and only if $J(\psi^{-1})^{(p-1)} \equiv 1 \pmod{p^2 \mathbb{Z}_p}$. The result now follows from Gold's criterion and the fact that $u^{p-1} = 1$.

Remark 3.8. When D = 6, the combination of Theorems 2.4 and 2.5 imply that $\lambda_p(\mathbb{Q}(\sqrt{-3})) > 1$ if and only if p is 1-exceptional for m = 3. When D = 4, we have that $\left(\frac{p^2-1}{2}\right)_p^{p-1}! \equiv 1 \pmod{p^2}$ (a corollary of Wilson's theorem), so $\lambda_p(\mathbb{Q}(i)) > 1$ if and only if p is 1-exceptional for m = 4.

Theorem 1.1 now follows as a special case of Theorem 3.7. Computations show that $K = \mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-3})$, $\mathbb{Q}(\sqrt{-5})$ and $\mathbb{Q}(\sqrt{-6})$ are the only imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-d})$ with d < 10,000 having maximal class number. In fact,

Theorem 3.9. Assuming the generalized Riemann hypothesis (GRH) holds for every non-principle primitive imaginary quadratic character, the only imaginary quadratic fields with maximal class number are $K = \mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-3})$, $\mathbb{Q}(\sqrt{-5})$ and $\mathbb{Q}(\sqrt{-6})$.

Proof. Let d > 0 be a square free integer and let D and $\chi_D = \chi$ be as above. Denote $K = \mathbb{Q}(\sqrt{-d})$ and h_K to be the class number of K, and assume that $h_K = \varphi(D)/2(2-\chi(2))$ (i.e. h_K is maximal). From Theorem 15 in [15], we have

$$\varphi(D) > \frac{D}{e^{\gamma} \log \log(D) + \frac{3}{\log \log(D)}}$$

where $e = \exp(1)$ and $\gamma = 0.577215665...$ is Euler's constant. On the other hand, under the assumption of the generalized Riemann hypothesis, Littlewood [14] gave the inequality $h_K < ce^{\gamma} \log \log(D)\sqrt{D}$, where c is an absolute constant. Recently, this bound has been improved (see [11] and [12]) to

$$h_K \le \frac{2e^{\gamma}}{\pi} \sqrt{D} \left(\log \log(D) - \log(2) + \frac{1}{2} + \frac{1}{\log \log(D)} \right)$$

for $D \ge 5$, and assuming GRH holds. Thus, when h_K is maximal and $D \ge 5$, the two inequalities above imply

$$\sqrt{D} < \frac{12e^{\gamma}}{\pi} \left(e^{\gamma} (\log \log(D))^2 + \frac{3}{(\log \log(D))^2} + e^{\gamma} + 3 \right) < 14 (\log \log(D))^2 + 140$$

This inequality does not hold for long. Indeed, set $f(x) = \sqrt{x} - 14(\log \log(x))^2 + 140$ and notice that $f'(x) = \frac{1}{2\sqrt{x}} - \frac{28\log \log(x)}{x\log(x)} > 0$ precisely when $x\log(x) > 56\sqrt{x}\log\log(x)$, which will eventually hold for all x sufficiently large (e.g. for all x > 300). Therefore, we have that f(x) is strictly increasing on $[300, \infty)$. We also have that f(300) > 0, so the inequality $\sqrt{D} > 14(\log \log(D))^2 + 140$ holds for all D > 300. Therefore, there are no imaginary quadratic fields with D > 300 having maximal class number. It is easy to check that the only imaginary quadratic fields with $D \le 300$ and maximal class number are the ones listed above.

4 Proof of Corollaries

We now turn to the proofs of Corollaries 1.2, 1.3 and 1.4 for which we will need some preliminary results. For a co-prime to p the Fermat quotient is defined as $q_p(a) = (a^{p-1} - 1)/p$, which is an integer by Fermat's little Theorem. The Fermat quotient has logarithmic properties, that is, for a and b co-prime to p,

$$q_p(a) + q_p(b) \equiv q_p(ab) \pmod{p}$$
 and $q_p(a) - q_p(b) \equiv q_p(a/b) \pmod{p}$

as well as

$$q_p(a+p) \equiv q_p(a) - \frac{1}{a} \pmod{p}.$$

Denote $H_n = \sum_{a=1}^n 1/a$ to be the *n*-th harmonic number and $w_p = ((p-1)!+1)/p$ to be the Wilson quotient (also an integer by Wilson's Theorem). It is well known that $w_p \equiv \sum_{a=1}^{p-1} q_p(a) \pmod{p}$.

