FUNCTORIALITY OF BOSE-MESNER ALGEBRAS AND PROFINITE ASSOCIATION SCHEMES

MAKOTO MATSUMOTO, KENTO OGAWA, AND TAKAYUKI OKUDA

ABSTRACT. We show that taking the set of primitive idempotents of commutative association schemes is a functor from the category of commutative association schemes with surjective morphisms to the category of finite sets with surjective partial functions. We then consider projective systems of commutative association schemes consisting of surjections (which we call profinite association schemes), for which Bose-Mesner algebra is defined, and describe a Delsarte theory on such schemes. This is another method for generalizing association schemes to those on infinite sets, related with the approach by Barg and Skriganov. Relation with (t, m, s)-nets and (t, s)-sequences is studied. We reprove some of the results of Martin-Stinson from this viewpoint.

1. INTRODUCTION

Association schemes are central objects in algebraic combinatorics, with many interactions with other areas of mathematics. It is natural to try to take projective limits of a system of association schemes, but there seems to be some obstruction to have a natural Bose-Mesner algebra. We show that a surjective morphism between commutative association schemes behaves well with respect to the two product structures of Bose-Mesner algebras and primitive idempotents, which gives rise to a notion of profinite association schemes and their Delsarte theory. They are closely related with profinite groups. We study kernel schemes and ordered Hamming schemes introduced by Martin-Stinson[16] as examples. We reprove their characterization of (t, m, s)-net as a design in an ordered Hamming scheme, and give a characterization of (t, s)-sequences in terms of profinite association schemes.

2. SURJECTIONS AMONG ASSOCIATION SCHEMES AND FUNCTORIALITY

Let $\mathbb{N}$ denote the set of natural numbers including 0, and $\mathbb{N}_{>0}$ the set of positive integers.

2.1. **Category of association schemes.** Let us recall the notion of association schemes briefly. See [1][6] for details. We summarize basic terminologies.

Definition 2.1. Let X be a set. By $\#X$ we denote the cardinality of X. If X has a topology, $C(X)$ denotes the vector space of continuous functions from X to $\mathbb{C}$. If a topology is not specified, X is regarded to be a discrete space. For $S \subset X$, we define the indicator function of S as a function $\chi_S : X \to \{0, 1\}$ where $\chi(x) = 1$ if and only if $x \in S$. If S is a singleton $\{x\}$, we denote the indicator function by δ_x. For a map $f : X \to Y$ between finite sets, $f^\dagger : C(Y) \to C(X)$ denotes the pull back: $g \in C(Y) \mapsto g \circ f$. It is injective (respectively surjective) if f is surjective (respectively injective). By the multiplication of functions, $C(X)$ is a unital commutative ring. This multiplication is called the Hadamard product. For a finite set X, the set $C(X \times X)$ is naturally identified with the set of complex valued square matrices of size $\#(X)$, and the matrix product is given by $AB(x, z) = \sum_{y \in X} A(x, y) B(y, z)$. The Hadamard product $\circ$ is given by the component wise product, namely, $(A \circ B)(x, z) = A(x, z) B(x, z)$. (Note that $\circ$ may denote the composition of mappings, but no confusions would occur.)

Definition 2.2. Let X, I be finite sets, and $R : X \times X \to I$ a surjection. We call (X, R, I) an association scheme, if the following properties are satisfied. For each $i \in I$, $R^{-1}(i)$ may be regarded as a relation on X, denoted by R_i. Let A_i be the corresponding adjacency matrix in $C(X \times X)$.

(1) There is an $i_0 \in I$ such that A_{i_0} is an identity matrix.
(2) Consider the injection $R^\dagger : C(I) \to C(X \times X)$. Its image A_X is closed by the matrix product.
(3) A_X is closed under the transpose of $C(X \times X)$.

The algebra A_X with the two multiplications (the Hadamard product and the matrix product) is called the Bose-Mesner algebra of (X, R, I). We may use the same notation i_0 for distinct association schemes. The number of 1 in a row of A_i is called the i-th valency and denoted by k_i. If A_X is commutative with respect to the matrix product, then (X, R, I) is said to be commutative.

In the following, we write simply an "association scheme X" for an association (X, R, I) by an abuse of language. We follow MacLane [15] for the terminologies of the category theory, in particular, functors, projective systems, and limits. The association schemes form a category, by the following[11][25].

Definition 2.3. Let (X, R, I), (X', R', I') be association schemes. A morphism of association schemes from X to X' is a pair of mappings $f : X \to X'$ and $g : I \to I'$ such that the following diagram commutes:

$$
\begin{array}{ccc}
X \times X & \to & I \\
f \times f \downarrow & \circlearrowleft & \downarrow g \\
X' \times X' & \to & I'.
\end{array}
\tag{2.1}
$$

It is clear that the surjectivity of f implies that of g. The morphism (f, g) is said to be surjective if f is surjective.

2.2. **Functoriality of Bose-Mesner algebra.**

Proposition 2.4. *If $(f, g) : X \to X'$ is a surjective morphism of association schemes, then for any $x' \in X'$, the cardinality of the fiber $\#f^{-1}(x')$ is $\#X/\#X'$.*

Proof. The cardinality of the fiber is the summation of the valencies of R_i for i with $g(i) = i_0'$, hence independent of the choice of x', which implies the result. This is

well-known [25], and it holds under a weaker condition (for unital regular relation partitions, see [12]). □

Theorem 2.5.

(1) Let $(f, g) : (X, R, I) \to (X', R', I')$ be a morphism. It induces an injection $C(I') \to C(I)$ and hence $\Psi : A_{X'} \to A_X$. Then Ψ is a morphism of unital rings with respect to the Hadamard product.

(2) Suppose that f is surjective. We define the convolution product $\bullet_X$ on A_X by

$$(A \bullet_X B)(x, z) = \frac{1}{\#X} \sum_{y \in X} A(x, y) B(y, z)$$

(i.e., the matrix product normalized by the factor of $\#X$). Then, A_X has a unit $E_X := \#X A_{i_0}$, and Ψ is a morphism of rings with respect to the convolution product (which may not map the unit of $A_{X'}$ to the unit of A_X).

(3) The vector space $C(X)$ is a left A_X-module by the following action:

$$A_X \times C(X) \to C(X), (A, h) \mapsto (A \bullet h)$$

given by

$$(A \bullet h)(x) = \frac{1}{\#X} \sum_{y \in X} A(x, y) h(y),$$

and the unit E_X acts trivially. This module structure is compatible with Ψ in the sense that the following diagram commutes:

$$
\begin{array}{ccc}
A_X \times C(X) & \to & C(X) \\
\Psi \times f^\dagger \uparrow & \circlearrowleft & \uparrow f^\dagger \\
A_{X'} \times C(X') & \to & C(X').
\end{array}
\tag{2.2}
$$

If we consider $C(X') \subset C(X)$, then the element $\Psi(E_{X'}) \in A_X$ acts on $C(X)$ as a projector $C(X) \to C(X')$.

Proof. For a morphism of finite sets $X \to Y$, the induced map $C(Y) \to C(X)$ is a morphism of unital rings with respect to the Hadamard product. The first statement follows from this and the commutative diagram (2.1). For the second part, take $A, B \in A'_X$. (In fact, one may take $A, B \in C(X' \times X')$; the following arguments depend only on the fact that the cardinality of the fiber of f is constant.)

Then

$$
\begin{aligned}
\Psi(A \bullet_{X'} B)(x,z) &= (A \bullet_{X'} B)(f(x), f(z)) \\
&= \frac{1}{\#X'} \sum_{y' \in X'} A(f(x), y') B(y', f(z)) \\
&= \frac{1}{\#X'} \frac{1}{\#f^{-1}(y')} \sum_{y \in X} A(f(x), f(y)) B(f(y), f(z)) \\
&= \frac{1}{\#X'} \frac{\#X'}{\#X} \sum_{y \in X} \Psi(A)(x,y)) \Psi(B)(y,z) \\
&= \frac{1}{\#X} \sum_{y \in X} \Psi(A)(x,y)) \Psi(B)(y,z) \\
&= (\Psi(A) \bullet_X \Psi(B))(x,z).
\end{aligned}
$$

For the third part, it is a routine calculation to check that A_X acts on $C(X)$ as a unital ring. For $A \in A_{X'}$ and $h \in C(X')$, the compatibility follows from:

$$
\begin{aligned}
f^\dagger((A \bullet h))(x) &= (A \bullet h)(f(x)) \\
&= \frac{1}{\#X'} \sum_{y' \in X'} A(f(x), y') h(y') \\
&= \frac{1}{\#X'} \frac{1}{\#f^{-1}(y')} \sum_{y \in f^{-1}(y'), y' \in X'} A(f(x), f(y)) h(f(y)) \\
&= \frac{1}{\#X'} \frac{\#X'}{\#X} \sum_{y \in Y} \Psi(A)(x,y)) (f^\dagger h)(y) \\
&= \frac{1}{\#X} \sum_{y \in X} \Psi(A)(x,y)) (f^\dagger h)(y) \\
&= (\Psi(A) \bullet (f^\dagger h))(x).
\end{aligned}
$$

The compatibility implies that $\Psi(E_{X'})$ trivially acts on the image of $C(X')$ in $C(X)$. For $A \in A_{X'}$ and $h \in C(X)$,

$$
\begin{aligned}
(\Psi(A) \bullet h))(x) &= \frac{1}{\#X} \sum_{y \in X} \Psi(A)(x,y) h(y) \\
&= \frac{1}{\#X} \sum_{y \in X} A(f(x), f(y)) h(y),
\end{aligned}
$$

which depends only on $f(x)$, and hence $\Psi(A) \bullet h$ lies in $f^\dagger(C(X')) \subset C(X)$. Hence $\Psi(E_{X'})$ is a projection $C(X) \to C(X')$. $\qquad\square$

Definition 2.6. Let $\mathrm{AS}_{\mathrm{surj}}$ be the category of association schemes with surjective morphisms.

Corollary 2.7. *Let* Alg_{HC} *(HC means Hadamard and convolution) be the category of finite dimensional $\mathbb{C}$-vector spaces A with:*

(1) one associative multiplication (Hadamard product) which gives a commutative unital semi-simple $\mathbb{C}$-algebra structure to A,

(2) one (possibly non-commutative) associative multiplication (convolution prod-
uct) which gives a unital semi-simple $\mathbb{C}$-algebra structure to A.

(Note that in this case, semi-simplicity is equivalent to that A is a direct product of
a finite number of matrix algebras over $\mathbb{C}$). Morphisms are injective $\mathbb{C}$-linear maps
preserving the both two products and the unit for the Hadamard product (we don't
require preservation of unit for the convolution product).

Then, the correspondence $X \mapsto A_X$ gives a contravariant functor from $\mathrm{AS}_{\mathrm{surj}}$ to
Alg_{HC}.

Proof. It is proved that A_X has two products. Semi-simplicity is well-known, c.f.
[25]. (It is enough to show that there is no nilpotent ideal, but for any nonzero
$A \in A_X$, the product with its unitary conjugate $A \bullet A^* = \frac{1}{\#X} A A^*$ is not nilpotent,
hence A_X is semi-simple.) Functoriality is easy to check, using Theorem 2.5. $\square$

Corollary 2.8. *Let us consider the category* Mod *of R-modules, whose object is a
pair of a unital commutative ring R and an R-module M, and a morphism from
(R, M) to (R', M') is a pair of a ring morphism $f : R \to R'$ and a $\mathbb{Z}$-module
morphism $g : M \to M'$ which makes the following diagram commute:*

$$\begin{array}{ccc} R \times M & \to & M \\ f \times g \downarrow & \circlearrowleft & \downarrow g \\ R' \times M' & \to & M'. \end{array}$$

*Then, the correspondence $X \mapsto (A_X, C(X))$ is a contravariant functor from $\mathrm{AS}_{\mathrm{surj}}$
to* Mod.

Proof. The correspondence is given in Theorem 2.5. It is a contravariant functor,
since each of the correspondences $X \mapsto A_X$, $X \mapsto C(X)$ is a contravariant functor.
$\square$

Remark 2.9. In [8], French constructed a sub-category of association schemes, and
a covariant functor from it to the category of algebra. Our construction produces
a contravariant functor, which seems to be of different nature.

2.3. **Commutative association schemes and primitive idempotents.** By
semisimplicity and Artin-Wedderburn Theorem, any Bose-Mesner algebra A_X (with
convolution product) is isomorphic to a product of matrix algebras over $\mathbb{C}$. We as-
sume that X is commutative. Then, A_X is, as a unital ring, isomorphic to a direct
product of copies of $\mathbb{C}$. Namely, $A_X \cong \mathbb{C} \times \cdots \times \mathbb{C} =: \mathbb{C}^n$ for some $n \in \mathbb{N}$, and this
decomposition is unique as a direct product of rings.

Proposition 2.10. *Let A be a ring isomorphic to $\mathbb{C}^n$ (with componentwise multi-
plication). An element $j \in A$ corresponding to an element of $\mathbb{C}^n$ with one coordinate
1 and the other coordinates 0 is called a primitive idempotent of A_X. It is char-
acterized by the idempotent property $j^2 = j$, $j \neq 0$ (this is equivalent to that the
each coordinate is 0 or 1 and at least one 1 exists) and that there is no nonzero
idempotent $j' \neq j$ such that $jj' = j'$ holds (this says only one coordinate is 1). Any
idempotent is uniquely a sum of primitive idempotents.*

Definition 2.11. Let FinSets be the category of finite sets. Let $\mathrm{FinSets}_{\mathrm{ps}}$ be the
category of finite sets and partial surjective functions.

Recall that for sets X and Y, a partial function f from X to Y consists of a pair
of a subset $\mathrm{dom}(f) \subset X$ (which may be empty) and a function $f|_{\mathrm{dom}(f)} : \mathrm{dom}(f) \to$

Y. Two partial functions are equal if they have the same domain and the same function on the domain. A partial function is denoted by $f : X \rightharpoonup Y$. For $S \subset Y$, $f^{-1}(S) := \{x \in X \mid x \in \mathrm{dom}(f), f(x) \in S\}$. Composition $g \circ f$ of partial functions $f : X \rightharpoonup Y$ and $g : Y \rightharpoonup Z$ has domain $f^{-1}(\mathrm{dom}(g))$. A partial function f is surjective if $\mathrm{Im}(f) := \{f(x) \mid x \in \mathrm{dom}(f)\} \subset Y$ is Y.

Proposition 2.12. *Let Alg_{cs} be the category of rings isomorphic to $\mathbb{C}^n$ for some n, with injective $\mathbb{C}$-linear ring morphisms which may not preserve the unit. For $A \in \mathrm{Alg}_{cs}$, we define $J(A)$ as the finite set of primitive idempotents of A. For a finite set I, we define $C(I)$ the set of maps from I to $\mathbb{C}$ as defined above. These are contravariant functors, and give contra-equivalence between Alg_{cs} and $\mathrm{FinSets}_{ps}$.*

Proof. Note first that the notion of $f : X \rightharpoonup Y$ is the notion of a family of disjoint subsets $S_y \subset X$ $(y \in Y)$, where $S_y = f^{-1}(y)$, and the surjectivity of f is equivalent to that $S_y \neq \emptyset$ for all $y \in Y$. To show that J is a functor, let $\varphi : A \to B$ be an injective $\mathbb{C}$-algebra homomorphism. For $j \in J(A)$, $\varphi(j)$ is a nonzero idempotent. Thus, it is a non-empty sum of elements of $J(B)$, which gives a non-empty subset S_j of $J(B)$ by Proposition 2.10. We construct a partial surjection $J(B) \rightharpoonup J(A)$ by: for $j \in J(A)$, elements of S_j is mapped to j. Note that if $j \neq j' \in J(A)$, then $jj' = 0$ and hence $S_j \cap S_{j'} = \emptyset$. Since $S_j \neq \emptyset$, the partial function is surjective. To check the functoriality, we consider $A \xrightarrow{\varphi} B \xrightarrow{\phi} C$. The construction of partial surjection for φ is done by assigning to $j \in J(A)$ the set $S_j \subset J(B)$ such that summation of elements in S_j is $\varphi(j)$. Similarly, for each $j' \in S_j$, we have $S_{j'} \subset J(C)$. Thus, $J(\varphi) \circ J(\phi)$ maps $\coprod_{j' \in S_j} S_{j'}$ to j. This is the same with $J(\phi \circ \varphi)$.

The functoriality of $I \to C(I)$ can be also checked in an elementary manner. It is easy to check that the two functors give contra-equivalence. We omit the detail. $\square$

Definition 2.13. Let AS_{csurj} be a full subcategory of the commutative association schemes of AS_{surj}, and Alg_{cHC} a full subcategory of Alg_{HC} consisting of algebras with commutative convolution multiplication. By the remark at the beginning of Section 2.3, we have a contravariant functor $\mathrm{AS}_{csurj} \to \mathrm{Alg}_{cHC}$.

Corollary 2.14. *The composite of the three functors $\mathrm{AS}_{csurj} \to \mathrm{Alg}_{cHC}$, $\mathrm{Alg}_{cHC} \to \mathrm{Alg}_{cs}$ forgetting the Hadamard product, and $\mathrm{Alg}_{cs} \to \mathrm{FinSets}_{ps}$ gives a covariant functor $\mathrm{AS}_{surj} \to \mathrm{FinSets}_{ps}$. We denote this composition functor by J. Thus, $(X, R, I) \mapsto J(X, R, I)$ is a functor, which corresponds X to the sets of primitive idempotents of A_X with respect to (the normalized) convolution products.*

Definition 2.15. Let j_0 denote the element of $J(X, R, I)$, corresponding to J_X, where J_X denotes the matrix in $C(X \times X)$ with components all 1. This element acts on $C(X)$ as a projection to the space of constant functions. We use the symbol j_0 for different association schemes, similarly to i_0.

2.4. **Trivial examples and remarks.** In the theory of commutative association schemes, the set I and the set J are considered to be "dual." Thus, it is natural to seek the relation between J's for a morphism of association schemes (while a morphism between I's is given by definition). The previous result says that there is a natural relation, if the morphism of association schemes is surjective. We couldn't generalize the result for general (non-surjective) morphisms.

