SYMPLECTIC CACTI, VIRTUALIZATION AND BERENSTEIN–KIRILLOV GROUPS

OLGA AZENHAS, MOJDEH TARIGHAT FELLER, AND JACINTA TORRES

ABSTRACT. We explicitly realize an internal action of the *symplectic* cactus group, recently defined by Halacheva for any complex, reductive, finite-dimensional Lie algebra, on crystals of Kashiwara–Nakashima tableaux. Our methods include a symplectic version of jeu de taquin due to Sheats and Lecouvey, symplectic reversal, and virtualization due to Baker. As an application, we define and study a symplectic version of the Berenstein–Kirillov group and show that it is a quotient of the symplectic cactus group. In addition two relations for symplectic Berenstein–Kirillov group are given that do not follow from the defining relations of the symplectic cactus group.

CONTENTS

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1. Introduction

The *cactus group* was originally defined by Henriques–Kamnitzer [HeKa06-1] in the context of coboundary categories defined by Drinfeld [Dr90]. Coboundary categories are monoidal categories equipped with a *commutor*, that is, a collection of natural isomorphisms $\sigma_{A,B} : A \otimes B \to B \otimes A$ satisfying certain properties. The idea of studying the cactus group was originally due to A. Berenstein and was taken up by Henriques–Kamnitzer in [HeKa06-1], who defined it and further showed that it can be realized as the fundamental group of the moduli space of marked real genus zero stable curves. The original idea of Berenstein was to construct a commutor in the category of crystals of a complex, reductive, finite-dimensional Lie algebra, by first defining an involution $\xi_{\mathsf{B}} : \mathsf{B} \to \mathsf{B}$ for each crystal B which flips the crystal by exchanging highest weight elements with lowest weight elements. In the case of $\mathfrak{sl}(n, C)$ with the tableau model for the highest weight crystal $\mathsf{B}(\lambda)$ it was known that $\xi_{\mathsf{B}(\lambda)}$ coincides with the Schützenberger involution on semi-standard Young tableaux of shape λ [BerZel96]. See [BuSc17, Sections 4.3, 14.3.3] and the references therein.

Let $\mathfrak{g}$ be a complex, semisimple Lie algebra with Dynkin diagram X. There is a Dynkin diagram automorphism $\theta : X \to X$ defined by $\alpha_{\theta(i)} = -w_0\alpha_i$, where w_0 is the longest element of the Weyl group of $\mathfrak{g}$. The *cactus group* $J_{\mathfrak{g}}$, defined by Halacheva in [Ha20, Ha16], is the group generated by σ_I, where I runs over all connected sub-Dynkin diagrams of X, subject to the following relations:

$$\sigma_I^2 = 1, \tag{1}$$
$$\sigma_I\sigma_J = \sigma_J\sigma_I \text{ if } J \subseteq X,\ J \cup I \text{ is disconnected} \tag{2}$$
$$\sigma_I\sigma_J = \sigma_{\theta_I(J)}\sigma_I \text{ if } J \subset I \tag{3}$$

where θ_I is the automorphism on I defined by the longest element of the parabolic group W^I. Halacheva has defined an internal action of the cactus group $J_{\mathfrak{g}}$ on a normal $\mathfrak{g}$–crystal by partial Schützenberger–Lusztig involutions ξ_I. From this action we know that partial Schützenberger–Lusztig involutions satisfy the cactus group $J_{\mathfrak{g}}$ relations [HaKaRyWe20]. Halacheva [Ha20] initiated a combinatorial study of the cactus group for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ by comparing the action of $J_n = J_{\mathfrak{sl}(n,\mathbb{C})}$ on a normal $\mathfrak{sl}(n, \mathbb{C})$-crystal with that of the Berenstein–Kirillov group on Gelfand-Tsetlin patterns (or semi-standard Young tableaux) [BerKir95]. Using a different approach, Chmutov, Glick and Pylyavskyy [CGP16] have also found relationships between those two groups.

Our results compose a combinatorial study of the cactus group for the symplectic Lie algebra $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$. There are many combinatorial models for $\mathfrak{sp}(2n, \mathbb{C})$-crystals: De Concini tableaux [DeCo79], King tableaux [Ki75], Lakshmibai-Seshadri [LakSes91] and Littelmann paths [Lit95, Lit97], the alcove path model of Gaussent–Littelmann [GL05] and the one of Lenart–Postnikov [LenPos08], but we work with Kashiwara–Nakashima tableaux, for which a rich combinatorial structure exists [KasNak91, HonKan02, Lec02, Lec07]. We review the basics in Sections 3 and 4. For each connected sub-Dynkin diagram I of X, we define the explicit action of ξ_I on a given Kashiwara–Nakashima tableau. The algorithmic procedure for that action is given by virtualization. In the case when I forms a Dynkin diagram of type C_{n-k}, it is also given by the *I-partial symplectic reversal*, a symplectic analogue of partial reversal on A_{n-1} tableaux. Thereby we provide a combinatorial action of the cactus generators σ_I on the set of Kashiwara–Nakashima tableaux on the alphabet $\mathcal{C}_n$. This is addressed in Sections 8 and 9. The case of $I = X$ has already been developed by Santos in [Sa21a], where he defines an operation on straight shaped Kashiwara–Nakashima tableaux which is a *symplectic analogue* of the Schützenberger involution operation, also known as evacuation, on straight shaped A_{n-1} semi-standard Young tableaux. This procedure includes the symplectic jeu-de-taquin defined by Sheats in [Sh99], and further developed by Lecouvey [Lec02] using crystal isomorphisms. This is the content of Section 7.

For $I \subseteq X$ such that I forms a Dynkin diagram of type C_{n-k}, we define an algorithm for *I-partial symplectic reversal* which generalizes Santos' algorithm in the sense that, when $I = X$, our algorithm is exactly the same. The symplectic C_{n-k} reversal extends symplectic C_{n-k} evacuation to arbitrary semi-standard skew tableaux on the alphabet $\mathcal{C}_{n-k}$ whose shift of the entries by k are admissible on the alphabet $\mathcal{C}_n$. The C_{n-k} reversal of a such semi-standard skew tableau P on the alphabet $\mathcal{C}_{n-k}$, is characterized to be the unique skew tableau coplactic equivalent to P and plactic equivalent to the C_{n-k} evacuation of the symplectic rectification of P.

An important inspiration behind our generalization is the operation of *tableau-switching* [BSS96] of Benkart, Sottile and Stroomer on A_{n-1} semi-standard tableaux. Given an admissible tableau on the alphabet $\mathcal{C}_n$, we start off by freezing the entries corresponding to nodes not appearing in I, creating at the same time a new Young tableau U with Young shape defined by the positive frozen entries as well as a skew tableau P consisting of the non-frozen entries. The tableau pair (U, P), sharing a common border, pass through each other via *symplectic jeu de taquin* (SJDT for short). After performing this procedure, a new pair (R, V) with R the symplectic rectification of P and V consisting of the entries of U as well as some new, *colored* letters. Each color records a precise instance of the symplectic rectification of P. Our *symplectic colourful tableau switching* is reversible since SJDT is reversible. It reduces to the A_{n-1} tableau switching on tableaux in the alphabet $[n]$. This work is carried out in detail in Subsection 9.2 of this paper, yielding the formula (52), and illustrated in Subsection 9.3.

For the general case we use the virtualization map defined by Baker [Ba00a], that is, an injective map

$$E : \mathsf{KN}(\lambda, n) \longrightarrow \mathsf{SSYT}(\lambda^A, n, \bar{n})$$

which assigns to the $\mathfrak{sp}(2n, \mathbb{C})$-crystal $\mathsf{KN}(\lambda, n)$, a subset of the $\mathfrak{sl}(2n, \mathbb{C})$-crystal $\mathsf{SSYT}(\lambda^A, n, \bar{n})$ in a reversible way. This is discussed in Section 5. We show that one may apply the map E, then perform a certain partial Schützenberger–Lusztig involution in the type $\mathfrak{sl}(2n, \mathbb{C})$-crystal without leaving the image of E, reverse the virtualization map E and obtain our desired result. Subsection 9.4.1 provides Theorem 5 and Theorem 6 with such algorithmic procedures. Additionally, in Definition 3, Section 6, we define the *virtual symplectic cactus group* $\widetilde{J}_{2n}$ and show that it is a subgroup of J_{2n} isomorphic to the symplectic cactus group $J_{\mathfrak{sp}(2n,\mathbb{C})}$. In Theorem 3, Section 8, an action of the virtual symplectic cactus group on the set $\mathsf{SSYT}(\lambda^A, n, \bar{n})$ is defined. The subset $E(\mathsf{KN}(\lambda, n))$ is preserved under this action as shown in Subsections 9.4.1 and 9.5. In particular, in Subsection 9.5, we realize such action of the virtual symplectic cactus group on the virtual images of Kashiwara–Nakashima tableaux and show that it *virtualizes* the action of the symplectic cactus group on Kashiwara–Nakashima tableaux. This work is illustrated in Section 9.6.

As an application, in Section 10, we define *symplectic* Bender–Knuth involutions combinatorially (Definition 7). We start off by defining the type C_n Berenstein–Kirillov group $\mathcal{BK}^{C_n}$ as the free group generated by the partial symplectic Schützenberger–Lusztig involutions with respect to connected subdiagrams of the type C_n Dynkin diagram of the form $I = [n]$ modulo the relations they satisfy on Kashiwara–Nakashima tableaux of any straight shape in the alphabet $\mathcal{C}_n$. These generators of $\mathcal{BK}^{C_n}$ satisfy the relations of the symplectic cactus group (Theorem 8). We show that symplectic Bender–Knuth involutions are also generators of $\mathcal{BK}^{C_n}$.

We study relations for $\mathcal{BK}^{C_n}$ under the virtualization map E. More precisely, we consider the relations satisfied by the embedding of generators of $\mathcal{BK}^{C_n}$ in $E(\mathsf{KN}(\lambda, n))$

$\subseteq \mathsf{SSYT}(\lambda^A, n, \bar{n})$; we call this group (Definition 8) the *virtual symplectic Berenstein–Kirillov group* $\widetilde{\mathcal{BK}}_{2n}$, a subgroup of the type A_{2n-1} Berenstein–Kirillov group $\mathcal{BK}_{2n}$ satisfying, in particular, the relations of the virtual cactus group $\widetilde{J}_{2n}$ (Theorem 9). Proposition 12 gives the virtual symplectic Bender–Knuth involutions generators of $\widetilde{\mathcal{BK}}_{2n}$ which are shown in Theorem 10 to be the virtualization of the symplectic Bender–Knuth involutions. The virtual image of the group $\mathcal{BK}^{C_n}$ satisfies the relations of $\widetilde{\mathcal{BK}}_{2n}$. Some of the ones listed in Proposition 13 are obtained by applying the partial inverse to the virtualization map. Relations (9) and (10) in Proposition 13 are the only ones that do not follow from the the symplectic cactus group $J_{\mathfrak{sp}(2n,\mathbb{C})}$. They are instead equivalent to the braid relations of type C_n Weyl group. In particular, relation (10) is not similar to any relation in previous Berenstein–Kirillov groups.

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3. Basics

Let $\mathfrak{g}$ be a finite dimensional, complex, semisimple Lie algebra. Let I be the Dynkin diagram associated to the root system of $\mathfrak{g}$, $\Delta = \{\alpha_i : i \in I\}$ the set of simple roots, W its Weyl group, generated by the simple reflections $\{r_i : i \in I\}$, and $w_0 \in W$ the longest Weyl group element. We will use the numbering of the vertices of I given by [Bo VI]. The Dynkin diagram has an automorphism, a permutation of its nodes which leaves the diagram invariant, $\theta : I \to I$ defined by $\alpha_{\theta(i)} = -w_0\alpha_i$, for any node $i \in I$, where w_0 is the longest element of W. We will also denote by Λ the integral weight lattice associated to the root system of $\mathfrak{g}$. It is generated by the fundamental weights ω_i, $i \in I$. For a connected sub-diagram of I, $J \subseteq I$, denote by $\theta_J : J \to J$ the Dynkin diagram automorphism that satisfies $\alpha_{\theta_J(j)} = -w_0^J\alpha_j$, for any node $j \in J$, where w_0^J is the longest element of the parabolic subgroup $W^J \subseteq W$ (the Weyl group for $\mathfrak{g}$ restricted to J) [BjBr05]. When $J = I$ one has the original notation $\theta_I = \theta$. We focus on the cases where $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C}), \mathfrak{sp}(2n,\mathbb{C})$. We will often abuse notation and write a Dynkin diagram I with n nodes as the interval $[n]$. The corresponding Weyl groups are the symmetric group $\mathfrak{S}_n$ on n letters and the hyperoctahedral group B_n respectively, where B_n is the free group generated by $r_1, \dots, r_{n-1}, r_n$ subject to the relations

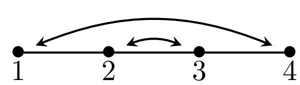
$$r_i^2 = 1,\ 1 \le i \le n, \tag{4}$$
$$(r_i r_j)^2 = 1,\ 1 \le i < j \le n,\ |i-j| > 1, \tag{5}$$
$$(r_i r_{i+1})^3 = 1, 1 \le i \le n-2, \tag{6}$$
$$(r_{n-1} r_n)^4 = 1. \tag{7}$$

The free group generated by $r_1, \ldots, r_{n-1}$, subject to the relations above, for $1 \le i, j < n$, is $\mathfrak{S}_n$ realized by the simple transpositions $r_i = (i, i+1)$ on the set $[n]$. The group B_n has $2^n n!$ elements and is realized by the signed transpositions $r_i = (i, i{+}1)(\bar{i}, \overline{i+1})$, $i = 1, \ldots, n-1$, and $r_n = (n, \overline{n})$ on the set $\{1 < \cdots < n < \bar{n} < \cdots < \bar{2} < \bar{1}\}$. That is, we may see B_n embedded in $\mathfrak{S}_{2n}$ by folding $\{1 < \cdots < n < \bar{n} < \cdots < \bar{2} < \bar{1}\}$ through a central symmetry. The long element of B_n has length n^2, while the long element of $\mathfrak{S}_n$ has length $n(n-1)/2$. For instance, $r_1 r_2 r_1 r_2 = r_2 r_1 r_2 r_1$ is the long element of B_2, and, more generally, $(r_n \cdots r_2 r_1)^n = (r_1 r_2 \cdots r_n)^n$ is the long element of B_n [BjBr05].

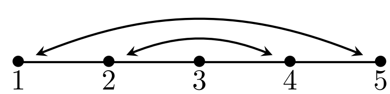
Occasionally, for the sake of clarity, we write w_0^A and w_0^C for the corresponding longest elements of $\mathfrak{S}_n$ and B_n respectively, or simply w_0 when there is no room for confusion. Given a vector $v \in \mathbb{Z}^n$, we have that r_i, with $i \in [n-1]$, acts on v, $r_i v$, swapping the i-th and the $(i+1)$-th entries, and r_n acts on v, $r_n v$, changing the sign of the last entry. Henceforth, w_0^A reverses v, $w_0^A(v_1, \ldots, v_n) = (v_n, \ldots, v_1)$, and w_0^C changes the sign of the entries of v, $w_0^C v = -v$.

Recall the $\mathfrak{sl}(n, \mathbb{C})$ simple roots $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$, $i \in [n-1]$, and the $\mathfrak{sp}(2n, \mathbb{C})$ simple roots $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$, $i \in [n-1]$ and $\alpha_n = 2\mathbf{e}_n$, where $\mathbf{e}_i$, $i \in [n]$, is the $\mathbb{R}^n$ standard basis. The A_{n-1} Dynkin diagram automorphisms above, since $-w_0 \alpha_i = -(-\alpha_{n-i}) = \alpha_{n-i}$, is given by $\theta(i) = n - i$, with $i \in I = [n-1]$. For instance,

$n = 5$ A_4



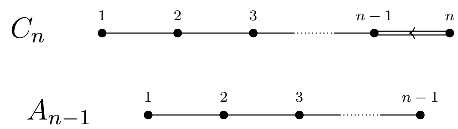
$n = 6$ A_5



The C_n Dynkin diagram automorphisms above, since for $w_0 \in B_n$, $-w_0 \alpha_i = -(-\alpha_i) = \alpha_i$, is given by $\theta(i) = i$, with $i \in I = [n]$. The weight lattices are $\Lambda = \mathbb{Z}^n$ for $\mathfrak{sp}(2n, \mathbb{C})$ and $\Lambda = \mathbb{Z}^n/(1, ..., 1)$ for $\mathfrak{sl}(n, \mathbb{C})$. We will often work with representatives in the case of $\mathfrak{sl}(n, \mathbb{C})$. The fundamental weights are $\omega_i = \sum_{j=1}^{i} e_i$, $1 \le i \le n$ and respectively have representatives ω_i, $1 \le i \le n-1$.

3.1. **Levi sub-diagrams.** Let I be a finite Dynkin diagram. A Levi sub-diagram J of I obtained by deleting from I a subset of its nodes is the Dynkin diagram of a semi-simple Lie algebra $\mathfrak{g}_J \subset \mathfrak{g}$ known as a *Levi sub-algebra* which is the Levi component of the parabolic Lie sub-algebra of $\mathfrak{g}$ generated by the Chevalley generators associated to the nodes of J.

Example 1. *If we remove the last node (the one labelled by n) from the Dynkin diagram of type C_n, we obtain a Dynkin diagram of type A_{n-1}. which corresponds to the Levi sub-algebra $\mathfrak{sl}(n, \mathbb{C})$ of $\mathfrak{sp}(2n, \mathbb{C})$.*



Example 2. *The semisimple Lie algebra $\mathfrak{sl}(3, \mathbb{C}) \times \mathfrak{sp}(4, \mathbb{C})$ is a Levi sub-algebra of $\mathfrak{sp}(12, \mathbb{C})$. Note that the semisimple Lie algebra $\mathfrak{sl}(n, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$ is not a Levi sub-algebra of $\mathfrak{sp}(2n, \mathbb{C})$, as its Dynkin diagram of type $A_{n-1} \times A_1$ cannot be obtained from the type C_n diagram by deleting some of its vertices.*

4. Normal $\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{sp}(2n, \mathbb{C})$-crystals and Levi restrictions

Crystals corresponding to finite-dimensional (quantum group) $U_q(\mathfrak{g})$-representations belong to a family of crystals called *normal crystals* [BuSc17, HaKaRyWe20]. In classical types, these crystals may be realized by a tableau model [KasNak91] and have nice combinatorial properties. Normal crystals arise as the crystals associated to the finite-dimensional representations of a quantum group $U_q(\mathfrak{g})$ for some Lie algebra $\mathfrak{g}$ [BuSc17]. These crystals decompose into connected components, one for each irreducible component to the representation at hand. The Levi restriction of a normal crystal is still a normal crystal, and the union of some connected components of a normal crystal is also a normal crystal [BuSc17, HaKaRyWe20]. The crystals that we deal with are tableau crystals for finite-dimensional representations of $U_q(\mathfrak{sl}(n, \mathbb{C}))$ and $U_q(\mathfrak{sp}(2n, \mathbb{C}))$.

A $\mathfrak{g}$-crystal is a finite set B along with maps

$$\mathsf{wt} : \mathsf{B} \to \Lambda, \quad e_i, f_i : \mathsf{B} \to \mathsf{B} \cup \{0\}, \, \varepsilon_i, \varphi_i : \mathsf{B} \to \mathbb{Z},$$

obeying the following axioms for any $b, b' \in \mathsf{B}$ and $i \in I$,

- $b' = e_i(b)$ if and only if $b = f_i(b')$,
- if $f_i(b) \neq 0$ then $\mathsf{wt}(f_i(b)) = \mathsf{wt}(b) - \alpha_i$;
  if $e_i(b) \neq 0$, then $\mathsf{wt}(e_i(b)) = \mathsf{wt}(b) + \alpha_i$, and
- $\varepsilon_i(b) = \max\{a \in \mathbb{Z}_{\geq 0} : e_i^a(b) \neq 0\}$ and $\varphi_i(b) = \max\{a \in \mathbb{Z}_{\geq 0} : f_i^a(b) \neq 0\}$.
- $\varphi_i(b) - \varepsilon_i(b) = \langle \mathsf{wt}(b), \alpha_i^\vee \rangle$,

where $\alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}$ are the coroots.

Remark 1. *Our abstract $\mathfrak{g}$-crystals are defined with the additional condition that they are seminormal* [BuSc17].

The *crystal graph* of B is the directed graph with vertices in B and edges labelled by $i \in I$. If $f_i(b) = b'$ for $b, b' \in \mathsf{B}$, then we draw an edge $b \to b'$. See Example 4. Given an arbitrary subset $J \subseteq I$, B_J is defined to be the crystal B restricted to the sub-diagram J of I, the Levi branched crystal. The crystal graph of B_J has the same vertices as B, but the arrows are only those labelled in J; that is, we forget the maps e_i, f_i, φ_i, and ε_i, for $i \notin J$ [BuSc17]. The weight map is $\mathsf{B} \overset{\mathsf{wt}}{\to} \Lambda \overset{can}{\to} \Lambda_J$, where wt is the weight map of B, Λ is the weight lattice of $\mathfrak{g}$, $\Lambda_J = \Lambda / < \omega_i : i \notin J >$ is the weight lattice of $\mathfrak{g}_J$, and $\Lambda \overset{can}{\to} \Lambda_J$ is the canonical projection. If $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ and we restrict to $J = [n-1]$, then we obtain a $\mathfrak{sl}(n, \mathbb{C})$-crystal. For instance, if we restrict an $\mathfrak{sp}(2n, \mathbb{C})$-crystal to $J = [n-1]$, then we obtain an $\mathfrak{sl}(n, \mathbb{C})$-crystal. Given $b \in \mathsf{B}$, $\mathsf{B}(b)$ denotes the connected component of B containing b.

A $\mathfrak{g}$-crystal is *normal* if it is isomorphic to a disjoint union of the crystals $\mathsf{B}(\lambda)$, where $\mathsf{B}(\lambda)$ is the crystal associated to an irreducible, finite-dimensional $\mathfrak{g}$-representation *of highest weight* λ, where $\lambda \in \Lambda$ is a *dominant weight*. In this work, where we focus on $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$, respectively $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, dominant weights in $\mathbb{Z}^n$, respectively in $\mathbb{Z}^n / < (1, ..., 1) >$, correspond precisely to *partitions*, that is, weakly decreasing vectors in $\mathbb{Z}^n$ with non-negative entries, respectively to weakly decreasing vectors in $\mathbb{Z}^n$, and each such representative is equivalent to a unique partition in $\mathbb{Z}^{n-1} \hookrightarrow \mathbb{Z}^n$, where the last entry is fixed as zero. An important property of normal crystals B is the existence of a unique highest weight vertex for each connected component of B, that is, an element which is a source in the corresponding crystal graph, whose weight is dominant. In $\mathsf{B}(\lambda)$, the highest weight vertex x has weight $\mathsf{wt}(x) = \lambda$. Note that we work solely with *highest weight crystals*, namely, crystals B such that for each $b \in \mathsf{B}$, there exists a finite sequence $a_1, a_2, \ldots, a_l \in I$ and a highest weight element $u_b \in \mathsf{B}(b)$ such that $b = f_{a_l} \cdots f_{a_2} f_{a_1}(u_b)$. For $b, b' \in \mathsf{B}$, we have $B(b) = B(b')$ if and only if $u_b = u_{b'}$. From now on, we will refer to $\mathfrak{sp}(2n, \mathbb{C})$-crystals by C_n-crystals, and $\mathfrak{sl}(n, \mathbb{C})$-crystals by A_{n-1}-crystals.

4.1. **Kashiwara–Nakashima tableaux.** Let $\mathsf{B}(\lambda)$ be the irreducible C_n-crystal with highest weight a partition λ of at most n parts. We realize $\mathsf{B}(\lambda)$ as the crystal $\mathsf{KN}(\lambda, n)$ of Kashiwara–Nakashima tableaux [KasNak91] of shape λ on the alphabet

$$\mathcal{C}_n = \{1 < \cdots < n < \bar{n} < \cdots < \bar{1}\}.$$

The irreducible A_{n-1}-crystal with highest weight a partition λ of at most n parts is realized as the crystal $\mathsf{SSYT}(\lambda, n)$ of semi-standard tableaux of shape λ on the alphabet $[n]$. We also will refer to these tableaux as the A_{n-1} tableaux of shape λ. The crystal $\mathsf{SSYT}(\lambda, n)$ is a connected sub-crystal of $\mathsf{KN}(\lambda, n)$. The *weight* of an A_{n-1} tableau T, respectively a Kashiwara–Nakashima tableau U, is represented by, respectively is, the vector $(\mu_1, ..., \mu_n) \in \mathbb{Z}^n$, where μ_i denotes the number of i's in T, respectively the number of i's minus the number of $\bar{i}$'s in U.

Kashiwara–Nakashima tableaux (KN for short) are semi-standard Young tableaux in the alphabet $\mathcal{C}_n$ which satisfy some extra conditions. They are a variation of De Concini symplectic tableaux [DeCo79]. A semi-standard Young tableau of any shape (skew or straight) with entries in $\mathcal{C}_n$ is KN if and only if the following two conditions hold.

- Each one of its columns is *admissible.*
- *Its splitting is a semi-standard Young tableau.*

Definition 1. *Let C be a semi-standard column in the alphabet $\mathcal{C}_n$ of length at most n. Let $Z = \{z_1 > ... > z_m\}$ be the set of non-barred letters z in $\mathcal{C}_n$ such that both*

z and $\bar{z}$ appear in C. We say that the column C is admissible if there exists a set $T = \{t_1 > ... > t_m\}$ of unbarred letters t that satisfies:

- *$t, \bar{t} \notin C$;*
- *$t_1 < z_1$ and is maximal with this property;*
- *$t_i < min(t_{i-1}, z_i)$ and is maximal with this property.*

The split of a column is the two-column tableau $lCrC$ where lC is the column obtained from C by replacing z_i by t_i and possibly re-ordering, and rC is obtained from C by replacing $\bar{z}_i$ by $\bar{t}_i$ and possibly re-ordering. The splitting of a tableau consisting of admissible columns is the concatenation of the splits of its columns.

Given $\mu \subseteq \lambda$ partitions with at most n parts, $\mathsf{KN}(\lambda/\mu, n)$ denotes the normal C_n-crystal of KN tableaux of skew shape λ/μ on the alphabet $\mathcal{C}_n$ [Lec02, Lemma 6.1.3, Corollary 6.3.9].

Example 3. *Let $n = 2$. The column $\begin{array}{|c|}\hline 2\\\hline \bar{2}\\\hline\end{array}$ is admissible, however, $\begin{array}{|c|}\hline 1\\\hline \bar{1}\\\hline\end{array}$ is not. Notice that although each one of its columns is admissible, the tableau $\begin{array}{|c|c|}\hline 2 & 2\\\hline \bar{2} & \bar{2}\\\hline\end{array}$ is not KN, because its split,*

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 1 & 2\\\hline \bar{2} & \bar{1} & \bar{2} & \bar{1}\\\hline\end{array}$$

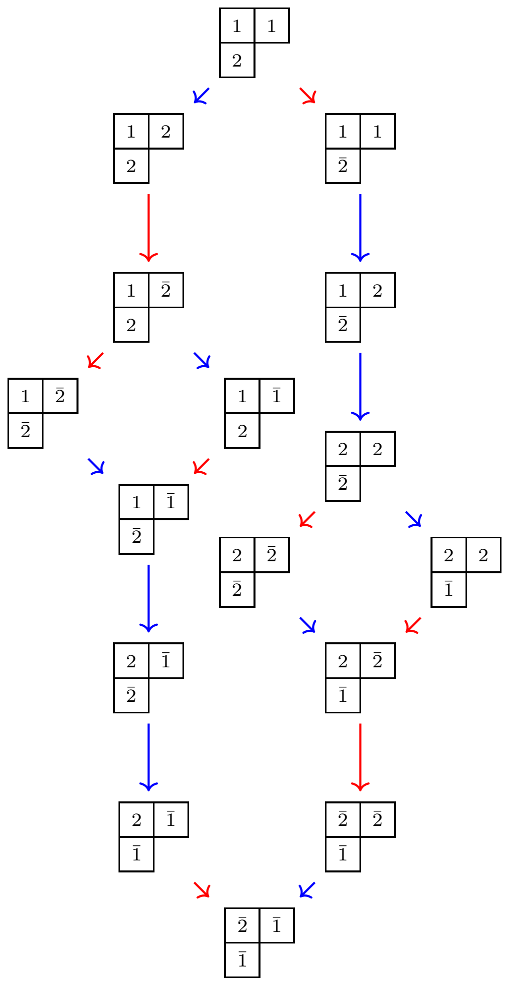
is not semi-standard.

We will mostly use the notation and definitions from [Lec02, Lec07]. We also refer the reader to the references therein.

Remark 2. [Lec02, Remark 2.2.2] *The maximal height of an admissible column is n. Moreover, a column C is admissible if and only if, for any $m \in [n]$,* ***the number*** *$N(m)$* ***of letters*** *x* ***in*** *C* ***such that either*** *$x \leq m$* ***or*** *$x \geq \bar{m}$ satisfies $N(m) \leq m$. Moreover, if there exists in C a letter $m \leq n$ such that $N(m) > m$, then C contains a pair $(z, \bar{z})$ satisfying $N(z) > z$.*

Remark 3. *In* [Lec02]*,* ***coadmissible*** *columns are defined as well (see* [Lec02, p.301]*). We will not delve into the details here, however, we remark that there exists a bijection between admissible and coadmissible columns given by filling in the shape of the given admissible column C with the unbarred letters of lC from top to bottom in increasing order, followed by the barred letters of rC in the same fashion. We will denote this bijection by Φ and use it in 7.3.*

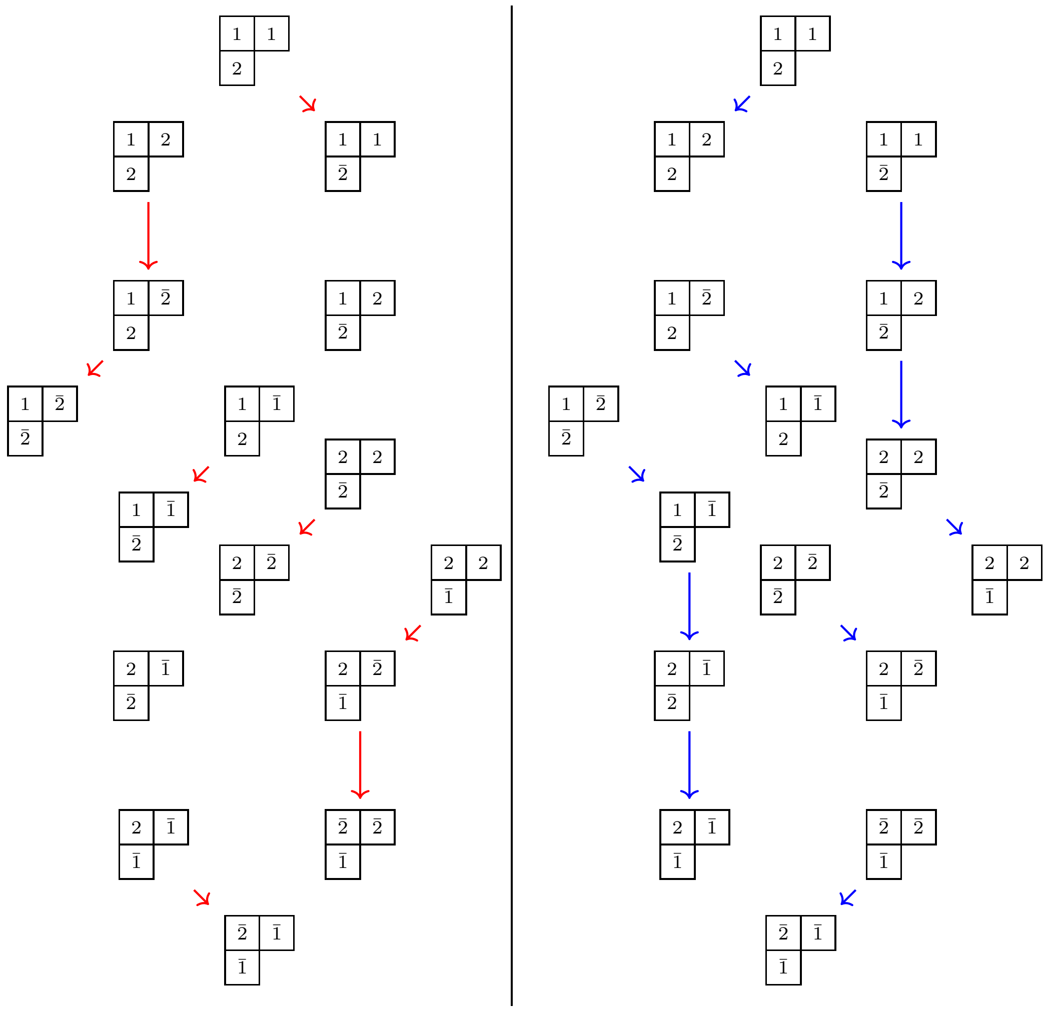
Example 4. *The C_2 crystal $\mathsf{KN}(\lambda, 2)$ of shape $\lambda = (2, 1)$. Each node in the graph represents an element of the crystal. There is a blue, respectively red, arrow connecting* *an element a to an element b whenever* $f_1(a) = b$*, respectively* $f_2(a) = b$*.*



4.1.1. *Levi branching of KN tableau crystals.* For $J \subseteq I$, $\mathsf{KN}_J(\lambda, n)$ is the restriction of $\mathsf{KN}(\lambda, n)$ to the sub-diagram J of I: as a crystal graph it has the same set of vertices as $\mathsf{KN}(\lambda, n)$ but only contains the arrows labelled by J, and it is also a normal crystal. The highest weight elements of $\mathsf{KN}_J(\lambda, n)$ are those C_n tableaux in $\mathsf{KN}(\lambda, n)$ where the only incoming edges are colored in $[n] \setminus J$.

Example 5. *We have the Levi-branched crystals* $\mathsf{KN}_{\{2\}}(\lambda, 2)$ *and* $\mathsf{KN}_{\{1\}}(\lambda, 2)$ *respectively from left to right for* $\lambda = (2, 1)$*. Both are* A_1*-crystals. The highest weights, with multiplicity, in the LHS have representatives* $(2, 1), (1, 2), (1, 0),$ $(0, 1), (0, 1), (-1, 2),$ $(-1, 0), (-2, 1)$*. In the quotient of* $\mathbb{Z}^2$ *by the fundamental weight* $\omega_1 = (1, 0)$*, these are equivalent to the vectors* $(0, 1), (0, 2), (0, 0),$ $(0, 1), (0, 1), (0, 2), (0, 0), (0, 1)$*, respectively. In practice, this means that we have ignored the multiplicity of the letters* $\{1, -1\}$ *in the tableaux of the LHS to compute the highest weights. On the RHS, we consider another embedding of* $\mathbb{Z} \hookrightarrow \mathbb{Z}^2$ *given by the quotient* $\mathbb{Z}^2 / < (1, 1) >$*, since*

$\omega_2 = (1,1)$. *The computation of the highest weights on the RHS is similar to that of the LHS, and we thus leave it as an exercise for the reader.*



If $J = [p,q]$, $1 \le p < q \le n$, the crystal graph $\mathsf{KN}_J(\lambda, n)$ consists of the KN tableaux of $\mathsf{KN}(\lambda, n)$ with arrows colored in J. Recall the C_n signature rule [KasNak91, Lec02, BuSc17] to compute the action of the crystal operators on a word in the alphabet $\mathcal{C}_n$.

If $q < n$, the Levi branched crystal $\mathsf{KN}_{[p,q]}(\lambda, n)$ is a type A_{q-p+1} normal crystal. The Weyl group is $W^J = \mathfrak{S}_{[p,q+1]}$, the symmetric group on the letters $\{p, \dots, q+1\}$ and generators $r_j = (j, j+1)(\bar{j}, \overline{j+1})$, $j \in J$. We say that the entries outside of $[\pm p, q+1] = \{p < \cdots < q+1\} \cup \{\overline{q+1} < \cdots < \overline{p}\}$ are *frozen*, which amounts to saying that the KN tableaux of the set $\mathsf{KN}(\lambda, n)$ in the same connected component of $\mathsf{KN}_{[p,q]}(\lambda, n)$ are stable in the entries over $\mathcal{C}_n \setminus [\pm p, q+1]$ under the action of the Kashiwara operators f_i, e_i, $i \in [p,q]$. That is, if $q < n$, in the same connected component of $\mathsf{KN}_{[p,q]}(\lambda, n)$, the subtableaux consisting of the letters $\{1 < \cdots < p-1\}$, $\{\overline{p-1} < \cdots < \overline{1}\}$ or $\{q+2 < \cdots < n < \bar{n} < \cdots < \overline{q+2}\}$ are the same.