Lemma 4.1. Let p > 2 be a prime. For any $b \in (\mathbb{Z}/p^2\mathbb{Z})^{\times}$ such that $b = b_0 + b_1p$ with $1 \le b_0 \le p-1$ and $0 \le b_1 \le p-1$, we can write $b \equiv b_0^p \left(1 + \left(\frac{b_1}{b_0} - q_p(b_0)\right)p\right) \pmod{p^2}$.

Proof. Let $b \in (\mathbb{Z}/p^2\mathbb{Z})^{\times}$ such that $b = b_0 + b_1p$ with $1 \leq b_0 \leq p-1$ and $0 \leq b_1 \leq p-1$. Then setting $x = b_1/b_0$, we see that $1 + px \equiv (pq_p(b_0) + 1)(1 + px - pq_p(b_0)) \pmod{p^2}$. Since $b_0^{p-1} = 1 + pq_p(b_0)$, we obtain the result by multiplying through by b_0 .

Proposition 4.2. Suppose $m \in \mathbb{Z}$ with $m \ge 2$ and $p \equiv 1 \pmod{m}$ is a prime. Then

$$\left(\frac{p^2 - 1}{m}\right)_p^{p-1}! \equiv 1 \pmod{p^2} \iff \frac{1}{m}(w_p - H_{\frac{p-1}{m}}) - \sum_{a=1}^{\frac{p-1}{m}} q_p(a) \equiv 0 \pmod{p}.$$

Proof. Using Lemma 4.1, we have

$$\left(\frac{p^2-1}{m}\right)_p^{p-1}! = \prod_{\substack{a=1\\ \gcd(a,p)=1}}^{\frac{p^2-1}{m}} a^{p-1} = \left(\prod_{a=1}^{p-1} \prod_{b=0}^{\frac{p-1}{m}-1} (a+bp)^{p-1}\right) \left(\prod_{a=1}^{\frac{p-1}{m}} \left(a+\frac{p-1}{m}p\right)^{p-1}\right)$$

$$\equiv \left(\prod_{a=1}^{p-1} \prod_{b=0}^{\frac{p-1}{m}-1} \left(1+\left(\frac{b}{a}-q_p(a)\right)p\right)\right) \left(\prod_{a=1}^{\frac{p-1}{m}} \left(1+\left(\frac{\frac{p-1}{m}}{a}-q_p(a)\right)p\right)\right) \pmod{p^2}$$

$$\equiv \left(\prod_{a=1}^{p-1} \prod_{b=0}^{\frac{p-1}{m}-1} (1+p)^{\frac{b}{a}-q_p(a)}\right) \left(\prod_{a=1}^{\frac{p-1}{m}} (1+p)^{\frac{p-1}{a}-q_p(a)}\right) \pmod{p^2}.$$

Combining all the factors of (1+p) we get the desired sum in the exponent which is taken modulo p (since 1+p is a p-th root of unity modulo p^2). It is known that $H_{p-1} \equiv 0 \pmod{p}$. Hence, $\sum_{a=1}^{p-1} \sum_{b=0}^{\frac{p-1}{m}-1} \frac{b}{a} \equiv 0 \pmod{p}$. The result now follows.