For a surjective morphism between association schemes, the corresponding partial surjection may be a properly partial function, and may be not an injection, as follows.

Definition 2.16. Let G be a finite group. The triple (G, R, I) where $I = G$ and $R(x, y) = x^{-1}y$ is a (possibly non-commutative) association scheme called the thin scheme of G.

Remark 2.17. Let G be a finite abelian group, and $f : G \to H$ be a surjective homomorphism of abelian groups. Then we have the thin scheme of G and the thin scheme of H, and f is a surjective morphism of the association schemes. In this case, J for G is naturally isomorphic to the character group $G^\vee$, and one can check that the functor J gives $G^\vee \rightharpoonup H^\vee$, which comes from the natural inclusion $\iota : H^\vee \to G^\vee$ ($x \in G^\vee$ maps to $y \in H^\vee$ if $\iota(y) = x$, and x is outside the domain if there is no such y). This gives an example that $J(X) \rightharpoonup J(X')$ is not a map but a partial function.

Remark 2.18. For any association scheme (X, R, I), the mappings $f = \mathrm{id}_X$ and $g : I \to \{0, 1\}$, $g(i_0) = 0$ and $g(i) = 1$ if $i \neq i_0$, gives a morphism of association schemes $(X, R, I) \to (X, g \circ R, \{0, 1\})$. In this case, $J(X, g \circ R, \{0, 1\})$ consists of j_0 and $E_X - j_0$. If (X, R, I) is commutative, then the image of $j \in J(X, R, I)$ in $J(X, g \circ R, \{0, 1\})$ is j_0 if $j = j_0$, and $E_X - j_0$ if $j \neq j_0$. This gives an example that $J(X) \rightharpoonup J(X')$ is not injective.

3. Profinite association schemes

3.1. **Projective system of association schemes.**
We recall the notion of projective system.

Definition 3.1. (Projective system)

We work in a fixed category $\mathcal{C}$. Let Λ be a directed ordered set, namely, Λ is non-empty and for any $\alpha, \beta \in \Lambda$, there is a $\gamma \in \Lambda$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$. Let $X_\lambda (\lambda \in \Lambda)$ be a family of objects. For any $\lambda \leq \lambda'$, a morphism $p_{\lambda', \lambda} : X_{\lambda'} \to X_\lambda$ is specified, and they satisfy the commutativity condition

$$p_{\lambda', \lambda} p_{\lambda'', \lambda'} = p_{\lambda'', \lambda}$$

for any $\lambda \leq \lambda' \leq \lambda''$, and $p_{\lambda, \lambda} = \mathrm{id}_{X_\lambda}$. Then the system $(X_\lambda, p_{\lambda, \lambda'})$ is called a projective system in $\mathcal{C}$. The morphisms $p_{\lambda', \lambda}$ are called the structure morphisms of the projective system. The dual notion (i.e. the direction of morphism is inverted in the definition) is called an inductive system.

Let X_λ be a projective system of association schemes. Then, each of X_λ, I_λ is a projective system of finite sets. If all morphisms are surjective, then by Theorem 2.5, we have an inductive system of Bose-Mesner algebras A_{X_λ}, and if they are commutative, by Corollary 2.14 a projective system $J_{X_\lambda} := J(X_\lambda, R_\lambda, I_\lambda)$ (with structure morphisms being partially surjective maps).

Definition 3.2. Let Λ be a directed ordered set. A projective system $(X_\lambda, R_\lambda, I_\lambda)$ $(\lambda \in \Lambda)$ in $\mathrm{AS}_{\mathrm{surj}}$ is called a profinite association scheme. (This includes the notion of usual association scheme, as the case where $\#\Lambda = 1$.) We define profinite sets (see Lemma 3.4) $X^\wedge := \varprojlim X_\lambda$ and $I^\wedge := \varprojlim I_\lambda$ with projective limit topologies. A commutative profinite association scheme is a projective system of commutative association schemes. In this case, we define $J^\wedge$ similarly: an element $\hat{j} \in J^\wedge$ is the set of pair $\lambda \in \Lambda$ and compatible family of elements $j_\mu \in J_\mu$ for all $\mu \geq \lambda$, divided by the equivalence relation $(\lambda, j_\mu) \sim (\lambda', j'_{\mu'})$ defined by the existence of $\lambda'' \geq \lambda$, $\lambda'' \geq \lambda'$ such that for all $\mu \geq \lambda''$ $j_\mu = j'_\mu$ holds. An open basis of $J^\wedge$ is given by the inverse image of $j_\lambda \in J_\lambda$ through $J^\wedge \rightharpoonup J_\lambda$.

Proposition 3.3. *The natural projections $X^\wedge \to X_\lambda$ and $I^\wedge \to I_\lambda$ are surjective, the induced map $R^\wedge : X^\wedge \times X^\wedge \to I^\wedge$ is surjective, and $J^\wedge \rightharpoonup J_\lambda$ is a partial surjection.*

The first two statements are well-known for projective systems of compact Hausdorff spaces with continuous morphisms, [22, Proposition 1.1.10, Lemma 1.1.5]. We need to prove the third statement, for partial surjections. It follows from the first statement, but for the self-containedness and for explaining the natural (projective limit) topology, we give a proof for the both.

Lemma 3.4.

(1) *Let X_λ be a projective system of finite sets and surjections. Define $X^\wedge$ as $\varprojlim X_\lambda$. Then, the natural map $X^\wedge \to X_\lambda$ is surjective. The set $X^\wedge$ is called a profinite set, and has a natural topology, called profinite topology, which is compact Hausdorff. For every $x_\lambda \in X_\lambda$, its inverse image in $X^\wedge$ is clopen, and these form an open basis.*

(2) *Let J_λ be a projective system of finite sets and partial surjections. Define $J^\wedge$ as above. Then, the natural partial map $J^\wedge \rightharpoonup J_\lambda$ is a partial surjection. The set $J^\wedge$ has a natural topology, which is locally compact and Hausdorff. For every $j_\lambda \in J_\lambda$, its inverse image in $J^\wedge$ is clopen, and these form an open basis.*

(3) *Let X_λ, Y_λ be projective systems of finite sets over the same directed ordered set Λ, and let $f_\lambda : X_\lambda \to Y_\lambda$ be a family of maps which commute with the structure morphisms. This induces a continuous map*

$$f^\wedge : \varprojlim X_\lambda \to \varprojlim Y_\lambda.$$

If every f_λ is surjective, then $f^\wedge$ is surjective.

Proof. (1): Recall the construction of the projective limit $\varprojlim X_\lambda$ in the category of set. It is a subset of the direct product $\prod_{\lambda \in \Lambda} X_\lambda$ defined as the intersection of

$$S_{\mu',\mu} := \{(s_\lambda)_\lambda \in \prod_{\lambda \in \Lambda} X_\lambda \mid s_\mu = p_{\mu',\mu}(s_{\mu'})\}$$

for all $\mu', \mu \in \Lambda$ with $\mu' \geq \mu$. (This means that $(s_\lambda)_\lambda$ belongs to $\varprojlim X_\lambda$ if and only if they satisfy $p_{\mu',\mu}(s_{\mu'}) = s_\mu$ for every $\mu' \geq \mu$.) Each finite set is equipped with the discrete topology, which is compact and Hausdorff. By Tychonoff's theorem, the product is compact and Hausdorff. Because each component is Hausdorff, $S_{\mu',\mu}$ is a closed subset, and the intersection $\varprojlim X_\lambda$ is compact and Hausdorff. Take any $x_\delta \in X_\delta$. We shall show that this is in the image from $\varprojlim X_\lambda$. For each $\alpha \in \Lambda$, $\alpha \geq \delta$, we consider the closed set

$$T_\alpha := \{(s_\lambda)_\lambda \in \prod_{\lambda \in \Lambda} X_\lambda \mid s_\delta = x_\delta, s_\beta = p_{\alpha,\beta}(s_\alpha) \text{ for all } \beta \leq \alpha\}.$$

There is an element $y_\alpha \in X_\alpha$ such that $p_{\alpha,\delta}(y_\alpha) = x_\delta$, since p is surjective. Then, there is a system $p_{\alpha,\beta}(y_\alpha)$ which is an element of T_α, and thus T_α is nonempty. Now we take the intersection of T_α for all $\alpha \geq \delta$. For any finite number of α_i, we may take γ as an upper bound of all α_i. Then the finite intersection contains T_γ, which is nonempty. By compactness, the intersection of T_α for $\alpha \geq \delta$ is nonempty. Take an element $(g_\lambda)_\lambda$ in the intersection. It lies in $\varprojlim X_\lambda$, and it projects to x_δ via $\varprojlim X_\lambda \to X_\delta$. From the definition of the direct product topology, the inverse

image of $x_\lambda \in X_\lambda$ for various λ forms an open basis. They are clopen, since $x \in X_\lambda$ is clopen.

(2): Choose an element $\star$ which is contained in no J_λ's. Let $J_\lambda^\star$ be $J_\lambda \cup \{\star\}$. For a partial surjection $p : J_\lambda \rightharpoonup J_\mu$, we associate a surjection $p^\star : J_\lambda^\star \to J_\mu^\star$ as follows. For $x \in \mathrm{dom}(p)$, $p^\star(x) = p(x)$. For $x \notin \mathrm{dom}(p)$, $p^\star(x) = \star$. In particular, $p^\star(\star) = \star$. Let us take the projective limit as in the previous case to obtain $J^{\wedge^\star} := \varprojlim J_\lambda^\star$. Then for any λ, by (1), we have a surjection

$$p_\lambda^\star : J^{\wedge^\star} \to J_\lambda^\star.$$

It is clear that there is a unique element $\widehat{\star} \in J^{\wedge^\star}$ that is mapped to $\star$ for every $J_\lambda^\star$. We define $J^{\wedge'} := (\varprojlim J_\lambda^\star) \setminus \{\widehat{\star}\}$, and a partial surjection $p_\lambda : J^{\wedge'} \rightharpoonup J_\lambda$ by: for $x \in J^{\wedge'}$, x is outside of the domain of p_λ if $p_\lambda^\star(x) = \star$, and otherwise $p_\lambda(x) = p_\lambda^\star(x)$. It is easy to check that p_λ is a partial surjection (using the surjectivity of $p_\lambda^\star$), and p_λ for various λ is compatible with the structure morphisms $p_{\mu',\mu}$.

Since $J^{\wedge^\star}$ is compact Hausdorff, its open subset $J^{\wedge'}$ is locally compact Hausdorff, and has an open basis as in the case (1). We need to prove that $J^\wedge$ in Definition 3.2 is canonically isomorphic to $J^{\wedge'}$. Let $\tilde{J}$ be the set of pairs $(\lambda, (j_\mu)_\mu)$ where $(j_\mu)_\mu$ is a compatible system for $\mu \geq \lambda$. For such a pair, we assign an element of $J^{\wedge^\star}$ as follows. For any $\alpha \in \Lambda$, take any β with $\beta \geq \alpha$ and $\beta \geq \lambda$. Define j_α as $p_{\beta,\alpha}^\star(j_\mu)$. This is well-defined by the commutativity of the structure morphisms, and gives a map $f : \tilde{J} \to J^{\wedge'}$. This is surjective. Two pairs $(\lambda, (j_\mu)_\mu)$, $(\lambda', (j'_{\mu'})_{\mu'})$ have the same image if and only if there is a λ'' with $\lambda'' \geq \lambda$, $\lambda'' \geq \lambda'$ such that for any $\mu \geq \lambda''$, $j_\mu = j'_\mu$ holds. This gives a bijection $J^\wedge := \tilde{J}/\sim \to J^{\wedge'}$.

(3): See [22, Lemma 1.1.5]. $\square$

Remark 3.5. The above construction gives a category equivalence between the category of sets with partial maps and the category of sets with one base point. Using this, it is not difficult to show that $J^\wedge$ is the projective limit in the category of sets with partial surjections.

Later it will be proved that, for any commutative profinite association scheme, $J^\wedge$ has the discrete topology (Proposition 3.10).

Definition 3.6. For a profinite association scheme $(X_\lambda, R_\lambda, I_\lambda)$, its Bose-Mesner algebra is defined by $A_{X^\wedge} := \varinjlim A_{X_\lambda}$. For a commutative profinite association scheme, through the natural isomorphism $C(J_\lambda) \to A_{X_\lambda}$, we have an isomorphism $\varinjlim C(J_\lambda) \to A_{X^\wedge}$ which maps the point-wise product to the convolution product, and an isomorphism $\varinjlim C(I_\lambda) \to A_{X^\wedge}$ which maps the point-wise product to the Hadamard product.

Let $C(X^\wedge \times X^\wedge)$ denote the set of complex valued continuous functions on $X^\wedge \times X^\wedge$ (and similarly $C(I^\wedge)$ be the set of continuous functions on $I^\wedge$). Since we have inductive families of injections

$$A_{X_\lambda} \to C(X_\lambda \times X_\lambda) \to C(X^\wedge \times X^\wedge),$$

by universality, we have a canonical injective morphism $A_{X^\wedge} \to C(X^\wedge \times X^\wedge)$. similarly, we have an injection $\varinjlim C(I_\lambda) \to C(I^\wedge)$.

Definition 3.7. For $j_\lambda \in J_\lambda$, we denote by $E_{j_\lambda} \in A_{X_\lambda}$ the primitive idempotent j_λ (the notation introduced to distinguish an element of J with an element of A_{X_λ}). It is an element of $C(X_\lambda \times X_\lambda)$, and acts on $C(X_\lambda)$ by the normalized convolution defined in Theorem 2.5. Let $C(X_\lambda)_{j_\lambda}$ be the image of E_{j_λ} by this action.

Proposition 3.8. *Take $j_\lambda \in J_\lambda$. For $\mu \geq \lambda$, we may regard $C(X_\lambda) \subset C(X_\mu)$. Then $\Psi(E_{j_\lambda})$ given in Theorem 2.5 acts on $g \in C(X_\mu)$ by $\Psi(E_{j_\lambda}) \bullet g$. The operator $\Psi(E_{j_\lambda})$ gives a splitting $C(X_\mu) \to C(X_\lambda)_{j_\lambda}$ to $C(X_\lambda)_{j_\lambda} \subset C(X_\mu)$.*

Proof. By Theorem 2.5, $\Psi(E_{X_\lambda}) \in \mathrm{End}(C(X_\mu))$ is a projection to $C(X_\lambda)$. Then

$$\Psi(E_{j_\lambda}) = \Psi(E_{j_\lambda} \bullet E_{X_\lambda}) = \Psi(E_{j_\lambda}) \bullet \Psi(E_{X_\lambda})$$

is a projection $C(X_\mu) \to C(X_\lambda) \to C(X_\lambda)_{j_\lambda}$. $\qquad\square$

Definition 3.9. Let j be an element of $J^\wedge$. Then j is said to be isolated at λ to j_λ, if there is a $j_\lambda \in J_\lambda$ such that the inverse image of j_λ is the singleton $\{j\}$.

Proposition 3.10. *The cardinality of the inverse image of j_λ in J_μ for $\mu \geq \lambda$ does not exceed the dimension of $C(X_\lambda)_{j_\lambda}$, which is finite. Consequently, for each $j \in J^\wedge$, there exists a λ at which j is isolated. Thus, $J^\wedge$ has the discrete topology.*

Proof. Suppose that the inverse image of j_λ in J_μ is $\{j_1, j_2, \ldots, j_n\}$. Then the operator $\Psi(E_{j_\lambda}) = \sum_{i=1}^n \Psi(E_{j_i})$ on $C(X_\mu)$ is a projection to $C(X_\lambda)_{j_\lambda}$. Thus $C(X_\mu)_{j_i}$ $(i = 1, \ldots, n)$ are nonzero direct summands of $C(X_\lambda)_{j_\lambda}$, and hence n does not exceed the dimension of $C(X_\lambda)_{j_\lambda}$. This means that for any $\mu \geq \lambda$, the cardinality of the inverse image of j_λ is bounded, and hence there is a μ' such that the cardinality is constant for all $\mu \geq \mu'$. Let us fix $j \in J^\wedge$. Take any λ_0 such that j is in the domain of the partial map to J_{λ_0}. We put j_{λ_0} to the image of j. Then by the arguments above, we can find a λ such that the cardinality of the inverse image of j_{λ_0} in $J_{\lambda'}$ is constant for all $\lambda' \geq \lambda$. We put j_λ to the image of j in J_λ. Then j_λ is in the inverse image of j_{λ_0} and the cardinality of the inverse image of j_λ in $J_{\lambda'}$ should be one for any $\lambda' \geq \lambda$. This implies that the inverse image of j_λ in $J^\wedge$ is just $\{j\}$, namely, j is isolated at λ. We have proved that $\{j\}$ is a clopen subset of $J^\wedge$ for any $j \in J^\wedge$, and hence $J^\wedge$ has the discrete topology. $\qquad\square$

Corollary 3.11. *We have $\varinjlim C(J_\lambda) = \bigoplus_{j \in J^\wedge} \mathbb{C} \cdot j$. We write E_j for the image of $j \in J^\wedge$ in $A_{X^\wedge}$. Then $\bigoplus_{j \in J^\wedge} \mathbb{C} \cdot E_j = A_{X^\wedge}$. We may use the notation $C_c(J^\wedge)$ for $\varinjlim C(J_\lambda)$, since it is the set of functions (automatically continuous) on $J^\wedge$ with finite (=compact) support. Through the isomorphism $C_c(J^\wedge) \to A_{X^\wedge}$, $E_j \in A_{X^\wedge}$ corresponds to the indicator function $\delta_j \in C_c(J^\wedge)$.*

Remark 3.12. In the Pontryagin duality, the dual of a compact abelian group is a discrete group. Since $I^\wedge$ is profinite and compact, the above discreteness of $J^\wedge$ is an analogue to this fact.

We summarize the results of this subsection, to compare with the case of finite commutative association schemes.