If $q = n$, the Levi branched crystal $\mathsf{KN}_{[p,n]}(\lambda, n)$ is isomorphic to a type C_{n-p+1} normal crystal. The Weyl group is $W^J = B_{[p,n]}$ generated by the signed permutations on the subset $\{p < \cdots < n < \bar{n} < \cdots < \bar{i}\}$. The entries outside of $[\pm p, n] = \{p < \cdots < n < \overline{n} < \cdots < \overline{p}\}$ are *frozen*; within the same connected component of $\mathsf{KN}_J(\lambda, n)$, the subtableaux either consisting of the letters $\{1 < \cdots < p-1\}$ or $\{\overline{p-1} < \cdots < \overline{1}\}$ are the same. In Example 5, since $\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{sp}(2, \mathbb{C})$, we get two crystals of types $A_1 = C_2$.

5. Virtualization

In this section we closely follow Baker [Ba00a, Section 2] and adopt the notation used there. In Example 9.6, we present a detailed example of the content in this section. We include it later rather than earlier because it includes some more information which is not yet presented up to the end of this section.

5.1. **Baker embedding and Baker recording tableau.** Let

$$\lambda = \lambda_1\omega_1 + \cdots + \lambda_n\omega_n \in \mathbb{Z}^n$$

with $\omega_j = \sum_{i=1}^{j} e_i \in \mathbb{Z}^n$, $1 \leq j \leq n$ the fundamental weights of type C_n. Let

$$\omega_j^A = \sum_{i=1}^{j} e_i \in \mathbb{Z}^{2n} \text{ for } 1 \leq j \leq n \tag{8}$$

$$\omega_{\bar{j}}^A = \omega_{2n-j+1}^A = \sum_{i=1}^{2n-j+1} e_i \in \mathbb{Z}^{2n} \text{ for } 1 < j \leq n \tag{9}$$

be the A_{2n-1} fundamental weights, and consider as well the $\mathbb{Z}^{2n}$ partition

$$\lambda^A = 2\lambda_n\omega_n^A + \sum_{i=1}^{n-1} \lambda_i(\omega_i^A + \omega_{\overline{i+1}}^A).$$

Let $\mathsf{SSYT}(\lambda^A, n, \bar{n})$ be the type A_{2n-1} crystal of semi-standard Young tableaux in the alphabet $\mathcal{C}_n$ of shape λ^A. We will denote the corresponding crystal operators by f_i^A for $i \in \mathcal{C}_n$ and consider, for $1 \leq i \leq n$, the operators $f_i^E = f_i^A f_{\overline{i+1}}^A$, $i < n$, and $f_n^E = (f_n^A)^2$. Let E denote the *virtualization map* defined on type C_n Kashiwara–Nakashima tableaux defined by Baker [Ba00a, Proposition 2.2, Proposition 2.3]. More precisely, E is an injective map

$$E : \mathsf{KN}(\lambda, n) \hookrightarrow \mathsf{SSYT}(\lambda^A, n, \bar{n}) \tag{10}$$

such that $E(f_i(T)) = f_i^E(E(T))$ for $T \in \mathsf{KN}(\lambda, n), 1 \leq i \leq n$. We will denote by E^{-1} the restriction of any left inverse of E to the image of $\mathsf{KN}(\lambda, n)$ under E.
Given an admissible column C in the alphabet $\mathcal{C}_n$ of shape ω_i, $1 \leq i \leq n$, denote by $\psi(C)$ its *Baker virtual split* [Ba00a, Proposition 2.2], a two column type A_{2n-1} tableau of shape $\omega_i^A + \omega_{2n-i}^A$. The map ψ is injective and embeds admissible columns of length i, in the alphabet $\mathcal{C}_n$, into $\mathsf{SSYT}(\omega_i^A + \omega_{2n-i}^A, n, \bar{n}), 1 \leq i \leq n$. We define

ψ^{-1} analogously to E^{-1}. From [Ba00a, Proposition 2.3] we know that, if we write T as a concatenation of its columns, that is, $T = C_k \cdots C_1$, then

$$E(T) = [\emptyset \leftarrow w(\psi(C_1)) \leftarrow \cdots \leftarrow w(\psi(C_k))],$$

where the word $w(\psi(C))$ of the type A_{2n-1} two-column tableau $\psi(C)$ is given by the Japanese reading of its two columns (from top to bottom and right to left), and $P \leftarrow w$ is the Schensted column insertion of a word w into a type A_{2n-1} semi-standard Young tableau P in the alphabet $\mathcal{C}_n$ [Fu97, St01].

Let $T_\lambda \in \mathsf{KN}(\lambda, n)$ be the highest weight element; that is, T_λ is the Yamanouchi tableau of shape and weight λ on the alphabet $[n]$ (each row i is solely filled with the letter i). Then $E(T_\lambda) = T_{\lambda^A}$ is the highest weight element of $\mathsf{SSYT}(\lambda^A, n, \bar{n})$, that is, the A_{2n-1} Yamanouchi tableau of shape and weight λ^A in the alphabet $\mathcal{C}_n$. The image of $\mathsf{KN}(\lambda, n)$ by E in $\mathsf{SSYT}(\lambda^A, n, \bar{n})$ is the crystal generated by acting with the lowering operators f_i^E on the highest weight element T_λ^A of $\mathsf{SSYT}(\lambda^A, n, \bar{n})$. For $T \in \mathsf{KN}(\lambda, n)$, where $T = C_k \cdots C_1$, we write

$$w_T = w(\psi(C_1)) \cdots w(\psi(C_k)).$$

Then w_T is a word in $\mathcal{C}_n^*$, the monoid of words in the alphabet $\mathcal{C}_n$, and $E(T) = [\emptyset \leftarrow w_T]$. We will call the *recording tableau of the column insertion of* w_T, $Q(w_T)$, the *Baker recording tableau* associated to T.

Proposition 1. *For $T \in \mathsf{KN}(\lambda, n)$, the Baker recording tableau $Q(w_T)$ depends only on λ. From now on, we will denote by Q_λ the Baker recording tableau associated to λ.*

Proof. By abuse of notation, we will denote by the same symbols the type A_{2n-1} crystal operators on the A_{2n-1} crystal $\mathcal{C}_n^*$ of words and those on semi-standard Young tableaux in the same alphabet. Now, we know that there exists a sequence $1 \leq i_1, ..., i_k \leq n$ such that $f_{i_k} \cdots f_{i_1}(T_\lambda) = T$. Therefore $f_{i_k}^E \cdots f_{i_1}^E(E(T_\lambda)) = E(T)$, where $E(T_\lambda) = T_{\lambda^A}$, the highest weight element of $\mathsf{SSYT}(\lambda^A, n, \bar{n})$, and so

$$f_{i_k}^E \cdots f_{i_1}^E(w_{T_\lambda}) = w_T$$

(recall that $f_n^E = (f_n^A)^2$) because the connected components of the crystal $\mathcal{C}_n^*$ of words of type A_{2n-1} with highest weight elements w_{T_λ} and $w(E(T_\lambda)) = w(T_{\lambda^A})$ have the same highest weight λ^A and are hence isomorphic. In particular, both w_T and w_{T_λ} belong to the same connected component of the crystal $\mathcal{C}_n^*$ of words of type A_{2n-1}, namely, the connected component containing the Yamanouchi word w_{T_λ} of weight λ^A (recall that all words w_T have the same rectification shape λ^A and that all A_{2n-1} crystal operators commute with *jeu de taquin*). Now, we consider a version of the RSK correspondence [Fu97, St01, Kwo09, BuSc17] which is a bijection

$$\mathcal{C}_n^* \overset{1:1}{\longleftrightarrow} \bigcup_{\substack{\mu \\ \ell(\mu) \leq 2n}} \mathsf{SSYT}(\mu, n, \bar{n}) \times \mathsf{SYT}(\mu) \tag{11}$$

$$w \overset{\text{RSK}}{\mapsto} (P(w), Q(w)) \tag{12}$$

where $\mathsf{SYT}(\mu)$ is the set of standard Young tableaux of shape μ, $P(w) = [\emptyset \leftarrow w]$ and $Q(w)$ is the corresponding recording tableau which encodes the sequence of shapes produced by the column insertion of w. In particular for each standard Young tableau Q of shape μ the pre-image $\mathrm{RSK}^{-1}(\mathsf{SSYT}(\mu, n, \bar{n}) \times \{Q\})$ is a crystal isomorphic to $\mathsf{SSYT}(\mu, n, \bar{n})$, and all of these pre-images are disjoint and cover $\mathcal{C}_n^*$. In particular this means that all the words w_T for $T \in \mathsf{KN}(\lambda, n)$ are contained in the same connected component of $\mathcal{C}_n^*$ defined by:

$$\mathrm{RSK}^{-1}(\mathsf{SSYT}(\lambda^A, n, \bar{n}) \times \{Q(w_{T_\lambda})\}).$$

Thereby, $Q(w_T) = Q(w_{T_\lambda})$ for all $T \in \mathsf{KN}(\lambda, n)$. □

Corollary 1. *Let $\lambda = \omega_{m_1} + \cdots + \omega_{m_k}$, $1 \leq m_1 \leq \cdots \leq m_k \leq n$, and let*

$$\lambda^A = \omega^A_{2n-m_1} + \omega^A_{m_1} + \cdots + \omega^A_{2n-m_k} + \omega^A_{m_k} \in \mathbb{Z}^{2n}.$$

Then Q_λ can be written out of the shape λ^A as a sequence of shapes by adding successively the columns $\omega^A_{m_1}, \omega^A_{2n-m_1}, \ldots, \omega^A_{m_k}, \omega^A_{2n-m_k}$, whose boxes are filled along columns, top to bottom with consecutive numbers from 1 to $|\lambda^A|$:

$$\emptyset \subset \omega^A_{m_1} \subset \omega^A_{2n-m_1} + \omega_{m_1} \subset \omega^A_{m_2} + \omega^A_{2n-m_1} + \omega^A_{m_1}$$

$$\subset \omega^A_{2n-m_2} + \omega^A_{m_2} + \omega^A_{2n-m_1} + \omega^A_{m_1}$$

$$\subset \cdots \subset \omega^A_{m_k} + \cdots + \omega^A_{2n-m_2} + \omega^A_{m_2} + \omega^A_{2n-m_1} + \omega^A_{m_1}$$

$$\subset \omega^A_{2n-m_k} + \omega^A_{m_k} + \cdots + \omega^A_{2n-m_1} + \omega^A_{m_1} = \lambda^A.$$

Given a partition λ with at most n parts, and $T = C_k \cdots C_1 \in \mathsf{KN}(\lambda, n)$, let $\Psi(T) = (w(\psi(C_1)), \ldots, w(\psi(C_k))) \in \mathcal{C}_n^*$ (here the word is presented as a k-tuple) and $\Psi^{-1} = \underbrace{(\psi^{-1}, \ldots, \psi^{-1})}_{k}$. Then $(E(T), Q_\lambda) = \mathrm{RSK}\Psi(T) = (P(w_T), Q_\lambda)$ and

$$E^{-1} = \Psi^{-1}\mathrm{RSK}^{-1}_{|E(\mathsf{KN}(\lambda,n))\times\{Q_\lambda\}}$$

where $\mathrm{RSK}^{-1}_{|\mathsf{KN}(\lambda,n)\times\{Q_\lambda\}}$ denotes the inverse of RSK restricted to $E(\mathsf{KN}(\lambda, n)) \times \{Q_\lambda\}$.

The computation of $\mathrm{RSK}^{-1}_{|\mathsf{KN}(\lambda,n)\times\{Q_\lambda\}}$ uses Q_λ to perform the inverse of column Schensted insertion. See Example 9.6.

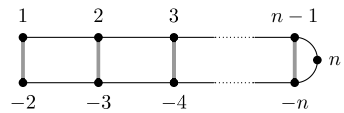
Remark 4. *Let $T \in \mathsf{KN}(\lambda, n)$ and $E(T) \in \mathsf{SSYT}(\lambda^A, n, \bar{n})$. Then*

$$\mathsf{wt}(E(T)) = \mathsf{wt}(w_T) = (\alpha_1, \ldots, \alpha_n, \alpha_{\bar{n}}, \ldots, \alpha_{\bar{1}}) \in \mathbb{Z}^{2n}_{\geq 0}$$

is such that

$$2\mathsf{wt}(T) = (\alpha_1 - \alpha_{\bar{1}}, \ldots, \alpha_n - \alpha_{\bar{n}}) \in \mathbb{Z}^n. \tag{13}$$

5.2. **The Levi branched crystal and virtualization.** Recall that a Levi branched crystal B_J, $J \subseteq I$, I a Dynkin diagram, is obtained by ignoring the maps $f_i, e_i, \varphi_i, \varepsilon_i$, for $i \notin J$. Let I be the A_{2n-1} Dynkin diagram with nodes $\{1, \dots, n, \bar{n}, \dots, \bar{2}\}$.



For each connected sub-diagram $J = [p, q]$ or $[k, n]$ with $1 \le p \le q < n$ and $k \le n$, of $[n]$, let $\bar{J} = [\overline{q+1}, \overline{p+1}]$ respectively $[\bar{n}, \overline{k+1}]$, if $k < n$, and $\bar{J} = \emptyset$ if $k = n$ be the corresponding connected sub-diagram of $[\bar{n}, \bar{2}]$.

Each connected component of the Levi branched crystal $\mathsf{KN}_{J \cup \bar{J}}(\lambda, n)$ with $J = [p, q]$, $[k, n]$, $1 \le p \le q < n$, $k \le n$, is embedded via E into a connected component of the Levi branched crystal $\mathsf{SSYT}_{J \cup \bar{J}}(\lambda, n)$ such that $J \cup \bar{J}$ is a disconnected diagram of $[1, \dots, n, \bar{n}, \dots \bar{2}]$ if $q < n$, and otherwise, $J \cup \bar{J} = [k, \overline{k+1}]$ or $\overline{J} = J$ if $J = \{n\}$. Consider the Levi branching of the type C_n crystal $\mathsf{KN}(\lambda, n)$ to $A_{q-p+1}, 1 \le p \le q < n$, and C_{n-k+1}, $k \le n$. The Levi type A_{q-p+1} Dynkin diagram is obtained via folding from the Levi subtype $A_{q-p+1} \times A_{q-p+1}$ of A_{2n-1} which is obtained by removing the nodes $1, \dots, p-1, q+1, \dots, n, \bar{n}, \dots, \overline{q+2}, \overline{p+2}, \dots, \bar{2}$ from the A_{2n-1} Dynkin diagram. The Levi type $C_{n-k+1}, k \le n$, is obtained via folding from the Levi subtype $A_{2n-2k+1}$ of A_{2n-1} obtained by removing the nodes $1, \dots, k-1, \overline{k}, \dots, \bar{2}$ from the A_{2n-1} Dynkin diagram [BuSc17].

In [Ba00a, Proposition 2.3 (ii)], it is shown that given $b \in \mathsf{KN}(\lambda, n)$, the C_n crystal length functions ε_i^C, φ_i^C, $1 \le i \le n$, on b, and the A_{2n-1} crystal length functions $\varepsilon_i^A, \varepsilon_{\overline{i+1}}^A$, $1 \le i < n$, ε_n^A, and φ_i^A, $\varphi_{\overline{i+1}}^A$,$1 \le i < n$, φ_n^A, on $E(b)$ are nicely related:

$$\varepsilon_i^C(b) = \varepsilon_i^A(E(b)) = \varepsilon_{\overline{i+1}}^A(E(b)),\ 1 \le i < n, \text{ and } \varepsilon_n^C(b) = 1/2\varepsilon_n^A(E(b)),$$

and similarly for $\varphi_i^C(b)$, $1 \le i \le n$, where $\varepsilon_i(b) = \max\{k \in \mathbb{Z}_{\ge 0} : e_i^k(b) \ne 0\}$ and $\varphi_i(b) = \max\{k \in \mathbb{Z}_{\ge 0} : f_i^k(b) \ne 0\}$. This means that b is the highest weight element of a connected component U of $\mathsf{KN}_J(\lambda, n)$ if and only if, for all $i \in J$,

$$\varepsilon_i^A(E(b)) = \varepsilon_{\overline{i+1}}^A(E(b)) = \varepsilon_i^C(b) = 0, \text{ for all } i \in J \setminus \{n\}$$

and

$$\varepsilon_n^A(E(b)) = \varepsilon_n^C(b) = 0,\ \text{ if } n \in J.$$

(Similarly, in the case b is the lowest weight element, by replacing appropriately, ε_i^C with φ_i^C and ε_i^A, $\varepsilon_{\overline{i+1}}^A$ with φ_i^A, respectively $\varphi_{\overline{i+1}}^A$, and ε_n^A with φ_n^A.)

Henceforth,

$$\varepsilon_i^C(b) = 0,\, i \in J \Leftrightarrow \varepsilon_i^A(E(b)) = 0,\ i \in J \cup \bar{J}$$

and

$$\varphi_i^C(b) = 0,\, i \in J \Leftrightarrow \varphi_i^A(E(b)) = 0,\ i \in J \cup \bar{J}.$$

In other words, because our crystals are seminormal, $E(b)$ is the highest weight element of the connected component V of $\mathsf{SSYT}_{J \cup \bar{J}}(\lambda, n)$ containing $E(b)$ and $E(U)$. It is therefore unique. A similar statement holds for the lowest weight element. The next proposition now easily follows.

Proposition 2. *Let $J \subseteq [n]$ be a connected sub-diagram of the type C_n Dynkin diagram. Let U be a connected component of the Levi branched crystal $\mathsf{KN}_J(\lambda, n)$ with highest and lowest weight elements u^{high} and u^{low} respectively. Then $E(U)$ is contained in a connected component of the Levi branched crystal $\mathsf{SSYT}_{J\cup\bar{J}}(\lambda, n)$ with highest and lowest weight elements $E(u^{\mathsf{high}})$ and $E(u^{\mathsf{low}})$ respectively.*

Remark 5. *Given $T \in \mathsf{SSYT}(\mu, n, \bar{n})$, with μ a partition with at most $2n$ parts, T may be decomposed into two disjoint semi-standard tableaux T^+ and T^-, $T = (T^+, T^-)$, where T^+ is the semi-standard tableau of shape $\mu+$ on the alphabet $[n]$ defined by the entries of T in $[n]$, that is, $T^+ \in \mathsf{SSYT}(\mu_+, n)$, called the positive part of T, and T^- is the semi-standard tableau of skew shape μ/μ_+ on the alphabet $[\bar{n}, \bar{1}]$ defined by the entries of T in $[\bar{n}, \bar{1}]$, that is, $T^- \in \mathsf{SSYT}(\mu/\mu_+, \bar{n})$, called the negative part of T. Provided that we only apply f_i^A, e_i^A with $i \in J \cup J'$ disconnected such that $J \subseteq [n-1]$ and $J' \subseteq [\bar{n}, \bar{2}]$, respectively, this shape decomposition is preserved. Those crystal operators preserve the shape decomposition above because, according to the type A_{2n-1} signature rule, they only change positive (resp. negative) letters into positive (resp. negative) letters.*

For $J \cup J'$ disconnected, $f_j^A f_{j'}^A = f_{j'}^A f_j^A$, with $j \in J$, $j' \in J'$. We then write, for $\{j_1, \ldots, j_r\} \subseteq J$ and $\{j_1', \ldots, j_m'\} \subseteq J'$,

$$f_{j_r}^A \cdots f_{j_1}^A f_{j_m'}^A \cdots f_{j_1'}^A(T) = (f_{j_r}^A \cdots f_{j_1}^A(T^+), f_{j_m'}^A \cdots f_{j_1'}^A(T^-)). \tag{14}$$

6. The cactus group and virtualization

Halacheva [Ha16, HaKaRyWe20] has defined a more general version of the cactus group J_n originally defined by Henriques–Kamnitzer [HeKa06-1] in terms of generators and relations.

Definition 2 ([Ha16, HaKaRyWe20])**.** *Let $\mathfrak{g}$ be a finite-dimensional, semisimple Lie algebra with Dynkin diagram I. The cactus group $J_{\mathfrak{g}}$ has generators s_J where J runs over the connected sub-diagrams of the Dynkin diagram I of $\mathfrak{g}$, and relations:*

1$\mathfrak{g}$. $s_J^2 = 1$, for all $J \subseteq I$,

2$\mathfrak{g}$. $s_J s_{J'} = s_{J'} s_J$, for all $J, J' \subseteq I$ such that $J \cup J'$ is disconnected,

3$\mathfrak{g}$. $s_J s_{J'} = s_{\theta_J(J')} s_J$, for all $J' \subseteq J \subseteq I$.

Remark 6. *Note that when $J' \subseteq J$, 3$\mathfrak{g}$. says that s_J commutes with $s_{J'}$ by reversing J' with respect to J. We also have a group epimomorphism $J_{\mathfrak{g}} \to W$ taking s_J to w_0^J (*[HaKaRyWe20], [Ha16, Remark 10.0.1]*). Together with 3$\mathfrak{g}$, this implies the relations $w_0 w_0^J w_0 = w_0^{\theta(J)}$ and $w_0^J w_0^{J'} w_0^J = w_0^{\theta_J(J')}$.*

If I is the A_{n-1} Dynkin diagram, θ_J acts on J by reversing the connected interval of nodes J, whereas in the C_n type it depends on whether J contains the node with label n or not.

Lemma 1. *The cactus group $J_{\mathfrak{sl}(n,\mathbb{C})} = J_n$ is the group with generators s_J, where J runs over all connected sub-diagrams of $I = [n-1]$, the A_{n-1} Dynkin diagram, subject to the relations*

1A. $s_J^2 = 1, J \subseteq [n-1]$,

2A. $s_J s_{J'} = s_{J'} s_J$, *for all* $J, J' \subseteq [n-1]$ *such that* $J \cup J'$ *is disconnected.*

3A. $s_{[p,q]} s_{[k,l]} = s_{[p+q-l,p+q-k]} s_{[p,q]}$ *for* $[k,l] \subset [p,q] \subseteq [n-1]$.

Proof. Relations $1\mathfrak{g}$ and $2\mathfrak{g}$ translate directly to 1A. and 2A. Consider two nested intervals $[k,l] \subset [p,q] \subseteq [n-1]$. The Weyl group $W^{[p,q]}$ is the quotient

$$W^{[p,q]} = W/\operatorname{Stab}_W([1,p-1] \cup [q+1,n])$$

and $w_0^J(\alpha_j) = -\alpha_{p+q-j}, \quad j \in J$. Then $\theta_{[p,q]}(d) = p+q-d$ for $d \in [p,q]$. After this observation one sees that Relation $3\mathfrak{g}$ translates directly into 3A above.

□

Remark 7. *The first and third relations ensure that the* $n-1$ *elements of the form*

$$s_{[1,k]},\ 1 \leq k \leq n-1, \tag{15}$$

generate J_n*, since any* $s_{[i,j]}$ *may be written as*

$$s_{[i,j]} = s_{[1,j]} s_{[1,j-i+1]} s_{[1,j]}. \tag{16}$$

By conjugation with $s_{[1,n-1]}$*, the elements* $s_{[i,n-1]}, 1 \leq i \leq n-1$*, also form a set of generators.*

Lemma 2. *The cactus group* $J_{\mathfrak{sp}(2n,\mathbb{C})}$ *is the group with generators* s_J*, where* J *runs over all connected sub-diagrams of the* C_n *Dynkin diagram* $I = [n]$*, subject to the relations*

1C. $s_J^2 = 1, J \subseteq [n]$,

2C. $s_J s_{J'} = s_{J'} s_J$, *for all* $J, J' \subseteq [n]$ *such that* $J \cup J'$ *is disconnected,*

3C. *(i)* $s_{[p,n]} s_{[k,l]} = s_{[k,l]} s_{[p,n]}$, $[k,l] \subseteq [p,n] \subseteq [n]$,

(ii) $s_{[p,q]} s_{[k,l]} = s_{[p+q-l,p+q-k]} s_{[p,q]}$, $[k,l] \subseteq [p,q] \subseteq [n-1]$.

Proof. Relations $1\mathfrak{g}$ and $2\mathfrak{g}$ translate directly to 1A. and 2A. Consider two nested intervals $[k,l] \subset [p,q]$. If $[p,q] \subset [n-1]$, we are in type A_{q-p}, hence 3C.(ii) holds, which is just relation 3A. If $q = n$, then we are in type C_{n-p+1}. The Weyl group $W^{[p,n]}$ is the restriction of the hyperoctahedral group B_n to the generators $r_p, \dots, r_n$, (as a group of signed permutations, it is the restriction to the set

$$[\pm p, n] := \{p < \cdots < n < \bar{n} < \cdots < \bar{p}\}),$$

and $w_0^J(\alpha_j) = -\alpha_j$ for $j \in J$. Therefore $\theta_{[p,n]}(d) = d$ for $d \in [k,l]$ and Relation 3C. (i) follows directly from $3\mathfrak{g}$.

□

Remark 8. *Note that the elements* s_J, $J \subseteq [n-1]$*, subject to the relations above, generate the cactus group* J_n*. As in* (16)*, the following are alternative* $2n-1$ *generators of* $J_{\mathfrak{sp}(2n,\mathbb{C})}$*:*

$$s_{[1,j]},\ 1 \leq j \leq n-1, \tag{17}$$

$$s_{[j,n]},\ 1 \leq j \leq n. \tag{18}$$

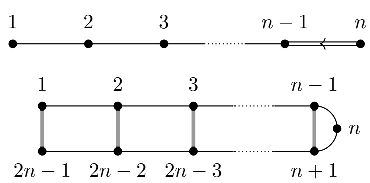
Remark 9. *We may observe that J_n is a subgroup of $J_{\mathfrak{sp}(2n,\mathbb{C})}$ defined by the subset of generators s_J, $J \subseteq [n-1]$, indexed by connected sub-diagrams of the A_{n-1} connected sub-diagram $[n-1]$ of the C_n Dynkin diagram $I = [n]$, subject to the relations above 1.C, 2.C and 3.C, (ii).*

Proposition 3. *If $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra, and $\mathfrak{l} \subset \mathfrak{g}$ is a Levi sub-algebra, then $J_{\mathfrak{l}}$ is a subgroup of $J_{\mathfrak{g}}$.*

Proof. Let I be the Dynkin diagram corresponding to $\mathfrak{g}$ and $J \subset I$ the sub-diagram corresponding to the Levi sub-algebra $\mathfrak{l}$. Any connected sub-diagram K of J is also a connected sub-diagram of I, hence one can define a map on generators by $s^J_K \mapsto s^I_K$. Here generators of $J_{\mathfrak{g}}$ are denoted by s^I_K, and generators of $J_{\mathfrak{l}}$ by s^J_K. Remark 6 implies that this map is a morphism of groups. The map is clearly injective because the generators of $J_{\mathfrak{g}}$ are all distinct.

□

6.1. **Embedding of $J_{\mathfrak{sp}(2n,\mathbb{C})}$ into J_{2n}.** We have observed that J_n is a subgroup of $J_{\mathfrak{sp}(2n,\mathbb{C})}$. We now show that there is a group embedding of $J_{\mathfrak{sp}(2n,\mathbb{C})}$ into J_{2n} by folding the A_{2n-1} Dynkin diagram through the middle node n:



Why should such an embedding exist? Let us consider the following elements of J_{2n}:

$$s'_{[p,q]} := s_{[p,q]}s_{[2n-q,2n-p]} = s_{[2n-q,2n-p]}s_{[p,q]}, \text{ for all } [p,q] \subseteq [n-1].$$

In Lemma 4 we show that these elements together with the generators $s_{[p,2n-p]}$ for $p \leq n$ generate a subgroup of J_{2n} isomorphic to $J_{\mathfrak{sp}(2n,\mathbb{C})}$. Notice the similarity between this and the construction of $\mathfrak{sp}(2n,\mathbb{C})$ as a sub-algebra of $\mathfrak{sl}(2n,\mathbb{C})$ by folding [Kac83, Chapter 8, pp. 89 – 102]. Moreover, the following lemma provides not only concrete combinatorial motivation for Lemma 4, but will also be the main ingredient in its proof.

Lemma 3. *The following relations hold in J_{2n}:*

$$s'^2_{[p,q]} = 1,\ 1 \leq p \leq q < n, \tag{19}$$

$$s^2_{[p,2n-p]} = 1,\ 1 \leq p < n, \tag{20}$$

$$s'_{[p,q]}s'_{[k,l]} = s'_{[k,l]}s'_{[p,q]},\quad [p,q] \cup [k,l] \subseteq [n-1] \text{ disconnected}, \tag{21}$$

$$s_{[p,2n-p]}s_{[k,2n-k]} = s_{[k,2n-k]}s_{[p,2n-p]},\ 1 \leq p < k < n, \tag{22}$$

$$s_{[p,2n-p]}s'_{[k,l]} = s'_{[k,l]}s_{[p,2n-p]},\ 1 \leq p \leq k \leq l < n, \tag{23}$$

$$s'_{[p,q]}s'_{[k,l]} = s'_{[p+q-l,p+q-k]}s'_{[p,q]},\ 1 \leq p \leq k \leq l \leq q < n. \tag{24}$$

There are no more relations among the elements $s'_{[p,q]}$ *and* $s_{[k,2n-k]}$*, for all* $[p,q] \subseteq [n-1]$ *and* $[k,n] \subseteq [n]$*.*

Proof. We have the relations (19), (20) and (23),

$$s'_{[p,q]}{}^2 = (s_{[p,q]}s_{[2n-q,2n-p]})^2 = s^2_{[p,q]}s^2_{[2n-q,2n-p]} \overset{1A.}{=} 1.$$

$$s^2_{[p,2n-p]} \overset{1A.}{=} 1.$$

For $1 \le p \le q < n$ and $1 \le k \le l < n$ such that $[p,q] \cup [k,l]$ is disconnected, the sub-diagrams $[p,q] \cup [2n-q,2n-p]$, and $[k,l] \cup [2n-l,2n-k]$ of $[2n-1]$ are disconnected, hence

$$s'_{[p,q]}s'_{[k,l]} \overset{2A.}{=} s'_{[k,l]}s'_{[p,q]}.$$

Additionally, if $q = n$, the sub-diagram $[k,l] \cup [p,2n-p] \cup [2n-l,2n-k]$ in $[2n-1]$ is disconnected, hence

$$s_{[p,2n-p]}s'_{[k,l]} \overset{2A.}{=} s'_{[k,l]}s_{[p,2n-p]}.$$

Moreover,

$$s_{[p,2n-p]}s_{[k,2n-k]} \overset{3A.}{=} s_{[2n-(2n-k),2n-k]}s_{[p,2n-p]} = s_{[k,2n-k]}s_{[p,2n-p]}$$

for $1 \le p < k < n$, hence relation (22) holds. Now, for $1 \le p < k < l < n$ we have:

$$\begin{aligned} s_{[p,2n-p]}s_{[k,l]}s_{[2n-l,2n-k]} &\overset{3A.}{=} s_{[2n-l,2n-k]}s_{[p,2n-p]}s_{[2n-l,2n-k]} \\ &\overset{3A.}{=} s_{[2n-l,2n-k]}s_{[2n-(2n-k),2n-(2n-l)]}s_{[p,2n-p]} \\ &= s_{[2n-l,2n-k]}s_{[k,l]}s_{[p,2n-p]} \end{aligned}$$

which establishes relation (23). Finally, for $1 \le p < k < l < q < n$ the following holds:

$$\begin{aligned} s'_{[p,q]}s'_{[k,l]} &= s_{[p,q]}s_{[2n-q,2n-p]}s_{[k,l]}s_{[2n-l,2n-k]} \\ &\overset{2A.}{=} s_{[p,q]}s_{[k,l]}s_{[2n-q,2n-p]}s_{[2n-l,2n-k]} \\ &\overset{3A.}{=} s_{[p+q-l,p+q-k]}s_{[p,q]}s_{[2n-(p+q-k),2n-(p+q-l)]}s_{[2n-q,2n-p]} \\ &\overset{2A.}{=} s_{[p+q-l,p+q-k]}s_{[2n-(p+q-k),2n-(p+q-l)]}s_{[p,q]}s_{[2n-q,2n-p]} \\ &= s'_{[p+q-l,p+q-k]}s'_{[p,q]}. \end{aligned}$$

This establishes relation (24). Any relation $R' = 1$ with the elements $s'_{[p,q]} = s_{[p,q]}s_{[2n-q,2n-p]}$ and $s_{[k,2n-k]}$, for some $[p,q] \subseteq [n-1]$ and $[k,n] \subseteq [n]$, translates to a relation $R = 1$ involving generators of J_{2n}, of the form $s_{[p,q]}$, $s_{[2n-q,2n-p]}$ in pairs, and $s_{[k,2n-k]}$, for some $[p,q] \subseteq [n-1]$ and $[k,n] \subseteq [n]$, which satisfy the same kind of relations as $s'_{[p,q]}$ and $s_{[k,2n-k]}$. Therefore from $R = 1$ we don't get new relations $R' = 1$. □

Definition 3. *The virtual symplectic cactus group* $\widetilde{J}_{2n}$ *is the group with generators* $\tilde{s}_J$, *where* J *runs over all sub-diagrams of* $I = [2n-1]$, *the* A_{2n-1} *Dynkin diagram, of the form* $J = [p, 2n-p]$ *for all* $[p, n] \subseteq [n]$, *or* $J = [p, q] \cup [2n-q, 2n-p]$ *for all* $[p, q] \subseteq [n-1]$ *subject to the relations*

1Ã. $\tilde{s}_J^2 = 1, J \subseteq [2n-1]$,

2Ã. $\tilde{s}_J \tilde{s}_{J'} = \tilde{s}_{J'} \tilde{s}_J$, *such that* $J \cup J'$ *is disconnected with respect to all* $[p, q] \subseteq [n]$,

3Ã. *(i)* $\tilde{s}_{[p,2n-p]} \tilde{s}_{[q,l] \cup [2n-l, 2n-q]} = \tilde{s}_{[q,l] \cup [2n-l,2n-q]} \tilde{s}_{[p,2n-p]}, [q, l] \subseteq [p, n] \subseteq [n]$,

(ii) for $[k, l] \subseteq [p, q] \subseteq [n-1]$,

$$\tilde{s}_{[p,q]\cup[2n-q,2n-p]}\tilde{s}_{[k,l]\cup[2n-l,2n-k]} =$$
$$\tilde{s}_{[q+p-l,q+p-k]\cup[2n-p+2n-q-(2n-k),2n-p+2n-q-(2n-l)]}\tilde{s}_{[p,q]\cup[2n-q,2n-p]} =$$
$$\tilde{s}_{[q+p-l,q+p-k]\cup[2n-(p+q)+k,2n-(p+q)+l]}\tilde{s}_{[p,q]\cup[2n-q,2n-p]}.$$

The following are $2n-1$ alternative generators of $\widetilde{J}_{2n}$:

$$\tilde{s}_{[1,j]\cup[2n-j,2n-1]},\ 1 \le j \le n-1, \tag{25}$$
$$\tilde{s}_{[j,2n-j]},\ 1 \le j \le n. \tag{26}$$

Proposition 4. *There is an isomorphism* $\widetilde{J}_{2n} \simeq J_{\mathfrak{sp}(2n,\mathbb{C})}$.

Proof. Clearly $\widetilde{J}_{2n}$ and $J_{\mathfrak{sp}(2n,\mathbb{C})}$ satisfy the same relations corresponding to all connected sub-diagrams $[p, q] \subseteq [n]$. Furthermore, the maps

$$\begin{aligned} J_{\mathfrak{sp}(2n,\mathbb{C})} &\to \widetilde{J}_{2n} \\ s_{[p,q]} &\mapsto \tilde{s}_{[p,q]\cup[2n-q,2n-p]}, \\ s_{[p,n]} &\mapsto \tilde{s}_{[p,2n-p]}, \end{aligned}$$

$$\begin{aligned} \widetilde{J}_{2n} &\to J_{\mathfrak{sp}(2n,\mathbb{C})} \\ \tilde{s}_{[p,q]\cup[2n-q,2n-p]} &\mapsto s_{[p,q]}, \\ \tilde{s}_{[p,2n-p]} &\mapsto s_{[p,n]} \end{aligned}$$

are epimorphisms inverse to each other. This follows directly from the definitions of $\widetilde{J}_{2n}$ and $J_{\mathfrak{sp}(2n,\mathbb{C})}$ (Definition 3 and Lemma 2 respectively). Therefore, $J_{\mathfrak{sp}(2n,\mathbb{C})} \simeq \widetilde{J}_{2n}$. □

Lemma 4. *The following assignment defines a group injection from* $J_{\mathfrak{sp}(2n,\mathbb{C})}$ *to* J_{2n}:

$$\begin{array}{llll} \Gamma: & J_{\mathfrak{sp}(2n,\mathbb{C})} & \hookrightarrow & J_{2n} \\ & s_{[p,q]} & \mapsto & s'_{[p,q]}, \quad 1 \le p \le q < n, \\ & s_{[p,n]} & \mapsto & s_{[p,2n-p]}, \quad 1 \le p \le n. \end{array}$$

Proof. We begin by showing that the map induced by Γ is indeed a group homomorphism. We check the relations $1C. - 3C.$ from Lemma 2.

1*C*. We have that for $1 \leq p \leq q < n$,

$$\Gamma({s_{[p,q]}}^2) = {s'_{[p,q]}}^2 \overset{(19)}{=} 1,$$

while for $1 \leq p < n$, we have $\Gamma({s_{[p,n]}}^2) = s^2_{[p,2n-p]} \overset{(20)}{=} 1$.