Recall that the Bernoulli numbers $\{B_n\}$ and the Bernoulli polynomials $\{B_n(t)\}$ are defined by

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1} \quad \text{and} \quad \sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!} = \frac{xe^{xt}}{e^x - 1}$$

Lemma 4.3. Let p be a prime such that $p \equiv 1 \pmod{2m}$. Then

$$\left(\frac{\left(\frac{p^2-1}{m}\right)_p!}{\left(\frac{p^2-1}{2m}\right)_p^2!}\right)^{p-1} \equiv 1 \pmod{p^2} \iff \frac{B_p(1/m) - 2^p B_p(1/2m)}{p^2} \equiv 0 \pmod{p}.$$

Proof. For any $n \in \mathbb{Z}^+$ with $p \equiv 1 \pmod{n}$, we use the relation $B_p(x+1) - B_p(x) = px^{p-1}$ along with the properties of the Fermat quotient to obtain

$$\sum_{a=1}^{\frac{p-1}{n}} q_p(a) \equiv \left(\frac{n^{p-1}}{p} \left(\frac{p^2}{n} B_{p-1} - B_p(1/n)\right) - \frac{p-1}{n}\right) + \frac{1}{n} q_p(n) - \frac{1}{n} H_{\frac{p-1}{n}} \pmod{p}.$$

Then for $p \equiv 1 \pmod{2m}$, a straightforward computation gives

$$\sum_{a=1}^{\frac{p-1}{m}} q_p(a) - 2\sum_{a=1}^{\frac{p-1}{2m}} q_p(a) \equiv -\frac{B_p(1/m) - 2^p B_p(1/2m)}{p^2} - \frac{1}{m} H_{\frac{p-1}{m}} + \frac{1}{m} H_{\frac{p-1}{2m}} \pmod{p}$$

From Proposition 4.2 we know that $\left(\frac{\left(\frac{p^2-1}{m}\right)_p!}{\left(\frac{p^2-1}{2m}\right)_p^2!}\right)^{p-1} \equiv (1+p)^{\xi} \pmod{p^2}$, where

$$\xi = \frac{1}{m} (w_p - H_{\frac{p-1}{m}}) - \sum_{a=1}^{\frac{p-1}{m}} q_p(a) - 2\left(\frac{1}{2m} (w_p - H_{\frac{p-1}{2m}}) - \sum_{a=1}^{\frac{p-1}{2m}} q_p(a)\right)$$
$$\equiv -\frac{B_p(1/m) - 2^p B_p(1/2m)}{p^2} \pmod{p}.$$

The result now follows.

Corollary 1.2] is an immediate consequence of Lemma 4.3. We also have,

Proof of Corollary 1.3. Let $p \equiv 1 \pmod{4}$. We have seen from Lemma 4.3 that p is 1-exceptional for 4 if and only if $B_p(1/2) - 2^p B_p(1/4) \equiv 0 \pmod{p^3}$. But from [13] we know that $B_p(1/2) = 0$ and $B_p(1/4) = -pE_{p-1}/4^p$. Corollary 1.3 now follows from Theorem 1.1.

Remark 4.4. The proof also shows that $E_{p-1} \equiv 0 \pmod{p}$ when $p \equiv 1 \pmod{4}$, although this was already observed by Zhang in [19].

The proof of Corollary 1.4 will be similar to that of Corollary 1.3, but will instead involve the Glaisher numbers $\{G_n\}$. Since these numbers are less well known we will take a moment to view some of their properties. In particular, we will see that for odd $n \ge 1$, $B_n(1/3) = -(n+1)G_{n-1}/3^{n-1}$. Recall the Glaisher numbers $\{G_n\}$ are defined by

$$\frac{3/2}{e^x + e^{-x} + 1} = \sum_{n=0}^{\infty} G_n \frac{x^n}{n!}.$$

Notice that $2\sum_{n=0}^{\infty} G_{2n+1} \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} G_n \frac{x^n}{n!} - \sum_{n=0}^{\infty} G_n \frac{(-x)^n}{n!} = 0$ so that $G_n = 0$ whenever n is odd, and $\sum_{n=0}^{\infty} G_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} G_{2n} \frac{x^{2n}}{(2n)!}$. We also know from [8] that G_n can only have powers of 3 in the denominator.

Example 4.5. In the following table we list all primes $p \equiv 1 \pmod{3}$ and $7 \le p \le 193$ in the first column, along with the reduced values of $G_{p-1} \pmod{p}$ in the second column and $G_{p-1} \pmod{p^2}$ in the third column:

7	0	42	97	0	1940
13	0	0	103	0	1133
19	0	342	109	0	7521
31	0	434	127	0	16002
37	0	1332	139	0	5282
43	0	559	151	0	15855
61	0	3660	157	0	785
67	0	3685	163	0	24939
73	0	803	181	0	0
79	0	2844	193	0	26441

Notice that 13 and 181 are the first two 1-exceptional primes for m = 3. It also appears that $G_{p-1} \equiv 0 \pmod{p}$ for all $p \equiv 1 \pmod{3}$, which we will soon see is true.