Theorem 3.13. *Let X_λ be a commutative profinite association scheme. Notations are as in Definition 3.6 and Corollary 3.11. We have the following commutative diagram, where all the vertical arrows are injections:*

$$
\begin{array}{ccccc}
C(J^\wedge) & & C(X^\wedge \times X^\wedge) & \leftarrow & C(I^\wedge) \\
\uparrow & & \uparrow & & \uparrow \\
C_c(J^\wedge) \;=\; \varinjlim C(J_\lambda) & \simeq & A_{X^\wedge} & \simeq & \varinjlim C(I_\lambda) \;=\; C_{lc}(I^\wedge),
\end{array}
\tag{3.1}
$$

where:

(1) The set $J^\wedge$ is discrete. The image of the left vertical arrow is the set of compact support functions $C_c(J^\wedge)$ (i.e. takes 0 except but finite points in $J^\wedge$).

(2) The image of the right vertical arrow is the set of locally constant functions $C_{lc}(I^\wedge)$, which is dense with respect to the supremum norm, see the next lemma.

Lemma 3.14. *Let $X^\wedge$ be the projective limit of a projective system of finite sets X_λ for a directed ordered set Λ.*

(1) *Let C be a compact subspace of $X^\wedge$, and O_a $(a \in A)$ be an open covering of C. Then, there exists a $\mu \in \Lambda$ such that by putting S to the image of C in X_μ, the inverse image of $s \in S$ in $X^\wedge$ constitutes a (finite and disjoint) open covering of C, and each of these open sets is contained in one of O_a.*

(2) *The image of $\varinjlim C(X_\lambda) \to C(X^\wedge)$ is the set of locally constant functions on $X^\wedge$, and is dense with respect to the supremum norm.*

Proof. (1): For $x_\lambda \in X_\lambda$, we denote by B_{x_λ} the inverse image of x_λ in $X^\wedge$, which we call a λ-ball. This gives an open basis, and thus every O_a is a union of λ-balls for various λ. Since C is compact and covered by these balls, we may find a finite number of these balls which covers C and each contained in some O_a. We take an upper bound μ for these finite number of λ. Let S be the image of C in X_μ. Then, $B_{s_\mu}(s_\mu \in S)$ are finite, clopen, and mutually disjoint balls covering C, such that each ball is contained in some O_a.

(2) Let f be a locally constant function. Then, for any $x \in X^\wedge$, there is an open neighborhood O_x so that f is constant on O_x. These constitute an open covering of $X^\wedge$. Applying (1) for $C = X^\wedge$ to find μ with $S = X_\mu$, where each μ-ball is contained in some O_x, and hence f is constant on each ball. This means that f is an image of an element in $C(X_\mu)$. Conversely, any function in $C(X_\mu)$ clearly maps to a locally constant function on $X^\wedge$.

We shall show the density. For any $f \in C(X^\wedge)$ and $\epsilon > 0$, by continuity, for all $x \in X^\wedge$ we have an open neighborhood O_x such that $y \in O_x$ implies $|f(x) - f(y)| < \epsilon$. By applying (1) for $C = X^\wedge$ to find a μ with $S = X_\mu$. We see that the μ-balls cover $X^\wedge$, and for each $x_\mu \in X_\mu$, we define $g(x_\mu) = f(x')$ by choosing any x' in B_{x_μ}. Denote the projection by $\mathrm{pr} : X^\wedge \to X_\mu$. Then for any $y \in X^\wedge$, $g(\mathrm{pr}(y)) = f(x')$ for some $x' \in B_{\mathrm{pr}(y)}$. Since $B_{\mathrm{pr}(y)} \subset O_x$ for some x, the values of f in this ball differ at most by 2ϵ, namely,

$$|g(\mathrm{pr}(y)) - f(y)| = |f(x') - f(y)| < 2\epsilon.$$

This shows that

$$\|\mathrm{pr}^\dagger(g) - f\|_{\sup} < 2\epsilon,$$

which is the desired density. $\qquad\square$

3.2. **Inner product and orthogonality.**

Proposition 3.15. *For a profinite association scheme, $X^\wedge$ has a natural probability regular Borel measure μ, whose push-forward is the normalized (probability) counting measure on X_λ for each λ. If $X^\wedge$ is a profinite group G considered as a projective system of G/N_λ for (finite index) open normal subgroups, then the above μ coincides with the probability Haar measure.*

Proof. This is a direct consequence of Choksi [5] and a general property of measures (see Halmos [10]), as follows. Each X_λ is finite discrete (hence compact Hausdorff) with normalized counting measure, and the measure is preserved by the uniformness of the cardinality of each fiber as in Proposition 2.4. This means that the system is an inverse family of measured spaces [5, Definition 2], and hence $X^\wedge$ has a weak measure structure: a set M of ring of sets and a function μ from M to non-negative real numbers, such that $X^\wedge \to X_\lambda$ preserves the measure. In this case, M is the set of open sets, and $\mu(X^\wedge) = 1$. Then [5, Theorem 2.2] tells that this μ on $X^\wedge$ is extended to a measure on σ-ring generated by M, namely, the Borel sets. There it is also proved that for any measurable set E with finite measure and any $\epsilon > 0$, there is a compact set $C \subset E$ with $\mu(E) < \mu(C) + \epsilon$. This proves the inner regularity of μ ([10, §52, P.224]). By taking the complement, for any $\epsilon > 0$ and measurable set E^c, there is an open $U \supset E^c$ with $1 - \mu(E^c) < 1 - \mu(U) + \epsilon$, i.e., $\mu(E^c) > \mu(U) - \epsilon$, which proves the outer regularity of μ, and hence μ is regular.

Let ν be the probability Haar measure of G. Then, for any (finite index) open normal subgroup N, $\nu(N)$ is $\frac{1}{\#(G/N)}$, since $\#(G/N)$ cosets of N disjointly covers G. This gives an inverse family of measured spaces, and the above construction gives a regular Borel measure μ. The Haar measure is regular [10, §58, Theorem B]. For the coset gN, $\mu(gN) = \nu(gN)$. The cosets gN for g and N varying make an open basis. By the outer regularity, for any compact set E and any $\epsilon > 0$, there is an open set $O \supset E$ with $\mu(E) > \mu(O) - \epsilon$. Use Lemma 3.14 for $C = E$ and the open covering O, to find μ, $S \subset X_\mu$. Then $B_{s_\mu}(s_\mu \in S)$ constitutes an open cover $\mathcal{C}$ of E consisting of a finite number of disjoint cosets (contained in O) such that the sum of the measure of each coset in $\mathcal{C}$, denoted by $\mu(\mathcal{C})$ ($\mu(\mathcal{C}) \leq \mu(O)$), satisfying

$$\mu(\mathcal{C}) \geq \mu(E) \geq \mu(\mathcal{C}) - \epsilon, \tag{3.2}$$

but if the same argument simultaneously applied for both μ and ν for the O satisfying both $\mu(E) > \mu(O) - \epsilon$ and $\nu(E) > \nu(O) - \epsilon$, we may take the same $\mathcal{C}$ for the both and the same inequality (3.2) holds for ν, and because consisting of a finite number of disjoint cosets, $\nu(\mathcal{C}) = \mu(\mathcal{C})$ holds, which implies $\mu(E) = \nu(E)$. This means that the restrictions of ν and μ to compact sets give the same regular content, which implies $\nu = \mu$ by [10, §54, Theorem B]. □

Proposition 3.16. *For a finite set X, we define a (normalized) Hermitian inner product on $C(X)$ by*

$$(f, g) := \frac{1}{\#X} \sum_{x \in X} f(x)\overline{g(x)}.$$

For a profinite association scheme, it is defined on $C(X^\wedge)$ by

$$(f, g) := \int_{x \in X^\wedge} f(x)\overline{g(x)} d\mu(x).$$

Then $C(X_\lambda) \to C(X^\wedge)$ preserves the inner products.

This is clear since $X^\wedge \to X_\lambda$ preserves the measures.

Definition 3.17. Let V, W be $\mathbb{C}$-vector spaces with Hermitian inner products. For a linear morphism $A : V \to W$, its conjugate $A^* : W \to V$ (which may not exists) is characterized by the property

$$(Af, g)_W = (f, A^*g)_V$$

for any $f \in V$, $g \in W$. It is unique if exists. An endomorphism $A : W \to W$ is said to be Hermitian if $A = A^*$, as usual.

Let W be again a $\mathbb{C}$-vector space with Hermitian inner product, and V is a subspace with restricted inner product. Let $V^\perp$ be the space of vectors which is orthogonal to any vector in V. If $V \oplus V^\perp$ is equal to W, then V is said to be an orthogonal component of W.

Definition 3.18. Let W be a $\mathbb{C}$-linear space with Hermitian inner product. For a linear subspace $V \subset W$, let $p : W \to W$ be a projection to V, i.e., the image of p is V and $p|_V = \mathrm{id}_V$. Then by linear algebra W is decomposed as the direct sum, $W = V \oplus \mathrm{Ker}\, p$. We say that p is an orthogonal projector to V if $V \oplus \mathrm{Ker}\, p$ is an orthogonal direct sum. An orthogonal projector to V exists uniquely, if V is an orthogonal component.

Proposition 3.19. *Let W be a $\mathbb{C}$-linear space with Hermitian inner product, $V \subset W$ be an orthogonal component with orthogonal projector p, and $\iota : V \hookrightarrow W$ be the inclusion. We denote by $q : W \to V$ the restriction of the codomain of p. Then we have $\iota^* = q$ and $q^* = \iota$. For $A : V \to V$, if A^* exists, then*

$$(\iota \circ A \circ q)^* = \iota \circ A^* \circ q : V \to V. \tag{3.3}$$

We sometimes denote $\iota \circ A \circ q$ simply by A. We make a remark when this abuse of notation is used.

Proof. For $w \in W$ and $v \in V$, write $w = w_V + w'_V$ with $w_V \in V$ and $w'_V \in V^\perp$. Then

$$(qw, v)_V = (w_V, v)_V = (\iota w_V, \iota v)_W = (w_V + w'_V, \iota v)_W = (w, \iota v)_W,$$

which shows that $\iota^* = q$ and $q^* = \iota$. Thus (3.3) follows. $\square$

Proposition 3.20. *Let W be a $\mathbb{C}$-vector space with Hermitian inner product, and $p : W \to W$ an idempotent, namely, $p^2 = p$. By linear algebra, $W = \mathrm{Im}(p) \oplus \mathrm{Ker}(p)$ and p is a projector to $V := \mathrm{Im}(p)$. Then, p is an orthogonal projector if and only if $p = p^*$. If $A \in \mathrm{End}(V)$ is an orthogonal projector to $U \subset V$, then $\iota A q : W \to W$ is a projector to $U \subset W$.*

Proof. If $p^2 = p$, then $W = V \oplus \mathrm{Ker}\, p$ for $V = \mathrm{Im}(p)$. Thus any element $w, w' \in W$ is written as $v + k, v' + k'$, respectively. If $V \oplus \mathrm{Ker}\, p$ is orthogonal,

$$(p(v + k), v' + k') = (pv, v' + k') = (v, v') = (v + k, pv') = (v + k, p(v' + k')),$$

which shows $p = p^*$. For the converse, assume $p = p^*$. Then for $v \in V$, $k \in \mathrm{Ker}\, p$, $(v, k) = (pv, k) = (v, pk) = (v, 0) = 0$, and hence W and $\mathrm{Ker}\, p$ are mutually orthogonal. For the last, (3.3) implies that $\iota A q$ is an Hermitian operator, and by $q \circ \iota = \mathrm{id}_V$ is an idempotent, whose image is U, hence is an orthogonal projector to U. $\square$

Definition 3.21. Let (X_λ) be a profinite association scheme, fix $\lambda \in \Lambda$ and $x_\lambda \in X_\lambda$. For the surjection $\mathrm{pr} : X^\wedge \to X_\lambda$, the fiber $\mathrm{pr}^{-1}(x_\lambda)$ at x_λ is a clopen subset of $X^\wedge$. Thus, we have the restriction of the measure μ on $X^\wedge$ to $\mathrm{pr}^{-1}(x_\lambda)$, denoted by the same symbol $(\mathrm{pr}^{-1}(x_\lambda), \mu)$. Note that the volume of the fiber $\mathrm{pr}^{-1}(x_\lambda)$ is $1/\#X_\lambda$.

Since $X^\wedge = \coprod_{x_\lambda \in X_\lambda} \mathrm{pr}^{-1}(x_\lambda)$, the following proposition holds.

Proposition 3.22. *For any λ and $f \in C(X^\wedge)$, we have*

$$\int_{x \in X^\wedge} f(x)d\mu(x) = \sum_{x_\lambda \in X_\lambda} \int_{y \in \mathrm{pr}^{-1}(x_\lambda)} f(y)d\mu(y).$$

Proposition 3.23. *Let (X_λ) be a profinite association scheme. Then, by the surjection $\mathrm{pr} : X^\wedge \to X_\lambda$, $C(X_\lambda)$ is identified with a subspace of $C(X^\wedge)$ with the inner product given by restriction. There is an orthogonal projection $q : C(X^\wedge) \to C(X_\lambda)$ given by averaging over each fiber: for $f \in C(X^\wedge)$ define*

$$q(f)(x_\lambda) = \frac{1}{\mu(\mathrm{pr}^{-1}(x_\lambda))} \int_{x \in \mathrm{pr}^{-1}(x_\lambda)} f(x)d\mu(x),$$

where $\mu(\mathrm{pr}^{-1}(x_\lambda)) = \frac{1}{\#X_\lambda}$. Thus, $C(X_\lambda)$ is an orthogonal component of $C(X^\wedge)$.

Proof. The preservation of the inner products is shown in Proposition 3.16. For any $g \in C(X_\lambda)$ its image is $g \circ \mathrm{pr} \in C(X^\wedge)$ and we have

$$
\begin{aligned}
q(g \circ \mathrm{pr})(x_\lambda) &= \frac{1}{\mu(\mathrm{pr}^{-1}(x_\lambda))} \int_{y \in \mathrm{pr}^{-1}(x_\lambda)} g(\mathrm{pr}(y))d\mu(y) \\
&= g(x_\lambda)\frac{1}{\mu(\mathrm{pr}^{-1}(x_\lambda))} \int_{y \in \mathrm{pr}^{-1}(x_\lambda)} 1 \cdot d\mu(y) \\
&= g(x_\lambda).
\end{aligned}
$$

This implies that q is a projection to $C(X_\lambda)$.

To show that q is an orthogonal projection, we take $f \in \mathrm{Ker}\, q$. This means that the average of f on each fiber of pr is zero. To show that q is orthogonal, it suffices to show that $(f, g \circ \mathrm{pr})_{X^\wedge} = 0$ for $g \in C(X_\lambda)$ by Definition 3.18, but by applying Proposition 3.22, we have

$$
\begin{aligned}
(f, g \circ \mathrm{pr})_{X^\wedge} &= \int_{x \in X^\wedge} f(x)\overline{g(\mathrm{pr}(x))}d\mu(x) \\
&= \sum_{x' \in X_\lambda} \int_{x \in \mathrm{pr}^{-1}(x')} f(x)\overline{g(\mathrm{pr}(x))}d\mu(x) \\
&= \sum_{x' \in X_\lambda} \overline{g(x')} \int_{x \in \mathrm{pr}^{-1}(x')} f(x)d\mu(x) \\
&= \sum_{x' \in X_\lambda} \overline{g(x')} \cdot 0 = 0.
\end{aligned}
$$

$\square$

Proposition 3.24.
  (1) *The (convolution) algebra $A_{X^\wedge}$ acts on $C(X^\wedge)$, which makes $C(X^\wedge)$ an $A_{X^\wedge}$-module. This is compatible with the action of A_{X_λ} on $C(X_\lambda)$, in the sense of Corollary 2.8.*
  (2) *The convolution unit $E_{X_\lambda} \in A_{X_\lambda} \subset A_{X^\wedge}$ acts as an orthogonal projector to $C(X_\lambda)$.*
  (3) *Let $j \in J^\wedge$. Take a λ where j is isolated to j_λ, and consider $E_{j_\lambda} \in A_{X_\lambda}$. By the abuse of notation in Proposition 3.19, E_{j_λ} extends to the orthogonal projector*

$$E_j : C(X^\wedge) \to C(X^\wedge)$$

that projects to the domain of the composition

$$C(X_\lambda)_{j_\lambda} \to C(X_\lambda) \to C(X^\wedge).$$

We define

$$C(X^\wedge)_j := C(X_\lambda)_{j_\lambda}.$$

This is independent of the choice of the λ.

(4) *For $j \neq j'$, $E_j \bullet E_{j'} = 0$.*

(5) *We have an orthogonal decomposition*

$$C(X_\lambda) = \bigoplus_{j \in J'} C(X^\wedge)_j$$

for the domain J' of the partial map from $J^\wedge$ to J_λ. Note that J' is finite.

Proof. The action is given by: $A \in A_{X_\lambda}$ acts on $f \in C(X^\wedge)$ by

$$(A \bullet f)(x) := \int_{y \in X^\wedge} A(\mathrm{pr}_\lambda(x), \mathrm{pr}_\lambda(y)) f(y) d\mu(y).$$

By Proposition 3.22, this is the same with

$$\sum_{y' \in X_\lambda} A(\mathrm{pr}_\lambda(x), y') \left(\int_{y \in \mathrm{pr}_\lambda^{-1}(y')} f(y) d\mu(y) \right). \tag{3.4}$$

Comparing with q in Proposition 3.23, we have

$$
\begin{aligned}
(A \bullet_{X^\wedge} f)(x) &= \sum_{y' \in X_\lambda} A(\mathrm{pr}_\lambda(x), y') \left(\int_{y \in \mathrm{pr}_\lambda^{-1}(y')} f(y) d\mu(y) \right) \\
&= \sum_{y' \in X_\lambda} A(\mathrm{pr}_\lambda(x), y') \frac{1}{\#X_\lambda} q(f)(y') \\
&= (A \bullet_{X_\lambda} q(f))(\mathrm{pr}_\lambda(x)),
\end{aligned}
$$

which shows that

$$A = \iota A q, \tag{3.5}$$

where $\iota : C(X_\lambda) \to C(X^\wedge)$ is the injection $\mathrm{pr}_\lambda^\dagger$ and A in the right hand side is an endomorphism of $C(X_\lambda)$. This and $q \circ \iota = \mathrm{id}_{C(X_\lambda)}$ show that $C(X^\wedge)$ is an $A_{X^\wedge}$-module. The well-definedness (independence of the choice of λ) follows by taking a sufficiently large μ.