2*C*. For $1 \leq p \leq q < n$ and $1 \leq k \leq l < n$ such that $[p,q] \cup [k,l]$ is disconnected, the sub-diagrams $[p,q] \cup [2n-q, 2n-p]$, and $[k,l] \cup [2n-l, 2n-k]$ of $[2n-1]$ are disconnected, hence

$$\Gamma(s_{[p,q]}s_{[k,l]}) = s'_{[p,q]}s'_{[k,l]} \overset{(21)}{=} s'_{[k,l]}s'_{[p,q]} = \Gamma(s_{[k,l]}s_{[p,q]}).$$

Additionally, if $q = n$, the sub-diagram $[k,l] \cup [p, 2n-p] \cup [2n-l, 2n-k]$ in $[2n-1]$ is disconnected, hence

$$\Gamma(s_{[p,n]}s_{[k,l]}) = s_{[p,2n-p]}s'_{[k,l]} \overset{(23)}{=} s'_{[k,l]}s_{[p,2n-p]} = \Gamma(s_{[k,l]}s_{[p,n]}).$$

3*C*. (*i*) We have that for $1 \leq p < k < n$ and $1 \leq p < k < l < n$ respectively:

$$\Gamma(s_{[p,n]}s_{[k,n]}) = s_{[p,2n-p]}s_{[k,2n-k]} \overset{(22)}{=} s_{[k,2n-k]}s_{[p,2n-p]} = \Gamma(s_{[k,n]}s_{[p,n]})$$

$$\Gamma(s_{[p,n]}s_{[k,l]}) = s_{[p,2n-p]}s_{[k,l]}s_{[2n-l,2n-k]} \overset{(23)}{=} s_{[2n-l,2n-k]}s_{[k,l]}s_{[p,2n-p]} = \Gamma(s_{[k,l]}s_{[p,n]}).$$

(*ii*) Let $1 \leq p < k < l < q < n$. Then

$$\Gamma(s_{[p,q]}s_{[k,l]}) = s'_{[p,q]}s'_{[k,l]} \overset{(24)}{=} s'_{[p+q-l,p+q-k]}s'_{[p,q]} = \Gamma(s_{[p+q-l,p+q-k]}s_{[p,q]}).$$

We have now finished proving that Γ is a group morphism. To show that it is injective, one needs to show that its left inverse defined by the assignment

$$\begin{array}{rcccl} \Gamma^{-1}_{left}: & \mathrm{im}(\Gamma) \subset J_{2n} & \hookrightarrow & J_{\mathfrak{sp}(2n,\mathbb{C})} & \\ & s'_{[p,q]} & \mapsto & s_{[p,q]}, & 1 \leq p \leq q < n, \\ & s_{[p,2n-p]} & \mapsto & s_{[p,n]}, & 1 \leq p \leq n. \end{array}$$

is also a group morphism. This however follows from Lemma 3 the previous calculations: the generators of $\mathrm{im}(\Gamma)$ satisfy the relations from Lemma 2, and there are no more relations between them (all possible cases have been already covered above). ☐

Proposition 5. *The group $\widetilde{J}_{2n}$ is isomorphic to a subgroup of J_{2n}.*

Proof. The map

$$\begin{aligned} \widetilde{J}_{2n} &\hookrightarrow J_{2n} \\ \tilde{s}_{[p,q]\cup[2n-q,2n-p]} &\mapsto s'_{[p,q]}, 1 \leq p \leq q < n, \\ \tilde{s}_{[p,2n-p]} &\mapsto s_{[p,2n-p]}, 1 \leq p \leq n. \end{aligned}$$

is a a group injection. This follows directly after composing the maps from Proposition 4 and Lemma 4.

☐

We may also think of $\widetilde{J}_{2n}$ as the *unfolding* of $J_{\mathfrak{sp}(2n,\mathbb{C})}$ in J_{2n}.

7. Full Schützenberger–Lusztig involutions and algorithms

7.1. **Full Schützenberger–Lusztig involution.** Let $\mathsf{B}(\lambda)$ be the normal $\mathfrak{g}$-crystal with highest weight λ. Let u_λ and u_λ^{low} be the highest, respectively lowest, weight elements of $\mathsf{B}(\lambda)$. The *Schützenberger–Lusztig involution* $\xi : B(\lambda) \to B(\lambda)$ *is the unique map of sets (hence set involution)* such that, for all $b \in \mathsf{B}(\lambda)$, and $i \in I$,

- $e_i\xi(b) = \xi f_{\theta(i)}(b)$
- $f_i\xi(b) = \xi e_{\theta(i)}(b)$
- $\mathsf{wt}(\xi(b)) = w_0\mathsf{wt}(\mathsf{b})$

where w_0 is the long element of the Weyl group W. (For the existence and uniqueness of ξ, ξ^2 is a map of crystals and hence $\xi^2 = 1$, see [HeKa06-1, BuSc17].) The involution ξ acts by w_0 on the weights and interchanges the action of e_i and $f_{\theta(i)}$. For A_{n-1}, ξ acts by reversing the weight and interchanges the action of e_i and f_{n-i}; for C_n, ξ acts by changing the sign of the weight and interchanges the action of e_i and f_i.

If B is a normal $\mathfrak{g}$-crystal, B is the disjoint union of connected components, each of which is a crystal isomorphic to $\mathsf{B}(\lambda)$ for some dominant integral weight λ. We define ξ_{B} on B by applying ξ to each one of its connected components. Each element of $\mathsf{B}(\lambda)$ is generated by u_λ (resp. u_λ^{low}) by applying f_i's (resp. e_i's). Hence the same sequence of f_i's (resp. e_i's) applies to the highest weight (resp. lowest weight) of any connected component of B isomorphic to $\mathsf{B}(\lambda)$.

The elements u_λ and u_λ^{low} are the unique elements of $\mathsf{B}(\lambda)$ of weight λ, respectively $w_0\lambda$. Hence, since $\mathsf{wt}(\xi(u_\lambda)) = w_0\lambda$, and $\mathsf{wt}(\xi(u_\lambda^{\mathsf{low}})) = \lambda$, ξ interchanges highest and weight elements of $B(\lambda)$, and so $u_\lambda^{\mathsf{low}} = \xi(u_\lambda)$, $\xi(u_\lambda^{\mathsf{low}}) = u_\lambda$. This implies that, $u_\lambda = e_{j_r}\cdots e_{j_1}(u_\lambda^{\mathsf{low}})$, for some sequence $j_1,\ldots,j_r \in I$, and

$$u_\lambda^{\mathsf{low}} = \xi(u_\lambda) = \xi(e_{j_r}\cdots e_{j_1}(u_\lambda^{\mathsf{low}})) = f_{\theta(j_r)}\cdots f_{\theta(j_1)}(\xi u_\lambda^{\mathsf{low}}) = f_{\theta(j_r)}\cdots f_{\theta(j_1)}(u_\lambda).$$

Corollary 2. *Let $b \in B(\lambda)$ and $b = f_{j_r}\cdots f_{j_1}(u_\lambda)$, for $j_r,\ldots,j_1 \in I$. Then*

$$\xi(b) = e_{\theta(j_r)}\cdots e_{\theta(j_1)}(u_\lambda^{low}), \quad wt(\xi(b)) = w_0 wt(b)$$

In particular,

- *in type A_{n-1}, $\xi(b) = e_{n-j_r}\cdots e_{n-j_1}(u_\lambda^{low})$, and $\mathsf{wt}(\xi(b)) = rev\,\mathsf{wt}(b)$, where rev is the reverse permutation (long element) of $\mathfrak{S}_n$,*
- *in type C_n, $\xi(b) = e_{j_r}\cdots e_{j_1}(u_\lambda^{low})$, and $\mathsf{wt}(\xi(b)) = -\mathsf{wt}(b)$.*

7.2. **The full $\mathfrak{sl}(n,\mathbb{C})$ reversal.** For $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$, ξ coincides with the Schützenberger involution [Len07, BerZel96] also known as $\mathsf{evacuation}$ (evac for short) on $\mathsf{SSYT}(\lambda, n)$ [Fu97, St01], and as $\mathsf{reversal}$ on the set $\mathsf{SSYT}(\lambda/\mu, n)$ of type A_{n-1} tableaux of skew-shape λ/μ in the alphabet $[n]$ [BSS96].

Let $T \in \mathsf{B} = \mathsf{SSYT}(\lambda/\mu, n)$ and let $\mathsf{B}(T)$ be the connected component of the crystal $\mathsf{SSYT}(\lambda/\mu, n)$ containing T. Then $\mathsf{B}(T) \simeq \mathsf{B}(\nu)$ for some partition ν and $\mathsf{rectification}(T) \in \mathsf{B}(\nu)$. Thereby, $\xi(T)$ is the unique tableau in $\mathsf{B}(T)$ such that

$$\mathsf{rectification}\,\xi(T) = \mathsf{evacuation}(\mathsf{rectification}(T)),$$

$$\xi(T) = \mathsf{arectification}(\mathsf{evacuation}(\mathsf{rectification}(T))), \tag{27}$$

where arectification denotes the inverse process of rectification [BSS96, ACM19]. More precisely, the rectification (rect for short) procedure is recorded by assigning to the inner shape μ of T a standard tableau S to form the tableau pair (S, T). The entries of S govern the *jeu de taquin* on T by sliding out all letters in the S filling, from the largest to the smallest, to get a new tableau pair $(\mathsf{rect}(T), S')$ where S' is the skew standard tableau consisting of the slid letters from S. The anti-rectification procedure, arectification, is defined by the reverse *jeu de taquin* to $\mathsf{evacuation}(\mathsf{rectification}(T)$ and is governed by the slid letters in S' in the tableau pair $(\mathsf{evacuation}(\mathsf{rectification}(T)), S')$ from the smallest to the largest. Eventually one obtains the tableau pair $(S, \mathsf{reversal}(T))$ where

$$\mathsf{reversal}(T) := \mathsf{arectification}(\mathsf{evacuation}(\mathsf{rectification}(T))). \tag{28}$$

Next we will discuss $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$.

7.3. **Lecouvey–Sheats symplectic jeu de taquin and symplectic Knuth equivalence.** If T is a KN tableau, we consider its word $w(T) \in \mathcal{C}_n^*$ obtained by reading in the Japanese way the columns of T from rightmost to leftmost, each column read from top to bottom.

7.3.1. *Lecouvey–Sheats symplectic jeu de taquin.* [Sh99, Lec02]

Let T be a punctured KN tableau with two columns C_1 and C_2 and split form $spl(T) = lC_1rC_1lC_2rC_2$, and let C_1 have the puncture $*$. Let α be the entry under the puncture of rC_1 and β the entry to the right of the puncture of rC_1,

$$spl(T) = lC_1rC_1lC_2rC_2 = \begin{array}{|c|c|c|c|}\hline \cdots & \cdots & \cdots & \cdots \\ \hline * & * & \beta & \cdots \\ \hline \cdots & \alpha & \cdots & \cdots \\ \hline \cdots & \cdots \\ \cline{1-2} \end{array} \, ,$$

where α or β may not necessarily exist. The elementary steps of the symplectic jeu de taquin, or SJDT for short, are the following:

A. If $\alpha \le \beta$ or β does not exist, then the puncture of T will change its position with the cell beneath it. This is a vertical slide.

B. If the slide is not vertical, then it is horizontal. We then have $\alpha > \beta$ or that α does not exist. Let C_1' and C_2' be the columns obtained after the slide. We have two subcases, depending on the sign of β:

1. If β is barred, we are moving a barred letter, β, from lC_2 to the punctured box of rC_1, and the puncture will occupy β's place in lC_2. Note that lC_2 has the same barred part as C_2 and that rC_1 has the same barred part as $\Phi(C_1)$. Looking at T, we will have an horizontal slide of the puncture, getting $C_2' = C_2 \setminus \{\beta\} \sqcup \{*\}$ and $C_1' = \Phi^{-1}(\Phi(C_1) \setminus * \sqcup \{\beta\})$. In a sense, β went from C_2 to $\Phi(C_1)$.

2. If β is unbarred, the procedure is similar, but this time β will go from $\Phi(C_2)$ to C_1; hence $C_1' = C_1 \setminus * \cup \{\beta\}$ and $C_2' = \Phi^{-1}(\Phi(C_2) \setminus \{\beta\} \sqcup *)$. However, in this case it may happen that C_1' is no longer admissible. In this situation, if i is the lowest entry such that $i, \overline{i}$ appear in C_1' and $N(i) > i$, we erase both i and $\overline{i}$ from the column and remove a cell from the bottom and from the top of the column, and place all the remaining cells orderly.

Applying elementary SJDT slides successively, eventually, the puncture will be a cell such that α and β do not exist. In this case we redefine the shape to not include this cell and the *jeu de taquin* ends. The SJDT when applied to semi-standard tableaux in the alphabet $[n]$ reduces to the ordinary *jeu de taquin.*

The SJDT is reversible, meaning that we can move $*$, the empty cell outside of μ, to the inner shape ν of a skew tableau T of shape μ/ν, simultaneously increasing both the inner and outer shapes of T by one cell. The slides work similarly to the previous case: the vertical slide means that an empty cell is going up, and a horizontal slide means that an entry goes from $\Phi(C_1)$ to C_2 or from C_1 to $\Phi(C_2)$, depending on whether the slid entry is barred or not, respectively.

7.3.2. *Symplectic Knuth equivalence.* In this section we gather the necessary tools from [LLT95, Lec02]. For $w \in \mathcal{C}_n^*$, let $P(w)$ be the Kashiwara–Nakashima tableau obtained by performing the Baker–Lecouvey insertion algorithm on w. We do not need the algorithm in this paper, but refer the reader to [Ba00b, Lec02] for the original descriptions. A detailed account can also be found in [Sa21b]. Given $w_1, w_2 \in \mathcal{C}_n^*$, the relation $w_1 \sim w_2 \Leftrightarrow P(w_1) = P(w_2)$ defines an equivalence relation on $\mathcal{C}_n^*$ known as *symplectic plactic equivalence.* It is the analogous relation defined by Knuth relations in the alphabet $[n]$ [Fu97]. The *symplectic plactic monoid* is the quotient $\mathcal{C}_n^*/\sim$. Each *symplectic plactic class* is uniquely identified with a KN tableau.

The plactic monoid $\mathcal{C}_n^*/\sim$ can also be described as the quotient of C_n^* by the following *symplectic plactic relations* (we use the notation from [Lec02]):

R1

$$yzx \cong yxz \text{ for } x \leq y < z \text{ with } z \neq \bar{x}$$
$$xzy \cong zxy \text{ for } x < y \leq z \text{ with } z \neq \bar{x}$$

R2

$$y\overline{x-1}(x-1) \cong yx\overline{x} \text{ and } x\overline{x}y \cong \overline{x-1}(x-1)y \text{ for } 1 < x \leq n \text{ and } x \leq y \leq \bar{x}$$

R3 (Symplectic contraction/dilation relation) $w \sim w \setminus \{z, \overline{z}\}$, where $w \in \mathcal{C}_n^*$ and $z \in [n]$ are such that w is a non-admissible column, z is the lowest non-barred letter in w such that $N(z) = z + 1$ and any proper factor of w is an admissible column.

Remark 10. [Sa21a] *It can be proven that given a column word $w \in \mathcal{C}_n^*$, any proper factor is admissible if and only if any proper prefix of w is admissible. Thus, in order to be able to apply the plactic relation* **R3** *to a non-admissible column word w, we need only check that all proper prefixes of w are admissible, instead of all proper factors.*

For example,

$$234\bar{4}\bar{3} \overset{\mathbf{R3}}{\equiv} 23\bar{3}, \quad 1234\bar{4}\bar{3} \overset{\mathbf{R3}}{\equiv} 123\bar{3} \overset{\mathbf{R3}}{\equiv} 12. \tag{29}$$

When Knuth relations are applied to factors of a word, the weight is preserved while the length may not be. Knuth relations can be seen as *jeu de taquin* moves on words or diagonally shaped tableaux, and each symplectic *jeu de taquin* slide preserves the Knuth class of the reading word of a tableau [Lec02, Theorem 6.3.8]. The words $23\overline{2}\overline{3}1$ and $\overline{1}113\overline{3}$ are Knuth related: $\overline{1}113\overline{3} \overset{R1}{\sim} \overline{1}131\overline{3} \overset{R1}{\sim} \overline{1}13\overline{3}1 \overset{R2}{\sim} 2\overline{2}3\overline{3}1 \overset{R1}{\sim} 23\overline{2}\overline{3}1$.

7.4. **Full symplectic reversal.**

7.4.1. *Symplectic evacuation algorithm.* In [Sa21a], Santos introduced a symplectic evacuation algorithm on tableaux in $\mathsf{KN}(\lambda, n)$ denoted by evac^{C_n} which he proved coincides with the full Lusztig–Schützenberger involution on a given $U_q(\mathfrak{sp}(2n,\mathbb{C}))$-crystal $\mathsf{B}(\lambda)$ associated to a representation of highest weight λ. The algorithm is defined on a given tableau $T \in \mathsf{KN}(\lambda, n)$ as follows. First, one complements its entries, that is, replaces all unbarred i's by $\bar{i}$'s and all $\bar{i}$'s by i's (this amounts to the action of $w_0^C = -id$ on the entries of the tableau). Second, one performs a rotation by π to obtain a skew tableau. Finally, one performs symplectic rectification or insertion using Lecouvey–Sheats symplectic jeu de taquin [Sh99, Lec02, Lec07], or Baker–Lecouvey insertion [Ba00b, Lec02, Lec07] respectively. The resulting tableau is defined to be $\mathsf{evac}^{C_n}(T)$. We refer the reader to [Sa21a, Section 5] for detailed examples of the algorithm. Santos' evacuation mimics the Schützenberger $\mathsf{evacuation}$ on $\mathsf{SSYT}(\lambda, n)$ by replacing the action of the long element of $\mathfrak{S}_n$ with that of the long element of B_n.

7.4.2. *Full symplectic reversal on KN skew tableaux.* The set $\mathsf{KN}(\lambda/\mu, m)$ is a normal C_m crystal whose connected components are isomorphic to $\mathsf{KN}(\nu, m)$ for some partition ν whose number of boxes $|\nu|$ might be less than $|\lambda| - |\mu|$. Let $n = m + j - 1$, where $1 \le j - 1 < n$ is the number of parts of μ and $J = [j, n]$. Shifting the entries of the skew KN tableaux in $\mathsf{KN}(\lambda/\mu, m)$ by $j-1$, we may identify $\mathsf{KN}(\lambda/\mu, m)$ with the (normal) full sub-crystal $\mathsf{B}(\lambda, \mu) \subset \mathsf{KN}_J(\lambda, n)$ consisting of the tableaux in $\mathsf{KN}(\lambda, n)$ with entries exclusively in $1 < \cdots < j < j+1 < \cdots < j+m < \overline{j+m} < \cdots < \overline{j}$ and whose sub-tableaux on the alphabet $\{1, \ldots, j-1\}$ is the fixed Yamanouchi tableau of shape μ [Lec02, Lemma 6.1.3]. $\mathsf{B}(\lambda, \mu)$ is stable under the action of f_{i+j-1}, e_{i+j-1}, $i = 1, \ldots, m$, and it decomposes into connected components of $\mathsf{KN}_J(\lambda, n)$. That is, the crystal operators, f_i, e_i, $i = 1, \ldots, m$ do not change the skew-shape of a KN tableau on the alphabet $\mathcal{C}_m$, and $\mathsf{KN}(\lambda/\mu, m)$ decomposes into connected components that can be identified with the connected components of $\mathsf{B}(\mu, \lambda)$.

In both type A_{n-1} and type C_n, Kashiwara operators e_i and f_i commute with SJDT slides. Let $T \in \mathsf{B} = \mathsf{KN}(\lambda/\mu, n)$. An inner corner in T is a box of μ such that the boxes below and to the right are not in μ; an outer corner in T is a box of λ such that the boxes below and to the right are not in λ. Let c be a fixed inner/outer corner of T. An *SJDT slide* or a *complete SJDT slide* to the inner corner c means a slide of the box c from an inner corner to an outer corner, or vice-versa. An SJDT slide to the inner/outer corner c of T gives a new KN skew tableau $SJDT(T, c)$, possibly with fewer/more boxes. Applying an SJDT slide to the same inner corner c in all vertices of $\mathsf{B}(T)$ defines an isomorphic crystal $\mathsf{B}(SJDT(T,c))$ [Lec02, Theorem 6.3.8]. The images of the KN tableaux in the same connected component of $\mathsf{KN}(\lambda/\mu,m)$ under this crystal isomorphism have the same skew shape [Lec02, Theorem 6.3.8]. Iterating the SJDT to all inner corners of T rectifies T, producing $\mathsf{rect}(T)$ [Sh99, Proposition 9.2], [Lec02, Theorem 6.1.9, Theorem 6.3.9].

At the end of each SJDT slide, the inner corner (outer corner) where the slide started is filled, or the column where the slide started has 2 fewer (more) boxes [Sh99, Proposition 9.2], [Lec02, Theorem 6.1.9]. The SJDT step where the tableau loses two boxes in a column has a previous step where this column is non- admissible but Knuth equivalent to the new column which is admissible. The step in reverse SJDT where the tableau gains two boxes in a column is **R3** Knuth equivalent to the previous one which is admissible. Therefore, in each step of SJDT we get crystals which are isomorphic. This allows, in the vein of reversal for A_{n-1} skew semi-standard tableaux, the definition of symplectic reversal, $\mathsf{reversal}^{C_n}$, on type C_n skew tableaux as a coplactic extension of $\mathsf{evacuation}^{C_n}$.

Lemma 5. *Let $T \in B = KN(\lambda/\mu, n)$. Then $\xi^{C_n}(T)$ is the unique KN tableau in $B(T)$ that is symplectic Knuth equivalent to $evac^{C_n}\, rect(T)$, and*

$$\mathit{rectification}\,\xi^{C_n}(T) = \mathit{evacuation}^{C_n}(\mathit{rectification}(T)). \tag{30}$$

Proof. The crystal $\mathsf{B}(T) \simeq \mathsf{B}(\nu)$ for some partition ν and $\mathsf{rectification}(T) \in \mathsf{B}(\nu)$. The full Schützenberger-Lusztig involution on KN tableaux of straight shape satisfies $\xi^{C_n}(\mathsf{rect}(T)) = \mathsf{evacuation}^{C_n}(\mathsf{rect}(T))$, and crystal operators commute with SJDT when passing from $\mathsf{B}(T)$ to $\mathsf{B}(\nu)$. Therefore, (30) holds. □

In Subsection 9.2.1 we will provide an algorithm for partial symplectic reversal on $\mathsf{KN}_J(\lambda, n)$ with $J = [j, n]$. An algorithm for full C_n reversal on $\mathsf{KN}(\lambda/\mu, n)$ will result as a special case by considering the normal full sub-crystal $\mathsf{B}(\mu, \lambda)$ of $\mathsf{KN}_J(\lambda, n)$.

8. Internal cactus group action on a normal crystal

8.1. **Partial Schützenberger-Lusztig involutions.** Partial Schützenberger involutions were first studied in the case $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ by Berenstein and Kirillov [BerKir95] but have been defined by Halacheva in general for $\mathfrak{g}$: given $J \subseteq I$ any sub-diagram, the partial Schützenberger–Lusztig involution ξ_{J} is defined to be the Schützenberger–Lusztig involution ξ_{B_J} on the normal crystal B_J [HaKaRyWe20]. The crystal B_J decomposes into connected components, and we apply the Schützenberger–Lusztig involution to each connected component. Let $b \in \mathsf{B}$, and let u^{high}, u^{low} be the highest and lowest weight elements of the connected component of B_J containing b. Let $b = f_{j_r} \cdots f_{j_1}(u^{\mathsf{high}})$, with $j_r \cdots j_1 \in J$. Then, for $j \in J$,

$$\xi_{\mathsf{J}} e_j(b) = f_{\theta_J(j)} \xi_{\mathsf{J}}(b), \tag{31}$$

$$\xi_{\mathsf{J}} f_j(b) = e_{\theta_J(j)} \xi_{\mathsf{J}}(b), \tag{32}$$

$$\mathsf{wt}_J(\xi_{\mathsf{J}}(b)) = w_0^J \mathsf{wt}_J(b), \tag{33}$$

and

$$\xi_{\mathsf{J}}(b) = e_{\theta_J(j_r)} \cdots e_{\theta_J(j_1)}(u^{\mathsf{low}}).$$

Remark 11. *If $J = K \cup K' \subseteq I$ is disconnected with K and K' connected sub-diagrams of I, we have the sub-type Dynkin diagram $K \times K'$, and the Weyl group is $W^K \times W^{K'}$ with longest elements w_0^K and $w_0^{K'}$, respectively, such that $w_0^J = w_0^K w_0^{K'} = w_0^{K'} w_0^K$. The weight lattice of $\mathfrak{g}_K \oplus \mathfrak{g}_{K'}$ is $\Lambda_{K\cup K'} := \Lambda_K \oplus \Lambda_{K'}$ (see* [BuSc17]*). Then if θ_K and $\theta_{K'}$ are the graph automorphisms defined by w_0^K and $w_0^{K'}$ in K and K', respectively, $\theta_J = \theta_K\theta_{K'} = \theta_{K'}\theta_K$ is a graph automorphism of the Dynkin graph $K \times K'$ and hence preserves the connected sub-diagrams K and K' of I as defined in Section 3. Thanks to* [Ha16, Lemmas 10.1.3, 10.1.4], [HaKaRyWe20, 2368–2369]*, the crystal operators act componentwise on the normal crystal $\mathsf{B}_{K\cup K'}$ (a normal $\mathfrak{g}_K \oplus \mathfrak{g}_{K'}$–crystal), and satisfy the following properties*

$$f_k f_{k'} = f_{k'} f_k,\ f_k e_{k'} = e_{k'} f_k,\ e_k e_{k'} = e_{k'} e_k,\ e_k f_{k'} = f_{k'} e_k,\ \textit{for } k \in K, k' \in K', \tag{34}$$

$$\varepsilon_k(e_{k'}(b)) = \varepsilon_k(b),\ \varphi_k(e_{k'}(b)) = \varphi_k(b),\ \ \textit{for } k \in K, k' \in K',\ \textit{and } e_{k'}(b) \neq 0$$

$$e_{k'}\xi_{\mathsf{B}_K} = \xi_{\mathsf{B}_K} e_{k'},\ f_{k'}\xi_{\mathsf{B}_K} = \xi_{\mathsf{B}_K} f_{k'},\ \textit{for } k \in K, k' \in K', \tag{35}$$

$$e_k\xi_{\mathsf{B}_{K'}} = \xi_{\mathsf{B}_{K'}} e_k,\ f_k\xi_{\mathsf{B}_{K'}} = \xi_{\mathsf{B}_{K'}} f_k,\ \textit{for } k \in K, k' \in K'. \tag{36}$$

This extends to a disconnected sub-diagram with more than two connected sub-diagrams. Henceforth, from [Ha16, HaKaRyWe20]*, ξ_K and $\xi_{K'}$ commute*

$$\xi_K\xi_{K'} = \xi_{K'}\xi_K.$$

Lemma 6. *Let $J = K \cup K' \subseteq I$ be a disconnected sub-diagram of I with K and K' connected. Then $\mathsf{B}_{K\cup K'}$ is a normal crystal, and the Schützenberger–Lusztig involution on $\mathsf{B}_{K\cup K'}$, $\xi_{K\cup K'}$ satisfies*

$$\xi_{K\cup K'} = \xi_K\xi_{K'} = \xi_{K'}\xi_K.$$

Proof. The result follows from the previous remark: $\xi_K\xi_{K'} = \xi_{K'}\xi_K$ is an involution, and from (34), (35) and (36), it satisfies the conditions (31), (32) above. In addition, the weight map $\mathsf{wt}_{k\cup k'} : \mathsf{B} \overset{\mathsf{wt}}{\to} \Lambda \to \Lambda_{K\cup K'} = \Lambda_K \oplus \Lambda_{K'}$ and therefore, $\mathsf{wt}_{k\cup k'}(\xi_K\xi_{K'}(b)) = (\mathsf{wt}_K(\xi_K(b)), \mathsf{wt}_{K'}(\xi_{K'}(b)) = w_0^K w_0^{K'}(\mathsf{wt}_K(b), \mathsf{wt}_{K'}(b))$. Since there is only one set involution on $\mathsf{B}_{K\cup K'}$ satisfying (31), (32), and (33), we have that $\xi_{K\cup K'} = \xi_K\xi_{K'} = \xi_{K'}\xi_K$. □

The partial Schützenberger–Lusztig involutions ξ_{J}, for any $J \subseteq I$ a *connected Dynkin sub-diagram* of I, satisfy the $J_{\mathfrak{g}}$ cactus relations.

Theorem 1 ([Ha16])**.** *The map $s_J \mapsto \xi_J$, for all $J \subseteq I$ connected Dynkin sub-diagrams of I, defines an action of the cactus group $J_{\mathfrak{g}}$ on the set B; that is, the following is a group homomorphism*

$$\begin{array}{cccc} \Phi_{\mathfrak{g}}: & J_{\mathfrak{g}} & \to & \mathfrak{S}_{\mathsf{B}} \\ & s_J & \mapsto & \xi_J. \end{array}$$

Moreover $\mathsf{wt}_J(\xi_J(b)) = w_0^J \mathsf{wt}_J(b)$, $b \in \mathsf{B}$.

In other words, s_J acts on each connected component of B by permuting its vertices via ξ_{J} exchanging highest weight and lowest weight elements.

Remark 12. (1) *The action of $J_{\mathfrak{g}}$ factorizes through the quotient by the braid relations of $W^{\mathfrak{g}}$.*

(2) *The partial Schützenberger–Lusztig involutions satisfy the cactus relations, and, in particular, for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, and $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C}), \mathfrak{sp}(2n, \mathbb{C})$, it holds that*

$$\xi^{A_{n-1}}_{[n-i,n-1]} = \xi^{A_{n-1}}_{[1,n-1]}\xi^{A_{n-1}}_{[1,i]}\xi^{A_{n-1}}_{[1,n-1]},\ 1 \le i < n,\ \ \xi^{C_n}_{[i,n]} = \xi^{C_n}_{[1,n]}\xi^{C_n}_{[i,n]}\xi^{C_n}_{[1,n]},\ 1 \le i \le n,$$

respectively.

The following corollary motivates what comes in the next section.

Corollary 3. (*a*) *For the $\mathfrak{sl}(n, \mathbb{C})$-crystal $\mathsf{SSYT}(\lambda, n)$, the map*

$$s_{[1,j]} \mapsto \xi_{[1,j]} = \mathsf{evac}_{j+1},\ 1 \le j \le n-1,$$

where evac_{j+1} denotes the evacuation on the sub-tableaux of straight shape obtained by restricting the entries to $\{1, \dots, j+1\}$ and fixing the remaining ones, defines an action of the cactus group J_n on the set $\mathsf{SSYT}(\lambda, n)$.

(*b*) *For the $\mathfrak{sp}(2n, \mathbb{C})$-crystal $\mathsf{KN}(\lambda, n)$, the map*

$$s_{[1,j]} \mapsto \xi^{C_n}_{[1,j]},\quad 1 \le j \le n-1, \tag{37}$$

$$s_{[j,n]} \mapsto \xi^{C_n}_{[j,n]},\quad 1 \le j \le n, \tag{38}$$

defines an action of $J_{\mathfrak{sp}(2n,\mathbb{C})}$ on the set $\mathsf{KN}(\lambda, n)$, where $\xi^{C_n}_{[1,n]} = \xi^{C_n} = \mathsf{evac}^{C_n}$, $\xi^{C_n}_{[1,j]}$, $1 \le j \le n-1$ is given by the Baker embedding, Theorem 5, and $\xi^{C_n}_{[j,n]}, 1 \le j \le n-1$ is given either by the partial symplectic reversal in (52)*, Subsection 9.2.1 or by the Baker embedding, Theorem 6.*

8.2. **The virtual symplectic cactus group action on an $\mathfrak{sl}(2n, \mathbb{C})$-crystal and the virtualization of an $\mathfrak{sp}(2n, \mathbb{C})$-crystal.** On A_{n-1} semi-standard tableaux, there is a straightforward algorithm to compute the action of a partial Schützenberger–Lusztig involution ξ_{J} with J a connected A_{n-1} Dynkin sub-diagram. Let $I = [n-1]$ and $J = [p, q] \subset I$, $1 \le p \le q < n$, be a connected sub-diagram. The *J-partial reversal*, $\mathsf{reversal}_J$, is the $\mathsf{reversal}$ on $\mathsf{SSYT}_J(\lambda, n)$ which means the $\mathsf{reversal}$ or Schützeberger involution ξ applied to each connected component of $\mathsf{SSYT}_J(\lambda, n)$. Let $T \in \mathsf{SSYT}(\lambda, n)$, then, from (27) and (28):

$$\begin{aligned}\xi_{\mathsf{J}}(T) &= \mathsf{reversal}_J(T)\\ &:= (T_{[1,p-1]}, \mathsf{reversal}(T_{[p,q+1]}), T_{[q+2,n]})\\ &= (T_{[1,p-1]}, \mathsf{arectification}(\mathsf{evacuation}(\mathsf{rectification}(T_{[p,q+1]}))), T_{[q+2,n]}),\end{aligned} \tag{39}$$

where $T = (T_{[1,p-1]}, T_{[p,q+1]}, T_{[q+2,n]})$ is such that $T_{[1,p-1]}$ is the tableau obtained by restricting T to the alphabet $[1, p-1]$, $T_{[p,q+1]}$ is the skew tableau obtained by restricting to the alphabet $[p, q+1]$, and $T_{[q+2,n]}$ is obtained by restricting to the alphabet $[q+2, n]$. Indeed, if $J = [1, q]$, $\mathsf{reversal}_{[1,q]}(T) = \mathsf{evac}_{q+1}(T)$. The case where J is a disconnected sub-diagram of I will be a consequence of Lemma 6.

To define an internal action of the *virtual symplectic cactus group* $\widetilde{J}_{2n}$ on a crystal $\mathsf{SSYT}(\mu, n, \bar{n})$ with μ a partitition with at most $2n$ parts, thanks to Lemma 6, we now explicitly characterize the partial Schützenberger-Lusztig involution on a disconnected sub-diagram $J \cup J'$ of the A_{2n-1} Dynkin diagram such that $J \subseteq [n-1]$ and $J' \subseteq [\bar{n}, \bar{2}]$ are connected sub-diagrams. In the case of the A_{2n-1} Dynkin diagram, we label its nodes either in $[2n-1]$ or in $\{1, \dots, n, \bar{n}, \dots, \bar{2}\}$.

Theorem 2. *Let $J \cup J'$ be a disconnected sub-diagram of the A_{2n-1} Dynkin diagram $I = \{1, \dots, n, \bar{n}, \dots, \bar{2}\}$ such that $J \subseteq [n-1]$ and $J' \subseteq [\bar{n}, \bar{2}]$ are connected sub-diagrams. Then $\xi^{A_{2n-1}}_{J \cup J'}$, the Schützenberger-Lusztig involution on $\mathsf{SSYT}_{J \cup J'}(\mu, n, \bar{n})$, with μ a partition with at most $2n$ parts, satisfies*

$$\xi^{A_{2n-1}}_{J \cup J'} = \xi^{A_{2n-1}}_{J} \xi^{A_{2n-1}}_{J'} = \xi^{A_{2n-1}}_{J'} \xi^{A_{2n-1}}_{J} \tag{40}$$

$$= \mathsf{reversal}^{A_{2n-1}}_{J} \mathsf{reversal}^{A_{2n-1}}_{J'} = \mathsf{reversal}^{A_{2n-1}}_{J'} \mathsf{reversal}^{A_{2n-1}}_{J}. \tag{41}$$

where $\xi^{A_{2n-1}}_{J} = \mathsf{reversal}^{A_{2n-1}}_{J}$ and $\xi^{A_{2n-1}}_{J'} = \mathsf{reversal}^{A_{2n-1}}_{J'}$ are the Schützenberger-Lusztig involutions on $\mathsf{SSYT}_J(\mu, n, \bar{n})$ and $\mathsf{SSYT}_{J'}(\mu, n, \bar{n})$, respectively.

Remark 13. *This statement is indeed also valid for the Schützenberger-Lusztig involution on $\mathsf{SSYT}_{J \cup J'}(\mu, n)$ where $J \cup J'$ is a disconnected sub-diagram of the A_{n-1} Dynkin diagram with n odd.*

The cactus group J_{2n} acts on an A_{2n-1}-crystal of semi-standard tableaux via partial reversals. We now conclude that the virtual symplectic cactus $\widetilde{J}_{2n}$, Definition 3, a subgroup of J_{2n}, also does. In the next section, Subsection 9.5, we establish that this action has the feature to preserve the subset $E(\mathsf{KN}(\lambda, n))$.