We will now show that $B_n(1/3) = -(n+1)G_{n-1}/3^{n-1}$ for odd $n \ge 1$. It should be noted that this result is already known (see page 352 in [13]), but not commonly stated or proven in the literature. Observe that

$$\frac{-x}{e^{\frac{1}{3}x} + e^{-\frac{1}{3}x} + 1} = -\frac{2}{3}x\frac{3/2}{e^{\frac{1}{3}x} + e^{-\frac{1}{3}x} + 1} = -\frac{2}{3}x\sum_{n=0}^{\infty}G_{2n}\frac{\left(\frac{1}{3}x\right)^{2n}}{(2n)!}$$
$$= 2\sum_{n=0}^{\infty}-\frac{(2n+1)G_{2n}}{3^{2n+1}}\frac{x^{2n+1}}{(2n+1)!}$$

and at the same time

$$2\sum_{n=0}^{\infty} B_{2n+1}(1/3) \frac{x^{2n+1}}{(2n+1)!} = \frac{x(e^{\frac{1}{3}x} - e^{\frac{2}{3}x})}{e^x - 1} = \frac{xe^{\frac{1}{3}x}(1 - e^{\frac{1}{3}x})}{(e^{\frac{1}{3}x} - 1)(e^{\frac{2}{3}x} + e^{\frac{1}{3}x} + 1)} = \frac{-x}{e^{\frac{1}{3}x} + e^{-\frac{1}{3}x} + 1}$$

Therefore,

$$\sum_{n=0}^{\infty} -\frac{(2n+1)G_{2n}}{3^{2n+1}} \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} B_{2n+1}(1/3) \frac{x^{2n+1}}{(2n+1)!}$$

which implies,

$$B_{2n+1}(1/3) = -\frac{(2n+1)G_{2n}}{3^{2n+1}}$$

For $k, n \in \mathbb{Z}^+$, we also have Raabe's multiplication formula $B_n(kx) = k^{n-1} \sum_{j=0}^{n-1} B_n(x+j/k)$. So, with x = 1/6 and k = 2 we have

$$B_{2n+1}(1/6) = \frac{2^{2n} + 1}{2^{2n}} B_{2n+1}(1/3)$$

Proof of Corollary 1.4. Let $p \equiv 1 \pmod{3}$. Then from Lemma 4.3 p is 1-exceptional for m = 3 if and only if

$$\frac{B_p(1/3) - 2^p B_p(1/6)}{p^2} = -\frac{(1+2^p) B_p(1/3)}{p^2} = \left(\frac{1+2^p}{3^p}\right) \frac{G_{p-1}}{p} \equiv 0 \pmod{p}.$$

The result now follows from Theorem 1.1.

Remark 4.6. From the proof of Corollary 1.4 we also have that $G_{p-1} \equiv 0 \pmod{p}$ for all primes $p \equiv 1 \pmod{3}$.

5 Some further questions

Dummit, Ford, Kisilevsky and Sands conjecture in [4] that given a fixed imaginary quadratic field K, there are infinitely many primes such that $\lambda_p(K) > 1$. We can now restate this conjecture in the case of $K = \mathbb{Q}(i)$ and $K = \mathbb{Q}(\sqrt{-3})$ in a way that may be of interest to those who study Euler and Glaisher numbers, as well as Gauss factorials:

Conjecture 5.1. There are infinitely many primes $p \equiv 1 \pmod{3}$ such that $G_{p-1} \equiv 0 \pmod{p^2}$. Equivalently, there are infinitely many primes $p \equiv 1 \pmod{3}$ such that p is 1-exceptional for m = 3.

Conjecture 5.2. There are infinitely many primes $p \equiv 1 \pmod{4}$ such that $E_{p-1} \equiv 0 \pmod{p^2}$. Equivalently, there are infinitely many primes $p \equiv 1 \pmod{4}$ such that p is 1-exceptional for m = 4.

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