By (3.5) and $E_{X_\lambda} = \mathrm{id}_{C(X_\lambda)}$ on $C(X_\lambda)$ show that

$$E_{X_\lambda} = \iota \circ q,$$

which is the orthogonal projection to $C(X_\lambda)$.

For $j \in J$, take a λ where j is isolated to $j_\lambda \in J_\lambda$. By (3.5) we have

$$E_{j_\lambda} = \iota E_{j_\lambda} q,$$

where E_{j_λ} in the right hand side is an endomorphism of $C(X_\lambda)$. Hence to prove that E_{j_λ} is an orthogonal projector $C(X^\wedge) \to C(X_\lambda)_{j_\lambda}$, it suffices to show that $E_{j_\lambda} \in A_{X_\lambda}$ is an orthogonal projector $C(X_\lambda) \to C(X_\lambda)_{j_\lambda}$ by Proposition 3.20. The orthogonality of $E_{j_\lambda} \in A_{X_\lambda}$ is well known [1]. (Since A_{X_λ} is closed under the anti-isomorphism $*$ of ring, $E_{j_\lambda}^*$ must be a primitive idempotent, and since $E_{j_\lambda} E_{j_\lambda}^* \neq 0$, they must coincide, hence E_{j_λ} is Hermitian and an orthogonal projection by Proposition 3.20.) We define $E_j := E_{j_\lambda}$, where

$$E_j : C(X^\wedge) \to C(X^\wedge)_j := C(X_\lambda)_{j_\lambda}.$$

For the well-definedness, we take λ, λ' where j is isolated. There exists a μ with $\mu \geq \lambda$, $\mu \geq \lambda'$. Then j is isolated to j_μ at μ, and by (3.5) it is not difficult to see that the E_j defined using μ is the same as those defined using λ, λ'. For $E_j \bullet E_{j'} = 0$, take a λ where both j, j' are isolated to j_λ, j'_λ, respectively. Then, $j_\lambda \neq j'_\lambda$, $E_{j_\lambda}, E_{j'_\lambda} \in A_{X_\lambda}$, and $E_{j_\lambda} \bullet E_{j'_\lambda} = 0$. Thus, the relation holds in the inductive limit $A_{X^\wedge}$.

For fixed λ, it is well-known [1] that we have an orthogonal decomposition

$$C(X_\lambda) = \bigoplus_{j_\lambda \in J_\lambda} C(X_\lambda)_{j_\lambda}.$$

(This follows from the orthogonality of E_{j_λ}.) Since each $C(X_\lambda)_{j_\lambda}$ is an orthogonal direct sum of $C(X^\wedge)_j$ over the finite number of $j \in J^\wedge$ whose image is j_λ, the claim follows. $\qquad\square$

The following is an analogue to Peter-Weyl Theorem.

Theorem 3.25. *We have an orthogonal direct decomposition*

$$\varinjlim C(X_\lambda) = \bigoplus_{j \in J} C(X^\wedge)_j.$$

Hence the right hand side is dense in $C(X^\wedge)$ with respect to the supremum norm.

Proof. Proposition 3.24 implies that $C(X^\wedge)_j$ are mutually orthogonal, and their orthogonal sum is a subspace of $C(X^\wedge)$. For any λ, $C(X_\lambda)$ is an orthogonal sum of a finite number of $C(X^\wedge)_j$. Thus $\varinjlim C(X_\lambda)$ is the direct sum of $C(X^\wedge)_j$ for $j \in J^\wedge$. The density follows from Lemma 3.14. $\qquad\square$

3.3. **Measures and Fourier analysis.** Since $X^\wedge$ has a natural probability measure, $I^\wedge$ has a pushforward measure via $X^\wedge \times X^\wedge \to I^\wedge$. It will be shown that $J^\wedge$ has a natural measure, and these measures are closely related to the classical notions of the multiplicity m_{j_λ} for J_λ, and the valency k_{i_λ} for I_λ (see [1][6]).

Definition 3.26. Let $\mu_{I^\wedge}$ be the pushforward measure on $I^\wedge$ from $X^\wedge \times X^\wedge$. This is characterized as follows. For $i_\lambda \in I_\lambda$, let $\mathrm{pr}_{I_\lambda}^{-1}(i) \subset I^\wedge$ be the open set. Then

$$\mu_{I^\wedge}(\mathrm{pr}_{I_\lambda}^{-1}(i_\lambda)) = \mu_{X^\wedge \times X^\wedge}(R_\lambda \circ \mathrm{pr}_{X_\lambda \times X_\lambda})^{-1}(i_\lambda).$$

Proposition 3.27. *In the above definition,*

$$\mu_{I^\wedge}(\mathrm{pr}_{I_\lambda}^{-1}(i_\lambda)) = \frac{k_{i_\lambda}}{\#X_\lambda}$$

holds, where k_{i_λ} is the valency of the adjacency matrix $A_{i_\lambda} \in A_{X_\lambda}$.

Proof. Recall that A_{i_λ} is the indicator function of $R_\lambda^{-1}(i_\lambda)$, and the sum of the rows is an integer k_{i_λ} called the valency [1]. Thus, the volume $\mu_{I^\wedge}(\mathrm{pr}_{I_\lambda}^{-1}(i_\lambda))$ is the volume of $\mu_{X^\wedge}(\mathrm{pr}_{X_\lambda \times X_\lambda})^{-1} R_\lambda^{-1}(i_\lambda)$, which is

$$
\begin{aligned}
\int_{x,y \in X^\wedge \times X^\wedge} A_{i_\lambda}(\mathrm{pr}_{X_\lambda}(x), \mathrm{pr}_{X_\lambda}(y)) d\mu(x) d\mu(y) &= \frac{1}{\#X_\lambda^2} \sum_{x',y' \in X_\lambda \times X_\lambda} A_{i_\lambda}(x', y') \\
&= \frac{k_{i_\lambda}}{\#X_\lambda}.
\end{aligned}
$$

$\square$

Definition 3.28. The algebra A_{X_λ} is closed under the adjoint *. We define the Hilbert-Schmidt inner product, which is Hermitian, as follows. For $A, B \in A_{X_\lambda}$,

$$(A, B)_{\mathrm{HS}} := \sum_{x \in X_\lambda} (A \bullet B^*)(x, x).$$

For $A, B \in A_{X^\wedge}$, we define its Hilbert-Schmidt inner product by

$$(A, B)_{\mathrm{HS}} := \int_{x \in X^\wedge} (A \bullet B^*)(x, x) d\mu(x).$$

Then inclusion $A_{X_\lambda} \to A_{X^\wedge}$ preserves the inner products.

Proof. For $B \in A_{X_\lambda}$, $B^* \in A_{X_\lambda}$ follows because of the definition of association schemes, and the Hilbert-Schmidt inner product is defined. For $B \in A_{X^\wedge}$, we must show that B^* exists in $A_{X^\wedge}$ and the integration converges. However, $B \in A_{X_\lambda}$ for some λ, and $B^* \in A_{X_\lambda}$. Since $C(X_\lambda)$ is an orthogonal component of $C(X^\wedge)$, $B^* = \iota B^* q$ is the adjoint as an operator on $C(X^\wedge)$, by the abuse of notation in Proposition 3.19. The notation $(A \bullet B^*)(x, x)$ comes from the middle vertical injection in (3.1). Since $A, B^* \in A_{X_\lambda} \subset C(X_\lambda \times X_\lambda)$,

$$\int_{x \in X^\wedge} (A \bullet B^*)(x, x) d\mu(x) = \sum_{x' \in X_\lambda} (A \bullet B^*)(x', x'),$$

which is finite, satisfying the axioms of Hermitian inner product, and compatible with $A_{X_\lambda} \to A_{X^\wedge}$. $\qquad\square$

Proposition 3.29. *Through the natural isomorphism*

$$\varinjlim C(I_\lambda) \to A_{X^\wedge},$$

we obtain Hermitian inner product on $\varinjlim C(I_\lambda)$, which is denoted by $(-, -)_{\mathrm{HS}}$. Then for $f, g \in \varinjlim C(I_\lambda)$, we have

$$(f, g)_{\mathrm{HS}} = \int_{i \in I^\wedge} f(i) \overline{g(i)} d\mu_{I^\wedge}(i).$$

Proof. We may assume $f, g \in C(I_\lambda)$ for some λ. Then by bilinearity it suffices to check the equality for $f = \delta_{i_\lambda}$, $g = \delta_{i'_\lambda}$. Then, the right hand side is $\delta_{i_\lambda, i'_\lambda} k_{i_\lambda} / \# X_\lambda$ by Proposition 3.27. In $A_{X^\wedge}$, these functions correspond to $f \mapsto A_{i_\lambda}, g \mapsto A_{i'_\lambda}$, and their inner product is

$$\int_{x,y \in X^\wedge \times X^\wedge} A_{i_\lambda}(\mathrm{pr}_{X_\lambda}(x), \mathrm{pr}_{X_\lambda}(y)) A_{i'_\lambda}(\mathrm{pr}_{X_\lambda}(y), \mathrm{pr}_{X_\lambda}(x))^* d\mu(x) d\mu(y)$$

$$= \frac{1}{\# X_\lambda^2} \sum_{x',y' \in X_\lambda \times X_\lambda} A_{i_\lambda}(x', y') A_{i'_\lambda}(x', y')$$

$$= \delta_{i_\lambda, i'_\lambda} \frac{k_{i_\lambda}}{\# X_\lambda}.$$

This uses a well-known orthogonality of A_{i_λ}: for any $x \in X_\lambda$ and $i_1, i_2 \in I_\lambda$, the (x, x)-component of the matrix product $A_{i_1} \cdot^t A_{i_2}$ is the number of $y \in X_\lambda$ with $R(x, y) = i_1$ and $R(x, y) = i_2$, which is zero unless $i_1 = i_2$, and in the equal case k_{i_1}. $\qquad\square$

Lemma 3.30. *For a commutative association scheme (X, R, I), we defined J as the set of primitive idempotents E_j with respect to the product $\bullet$ in Theorem 2.5. We put a positive measure m_j for $j \in J$, so that the inner product with respect to m_j in $C(J)$ is isometric to that of A_X with respect to HS inner product. Then, m_j is the multiplicity of j, defined as $\dim(C(X)_j)$ [1].*

Proof. Let $E_j, E_{j'} \in A_X$ be two primitive idempotents. The computation

$$(E_j, E_{j'})_{\text{HS}} = \sum_{x \in X} E_j \bullet E_{j'}^*(x, x) = \delta_{jj'} \text{tr}(E_j) = \delta_{jj'} m_j$$

implies that if we define $\mu_J(j) := m_j$, then the corresponding inner product is compatible with $C(J) \to A_X$ with respect to the HS-inner product on A. $\qquad\square$

Proposition 3.31. *For a projective association scheme $(X_\lambda, R_\lambda, I_\lambda)$, we define a measure $\mu_{J^\wedge}$ on the discrete set $J^\wedge$ by $\mu_{J^\wedge}(j) = \dim C(X^\wedge)_j$. Then, the inner product on $\varinjlim C(J_\lambda)$ defined by*

$$(f, g)_{\text{HS}} = \sum_{j \in J^\wedge} f(j)\overline{g(j)}\mu_{J^\wedge}(j)$$

makes $\varinjlim C(J_\lambda) \cong A_{X^\wedge}$ isometric with respect to HS inner product.

Proof. Any $f, g \in \varinjlim C(J_\lambda)$ are contained in some $C(J_\lambda)$, where any j in the support of f or g is isolated at λ. The functions f, g are linear combinations of orthogonal basis E_j of $C(J_\lambda)$, namely, $f = \sum_j f(j)E_j$ and $g = \sum_j g(j)E_j$ in A_{X_λ}. The previous lemma says that $(E_j, E_{j'})_{\text{HS}} = \delta_{jj'} \dim C(X^\wedge)_j$, which proves the proposition. $\qquad\square$

In sum, we have the following form of Fourier transform.

Theorem 3.32. *The isomorphism of vector spaces $\varinjlim C(J_\lambda) \cong A_{X^\wedge} \cong \varinjlim C(I_\lambda)$ is an isometry between the space of compact support functions $C_c(J^\wedge)$ with inner product defined in Proposition 3.31 and the space of locally constant functions $C_{lc}(I^\wedge)$ with inner product defined in Proposition 3.29, which maps the componentwise product in $C_c(J^\wedge)$ to the convolution product in $C_{lc}(I^\wedge)$.*

Remark 3.33. For a finite association scheme (X, R, I), both E_j $(j \in J)$ and A_i $(i \in I)$ are orthogonal bases of A_X with respect to HS-inner product. The isometry

$$C(I) \to C(J)$$

is given by

$$A_i \quad \mapsto \quad \sum_{j \in J} p_i(j)E_j.$$

The isometry shows that

$$(A_i, A_{i'})_{\text{HS}} = \delta_{ii'}\mu_I(i) = \delta_{ii'}\frac{k_i}{\#X}$$

is equal to

$$\sum_{j \in J} p_i(j)\overline{p_{i'}(j)}m_j,$$

which is one of the two orthogonalities of eigenmatrices as classically known [1]. (Here the factor $\#X$ appears in a different manner because of the normalization in

Theorem 2.5. Our E_j is $\#X$ times the classical primitive idempotent, which makes $p_i(j) = \frac{1}{\#X}P_i(j)$ where $P_i(j)$ are the components of the standard eigen matrices.)

The next is an analogue of the Fourier transform for L^2 spaces.

Proposition 3.34. *The isometry $\varinjlim C(J_\lambda) \to \varinjlim C(I_\lambda)$ extends to a unique isometry*

$$L^2(J^\wedge, \mu_{J^\wedge}) \to L^2(I^\wedge, \mu_{I^\wedge}).$$

Proof. We consider L^2-norm for all the spaces. The map $\varinjlim C(J_\lambda) \to L^2(J^\wedge)$ is isometric, injective, and has a dense image, by considering the orthogonal basis E_j. Since $L^2(J^\wedge)$ is complete, this is a completion of $\varinjlim C(J_\lambda)$. The map $\varinjlim C(I_\lambda) \to L^2(I^\wedge)$ is also isometric, hence injective, and has a dense image (by Lemma 3.14 it is dense with respect to supremum norm in $C(I^\wedge)$, and since $I^\wedge$ is compact, dense with respect to the L^2-norm in $C(I^\wedge)$, and thus dense in $L^2(I^\wedge)$), so $L^2(I^\wedge)$ is a completion of $\varinjlim C(I_\lambda)$. By the universality of the completion, they are isometric. $\square$

4. Some examples and relation with Barg-Skriganov theory

The results of this paper are closely related to the study by Barg and Skriganov [2]. They defined the notion of association schemes on a set X with a σ-additive measure. See [2, Definition 1], where I is denoted by Υ. Association scheme structure is given by a surjection

$$R : X \times X \to I,$$

with axioms similar to the finite case. The inverse image $R^{-1}(i) \subset X \times X$ of $i \in I$ is required to be measurable in $X \times X$, and intersection numbers p_{ij}^k are defined as the measure of the set

$$\{z \in X \mid R(x,z) = i, R(z,y) = j\}$$

where $R(x,y) = k$, which is required to be independent of the choice of x, y. The set I is required to be at most countably infinite. This countability assumption seems to be necessary, for example to avoid the case where p_{ij}^k is all zero for any i, j, k.

Let $\chi_i(x,y)$ be the indicator function of $R^{-1}(i)$. They define adjacency algebras (Bose-Mesner algebras in our terminology) in [2, Section 3] as a set of finite linear combination of $\chi_i(x,y)$ with complex coefficient, with convolution product given by the integration

$$(a * b)(x,y) := \int_X a(x,z)b(z,y)d\mu(z).$$

However, it is not clear whether the adjacency algebra is closed under the convolution product. In fact, the authors remarked in the last of [2, Section 3.1] "our arguments in this part are of somewhat heuristic nature. We make them fully rigorous for the case of association schemes on zero-dimensional groups; see Section 8." Later in Section 8, they define the adjacency algebra with an aid of locally compact zero-dimensional abelian groups and their duality. Some of the treated objects (compact cases) are examples of profinite association schemes in this paper.

4.1. **Profinite groups.** Let G be a profinite group, that is, a projective limit of finite groups G_λ with projective limit topology. Absolute Galois groups, arithmetic fundamental groups and p-adic integers are some examples appear in arithmetic geometry [22]. There are significant amounts of studies on analysis on abelian profinite groups, see for example [14].

4.1.1. *Schurian profinite association schemes.* We describe Schurian association schemes, and then profinite analogues. When we say an action of a group G, it means the left action, unless otherwise stated.

Lemma 4.1. *Let X be a topological space and G a group acting on X such that every element of G induces a homeomorphism on X. Then the space of continuous functions $C(X)$ is a left G-module by $f(x) \mapsto f(g^{-1}x)$. Let $G\backslash X$ denote the quotient space with quotient topology. Let*

$$C(X)^G \subset C(X)$$

denote the subspace of functions invariant by the action of G. There is a canonical injection

$$C(G\backslash X) \hookrightarrow C(X)$$

associated with the continuous surjection $X \to G\backslash X$. Then, there is a canonical isomorphism of $\mathbb{C}$-algebras

$$C(X)^G \to C(G\backslash X),$$

and hence we identify them.