Theorem 3. *For the $\mathfrak{sl}(2n, \mathbb{C})$-crystal of tableaux $\mathsf{SSYT}(\mu, 2n)$, with μ a partition with at most $2n$ parts, the map*

$$\tilde{s}_{[1,q] \cup [2n-q, 2n-1]} \mapsto \xi^{A_{2n-1}}_{[1,q] \cup [2n-q, 2n-1]} = \xi^{A_{2n-1}}_{[1,q]} \xi^{A_{2n-1}}_{[2n-q, 2n-1]} = \mathsf{evac}_{q+1} \mathsf{evac}_{2n} \mathsf{evac}_{q+1} \mathsf{evac}_{2n}, \quad 1 \le q < n, \tag{42}$$

$$\tilde{s}_{[q, 2n-q]} \mapsto \xi^{A_{2n-1}}_{[q, 2n-q]} = \mathsf{reversal}^{A_{2n-1}}_{[q, 2n-q]}, \quad 1 \le q \le n, \tag{43}$$

defines an action of the virtual symplectic cactus group $\widetilde{J}_{2n}$ on the set $\mathsf{SSYT}(\mu, 2n)$. That is, the following is a group homomorphism

$$\begin{array}{rccc} \tilde{\Phi}_{\mathfrak{sl}(2n,\mathbb{C})}: & \widetilde{J}_{2n} & \to & \mathfrak{S}_{B} \\ & \tilde{s}_J & \mapsto & \xi^{A_{2n-1}}_{J}, \end{array}$$

where $B = \mathsf{SSYT}(\mu, 2n)$ and J as in (42)or (43). Moreover, the action of $\widetilde{J}_{2n}$ on $\mathsf{SSYT}(\lambda^A, n, \bar{n})$ preserves the subset $E(\mathsf{KN}(\lambda, n))$.

Proof. Since J_{2n} acts on $\mathsf{SSYT}(\mu, 2n)$, the partial Schützenberger involutions ξ_{J}, with J a connected sub-diagram of the A_{2n-1} Dynkin diagram $I = [2n-1]$, satisfy the J_{2n}

cactus relations namely the ones in Lemma 3 which are the $\widetilde{J}_{2n}$ relations. We consider $\widetilde{J}_{2n}$ with generators (25), (26). In Subsections 9.4.1 and 9.5, (58), we conclude that $\widetilde{J}_{2n}$ acts on the set $\mathsf{SSYT}(\lambda^A, n, \bar{n})$ permuting its elements in a way that the subset $E(\mathsf{KN}(\lambda, n))$ is preserved. □

Therefore, the partial Schützenberger involutions ξ_J, with J any connected sub-diagram of the A_{2n-1} Dynkin diagram of the form $J = [q, 2n-q]$, $[q, n] \subseteq [n]$, or $J = [1, q] \cup [2n-q, 2n-1]$, $[1, q] \subseteq [n-1]$, satisfy the virtual symplectic cactus $\widetilde{J}_{2n}$ relations.

9. Partial symplectic Schützenberger–Lusztig involutions and algorithms

For J a connected sub-diagram of the Dynkin diagram $I = [n-1]$ of type A_{n-1}, the partial Schützenberger involution ξ_J coincides with J-*partial reversal*, that is, $\mathsf{reversal}_J$ (39). The case wherein J is a disconnected sub-diagram of I has been studied in Theorem 2 and Remark 13.

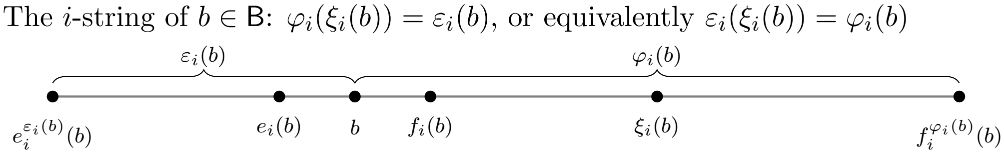
So far, there is no known form of tableau-switching for KN tableaux. The algorithm to compute J-*partial symplectic reversal*, $\mathsf{reversal}_J^{C_n}$, with $J = [p, n]$ a sub-diagram in the Dynkin diagram I of type C_n, presented in subsection 9.2 and summarized in (52), is inspired by this problem and mimics the type A partial reversal algorithm on type A_{n-1} tableaux, summarized in (39). The case $J = [p, q] \subseteq I$, $p < q < n$, is solved by virtualization in Subsection 9.4.1. In fact, all partial symplectic reversals can be virtualized as shown in Subsection 9.4.1.

9.1. **Dynkin sub-diagram with a sole node and the Weyl group action.** Let B be a normal crystal. If J has a sole node i of I, $\xi_i := \xi_{\{i\}}$, the Schützenberger–Lusztig involution to the i-strings (the connected components) of $\mathsf{B}_{\{i\}}$, agrees with the Kashiwara $\mathfrak{g}$-crystal reflection operator, originally studied by Lascoux and Schützenberger in the $\mathfrak{gl}(n, \mathbb{C})$ case [LSü81] and rediscovered by Kashiwara for any Cartan type [Kas94].

Theorem 4. [Kas94, Section 7] $\{\xi_i : i \in I\}$ *define an action of the Weyl group W on the underlying set of the normal crystal B, $r_i.b = \xi_i(b)$, for $b \in \mathsf{B}$, such that*

(1) $e_i \xi_i = \xi_i f_i$,
(2) $r_i.\mathsf{wt}(b) = \mathsf{wt}(\xi_i(b))$,
(3) $u_\lambda^{\mathsf{low}} = w_0.u_\lambda^{\mathsf{high}}$, *if* $\mathsf{B} = \mathsf{B}(\lambda)$.

The i-string of $b \in \mathsf{B}$: $\varphi_i(\xi_i(b)) = \varepsilon_i(b)$, or equivalently $\varepsilon_i(\xi_i(b)) = \varphi_i(b)$



Next propositions are a follow-up of the action of the Weyl group on i-strings where useful information is gathered.

Proposition 6. *(1) For $U_q(\mathfrak{sp}(2n, \mathbb{C}))$ and the alphabet $\mathcal{C}_n$: given $i \in [n-1]$, let u^- be a word in the alphabet $\{\bar{i}, i+1\}$ with length $\ell(u^-) = r$, and let v^+ be a* *word in the alphabet* $\{i, \overline{i+1}\}$ *with length* $\ell(v^+) = s$. *Then, for all* $r_i \in B_n$, $1 \le i \le n-1$,

$$r_i.(u^- v^+) = \xi_i(u^- v^+) = \begin{cases} u_1^- e_i^{r-s}(u_2^-)v^+, \ r > s \\ u^- v^+, \ r = s \\ u^- f_i^{s-r}(v_1^+)v_2^+, \ r < s \end{cases}, \tag{44}$$

such that when $r > s$, $u^- = u_1^- u_2^-$, *with* $\ell(u_2^-) = r - s$, *and when* $r < s$, $v = v_1^+ v_2^+$ *with* $\ell(v_1^+) = s - r$.

When $i = n$,

$$r_n.\overline{n}^r n^s = \xi_n(\overline{n}^r n^s) = \overline{n}^s n^r. \tag{45}$$

(2) *If* $b \in B(\lambda)$ *with* $b = f_{j_r} \cdots f_{j_1}(u_\lambda)$, *and* $r_k \cdots r_i r_j$ *is a reduced word for* $w_0 \in W$, *then*

$$\xi(b) = e_{\theta(j_r)} \cdots e_{\theta(j_1)}(r_k \cdots r_i r_j.u_\lambda).$$

(3) *For* $U_q(\mathfrak{sp}(2n, \mathbb{C}))$: *the crystal reflection operators* ξ_i *satisfy the relations of the Weyl group* B_n:
- $\xi_i^2 = 1$, $1 \le i \le n$,
- $\xi_i \xi_j = \xi_j \xi_i$, $|i-j| > 1$, $1 \le i, j \le n$,
- $(\xi_i \xi_{i+1})^3 = 1$, $1 \le i \le n-2$,
- $(\xi_{n-1}\xi_n)^4 = 1$.

Example 6. *From* (44), (45), *the action of* ξ_i *on a KN tableau is given by the signature rule on its reading word* [KasNak91, Lec02]:

(1)

$$T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 2 & 3 & \bar2 & \bar1 \\ \hline 2 & \bar4 & \bar3 & \bar3 & \bar1 \\ \cline{1-5} 4 & \bar2 & \bar1 \\ \cline{1-3} \bar4 \\ \cline{1-1} \end{array} \to \begin{array}{|c|c|c|c|c|c|} \hline + & - & - & 3 & + & - \\ \hline - & \bar4 & \bar3 & \bar3 & - \\ \cline{1-5} 4 & + & - \\ \cline{1-3} \bar4 \\ \cline{1-1} \end{array} \to \begin{array}{|c|c|c|c|c|c|} \hline + & - & - & 3 & + & - \\ \hline - & \bar4 & \bar3 & \bar3 & - \\ \cline{1-5} 4 & + & - \\ \cline{1-3} \bar4 \\ \cline{1-1} \end{array} \to \begin{array}{|c|c|c|c|c|c|} \hline + & 1 & 1 & 3 & + & - \\ \hline - & \bar4 & \bar3 & \bar3 & - \\ \cline{1-5} 4 & + & \bar2 \\ \cline{1-3} \bar4 \\ \cline{1-1} \end{array}$$

$$\to \xi_1(T) = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 3 & \bar2 & \bar1 \\ \hline 2 & \bar4 & \bar3 & \bar3 & \bar1 \\ \cline{1-5} 4 & \bar2 & \bar2 \\ \cline{1-3} \bar4 \\ \cline{1-1} \end{array}, \quad \mathit{wt}(\xi_1(T)) = r_1.\mathit{wt}(T) = r_1(-2,1,-1,-1) = (1,-2,-1,-1)$$

The reading word of T *is* $\overline{121}2\bar{1}2\bar{2}12$ *and*

$$\xi_1(\overline{121}2\bar{1}2\bar{2}12) = -(+-)---+(+-) \to -(+-)++++(+-) = \overline{121}1\bar{2}1\bar{2}12$$

$$\mathit{wt}(\overline{121}1\bar{2}1\bar{2}12) = (1,-2) = r_1.\mathit{wt}(\overline{121}2\bar{1}2\bar{2}12) = (-2,1)$$

$$S = \begin{array}{|c|c|c|c|c|} \hline 2 & 3 & 3 & \bar2 & \bar1 \\ \hline 4 & \bar4 & \bar1 & \bar1 \\ \cline{1-4} \bar4 & \bar2 \\ \cline{1-2} \end{array} \to \begin{array}{|c|c|c|c|c|} \hline - & 3 & 3 & + & - \\ \hline 4 & \bar4 & - & - \\ \cline{1-4} \bar4 & + \\ \cline{1-2} \end{array} \to \begin{array}{|c|c|c|c|c|} \hline - & 3 & 3 & + & + \\ \hline 4 & \bar4 & + & - \\ \cline{1-4} \bar4 & + \\ \cline{1-2} \end{array} \to \begin{array}{|c|c|c|c|c|} \hline - & 3 & 3 & + & \bar2 \\ \hline 4 & \bar4 & \bar2 & - \\ \cline{1-4} \bar4 & + \\ \cline{1-2} \end{array}$$

$$\to \xi_1(S) = \begin{array}{|c|c|c|c|c|}\hline 2 & 3 & 3 & \bar{2} & \bar{2} \\ \hline 4 & \bar{4} & \bar{2} & \bar{1} & \\ \cline{1-4} \bar{4} & + & & & \\ \cline{1-2}\end{array}, \quad \mathsf{wt}(\xi_1(S)) = r_1.\mathsf{wt}(S) = r_1(-3,-1,2,-1) = (-1,-3,2,-1)$$

$$\text{(2)}\ \xi_4(T) = \xi_4 \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 2 & 3 & \bar{2} & \bar{1} \\ \hline 2 & \bar{4} & \bar{3} & \bar{3} & \bar{1} & \\ \cline{1-5} 4 & \bar{2} & \bar{1} & & & \\ \cline{1-3} \bar{4} & & & & & \\ \cline{1-1}\end{array} = \xi_4 \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 2 & 3 & \bar{2} & \bar{1} \\ \hline 2 & \bar{4} & \bar{3} & \bar{3} & \bar{1} & \\ \cline{1-5} 4 & \bar{2} & \bar{1} & & & \\ \cline{1-3} \bar{4} & & & & & \\ \cline{1-1}\end{array} = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 2 & 3 & \bar{2} & \bar{1} \\ \hline 2 & 4 & \bar{3} & \bar{3} & \bar{1} & \\ \cline{1-5} 4 & \bar{2} & \bar{1} & & & \\ \cline{1-3} \bar{4} & & & & & \\ \cline{1-1}\end{array}$$

$$\mathsf{wt}\xi_4(T) = r_4.\mathsf{wt}(T) = r_4(-2,1,-1,-1) = (-2,1,-1,1).$$

Proposition 7. *Let $\mathsf{B}(\lambda)$ be a type C_n crystal, $J \subseteq I$ and $\mathsf{B}_J = \mathsf{B}_J(\lambda)$. Let $b \in \mathsf{B}(\lambda)$. The connected component of B_J containing b has highest weight element b_J^{high} and lowest weight element b_J^{low}. Then*

(1) $b_J^{\mathsf{low}} = r_a \cdots r_d.b_J^{\mathsf{high}} = \xi_a \cdots \xi_d(b_J^{\mathsf{high}})$ where $r_a \cdots r_d$ is a short word for $w_0^J \in W^J$ with $a, \dots, d \in J$, and $b = f_{j_r} \cdots f_{j_1}(b_J^{\mathsf{high}})$ for some $j_r, \dots, j_1 \in J$.

(2) If $J = [p,n]$, $\mathsf{B}_{[p,n]}$ is a type C_{n-p+1} crystal, then

$$\xi_J(b) = e_{j_r} \cdots e_{j_1}(r_a \cdots r_d.b_J^{\mathsf{high}}),\ \mathsf{wt}_J(\xi_J(b)) = -\mathsf{wt}_J(b),$$

where $\mathsf{wt}_J(x) \in \mathbb{Z}^{n-p+1}$, $x \in \mathsf{B}$, denotes $\mathsf{wt}(x) \in \mathbb{Z}^n$ restricted to the entries in $[p,n]$.

(3) If $J = [p,q]$, $1 \le p \le q < n$, $\mathsf{B}_{[p,q]}$ is a type A_{q-p+1}, crystal, and

$$\xi_J(b) = e_{q-p-j_r+1} \cdots e_{q-p-j_1+1}(r_a \cdots r_d.b_J^{\mathsf{high}}),\ \mathsf{wt}_J(\xi_J(b)) = \mathsf{reverse}(\mathsf{wt}_J(b)),$$

where $\mathsf{wt}_J(x) \in \mathbb{Z}^{q-p+1}$, $x \in \mathsf{B}$, denotes $\mathsf{wt}(x) \in \mathbb{Z}^n$ restricted to the entries in $[p,q+1]$.

9.2. **Dynkin sub-diagram $J = [j,n]$: J-symplectic reversal.** On the set $\mathsf{KN}(\lambda,n)$, $\xi^{C_n} = \xi^{C_n}_{[1,n]}$ coincides with Santos' symplectic evacuation evac^{C_n} (see 7.4.1 or [Sa21a, Section5]). The partial Schützenberger–Lusztig involution $\xi^{C_n}_{[j,n]}$ is the Schützenberger–Lusztig involution on each connected component of $\mathsf{KN}_{[j,n]}(\lambda,n)$.

9.2.1. *The action of the Knuth operator* **R3** *on a skew tableau.* Given $1 < j \le n$, the Levi branched crystal $\mathsf{KN}_{[j,n]}(\lambda,n)$ decomposes into connected components. Let $T \in \mathsf{KN}(\lambda,n)$, which belongs to some connected component of $\mathsf{KN}_{[j,n]}(\lambda,n)$, and let $T_{[\pm j,n]}$ denote the restriction of T to the alphabet $[\pm j,n]$. $T_{[\pm j,n]}$ is a KN skew tableau on the alphabet $\mathcal{C}_n$ (Lemma 7). However, $T_{[\pm j,n]}$ might have non-admissible columns with respect to the alphabet $[\pm j,n]$. This means that by doing a shift of $-(j-1)$ to the entries of $T_{[\pm j,n]}$, we might produce a non-admissible skew tableau on the alphabet $\mathcal{C}_{n-j+1}$ (recall Definition 1 and Remark 2). We show that under the action of the contractor operator **R3**, $T_{[\pm j,n]}$ is symplectic Knuth equivalent to a KN skew tableau on the alphabet $[\pm j,n]$. Consequently, the connected component containing $T_{[\pm j,n]}$ is symplectic Knuth equivalent to a crystal connected component of admissible skew tableaux on the alphabet $[\pm j,n]$ (of the same skew shape).

Proposition 8. [Lec02, Proposition 2.3.3] *Let $C_1, \dots, C_k$ be admissible columns on the alphabet $\mathcal{C}_n$. Then $T = C_1C_2 \cdots C_k$ is a KN tableau on the alphabet $\mathcal{C}_n$ if and only if $l(C_i) \le r(C_{i+1})$, that is, if $l(C_i)r(C_{i+1})$ is a type A_{2n-1} semi-standard tableau for $i = 1, \dots, k-1$.*

Lemma 7. *Let $T \in \mathsf{KN}(\lambda, n)$. The restriction of T to the alphabet $[\pm j, n] = [n] \setminus \{1 < \cdots < j-1 < \overline{j-1} < \cdots < \bar{2} < \bar{1}\}$, $T_{\pm j,n}$, is a KN skew tableau on the alphabet $\mathcal{C}_n$ where $T_{\pm j,n}$ might have non-admissible columns with respect to the alphabet $[\pm j, n]$.*

Proof. If a cell of T has a barred letter in $[\pm j-1]$, then the cells to the southeast have barred entries in $[\pm j-1]$, and if a cell of T has a non-barred letter in $[\pm j-1]$, then the cells to the northwest are non-barred and belong to $[j-1]$. Therefore, the non-barred letters of T in $[\pm j-1]$ define a partition shape, say μ, in T, and the barred letters in $[\pm j-1]$ define a skew shape λ/ν where $\mu \subseteq \nu \subseteq \lambda$. Hence the cells of T filled in $[\pm j, n] = [n] \setminus \{1 < \cdots < j-1 < \overline{j-1} < \cdots < \bar{2} < \bar{1}\}$ define the skew shape ν/μ. □

Lemma 8. *Let C_1 and C_2 be two columns with entries on the alphabet $[\pm j, n]$ such that C_1C_2 is a skew KN tableau on the alphabet $\mathcal{C}_n$. Assume that C_1 and C_2 have exactly $m \ge 0$ and $t \ge 0$ pairs of symmetric entries $(x, \bar{x})$, respectively, with $N(x) > x$ with respect to the alphabet $[\pm j, n]$. The columns are admissible skew tableaux on the alphabet $\mathcal{C}_n$ but not necessarily on the alphabet $[\pm j, n]$ when $j > 1$. Then C_1 has at least m boxes strictly below the row containing the last box of C_2, and C_2 has at least t boxes strictly above the row containing the top box of column C_1.*

Proof. Consider $spl(C_1C_2) = lC_1rC_1lC_2rC_2$ in $[\pm n]$, which is a type A_{2n-1} semi-standard tableau on the alphabet $\mathcal{C}_n$. Under the lemma's assumptions, when $m > 0$, C_1 is not-admissible in $[\pm j, n]$ and has $m > 0$ pairs of symmetric entries $(\alpha_i, \bar{\alpha}_i)$ where $N(\alpha_i) > \alpha_i$, $i = 1, \dots, m$; when $t > 0$, C_2 is not admissible in $[\pm j, n]$ and has $t > 0$ pairs of symmetric entries $(\beta_i, \bar{\beta}_i)$ where $N(\beta_i) > \beta_i$, $i = 1, \dots, t$. Therefore, from the definition of $spl(C_1C_2)$, the top box of $r(C_1)$ is filled in the interval $[\pm j, n]$, and the first t entries of lC_2 are filled in $[j-1]$. Since rC_1lC_2 is a type A_{2n-1} semi-standard tableau on the alphabet $\mathcal{C}_n$, it follows that the first t entries of column C_2 are strictly above the row containing the top box of column C_1. On the other hand, from the definition of $spl(C_1C_2)$, the last m boxes of $r(C_1)$ are filled in $\{\overline{(j-1)} < \cdots < \bar{2} < \bar{1}\}$, and the bottom box of lC_2 is filled in $[\pm j, n]$. Similarly, since rC_1lC_2 is a type A_{2n-1} semi-standard tableau on the alphabet $\mathcal{C}_n$, it follows that the last m entries of column C_1 are strictly below the row containing the bottom box of column C_2. □

Let $(\mathbf{R3})^m$ denote the iteration of the Knuth operator $\mathbf{R3}$, $m \ge 0$ times.

Proposition 9. *Let C_1, C_2 be two columns on the alphabet $[\pm j, n]$ such that C_1C_2 is a skew KN tableau on the alphabet $\mathcal{C}_n$ under the conditions of the previous lemma. Let $C_1 \overset{(\mathbf{R3})^m}{\equiv} X$, where X is an admissible column on $[\pm j, n]$, and $C_2 \overset{(\mathbf{R3})^t}{\equiv} Y$, where Y is an admissible column on $[\pm j, n]$. Then $C_1C_2 \overset{(\mathbf{R3})^m}{\equiv} XC_2 \overset{(\mathbf{R3})^t}{\equiv} XY$ is a skew KN tableau on $[\pm j, n]$.*

Proof. Under our assumptions, C_1 has $m \geq 0$ pairs of symmetric entries $(\alpha_i, \bar{\alpha}_i)$, where $N(\alpha_i) > \alpha_i$, $i = 1, \ldots, m$, with respect to the interval $[\pm j, n]$ (a column is non-admissible on $[\pm j, n]$ if $m > 0$), and the **R3** contraction is applied $m \geq 0$ times to C_1, $C_1 \overset{(\mathbf{R3})^m}{\equiv} X$, where X is admissible on $[\pm j, n]$. Henceforth, after applying the contraction **R3** m times to C_1, the relevant m pairs of symmetric entries are deleted, the top m entries and the bottom m entries of column C_1 are made empty and the remaining entries of C_1 are put in order in the remaining $|C_1| - 2m$ boxes of C_1 to define the admissible column X on $[\pm j, n]$. Similarly, under our assumptions about C_2, C_2 has $t \geq 0$ pairs of symmetric entries $(\beta_i, \bar{\beta}_i)$ such that $N(\beta_i) > \beta_i$, $i = 1, \ldots, t$, with respect to the interval $[\pm j, n]$. After applying the contraction **R3** t to C_2, the relevant t pairs of symmetric entries are deleted, the top t entries and the bottom t entries of column C_2 are made empty and the remaining entries of C_2 are put in order in the remaining $|C_2| - 2t$ boxes of C_2 to define the admissible column Y on $[\pm j, n]$. Thus, from Lemma 8, the resulting pair XY of admissible columns has skew shape.

Moreover, XY is a KN skew tableau on the alphabet $[\pm j, n]$, that is, $rX\, lY$, with entries on the alphabet $[\pm j, n]$, is a type A_{2n-1} semi-standard tableau. By definition of $spl(X)spl(Y)$, $rX\, lY$ has the same skew shape as XY. Note that

$$rX = rC_1 \setminus (\{\alpha_1, \ldots \alpha_m\} \sqcup [\pm j - 1])$$

is obtained from rC_1 by emptying the top m boxes and bottom m boxes of rC_1 and by filling in order the remaining boxes of rC_1 with $rC_1 \setminus (\{\alpha_1, \ldots \alpha_m\} \sqcup [\pm j - 1])$. Indeed, from Lemma 8,

$$rX \leq lC_2 \setminus [j - 1],$$

and in particular, $rXlC_2 \setminus [j-1])$ is a semi-standard tableau. Recall that the top box of rC_1 is strictly below the top t boxes of rC_2 (exactly the ones in lC_2 filled in $[j-1]$), and the bottom box of lC_2 is strictly above the bottom m boxes of rC_1 (exactly the ones in rC_1 filled in $\overline{(j-1)} < \cdots < \bar{1}$).

Finally, note that

$$lY = lC_2 \setminus ([j - 1] \sqcup \{\bar{\beta}_1, \ldots, \bar{\beta}_t\}),$$

and if $rXlC_2\setminus[j-1])$ is a semi-standard tableau, then $rXl(C_2\setminus([j-1]\sqcup\{\bar{\beta}_1, \ldots, \bar{\beta}_t\}) = rX\, lY$ is also a semi-standard tableau. ☐

9.2.2. *Reduced symplectic jeu de taquin.* Given $T \in \mathsf{KN}(\lambda/\mu, n)$ and $j \in [n]$ such that T has all entries in $[\pm j, n]$, the following is an algorithm to compute the *reduced symplectic jeu de taquin* on T on the interval $[\pm j, n]$, denoted $SJDT_j$. The skew tableau T might not be admissible on the alphabet $[\pm j, n]$. This means that we apply the SJDT after shifting all entries in T by $-(j-1)$ and iterating on T the contractor operator **R3** the needed number of times to get an admissible skew tableau on the alphabet $\mathcal{C}_{n-j+1}$. When $j = 1$, we recover the ordinary SJDT.

Definition 4. *Reduced SJDT ($SJDT_j$)*

- *Let T_j be the tableau obtained by replacing each non-barred entry c and barred entry $\bar{c}$ in T by $c - j + 1$ and $\overline{c - j + 1}$, respectively.*

- *If T_j is not a KN tableau in $\mathsf{KN}(\lambda/\mu, n-j+1)$, we have some columns containing pairs of the form $b, \overline{b}$ such that $b \in [n-j+1]$ is lowest in the column and $N(b) > b$. Iteratively, we apply the Knuth contractor* **R3** *operator to T_j until we make all columns admissible. Define T_j to be the resulting tableau with all admissible columns.*
- *Compute SJDT on T_j as usual.*
- *Replace each non-barred entry m and $\bar{m}$ in $SJDT(T_j)$ by $m+j-1$ and $\overline{m-j+1}$, respectively.*

The *reduced rectification* to the alphabet $[\pm j, n]$, denoted $\mathsf{rectification}_j$ (rect_j), of T is the iteration of the $SJDT_j$ to all inner corners in T. Indeed, $\mathsf{rect}_j(T)$ is the shift by $j-1$ of all entries of $\mathsf{rect}(T_j)$. When $j=1$ we recover the ordinary rectification.

Here is an illustrative example- first, we compute a complete SJDT slide on the interval $[\pm 1, 3]$:

$$\begin{array}{|c|c|}\hline * & 2 \\ \hline 3 & \bar{2} \\ \hline \bar{3} & \\ \cline{1-1}\end{array} \overset{\mathsf{SJDT}}{\to} \begin{array}{|c|c|}\hline 1 & \bar{1} \\ \hline 3 & * \\ \hline \bar{3} & \\ \cline{1-1}\end{array}.$$

Whereas, complete the $SJDT_2$ slide, the complete SJDT slide reduced to the interval $[\pm 2, 3]$, is such that:

$$\begin{array}{|c|c|}\hline * & 2 \\ \hline 3 & \bar{2} \\ \hline \bar{3} & \\ \cline{1-1}\end{array} \to T_2 = \begin{array}{|c|c|}\hline * & 1 \\ \hline 2 & \bar{1} \\ \hline \bar{2} & \\ \cline{1-1}\end{array} \overset{\mathbf{R3}}{\to} \begin{array}{|c|}\hline * \\ \hline 2 \\ \hline \bar{2} \\ \hline\end{array} \overset{SJDT}{\to} \begin{array}{|c|}\hline 2 \\ \hline \bar{2} \\ \hline * \\ \hline\end{array}.$$

Therefore,

$$\begin{array}{|c|c|}\hline * & 2 \\ \hline 3 & \bar{2} \\ \hline \bar{3} & \\ \cline{1-1}\end{array} \overset{SJDT_2}{\to} \begin{array}{|c|}\hline 3 \\ \hline \bar{3} \\ \hline * \\ \hline\end{array}.$$

Another illustration: first we compute an ordinary complete SJDT slide,

$$\begin{array}{|c|c|}\hline * & 3 \\ \hline 3 & \bar{3} \\ \hline \bar{3} & \\ \cline{1-1}\end{array} \to spl(T) = \begin{array}{|c|c|c|c|}\hline * & * & 2 & 3 \\ \hline 2 & 3 & \bar{3} & \bar{2} \\ \hline \bar{3} & \bar{2} & & \\ \cline{1-2}\end{array} \overset{\mathsf{SJDT}}{\to} \begin{array}{|c|c|}\hline 2 & * \\ \hline 3 & \bar{2} \\ \hline \bar{2} & \\ \cline{1-1}\end{array} \overset{\mathsf{SJDT}}{\to} \begin{array}{|c|c|}\hline 2 & \bar{2} \\ \hline 3 & * \\ \hline \bar{2} & \\ \cline{1-1}\end{array}.$$

On the other hand, a complete $SJDT_2$ slide means:

$$\begin{array}{|c|c|}\hline * & 3 \\ \hline 3 & \bar{3} \\ \hline \bar{3} & \\ \cline{1-1}\end{array} \to T_2 = \begin{array}{|c|c|}\hline * & 2 \\ \hline 2 & \bar{2} \\ \hline \bar{2} & \\ \cline{1-1}\end{array} \overset{SJDT}{\to} \begin{array}{|c|c|}\hline 1 & * \\ \hline 2 & \bar{1} \\ \hline \bar{1} & \\ \cline{1-1}\end{array} \overset{SJDT}{\to} \begin{array}{|c|c|}\hline 1 & \bar{1} \\ \hline 2 & * \\ \hline \bar{1} & \\ \cline{1-1}\end{array} \overset{\mathbf{R3}}{\to} \begin{array}{|c|c|}\hline & \bar{1} \\ \hline 2 & * \\ \hline\end{array}$$

Therefore,

$$\begin{array}{|c|c|}\hline * & 3 \\ \hline 3 & \bar{3} \\ \hline \bar{3} & \\ \cline{1-1}\end{array} \overset{SJDT_2}{\to} \begin{array}{|c|c|}\hline 2 & * \\ \hline 3 & \bar{2} \\ \hline \bar{2} & \\ \cline{1-1}\end{array} \overset{SJDT_2}{\to} \begin{array}{|c|c|}\hline 2 & \bar{2} \\ \hline 3 & * \\ \hline \bar{2} & \\ \cline{1-1}\end{array} \overset{\mathbf{R3}}{\to} \begin{array}{|c|c|}\hline & \bar{2} \\ \hline 3 & * \\ \hline\end{array} \overset{SJDT_2}{\to} \begin{array}{|c|c|}\hline 3 & \bar{2} \\ \hline & \\ \hline\end{array} = \mathsf{rect}_2(T).$$

9.2.3. *Partial symplectic reversal: colorful symplectic tableau switching.* Let $T \in \mathsf{KN}(\lambda, n)$ and $j \geq 1$. Let B be the crystal connected component of $\mathsf{KN}_{[j,n]}(\lambda, n)$ containing T. B is a highest weight crystal and all vertices of B are KN tableaux on the alphabet $\mathcal{C}_n$, with the letters in $[\pm j-1]$ frozen, as the crystal operators in B are indexed by $[j, n]$ and do not act on the entries filled in $[\pm j-1]$.

Let H be the highest weight element of B, and let $\mathsf{wt}(H_{[\pm j,n]}) \in \mathbb{Z}^{n-j+1}$ be its highest weight, where $H_{[\pm j,n]}$ is the restriction of H to the alphabet $[\pm j, n]$. The restriction of H to the alphabet $[\pm j, n]$ is a skew KN tableau on the alphabet $\mathcal{C}_n$. The entries of H in $[j-1]$ define a semi-standard tableau $T^+_{[j-1]}$ of shape, say μ, and the entries in $[\overline{j-1}, \bar{1}]$ define a skew semi-standard tableau $T^-_{[\overline{j-1},\bar{1}]}$ of shape λ/ν, where $\mu \subseteq \nu \subseteq \lambda$. Hence the cells of H filled in $[\pm j, n] = [n] \setminus \{1 < \cdots < j-1 < \overline{j-1} < \cdots < \bar{2} < \bar{1}\}$ define the skew shape ν/μ, and because the crystal operators in B are indexed by $[j, n]$, they do not change the skew shape ν/μ either. Therefore, since all the vertices of B are connected to H through those crystal operators, the vertices of B restricted to the alphabet $[\pm j, n]$ have the same skew shape ν/μ and the same semi-standard tableaux $T^+_{[j-1]}$ and $T^-_{[\overline{j-1},\bar{1}]}$ [Lec02, Lemma 6.1.3].

Step I. The sequence of isomorphic crystals from $T_{[\pm j,n]}$ to its reduced rectification. I.1 - THE C_{n-j+1} CONNECTED CRYSTAL B^0 CONTAINING $T_{[\pm j,n]}$.

Erase in the vertices of B the entries in $[\pm j-1]$; that is, erase the semi-standard tableaux $T^+_{[j-1]}$ and $T^-_{[\overline{j-1},\bar{1}]}$. We obtain the connected C_{n-j+1} crystal B^0 of semi-standard skew tableaux of shape ν/μ with entries in the alphabet $[\pm j, n]$, possibly with some non-admissible columns, containing $T_{[\pm j,n]}$. These KN skew tableaux over $\mathcal{C}_n$ might have non-admissible columns over $[\pm j, n]$. More precisely, B^0 is the connected crystal of words on the alphabet $[\pm j, n]$, with highest element the word of $H_{[\pm j,n]}$.

The set B^0 bijects the set B, with the same crystal graph structure and the same weight vertices as B. Hence, B^0 and B are isomorphic crystals.

I.1.1- THE GREEN INNER STANDARD TABLEAU U_0 FOR ANY VERTEX OF B^0.

Define a standard tableau U_0 of shape μ filled in a completely ordered alphabet of *green letters* $\{g_1 < \cdots < g_{|\mu|}\}$ where $|\mu|$ is the number of boxes of μ. Assign the inner standard tableau U_0 the inner shape of each vertex of B^0. Recall $T_{[\pm j,n]}$ is the image of T in B^0; see the tableau pair $(U_0, T_{[\pm j,n]})$ in Figure 1.

I.2 - THE C_{n-j+1} CRYSTAL B^x OF KN SKEW TABLEAUX **R3** ISOMORPHIC TO B^0.

Let $H^0 := H_{[\pm j,n]}$ be the highest weight element of the C_{n-j+1} crystal B^0. The skew tableau H^0 of shape ν/μ may have non-admissible columns on the alphabet $[\pm j, n]$. Let $r < s < \cdots < q < t$ be the non-admissible columns of H^0. Then exactly the same columns in all vertices of B^0 are non-admissible. The Knuth contraction **R3** relation, Subsection 7.3.2, defines a crystal isomorphism; it commutes with the crystal operators and preserves the weight. Moreover, each time **R3** is applied to a column of some vertex of B^0, it is also applied to the same column in every vertex of B^0 (see [Lec02, Proposition 3.2.4, Corollary 3.2.5]).

$(U_0, T_{[\pm j,n]}) =$

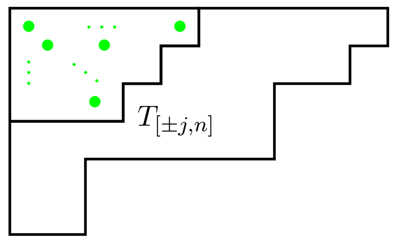


FIGURE 1. $T_{[\pm j,n]}$ in the crystal B^0 and the inner tableau U_0.

In each vertex of B^0, apply the **R3** contraction operation to column i, for $i = r, s, \dots, q, t$, until column i becomes admissible. For $i = r, s, \dots, q, t$, each time we apply **R3** to column i, a pair of entries $(k, \bar{k})$ is erased (whenever $k \in [n]$ is minimal for $N(k) > k$, k and $\bar{k}$ appear in the column and all prefixes are admissible). Then the cells from the top and the bottom of the current column i are emptied; the remaining entries are placed in order in the remaining cells between those erased. We obtain a new crystal of KN skew tableaux on the alphabet $[\pm j, n]$ isomorphic to the crystal B^0.

Let x be the total times **R3** has to be applied to H^0, from column r to column t as explained above, to get a KN skew tableau on alphabet $[\pm j, n]$. Denote the resulting KN skew tableau by H^x. Note that for each column of any vertex of B^0, the number of times **R3** is applied is the same. We then obtain the sequence of isomorphic crystals

$$\mathsf{B}^0 \overset{\mathbf{R3}}{\simeq} \mathsf{B}^1 \overset{\mathbf{R3}}{\simeq} \cdots \overset{\mathbf{R3}}{\simeq} \mathsf{B}^{x_r} \overset{\mathbf{R3}}{\simeq} \mathsf{B}^{x_r+1} \overset{\mathbf{R3}}{\simeq} \cdots \overset{\mathbf{R3}}{\simeq} \mathsf{B}^{x_r+x_s}$$
$$\overset{\mathbf{R3}}{\simeq} \cdots \overset{\mathbf{R3}}{\simeq} \mathsf{B}^{x_r+x_s+\cdots+x_q+x_t} = \mathsf{B}^x,$$

where $x = x_r + x_s + \cdots + x_q + x_t$ and x_i is the number of times we apply **R3** to column i of H^0, for $i = r, s, \dots, q, t$. The crystal B^x, isomorphic to B^0, is obtained by applying **R3** x times to each vertex of B^0, namely, x_i times to column i, for $i = r, \dots, q, t$, of each vertex of B^x. Equivalently, B^x is the crystal whose highest weight element is the KN skew tableau H^x of shape ν^x/μ^x, where $\nu^x \subseteq \nu$, $\mu \subseteq \mu^x$ and $|\mu^x| - |\mu| = |\nu| - |\nu^x| = x$ is the number of times **R3** has been applied to H^0 (or $T_{[\pm j,n]}$).