Proof. An element f of $C(X)^G$ is a continuous function which takes one same value for each G-orbit of X. Thus, it gives a mapping $G\backslash X \to \mathbb{C}$. By the universality of the quotient topology, this is continuous, hence lies in $C(G\backslash X)$. Conversely, a function in $C(G\backslash X)$ gives a continuous function in $C(X)$, which is invariant by G. $\qquad\square$

Proposition 4.2. *(Schurian association scheme)*
Let G be a finite group, H a subgroup, and let $X := G/H$ be the quotient. Then, we have a natural bijection

$$G\backslash(X \times X) \to H\backslash G/H, \tag{4.1}$$

where G acts diagonally. We define

$$I := G\backslash(X \times X).$$

The quotient mapping

$$R : X \times X \to I$$

is an association scheme, which is called a Schurian scheme.

We give a proof, since descriptions used in the proof are necessary for the profinite case. The equality (4.1) is given by a surjection

$$(g_1 H, g_2 H) \mapsto H g_1^{-1} g_2 H,$$

which certainly factors through

$$G\backslash(g_1 H, g_2 H) \mapsto H g_1^{-1} g_2 H,$$

and the converse map is given by

$$HgH \mapsto (H, gH),$$

where the check of the well-definedness is left to the reader.

Lemma 4.3. *Let $C(X)^\vee := \mathrm{Hom}_{\mathbb{C}}(C(X), \mathbb{C})$ be the dual space, which is regarded as a left G-module, by letting $g \in G$ act on $\xi \in C(X)^\vee$ by $g(\xi)f = \xi(g^{-1}(f))$. Then we have an isomorphism of G-modules*

$$C(X) \overset{\vee}{\to} C(X)^\vee, \quad \delta_x \mapsto \mathrm{ev}_x$$

for every $x \in X$, where δ_x is the indicator function of x and ev_x is the evaluation at x.

Proof. It is well-known that $\{\delta_x \mid x \in X\}$ is a linear basis of $C(X)$, and $\{\mathrm{ev}_x \mid x \in X\}$ is a linear basis of $C(X)^\vee$. We compute

$$g(\delta_x)(y) = \delta_x(g^{-1}y) = \delta_{gx}(y).$$

On the other hand,

$$g(\mathrm{ev}_x)(f) = \mathrm{ev}_x(g^{-1}(f)) = (g^{-1}(f))(x) = f(gx) = \mathrm{ev}_{gx}(f).$$

$\square$

Proposition 4.4. *The identification of a matrix and a linear map is obtained by an isomorphism of G-modules*

$$C(X \times X) = C(X) \otimes C(X) \overset{\mathrm{id} \otimes \vee}{\to} C(X) \otimes C(X)^\vee \cong \mathrm{Hom}_{\mathbb{C}}(C(X), C(X)). \quad (4.2)$$

Proof. An element $A \in C(X \times X)$ is mapped as

$$A \;\mapsto\; \sum_{x,y \in X^2} A(x,y)\delta_x \otimes \delta_y$$

$$\mapsto\; \sum_{x,y \in X^2} A(x,y)\delta_x \otimes \mathrm{ev}_y.$$

The last element is regarded as an element of $\mathrm{Hom}_{\mathbb{C}}(C(X), C(X))$ by mapping $f = \sum_{y \in X} f(y)\delta_y \in C(X)$ to

$$\Big(\sum_{x,y \in X^2} A(x,y)\delta_x \otimes \mathrm{ev}_y \Big)(f) = \sum_{x \in X} \sum_{y \in Y} A(x,y)f(y)\delta_x,$$

which is the multiplication of a matrix to a vector. $\square$

Proposition 4.5. *By taking the G-invariant part of (4.2), we have*

$$\mathrm{Hom}_G(C(X), C(X)) \cong C(X \times X)^G = C(G \backslash (X \times X)) = C(H \backslash G / H), \quad (4.3)$$

where the left Hom_G denotes the endomorphism as G-modules.

Proof. This follows from (4.1), Lemma 4.1 and the definition of G-homomorphisms.
$\square$

Corollary 4.6. *The construction in Proposition 4.2 gives an association scheme. The endomorphism ring $\mathrm{Hom}_G(C(X), C(X))$ is isomorphic to the Bose-Mesner algebra A_X.*

Proof. By the construction of $I = G \backslash (X \times X)$, the set of the adjacency matrices is closed under the transpose and contains the identity matrix. Their linear span is closed by the matrix product, since the adjacency matrices form a linear basis of (4.3) and the matrix product corresponds to the composition of the endomorphism ring $\mathrm{Hom}_G(C(X), C(X))$. This gives an association scheme with Bose-Mesner algebra $\mathrm{Hom}_G(C(X), C(X))$. $\square$

Definition 4.7. Let G be a finite group acting on a $\mathbb{C}$-linear space V. By the representation of finite groups, V is decomposed into irreducible representations. For a fixed irreducible representation W, the multiplicity of W in V is the dimension of the vector space

$$\mathrm{Hom}_G(W, V).$$

We say V is multiplicity free, if the multiplicity is at most one for any irreducible representation W.

Corollary 4.8. *The Schurian association scheme is commutative, if and only if $C(X)$ is multiplicity free. In this case, the set of the primitive idempotents is canonically identified with the set of the irreducible representations of G in $C(X)$.*

Proof. If the multiplicity is one for each irreducible representation, Schur's lemma implies that $\mathrm{Hom}_G(C(X), C(X))$ is a direct sum of copies of $\mathbb{C}$ (each copy corresponding to each irreducible representation in $C(X)$) as a ring, and hence is commutative. If there is an irreducible component with multiplicity $m \geq 2$, $\mathrm{Hom}_G(C(X), C(X))$ contains a matrix algebra $M_m(\mathbb{C})$, and thus is not commutative. $\qquad\square$

See [22] for the following definition and properties (c.f. Section 3.1 above).

Definition 4.9. Let $G_\lambda(\lambda \in \Lambda)$ be a projective system of surjections of finite groups. A profinite group is a topological group which is isomorphic to

$$G^\wedge := \varprojlim G_\lambda,$$

where each G_λ has the discrete topology. Because the projection $\mathrm{pr}_\lambda : G^\wedge \to G_\lambda$ is surjective and continuous, we identify G_λ with $G^\wedge/N_\lambda$, where N_λ is the kernel of the pr_λ, which is a clopen normal subgroup of $G^\wedge$, since G_λ is finite and discrete. By Tychonoff's theorem, $G^\wedge$ is a compact Hausdorff group, with a basis of open neighborhood of the unit element e consisting of N_λ. Being Hausdorff, the intersection of N_λ is $\{e\}$.

We prepare a well-known lemma for describing profinite Schurian association schemes.

Lemma 4.10. *Let $G^\wedge = \varprojlim G_\lambda = \varprojlim G^\wedge/N_\lambda$ be a profinite group. Let H and K be closed subgroups of $G^\wedge$. Give a quotient topology on $H\backslash G^\wedge/K$ by*

$$q : G^\wedge \to H\backslash G^\wedge/K.$$

(1) *The image of K in G_λ is KN_λ (which is a union of cosets of N_λ, and hence a subset of G_λ). The same statement holds for H.*

(2) *We have the following commutative diagram of continuous surjections, where α is a bijection.*

$$
\begin{array}{ccc}
H\backslash G^\wedge/K & & \\
\downarrow & \searrow & \\
HN_\lambda\backslash G^\wedge/KN_\lambda & \overset{\alpha}{\to} & HN_\lambda\backslash G_\lambda/KN_\lambda.
\end{array}
\tag{4.4}
$$

(3) *The $q : G^\wedge \to H\backslash G^\wedge/K$ is an open morphism, and $H\backslash G^\wedge/K$ is compact Hausdorff.*

(4) *The finite sets $HN_\lambda\backslash G_\lambda/KN_\lambda$ constitute a projective system, with a canonical homeomorphism*

$$H\backslash G^\wedge/K \overset{\sim}{\to} \varprojlim HN_\lambda\backslash G_\lambda/KN_\lambda.$$

Proof. (1) When we consider G_λ as a quotient of $G^\wedge$, G_λ is a set of cosets gN_λ. The image of K is the union of kN_λ for $k \in K$, which is nothing but KN_λ.

(2) Every set in the diagram is a quotient of $G^\wedge$ with various equivalence relation. The vertical arrow is obviously well-defined. The mapping α is bijective, since as a quotient of $G^\wedge$, they are the same. The slanting arrow is merely the composition of the two. We recall that $G^\wedge \to H\backslash G^\wedge/K$ is a quotient map. Thus, every map in the diagram is continuous, since the maps from $G^\wedge$ are continuous (where the mappings to the two sets at the bottom of the diagram factors through $G^\wedge/N_\lambda$, which is finite and discrete).

(3) The open sets sN_λ for $s \in G^\wedge$ and various λ constitute an open basis of $G^\wedge$. The image of sN_λ in $H\backslash G^\wedge/K$ is $HsKN_\lambda$ (since N_λ is normal), whose inverse image in $G^\wedge$ is again $HsKN_\lambda$, which is open. Thus, by the definition of quotient topology, $HsKN_\lambda$ is open in $H\backslash G^\wedge/K$, and hence q is open.

Take two distinct elements HgK, $Hg'K$ in $H\backslash G^\wedge/K$. These are compact disjoint subsets of the Hausdorff space $G^\wedge$. It is well known that there are an open set $O_a \subset G^\wedge$ containing HgK and an open set $O_b \supset Hg'K$ that separate HgK and $Hg'K$. By (1) of Lemma 3.14 for the union of HgK and $Hg'K$, we may take μ such that open sets of the form $sN_\mu \subset O_a$ with $sN_\mu \cap HgK \neq \emptyset$ cover HgK and open sets $sN_\mu \subset O_b$ with $sN_\mu \cap Hg'K \neq \emptyset$ cover $Hg'K$, that is, $HgKN_\mu \subset O_a$ and $Hg'KN_\mu \subset O_b$ are disjoint. Since q is open, $HgKN_\mu$ and $Hg'KN_\mu$ are open sets in $H\backslash G^\wedge/K$ which have no intersection and are neighborhoods of HgK, $Hg'K$ (which are considered as elements of $H\backslash G^\wedge/K$), respectively. Thus $H\backslash G^\wedge/K$ is Hausdorff.

(4) If we see these sets as quotients of $G^\wedge$ as in (4.4), it is clear that these form a projective system. By the universality of projective limit, we have a continuous map

$$f : H\backslash G^\wedge/K \to \varprojlim HN_\lambda\backslash G_\lambda/KN_\lambda$$

by (2). The image of f is dense, since for any element ξ in the projective limit and its open neighborhood, there is a smaller open neighborhood which has the form $\mathrm{pr}_\lambda^{-1}\mathrm{pr}_\lambda(\xi)$ for some λ, since these form a basis of open neighborhoods of ξ. Because

$$H\backslash G^\wedge/K \to HN_\lambda\backslash G_\lambda/KN_\lambda$$

is surjective, the density of the image follows. Since $H\backslash G^\wedge/K$ is compact (being a quotient of a compact set $G^\wedge$), the image is compact and closed, hence f is surjective. Since $H\backslash G^\wedge/K$ is Hausdorff, any two distinct points HgK, $Hg'K$ have disjoint open neighborhoods $HgKN_\lambda$, $Hg'KN_\lambda$, respectively, as proved above, hence their images in $HN_\lambda\backslash G_\lambda/KN_\lambda$ are distinct, which implies the injectivity of f. By the compactness of $H\backslash G^\wedge/K$, f is closed, hence f^{-1} is continuous, thus a homeomorphism. $\square$

Proposition 4.11. *(profinite Schurian association scheme) Let $G^\wedge = \varprojlim G_\lambda = \varprojlim G^\wedge/N_\lambda$ be a profinite group. We denote by G_λ the quotient $G^\wedge/N_\lambda$. For a closed subgroup $H < G$, its image in G_λ is $H_\lambda := HN_\lambda$. Then we have a projective system of Schurian association schemes*

$$X_\lambda := G_\lambda/H_\lambda$$

and

$$I_\lambda := H_\lambda\backslash G_\lambda/H_\lambda.$$

These systems give a profinite association scheme, which may be non-commutative. We call this type of profinite association scheme Schurian. We have canonical homeomorphisms

$$G^\wedge/H \overset{\sim}{\to} X^\wedge = \varprojlim X_\lambda = \varprojlim G^\wedge/HN_\lambda$$

and

$$H\backslash G^\wedge/H \overset{\sim}{\to} I^\wedge = \varprojlim I_\lambda = \varprojlim H_\lambda\backslash G^\wedge/H_\lambda.$$

Proof. Everything is proved in Proposition 4.2 and Lemma 4.10, except that these mappings give surjective morphisms of association schemes. Take $\lambda \geq \mu$. Since the surjectivity is proved in (2), it suffices to show the commutativity of

$$\begin{array}{ccc} X_\lambda \times X_\lambda & \to & I_\lambda \\ \downarrow & & \downarrow \\ X_\mu \times X_\mu & \to & I_\mu, \end{array}$$

which is easy to check: an element $(gHN_\lambda, g'HN_\lambda)$ at the left top is mapped to the right $Hg^{-1}g'HN_\lambda$, then to the bottom $Hg^{-1}g'HN_\mu$. This is the same via the left bottom corner. $\qquad\square$

Proposition 4.12. *Let $G^\wedge/H$ be a Schurian profinite association scheme as in 4.11. Suppose that each finite association scheme X_λ is commutative. Then, $J^\wedge$ is the set of irreducible representations of $G^\wedge$ appearing in $C(X_\lambda)$ for some λ, where the multiplicity is at most one for any $C(X_\lambda)$.*

Proof. We have an inductive system of injections $C(X_\lambda \times X_\lambda)$, which gives via (4.2) an inductive system of injections

$$\mathrm{Hom}_\mathbb{C}(C(X_\mu), C(X_\mu)) \to \mathrm{Hom}_\mathbb{C}(C(X_\lambda), C(X_\lambda))$$

for $\lambda \geq \mu$. To obtain A_{X_μ} and A_{X_λ}, we take $G^\wedge$-invariant part (4.3)

$$\mathrm{Hom}_{G_\mu}(C(X_\mu), C(X_\mu)) \to \mathrm{Hom}_{G_\lambda}(C(X_\lambda), C(X_\lambda))$$

(note that there is a canonical surjection $G^\wedge \to G_\lambda$, hence a G_λ-module is a $G^\wedge$-module.) By the surjectivity of $G^\wedge \to G_\lambda$, an irreducible representation in $C(X_\mu)$ of G_μ lifts to an irreducible representation in $C(X_\lambda)$ of G_λ.

We assume that every finite association scheme appeared is commutative. Then, the multiplicity of each irreducible representation $V \subset C(X_\lambda)$ is one by Corollary 4.8 for any λ. Thus, the partial surjection given in Corollary 2.14 is in this case the inverse to the injection $J_\mu \to J_\lambda$, induced by $G_\lambda \to G_\mu$, and

$$J^\wedge = \varinjlim J_\lambda$$

is identified with the set of irreducible representations of $G^\wedge$ appeared in some $C(X_\lambda)$. $\qquad\square$

Indeed, the above irreducible representations are exactly the irreducible representations of $G^\wedge$ appearing in $C(G^\wedge/H)$. To show this, we recall a well known "no small subgroups" lemma.

Lemma 4.13. *Let G be a Lie group over $\mathbb{R}$ or $\mathbb{C}$. Then, there is an open neighborhood of the unit e, which contains no subgroup except $\{e\}$.*

Proof. We consider the Lie algebra $\mathfrak{g}$ of G. Give an arbitrary (Hermitian or positive definite) metric to $\mathfrak{g}$. It is known that

$$\exp : \mathfrak{g} \to G$$

is homeomorphism when restricted to a small enough neighborhood of 0. We take the homeomorphic neighborhoods U of $e \in G$ and V of $0 \in \mathfrak{g}$. We may assume that V is bounded with respect to the metric. Assume that U contains non-trivial subgroup of G. Take an $e \neq g \in G$ in the subgroup. Consequently, for any integer n, $g^n \in U$. If $h = \log(g) \in V \subset \mathfrak{g}$, $nh = \log(g^n)$ corresponds to g^n and hence $nh \in V$ for any n. However, V is bounded, which contradicts the assumption. $\square$

Corollary 4.14. *Let $G^\wedge$ be a profinite group, and $f : G^\wedge \to G$ a group morphism to a real or complex Lie group G. Then, $\mathrm{Ker}(f)$ is an open normal subgroup of $G^\wedge$, and the image $f(G^\wedge)$ in G is a finite subgroup of G.*

Proof. Take a small enough neighborhood of U in Lemma 4.13. Consider $H = f^{-1}(U) \subset G^\wedge$, which is open in $G^\wedge$. Thus, there exists λ such that the open normal subgroup $N = N_\lambda$ is contained in H. The image $f(N) < G$ is a subgroup in U, which must be $\{e\} \subset G$ by definition of U. Thus, $\mathrm{Ker}(f)$ contains N, and is a union of cosets of N, hence an open normal subgroup of $G^\wedge$. Since $G^\wedge$ is compact and $\mathrm{Ker}\, f$ is open, $f(G^\wedge) \cong G^\wedge / \mathrm{Ker}\, f$ is a finite group. $\square$

Proposition 4.15. *Let G be a compact Hausdorff group, and H a closed subgroup. Consider the representation of G on $C(G/H)$. Then, each irreducible component of $C(G/H)$ is finite dimensional.*

Proof. This follows from a variant of the Peter-Weyl theorem, see Takeuchi[24, Section 1]. $\square$

Proposition 4.16. *Let $G^\wedge$ be a profinite group and H a closed subgroup. Consider the profinite association scheme arising from $G^\wedge/H$. Let $J(G^\wedge/H)$ be the set of the irreducible representations of $G^\wedge$ on $C(G^\wedge/H)$. Then, all finite association schemes X_λ in Proposition 4.11 are commutative, if and only if the representation $C(G^\wedge/H)$ is multiplicity free. In this case, there is a canonical bijection*

$$f : J(G^\wedge/H) \xrightarrow{\sim} J^\wedge = \varinjlim J_\lambda. \tag{4.5}$$

Proof. Put $X^\wedge := G^\wedge/H$. Let $V \subset C(X^\wedge)$ be an irreducible representation of $G^\wedge$. By Proposition 4.15, V is finite dimensional. By Lemma 4.13, the representation factors through

$$G \to G/N_\lambda \to \mathrm{GL}(V)$$

for some λ. By Lemma 4.1, this means that $V \subset C(X^\wedge)$ lies in

$$V \subset C(N_\lambda \backslash X^\wedge) = C(X_\lambda) \subset C(X^\wedge).$$

Thus, $C(X^\wedge)$ is multiplicity free, if and only if $C(X_\lambda)$ is multiplicity free for every $\lambda \in \Lambda$, in other words, if and only if every X_λ is a commutative association scheme, by Proposition 4.12. In the commutative case, the following observation shows that every irreducible component of $C(X^\wedge)$ appears in $C(X_\lambda)$ for some λ. Because the multiplicity is one, for any V above, there is a unique representation isomorphic to V in $\varinjlim C(X_\lambda)$. This gives a map f in (4.5). The injectivity follows from that $C(X^\wedge)$ is multiplicity-free. The surjectivity follows from $C(X_\lambda) \subset C(X^\wedge)$ is a $G^\wedge$-submodule, and any G_λ-irreducible component of $C(X_\lambda)$ is a $G^\wedge$-irreducible component since $G^\wedge \to G_\lambda$ is surjective. $\square$

Remark 4.17. Kurihara-Okuda[13] gives the notion of Bose-Mesner algebra for general homogeneous space G/H, where G is compact Hausdorff and H is a closed subgroup. In the case where G is a profinite group and $C(G/H)$ is multiplicity free, their construction coincides with our definition, by Proposition 4.16.

4.1.2. *Profinite abelian groups.* For any finite abelian group X, we consider its thin scheme. Namely, $R : X \times X \to I = X$ with $(x,y) \mapsto y - x$. Then, the i-th adjacency matrix $R^{-1}(i)$ is the representation matrix of addition of i on $C(X)$, and the Bose-Mesner algebra is isomorphic to the group ring $C(X)$, with usual Hadamard product and the convolution product

$$(f * g)(x) = \frac{1}{\#X} \sum_y f(x - y)g(y)$$

(the factor $\frac{1}{\#X}$ is not the standard, incorporated to be compatible with Theorem 2.5). Primitive idempotents are exactly the characters $\xi \in \check{X}$, where $\check{X} \subset C(X)$ is the group of characters of X.

For a profinite abelian group $X^\wedge$, it is a projective limit of finite abelian groups X_λ. We have an inductive system of the character groups $J_\lambda := \check{X}_\lambda$. For $\lambda \geq \mu$, the partial map $p_{\lambda,\mu} : J_\lambda \rightharpoonup J_\mu$ is given by the injection $i_{\mu,\lambda} : J_\mu \to J_\lambda$, where the domain of $p_{\lambda,\mu}$ is the image of $i_{\mu,\lambda}$, and in the domain $p_{\lambda,\mu}$ is the inverse of $i_{\mu,\lambda}$. Thus, $I^\wedge$ is isomorphic to $X^\wedge$, $J^\wedge$ is the inductive limit of the character groups $\check{X}_\lambda$. The Bose-Mesner algebra is isomorphic to $C_c(J^\wedge)$, the space of finite linear combinations of characters of $X^\wedge$, which is isomorphic to $C_{lc}(I^\wedge)$, the space of locally constant functions on $I^\wedge \simeq X^\wedge$. This example is outside of the scope of Barg-Skriganov theory, if $I^\wedge \simeq X^\wedge$ is uncountable.

Remark 4.18. Since $J^\wedge$ is discrete, $C_c(J^\wedge)$ has a natural (and orthogonal) basis consisting of E_j ($j \in J^\wedge$), but for the canonically isomorphic vector space $C_{lc(I^\wedge)}$, we do not have a natural basis, since $I^\wedge$ is compact with (possibly) uncountable cardinality. This makes a definition of eigenmatrices difficult.

4.2. **Kernel schemes.** Another example of profinite association schemes is a projective system of the kernel schemes. The kernel schemes are finite association schemes defined in Martin-Stinson[16] to study (t, m, s)-nets. The (t, m, s)-nets are introduced by Niederreiter for quasi-Monte Carlo integrations, see [18].

We recall the kernel scheme, with a slight modification on the notation for I, to make a description as a projective system easier.

Definition 4.19. Let n be a positive integer, and V be a finite set of alphabet with cardinality $v \geq 2$. Let X_n be V^n, and $I_n := \{1, 2, \ldots, n\} \cup \{\infty\}$. Define $R_n : X_n \times X_n \to I_n$ as follows. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be elements of X_n. Let $R(x, y)$ be the smallest index i for which $x_i \neq y_i$. If $x = y$, then $R(x, y) = \infty$. This is a symmetric (and hence commutative) association scheme, with $R^{-1}(\infty)$ being the identity relation. This is called a kernel scheme, and denoted by $\overrightarrow{k(n, v)}$.

A direct computation of the intersection numbers shows that this is a commutative association scheme, as shown in [16]. We shall give an indirect proof in Corollary 4.27. (This is only for the self-containedness.)

Proposition 4.20. *The kernel schemes $\overrightarrow{k(n,v)}$ forms a projective system, where $X_{n+1} \to X_n$ is given by deleting the right most component, and $I_{n+1} \to I_n$ is given by $m \mapsto m$ for $m \leq n$, and the both $n+1, \infty$ are mapped to ∞. Then $X^\wedge = V^{\mathbb{N}_{>0}}$ holds, and $\mathrm{pr}_n : X^\wedge \to X_n$ is obtained by taking the left (first) n components, while $I^\wedge$ is $\mathbb{N}_{>0} \cup \{\infty\}$, with topology obtained by: each $n \in \mathbb{N}_{>0}$ is clopen, and an open set containing ∞ is a complement of a finite subset of $\mathbb{N}_{>0}$, that is, $I^\wedge$ is the one-point compactification of $\mathbb{N}_{>0}$. We call this a pro-kernel scheme and denote by $\overrightarrow{k(\infty,v)}$.*

Proof. It is easy to check that these make a projective system of association schemes. The topology of $I^\wedge$ comes from the definition of projective limit topology. $\qquad\square$

Remark 4.21. In this case $I^\wedge$ is countable, and this is a special case of infinite association schemes examined in detail, named metric schemes, by Barg-Skriganov [2, Section 8], for which the adjacency algebra is defined. In particular, in (8.33) they showed that it is an inductive limit of finite dimensional adjacency algebra, which indeed coincides with our definition.

Remark 4.22. This is merely an observation. For the set of p-adic integers $\mathbb{Z}_p$, we have a valuation $v : \mathbb{Z}_p \to \mathbb{N} \cup \{\infty\}$ giving an ultrametric. The above pro-kernel scheme is isomorphic to this metric, up to the difference by one on the metric, i.e.,

$$R : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{N} \cup \{\infty\}, (x,y) \mapsto v(x-y).$$

The same holds for a ring of formal power series $\mathbb{F}_q[[t]]$, where $\mathbb{F}_q$ denotes the finite field of q elements. The both yield the isomorphic pro-kernel schemes if $p = q$.

To see $J^\wedge$ of the pro-kernel scheme, we need to work with primitive idempotents. All necessary results are in Martin-Stinson[16], but because of the choice of the index I and normalization of the products necessary for making the projective system, the obtained constants slightly differ. To help the readers' understandings, we recall the methods of Martin-Stinson. For the chosen integer $v \geq 2$, let $V := Z_v$ be the additive group $\mathbb{Z}/v$. Then $X = Z_v^n$ is an additive group, and the kernel scheme is a translation scheme [4, §2.10].

Definition 4.23. Let X be a finite abelian group. A translation scheme is an association scheme $R : X \times X \to I$, which factors through $X \times X \to X, (x,y) \mapsto y - x$, namely, there is an $S : X \to I$ such that $R(x,y) = S(y-x)$.

Definition 4.24. Return to the kernel scheme. We define $\mathrm{bot} : X_n \to I_n$ by

$$\mathrm{bot}(x_1, \ldots, x_n) := \min\{i \mid x_i \neq 0\},$$

and $\mathrm{bot}(0, \cdots, 0) = \infty$. Then we have

$$R_n : X_n \times X_n \to I_n, \quad (x,y) \mapsto \mathrm{bot}(x-y).$$

By $S := \mathrm{bot}$, the kernel scheme is a translation scheme.

Let (X, R, I) be a translation scheme. For $i \in I$, let $S_i \subset X$ be $S^{-1}(i)$ (S for the sphere). Let $\check{X}$ be the dual group of X. Then, $\xi \in \check{X}$ is an orthonomal basis of $C(X)$ under the normalized inner product in Definition 3.16. Let A_i denote the i-th adjacency matrix, and $\chi_i : X \to \{0,1\}$ be the indicator function for S_i. Our

action in Theorem 2.5 shows that for $x \in X$

$$
\begin{aligned}
(A_i \bullet \xi)(x) &= \frac{1}{\#X} \sum_{y \in X} A_i(x, y)\xi(y) \\
&= \frac{1}{\#X} \sum_{y \in X} \chi_i(y - x)\xi(y) \\
&= \frac{1}{\#X} \sum_{y \in x + S_i} \xi(y) \\
&= \frac{1}{\#X} \sum_{a \in S_i} \xi(x + a) \\
&= \frac{1}{\#X} \left(\sum_{a \in S_i} \xi(a) \right)\xi(x).
\end{aligned}
$$

Thus ξ is an eigenvector with eigenvalue

$$
\frac{1}{\#X} \left(\sum_{a \in S_i} \xi(a) \right). \tag{4.6}
$$

The primitive idempotents are projectors to the common eigen spaces of A_i for all $i \in I$. Let us assume $X = Z_v^n$, and then $\check{X} = \check{Z_v}^n$, where the product of $\check{Z_v}$ is written multiplicatively with unit 1. Thus,

$$
(\xi_1, \ldots, \xi_n)(x_1, \ldots, x_n) = \prod_{i=1}^{n} \xi_i(x_i). \tag{4.7}
$$

Definition 4.25. For $\xi = (\xi_1, \ldots, \xi_n) \in \check{X}$, define $\text{top}(\xi)$ as the max i such that $\xi_i \neq 1$. If $\xi = 1$, then $\text{top}(\xi) = 0$. Thus $\text{top} : \check{X} \to \{0, 1, \ldots, n\}$. We shall denote $J_n := \{0, 1, \ldots, n\}$, because we shall soon show a natural correspondence between the set of primitive idempotents and J_n.

Lemma 4.26. *For ξ with $j = \text{top}(\xi)$, we denote by $p_i(j)$ the eigenvalue of A_i for ξ. Then*

$$
p_i(j) = \begin{cases} v^{-n} & \text{if } i = \infty, \\ (v - 1)v^{-i} & \text{if } j < i < \infty, \\ -v^{-i} & \text{if } j = i, \\ 0 & \text{if } j > i. \end{cases}
$$

Thus, for any A_i, the eigenvalue of $\xi \in \check{X}$ depends only on $\text{top}(\xi)$, and if $\text{top}(\xi) \neq \text{top}(\xi')$, then there is an i such that A_i has different eigenvalues for ξ and ξ'.

Proof. Recall that $I = \{1, 2, \ldots, n\} \cup \{\infty\}$. Put

$$
B_i := \bigcup_{i \leq \ell \leq \infty} S_\ell.
$$

(B_i is the ball consisting of all the elements $(x_1, \ldots, x_n)$ with $x_\ell = 0$ for all $\ell < i$, with the cardinality v^{n-i+1} for $i < \infty$ and 1 for $i = \infty$.) Fix i and $\xi \in \check{X}$ with

$\text{top}(\xi) = j$, then by (4.6) we have

$$
\begin{aligned}
\tau(i,j)\xi \quad &:= \quad \sum_{i \le \ell \le \infty} A_i \bullet \xi \\
&= \quad \frac{1}{v^n}(\sum_{a \in B_i} \xi(a))\xi \\
&= \quad \begin{cases} v^{-n}\xi & \text{if } i = \infty, \\ v^{-i+1}\xi & \text{if } j < i < \infty, \\ 0 & \text{if } j \ge i. \end{cases}
\end{aligned}
$$

The last equality follows from that for $i = \infty$, $B_\infty = \{0\}$ and hence the summation is $\sum_{a \in B_i} \xi(a) = 1$, and if $j < i < \infty$, then (4.7) is always 1 and the cardinality of B_i is v^{n-i+1}, while if $j \ge i$, $\xi(a)$ depends on a and the summation over $a \in B_i$ is zero, since ξ is a non-trivial character on the abelian group B_i. Thus, the eigenvalue of A_i for ξ is the summation over $S_i = B_i \setminus B_{i+1}$, that is,

$$
\begin{cases} v^{-n} & \text{if } i = \infty, \\ (\tau(i,j) - \tau(i+1,j)) = (v-1)v^{-i} & \text{if } j < i < \infty, \\ (\tau(i,j) - \tau(i+1,j)) = -v^{-i} & \text{if } i = j, \\ 0 & \text{if } j > i, \end{cases}
$$

as desired. □

Corollary 4.27. *The kernel scheme $\overrightarrow{k(n,v)}$ is a commutative association scheme. A primitive idempotent is the projection to the subspace $C(X)_j$ of $C(X)$ spanned by $\{\xi \mid \text{top}(\xi) = j\}$ for each $j \in J_n$, which identifies the set of primitive idempotents with J_n.*

Proof. Recall that we did not use the property of association schemes for the kernel schemes so far. Every A_i has $C(X)_j$ as an eigenspace with an eigenvalue. Since $C(X)$ is a direct sum of $C(X)_j$ for $j \in J_n$, this implies that the matrices A_i mutually commute. We consider the algebra generated by A_i by the matrix product. Each element of this algebra is determined by the $n+1$ eigenvalues corresponding to J_n. Thus, the dimension of the algebra as a vector space does not exceed $n+1$, but A_i's are $n + 1$ linearly independent matrices, hence their linear combination is closed under the matrix multiplication. The rest of the axioms of association schemes are easy to verify. The description of primitive idempotents immediately follows.   □

Corollary 4.28. *The multiplicity m_j is 1 for $j = 0$ and $m_j = (v-1)v^{j-1}$ for $j > 0$. The valency k_i is 1 for $i = \infty$ and $k_i = (v-1)v^{n-i}$ for $i < \infty$.*

Proof. The multiplicity is the number of ξ with $\text{top}(\xi) = j$, which implies the first statement. The valency is the cardinality of the sphere S_i, which implies the second.   □

Corollary 4.29. *For $j \in J_n$, the projector E_j is given by*

$$
\begin{aligned}
E_j(x,y) \quad &= \quad \sum_{\xi \in \check{X}, \text{top}(\xi) = j} \xi(x)\overline{\xi(y)} \\
&= \quad \sum_{\xi \in \check{X}, \text{top}(\xi) = j} \xi(x - y).
\end{aligned}
$$

We have

$$A_i = \sum_{j \in J_n} p_i(j) E_j$$

where $p_i(j)$ is defined in Lemma 4.26.

Proof. The first half describes the projection to the eigenspace corresponding to j by an orthonomal basis. The second is because A_i has the eigenvalue $p_i(j)$ for that eigenspace. $\qquad\square$

For $x - y \in S_i$, we define

$$
\begin{aligned}
q_j(i) \quad &:= \quad E_j(x, y) = \sum_{\xi \in \check{X}, \text{top}(\xi) = j} \xi(x - y) \\
&= \quad \begin{cases} 1 & \text{if } 0 = j \\ (v - 1)v^{j-1} & \text{if } 0 < j < i \le \infty \\ -v^{j-1} & \text{if } j = i \\ 0 & \text{if } j > i. \end{cases}
\end{aligned}
$$

This computation is dual to that for $p_i(j)$, so omitted.

Corollary 4.30.

$$E_j = \sum_{i \in I} q_j(i) A_i.$$

Proof. We evaluate the both sides at (x, y). The left hand side is $q_j(i)$ for $x - y \in S_i$, which is equal to the right hand side, since $A_i(x, y) = \chi_i(x - y)$. $\qquad\square$

Corollary 4.31. *In $\overrightarrow{k(\infty, v)}$, the set $J^\wedge$ is the inductive limit of $J_n \to J_{n+1}$, and hence naturally identified with $\mathbb{N}$.*

Now we pass to the pro-kernel scheme, i.e., take the inductive limit of $A_{\overrightarrow{k(n,v)}}$ to obtain $A_{X^\wedge} = A_{\overrightarrow{k(\infty,v)}}$.

Proposition 4.32. *Let X_n denote the kernel scheme $\overrightarrow{k(n, v)}$ with I_n and primitive idempotents J_n, and $X^\wedge$ the pro-kernel scheme $\overrightarrow{k(\infty, v)}$ with $I^\wedge$ and the primitive idempotents $J^\wedge$. For $i \in I^\wedge \setminus \{\infty\}$, we denote by A_i the image of $\delta_i \in C_{lc}(I^\wedge)$ by the isomorphism $C_{lc}(I^\wedge) \to A_{X^\wedge}$. Note that $\delta_\infty \notin C_{lc(I^\wedge)}$. Through $A_{X_n} \to A_{X^\wedge}$, E_j is mapped to E_j, and A_i for $i \in \{1, 2, \ldots, n\}$ is mapped to A_i, while $A_\infty \in A_{X_n}$ is mapped to an element corresponding to the indicator function $\chi_{\{n+1,\ldots,\infty\}} \in C_{lc}(I^\wedge)$. It is convenient to denote the corresponding element in $A_{X^\wedge}$ by the symbol*

$$\sum_{i \in \{n+1,\ldots,\infty\}} A_i,$$

since

$$\chi_{\{n+1,\ldots,\infty\}} = \sum_{i \in \{n+1,\ldots,\infty\}} \delta_i$$

holds. (Note that A_∞ does not exist in $A_{X^\wedge}$.)