I.2.1 - THE PAIR (U_x, V_x) OF GREEN-PURPLE INNER AND PURPLE OUTER STANDARD TABLEAUX FOR ANY VERTEX OF B^x.

Let

$$\{g_1 < \cdots < g_{|\mu|} < p_1 < p_2 < \cdots < p_x < p'_x < \cdots < p'_2 < p'_1\} \tag{46}$$

be a completely ordered alphabet of $|\mu| + 2x$ letters consisting of $|\mu|$ green letters and x unprimed and x primed *purple letters*.

Define the standard tableau U_x of shape μ^x, where $\mu \subseteq \mu^x$ and $|\mu^x| = |\mu| + x$, to be an extension of U_0 filled with the $|\mu|$ green letters by filling the extra x cells, the total number of cells made empty at the top of each non-admissible column in a vertex of B^0, with the unprimed purple letters $\{p_1 < \cdots < p_{x_r} < \cdots < p_x\}$. Define the standard tableau V_x of shape ν/ν^x by filling the x cells made empty at the bottom of each non-admissible column in a vertex of B^0 with the primed purple letters $p'_x < \cdots < p'_{x_r} < \cdots < p'_1$. The filling rule is as follows.

Fill successively the pair of cells made empty each time **R3** is applied, with one unprimed purple letter and one primed purple letter, $p_1 < p'_1, \ldots, p_{x_r} < p'_{x_r}, p_{x_r+1} < p'_{x_r+1}, \ldots, p_{x_r+x_s} < p'_{x_r+x_s}, \ldots, p_x < p'_x$, with the unprimed letter at the top of the column and the primed letter at the bottom of the column. We impose the order

$$g_1 < \cdots < g_{|\mu|} < p_1 < \cdots < p_{x_r} < p_{x_r+1} < \cdots < p_{x_r+x_s} < \cdots < p_x <$$

$$< p'_x < \cdots < p'_{x_r+x_s} < \cdots < p'_{x_r+1} < p'_{x_r} < \cdots < p'_1.$$

That is, each time an unprimed purple letter and a primed purple letter are added to U_x and V_x, respectively, the unprimed letter is strictly larger than any green letter and any unprimed purple letter already added to U_x, and simultaneously, the primed purple letter is strictly smaller than any primed purple letter already added to V_x.

By construction, the pair (U_x, V_x) of inner and outer standard tableaux is the same for any vertex of B^x. More precisely, U_x of shape μ^x is the extension of U_0 filled with the alphabet $\{g_1 < \cdots < g_{|\mu|} < p_1 < p_2 < \cdots < p_x\}$; V_x of skew shape ν/ν^x is filled with the alphabet of primed purple letters $p'_x < \cdots < p'_{x_r+\cdots+x_q} < \cdots < p'_{x_r+x_s} < \cdots < p'_{x_r+1} < p'_{x_r} < \cdots < p'_1$. Regarding U_x, extend the column r of U_0 with the x_r unprimed purple letters $p_1 < \cdots < p_{x_r}$, the column s with the x_s unprimed purple letters $p_{x_r+1} < \cdots < p_{x_r+x_s}$, and finally the column t with the x_t unprimed purple letters $p_{x_r+\cdots+x_q+1} < \cdots < p_{x_r+\cdots+x_q+x_t} = p_x$; regarding V_x of skew shape ν/ν^x, start with the skew shape ν/μ_0, and fill the bottom x_r boxes of column r with the alphabet of primed purple letters $p'_{x_r} < \cdots < p'_1$, the bottom x_s of column s with the alphabet $p'_{x_r+x_s} < \cdots < p'_{x_r+1}$, and, finally, the bottom x_t boxes of column t with the alphabet $p'_x < \cdots < p'_{x_r+x_s+\cdots+x_q+1}$. See the triple (U_x, H^x, V_x) in Figure 2.

I.3 - Rectification of the C_{n-j+1} crystal B^x and reduced rectification of $T_{[\pm j,n]}$

Consider the triple of tableaux (U_x, H^x, V_x) previously defined. Apply complete $SJDT_j$ slides successively to the cells of U_x, from the largest entry to the smallest one, to rectify H^x. At the end of each complete $SJDT_j$ slide, we get an outer cell filled with the letter where the slide started in U_x. While H^x is being rectified, the cells of U_x are slid to end up as outer corners and added to the skew standard tableau V_x.

The rectification of H^x does not depend on the choice of the inner corner made in each step during the rectification process [Lec02, Corollary 6.3.9]. Applying $SJDT_j$ to any corner of U_x in an element of B^x (recall that for all elements of B^x, U_x is the same) gives a crystal isomorphism. This observation is equivalent to the fact that the rectification does not depend on the filling of U_x: U_x is a choice to keep track of the rectification of H^x (or of any other vertex of B^x). If a complete $SJDT_j$ slide applies to an inner corner of H^x, then a complete $SJDT_j$ slide also applies to the same inner

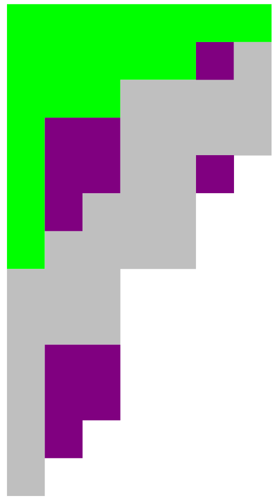


FIGURE 2. The triple (U_x, H^x, V_x) with H^x in gray, V_x in purple, and $U_0(\subseteq U_x)$ in green.

corner in every vertex of the crystal B^x and creates the same outer corner filled with the same letter.

However, if the number of boxes of H^x, $|H^x|$, exceeds the minimal number of boxes of its Knuth class, it will be necessary to apply $SJDT_j$ more than $|U_x| = |\mu| + x$ times to rectify H^x. When H^x has the minimal number of boxes of its Knuth class, only x unprimed purple letters and $|\mu|$ green letters will slide outwards and join the outer tableau V_x.

Let $2y \geq 0$ be the number of boxes of H^x that exceeds the minimal number of boxes of its Knuth class, that is,

$$2y = |H^x| - |\mathsf{rectification}_j(H^x)|.$$

When H^x has the minimal number of boxes of its Knuth class, $y = 0$. Necessarily $2y$ boxes of H^x will be lost in the $SJDT_j$ rectification process. Henceforth, the $SJDT_j$ B.2 case will be applied y times, each application creates a non- admissible column followed by the application of a contractor **R3** operation resulting in the loss of two boxes.

Remark 14. *Theorem 6.1.9 in Lecouvey's paper* [Lec02] *says: if the B.2 case appears with the creation of a non-admissible column when applying complete SJDT to an inner corner of a KN skew tableau, it has to be at the initial column where the inner corner was originally contained.*

This observation implies that each of the y mentioned non-admissible columns will only occur in the columns containing the inner corners where the slide started.

The complete $SJDT_j$ slides applied successively to the entries of U_x, as mentioned, will transform the crystal B^x into an isomorphic crystal of KN skew tableaux, as long as the $SJDT_j$ B.2 case does not create a non-admissible column. Otherwise, one has an isomorphic crystal where each vertex has a non-admissible column. In this case, we apply the contractor operator **R3** to that column in each vertex, erasing a pair $(k, \bar{k})$ if $k \in [\pm j, n]$ is the lowest entry such that $N(k) > k$. Then, as in **I.2** above, the cells from the top and the bottom of the current column are emptied and the remaining entries are placed in order. We get a new isomorphic crystal of KN skew tableaux where each vertex has two fewer boxes. As observed above, this may only happen in the y columns where $SJDT_j$ was applied, specifically, those containing the inner corners where the slides started; no other boxes are deleted in the rectification process of B^x.

Eventually, H^x is rectified to $\mathsf{rectification}(H^x)$, as are all vertices of B^x, and we get the crystal R of straight KN tableaux with highest weight element $\mathsf{rectification}(H^x)$,

$$\mathsf{B}^0 \simeq \mathsf{B}^x \simeq \mathsf{R}.$$

I.3.1 - The green-purple-red standard tableau V of every vertex in the C_{n-j+1} crystal R containing $\text{RECT}_j(T_{[\pm j,n]})$.

Let

$$\mathsf{B}^{x,1}, \mathsf{B}^{x,1,-}, \mathsf{B}^{x,2}, \mathsf{B}^{x,2,-} \dots, \mathsf{B}^{x,y}, \mathsf{B}^{x,y,-}$$

be the sequence of $2y$ isomorphic crystals appearing in the rectification process from B^x to R, tracking each complete $SJDT_j$ slide which triggers a B.2 case and the subsequent application of a contractor **R3** operator to that non-admissible column. In particular, for $i = 1, \dots, y$, $\mathsf{B}^{x,i}$ is the crystal where for the ith time in the complete $SJDT_j$ slide, the B.2 case appeared to create a non-admissible column in the column containing the inner corner where the slide started, and $\mathsf{B}^{x,i,-}$ is the crystal obtained by applying an **R3** contractor operator to that non-admissible column.

For $i = 1, \dots, y$, let $H^{x,i}$ and $H^{x,i,-}$ be the pair of highest weight elements of the crystal pair $\mathsf{B}^{x,i}$ and $\mathsf{B}^{x,i,-}$, respectively. Each $H^{x,i}$ has exactly one non-admissible column, and $H^{x,i,-}$ has non-admissible columns.

We have to store $2y$ new auxiliary letters to record the $2y$ empty cells created by the y applications of an **R3** contractor as a consequence of the creation of y non-admissible columns by the complete $SJDT_j$ slide where the B.2 case appeared and created a non-admissible column.

Consider the triple of tableaux (U_x, H^x, V_x) corresponding to the crystal B^x. Let $(U_{x,1},\ H^{x,1},\ V_{x,1})$ be the triple of tableaux obtained from (U_x, H^x, V_x) by applying complete $SJDT_j$ slides to the entries of U_x and transforming the KN skew tableau H^x into $H^{x,1}$, where for the first time in the complete $SJDT_j$ slide, the B.2 case appears and creates a non-admissible column; that is, $H^{x,1}$ has a non-admissible column, and the highest weight elements of all previous crystals obtained from B^x had all columns admissible. After the said complete $SJDT_j$ slides to U_x, $U_{x,1}$ is the inner standard tableau of $H^{x,1}$, and $V_{x,1}$ is obtained from V_x by adding the slid entries from U_x to V_x. $V_{x,1}$ is indeed a standard tableau because by construction, the entries of U_x are strictly smaller than the primed purple entries of V_x,

$$\{g_1 < \cdots < g_{|\mu|} < p_1 < p_2 < \cdots < p_x < p'_x < \cdots < p'_1\}.$$

The pair $(U_{x,1}, V_{x,1})$ of inner and outer standard tableaux is the same for every vertex of $\mathsf{B}^{x,1}$:

$$U_{x,1} \subseteq U_x, \quad V_{x,1} \supseteq V_x.$$

We have to apply an **R3** contractor operator to $H^{x,1}$ (and to every vertex of $\mathsf{B}^{x,1}$) to transform the non-admissible column into an admissible one: a pair of symmetric entries in each vertex of $\mathsf{B}^{x,1}$ will be deleted, the top and bottom cells of that column will be emptied and the remaining entries will be placed in order. Let $\mathsf{B}^{x,1,-}$ be the new crystal of KN skew tableaux isomorphic to $\mathsf{B}^{x,1}$, and let $H^{x,1,-}$ be its highest weight element (it two has fewer boxes than $H^{x,1}$). Note the number of the column where **R3** acts is the same for every vertex of $\mathsf{B}^{x,1}$. Fill the empty entries with *red letters* $r_1 < r_1'$, with r_1 on the top and r_1' on the bottom, where in the complete $SJDT_j$ slide, the $B.2$ case appears and has created a non-admissible column such that r_1 is strictly larger than any entry of $U_{x,1}$, and r_1' is strictly smaller than any entry of $V_{x,1}$. $V_{x,1}$ is filled with the entries of U_x already slid and with all primed purple letters. The cell with the red letter r_1 was the cell of U_x where the complete $SJDT_j$ slide started and the B.2 case appeared with the creation of a non-admissible column.

Let $U_{x,1,+}$ be the standard tableau obtained by adding the red letter r_1 to $U_{x,1}$, and let $V_{x,1,+}$ be the standard tableau obtained by adding the primed red letter r_1' to $V_{x,1}$ in the manner described,

$$U_{x,1} \subset U_{x,1,+} \subseteq U_x, \quad V_{x,1,+} \supset V_{x,1} \supseteq V_x.$$

We keep applying complete $SJDT_j$ slides to entries of $U_{x,1,+}$, from the largest to the smallest, to rectify $H^{x,1,-}$, so the cell r_1 will be the first to slide outwards and become an outer corner.

Let $(U_{x,2}, H^{x,2}, V_{x,2})$ be the triple of tableaux obtained from $(U_{x,1,+}, H^{x,1,-}, V_{x,1,+})$ by applying complete $SJDT_j$ slides to the entries of $U_{x,1,+}$ and transforming the KN skew tableau $H^{x,1,-}$ into $H^{x,2}$, where for the second time in the complete $SJDT_j$ slide, the B.2 case appears with the creation of a non-admissible column; that is, $H^{x,2}$ has a non-admissible column, and the highest weight elements of all previous crystals obtained from $\mathsf{B}^{x,1,-}$ had all columns admissible. After these complete $SJDT_j$ slides to $U_{x,1,+}$, $U_{x,2}$ is the inner standard tableau of $H^{x,2}$; $V_{x,2}$ is obtained from $V_{x,1,+}$ by adding the slid entries from $U_{x,1,+}$ to $V_{x,1,+}$. $V_{x,2}$ is indeed a standard tableau because by construction the entries of $U_{x,1,+}$ are strictly smaller than the entries of $V_{x,1,+}$. At this point, the red letter r_1 has already slid from $U_{x,1,+}$ to $V_{x,2}$; that is, r_1 is no longer in $U_{x,2}$ and instead belongs to $V_{x,2}$,

$$U_{x,2} \subseteq U_{x,1} \subset U_{x,1,+} \subseteq U_x, \quad V_{x,2} \supset V_{x,1,+} \supset V_{x,1} \supseteq V_x$$

Let $\mathsf{B}^{x,2}$ be the crystal with highest weight element $H^{x,2}$. We have to apply the **R3** contractor operator to $H^{x,2}$ (and to every vertex of $\mathsf{B}^{x,2}$) to transform the non-admissible column into an admissible one: a pair of symmetric entries in each vertex of $\mathsf{B}^{x,2}$ will be deleted, the top and bottom cells of that column will be emptied and the remaining entries will be placed in order. Let $\mathsf{B}^{x,2,-}$ be the new crystal of KN skew tableaux isomorphic to $\mathsf{B}^{x,2}$, and let $H^{x,2,-}$ be its highest weight element (it has two fewer boxes than $H^{x,2}$). Fill the empty entries with red letters $r_2 < r_2'$, r_2 on the top and r_2' on the bottom of the column, where in the complete $SJDT_j$ slide, the $B.2$ case appears and has created a non-admissible column such that r_2 is strictly larger than any entry of $U_{x,2}$, and r_2' is strictly smaller than any entry of $U_{x,2}$ already slid. The primed letters are considered to be slid because by the time of their creation, they are outer corners.

The cell with the red letter r_2 was the cell of $U_{x,1,+}$ where the complete $SJDT_j$ slide started and the B.2 case appeared with the creation of a non-admissible column. Let $U_{x,2,+}$ be the standard tableau obtained by adding the red letter r_2 to $U_{x,2}$, and let $V_{x,2,+}$ be the standard tableau obtained by adding the primed red letter r_2' to $V_{x,2}$. We keep applying complete $SJDT_j$ slides to the entries of $U_{x,2,+}$ from the biggest to the smallest to rectify $H^{x,2,-}$.

At this point, one has the following relative ordering of the red letters, where r_2 belongs to $U_{x,2,+}$ and $r_2' < r_1 < r_1'$ belong to $V_{x,2,+}$:

$$r_2 < r_2' < r_1 < r_1',$$

$$U_{x,2} \subset U_{x,2,+} \subseteq U_{x,1} \subset U_{x,1,+} \subseteq U_x, \quad V_{x,2,+} \supset V_{x,2} \supset V_{x,1,+} \supset V_{x,1} \supseteq V_x.$$

Continue in this fashion. Let $\mathsf{B}^{x,y}$ be the crystal obtained after a complete $SJDT_j$ slide to an entry of $U_{x,y-1,+}$, where the B.2 case arises and creates a non-admissible column for the y-th time. Let $U_{x,y}$ be the standard tableau obtained from $U_{x,y-1,+}$ after applying the complete SJDT slides to its entries so far. We have to apply the **R3** contractor operator to every vertex of $\mathsf{B}^{x,y}$ to transform the non-admissible column into an admissible one: a pair of symmetric entries in each vertex of $\mathsf{B}^{x,y}$ will be deleted, the top and bottom cells of that column will be emptied and the remaining entries will be placed in order. Let $\mathsf{B}^{x,y,-}$ be the new crystal of KN skew tableaux isomorphic to $\mathsf{B}^{x,y}$. Fill the empty entries with red letters $r_y < r_y'$, as before with r_y on the top and r_y' on the bottom of that column such that r_y is strictly larger than any entry of $U_{x,y}$, and r_y' is strictly smaller than any entry of $U_{x,y-1,+}$ already slid.

The cell with the red letter r_y was the cell of $U_{x,y-1,+}$ where the complete $SJDT_j$ slide started and the B.2 case appeared with the creation of a non-admissible column. Let $U_{x,y,+}$ be the standard tableau obtained by adding the red letter r_y to $U_{x,y}$, and let $V_{x,y,+}$ be the standard tableau obtained by adding the primed red letter r_y' to $V_{x,y}$. We keep applying complete $SJDT_j$ slides to the entries of $U_{x,y,+}$ from the largest to the smallest, and eventually, we rectify $H^{x,y,-}$ without further recourse of the contractor **R3**. We reach the crystal R, where every vertex is rectified. The crystal R is called the rectification of B^0 .

At this point one has the following relative ordering among the $2y$ red letters:

$$r_y < r_y' < \cdots < r_2 < r_2' < \cdots < r_1 < r_1'$$

and the rectification storing tableaux

$$\emptyset \subset U_{x,y} \subset U_{x,y,+} \subseteq U_{x,y-1} \subset \cdots \subset U_{x,2} \subset U_{x,2,+} \subseteq U_{x,1} \subset U_{x,1,+} \subseteq U_x,$$

$$V \supset V_{x,y,+} \supset V_{x,y} \supset V_{x,y-1,+} \supset V_{x,y-1} \supset \cdots \supset V_{x,2,+} \supset V_{x,2} \supset V_{x,1,+} \supset V_{x,1} \supseteq V_x,$$

where V is the standard tableau obtained by adding to $V_{x,y,+}$ via sliding the letters from $U_{x,y,+}$. We have the following ordering of all colored letters, green, purple (primed and unprimed), and red (primed and unprimed) in the skew standard tableau V:

$$g_1 < \cdots < r_y < r'_y < g_l < \cdots < r_d < r'_d < \cdots < g_{|\mu|} < \\ < p_1 < p_2 < \cdots < r_k < r'_k < \cdots < p_i < \cdots < r_1 < r'_1 < \cdots < p_x < p'_x < \cdots < p'_1. \tag{47}$$

We have constructed the following sequence of isomorphic crystals, stored in V via the slid colorful letters:

$$\mathsf{B}^0 \overset{\mathbf{R3}}{\simeq} \cdots \mathsf{B}^{x_r} \overset{\mathbf{R3}}{\simeq} \cdots \mathsf{B}^{x_r+x_s} \overset{\mathbf{R3}}{\simeq} \cdots \mathsf{B}^{x_r+x_s+\cdots+x_t} = \mathsf{B}^x \tag{48}$$

$$\mathsf{B}^x \overset{SJDT_j}{\simeq} \cdots \mathsf{B}^{x,1} \overset{\mathbf{R3}}{\simeq} \mathsf{B}^{x,1,-} \overset{SJDT_j}{\simeq} \cdots \mathsf{B}^{x,2} \overset{\mathbf{R3}}{\simeq} \mathsf{B}^{x,2,-} \overset{SJDT_j}{\simeq} \cdots \mathsf{B}^{x,y} \overset{\mathbf{R3}}{\simeq} \mathsf{B}^{x,y,-} \tag{49}$$

$$\mathsf{B}^{x,y,-} \overset{SJDT_j}{\simeq} \cdots \overset{SJDT_j}{\simeq} \mathsf{R}. \tag{50}$$

Remark 15. *In our construction, purple letters are larger than all green ones* (46)*. However, for the red ones together with the two other colors, we just write* (47)*.*

I.4 - The Schützenberger-Lusztig involution on the C_{n-j+1} crystals B^0 its rectification, the crystal R, and the reversal.

Let L^0 be the lowest weight element of the C_{n-j+1} connected normal crystal B^0. The crystal R with highest weight element $\mathsf{rectification}_j(H^0)$ is the rectification of the crystal B^0 and contains $\mathsf{rectification}_j(T_{[\pm j,n]})$. Let F be the composition of the sequence of lowering operators connecting H^0 to $T_{[\pm j,n]}$ in B^0, $F(H^0) = T_{[\pm j,n]}$. The Schützenberger-Lusztig involution ξ in B^0 gives $\xi(T_{[\pm j,n]}) = F^{-1}(L^0)$, where F^{-1} means the sequence obtained by replacing each lowering operator f_i in F with the corresponding raising operator e_i. In each crystal of the sequence (48), (49), (50) above, the same sequence F (F^{-1}) connects the corresponding highest (lowest) weight element to the corresponding coplactic image of $T_{[\pm j,n]}$ ($\xi(T_{[\pm j,n]})$). In particular, F connects $\mathsf{rectification}_j(H^0)$ to $\mathsf{rectification}_j(T_{\pm j,n})$, $F(\mathsf{rectification}_j H^0) = \mathsf{rectification}_j(T_{[\pm j,n]})$. By Santos [Sa21a], the Schützenberger-Lusztig involution in R guarantees that $\mathsf{evac}^{C_{n-j+1}}(\mathsf{rectification}_j(T_{[\pm j,n]}) = F^{-1}(\mathsf{rectification}_j(L^0)$ is in R. Thanks to the crystal isomorphisms and Lemma 5,

$$\mathsf{reversal}^{C_{n-j+1}}(T_{[\pm j,n]}) = F^{-1}(L^0) = \mathsf{arectification}_j\, \mathsf{evac}^{C_{n-j+1}}(\mathsf{rectification}_j(T_{[\pm j,n]})). \tag{51}$$

To compute the reversal of $T_{[\pm j,n]}$ in B^0 without using the sequence F of crystal operators and the highest/lowest weight elements H^0, L_0 of B^0, we use Santos' evacuation on $\mathsf{rect}_j(T_{[\pm j,n]})$ and the rectification sequence of crystals backwards in (48), (49),(50) stored in the standard skew tableau V.

Step II. Computation of symplectic evacuation of $\mathrm{RECT}_j(T_{[\pm j,n]})$ in the C_{n-j+1} crystal R.

The tableau $\mathsf{rect}_j(T_{[\pm j,n]})$ is admissible in the alphabet $[\pm j, n]$. Use Santos' algorithm as follows: take π-rotation and change the sign of $\mathsf{rect}_j(T_{[\pm j,n]})$; then, apply $SJDT_j$ to obtain $\mathsf{evac}^{C_{n-j+1}}(\mathsf{rectification}_j(T_{[\pm j,n]}))$ in the crystal R.

Replace the tableau pair $(\mathsf{rectification}_j(T_{[\pm j,n]})), V)$ with

$$(\mathsf{evac}^{C_{n-j+1}}(\mathsf{rectification}_j(T_{[\pm j,n]})), V).$$

Step III. Symplectic reversal of $T_{[\pm j,n]}$ in the C_{n-j+1} crystal B^0.

Consider the pair of tableaux $(\mathsf{evac}^{C_{n-j+1}}(\mathsf{rectification}_j(T_{[\pm j,n]})), V)$, where V is the standard tableau consisting of all the slid letters in the rectification sequence (48), (49),(50) on the alphabet of green, purple and red letters.

Apply the *reverse* $SJDT_j$, $RSJDT_j$, to the entries of V from the smallest to the largest to send $\mathsf{evac}^{C_{n-j+1}}(\mathsf{rect}_j(T_{[\pm j,n]}))$ to $\mathsf{reversal}(T_{[\pm j,n]}) = F^{-1}(L_0)$ in the C_{n-j+1} crystal B^0.

When the $SJDT_j$ applies to an unprimed red letter r_i, $i \in \{1, \dots, y\}$, in V, the letter r_i slides to the top of a column with the cell r'_i on the bottom. At this point, we have reached the crystal $\mathsf{B}^{x,i}$. Then we apply the dilation operator **R3** to the column containing the pair (r_i, r'_i) by erasing those entries and adding a pair of symmetric entries $(k, \bar{k})$ so that we get a non-admissible column on the alphabet $[\pm j, n]$. The $SJDT_j$ applies now to the next letter bigger than r'_i. In this complete reverse slide, the $SJDT_j$ B.2 case occurs.

When the reverse $SJDT_j$ slides have been applied to all non-primed purple letters, we have reached the crystal B^x, where the columns $r, s, \dots, t$ have x_i non-primed purple letters $p_{x_1+\cdots+x_{i-1}+1} < \cdots < p_{x_1+\cdots+x_i}$ on the top and the corresponding primed letters on the bottom for $i = r, s, \dots, t$. Then, for $i = t, \dots, s, r$, we apply the dilation operator **R3** to each such column i x_i times, and we reach the crystal B^0, where each vertex has non-admissible columns $r, s, \dots, t$. In particular, we obtain $\mathsf{reversal}^{C_{n-j+1}}(T_{[\pm j,n]})$.

Step IV. Partial symplectic reversal of T computes $\xi^{C_n}_{[j,n]}(T)$.

$\xi^{C_n}_{[j,n]}(T)$ is the Schützenberger–Lusztig involution of $T = (T^+_{[j-1]}, T_{[\pm j,n]}, T^-_{\overline{[j-1,\bar{1}]}})$ in the crystal connected component $\mathsf{B} \approx \mathsf{B}^0$ of $\mathsf{KN}_{[j,n]}(\lambda, n)$. Replace $T_{[\pm j,n]}$ with $\mathsf{reversal}^{C_{n-j+1}}(T_{[\pm j,n]})$, (51), in T,

which gives

$$\xi^{C_n}_{[j,n]}(T) = \mathsf{reversal}^{C_n}_{[j,n]}(T) = (T^+_{[j-1]}, \mathsf{arect}_j \mathsf{evac}^{C_{n-j+1}}(\mathsf{rect}_j(T_{[\pm j,n]})), T^-_{\overline{[j-1,\bar{1}]}}). \quad (52)$$

Remark 16.
- *If we put $j = 1$ in the colorful algorithm, we reduce to Step II of C_n evacuation.*
- *If $T \in \mathsf{SSYT}(\lambda, n) \subseteq \mathsf{KN}(\lambda, n)$, our colorful tableau switching algorithm reduces to just the green color, that is, to the* ordinary tableau switching *of the tableau-pair $(U_0, T_{[j,n]})$, with U_0 a standard tableau of shape μ and $T_{[j,n]}$ a semi-standard skew tableau of shape λ/μ on the alphabet $[j, n]$. The semi-standard skew-tableau $T_{[j,n]}$ on the alphabet $[j, n]$ is also a C_{n-j+1} admissible*

tableau. **R3** *does not apply, and $SJDT_j$ reduces to the ordinary JDT. Therefore, purple and red colors do not pop up in Step I. This means Step I returns the pair $(\mathsf{rect}_j(T_{[j,n]}, V)$, with $\mathsf{rect}_j(T_{[j,n]})$ a semi-standard tableau in the alphabet $[j,n]$ with $|\lambda| - |\mu|$ boxes and V completely green. Step II computes the symplectic C_{n-j+1} evacuation of $\mathsf{rect}_j(T_{[j,n]})$; this step produces a semi-standard tableau of the same shape on the alphabet $[\bar{n}, \bar{j}]$. Step III applies the ordinary reverse JDT to $\mathsf{evac}^{C_{n-j+1}}\mathsf{rect}_j(T_{[j,n]})$ governed by the green V and returns a semi-standard skew tableau of shape λ/μ on the alphabet $[\bar{n}, \bar{j}]$. Step IV computes the partial reversal $\mathsf{reversal}_J^{C_n}(T)$, $J = [j,n]$, with $T^{-}_{\overline{[j-1,\bar{1}]}}$ empty,*

$$\xi^{C_n}_{[j,n]}(T) = \mathsf{reversal}^{C_n}_{[j,n]}(T) = (T^{+}_{[j-1]}, \mathsf{arect}_j\, \mathsf{evac}^{C_{n-j+1}}(\mathsf{rect}_j(T_{[j,n]}))). \tag{53}$$

- *The algorithm for the full C_m reversal of a KN skew tableau $T \in \mathsf{KN}(\lambda/\mu, m)$ results from our colorful tableau switching algorithm by considering the image of T, $(T_\mu, \hat{T})$, in the full sub-crystal $\mathsf{B}(\mu,\lambda) \subseteq \mathsf{KN}_{[j,n]}(\lambda, n)$, where $n = m+j-1$ (Subsection 7.4.2). Let B be the crystal connected component of $\mathsf{B}(\mu,\lambda)$ containing $(T_\mu, \hat{T})$, where T_μ is the Yamanouchi tableau of shape μ and $\hat{T}$ is obtained by increasing each of the entries of T by $j-1$. Then, restricting $(T_\mu, \hat{T})$ to the alphabet $[\pm j, n]$, $\hat{T}$ is an admissible C_{n-j+1} skew tableau in the C_{n-j+1} crystal B^0. Our algorithm reduces to Step I with just green and red, Step II and Step III. Finally, we subtract $j-1$ from the entries of $\mathsf{reversal}^{C_{n-j+1}}(\hat{T})$ to get $\mathsf{reversal}^{C_m}(T)$. However, subtraction by $j-1$ cancels the last step in the reduced $SJDT_j$ (Definition 4), and therefore it is enough to apply SJDT.*

  This means that the algorithm for the full C_m reversal of the KN skew tableau T results from our algorithm with $\mathsf{B}^0 = \mathsf{B}(T)$ a type C_m crystal, $x = 0$ and applying SJDT to U_0 to get $(rect(T), V)$, where V is a skew standard tableau without purple letters. Then RSJDT applied to V gives

$$\mathsf{reversal}^{C_m}(T) = \mathsf{arectification}\, \mathsf{evac}^{C_m}(\mathsf{rectification}(T)).$$

9.3. Examples of full and partial symplectic reversal.

Example 7. *Full reversal of a skew tableau, $J = I$. In this case, we have no purple letters, as no letters are deleted at the beginning. Let* $T = \begin{array}{|c|c|c|c|} \hline & 2 & \bar{2} & \bar{1} \\ \hline \bar{2} & \bar{2} & \bar{1} & \\ \cline{1-3} \bar{1} & & & \\ \cline{1-1} \end{array} \in$ *$\mathsf{KN}((4,3,2)/(1),3)$. We compute $\xi^{C_3}(T)$ as follows. First, we fill in the empty box in T with a green letter (it defines the one box standard tableau U_0), to which we perform symplectic jeu de taquin until it becomes an outer corner.*

$$(U_0, T) = \begin{array}{|c|c|c|c|} \hline g & 2 & \bar{2} & \bar{1} \\ \hline \bar{2} & \bar{2} & \bar{1} \\ \cline{1-3} \bar{1} \\ \cline{1-1} \end{array} \overset{\mathsf{SJDT}}{\to} \begin{array}{|c|c|c|c|} \hline 1 & g & \bar{2} & \bar{1} \\ \hline \bar{2} & \bar{1} & \bar{1} \\ \cline{1-3} \bar{1} & \\ \cline{1-2} \end{array} \overset{\mathsf{SJDT}}{\to} \begin{array}{|c|c|c|c|} \hline r & g & \bar{2} & \bar{1} \\ \hline \bar{2} & \bar{1} & \bar{1} \\ \cline{1-3} \bar{r} \\ \cline{1-1} \end{array} \overset{\mathsf{SJDT}}{\to} \begin{array}{|c|c|c|c|} \hline r & \bar{2} & g & \bar{1} \\ \hline \bar{2} & \bar{1} & \bar{1} \\ \cline{1-3} \bar{r} \\ \cline{1-1} \end{array} \overset{\mathsf{SJDT}}{\to} \begin{array}{|c|c|c|c|} \hline r & \bar{2} & \bar{1} & \bar{1} \\ \hline \bar{2} & \bar{1} & g \\ \cline{1-3} \bar{r} \\ \cline{1-1} \end{array}$$

$$\overset{\mathsf{SJDT}}{\rightarrow} \begin{array}{|c|c|c|c|}\hline \bar{2}&\bar{2}&\bar{1}&\bar{1}\\\hline r&\bar{1}&g\\\cline{1-3} \bar{r}\\\cline{1-1}\end{array} \overset{\mathsf{SJDT}}{\rightarrow} \begin{array}{|c|c|c|c|}\hline \bar{2}&\bar{2}&\bar{1}&\bar{1}\\\hline \bar{1}&r&g\\\cline{1-3} \bar{r}\\\cline{1-1}\end{array} \Rightarrow rect(T) = \begin{array}{|c|c|c|c|}\hline \bar{2}&\bar{2}&\bar{1}&\bar{1}\\\hline \bar{1}\\\cline{1-1}\end{array}, V = \begin{array}{|c|c|c|c|}\hline \; & \; & \; & \;\\\hline & r & g\\\cline{1-3} \bar{r}\\\cline{1-1}\end{array}.$$

$$r < g < \bar{r}$$

Taking π-rotation and changing the signs of $rect(T)$, we again apply $SJDT$ to compute $\mathsf{evac}^{C_3}(\mathsf{rect}(T))$:

$$\begin{array}{|c|c|c|c|}\hline * & * & * & 1\\\hline 1&1&2&2\\\hline\end{array} \overset{\mathsf{SJDT}}{\rightarrow} \begin{array}{|c|c|c|c|}\hline * & * & 1 & *\\\hline 1&1&2&2\\\hline\end{array} \overset{\mathsf{SJDT}}{\rightarrow} \begin{array}{|c|c|c|c|}\hline * & * & 1 & 2\\\hline 1&1&2&*\\\hline\end{array} \overset{\mathsf{SJDT}}{\rightarrow} \begin{array}{|c|c|c|c|}\hline * & 1 & 1 & 2\\\hline 1&*&2&*\\\hline\end{array} \overset{\mathsf{SJDT}}{\rightarrow} \begin{array}{|c|c|c|c|}\hline * & 1 & 1 & 1\\\hline 1&2&*&*\\\hline\end{array}$$

$$\overset{\mathsf{SJDT}}{\rightarrow} \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2\\\hline *&2&*&*\\\hline\end{array} \overset{\mathsf{SJDT}}{\rightarrow} \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2\\\hline 2&*&*&*\\\hline\end{array} = \mathsf{evac}^{C_3}(\mathsf{rect}(T)).$$

We replace $\mathsf{rect}(T)$ with $\mathsf{evac}^{C_3}(\mathsf{rect}(T))$ in $(\mathsf{rect}(T), V)$ and apply reverse $SJDT$ to V to compute $\xi^{C_3}(T) = \mathsf{reversal}^{C_3}(T)$:

$$(\mathsf{evac}^{C_3}(\mathsf{rect}(T)), V) = \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2&r&g\\\cline{1-3} \bar{r}\\\cline{1-1}\end{array} \overset{\mathsf{RSJDT}}{\rightarrow} \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline r&2&g\\\cline{1-3} \bar{r}\\\cline{1-1}\end{array} \overset{\mathsf{RSJDT}}{\rightarrow} \begin{array}{|c|c|c|c|}\hline r&1&1&2\\\hline 1&2&g\\\cline{1-3} \bar{r}&\;\\\cline{1-2}\end{array} \overset{\mathbf{R3}}{\equiv} \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2&2&g\\\cline{1-3} \bar{2}\\\cline{1-1}\end{array}$$

$$\overset{\mathsf{RSJDT}}{\rightarrow} \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2&g&2\\\cline{1-3} \bar{2}\\\cline{1-1}\end{array} \overset{\mathsf{RSJDT}}{\rightarrow} \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline g&2&2\\\cline{1-3} \bar{2}\\\cline{1-1}\end{array} \overset{\mathsf{RSJDT}}{\rightarrow} \begin{array}{|c|c|c|c|}\hline g&1&1&2\\\hline 1&2&2\\\cline{1-3} \bar{2}\\\cline{1-1}\end{array} = (U_0, \xi^{C_3}(T))$$

$$\Rightarrow \xi^{C_3}(T) = \begin{array}{|c|c|c|c|}\hline \;&1&1&2\\\hline 1&2&2\\\cline{1-3} \bar{2}\\\cline{1-1}\end{array}.$$

Example 8. *Let* $P = \begin{array}{|c|c|c|c|}\hline 1&2&2&\bar{1}\\\hline 4&4&\bar{3}\\\cline{1-3} \bar{4}&\bar{2}&\bar{1}\\\cline{1-3} \bar{3}\\\cline{1-1}\end{array} \in \mathsf{KN}((4,3,3,1),4)$. *We have* $\mathsf{wt}(P) = (-1,1,-2,,1)$.

To compute $\xi^{C_4}_{[2,4]}(P) = \mathsf{reversal}^{C_4}_{[2,4]}(P)$, we freeze the letters $1, \bar{1}$ in P and consider $P_{[\pm 2,4]}$. $P_{[\pm 2,4]}$ is not an admissible C_3 tableau in the alphabet $[\pm 2, 4]$: the second column $24\bar{2}$ is not an admissible C_3 column; $24\bar{2} \overset{SJDT_2}{\rightarrow} 4$. The column reading of $P_{[\pm 2,4]}$ is $2\bar{3}24\bar{2}4\bar{4}\bar{3} \overset{R_3}{\equiv} 2\bar{3}44\bar{4}\bar{3}$. We include this non-admissible second column in the $SJDT_2$ sequence to rectify $P_{[\pm 2,4]}$.

***1.** Rectification of $P_{[\pm 2,4]}$*

$$(U_0, P_{[\pm 2,4]}) = \begin{array}{|c|c|c|c|}\hline g&2&2&\;\\\hline 4&4&\bar{3}\\\cline{1-3} \bar{4}&\bar{2}&\;\\\cline{1-3} \bar{3}\\\cline{1-1}\end{array} \overset{\mathsf{SJDT}_2}{\rightarrow} \begin{array}{|c|c|c|c|}\hline g&p&2&\;\\\hline 4&4&\bar{3}\\\cline{1-3} \bar{4}&p'&\;\\\cline{1-3} \bar{3}\\\cline{1-1}\end{array} \overset{\mathsf{SJDT}_2}{\rightarrow} \begin{array}{|c|c|c|c|}\hline g&2&\bar{3}&\;\\\hline 4&4&p\\\cline{1-3} \bar{4}&p'&\;\\\cline{1-3} \bar{3}\\\cline{1-1}\end{array} \overset{\mathsf{SJDT}_2}{\rightarrow} \begin{array}{|c|c|c|c|}\hline 2&4&\bar{3}&\;\\\hline 4&g&p\\\cline{1-3} \bar{4}&p'&\;\\\cline{1-3} \bar{3}\\\cline{1-1}\end{array}$$

$$\overset{SJDT_2}{\rightarrow} \begin{array}{cccc} c & 4 & \bar{3} & \square \\ 2 & g & p & \\ \bar{3} & p' & \square & \\ r' & & & \end{array} \overset{SJDT_2}{\rightarrow} \begin{array}{cccc} 2 & 4 & \bar{3} & \square \\ \bar{3} & g & p & \\ r & p' & \square & \\ r' & & & \end{array} = (\mathsf{rect}_2 P_{[\pm 2,4]}, V) \Rightarrow \mathsf{rect}_2 P_{[\pm 2,4]} = \begin{array}{ccc} 2 & 4 & \bar{3} \\ \bar{3} & & \end{array},$$

$$V = \begin{array}{cccc} \square & \square & \square & \square \\ \square & g & p & \\ r & p' & \square & \\ r' & & & \end{array}, r < r' < g < p < p'.$$

2. Computation of $\mathsf{evac}^{C_3}\,\mathsf{rect}_2(P_{[\pm 2,4]})$. *Taking* π*-rotation and changing the signs of* $\mathsf{rect}_2(P_{[\pm 2,4]})$*, we again apply* $SJDT_2$*:*

$$\begin{array}{ccc} \square & \square & 3 \\ 3 & \bar{4} & \bar{2} \end{array} \overset{SJDT}{\rightarrow} \begin{array}{ccc} \square & 3 & \bar{2} \\ 3 & \bar{4} & \end{array} \overset{SJDT}{\rightarrow} \begin{array}{ccc} 3 & 3 & \bar{2} \\ \square & \bar{4} & \end{array} \overset{SJDT}{\rightarrow} \begin{array}{ccc} 3 & 3 & \bar{2} \\ \bar{4} & & \end{array} = \mathsf{evac}^{C_3}\,\mathsf{rect}_2 P_{[\pm 2,4]}.$$

3. Reversal of $P_{[\pm 2,4]}$. *Replace* $\mathsf{rect}_2(P_{[\pm 2,4]})$ *with* $\mathsf{evac}^{C_3}(\mathsf{rect}_2(P_{[\pm 2,4]}))$ *in* $(\mathsf{rect}_2(P_{[\pm 2,4]}), V)$ *and apply* $RSJDT_2$ *to* V*.*

$$(\mathsf{evac}^{C_3}(\mathsf{rect}_2 P_{[\pm 2,4]}), V) = \begin{array}{cccc} 3 & 3 & \bar{2} & \square \\ \bar{4} & g & p & \\ r & p' & \square & \\ r' & & & \end{array} \overset{RSJDT_2}{\rightarrow} \begin{array}{cccc} r & 3 & \bar{2} & \square \\ 3 & g & p & \\ \bar{4} & p' & \square & \\ r' & & & \end{array} \overset{RSJDT_2}{\rightarrow} \begin{array}{cccc} 2 & 3 & \bar{2} & \square \\ 3 & g & p & \\ \bar{4} & p' & \square & \\ \bar{2} & & & \end{array} \overset{RSJDT_2}{\rightarrow} \begin{array}{cccc} g & 2 & \bar{2} & \square \\ 3 & 3 & p & \\ \bar{4} & p' & \square & \\ \bar{2} & & & \end{array}$$

$$\overset{RSJDT_2}{\rightarrow} \begin{array}{cccc} g & p & 3 & \square \\ 3 & 3 & \bar{3} & \square \\ \bar{4} & p' & \square & \\ \bar{2} & & & \end{array} \overset{RSJDT_2}{\rightarrow} \begin{array}{cccc} g & 2 & 3 & \square \\ 3 & 3 & \bar{3} & \square \\ \bar{4} & \bar{2} & \square & \\ \bar{2} & & & \end{array} = (U_0, \mathsf{reversal}^{C_3}(P_{[\pm 2,4]}))$$

$$\Rightarrow \mathsf{reversal}^{C_3}(P_{[\pm 2,4]}) = \begin{array}{cccc} \square & 2 & 3 & \square \\ 3 & 3 & \bar{3} & \square \\ \bar{4} & \bar{2} & \square & \\ \bar{2} & & & \end{array}.$$

4. Replace $P_{[\pm 2,4]}$ *with* $\mathsf{reversal}^{C_3}(P_{[\pm 2,4]})$ *in* P *to obtain*

$$\mathsf{reversal}^{C_4}_{[2,4]}(P) = \begin{array}{cccc} 1 & 2 & 3 & \bar{1} \\ 3 & 3 & \bar{3} & \\ \bar{4} & \bar{2} & \bar{1} & \\ \bar{2} & & & \end{array}, \mathsf{wt}_{[2,4]}(\xi^{C_4}_{[2,4]}(P)) = -\mathsf{wt}_{[2,4]}(P) = (-1, 2, -1).$$

9.4. **General Dynkin sub-diagram and virtualization.** Let $\xi^{C_n}_{[1,j]}$, $1 \leq j \leq n-1$, be the Schützenberger–Lusztig involution on $\mathsf{KN}_{[1,j]}(\lambda, n)$. $\mathsf{KN}_{[1,j]}(\lambda, n)$ is a type A_j crystal of Kashiwara–Nakashima tableaux of shape λ on the alphabet $[\pm n]$ with lowering and raising operators f_i and e_i, respectively, given by the type C_n signature rule with $i \in [1, j]$. Notice that the unique crystal operators which change the signs of the entries are f_n and e_n, which are forgotten. Next we give a computation of

$\xi^{C_n}_{[p,q]}$, for any $1 \le p \le q \le n$, via virtualization E and bring it back to $\mathsf{KN}_{[p,q]}(\lambda, n)$ by applying E^{-1}, see Theorem 5 and Theorem 6 below.

In the next section, we give a computation of $\xi^{C_n}_{[1,j]}$, $1 \le j \le n-1$, via virtualization E and bring it back to $\mathsf{KN}_{[1,j]}(\lambda, n)$ by applying E^{-1}. See Theorem 5 below.

9.4.1. *Embedding of a partial symplectic Schützenberger–Lusztig involution and back.* Let $J \subseteq [n]$ be a sub-Dynkin diagram of the type C_n Dynkin diagram $I = [n]$. Let U be a connected component of the Levi branched crystal $\mathsf{KN}_J(\lambda, n)$ with $J \subseteq [n]$ and with highest and lowest weight elements u^{high} and u^{low}, respectively. Recall from Subsection 5.2, Proposition 2, that each connected component U of the Levi branched crystal $\mathsf{KN}_J(\lambda, n)$ is embedded via E into a connected component of the Levi branched crystal $\mathsf{SSYT}_{J\cup\bar{J}}(\lambda, n)$ with highest and lowest weight elements $E(u^{\mathsf{high}})$ and $E(u^{\mathsf{low}})$, respectively.

Let $P = (P^+, P^-) \in \mathsf{SSYT}(\lambda^A, n, \bar{n})$. The crystals $\mathsf{SSYT}_{[p,q]}(\lambda^A, n, \bar{n})$ and $\mathsf{SSYT}_{[\overline{q+1},\overline{p+1}]}(\lambda^A, n, \bar{n})$ are isomorphic to $\mathsf{SSYT}_{[p,q]}(\lambda^A_+, n)$ and $\mathsf{SSYT}_{[\overline{q+1},\overline{p+1}]}(\lambda^A/\lambda^A_+, \bar{n})$, respectively (recall Remark 5). The corresponding pair of isomorphic crystals has the same multiset of highest weight vectors in $\mathbb{Z}^{q-p+1}$ respectively, regarding the sub-Dynkin diagram $[p,q]$. We may then write

$$\xi^{A_{2n-1}}_{[p,q]}\xi^{A_{2n-1}}_{[\overline{q+1},\overline{p+1}]}(P) = (\xi^{A_{n-1}}_{[p,q]}(P^+), \xi^{A_{n-1}}_{[\overline{q+1},\overline{p+1}]}(P^-)). \tag{54}$$

Theorem 5. *Let $T \in \mathsf{KN}_{[p,q]}(\lambda, n)$ of type $A_{p-q+1}, 1 \le p \le q < n$. Then*

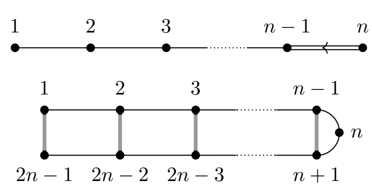
$$\xi^{A_{2n-1}}_{[p,q]\cup[\overline{q+1},\overline{p+1}]}(E(T)) = \xi^{A_{2n-1}}_{[p,q]}\xi^{A_{2n-1}}_{[\overline{q+1},\overline{p+1}]}(E(T)) = E(\xi^{C_n}_{[p,q]}(T)).$$

Moreover,

$$\xi^{C_n}_{[p,q]}(T) = E^{-1}\mathsf{reversal}^{A_{2n-1}}_{[p,q]}\mathsf{reversal}^{A_{2n-1}}_{[\overline{q+1},\overline{p+1}]}E(T). \tag{55}$$

Proof. Recall Proposition 2, Remark 5 and (14). Then it follows from Theorem 2. $\square$

It is now convenient to change the labeling of the A_{2n-1} Dynkin diagram. Instead of $[k, \overline{k+1}]$, we write $[k, 2n-k]$, and $\mathsf{SSYT}_{[k,2n-k]}(\lambda, n, \bar{n})$. This relabelling is illustrated in the picture below.



Theorem 6. *Let $T \in \mathsf{KN}_{[k,n]}(\lambda, n)$ of type C_{n-k+1} for some $1 \le k \le n$. Then*

$$\xi^{A_{2n-1}}_{[k,2n-k]}(E(T)) = E(\xi^{C_n}_{[k,n]}(T)).$$

Moreover, on $\mathsf{SSYT}(\lambda^A, n, \bar{n})$, $\xi^{A_{2n-1}}_{[k,2n-k]} = \mathsf{reversal}^{A_{2n-1}}_{[k,2n-k]}$, and

$$\xi^{C_n}_{[k,n]} = E^{-1}\mathsf{reversal}^{A_{2n-1}}_{[k,2n-k]}E. \tag{56}$$

Proof. Recall Corollary 2 and that, in the case of the branched crystal $\mathsf{SSYT}_{[k,2n-k]}(\lambda, n, \bar{n})$, $\theta(i) = 2n - i$, for $i \in [k, 2n-k]$. Let U be the connected component of $\mathsf{KN}_{[k,n]}(\lambda, n)$ containing T, and let the highest and lowest weight elements of U be u^{high} and u^{low}, respectively.

$$T = f_{i_r} \dots f_{i_1}(u^{\mathsf{high}}), i_1, \dots, i_r \in [k, n],$$
$$E(T) = f^A_{i_r} f^A_{2n-i_r} \dots f^A_{i_1} f^A_{2n-i_1}(E(u^{\mathsf{high}})), i_1, \dots, i_r \in [k, n], \tag{57}$$

and

$$\xi^{C_n}_{[k,n]}(T) = e_{i_r} \dots e_{i_1}(u^{\mathsf{low}}).$$

Then, from Subsection 5.1, (10),

$$E(\xi^{C_n}_{[k,n]}(T)) = E(e_{i_r} \dots e_{i_1}(u^{\mathsf{low}})) = e^A_{i_r} e^A_{2n-i_r} \dots e^A_{i_1} e^A_{2n-i_1} E(u^{\mathsf{low}})$$

and, from (57),

$$\begin{aligned} \xi^A_{[k,2n-k]}(E(T)) &= e^A_{\theta(i_r)} e^A_{\theta(2n-i_r)} \dots e^A_{\theta(i_1)} e^A_{\theta(2n-i_1)}(E(u^{\mathsf{low}})) \\ &= e^A_{2n-i_r} e^A_{i_r} \dots e^A_{2n-i_1} e^A_{i_1}(E(u^{\mathsf{low}})) \\ &= e^A_{i_r} e^A_{2n-i_r} \dots e^A_{i_1} e^A_{2n-i_1} E(u^{\mathsf{low}}) = E(\xi^C_{[k,n]}(T)). \end{aligned}$$

Finally, (56) follows from (39). ☐

Using a generalized form of Lemma 6, the following corollary is a generalization of the two theorems above.

Corollary 4. *Let $T \in \mathsf{KN}_{[p,q]\cup[k,n]}(\lambda, n)$ of subtype $A_{p-q+1} \times C_{n-k+1}$ for some $1 \leq p \leq q < k-1 < n$. Then*

$$\xi^{A_{2n-1}}_{[p,q]\cup[2n-q,2n-p]\cup[k,2n-k]}(E(T)) = \xi^{A_{2n-1}}_{[p,q]} \xi^{A_{2n-1}}_{[2n-q,2n-p]} \xi^{A_{2n-1}}_{[k,2n-k]}(E(T)) = E(\xi^{C_n}_{[p,q]\cup[k,n]}(T)).$$

Moreover, on $\mathsf{SSYT}(\lambda^A, n, \bar{n})$, $\xi^{A_{2n-1}}_{[p,q]} = \mathsf{reversal}^{A_{2n-1}}_{[p,q]}$, $\xi^{A_{2n-1}}_{[k,2n-k]} = \mathsf{reversal}^{A_{2n-1}}_{[k,2n-k]}$ and $\xi^{A_{2n-1}}_{[2n-q,2n-p]} = \mathsf{reversal}^{A_{2n-1}}_{[2n-q,2n-p]}$.

Remark 17. *Both $\xi^{A_{2n-1}}_{[p,q]\cup[2n-q,2n-p]}$ and $\xi^{A_{2n-1}}_{[k,2n-k]}$ act on the set $\mathsf{SSYT}(\lambda^A, n, \bar{n})$ to define a permutation such that the subset $E(\mathsf{KN}(\lambda, n))$ is preserved. In other words, each of these involutions defines a permutation on $E(\mathsf{KN}(\lambda, n))$ when their action is restricted to this subset.*

Corollary 5. *Let $\mathsf{SSYT}(\mu, 2n)$ with μ a partition with at most $2n$ parts, and let B_n be the Weyl group realized as $< r_i = (i\ i+1)(2n-i\ 2n-i+1)$, $r_n = (n\ n+1)$, $1 \leq i \leq n-1 >$. Then $\{\xi^{A_{2n-1}}_i \xi^{A_{2n-1}}_{2n-i}, \xi^{A_{2n-1}}_n, 1 \leq i \leq n-1\}$ define an action of B_n on $\mathsf{SSYT}(\lambda, 2n)$ by $r_i.b = \xi^{A_{2n-1}}_i \xi^{A_{2n-1}}_{2n-i} b$, $1 \leq i \leq n-1$, and $r_n.b = \xi^{A_{2n-1}}_n.b$, for $b \in \mathsf{SSYT}(\lambda, 2n)$ such that*

1. $e^{A_{2n-1}}_i e^{A_{2n-1}}_{2n-i} \xi^{A_{2n-1}}_i \xi^{A_{2n-1}}_{2n-i} = \xi^{A_{2n-1}}_i \xi^{A_{2n-1}}_{2n-i} f^{A_{2n-1}}_i f^{A_{2n-1}}_{2n-i}$, $1 \leq i < n$,
2. $e^{A_{2n-1}}_n e^{A_{2n-1}}_n \xi^{A_{2n-1}}_n = \xi^{A_{2n-1}}_n f^{A_{2n-1}}_n f^{A_{2n-1}}_n$,
3. $\mathsf{wt}(r_i.b) = r_{\mathsf{iwt}}(b)$, $1 \leq i \leq n$,
4. $w_0.T_\mu = T^{\mathsf{low}}_\mu$, $w_0 = (r_n \cdots r_1)^n$ *the long element of* B_n.

(5) if $\mu = \lambda^A$ is the virtual partition for some λ, it preserves the action of B_n on the underlying set of the crystal $E(\mathsf{KN}(\lambda, n))$ embedded in the crystal $\mathsf{SSYT}(\lambda^A, 2n)$.

Proof. (5) It follows from Theorem 5 and Theorem 6 with $p = q$ and respectively $k = n$ that $\xi_p^{A_{2n-1}}\xi_{\overline{p+1}}^{A_{2n-1}}(E(T)) = \xi_p^{A_{2n-1}}\xi_{2n-p}^{A_{2n-1}}(E(T)) = E(\xi_p^{C_n}(T))$ and $\xi_n^{A_{2n-1}}(E(T)) = E(\xi_n^{C_n}(T))$. From Proposition 6, 3, $\{\xi_n^{C_n} : 1 \leq i \leq n\}$ define an action of B_n on $\mathsf{KN}(\lambda, n)$. Therefore $\{\xi_i^{A_{2n-1}}\xi_{2n-i}^{A_{2n-1}}, \xi_n^{A_{2n-1}}, 1 \leq i \leq n-1\}$ is the translation of this action to the embedded crystal $E(\mathsf{KN}(\lambda, n))$ in $\mathsf{SSYT}(\lambda^A, 2n)$. □

9.5. **Virtualization of the action of $J_{\mathfrak{sp}(2n,\mathbb{C})}$ on the crystal $\mathsf{KN}(\lambda, n)$.** We have the following commutative diagram corresponding to the crystal embedding E and the partial C_n and A_{2n-1} Schützenberger–Lusztig involutions, where $[p, q] \subseteq [n-1]$ and $[p, n] \subseteq [n]$ are connected subintervals of the Dynkin diagram of C_n,

$$\begin{array}{ccc} \mathsf{KN}(\lambda, n) & \xrightarrow{E} & \mathsf{SSYT}(\lambda^A, n, \bar{n}) \\ \xi_{[p,n]}^{C_n} \Big\downarrow \xi_{[p,q]}^{C_n} & & \xi_{[p,\overline{p+1}]}^{A_{2n-1}} \Big\downarrow \xi_{[p,q]\cup[\overline{q+1},\overline{p+1}]}^{A_{2n-1}} \\ \mathsf{KN}(\lambda, n) & \xrightarrow[E]{} & \mathsf{SSYT}(\lambda^A, n, \bar{n}) \end{array}$$

$$\begin{array}{lcl} \widetilde{\Phi}_{\mathfrak{gl}_{2n}}^E : \widetilde{J}_{2n} & \longrightarrow & \mathfrak{S}_{E(\mathsf{KN}(\lambda, n))} \\ \tilde{s}_{[p,q]\cup[\overline{q+1},\overline{p+1}]} & \mapsto & \xi_{[p,q]\cup[\overline{q+1},\overline{p+1}]}^{A_{2n-1}} = \xi_{[p,q]}^{A_{2n-1}}\xi_{[\overline{q+1},\overline{p+1}]}^{A_{2n-1}} \\ \tilde{s}_{[p,\overline{p+1}]} & \mapsto & \xi_{[p,\overline{p+1}]}^{A_{2n-1}} \end{array} \tag{58}$$

Theorem 3 and Remark 17 imply that the action of $\widetilde{J}_{2n}$ on $\mathsf{SSYT}(\lambda^A, n, \bar{n})$ preserves the subset $E(\mathsf{KN}(\lambda, n)$, and thus, we have an action of $\widetilde{J}_{2n}$ on the set $E(\mathsf{KN}(\lambda, n))$ defined by

$$\begin{array}{lccl} \widetilde{\Phi}_{\mathfrak{sl}(2n,\mathbb{C})}^E : & \widetilde{J}_{2n} & \longrightarrow & \mathfrak{S}_{E(\mathsf{KN}(\lambda, n))} \\ & \tilde{s}_{[p,q]\cup[\overline{q+1},\overline{p+1}]} & \mapsto & \xi_{[p,q]\cup[\overline{q+1},\overline{p+1}]}^{A_{2n-1}} = \xi_{[p,q]}^{A_{2n-1}}\xi_{[\overline{q+1},\overline{p+1}]}^{A_{2n-1}} \\ & \tilde{s}_{[p,\overline{p+1}]} & \mapsto & \xi_{[p,\overline{p+1}]}^{A_{2n-1}} \end{array}$$

such that $\widetilde{\Phi}_{\mathfrak{sl}(2n,\mathbb{C})}^E(\tilde{s}_J) = \widetilde{\Phi}_{\mathfrak{sl}(2n,\mathbb{C})}(\tilde{s}_J)_{|E(\mathsf{KN}(\lambda, n))} \in \mathfrak{S}_{E(\mathsf{KN}(\lambda, n))}$. Let $\tilde{\imath} : J_{\mathfrak{sp}(2n,\mathbb{C})} \to \widetilde{J}_{2n}$ be the group isomorphism defined by $s_{[1,j]} \mapsto \tilde{s}_{[1,j]\cup[\overline{j+1},\bar{2}]}$, $1 \leq j < n$, and $s_{[j,n]} \mapsto \tilde{s}_{[j,\overline{j+1}]}$, $1 \leq j < n$, (see Proposition 4), and $\imath : \mathfrak{S}_{\mathsf{KN}(\lambda, n)} \to \mathfrak{S}_{E(\mathsf{KN}(\lambda, n))}$ the group isomorphism defined by $\imath(\sigma) = E\sigma E^{-1}$. The virtualization of the action of $J_{\mathfrak{sp}(2n,\mathbb{C})}$ on the crystal $\mathsf{KN}(\lambda, n)$ is then realized from the following commutative diagram

$$\begin{array}{ccc} J_{\mathfrak{sp}(2n,\mathbb{C})} & \xrightarrow{\Phi_{\mathfrak{sp}(2n,\mathbb{C})}} & \mathfrak{S}_{\mathsf{KN}(\lambda,n)} \\ \tilde{\imath}\downarrow & & \imath\downarrow \\ \widetilde{J}_{2n} & \xrightarrow[\widetilde{\Phi}^{E}_{\mathfrak{sl}(2n,\mathbb{C})}]{} & \mathfrak{S}_{E(\mathsf{KN}(\lambda,n))} \end{array} \qquad \widetilde{\Phi}^{E}_{\mathfrak{sl}(2n,\mathbb{C})}\tilde{\imath} = \imath\Phi_{\mathfrak{sp}(2n,\mathbb{C})} \tag{59}$$

From (59)

$$\begin{aligned} \widetilde{\Phi}^{E}_{\mathfrak{sl}(2n,\mathbb{C})}\tilde{\imath}(s_{[1,j]}) &= \widetilde{\Phi}^{E}_{\mathfrak{sl}(2n,\mathbb{C})}(\tilde{s}_{[1,j]\cup[\overline{j+1},\bar{2}]}) = \xi^{A_{2n-1}}_{[1,j]\cup[\overline{j+1},\bar{2}]} \\ = \imath\Phi_{\mathfrak{sp}(2n,\mathbb{C})}(s_{[1,j]}) &= \imath\xi^{C_n}_{[1,j]} = E\xi^{C_n}_{[1,j]}E^{-1} = \xi^{A_{2n-1}}_{[1,j]\cup[\overline{j+1},\bar{2}]}, \end{aligned}$$

$$\begin{aligned} \widetilde{\Phi}^{E}_{\mathfrak{sl}(2n,\mathbb{C})}\tilde{\imath}(s_{[j,n]}) &= \widetilde{\Phi}^{E}_{\mathfrak{sl}(2n,\mathbb{C})}(s_{[j,\overline{j+1}]}) = \xi^{A_{2n-1}}_{[j,\overline{j+1}]} \\ = \imath\Phi_{\mathfrak{sp}(2n,\mathbb{C})}(s_{[j,n]}) &= \imath\xi^{C_n}_{[j,n]} = E\xi^{C_n}_{[j,n]}E^{-1} = \xi^{A_{2n-1}}_{[j,\overline{j+1}]}. \end{aligned}$$

9.6. **Virtualization example.**

Example 9. *Consider $n = 6$, $J = [1,5]$ and the KN tableau T of shape $\lambda = 2\omega_6 + \omega_5 + \omega_2$:*

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & \bar{3} \\ \hline 2 & 4 & 5 & \bar{1} \\ \hline 3 & 6 & \bar{5} \\ \cline{1-3} 5 & \bar{6} & \bar{3} \\ \cline{1-3} 6 & \bar{5} & \bar{1} \\ \cline{1-3} \bar{5} & \bar{4} \\ \cline{1-2} \end{array}, \quad \mathit{wt}(T) = (-1, 2, 0, 0, -1, 1)$$

From λ we may immediately write $\lambda^A = 2\omega_6^A + 2\omega_6^A + \omega_7^A + \omega_5^A + \omega_{10}^A + \omega_2^A$, and from λ^A we write the Baker recording tableau Q_λ of shape λ^A as a sequence of shapes where we successively fill the boxes along columns, top to bottom, from 1 to $|\lambda^A| = 4|\omega_6^A| + |\omega_7^A| + |\omega_5^A| + |\omega_{10}^A| + |\omega_2^A| = 48$,

$$\omega_2^A \subseteq \omega_{10}^A + \omega_2^A \subseteq \omega_5^A + \omega_{10}^A + \omega_2^A \subseteq \omega_7^A + \omega_5^A + \omega_{10}^A + \omega_2^A \subseteq \omega_6^A + \omega_7^A + \omega_5^A + \omega_{10}^A + \omega_2^A$$
$$\subseteq 2\omega_6^A + \omega_7^A + \omega_5^A + \omega_{10}^A + \omega_2^A \subseteq 3\omega_6^A + \omega_7^A + \omega_5^A + \omega_{10}^A + \omega_2^A \subseteq 4\omega_6^A + \omega_7^A + \omega_5^A + \omega_{10}^A + \omega_2^A = \lambda^A$$

$Q_\lambda =$

1	3	*13*	18	*25*	31	*37*	43
2	4	*14*	19	*26*	32	*38*	44
5	*15*	20	*27*	33	*39*	45	
6	*16*	21	*28*	34	*40*	46	
7	*17*	22	*29*	35	*41*	47	
8	23	*30*	36	*42*	48		
9	24						
10							
11							
12							

.

Labelling the columns of T from left to right as C_4, C_3, C_2 and C_1, we have:

$$\psi(C_4) = \begin{array}{cc} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 5 \\ 6 & 6 \\ \overline{5} & \overline{4} \end{array}, \psi(C_3) = \begin{array}{cc} 1 & 2 \\ 2 & 4 \\ 3 & 6 \\ \overline{6} & \overline{5} \\ \overline{5} & \overline{3} \\ \overline{4} & \overline{1} \end{array}, \psi(C_2) = \begin{array}{cc} 2 & 3 \\ 4 & 5 \\ 6 & \overline{4} \\ \overline{6} & \overline{2} \\ \overline{5} & \overline{1} \\ \overline{3} & \\ \overline{1} & \end{array}, \psi(C_1) = \begin{array}{cc} 2 & \overline{3} \\ 4 & \overline{1} \\ 5 & \\ 6 & \\ \overline{6} & \\ \overline{5} & \\ \overline{4} & \\ \overline{3} & \\ \overline{2} & \\ \overline{1} & \end{array}.$$

Then $E(T)$ has shape $\lambda^A = 4\omega_6^A + \omega_7^A + \omega_5^A + \omega_{10}^A + \omega_2^A$,

$$\mathsf{wt}(E(T)) = \mathsf{wt}(w_T) = (3,6,4,4,3,5,3,5,4,4,2,5),$$

and $E(T) = [\emptyset \leftarrow w_T]$,

$$E(T) = [\emptyset \leftarrow w(\psi(C_1)) \leftarrow w(\psi(C_2)) \leftarrow w(\psi(C_3)) \leftarrow w(\psi(C_4))] = \begin{array}{cccccccc} 1 & 1 & 1 & 2 & 2 & 2 & 4 & \overline{3} \\ 2 & 2 & 2 & 3 & 4 & 5 & 5 & \overline{1} \\ 3 & 3 & 3 & 4 & 6 & 6 & \overline{3} & \\ 4 & 5 & 6 & \overline{6} & \overline{6} & \overline{5} & \overline{2} & \\ 6 & 6 & \overline{6} & \overline{5} & \overline{5} & \overline{3} & \overline{1} & \\ \overline{5} & \overline{5} & \overline{4} & \overline{4} & \overline{3} & \overline{1} & & \\ \overline{4} & \overline{1} & & & & & & \\ \overline{3} & & & & & & & \\ \overline{2} & & & & & & & \\ \overline{1} & & & & & & & \end{array}$$

which has recording tableau $Q(w_T) = Q_\lambda$.

Considering the barred and unbarred parts of $E(T)$ separately, we compute the evacuation, evac*, of the unbarred part and the reversal,* reversal*, of the barred part, yielding:*

$$\mathsf{evac} \begin{array}{ccccccc} 1 & 1 & 1 & 2 & 2 & 2 & 4 \\ 2 & 2 & 2 & 3 & 4 & 5 & 5 \\ 3 & 3 & 3 & 4 & 6 & 6 & \\ 4 & 5 & 6 & & & & \\ 6 & 6 & & & & & \end{array} = \begin{array}{ccccccc} 1 & 1 & 1 & 1 & 1 & 4 & 4 \\ 2 & 2 & 2 & 3 & 3 & 5 & 5 \\ 3 & 3 & 4 & 5 & 5 & 6 & \\ 4 & 5 & 6 & & & & \\ 5 & 6 & & & & & \end{array}$$

and

$$\mathsf{reversal}\ \begin{array}{cccccccc} * & * & * & * & * & * & * & \bar{3} \\ * & * & * & * & * & * & * & \bar{1} \\ * & * & * & * & * & * & \bar{3} & \\ * & * & * & \bar{6} & \bar{6} & \bar{5} & \bar{2} & \\ * & * & \bar{6} & \bar{5} & \bar{5} & \bar{3} & \bar{1} & \\ \bar{5} & \bar{5} & \bar{4} & \bar{4} & \bar{3} & \bar{1} & & \\ \bar{4} & \bar{1} & & & & & & \\ \bar{3} & & & & & & & \\ \bar{2} & & & & & & & \\ \bar{1} & & & & & & & \end{array} = \begin{array}{cccccccc} * & * & * & * & * & * & * & \bar{5} \\ * & * & * & * & * & * & * & \bar{2} \\ * & * & * & * & * & * & \bar{5} & \\ * & * & * & \bar{6} & \bar{6} & \bar{6} & \bar{2} & \\ * & * & \bar{6} & \bar{5} & \bar{4} & \bar{4} & \bar{1} & \\ \bar{6} & \bar{5} & \bar{4} & \bar{4} & \bar{2} & \bar{2} & & \\ \bar{4} & \bar{1} & & & & & & \\ \bar{3} & & & & & & & \\ \bar{2} & & & & & & & \\ \bar{1} & & & & & & & \end{array} .$$

Putting these tableaux together, one obtains

$$\xi^{A_{11}}_{[1,5]\sqcup[\bar{6},\bar{2}]}(E(T)) = (\mathsf{evac}(E(T)^+), \mathsf{reversal}(E(T)^-)).$$

Using Q_λ to perform the reverse column Schensted insertion on the resulting A_{11} tableau $\xi^{A_{11}}_{[1,5]\sqcup[\bar{6},\bar{2}]}(E(T))$ provides the image under ψ of four KN columns C'_1, C'_2, C'_3, C'_4:

$$\psi(C'_4) = \begin{array}{cc} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 5 \\ 5 & 6 \\ \bar{6} & \bar{4} \end{array}, \psi(C'_3) = \begin{array}{cc} 1 & 1 \\ 2 & 3 \\ 4 & 5 \\ \bar{6} & \bar{6} \\ \bar{5} & \bar{4} \\ \bar{4} & \bar{2} \end{array}, \psi(C'_2) = \begin{array}{cc} 3 & 4 \\ 5 & 5 \\ 6 & \bar{5} \\ \bar{6} & \bar{2} \\ \bar{4} & \bar{1} \\ \bar{2} & \\ \bar{1} & \end{array}, \psi(C'_1) = \begin{array}{cc} 1 & \bar{5} \\ 4 & \bar{2} \\ 5 & \\ 6 & \\ \bar{6} & \\ \bar{5} & \\ \bar{4} & \\ \bar{3} & \\ \bar{2} & \\ \bar{1} & \end{array}$$

and applying ψ^{-1} to each column results in:

$$C'_4C'_3C'_2C'_1 = \begin{array}{cccc} 1 & 1 & 4 & \bar{3} \\ 2 & 3 & 5 & \bar{2} \\ 3 & 5 & \bar{4} & \\ 5 & \bar{6} & \bar{2} & \\ 6 & \bar{5} & \bar{1} & \\ \bar{6} & \bar{3} & & \end{array} = \xi_{[1,5]}(T).$$

$$\mathsf{wt}_{[1,5]}(\xi^C_{[1,5]}(T)) = \mathsf{reverse}(\mathsf{wt}(T)) = (1, -1, 0, 0, 2, -1).$$

This solution has been verified in [SageMath].

10. The type C_n Berenstein–Kirillov group

10.1. The type A Berenstein–Kirillov group. The type *A Berenstein–Kirillov group $\mathcal{BK}$* (or *Gelfand–Tsetlin group*) [BerKir95] is the free group generated by the Bender–Knuth involutions [BeKn72] t_i, $i > 0$, modulo the relations they satisfy on semi-standard Young tableaux of any (straight) shape.

Definition 5. *The Bender–Knuth involution t_i, $i \geq 1$, is an operation that acts on a semi-standard tableau T of any shape (skew or straight) as follows:*

- *pairs $(i, i+1)$ within each column of T are considered fixed, and other occurrences of i's or $i+1$'s are considered free*
- *if a row within T has k free i's followed by l free $i+1$'s, then we replace these letters by l free i's followed by k free $i+1$'s.*

The t_i's have many known relations in $\mathcal{BK}$ [BerKir95, CGP16]:

$$t_i^2 = 1, \qquad \text{for } i \geq 1 \text{ [BerKir95, Corollary 1.1]} \tag{60}$$

$$t_i t_j = t_j t_i, \qquad \text{for } |i-j| > 1 \text{ [BerKir95, Corollary 1.1]}, \tag{61}$$

$$(t_1 q_{[1,i]})^4 = 1, \qquad \text{for } i > 2 \text{ [BerKir95, Corollary 1.1]}, \tag{62}$$

$$(t_1 t_2)^6 = 1, \qquad \text{[BerKir95, Corollary 1.1]}, \tag{63}$$

$$(t_i q_{[j,k-1]})^2 = 1, \qquad \text{for } i+1 < j < k, \text{[CGP16]}, \tag{64}$$

where

$$q_{[1,i]} := t_1(t_2t_1)\cdots(t_it_{i-1}\cdots t_1), \qquad \text{for } i \geq 1, \tag{65}$$

$$q_{[j,k-1]} := q_{[1,k-1]}q_{[1,k-j]}q_{[1,k-1]}, \qquad \text{for } j < k. \tag{66}$$

Remark 18. *(1) It is not known whether the latter forms a complete set of relations.*

(2) [BerKir95, Section 2] *On straight-shaped semi-standard Young tableaux,*

$$q_{[1,i]} = \xi_{[1,i]},\ i \geq 1,\ q_{[j,k-1]} = \xi_{[j,k-1]},\ j < k, \tag{67}$$

and $q_{[j,j]} = q_{[1,j]}q_{[1,1]}q_{[1,j]}$ computes the crystal reflection operator $\xi_j = \xi_{[j,j]}$, where $q_{[1,1]} = \xi_{[1,1]} = t_1$, for $j \geq 1$. In particular, $q_{[1,i]} = \xi_{[1,i]} = \mathsf{evac}_{i+1}$, the evacuation restricted to the alphabet $[i+1]$, and $q_{[j,k-1]}$ computes the Schützenberger evacuation restricted to the alphabet $[j,k]$, $\xi_{[j,k-1]} = \mathsf{evac}_k\mathsf{evac}_{k-j+1}\mathsf{evac}_k$, for $j < k$.