Proof. According to Proposition 4.20 we have $I^\wedge = \mathbb{N}_{>0} \cup \{\infty\}$ and $J^\wedge = \mathbb{N}$ by Corollary 4.31. Since $J_n \to J^\wedge$ is an injection, $E_j \in A_{X_n}$ is mapped to $E_j \in A_{X^\wedge}$ (see Proposition 3.24). The projection $I^\wedge \to I_n$ is obtained by mapping $i \mapsto i$ for

$1 \leq i \leq n$, and $i \mapsto \infty$ for $i > n$. Since the indicator function $\delta_i \in C(I_n)$ is mapped to $\delta_i \in C_{lc}(I^\wedge)$ for $1 \leq i \leq n$, A_i is mapped to A_i. The description of the image of A_∞ is by the definition of $A_{X_n} \to A_{X^\wedge}$, which comes from $I^\wedge \to I_n$. $\qquad \square$

Everything is well-defined, except $i_0 = \infty \in I^\wedge$. In fact, the operators A_i, E_j, the values $p_i(j)$, $q_j(i)$, $\mu_{J^\wedge}(j) = m_j$, $\mu_{I^\wedge}(i) = k_i/\#X = (v-1)v^{-i}$ are all well-defined, except that A_∞ and $p_\infty(j)$ are not well-defined, since there is no corresponding element to A_∞ in $A_{X^\wedge}$. Indeed, A_∞ would correspond to the indicator function of $\infty \in I^\wedge$, which is not continuous and hence is outside $A_{X^\wedge}$. This forces to choose something which replaces A_∞. One possible way is to replace it with E_0 because

$$E_0 = \sum_{i \in I^\wedge} A_i$$

holds as a function (constant value 1) on $I^\wedge$. then

$$\{A_i \mid i \in \mathbb{N}_{>0}\} \cup \{E_0\}$$

is a linear basis of $C_{lc}(I^\wedge)$, although this is not an orthogonal basis. This is the method chosen in [2, (8.3), (8.9)] to replace an infinite sum with a finite sum (α_0 in (8.8) in their paper is nothing but E_0), which gives a precise definition of the adjacency algebra. In our interpretation in Proposition 4.32, the expression in Corollary 4.29 becomes

$$A_i = \sum_{j \in J^\wedge} p_i(j) E_j$$

which is a finite sum (see Lemma 4.26) and holds for $i \neq \infty$ (since the equality holds in A_{X_i}), except the problem that for $i = \infty$, the value $p_\infty(j) = v^{-n}$ converges to 0, while the summation is over infinitely many E_j. In Corollary 4.30 the expression

$$E_j = \sum_{i \in I^\wedge} q_j(i) A_i$$

seems an infinite sum, but $q_j(i) = (v-1)v^{j-1}$ is constant for $i > j$, and thus the right hand side lies in $C_{lc}(I^\wedge)$, or more precisely, the seemingly infinite sum is a finite sum by the usage of the symbol in Proposition 4.32, and the equality does hold (since it holds in A_{X_j}.) This may be translated into a finite summation formula for $j \geq 1$:

$$
\begin{aligned}
E_j &= \sum_{i \leq j}(q_j(i) - (v-1)v^{j-1})A_i + (v-1)v^{j-1}E_0 \\
&= (\sum_{i < j} -(v-1)v^{j-1}A_i) - v^j A_j + (v-1)v^{j-1}E_0.
\end{aligned}
$$

4.3. **Ordered Hamming schemes.** Martin and Stinson [16, Section 1.3] introduced ordered Hamming schemes as Delsarte's extension of length s [6, Section 2.5] of the kernel schemes.

Proposition 4.33. *(Extension of length s)*
Let $R : X \times X \to I$ be a commutative association scheme. Let s be a positive integer, and S_s a symmetric group of degree s, acting on the direct product I^s. We obtain a quotient map $I^s \to I^s/S_s$. Then, the composition

$$X^s \times X^s \to I^s \to I^s/S_s$$

is a commutative association scheme. For $\bar{i} \in I^s/S_s$, the corresponding Hadamard primitive idempotent is the sum of

$$A_{i_1} \otimes \cdots \otimes A_{i_s}$$

over $(i_1, \ldots, i_s) \in I^s$ which maps to $\bar{i}$. The set of primitive idempotents are naturally identified with J^s/S_s, as follows. For $\bar{j} \in J^s/S_s$, the corresponding primitive idempotent is the sum of

$$E_{j_1} \otimes \cdots \otimes E_{j_s}$$

over $(j_1, \ldots, j_s) \in J^s$ which maps to $\bar{j}$. The corresponding eigenspace is the direct sum of the tensors of the eigenspaces

$$C(X)_{j_1} \otimes \cdots \otimes C(X)_{j_s}$$

over $(j_1, \ldots, j_s) \in J^s$ which maps to $\bar{j}$.

A proof is not given in the paper by Delsarte, but found in an unpublished (but reachable) paper by Godsil [9, 3.2 Corollary]. Because he did not mention the eigenspaces, which we need later, we shall recall a proof.

Proof. Let

$$X^s \times X^s \to I^s$$

be the direct product association scheme. Its Bose-Mesner algebra $A_{X^s} = A_X^{\otimes s}$ is generated by the s-fold kronecker products of A_i's for $i \in I$, which satisfies the axiom of association scheme. The symmetric group S_s acts on A_{X^s} by permutation. We take the fixed part $A_{X^s}^{S_s}$ of A_{X^s}. Since S_s preserves the two products and the transpose, fix the i_0 and the j_0, $A_{X^s}^{S_s}$ is a Bose-Mesner algebra. As a vector space, this is canonically isomorphic to $C(I^s)^{S_s} \cong C(I^s/S_s)$, induced by $I^s \to I^s/S_s$. For an $\bar{i} \in I^s/S_s$, its inverse image is the set of $(i_1, \ldots, i_s)$ mapped to $\bar{i}$, and the set of $A_{i_1} \otimes \cdots \otimes A_{i_s}$ are mutually disjoint Hadamard idempotent, whose sum is an Hadamard idempotent $A_{\bar{i}}$. This must be primitive in $A_{X^s}^{S_s}$, since the Hadamard product satisfies $A_{\bar{i}} \circ A_{\bar{i}'} = \delta_{\bar{i}\,\bar{i}'}$, and their sum is the Hadamard unit $J_{A_{X^s}}$. The same argument applies for the primitive idempotents $E_{\bar{j}}$. The description of the corresponding eigenspace follows. $\square$

This construction, when applied to the kernel schemes X_n, $I_n = \{1, 2, \ldots, n\} \cup \{\infty\}$, and $R_n : X_n \times X_n \to I_n$ given Definition 4.19), yields an ordered Hamming scheme.

Definition 4.34. The ordered Hamming scheme, denoted by $\overrightarrow{H}(s, n, v)$, is defined as the extension of length s of the kernel scheme $\overrightarrow{k(n, v)}$.

The following is easy to check.

Proposition 4.35. *The projections $X_{n+1}^s \to X_n^s$ and $I_{n+1}^s/S_s \to I_n^s/S_s$ coming from Proposition 4.20 give a projective system $\overrightarrow{H}(s, n+1, v) \to \overrightarrow{H}(s, n, v)$ of commutative association schemes, namely, a profinite association scheme, which we call the pro-ordered Hamming scheme and denote by $\overrightarrow{H}(s, \infty, v)$. The $I^\wedge$ of this projective association schemes is $(\mathbb{N}_{>0} \cup \{\infty\})^s/S_s$, and $J^\wedge$ is $\mathbb{N}^s/S_s$.*

We shall mention on this profinite scheme later in the last of Section 5.3.

5. Delsarte theory for profinite association schemes

Here we extend Delsarte theory introduced in [6]. The method here follows Kurihara-Okuda [13], which generalizes Delsarte theory to compact homogeneous spaces. In this section, let $(X_\lambda, R_\lambda, I_\lambda)$ be a profinite association scheme, and $X^\wedge$, $I^\wedge$, $J^\wedge$, $A_{X^\wedge}$ be those defined in Section 3.1.

5.1. **Multiset and Averaging functional.** We consider a finite multi-subset Y of $X^\wedge$, which means that Y is a set in which finite multiplicity of elements is allowed and taken into account. To make the notion rigorous, we consider a map from a finite set Z to a set X

$$g : Z \to X.$$

Then the image $Y := g(Z)$ in X has a natural finite multiset structure, where the multiplicity of $y \in Y$ is the cardinality of the fiber $\#g^{-1}(y)$. For $S \subset X$, we use the notation

$$\#(Y \cap S) := \#(Z \cap g^{-1}(S)).$$

This merely means to count the number of elements in $Y \cap S$ with taking the multiplicity into account. Thus, we call Y a "multi-subset" of X, and use the notation

$$\text{a finite multiset } Y \subset X$$

by an abuse of language.

For any non-empty finite multiset $Y \subset X^\wedge$, we would like to describe the notions of codes and designs. We begin with defining the averaging functional.

Definition 5.1. Let $Y \subset X^\wedge$ be a finite multi-subset in the sense above. Define the averaging functional

$$\text{avg}_Y : C(X^\wedge) \to \mathbb{C}, \quad f \mapsto \frac{1}{\#Y} \sum_{x \in Y} f(x) := \frac{1}{\#Z} \sum_{z \in Z} f(g(z)).$$

By (3.1), $C_c(J^\wedge) \cong A_{X^\wedge} \cong C_{lc}(I^\wedge)$ holds, and by

$$A_{X^\wedge} \to C(X^\wedge \times X^\wedge) \overset{\text{avg}_Y \otimes \text{avg}_Y}{\to} \mathbb{C},$$

$\text{avg}_Y^2 := \text{avg}_Y \otimes \text{avg}_Y$ defines a functional on $A_{X^\wedge}$, on $C_c(J^\wedge)$, and on $C_{lc}(I^\wedge)$. For example, for $f \in C_{lc}(I^\wedge)$, we have

$$
\begin{aligned}
\text{avg}_Y^2(f) &= \frac{1}{\#Y^2} \sum_{x,y \in Y} f \circ R^\wedge(x,y) \\
&= \frac{1}{\#Y^2} \sum_{i \in I^\wedge} \sum_{x,y \in Y, R^\wedge(x,y)=i} f(i) \\
&= \frac{1}{\#Y^2} \sum_{i \in I^\wedge} \#((Y \times Y) \cap R^{\wedge^{-1}}(i)) f(i).
\end{aligned}
$$

Note that these coefficients are the inner-distribution of Y (Delsarte [6, Section 3.1]) multiplied by $\frac{1}{\#Y}$. Let us denote by $\mathbb{C}^{\oplus I^\wedge}$ the vector space whose basis is $I^\wedge$, namely, the space of finite linear combinations of the elements of $I^\wedge$. We have a mapping

$$\mathbb{C}^{\oplus I^\wedge} \to C_{lc}(I^\wedge)^\vee, \tag{5.1}$$

where $\vee$ denotes the dual (i.e. $\text{Hom}(C_{lc}(I^\wedge), \mathbb{C})$), by the evaluation at i: $i \mapsto (f \mapsto f(i))$, which is injective since only a finite number of linear combinations appear in

$\mathbb{C}^{\oplus I^\wedge}$, and their support can be separated by clopen subsets. The above computation shows that $\mathrm{avg}_Y^2 \in C_{lc}(I^\wedge)^\vee$ lies in $\mathbb{C}^{\oplus I^\wedge}$, namely,

$$\mathrm{avg}_Y^2 = \sum_{i \in I} \frac{1}{\#Y^2}(\#((Y \times Y) \cap R^{\wedge-1}(i))) \cdot i \in \mathbb{C}^{\oplus I^\wedge}. \qquad (5.2)$$

Note that every coefficient is non-negative. Next we compute avg_Y^2 on $C_c(J^\wedge)$. For $j \in J^\wedge$, we have the orthogonal projector $E_j : C(X^\wedge) \to C(X^\wedge)_j$. For $f \in C(X^\wedge)_j$,

$$f(x) = \int_{y \in X^\wedge} E_j(x,y)f(y)d\mu(y) = (f, \overline{E_j(x,-)})_{\mathrm{HS}}.$$

Thus, we define $\mathrm{avg}_Y^j \in C(X^\wedge)_j$ by

$$\mathrm{avg}_Y^j(-) := \frac{1}{\#Y}\sum_{x \in Y} E_j(x,-), \qquad (5.3)$$

which represents the averaging functional in $C(X^\wedge)_j$:

$$(f, \overline{\mathrm{avg}_Y^j})_{\mathrm{HS}} = \mathrm{avg}_Y(f). \qquad (5.4)$$

Then

$$\begin{aligned}
\mathrm{avg}_Y^2(E_j) &= \frac{1}{\#Y^2}\sum_{x,y \in Y} E_j(x,y) \\
&= \frac{1}{\#Y}\sum_{y \in Y} \mathrm{avg}_Y^j(y) \\
&= (\mathrm{avg}_Y^j, \overline{\mathrm{avg}_Y^j}) = \|\mathrm{avg}_Y^j\|_{\mathrm{HS}}^2 \geq 0. \qquad (5.5)
\end{aligned}$$

These positivities make the LP method by Delsarte possible [6, Thorem 3.3].

Definition 5.2. Let us denote by $Q_j \in C_{lc}(I^\wedge)$ the image of $E_j \in A_{X^\wedge}$. This may be considered as a description of the canonical isomorphism

$$Q : C_c(J^\wedge) \to C_{lc}(I^\wedge), \quad \delta_j \mapsto Q_j,$$

where δ_j means the indicator function on $J^\wedge$ at $j \in J^\wedge$. (Note that $E_j \in A_{X^\wedge}$ corresponds to $\delta_j \in C_c(J^\wedge)$ in Theorem 3.13, see Corollary 3.11.) Then we have an injection

$$\mathbb{C}^{\oplus I^\wedge} \to C_{lc}(I^\wedge)^\vee \xrightarrow{Q^\vee} C_c(J^\wedge)^\vee,$$

obtained by

$$i \mapsto (f \mapsto f(i)) \mapsto (\delta_j \mapsto Q_j(i)).$$

Definition 5.3. The above morphism $Q^\vee|_{I^\wedge} : \mathbb{C}^{\oplus I^\wedge} \to C_c(J^\wedge)^\vee$ is called the Mac-Williams transform. In concrete, it maps

$$Q^\vee|_{I^\wedge} : \sum_{i \in I} a_i \cdot i \mapsto \sum_{j \in J^\wedge}\sum_{i \in I^\wedge} a_i Q_j(i)\mathrm{ev}_j,$$

where ev_j is the dual basis, i.e., $\mathrm{ev}_j(\delta_{j'}) = \delta_{jj'}$, or equivalently,

$$\mathrm{ev}_j : C_c(J^\wedge) \to \mathbb{C}, f \mapsto f(j).$$

Note that the left is a finite sum, but the right may be an infinite sum (which causes no problem, since the dual of a direct sum is the direct product).

The following is a formal consequence:

$$Q^\vee|_{I^\wedge} : \mathrm{avg}_Y^2 \in \mathbb{C}^{\oplus I^\wedge} \mapsto \mathrm{avg}_Y^2 \in C_c(J^\wedge)^\vee. \tag{5.6}$$

5.2. **Codes and designs.** We define codes and designs. Let $I_C \subset I^\wedge$ be a subset which does not contain i_0 (C for code). Let $J_D \subset J^\wedge$ be a subset which does not contain j_0 (D for design). We define a convex cone $(I_C; J_D) \subset \mathbb{C}^{\oplus I^\wedge}$ by

$$(I_C; J_D) \quad :=$$
$$\{ \quad \sum_{i \in I} a_i \cdot i \mid a_i = 0 \text{ for all but finite } i \ ,$$
$$a_i \geq 0 \text{ for all } i \in I^\wedge,$$
$$a_i = 0 \text{ for all } i \in I_C,$$
$$\sum_{i \in I^\wedge} a_i Q_j(i) \geq 0 \text{ for all } j \in J^\wedge,$$
$$\sum_{i \in I^\wedge} a_i Q_j(i) = 0 \text{ for all } j \in J_D \quad \}.$$

Definition 5.4. A non-empty finite multi-subset $Y \subset X^\wedge$ is called an I_C-free-code-J_D-design if avg_Y^2 lies in the cone $(I_C; J_D)$. Furthermore, an I_C-free-code-$\emptyset$-design [resp. an $\emptyset$-free-code-J_D-design] is simply called an I_C-free-code [resp. a J_D-design].

More explicitly, the i-component of avg_Y^2 in $\mathbb{C}^{\oplus I^\wedge}$ is

$$a_i(Y) := \frac{1}{\#Y^2} \#((Y \times Y) \cap R^{\wedge^{-1}}(i)), \tag{5.7}$$

which is non-negative (5.2) and required to be 0 for $i \in I_C$ (namely, there is no pair $(x, y) \in Y \times Y$ with relation $R^\wedge(x, y) \in I_C$), and its ev_j-component in $C_c(J^\wedge)^\vee$ is

$$b_j(Y) := \mathrm{avg}_Y^2(E_j) = || \mathrm{avg}_Y^j ||_{\mathrm{HS}}^2, \tag{5.8}$$

which is non-negative (5.5) and required to be 0 for $j \in J_D$ (namely, the j-component of avg_Y is zero for $j \in J_D$, or equivalently, for any $f \in C(X^\wedge)_j$, $\sum_{y \in Y} f(y) = 0$ for $j \in J_D$). Note that i_0 is removed from I_C, since

$$\frac{1}{\#Y^2} \#((Y \times Y) \cap R^{\wedge^{-1}}(i_0)) = \frac{1}{\#Y}$$

is positive, and j_0 is removed since $|| \mathrm{avg}_Y^{j_0} ||_{\mathrm{HS}} = 1$ (being the operator norm of the averaging for constant functions). Since

$$\#Y \, \mathrm{avg}_Y^2(i_0) = 1 \text{ and } \sum_{i \in I^\wedge} \#Y \, \mathrm{avg}_Y^2(i) = \#Y,$$

we may consider an LP problem: under the constraint that $(a_i)_{i \in I^\wedge}$ lies in the cone $(I_C; J_D)$ and $a_{i_0} = 1$, maximize/minimize $\sum_{i \in I^\wedge} a_i$, which gives an upper/lower bound on the cardinality of Y which is an I_C-free-code-J_D-design. The following is a formal consequence of (5.6), stating the relation with classic theory [6, Section 3].