(3) Relation (64) *implies that in particular, $(t_i\xi_j)^2 = 1$, $j > i+1$, which generalizes the relation $(t_1q_{[1,i]})^4 = 1$.*

(4) For a generic (straight or skew) shaped semi-standard Young tableau T, $\mathsf{wt}(t_i(T)) = \mathsf{wt}(\xi_i(T)) = r_i\mathsf{wt}(T)$, $r_i \in \mathfrak{S}_n$ for all $n \geq 1$. However, $t_i \neq \xi_i$, for $i > 1$; $t_1 = \xi_1$ needs only coincide on straight shaped semi-standard Young tableaux. Moreover, t_i, $1 \leq i < n$, do not need to satisfy the braid relations of $\mathfrak{S}_n$, however, they do on key tableaux, that is, straight shaped tableaux whose weight is a permutation of its shape [Fu97].

Let $\mathcal{BK}_n$ be the subgroup of $\mathcal{BK}$ generated by $t_1, \ldots, t_{n-1}$.

Proposition 10. [BerKir95, Remark 1.3] *As elements of* $\mathcal{BK}$,

$$t_1 = q_{[1,1]}, \qquad t_i = q_{[1,i-1]}q_{[1,i]}q_{[1,i-1]}q_{[1,i-2]},\ \textit{for } i \geq 2,\ \ q_0 := 1.$$

The elements $q_{[1,1]}, \ldots, q_{[1,n-1]}$ *are generators of* $\mathcal{BK}_n$.

The following result is both a consequence of the combinatorial action of the cactus group J_n via partial Schützenberger involutions $\xi_{[1,i]}$ on the straight-shape tableau crystal $\mathsf{SSYT}(\lambda, n)$, as defined by Halacheva [Ha16], and the cactus group J_n relations satisfied by the generators $q_{[i,j]} = \xi_{[i,j]}$ of $\mathcal{BK}_n$ when acting on $\mathsf{SSYT}(\lambda, n)$, as studied by Chmutov, Glick and Pylyavskyy via the growth diagram approach [CGP16].

Theorem 7. *The following are group epimorphisms from* J_n *to* $\mathcal{BK}_n$*:*

(1) $s_{[i,j]} \mapsto q_{[i,j]}$ [CGP16, Theorem 1.4],
(2) $s_{[1,j]} \mapsto q_{[1,j]}$ [BerKir95, Remark 1.3], [Ha16, Section 10.2], [Ha20, Remark 3.9].

The group $\mathcal{BK}_n$ *is isomorphic to a quotient of* J_n*. The generators* $q_{[1,1]}, \ldots, q_{[1,n-1]}$ *of* $\mathcal{BK}_n$ *(and therefore* $q_{[i,j]}$*) satisfy the relations of* J_n.

Remark 19. *It follows from* [CGP16] *that* (63) *is the only known relation which does not follow from the cactus group* J_n *relations. It is in fact equivalent to the braid relations satisfied by the crystal reflection operators* $\xi_i = \xi_{[1,i]}t_1\xi_{[1,i]}$, $1 \leq i < n$, *on a* $U_q(\mathfrak{sl}(n, \mathbb{C}))$ *crystal* [BerKir95, Proposition 1.4], [Ro21].

Remark 20. *We may define the two dual sets of generators*

$$\tilde{t}_{n-i} := q_{[1,n-1]}t_i q_{[1,n-1]},\ 1 \leq i < n, \tag{68}$$

called dual Bender–Knuth involutions, and

$$\tilde{q}_{[1,i]} := q_{[1,n-1]}q_{[1,i]}q_{[1,n-1]} = q_{[n-i,n-1]},\ 1 \leq i < n,$$

for $\mathcal{BK}_n$*. Indeed, from Proposition 10 and Theorem 7, one has*

$$\tilde{t}_{n-1} = q_{[n-1,n-1]}, \qquad \tilde{t}_{n-i} = q_{[n-i+1,n-1]}q_{[n-i,n-1]}q_{[n-i+1,n-1]}q_{[n-i+2,n-1]},$$

for $2 \leq i < n$, $q_{[n,n-1]} := 1$, *and* $\mathsf{wt}(\tilde{t}_{n-i}(T)) = r_{n-i}.\mathsf{wt}(T)$ *for* $T \in \mathsf{SSYT}(\lambda, n)$ *and* $r_i \in \mathfrak{S}_n$, $i < n$.

The dual generators satisfy a list of relations similar to (60) (61), (62), (63), (64)*:*

$$\tilde{t}_{n-i}^2 = 1,\ \textit{for } i \geq 1 \tag{69}$$

$$\tilde{t}_{n-i}\tilde{t}_{n-j} = \tilde{t}_{n-j}\tilde{t}_{n-i},\ \textit{for }\ |i-j| > 1, \tag{70}$$

$$(\tilde{t}_{n-1}\tilde{t}_{n-2})^6 = 1, \tag{71}$$

$$(\tilde{t}_{n-i}\tilde{q}_{[j,k-1]})^2 = (\tilde{t}_{n-i}q_{[n-k+1,n-j]})^2 = 1,\ \textit{for}\quad n-k < n-j < n-i-1, \tag{72}$$

where

$$\tilde{q}_{[1,i]} = \tilde{t}_{n-1}(\tilde{t}_{n-2}\tilde{t}_{n-1}) \cdots (\tilde{t}_{n-i}\tilde{t}_{n-i+1} \cdots \tilde{t}_{n-1}),\ \textit{for } i \geq 1, \tag{73}$$

$$\tilde{q}_{[j,k-1]} := q_{[n-k+1,n-j]} = q_{[n-k+1,n-1]}q_{[n-k+j,n-1]}q_{[n-k+1,n-1]},\ \ \textit{for } j < k. \tag{74}$$

Remark 21. *We note some features of the operators* (68) *when acting on straight shaped semi-standard tableaux. Set* $\mathsf{evac} := \mathsf{evac}_n$. *Let* $1 \leq i < n$, *and* $T = (A, B) \in \mathsf{SSYT}(\lambda, n)$ *where* A *of straight shape is the restriction of* T *to the alphabet* $[1, n-i-1]$ *and* B, *an extension of* A, *is the restriction of* T *to the alphabet* $[n-i, n]$. *One has* $\mathsf{evac}(A, B) = (\mathsf{evac}\,\mathsf{rect}(B), X)$ *with* X *such that* $\mathsf{rect}(X) = \mathsf{evac}(A)$. *Therefore,*

$$\begin{aligned}\tilde{t}_{n-i}(T) = \tilde{t}_{n-i}(A, B) &= \mathsf{evac}\, t_i\, \mathsf{evac}(A, B) \text{ by } (68)\\ &= \mathsf{evac}\, t_i(\mathsf{evac}\,\mathsf{rect}(B), X), \text{ such that } \mathsf{rect}(X) = \mathsf{evac}(A)\\ &= \mathsf{evac}(t_i\, \mathsf{evac}\,\mathsf{rect}(B), X)\\ &= (\mathsf{evac}\,\mathsf{rect}(X), Z)\\ &= (A, Z) \text{ such that } \mathsf{rect}(Z) = \mathsf{evac}\, t_i\, \mathsf{evac}\,\mathsf{rect}(B). \qquad (75)\end{aligned}$$

10.2. **The type C_n Berenstein–Kirillov group and virtualization.** Symplectic Bender–Knuth involutions $t_i^{C_n}$ are not known for KN tableaux. Motivated by the fact that for $n \geq 1$, $q_{[1,1]}, \ldots, q_{[1,n-1]}$ are generators for the Berenstein–Kirillov group $\mathcal{BK}_n$ in type A, and that on straight shaped semi-standard tableaux, they coincide with the action of the partial Schützenberger-Lusztig involutions $\xi_{[1,i]}$, $1 \leq i < n$, we use the action of the partial Schützenberger-Lusztig involutions $\xi_{[1,i]}^{C_n}$, $1 \leq i \leq n-1$, and $\xi_{[i,n]}^{C_n}$ $1 \leq i \leq n$, on KN tableaux of any straight shape on the alphabet $\mathcal{C}_n$ to define the type C_n *Berenstein–Kirillov group*, $\mathcal{BK}^{C_n}$.

Definition 6. *Given* $n \geq 1$, *the* symplectic Berenstein–Kirillov group $\mathcal{BK}^{C_n}$ *is the free group generated by the* $2n-1$ *partial Schützenberger-Lusztig involutions*

$$q_{[1,i]}^{C_n} := \xi_{[1,i]}^{C_n},\ 1 \leq i < n,$$

and

$$q_{[i,n]}^{C_n} := \xi_{[i,n]}^{C_n},\ 1 \leq i \leq n,$$

on straight shaped KN tableaux on the alphabet $\mathcal{C}_n$ *modulo the relations they satisfy on those tableaux. We also define* $q_{[1,0]}^{C_n} = q_{[0,n]}^{C_n} = q_{[n+1,n]}^{C_n} := 1$ *and* $q_{[j,k-1]}^{C_n} := q_{[1,k-1]}^{C_n} q_{[1,k-j]}^{C_n} q_{[1,k-1]}^{C_n}$, $1 \leq j < k \leq n$.

Thanks to Theorem 1, (37) and (38), one has that $\mathcal{BK}^{C_n}$ is a quotient of $J_{\mathfrak{sp}(2n,\mathbb{C})}$. The generators of $\mathcal{BK}^{C_n}$ satisfy the cactus group $J_{\mathfrak{sp}(2n,\mathbb{C})}$ relations.

Theorem 8. *The following is a group epimorphism from* $J_{\mathfrak{sp}(2n,\mathbb{C})}$ *to* $\mathcal{BK}^{C_n}$*:*

$$s_{[1,j]} \mapsto q_{[1,j]}^{C_n},\ 1 \leq j < n, \qquad s_{[j,n]} \mapsto q_{[j,n]}^{C_n},\ 1 \leq j \leq n.$$

$\mathcal{BK}^{C_n}$ *is isomorphic to a quotient of* $J_{\mathfrak{sp}(2n,\mathbb{C})}$.

We next define symplectic Bender–Knuth involutions $t_i^{C_n}$, $1 \leq i \leq 2n-1$, on straight shaped KN tableaux that in turn generate $\mathcal{BK}^{C_n}$.

Definition 7. *The* $2n-1$ *symplectic Bender–Knuth involutions* $t_i^{C_n}$ *on KN tableaux of straight shape on the alphabet* $\mathcal{C}_n$ *are defined by*

$$t_1^{C_n} := q_{[1,1]}^{C_n},\ t_i^{C_n} := q_{[1,i-1]}^{C_n} q_{[1,i]}^{C_n} q_{[1,i-1]}^{C_n} q_{[1,i-2]}^{C_n}, \qquad 2 \leq i \leq n-1, \qquad (76)$$

$$t_n^{C_n} := q_{[n,n]}^{C_n} = \xi_n^{C_n},\ t_{n-1+i}^{C_n} := q_{[n-i+1,n]}^{C_n} q_{[n-i+2,n]}^{C_n}, \qquad 2 \le i \le n. \tag{77}$$

Thanks to the $J_{\mathfrak{sp}(2n,\mathbb{C})}$ relations satisfied by the generators of $\mathcal{BK}^{C_n}$,

$$q_{[j,j]}^{C_n} = q_{[1,j]}^{C_n} q_{[1,1]}^{C_n} q_{[1,j]}^{C_n} = q_{[1,j]}^{C_n} t_1^{C_n} q_{[1,j]}^{C_n}$$

(Definition 6 with $j = k-1$) computes the symplectic crystal reflection operator $\xi_j^{C_n}$, for $1 \le j \le n$, on KN tableaux (see Proposition 6, (3)).

Remark 22. *The symplectic Bender–Knuth involutions $t_i^{C_n}$, $1 \le i \le n$, act on the weights of the elements in the crystal $\mathsf{KN}(\lambda, n)$, $\mathsf{wt}(t_i^{C_n}(T)) = \mathsf{wt}(\xi_i^{C_n}(T)) = r_i.\mathsf{wt}(T)$, $r_i \in B_n$, $1 \le i \le n$, inducing an action of the Weyl group B_n on these weights, although, as we shall see, in Subsection 10.3, they do not define an action of the hyperoctahedral group $B_n =< r_1, \dots, r_n >$ on the set $\mathsf{KN}(\lambda, n)$. Let $T \in \mathsf{KN}(\lambda, n)$ and $\mathsf{wt}(T) = (v_1, \dots, v_n) \in \mathbb{Z}^n$, then*

$$\begin{aligned}
\mathsf{wt}(t_i^{C_n}(T)) &= r_i.\mathsf{wt}(T),\ 1 \le i < n,\\
\mathsf{wt}(t_n^{C_n}(T)) &= (v_1, \dots, -v_n) = r_n.\mathsf{wt}(T)\\
\mathsf{wt}(t_{2n-i}^{C_n}(T)) &= (v_1, \dots, -v_i, \dots, v_n) = r_{n-1} \cdots r_{n-i} r_n r_{n-i} \cdots r_{n-1}(v_1, \dots, v_n)\\
&= t_{n-1} \cdots t_{n-i} t_n t_{n-i} \cdots t_{n-1}(v_1, \dots, v_n), \quad 1 \le i < n.
\end{aligned}$$

Proposition 11. *The symplectic Bender–Knuth involutions $t_i^{C_n}$, $1 \le i \le 2n-1$, generate $\mathcal{BK}^{C_n}$. In particular,*

(1) $q_{[1,i]}^{C_n} = p_1^{C_n} p_2^{C_n} \cdots p_i^{C_n}$, $1 \le i < n$, and
(2) $q_{[i,n]}^{C_n} = t_{2n-i}^{C_n} \cdots t_n^{C_n}$, $1 \le i \le n$,

where $p_i^{C_n} := t_i^{C_n} \cdots t_2^{C_n} t_1^{C_n}$ is the symplectic promotion, $1 \le i \le 2n-1$.

Proof. (1) We show by induction on i that $q_{[1,i]}^{C_n} = q_{[1,i-1]}^{C_n} p_i^{C_n}$. Note that $q_{[1,1]}^{C_n} = p_1^{C_n} = t_1^{C_n}$. Furthermore, for $i > 1$, $q_{[1,i]}^{C_n} = q_{[1,i-1]}^{C_n} t_i^{C_n} q_{[1,i-2]}^{C} q_{[1,i-1]}^{C_n}$. Then, assuming that for some fixed positive integer k, $q_{[1,j]}^{C_n} = q_{[1,j]}^{C_n} p_i^{C_n}$ for all $j \in [1, k-1]$, our inductive hypothesis implies

$$q_{[1,k]}^{C_n} = q_{[1,k-1]}^{C_n} t_k^{C_n} q_{[1,k-2]}^{C} q_{[1,k-1]}^{C_n} = q_{[1,k-1]}^{C_n} t_k^{C_n} q_{[1,k-2]}^{C} q_{[1,k-2]}^{C_n} p_{k-1}^{C_n}$$

$$= q_{[1,k-1]}^{C_n} t_k^{C_n} p_{k-1}^{C_n} = q_{[1,k-1]}^{C_n} p_k^{C_n}.$$

(2) We proceed by induction on i in the statement $q_{[n-i-1,n]}^{C_n} = t_{n+i}^{C_n} t_{n+i-1}^{C_n} \cdots t_n^{C_n}$. As a base case, when $i = 1$, we have $t_n^{C_n} = q_{[n,n]}^{C_n}$. As an inductive step, we assume the statement is true for all $j \in [1, k]$ for some fixed positive integer k, so

$$\begin{aligned} t^{C_n}_{n+k+1-1} &= t^{C_n}_{n+k} \\ &= q^{C_n}_{[n-k,n]} q^{C_n}_{[n-(k-1),n]} \\ \Rightarrow t^{C_n}_{n+k} q^{C_n}_{[n-(k-1),n]} &= q^{C_n}_{[n-k,n]} \\ \Rightarrow q^{C_n}_{[n-k,n]} &= t^{C_n}_{n+k} t^{C_n}_{n+k-1} \cdots t^{C_n}_{n}. \end{aligned}$$

□

By Theorem 7, the involutions $q^{A_{2n-1}}_{[i,j]} \in \mathcal{BK}_{2n}$, $i \le i \le j < 2n$, satisfy the cactus J_{2n} relations. Consider in $\mathcal{BK}_{2n}$ the involution $q^{A_{2n-1}}_{[1,i]}$ with its dual $\tilde{q}^{A_{2n-1}}_{[1,i]} := q^{A_{2n-1}}_{[2n-i,2n-1]}$, for $1 \le i < n$ (Remark 20), and $q^{A_{2n-1}}_{[i,2n-i]}$, $1 \le i \le n$.

Definition 8. *The* virtual symplectic Berenstein–Kirillov group $\widetilde{\mathcal{BK}}_{2n}$ *is the subgroup of* $\mathcal{BK}_{2n}$ *generated by the* $2n-1$ *involutions*

$$q^{A_{2n-1}}_{[1,i]\cup[2n-i,2n-1]} := q^{A_{2n-1}}_{[1,i]} q^{A_{2n-1}}_{[2n-i,2n-1]} = q^{A_{2n-1}}_{[2n-i,2n-1]} q^{A_{2n-1}}_{[1,i]}, \qquad 1 \le i < n, \tag{78}$$

$$q^{A_{2n-1}}_{[i,2n-i]}, \qquad 1 \le i \le n, \tag{79}$$

modulo the relations they satisfy when acting on semi-standard tableaux of any straight shape.

By Theorem 2, $q^{A_{2n-1}}_{[1,i]\cup[2n-i,2n-1]}$ coincides with $\xi^{A_{2n-1}}_{[1,i]\cup[2n-i,2n-1]}$ on semi-standard tableaux of any straight shape, $1 \le i < n$. (In particular, in $E(\mathsf{KN}(\lambda, n))$, for any partition λ with at most n parts.)

Remark 23. *The action of* $q^{A_{2n-1}}_{[1,i]\cup[2n-i,2n-1]}$ *on a semi-standard tableau is non trivial only in the entries* $\le 2n$. *Therefore it is enough to consider the sets* $\mathsf{SSYT}(\lambda, 2n)$ *with* λ *any partition with at most* $2n$ *parts.*

Proposition 12. *For* $1 \le i < n$, *consider the Bender–Knuth involution* $t^{A_{2n-i}}_i$ *with its dual* $\tilde{t}^{A_{2n-i}}_{2n-i}$ *in* $\mathcal{BK}_{2n}$. *The group* $\widetilde{\mathcal{BK}}_{2n}$ *also has the* $2n-1$ *generators*

$$t^{A_{2n-1}}_{[i]\cup[2n-i]} := t^{A_{2n-1}}_{i} \tilde{t}^{A_{2n-1}}_{2n-i} = \tilde{t}^{A_{2n-1}}_{2n-i} t^{A_{2n-1}}_{i}, 1 \le i < n, \tag{80}$$

$$t^{A_{2n-1}}_{n-i+1,n+i} := q^{A_{2n-1}}_{[n-i+1,n+i-1]} q^{A_{2n-1}}_{[n-i+2,n+i-2]} \tag{81}$$

$$= q^{A_{2n-1}}_{[n-i+2,n+i-2]} q^{A_{2n-1}}_{[n-i+1,n+i-1]}, 1 \le i \le n. \tag{82}$$

where $q^{A_{2n-1}}_{[n+1,n-1]} := 1$. *We call them the virtual symplectic Bender–Knuth involutions.*

Proof. The group $\mathcal{BK}_{2n}$ satisfies the J_{2n} relations and $\widetilde{\mathcal{BK}}_{2n} \subseteq \mathcal{BK}_{2n}$. Hence $q^{A_{2n-1}}_{[1,i]} q^{A_{2n-1}}_{[2n-i,2n-1]} = q^{A_{2n-1}}_{[2n-i,2n-1]} q^{A_{2n-1}}_{[1,i]}$, $1 \le i < n$, and by (23),

$$q^{A_{2n-1}}_{[n-i+1,n+i-1]} q^{A_{2n-1}}_{[n-i+2,n+i-2]} = q^{A_{2n-1}}_{[n-i+2,n+i-2]} q^{A_{2n-1}}_{[n-i+1,n+i-1]}, 1 \le i \le n.$$

In addition, from Remark 20 in $\mathcal{BK}_{2n}$,

$$\begin{aligned} t_i^{A_{2n-1}}\tilde{t}_{2n-i}^{A_{2n-1}} &= q_{[1,i-1]}^{A_{2n-1}}q_{[1,i]}^{A_{2n-1}}q_{[1,i-1]}^{A_{2n-1}}q_{[1,i-2]}^{A_{2n-1}}q_{[2n-i+1,2n-1]}^{A_{2n-1}}q_{[2n-i,2n-1]}^{A_{2n-1}}q_{[2n-i+1,2n-1]}^{A_{2n-1}}q_{[2n-i+2,2n-1]}^{A_{2n-1}} \\ &= q_{[2n-i+1,2n-1]}^{A_{2n-1}}q_{[2n-i,2n-1]}^{A_{2n-1}}q_{[2n-i+1,2n-1]}^{A_{2n-1}}q_{[1,i-1]}^{A_{2n-1}}q_{[1,i]}^{A_{2n-1}}q_{[1,i-1]}q_{[1,i-2]}^{A_{2n-1}}q_{[2n-i+2,2n-1]}^{A_{2n-1}} \\ &= \tilde{t}_{2n-i}^{A_{2n-1}}t_i^{A_{2n-1}},\ 1\le i<n. \end{aligned} \tag{83}$$

Again by Remark 20 in $\mathcal{BK}_{2n}$ and (83), for $1\le i<n$,

$$\begin{aligned} q_{[1,i]\cup[2n-i,2n-1]}^{A_{2n-1}} &= q_{[1,i]}^{A_{2n-1}}q_{[2n-i,2n-1]}^{A_{2n-1}} \\ &= p_1^{A_{2n-1}}\cdots p_i^{A_{2n-1}}q_{[1,2n-1]}^{A_{2n-1}}p_1^{A_{2n-1}}\cdots p_i^{A_{2n-1}}q_{[1,2n-1]}^{A_{2n-1}} \\ &= p_1^{A_{2n-1}}\cdots p_i^{A_{2n-1}}\tilde{p}_{2n-1}^{A_{2n-1}}\cdots\tilde{p}_{2n-i}^{A_{2n-1}} \\ &= t_1^{A_{2n-1}}\tilde{t}_{2n-1}^{A_{2n-1}}(t_2^{A_{2n-1}}\tilde{t}_{2n-2}^{A_{2n-1}}t_1^{A_{2n-1}}\tilde{t}_{2n-1}^{A_{2n-1}})\cdots(t_i^{A_{2n-1}}\tilde{t}_{2n-i}^{A_{2n-1}}\cdots \\ &\qquad\cdots t_2^{A_{2n-1}}\tilde{t}_{2n-2}^{A_{2n-1}}t_1^{A_{2n-1}}\tilde{t}_{2n-1}^{A_{2n-1}}) \\ &= t_{[1]\cup[2n-1]}^{A_{2n-1}}(t_{[2]\cup[2n-2]}^{A_{2n-1}}t_{[1]\cup[2n-1]}^{A_{2n-1}})\cdots(t_{[i]\cup[2n-i]}^{A_{2n-1}}\cdots \\ &\qquad\cdots t_{[2]\cup[2n-2]}^{A_{2n-1}}t_{[1]\cup[2n-1]}^{A_{2n-1}}), \end{aligned}$$

where $q_{[1,i]}^{A_{2n-1}}=p_1^{A_{2n-1}}\cdots p_i^{A_{2n-1}}$ with $p_i^{A_{2n-1}}:=t_i^{A_{2n-1}}\cdots t_2^{A_{2n-1}}t_1^{A_{2n-1}}$, and

$$\begin{aligned} \tilde{p}_{2n-i}^{A_{2n-1}} &:= q_{[1,2n-1]}^{A_{2n-1}}p_i^{A_{2n-1}}q_{[1,2n-1]}^{A_{2n-1}} \\ &= \tilde{t}_{2n-i}^{A_{2n-1}}\cdots\tilde{t}_{2n-2}^{A_{2n-1}}\tilde{t}_{2n-1}^{A_{2n-1}}, 1\le i<n. \end{aligned}$$

On the other hand, for $1\le i\le n$,

$$\begin{aligned} q_{[n-i+1,n+i-1]}^{A_{2n-1}} &= q_{[n,n]}^{A_{2n-1}}(q_{[n,n]}^{A_{2n-1}}q_{[n-1,n+1]}^{A_{2n-1}})(q_{[n-1,n+1]}^{A_{2n-1}}q_{[n-2,n+2]}^{A_{2n-1}})\cdots \\ &\qquad\cdots(q_{[n-(i-2),n+i-2]}^{A_{2n-1}}q_{[n-i+1,n+i-1]}^{A_{2n-1}}) \\ &= t_{n,n+1}^{A_{2n-1}}t_{n-1,n+2}^{A_{2n-1}}t_{n-2,n+3}^{A_{2n-1}}t_{n-i+2,n+i-1}^{A_{2n-1}}t_{n-i+1,n+i}^{A_{2n-1}}. \end{aligned}$$

□

Remark 24. *If T is an A_{2n-1} semi-standard tableau, $\mathsf{wt}(t_{[i]\cup[2n-i]}^{A_{2n-1}}(T))=r_ir_{2n-i}.\mathsf{wt}(T)$, where $r_i=(i\ i+1)$ and $r_{2n-i}=(2n-i\ \ 2n-i+1)$ are simple transpositions in $\mathfrak{S}_{2n}$, for $1\le i<n$, and $\mathsf{wt}(t_{n-i+1,n+i}(T))=(n-i+1\ \ n+i)\mathsf{wt}(T)$, where $(n-i+1\ \ n+i)$ is the transposition of $\mathfrak{S}_{2n}$ that swaps $n-i+1$ and $n+i$, for $1\le i\le n$. The virtual symplectic Bender–Knuth involutions $t_{[i]\cup[2n-i]}^{A_{2n-1}}$, $1\le i<n$, and $t_{n,n+1}^{A_{2n-1}}$ act on the weights in $\mathbb{Z}^{2n}$ of the elements in the crystal $\mathsf{SSYT}(\lambda,2n)$, inducing an action of the Weyl group B_n, realized as $<(i\ i+1)(2n-i\ \ 2n-i+1),\ (n,n+1),\ 1\le i<n>$, on these weights.*

Thanks to Theorem 3, we have that $\widetilde{\mathcal{BK}}_{2n}$ is a quotient of the virtual symplectic cactus $\widetilde{J}_{2n}$. The generators (78) and (79) of the group $\widetilde{\mathcal{BK}}_{2n}$ satisfy the relations of the cactus $\widetilde{J}_{2n}$, or equivalently, those of the cactus $J_{\mathfrak{sp}(2n,\mathbb{C})}$.

Theorem 9. *The following is a group epimorphism from $\widetilde{J}_{2n}$ to $\widetilde{\mathcal{BK}}_{2n}$:*

$$\tilde{s}_{[1,j]\cup[2n-j,2n-1]} \mapsto q^{A_{2n-1}}_{[1,j]\cup[2n-j,2n-1]},\ 1 \le j < n, \qquad \tilde{s}_{[j,2n-j]} \mapsto q^{A_{2n-1}}_{[j,2n-j]},\ 1 \le j \le n.$$

$\widetilde{\mathcal{BK}}_{2n}$ is isomorphic to a quotient of $\widetilde{J}_{2n}$, and via the isomorphism between $J_{\mathfrak{sp}(2n,\mathbb{C})}$ and $\widetilde{J}_{2n}$ that sends $s_{[1,j]} \mapsto \tilde{s}_{[1,j]\cup[2n-j,2n-1]}$, $1 \le j < n$, and $\tilde{s}_{[j,n]} \mapsto \tilde{s}_{[j,2n-j]}$, $1 \le j \le n$, is also isomorphic to a quotient of $J_{\mathfrak{sp}(2n,\mathbb{C})}$.

Because the action of $\widetilde{J}_{2n}$ on the set $\mathsf{SSYT}(\lambda^A, n, \bar{n})$ preserves the subset $E(\mathsf{KN}(\lambda, n))$, see Remark 17, we now relate the virtual symplectic and symplectic Bender–Knuth involutions by embedding the crystal $\mathsf{KN}(\lambda, n)$ into the crystal $\mathsf{SSYT}(\lambda^A, n, \bar{n})$.

Theorem 10. *The symplectic Bender–Knuth involutions $t_i^{C_n}$, $1 \le i \le 2n-1$, in $\mathcal{BK}^{C_n}$ can be realized by the virtual symplectic Bender–Knuth involutions $t_i^{A_{2n-i}}\tilde{t}_{2n-i}^{A_{2n-i}}$, $1 \le i < n$, and $t^{A_{2n-1}}_{n-i+1,n+i}$, $1 \le i \le n$, in $\widetilde{\mathcal{BK}}_{2n}$, and* vice-versa*,*

$$\begin{aligned} t_i^{C_n} &= E^{-1} t_i^{A_{2n-1}} \tilde{t}_{\overline{i+1}}^{A_{2n-1}} E = E^{-1} t_i^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-1}} E, && 1 \le i < n,\\ t^{C_n}_{n+i-1} &= E^{-1} t^{A_{2n-1}}_{n-i+1,n+i} E, && 1 \le i \le n. \end{aligned}$$

Proof. By Theorem 5, for $1 \le i < n$,

$$t_i^{C_n} = q^{C_n}_{[1,i-1]} q^{C_n}_{[1,i]} q^{C}_{[1,i-1]} q^{C_n}_{[1,i-2]} =$$

$$E^{-1}(\xi^{A_{2n-1}}_{[1,i-1]} \xi^{A_{2n-1}}_{[\bar{i},\bar{2}]}) E E^{-1} (q^{A_{2n-1}}_{[1,i]} q^{A_{2n-1}}_{[\overline{i+1},\bar{2}]}) E E^{-1} (\xi^{A_{2n-1}}_{[1,i-1]} q^{A_{2n-1}}_{[\bar{i},\bar{2}]}) E E^{-1} (\xi^{A_{2n-1}}_{[1,i-2]} \xi^{A_{2n-1}}_{[\overline{i-1},\bar{2}]}) E$$

$$= E^{-1}(\xi^{A_{2n-1}}_{[1,i-1]} \xi^{A_{2n-1}}_{[1,i]} \xi^{A_{2n-1}}_{[1,i-1]} \xi^{A_{2n-1}}_{[1,i-2]} \xi^{A_{2n-1}}_{[\bar{i},\bar{2}]} \xi^{A_{2n-1}}_{[\overline{i+1},\bar{2}]} \xi^{A_{2n-1}}_{[\bar{i},\bar{2}]} \xi^{A_{2n-1}}_{[\overline{i-1},\bar{2}]}) E$$

$$= E^{-1}(t_i^{A_{2n-1}} \tilde{t}^{A_{2n-1}}_{\overline{i+1}}) E.$$

By Theorem 6, for $2 \le i \le n$, $t^{C_n}_{n+i-1} = q^{C_n}_{[n-i+1,n]} q^{C_n}_{[n-i+2,n]}$, and $t_n^{C_n} = q^{C_n}_{[n,n]} = \xi^{C_n}_n$. ☐

10.3. **Symplectic Bender–Knuth involutions and the character of a KN tableau crystal.** The C_2 Weyl group is $B_2 =< r_1, r_2 : r_i^2 = 1, (r_1 r_2)^4 = 1 >$ with long element $r_2 r_1 r_2 r_1$, and the C_2 symplectic Bender–Knuth involutions are $t_1^{C_2} = \xi_1^{C_2}$, $t_2^{C_2} = \xi_2^{C_2}$, $t_3^{C_2} = \xi^{C_2}_{[1,2]} \xi_2^{C_2} = \xi_2^{C_2} \xi^{C_2}_{[1,2]}$ (see Example 10), and one has

$$\xi^{C_2} = t_3^{C_2} t_2^{C_2} \neq t_2^{C_2} t_1^{C_2} t_2^{C_2} t_1^{C_2} = t_1^{C_2} t_2^{C_2} t_1^{C_2} t_2^{C_2}.$$

From Proposition 6, (3), B_2 indeed acts on the C_2-crystal $\mathsf{KN}(\lambda, 2)$ via $t_1^{C_2} := \xi_1^{C_2}$ and $t_2^{C_2} := \xi_2^{C_2}$. Therefore, in this case, $t_1^{C_2}$ and $t_2^{C_2}$ define an action of Weyl group B_2 on the crystal $\mathsf{KN}(\lambda, 2)$. However, the action of the Schützenberger–Lusztig involution $\xi^{C_2}_{[1,2]} = \xi^{C_2}$ on $\mathsf{KN}(\lambda, n)$ does not coincide in general with the action of the long element of the Weyl group B_2, that is, $\xi^{C_2} \neq t_1^{C_2} t_2^{C_2} t_1^{C_2} t_2^{C_2} = t_2^{C_2} t_1^{C_2} t_2^{C_2} t_1^{C_2}$.

For instance, considering $T = \begin{array}{|c|c|}\hline 2 & \bar{1} \\ \hline \bar{2} \\ \cline{1-1}\end{array}$ in Example 4, despite

$$\mathsf{wt}(t_2^{C_2} t_1^{C_2} t_2^{C_2} t_1^{C_2}(T)) = (1,0) = w_0^C(-1,0) = \mathsf{wt}(\xi^{C_2}_{[1,2]}(T)),$$

the coincidence of the actions of the Schützenberger–Lusztig involution and of the long element of the Weyl group can only be ensured when T is the highest weight or lowest weight element in the crystal $\mathsf{KN}(\lambda, n)$, as Proposition 7 ensures. But T is not in that case, and in fact

$$\begin{aligned} r_2 r_1 r_2 r_1 . T &= t_2^{C_2} t_1^{C_2} t_2^{C_2} t_1^{C_2} \left(\begin{array}{|c|c|}\hline 2 & \bar{1} \\ \hline \bar{2} \\ \cline{1-1} \end{array} \right) \\ &= t_1^{C_2} t_2^{C_2} t_1^{C_2} t_2^{C_2} \left(\begin{array}{|c|c|}\hline 2 & \bar{1} \\ \hline \bar{2} \\ \cline{1-1} \end{array} \right) = \begin{array}{|c|c|}\hline 1 & \bar{2} \\ \hline 2 \\ \cline{1-1} \end{array} \neq \xi^{C_2}_{[1,2]}(T) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \bar{2} \\ \cline{1-1} \end{array} . \end{aligned}$$

In general, for $n \geq 3$, the symplectic Bender–Knuth involutions, $t_1^{C_n}, \dots, t_n^{C_n}$, do not define an action of the Weyl group $B_n =< r_1, \dots, r_{n-1}, r_n >$ on the set $\mathsf{KN}(\lambda, n)$. On the other hand, contrary to the A_{n-1} case, the Schützenberger–Lusztig involution ξ^{C_n} is not given by the long word of B_n in the first n symplectic Bender–Knuth involutions. One has in fact $\xi^{C_n} = t_{2n-1}^{C_n} \cdots t_n^{C_n}$, as stated in Proposition 11.