Theorem 5.5. *The Mac-Williams transformation $Q^\vee|_{I^\wedge}$ maps (5.7) to (5.8):*

$$\sum_{i \in I^\wedge} a_i(Y) \cdot i \mapsto \sum_{j \in J^\wedge} b_j(Y) \mathrm{ev}_j.$$

In concrete,

$$b_j(Y) = \sum_{i \in I^\wedge} a_i(Y) Q_j(i).$$

Remark 5.6. The meaning of the values (5.7) is clear, and called the inner distribution [6, Section 3.1] (up to a factor of $\#Y$), but there, the meaning of the values $b_j(Y) = \sum_{i \in I^\wedge} a_i(Y) Q_j(i)$ is not clear, just the semi-positivity is proved. Now the value $b_j(Y) = ||\operatorname{avg}_Y^j||_{\mathrm{HS}}^2$ has a clear interpretation. On the space $C(X^\wedge)_j$, the operator norm of avg_Y^j is $||\operatorname{avg}_Y^j||_{\mathrm{HS}}$, since

$$\sup_{f \in C(X^\wedge)_j \setminus \{0\}} \frac{|\operatorname{avg}_Y^j(f)|}{||f||} = \sup_{f \in C(X^\wedge)_j \setminus \{0\}} \frac{(f, \overline{\operatorname{avg}}_Y^j)}{||f||} \le \frac{||f|| \cdot ||\operatorname{avg}_Y^j||}{||f||} = ||\operatorname{avg}_Y^j||,$$

and the equality holds for $f = \operatorname{avg}_Y^j$. This may be interpreted as the worst-case error in approximating the integration for $C(X)_j$ by avg_Y (where the true integration value is zero for $j \ne j_0$, because of the orthogonality to the constant functions $C(X)_{j_0}$), see a comprehensive book by Dick-Pillichshammer [7, Definition 2.10] on quasi-Monte Carlo integration. Another remark: the injectivity (5.1) implies that Y and Y' have the same inner distribution

$$\#((Y \times Y) \cap R^{\wedge^{-1}}(i)) = \#((Y' \times Y') \cap R^{\wedge^{-1}}(i)) \text{ for all } i \in I^\wedge$$

if and only if

$$||\operatorname{avg}_Y^j||_{\mathrm{HS}} = ||\operatorname{avg}_{Y'}^j||_{\mathrm{HS}} \text{ for all } j \in J^\wedge.$$

5.3. (t, m, s)-nets and (t, s)-sequences. The (t, m, s)-nets are point sets for quasi-Monte Carlo integration, well-studied, see [18]. For (t, m, s)-nets, the LP bound by ordered Hamming schemes introduced by Martin-Stinson [16] yields strong lower bounds on the cardinality of the point set P, see [16] and Bierbrauer [3] for example. We do not have such an application, but profinite association schemes may be used to formalize the notion of (t, s)-sequence [18], as mentioned at the last of this section.

The purpose of this section is to reprove some results of Marin-Stinson, and see a relation to pro-ordered Hamming schemes. The main result Corollary 5.19 of this section is proved in Martin-Stinson[16, Theorem 3.4] by using the notion of dual schemes and weight enumerator polynomials. Here we shall give a self-contained proof avoiding the use of these tools. Our proof matches to the formalization in Section 5.2, relying on the average functional avg_Y introduced in Definition 5.1.

Recall the kernel scheme in Definition 4.19, where the set of alphabets is $Z_v := \mathbb{Z}/v$, and $X_n = Z_v^n$, $I_n = \{1, 2, \dots, n\} \cup \{\infty\}$,

$$\mathrm{bot} : X_n \to I_n$$

in Definition 4.24 give the kernel scheme as a translation scheme. We shall identify a point in X_n with a point in the real interval $[0, 1]$, by

$$(x_1, \dots, x_n) \mapsto 0.x_1 x_2 \cdots x_n \in [0, 1), \tag{5.9}$$

where the right expression means the v-adic decimal expansion. Then, we have a map

$$X_n^s \to [0, 1]^s.$$

We shall define (t, m, s)-nets following Niederreiter [17, Definition 4.1]. An elementary interval of type $(d_1, \ldots, d_s)$ is a subset of $[0, 1]^s$ of the form

$$\prod_{i=1}^{s} [a_i v^{-d_i}, (a_i + 1) v^{-d_i}),$$

where a_i and d_i are non-negative integers such that $a_i < v^{d_i}$. Its volume is

$$\prod_i^s v^{-d_i} = v^{-\sum_{i=1}^s d_i}.$$

Definition 5.7. Let $0 \leq t \leq m$ be integers. A (t, m, s)-net in base v is a multi-subset $P \subset [0, 1]^s$ consisting of v^m points, such that every elementary interval of type $(d_1, \ldots, d_s)$ with $d_1 + \cdots + d_s = m - t$ (hence volume v^{t-m}) contains exactly v^t points of P.

Our purpose is to state these conditions in terms of designs $Y \subset X_n^s$ for a finite multi-subset Y.

Definition 5.8. A finite multi-subset Y in X is said to be uniform on X if $\#(Y \cap \{x\})$ is constant for $x \in X$, that is, $\#g^{-1}(x)$ is constant for $x \in X$ if Y is defined by a map $g : Z \to X$.

Definition 5.9. For an s-tuple of non-negative integers $\mathbf{d} = (d_1, \ldots, d_s)$ with each component less than or equal to n, define

$$X_{\mathbf{d}} := \prod_{i=1}^{s} Z_v^{d_i},$$

and the projection

$$\mathrm{pr}_{\mathbf{d}} : X_n^s \to X_{\mathbf{d}}$$

by taking the left most d_i components of the i-th X_n in X_n^s, for $i = 1, \ldots, s$.

Definition 5.10. A non-empty multi-subset $Y \subset X_n^s$ is $\mathbf{d}$-balanced, if the image of Y in $X_{\mathbf{d}}$ by the map $X_n^s \to X_{\mathbf{d}}$ is uniform on $X_{\mathbf{d}}$.

Proposition 5.11. A multi-subset $Y \subset X_n^s$ of cardinality v^m gives a (t, m, s)-net if and only if it is $\mathbf{d}$-balanced for any non-negative tuples $\mathbf{d} = (d_1, \ldots, d_s)$ with $d_1 + \cdots + d_s = m - t$.

The proof is immediate when we consider the mapping (5.9), since then there is a natural one-to-one correspondence between $X_{\mathbf{d}}$ and the set of elementary intervals of type $(d_1, \ldots, d_s)$. We may state the conditions in terms of the dual group.

Lemma 5.12. Let X be a finite abelian group, and Y a finite multi-subset in X. For $f \in C(X)$, define

$$I(f) := \mathrm{avg}_X(f) = \frac{1}{\#X} \sum_{x \in X} f(x).$$

(This is the true integral of f over the finite set X). The following are equivalent.
  (1) $I(f) = \mathrm{avg}_Y(f)$ holds for any $f \in C(X)$.
  (2) The finite multi-subset Y is uniform on X.
  (3) $\mathrm{avg}_Y(\xi) = 0$ holds for any non-trivial $\xi \in \check{X}$.

Proof. Suppose that the first condition holds. Let χ_x be the indicator function of $x \in X$. The condition $I(\chi_x) = \mathrm{avg}_Y(\chi_x)$ implies $\frac{1}{\#X} = \frac{\#(Y \cap \{x\})}{\#Y}$, hence Y is uniform on X. The converse is obvious.

Suppose again the first condition. For any non-trivial ξ, $I(\xi) = 0$. This implies that $\mathrm{avg}_Y(\xi) = 0$. For the converse, we assume the third condition. Note that $\check{X}$ is a base of $C(X)$, so it suffices to show that $I(\xi') = \mathrm{avg}_Y(\xi')$ holds for any $\xi' \in \check{X}$. This holds for non-trivial ξ, and for the trivial $\xi = 1$, $I(1) = 1 = \mathrm{avg}_Y(1)$. $\square$

Lemma 5.13. *The dual $\check{X}_{\mathbf{d}}$ of $X_{\mathbf{d}}$ can be identified with the set of characters*

$$\{\xi := (\xi_1, \ldots, \xi_s) \in \check{X}_n^{\,s} \mid \mathrm{top}(\xi_i) \leq d_i \text{ for } i = 1, \ldots, s\}$$

(see Definition 4.25 for the notation top*).*

Corollary 5.14. *For a multi-subset $Y \subset X_n^s$, the composition $Y \to X_n^s \to X_{\mathbf{d}}$ is uniform if and only if*

$$\mathrm{avg}_Y(\xi) = 0$$

holds for any non-trivial $\xi \in \check{X}_{\mathbf{d}}$.

Corollary 5.15. *A multi-subset $Y \subset X_n^s$ of cardinality v^m is a (t, m, s)-net if and only if*

$$\mathrm{avg}_Y(\xi) = 0$$

holds for any $\xi \in \check{X}_{\mathbf{d}} \setminus \{1\}$ and any non-negative $\mathbf{d} = (d_1, \ldots, d_s)$ satisfying $\sum_{i=1}^s d_i = m - t$.

Niederreiter[17] and Rosenbloom-Tsfasman[23] defined the notion of NRT-weight, see also Niederreiter and Pirsic[20]:

Definition 5.16. Define the NRT-weight wt by the composition

$$\mathrm{wt} : \check{X}_n^s \overset{\mathrm{top}^s}{\to} J_n^s \overset{\mathrm{sum}}{\to} \mathbb{N},$$

where sum is a function taking the sum of the s coordinates and top is defined in Definition 4.25.

Theorem 5.17. *Let $Y \subset X_n^s$ be a multi-subset of cardinality v^m. Then it is a (t, m, s)-net if and only if for any $\xi \in \check{X}_n^s$ with $\mathrm{wt}(\xi) \leq m - t$ and $\xi \neq 1$, $\mathrm{avg}_Y(\xi) = 0$.*

This follows from Corollary 5.15. Following Martin-Stinson[16], let us define

$$\mathrm{shape} : J_n^s \to J_n^s / S_s.$$

In Proposition 4.33 and Lemma 4.26, we observed that for $\bar{j} \in J_n^s / S_s$, the corresponding eigenspace $C(X_n^s)_{\bar{j}}$ is spanned by the characters in the inverse image of $\bar{j}$ by

$$\check{X}_n^s \overset{\mathrm{top}^s}{\to} J_n^s \to J_n^s / S_s.$$

It is easy to see that wt factors as

$$\check{X}_n^s \overset{\mathrm{top}^s}{\to} J_n^s \overset{\mathrm{shape}}{\to} J_n^s / S_s \overset{\mathrm{sumup}}{\to} \mathbb{N},$$

where the definition of sumup is clear from the context. (We remark that sumup is denoted by "height" in Martin-Stinson[16, P.337], but we changed the notation to avoid the collision with a standard use of "height" in the context of the Young diagrams.) Thus,

$$\{\xi \in \check{X}_n^s \mid \mathrm{wt}(\xi) \leq m - t\} \quad = \quad \{\xi \in \check{X}_n^s \mid \mathrm{sumup}(\mathrm{shape}(\mathrm{top}^s(\xi))) \leq m - t\}.$$

The condition for $Y \subset X_n^s$ to be a (t, m, s)-net is that $\mathrm{avg}_Y(\xi)$ vanishes for any nontrivial ξ in the left hand side (Theorem 5.17), which is equivalent to the vanishing in the right hand side, namely, $\mathrm{avg}_Y(\xi) = 0$ for each of $\xi \in C(X_n^s)_{\bar{j}}$ for $\bar{j} \in J_n^s/S_s \setminus \{j_0\}$, $\mathrm{sumup}(\bar{j}) \leq m - t$. This is equivalent to $\mathrm{avg}_Y^{\bar{j}} = 0$ (the left hand side is the representation of avg_Y in the subspace $C(X_n^s)_{\bar{j}}$ defined in (5.3) with property (5.4)) for these $\bar{j}$.

Now we proved [16, Theorem 3.4]:

Theorem 5.18. *Let $Y \subset X_n^s$ be a multi-subset of cardinality v^m. Then, Y is a (t, m, s)-net if and only if $\mathrm{avg}_Y^{\bar{j}} = 0$ for any $\bar{j} \neq j_0$, $\mathrm{sumup}(\bar{j}) \leq m - t$.*

In terms of the designs (see Section 5.2), we may state

Corollary 5.19. *Define*
$$J_D := \{\bar{j} \in J_n^s/S_s \mid \bar{j} \neq j_0, \ \mathrm{sumup}(\bar{j}) \leq m - t\}.$$
Then, $Y \subset X_n^s$ is a J_D-design if and only if it is a (t, m, s)-net in X_n^s.

By this, Delsarte's LP method works, see [16]. The next is an example where we may use a profinite association scheme to make a concept of (t, s)-sequence inside the design theory, (see [18, Definition 4.2] and [19, Remark 2.2]).

Let us consider the pro-ordered Hamming scheme $\overrightarrow{H}(s, \infty, v) = (X^\wedge, R^\wedge)$ defined in Proposition 4.35. Note that $X^\wedge = (\varprojlim X_n)^s = \varprojlim(X_n^s) = (Z_v^{\mathbb{N}>0})^s$. For each n, we denote by π_n the surjection from $X^\wedge$ onto X_n^s. That is,
$$\pi_n(x^1, \ldots, x^s) := ([x^1]_n, \ldots, [x^s]_n) \tag{5.10}$$
for each $(x^1, \ldots, x^s) \in X^\wedge = (Z_v^{\mathbb{N}>0})^s$, where we put $[(x_1, \ldots)]_n := (x_1, \ldots, x_n)$ for each $(x_1, \ldots) \in Z_v^{\mathbb{N}>0}$.

Definition 5.20. A sequence of points $p_0, p_1, \ldots \in X^\wedge$ is a (t, s)-sequence in base v, if for any integers k and $m > t$, the point set $Y_{k,m} \subset X_m^s \subset [0, 1)^s$ consisting of the $\pi_m(p_j)$ with $kv^m \leq j < (k+1)v^m$ (considered as a multi-set) is a (t, m, s)-net in base v.

For a comparison with a standard Definition 5.22, see Remark 5.23. We may state the condition of (t, s)-sequences in terms of the designs in the pro-ordered Hamming scheme $\overrightarrow{H}(s, \infty, v)$. Recall that $J^\wedge = \mathbb{N}^s/S_s$, and we may define sumup so that the composition
$$\mathbb{N}^s \to J^\wedge = (\mathbb{N}^s/S_s) \overset{\mathrm{sumup}}{\to} \mathbb{N}$$
maps $(j_1, \ldots, j_s) \mapsto \sum_{i=1}^s j_i$.

Theorem 5.21. *In the pro-ordered Hamming scheme, we define*
$$J_{D_\ell} := \{h \in (\mathbb{N}^s/S_s) = J^\wedge \mid \mathrm{sumup}(h) \leq \ell, h \neq j_0\}.$$
A sequence of points $p_0, p_1, \ldots \in X^\wedge$ is a (t, s)-sequence in base v, if and only if for any integers k and $m > t$, the point set $Y_{k,m}$ consisting of the $\pi_m(p_j)$ with $kv^m \leq j < (k+1)v^m$ is a $J_{D_{m-t}}$-design.

In the original definition, the sequence is taken in an infinite set $[0, 1]^s$, where infinite precision is necessary. Because of this infiniteness, a fixed finite scheme would not be able to describe the notion of (t, s)-sequence. Our profinite association scheme is a tool to deal with the infiniteness.

We close our paper with a discussion on the definitions of (t, s)-sequences. The following is a definition by Niederreiter-Özbudak[19, Definition 2.2] (to be precise, its special case), which is a slight variant of the original Niederreiter's one: [18, Definition 4.2].

Definition 5.22. Let $p_0, p_1, \ldots \in [0, 1]^s$ be a sequence of points, with prescribed v-adic expansions. (In other words, each p_j is considered as an element in $X^\wedge$.) It is a (t, s)-sequence in base v, if for any integers k and $m > t$, the point set $P_{k,m}$ consisting of the $\pi_m(p_j)$ with $kv^m \leq j < (k+1)v^m$ is a (t, m, s)-net in base v.

Remark 5.23. It is easy to see that Definitions 5.20 and 5.22 are the same. In Definition 5.22, we use π_m in (5.10) for p_j, where p_j is an element of $X^\wedge$. In fact, π_m is not well-defined for $[0, 1]^s$, because of the non-injectivity of

$$(Z_v^{\mathbb{N}>0}) \to [0, 1],$$

namely, for base $v = 2$, for example, $0.0111\cdots = 0.1000\cdots$. One way to avoid this subtle problem may be to require the v-adic expansion of a real number to be of the latter type, namely, to avoid expansions consisting of all $(v-1)$'s after some digit. However, in important constructions such as in Niederreiter-Xing[21], points with the former type of expansions may appear. Thus, to define the notion of (t, s)-sequences as a sequence in $[0, 1]^s$ has a subtle problem. Definition 5.20 would be a natural definition of a (t, s)-sequence. In other words, the notion would be better defined in terms of $X^\wedge$ rather than $[0, 1]^s$. This is essentially stated in [19, Remark 2.2]. Of course, a large part of researchers would prefer to work in $[0, 1]^s$.

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Mathematics Program, Graduate School of Advanced Science and Engineering, Hiroshima University, 739-8526 Japan

Email address: m-mat@math.sci.hiroshima-u.ac.jp

Mathematics Program, Graduate School of Advanced Science and Engineering, Hiroshima University, 739-8526 Japan

Email address: knt-ogawa@hiroshima-u.ac.jp

Mathematics Program, Graduate School of Advanced Science and Engineering, Hiroshima University, 739-8526 Japan

Email address: okudatak@hiroshima-u.ac.jp