To show the former claim, recall that the first $n-1$ generators of the Weyl group B_n satisfy the braid relations (6) of $\mathfrak{S}_n$, but we claim that, in general, $t_i^{C_n} t_{i+1}^{C_n} t_i^{C_n} \neq t_{i+1}^{C_n} t_i^{C_n} t_{i+1}^{C_n}$, i.e., $(t_i^{C_n} t_{i+1}^{C_n})^3 \neq 1$ for $1 \leq i < n$. To show this inequality, note that by Theorem 10, it is enough to consider the virtual symplectic Bender–Knuth involutions and the corresponding virtual inequality

$$t_i^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-1}} t_{i+1}^{A_{2n-1}} \tilde{t}_{2n-(i+1)}^{A_{2n-1}} t_i^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-1}} \neq t_{i+1}^{A_{2n-1}} \tilde{t}_{2n-(i+1)}^{A_{2n-1}} t_i^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-1}} t_{i+1}^{A_{2n-1}} \tilde{t}_{2n-(i+1)}^{A_{2n-1}} . \tag{84}$$

From Proposition 10 and Remark 20, $t_i^{A_{2n-1}} \tilde{t}_{2n-(i+1)}^{A_{2n-1}} = \tilde{t}_{2n-(i+1)}^{A_{2n-1}} t_i^{A_{2n-1}}$, for $1 \leq i < n$. If we had equality in (84), then

$$\begin{gathered} \tilde{t}_{2n-i}^{A_{2n-1}} \tilde{t}_{2n-(i+1)}^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-1}} t_i^{A_{2n-1}} t_{i+1}^{A_{2n-1}} t_i^{A_{2n-1}} = \tilde{t}_{2n-(i+1)}^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-1}} \tilde{t}_{2n-(i+1)}^{A_{2n-1}} t_{i+1}^{A_{2n-1}} t_i^{A_{2n-1}} t_{i+1}^{A_{2n-1}} \\ \Leftrightarrow (\tilde{t}_{2n-(i+1)}^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-1}})^3 = (t_i^{A_{2n-1}} t_{i+1}^{A_{2n-1}})^3 . \end{gathered}$$

Applying this identity to the A_{11} tableau $E(T) = (P^+, P^-)$ in the virtualization Example 9.6 would imply that

$$\begin{gathered} (\tilde{t}_{\bar{6}}^{A_{11}} \tilde{t}_{\bar{5}}^{A_{11}})^3 (E(T)) = (t_4^{A_{11}} t_5^{A_{11}})^3 (E(T)) \Leftrightarrow \\ (P^+, (\tilde{t}_{\bar{6}}^{A_{11}} \tilde{t}_{\bar{5}}^{A_{11}})^3 (P^-)) = ((t_4^{A_{11}} t_5^{A_{11}})^3 (P^+), P^-), \end{gathered} \tag{85}$$

but this is impossible, as $(t_4^{A_{11}} t_5^{A_{11}})^3 (P^+) \neq P^+$. Note that the LHS of (85) follows from $\tilde{t}_{2n-i}(P^+, P^-) = \mathsf{evac}\, t_i \mathsf{evac}(P^+, P^-)$ and Remark 21.

Despite the fact that the symplectic Bender Knuth involutions $t_i^{C_n}$, $1 \leq i \leq n$, do not define an action of the Weyl group B_n on the set $\mathsf{KN}(\lambda, n)$, similarly to the type A_{n-1} case, they can be used to show that the character of the crystal $\mathsf{KN}(\lambda, n)$ is a symmetric Laurent polynomial with respect to the action of the Weyl group B_n. Let $\mathcal{E} := \mathbb{Z}[x_1^{\pm}, \dots, x_n^{\pm}]$ be the ring of Laurent polynomials on the variables $x_1, \dots, x_n$ over $\mathbb{Z}$, and let $\mathcal{E}^{B_n} = \{f \in \mathcal{E} : r_i . f = f,\ r_i \in B_n,\ 1 \leq i \leq n\}$, where $r_i . x^{\alpha} := x^{r_i . \alpha}$, for $x^\alpha := x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, $\alpha\in\mathbb{Z}^n$ and $r_i\in B_n$, be the subring of symmetric Laurent polynomials.

The character of $\mathsf{KN}(\lambda,n)$ is the symplectic Schur function $sp_\lambda(x)$ in the sequence of variables $x=(x_1,\dots,x_n)$. Thanks to Remark 22, $\mathsf{wt}(t_i^{C_n}.b)=r_i.\mathsf{wt}(b)$ for any $b\in\mathsf{KN}(\lambda,n)$ and $1\le i\le n$. Therefore, since $t_i^{C_n}$, $1\le i\le n$, is an involution on the set $\mathsf{KN}(\lambda,n)$, $sp_\lambda(x)$ is a symmetric Laurent polynomial

$$\begin{aligned} sp_\lambda(x) &= \sum_{b\in\mathsf{KN}(\lambda,n)} x^{\mathsf{wt}(b)} = \sum_{b\in\mathsf{KN}(\lambda,n)} x^{\mathsf{wt}(t_i^{C_n}b)},\ 1\le i\le n,\\ &= \sum_{b\in\mathsf{KN}(\lambda,n)} x^{r_i.\mathsf{wt}(b)} = sp_\lambda(r_i.x),\ 1\le i\le n. \end{aligned}$$

10.4. **Relations for the symplectic Berenstein–Kirilov group.** Thanks to Theorem 8 and Theorem 9, we now provide the following relations for $\mathcal{BK}^{C_n}$ equivalently $\widetilde{\mathcal{BK}}_{2n}$. The relations (9) and (10) below are the only ones known for $\mathcal{BK}^{C_n}$, equivalently, $\widetilde{\mathcal{BK}}_{2n}$ which do not follow from the cactus group $J_{\mathfrak{sp}(2n,\mathbb{C})}$ relations, equivalently, the virtual cactus group $\widetilde{J}_{2n}$ relations (see also Remark 25).

Proposition 13. *The symplectic Bender–Knuth involutions $t_i^{C_n}=1$, $i=1,\dots,2n-1$, satisfy the following relations:*

(1) $(t_i^{C_n})^2=1$, $i=1,\dots,2n-1$.
(2) $(t_{n+i-1}^{C_n}t_{n+j-1}^{C_n})^2=1$, $1\le i,j\le n$.
(3) $(t_i^{C_n}t_j^{C_n})^2=1$, $|i-j|>1$, $1\le i,j<n$.
(4) $(t_i^{C_n}t_{n+j-1}^{C_n})^2=1$, $i<n-j$.
(5) $(t_i^{C_n}q_{[j,k-1]}^{C_n})^2=1$, $i+1<j<k\le n$.
(6) $(t_i^{C_n}q_{[j,n]}^{C_n})^2=1$, $i+1<j\le n$.
(7) $(t_{n+i-1}^{C_n}q_{[j,n]}^{C_n})^2=1$, $1\le i,j\le n$.
(8) $(t_{n+i-1}^{C_n}q_{[j,k-1]}^{C_n})^2=1$, $n-i+1<j<k\le n$.
(9) $(t_1^{C_n}t_2^{C_n})^6=1$, $n\ge 3$.
(10) $(t_{n-1}^{C_n}\cdots t_2^{C_n}t_1^{C_n}t_2^{C_n}\cdots t_{n-1}^{C_n}t_n^{C_n})^4=1$.

The virtual symplectic Bender–Knuth involutions $t_i^{A_{2n-i}}\tilde{t}_{2n-i}^{A_{2n-i}}=Et_i^{C_n}E^{-1}$, $1\le i<n$, and $t_{n-i+1,n+i}^{A_{2n-1}}=Et_{n-i+1}^{C_n}E^{-1}$, $1\le i\le n$, in $\widetilde{\mathcal{BK}}_{2n}$ satisfy the same relations as those of $\mathcal{BK}^{C_n}$ by replacing $t_i^{C_n}$ by $t_i^{A_{2n-i}}\tilde{t}_{2n-i}^{A_{2n-i}}$, $1\le i<n$, and $t_{n+i-1}^{C_n}$ by $t_{n-i+1,n+i}^{A_{2n-1}}$, $1\le i\le n$.

Proof. Recall Theorems 8 and 9.

(1) $(t_1^{C_n})^2=(q_{[1,1]}^{C_n})^2=1$, $(t_n^{C_n})^2=(q_{[1,n]}^{C_n})^2=1$. For $2\le i\le n$, $(t_{n-1+i}^{C_n})^2=(q_{[n-i+1,n]}^{C_n}q_{[n-i+2,n]}^{C_n})^2=1$ is equivalent to the $J_{\mathfrak{sp}(2n,\mathbb{C})}$ relation $3C(i)$.

For $2\le i\le n-1$,

$$(t_i^{C_n})^2=q_{[1,i-1]}^{C_n}q_{[1,i]}^{C_n}q_{[1,i-1]}^{C_n}q_{[1,i-2]}^{C_n}q_{[1,i-1]}^{C_n}q_{[1,i]}^{C_n}q_{[1,i-1]}^{C_n}q_{[1,i-2]}^{C_n}=1$$

follows from the cactus $J_{\mathfrak{sp}(2n,\mathbb{C})}$ relation $3C(ii)$ and the observations that

$$q^{C_n}_{[1,i-1]}q^{C_n}_{[1,i-2]} = q^{C_n}_{[2,i-1]}q^{C_n}_{[1,i-1]},\ q^{C_n}_{[1,i-1]}q^{C_n}_{[2,i-1]} = q^{C_n}_{[1,i-2]}q^{C_n}_{[1,i-1]},$$
$$q^{C_n}_{[1,i]}q^{C_n}_{[2,i-1]} = q^{C_n}_{[2,i-1]}q^{C_n}_{[1,i]}.$$

(2) Let $i \neq j$. From 3C(i),

$$t^{C_n}_{n+i-1}t^{C_n}_{n+j-1} = q^{C_n}_{[n-i+1,n]}q^{C_n}_{[n-i+2,n]}q^{C_n}_{[n-j+1,n]}q^{C_n}_{[n-j+2,n]}$$
$$= q^{C_n}_{[n-j+1,n]}q^{C_n}_{[n-j+2,n]}q^{C_n}_{[n-i+1,n]}q^{C_n}_{[n-i+2,n]} = t^{C_n}_{n+j-1}t^{C_n}_{n+i-1}.$$

(3) Recall (60) and Remark 20. Then

$$\begin{aligned}(t^{C_n}_i t^{C_n}_j)^2 &= (E^{-1}t^{A_{2n-1}}_i \tilde{t}^{A_{2n-1}}_{2n-i} E E^{-1} t^{A_{2n-1}}_j \tilde{t}^{A_{2n-1}}_{2n-j} E)^2\\ &= E^{-1}(t^{A_{2n-1}}_i t^{A_{2n-1}}_j)^2(\tilde{t}^{A_{2n-1}}_{2n-i}\tilde{t}^{A_{2n-1}}_{2n-j})^2 E = 1,\ \text{for } |i-j|>1,\ 1\leq i,j\leq n-1.\end{aligned}$$

(4) For $i < n-j$,

$$t^{C_n}_i t^{C_n}_{n+j-1} = q^{C_n}_{[1,i-1]}q^{C_n}_{[1,i]}q^{C_n}_{[1,i-1]}q^{C_n}_{[1,i-2]}q^{C_n}_{[n-j+1,n]}q^{C_n}_{[n-j+2,n]}$$
$$= q^{C_n}_{[n-j+1,n]}q^{C_n}_{[n-j+2,n]}q^{C_n}_{[1,i-1]}q^{C_n}_{[1,i]}q^{C_n}_{[1,i-1]}q^{C_n}_{[1,i-2]}$$

due to the $J_{\mathfrak{sp}(2n,\mathbb{C})}$ relation $2C$.

(5) For $i+1<j<k\leq n$,

$$\begin{aligned}(t^{C_n}_i q^{C_n}_{[j,k-1]})^2 &= (E^{-1}t^{A_{2n-1}}_i\tilde{t}^{A_{2n-1}}_{2n-i}EE^{-1}\xi^{A_{2n-1}}_{[j,k-1]}\xi^{A_{2n-1}}_{[2n-k+1,2n-j]}E)^2\\ &= E^{-1}(t^{A_{2n-1}}_i\xi^{A_{2n-1}}_{[j,k-1]}\tilde{t}^{A_{2n-1}}_{2n-i}\xi^{A_{2n-1}}_{[2n-k+1,2n-j]})^2E,\\ &\qquad \text{for } 2n-k<2n-j<2n-i-1\\ &= E^{-1}t^{A_{2n-1}}_i\xi^{A_{2n-1}}_{[j,k-1]}\tilde{t}^{A_{2n-1}}_{2n-i}\xi^{A_{2n-1}}_{[2n-k+1,2n-j]}\\ &\qquad t^{A_{2n-1}}_i\xi^{A_{2n-1}}_{[j,k-1]}\tilde{t}^{A_{2n-1}}_{2n-i}\xi^{A_{2n-1}}_{[2n-k+1,2n-j]}E\\ &= E^{-1}t^{A_{2n-1}}_i\xi^{A_{2n-1}}_{[j,k-1]}t^{A_{2n-1}}_i\xi^{A_{2n-1}}_{[j,k-1]}\tilde{t}^{A_{2n-1}}_{2n-i}\xi^{A_{2n-1}}_{[2n-k+1,2n-j]}\\ &\qquad \tilde{t}^{A_{2n-1}}_{2n-i}\xi^{A_{2n-1}}_{[2n-k+1,2n-j]}E\\ &= E^{-1}(t^{A_{2n-1}}_i\xi^{A_{2n-1}}_{[j,k-1]})^2(\tilde{t}^{A_{2n-1}}_{2n-i}\xi^{A_{2n-1}}_{[2n-k+1,2n-j]})^2E\\ &= E^{-1}(\tilde{t}^{A_{2n-1}}_{2n-i}\xi^{A_{2n-1}}_{[2n-k+1,2n-j]})^2E = 1,\\ &\qquad \text{for } 2n-k<2n-j<2n-i-1.\end{aligned}$$

(6) For $i+1<j\leq n$,

$$\begin{aligned}t^{C_n}_i q^{C_n}_{[j,n]} &= E^{-1}t^{A_{2n-1}}_i\tilde{t}^{A_{2n-1}}_{2n-i}\xi^{A_{2n-1}}_{[j,2n-j]}E,\ 2n-i-1>2n-j\geq n\ \ (64),(72)\\ &= E^{-1}\xi^{A_{2n-1}}_{[j,2n-j]}t^{A_{2n-1}}_i\tilde{t}^{A_{2n-1}}_{2n-i}E,\ \ \text{by Theorem 6}\\ &= q^{C_n}_{[j,n]}t^{C_n}_i.\end{aligned}$$

(7) $(t^{C_n}_{n+i-1}q^{C_n}_{[j,n]})^2=1$, $1\le i,j\le n$.

$$\begin{aligned}t^{C_n}_{n+i-1}q^{C_n}_{[j,n]}&=E^{-1}\xi^{A_{2n-1}}_{[n-(i-1),n+(i-1)]}\xi^{A_{2n-1}}_{[n-(i-2),n+(i-2)]}\xi^{A_{2n-1}}_{[j,2n-j]}E\\&=E^{-1}\xi^{A_{2n-1}}_{[j,2n-j]}\xi^{A_{2n-1}}_{[n-(i-1),n+(i-1)]}\xi^{A_{2n-1}}_{[n-(i-2),n+(i-2)]}E,\text{ by Theorem 9}\\&=q^{C_n}_{[j,n]}t^{C_n}_{n+i-1}.\end{aligned}$$

(8) $(t^{C_n}_{n+i-1}q^{C_n}_{[j,k-1]})^2=1$, $n-i+1<j<k\le n$.

$$\begin{aligned}t^{C_n}_{n+i-1}q^{C_n}_{[j,k-1]}&=E^{-1}\xi^{A_{2n-1}}_{[n-(i-1),n+(i-1)]}\xi^{A_{2n-1}}_{[n-(i-2),n+(i-2)]}\xi^{A_{2n-1}}_{[j,k-1]}\xi^{A_{2n-1}}_{[2n-k+1,2n-j]}E\\&=E^{-1}\xi^{A_{2n-1}}_{[n-(i-1),n+(i-1)]}\xi^{A_{2n-1}}_{[j,k-1]}\xi^{A_{2n-1}}_{[2n-k+1,2n-j]}\xi^{A_{2n-1}}_{[n-(i-2),n+(i-2)]}E\\&\qquad\text{by Theorem 9}\\&=E^{-1}\xi^{A_{2n-1}}_{[j,k-1]}\xi^{A_{2n-1}}_{[2n-k+1,2n-j]}\xi^{A_{2n-1}}_{[n-(i-1),n+(i-1)]}\xi^{A_{2n-1}}_{[n-(i-2),n+(i-2)]}E,\\&\qquad\text{by Theorem 9}\\&=q^{C_n}_{[j,k-1]}t^{C_n}_{n+i-1}.\end{aligned}$$

(9) Recall (63) and Remark 20, (71). Then, for $n\ge 3$,

$$\begin{aligned}(t^{C_n}_1t^{C_n}_2)^6&=E^{-1}(t^{A_{2n-1}}_1\tilde t^{A_{2n-1}}_{\bar 2}t^{A_{2n-1}}_2\tilde t^{A_{2n-1}}_{\bar 3})^6E\\&=E^{-1}(t^{A_{2n-1}}_1t^{A_{2n-1}}_2\tilde t^{A_{2n-1}}_{\bar 2}\tilde t^{A_{2n-1}}_{\bar 3})^6E\\&=E^{-1}(t^{A_{2n-1}}_1t^{A_{2n-1}}_2)^6(\tilde t^{A_{2n-1}}_{\bar 2}\tilde t^{A_{2n-1}}_{\bar 3})^6E\\&=E^{-1}(\tilde t^{A_{2n-1}}_{\bar 2}\tilde t^{A_{2n-1}}_{\bar 3})^6E\\&=E^{-1}(\tilde t^{A_{2n-1}}_{2n-1}\tilde t^{A_{2n-1}}_{2n-2})^6E=1.\end{aligned}$$

(10) From Proposition 6, (3), we know that $(\xi^{C_n}_{n-1}\xi^{C_n}_n)^4=1$. Then, from Defintion 6 and (77), one has

$$\begin{aligned}(\xi^{C_n}_{n-1}\xi^{C_n}_n)^4&=(q^{C_n}_{[1,n-1]}t^{C_n}_1q^{C_n}_{[1,n-1]}t^{C_n}_n)^4\\&=(q^{C_n}_{[1,n-2]}p^{C_n}_{n-1}t^{C_n}_1q^{C_n}_{[1,n-2]}p^{C_n}_{n-1}t^{C_n}_n)^4,\quad\text{by Proposition 11.}\end{aligned}\tag{86}$$

From [BerKir95, Proposition 1.4, (a)] and mimicking its proof in conjunction with relation (5), we may write

$$q^{C_n}_{[1,n-1]}=q^{C_n}_{[1,n-2]}p^{C_n}_{n-1}=(p^{C_n}_{n-1})^{-1}q^{C_n}_{[1,n-2]}.\tag{87}$$

Therefore, using (87),

$$\begin{aligned}(\xi_{n-1}^{C_n}\xi_n^{C_n})^4 = (86) &= (q_{[1,n-2]}^{C_n}p_{n-1}^{C_n}t_1^{C_n}(p_{n-1}^{C_n})^{-1}q_{[1,n-2]}^{C_n}t_n^{C_n})^4\\ &= (q_{[1,n-2]}^{C_n}p_{n-1}^{C_n}t_1^{C_n}(p_{n-1}^{C_n})^{-1}t_n^{C_n}q_{[1,n-2]}^{C_n})^4 \quad \text{by relation (4)}\\ &= q_{[1,n-2]}^{C_n}(p_{n-1}^{C_n}t_1^{C_n}(p_{n-1}^{C_n})^{-1}t_n^{C_n})^4q_{[1,n-2]}^{C_n}\\ &= q_{[1,n-2]}^{C_n}(p_{n-1}^{C_n}t_1^{C_n}(p_{n}^{C_n})^{-1})^4q_{[1,n-2]}^{C_n}\end{aligned}$$

Since $(q_{[1,n-2]}^{C_n})^2=1$, we get

$$(p_{n-1}^{C_n}t_1^{C_n}(p_n^{C_n})^{-1})^4 = (t_{n-1}^{C_n}\cdots t_2^{C_n}t_1^{C_n}t_2^{C_n}\cdots t_n^{C_n})^4 = 1, \text{ where } p_0^{C_n} := 1.$$

In particular, for $n=2$, $(t_1^{C_n}t_2^{C_n})^4=1$. □

Remark 25. *(1) The relation* (9) *in* $\mathcal{BK}^{C_n}$*, respectively*

$$(t_1^{A_{2n-1}}t_2^{A_{2n-1}})^6(\tilde{t}_{2n-1}^{A_{2n-1}}\tilde{t}_{2n-2}^{A_{2n-1}})^6 = 1, \quad \text{in } \widetilde{\mathcal{BK}}_{2n},$$

is equivalent to the braid relations of B_n*,* $(\xi_i^{C_n}\xi_{i+1}^{C_n})^3=1$*, for* $1\le i<n-1$*, respectively*

$$(\xi_i^{A_{2n-1}}\xi_{2n-i}^{A_{2n-1}}\xi_{i+1}^{A_{2n-1}}\xi_{2n-i-1}^{A_{2n-1}})^3 = (\xi_i^{A_{2n-1}}\xi_{i+1}^{A_{2n-1}})^3(\xi_{2n-i}^{A_{2n-1}}\xi_{2n-i-1}^{A_{2n-1}})^3 = 1,$$

the braid relations of $\mathfrak{S}_{2n}$*, for* $1\le i<n-1$*,* [BerKir95, Proposition 1.4, (d)].

(2) The identity $(\xi_{n-1}^{C_n}\xi_n^{C_n})^4=1$ *in* $\mathcal{BK}^{C_n}$ *translates to* $\widetilde{\mathcal{BK}}_{2n}$ *as* $(\xi_{n-1}^{A_{2n-1}}\xi_{n+1}^{A_{2n-1}}\xi_n^{A_{2n-1}})^4=1$*. Thus relation* (10) *in* $\mathcal{BK}^{C_n}$ *translates to* $\widetilde{\mathcal{BK}}_{2n}$ *as*

$$(t_{n-1}^{A_{2n-1}}\cdots t_1^{A_{2n-1}}\tilde{t}_{n+1}^{A_{2n-1}}\cdots\tilde{t}_{2n-1}^{A_{2n-1}}\xi_n^{A_{2n-1}})^4 = 1. \tag{88}$$

The relation $\xi_{n-1}^{A_{2n-1}}\xi_n^{A_{2n-1}}\xi_{n-1}^{A_{2n-1}} = \xi_n^{A_{2n-1}}\xi_{n-1}^{A_{2n-1}}\xi_n^{A_{2n-1}}$*, although true in* $\mathcal{BK}_{2n}$*, does not hold in* $\widetilde{\mathcal{BK}}_{2n}$ *because* $\xi_n^{A_{2n-1}}\in\widetilde{\mathcal{BK}}_{2n}$ *and it would imply* $\xi_{n-1}^{A_{2n-1}}\in\widetilde{\mathcal{BK}}_{2n}$ *which is absurd. From Remark 24, we know that the generators of* $\widetilde{\mathcal{BK}}_{2n}$ *induce an action of* B_n *on the weights of* $\mathsf{SSYT}(\lambda,2n)$ *and one has differently* $\mathsf{wt}(\xi_{n-1}^{A_{2n-1}}(T)) = r_{n-1}\mathsf{wt}(T)$*. This means that the relation* (88) *in* $\widetilde{\mathcal{BK}}_{2n}$ *does not follow from the previous corresponding relations in Proposition 13. Alternatively we could have used Corollary 5 to obtain* (10).

10.5. **Example: the C_2 Bender–Knuth involutions and their virtual images.** The known relations for group the $\mathcal{BK}^{C_2}$ with generators $t_1^{C_2}$, $t_2^{C_2}$, $t_3^{C_2}$ are $t_i^2=(t_1t_2)^4=(t_2t_3)^2=1$, $i=1,2,3$.

Example 10. *We illustrate the symplectic Bender–Knuth involutions* $t_1^{C_2}$*,* $t_2^{C_2}$*,* $t_3^{C_2}$ *in* $\mathcal{BK}^{C_2}$ *as well their virtual images in* $\widetilde{\mathcal{BK}}_4$*:* $\Lambda=\mathbb{Z}^2$ *and* $B_2=<r_1,r_2>$*; if* $(a,b)\in\mathbb{Z}^2$*,* $r_1(a,b)=(b,a)$ *and* $r_2(a,b)=(a,\bar{b})$*;* $t_1^{C_2}=\xi_1^{C_2}$*,* $t_2^{C_2}=\xi_2^{C_2}$*,* $t_3^{C_2}=\xi_{[1,2]}^{C_2}\xi_2^{C_2}=\xi_2^{C_2}\xi_{[1,2]}^{C_2}$.

Using the type C_2 signature rule

$$\begin{aligned} 1,\bar{2} &\to + \\ 2,\bar{1} &\to - \\ 2 &\to + \\ \bar{2} &\to - \end{aligned}$$

we compute $t_1^{C_2}T = \xi_1^{C_2}T$, where

$$T = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 2 & \bar{2} & \bar{1} & \bar{1} \\ \hline 2 & \bar{2} & \bar{2} & \bar{1} & \bar{1} \\ \cline{1-5}\end{array}, \quad \mathit{wt}(T) = (-2, 1)$$

$$\bar{1}\bar{1}\bar{2}2\bar{1}2\bar{1}2\bar{2}1\bar{2}12$$

$$\bar{1}\bar{1}\bar{1}2\bar{1}2\bar{2}1\bar{2} \mapsto \bar{1}\bar{1}\bar{1}1\bar{2}1\bar{2}1\bar{2},$$

$$t_1^{C_2}T = \xi_1^{C_2}T = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 2 & \bar{2} & \bar{1} & \bar{1} \\ \hline 2 & \bar{2} & \bar{2} & \bar{2} & \bar{1} \\ \cline{1-5}\end{array} \quad \mathit{wt}(t_1^{C_2}(T)) = (1, -2).$$

Virtualization of $t_1^{C_2}$ in A_3: *$t_1^{C_2} = E^{-1}\xi_{\bar{2}}^{A_3}t_1^{A_3}E = E^{-1}t_1^{A_3}\xi_{\bar{2}}^{A_3}E$. The shape of T is $3\omega_1 + 5\omega_2$, and thus the shape of $E(T)$ and $E(t_1^{C_2}T)$ is $3(\omega_1 + \omega_3) + 10\omega_2$, where $\omega_k = (1^k)$, $1 \le k \le 3$,*

$$E(T) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & \bar{2} & \bar{2} & \bar{1} & \bar{1} \\ \hline 2 & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} \\ \cline{1-13} \bar{2} & \bar{1} & \bar{1} \\ \cline{1-3}\end{array},$$

and

$$E(t_1^{C_2}T) = E(\xi_1^{C_2}T) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & \bar{2} & \bar{2} & \bar{1} & \bar{1} \\ \hline 2 & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{1} & \bar{1} & \bar{1} \\ \cline{1-13} \bar{2} & \bar{1} & \bar{1} \\ \cline{1-3}\end{array}.$$

Using the A_3 signature rule

$$\begin{array}{llll} 1 \to + & 2 \to + & \textit{and } \bar{2} & \to + \\ 2 \to - & \bar{2} \to - & \quad\ \bar{1} & \to - \end{array} \tag{89}$$

we compute

$$\xi_{\bar{2}}^{A_3}t_1^{A_3}E(T) = \xi_{\bar{2}}^{A_3} \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & \bar{2} & \bar{2} & \bar{1} & \bar{1} \\ \hline 2 & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} \\ \cline{1-13} \bar{2} & \bar{1} & \bar{1} \\ \cline{1-3}\end{array}$$

$$= \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & \bar{2} & \bar{2} & \bar{1} & \bar{1} \\ \hline 2 & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{1} & \bar{1} & \bar{1} \\ \cline{1-13} \bar{2} & \bar{1} & \bar{1} \\ \cline{1-3}\end{array} = E(t_1^{C_2}T)$$

Therefore,

$$t_1^{C_2}(T) = E^{-1}\xi_{\bar{2}}^{A_3} t_1^{A_3} E(T).$$

Virtualization of $t_2^{C_2}$: $t_2^{C_2} = E^{-1}\xi_2^{A_3} E.$

$$T = \begin{array}{cccccccc} 1 & 1 & 2 & 2 & 2 & \bar{2} & \bar{1} & \bar{1} \\ 2 & \bar{2} & \bar{2} & \bar{1} & \bar{1} & & & \end{array},$$

$$t_2^{C_2}(T) = \xi_2^{C_2}(T) = \begin{array}{cccccccc} 1 & 1 & 2 & 2 & \bar{2} & \bar{2} & \bar{1} & \bar{1} \\ 2 & \bar{2} & \bar{2} & \bar{1} & \bar{1} & & & \end{array}$$

$$\mathsf{wt}(t_2^{C_2}(T)) = (-2,-1)$$
$$E(t_2^{C_2}(T)) = \xi_2^{A_3} E(T)$$
$$t_2^{C_2}(T) = E^{-1}\xi_2^{A_3} E(T)$$

$$\xi_2^{A_3} E(T) = \xi_2^{A_3} \begin{array}{cccccccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & \bar{2} & \bar{2} & \bar{1} & \bar{1} \\ 2 & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & & & \\ \bar{2} & \bar{1} & \bar{1} & & & & & & & & & & & & & \end{array}$$

$$= \begin{array}{cccccccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{1} & \bar{1} \\ 2 & 2 & 2 & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & & & \\ \bar{2} & \bar{1} & \bar{1} & & & & & & & & & & & & & \end{array} = E(t_2^{C}(T))$$

Virtualization of $t_3^{C_2}$: $t_3^{C_2} = E^{-1}\xi_2^{A_3} t_1^{A_3}(t_2^{A_3} t_1^{A_3})(t_{\bar{2}}^{A_3} t_2^{A_3} t_1^{A_3})E.$

$$T = \begin{array}{cccccccc} 1 & 1 & 2 & 2 & 2 & \bar{2} & \bar{1} & \bar{1} \\ 2 & \bar{2} & \bar{2} & \bar{1} & \bar{1} & & & \end{array}, \quad \mathsf{wt}(T) = (-2,1)$$

$$t_3^{C_2}(T) = \xi_2^{C_2}\xi_{[1,2]}^{C_2}(T) = \xi_{[1,2]}^{C_2}\xi_2^{C_2}(T) = \begin{array}{cccccccc} 1 & 1 & 1 & 1 & 2 & 2 & \bar{2} & \bar{1} \\ 2 & 2 & \bar{2} & \bar{2} & \bar{1} & & & \end{array}, \quad wt(t_3^{C_2}(T)) = (2,1)$$

$$E(t_3^{C_2}(T)) = E(\xi_2^{C_2}\xi_{[1,2]}^{C_2}(T)) = \xi_2^{A_3} E(\xi_{[1,2]}^{C_2}(T)) = \xi_2^{A_3}\mathsf{evac}^{A_3} E(T) = \mathsf{evac}^{A_3}\xi_2^{A_3} E(T),$$
$$t_3^{C_2}(T) = E^{-1}\xi_2^{A_3}\mathsf{evac}^{A_3} E(T) = E^{-1}\xi_2^{A_3} t_1^{A_3}(t_2^{A_3} t_1^{A_3})(t_{\bar{2}}^{A_3} t_2^{A_3} t_1^{A_3})E(T).$$

11. Open questions and final remarks

It remains to establish whether or not $\mathcal{BK}^{C_n}$ satisfies additional relations besides those listed in Proposition 13.

Chmutov, Glick and Pylyavskyy [CGP16] have determined relationships between subsets of relations in the groups $\mathcal{BK}_n$ and J_n which yield a presentation for the cactus group J_n in terms of Bender–Knuth generators. Rodrigues [Ro20, Ro21] has also introduced a shifted Berenstein–Kirillov group with many parallels with the original $\mathcal{BK}$ group. Following Halacheva she has defined a cactus group action of J_n via partial shifted Schützenberger–Lusztig involutions (partial shifted reversal) on the Gillespie-Levibnson-Puhrboo shifted tableau crystal [GLP17]. On the other hand, with the shifted tableau switching she has defined shifted Bender–Knuth involutions, and following Chmutov, Glick and Pylyavskyy she has yield a presentation for the cactus group J_n in terms of shifted Bender–Knuth generators. In the same vein, it is natural to seek precise relationships between subsets of relations in the two groups $\widetilde{\mathcal{BK}}_{2n}$ and the virtual symplectic cactus group $\tilde{J}_n$. It is also natural to seek a presentation of the virtual symplectic cactus group $\widetilde{J}_{2n}$ in terms of the virtual symplectic Bender–Knuth generators.

Glossary

$\mathcal{BK}^{C_n}$: The type C_n symplectic Berenstein–Kirillov group. 4, 5, 56, 57, 60, 62, 65, 67

$\mathcal{BK}_n$: The subgroup of $\mathcal{BK}$ generated by the first $n-1$ Bender–Knuth involutions $t_1, \ldots, t_{n-1}$. 54–56, 67

$\mathcal{BK}$: The Berenstein–Kirillov group (or Gelfand-Tsetlin group) [BerKir95]. 54, 55, 67

$\mathsf{KN}_J(\lambda, n)$: The Levi branched crystal of Kashiwara–Nakashima tableaux obtained by deleting in $\mathsf{KN}(\lambda, n)$ all the arrows not labelled in $J \subset I$. 10–12, 15, 16, 25, 26, 47, 48

$\mathsf{SSYT}(\lambda, n)$: The $U_q(\mathfrak{sl}(n, \mathbb{C}))$-crystal of semi-standard Young tableaux of shape λ and entries in $[n]$. 8, 22, 25, 28, 44, 55, 56

$\widetilde{J}_{2n}$: The virtual symplectic cactus group. 4, 5, 20–22, 28–30, 50, 59, 60, 62, 68

$\widetilde{\mathcal{BK}}_{2n}$: The virtual symplectic Berenstein–Kirillov group, a subgroup of $\mathcal{BK}_{2n}$ satisfying the relations of the virtual symplectic cactus group $\widetilde{J}_{2n}$. 5, 58–60, 62, 65, 68

E: The virtualization map defined by Baker [Ba00a, Proposition 2.2, Proposition 2.3] on type C_n Kashiwara–Nakashima tableaux. 4, 12–16, 29, 47–50, 52, 53, 58, 60, 66, 67

J_n: The cactus group $J_{\mathfrak{sl}(n,\mathbb{C})}$. 3, 16–18, 28, 55, 67, 68

$J_{\mathfrak{sp}(2n,\mathbb{C})}$: The symplectic cactus group with generators s_J for J any connected subdiagram of the C_n Dynkin diagram subject to the relations in Lemma 2. 1, 2, 4, 5, 17, 18, 20–22, 28, 50, 56, 57, 59, 60, 62, 63

R3: The symplectic contraction/dilation relation in the symplectic plactic monoid $\mathcal{C}_n^*/\sim$. 24, 26, 32–37, 39–44, 46

$\mathcal{C}_n^*$: The monoid of words in the alphabet $\mathcal{C}_n$. 13, 14, 23, 24

$\mathcal{C}_n$: $\{1 < \cdots < n < \bar{n} < \cdots < \bar{1}\}$. 3, 4, 8, 9, 11–13, 30, 32, 33, 35, 36, 56

$\mathfrak{g}$: Finite dimensional, complex, semisimple Lie algebra. 2, 3, 5–8, 16–18, 22, 23, 26–28, 30

$\mathsf{B}(\lambda)$: The normal $\mathfrak{g}$-crystal with highest weight λ. 2, 8, 22, 25, 30–32, 69

B_J: The Levi branched normal crystal B_J, the restriction of B to the sub-diagram J of I. 7, 15, 26, 32, 69

B: A normal crystal. 2, 7, 8, 22, 25–27, 29, 30, 32, 35, 36, 45

$\mathsf{KN}(\lambda, n)$: The $U(\mathfrak{sp}(2n, \mathbb{C}))$ crystal of Kashiwara–Nakashima tableaux of shape λ in the alphabet $\mathcal{C}_n$. 2, 4, 8, 10–15, 25, 28, 29, 32, 33, 35, 44, 49, 50, 57, 58, 60–62

$\mathsf{SSYT}(\lambda^A, n, \bar{n})$: The $U_q(\mathfrak{sl}(2n, \mathbb{C}))$-crystal of semi-standard Young tableaux of shape λ^A and entries in $\mathcal{C}_n$. 4, 12–14, 29, 48–50, 60

$\mathsf{SSYT}_J(\lambda, n)$: The Levi branched crystal, the restriction of $\mathsf{SSYT}(\lambda, n)$ to $J \subseteq [n-1]$. 28

reversal$_J^{C_n}$: J-partial symplectic reversal, the symplectic reversal $\mathsf{KN}_J(\lambda, n)$ with $J \subseteq [n]$ a connected sub-diagram containing the node n. 30, 45

reversal^{C_n}: Combinatorial procedure to compute the Schützenberger involution ξ on $\mathsf{KN}(\lambda, n)$. 26

reversal$_J$: J-partial reversal, the reversal on $\mathsf{SSYT}_J(\lambda, n)$ with $J \subseteq [n-1]$. 28, 30

reversal: Combinatorial procedure to compute the Schützenberger involution ξ on $\mathsf{SSYT}(\lambda, n)$. 22, 23, 28, 44, 52, 53

ξ_{B}: The Schützenberger–Lusztig involution on the normal crystal B. 2, 22

ξ_{J}: The partial Schützenberger–Lusztig involution to the sub-diagram $J \subseteq I$ is the Schützenberger–Lusztig involution ξ_{B_J} on the normal crystal B_J. 26–30, 32

ξ: The Schützenberger–Lusztig involution on $\mathsf{B}(\lambda)$. 22, 23, 28, 31, 43

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