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Sobolev Inequality on Manifolds With Asymptotically Nonnegative Bakry-Émery Ricci Curvature*

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ABSTRACT In this paper, inspired by [4, 9], we prove a Sobolev inequality on manifolds with density and asymptotically nonnegative Bakry-Émery Ricci curvature.

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1 Introduction

In 2019, S. Brendle [3] proved a Sobolev inequality for submanifolds in Euclidean space. Moreover, he obtained a sharp isoperimetric inequality for compact minimal submanifolds in Euclidean space with codimension at most 2. In 2020, he generalized the results in [3] to the case of manifolds with nonnegative curvature ([4]). Recently, we extended [4] to manifolds with asymptotically nonnegative curvature ([6]). In 2021, F. Johne [9] generalized the results of [4] to the case of manifolds with density and nonnegative Bakry-Émery Ricci curvature. In this note, we establish a Sobolev inequality in manifolds with density and asymptotically nonnegative Bakry-Émery Ricci curvature.

Let $(M, g, w\, d\mathrm{vol}_g)$ be a smooth complete noncompact n-dimensional Riemannian manifold with density, where w is a smooth positive density function on M and $d\mathrm{vol}_g$ is the Riemannian volume measure with respect to the metric g. As a generalization of Ricci curvature, the Bakry-Émery Ricci curvature [2] of $(M, g, w\, d\mathrm{vol}_g)$ is defined by

$$\mathrm{Ric}^w_\alpha = \mathrm{Ric} - D^2(\log w) - \frac{1}{\alpha} D\log w \otimes D\log w, \tag{1.1}$$

where Ric denotes the Ricci curvature of M, D is the Levi-Civita connection with respect to the metric g and $\alpha > 0$. If the density function w is constant, the Bakry-Émery Ricci curvature Ric^w_α reduces to the Ricci curvature.

Suppose $\lambda : [0, +\infty) \to [0, +\infty)$ is a nonnegative and nonincreasing continuous function satisfying

$$b_0 := \int_0^{+\infty} s\lambda(s)\, ds < +\infty, \tag{1.2}$$

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which implies

$$b_1 := \int_0^{+\infty} \lambda(s)\, ds < +\infty. \tag{1.3}$$

A complete noncompact Riemannian manifold $(M, g, w\, \mathrm{dvol}_g)$ with density of dimension n is said to have **asymptotically nonnegative Bakry-Émery Ricci curvature** if there exists a base point $o \in M$ such that

$$\mathrm{Ric}^w_\alpha(q) \geq -(n+\alpha-1)\, \lambda(d(o,q)), \quad \forall q \in M, \tag{1.4}$$

where d is the distance function of M. In particular, $\lambda \equiv 0$ in (1.4) corresponds to the case treated in [9].

Suppose $h : [0, T) \to \mathbb{R}$ is the unique solution of the initial value problem

$$\begin{cases} h''(t) = \lambda(t) h(t), \\ h(0) = 0, h'(0) = 1. \end{cases} \tag{1.5}$$

By the theory of ordinary differential equations [16], the solution exists for all time, i.e. $T = \infty$. We remark that h reduces to the radius function, if $\lambda = 0$. Similar to the work of F. Johne [9], we define the α**-asymptotic volume ratio** $\mathcal{V}_\alpha$ of $(M, g, w\, \mathrm{dvol}_g)$ by

$$\mathcal{V}_\alpha := \lim_{r \to +\infty} \frac{\int_{B_r(o)} w}{(n+\alpha) \int_0^r h^{n+\alpha-1}(t)\, dt}, \tag{1.6}$$

where o is the base point and $B_r(o)$ denotes the geodesic ball of radius r, i.e. $B_r(o) = \{q \in M : d(o,q) < r\}$. In Theorem 2.2, we will show a comparison theorem for weighted volumes, to be more precise we will show

$$\frac{\int_{B_r(o)} w}{(n+\alpha) \int_0^r h^{n+\alpha-1}(t)\, dt}$$

is a nonincreasing function for $r \in (0, +\infty)$, so $\mathcal{V}_\alpha$ is well defined.

By combining the ABP-method in [4, 9] with some comparison theorems for ordinary differential equations, we establish a Sobolev inequality for a compact domain in manifolds with density, under the asymptotically nonnegative Bakry-Émery Ricci curvature as follows.

Theorem 1.1. *Let $(M, g, w\, \mathrm{dvol}_g)$ be a smooth complete noncompact n-dimensional Riemannian manifold of smooth density $w > 0$ and asymptotically nonnegative Bakry-Émery Ricci curvature with respect to a base point $o \in M$. Let Ω be a compact domain in M with boundary $\partial\Omega$, and f be a positive smooth function on Ω. Then*

$$\begin{aligned} &\int_{\partial\Omega} wf + \int_\Omega w|Df| + 2b_1(n+\alpha-1) \int_\Omega wf \\ &\geq (n+\alpha)\mathcal{V}_\alpha^{\frac{1}{n+\alpha}} \left(\frac{1+b_0}{e^{2r_0 b_1 + b_0}}\right)^{\frac{n+\alpha-1}{n+\alpha}} \left(\int_\Omega w f^{\frac{n+\alpha}{n+\alpha-1}}\right)^{\frac{n+\alpha-1}{n+\alpha}}, \end{aligned}$$

where $r_0 = \max\{d(o,x)|x \in \Omega\}$, $\mathcal{V}_\alpha$ *is the* α*-asymptotic volume ratio of* M *given by* (1.6)*,* b_0, b_1 *are defined in* (1.2) *and* (1.3)*, respectively.*

Taking $f = 1$ in Theorem 1.1, we obtain an isoperimetric inequality:

Corollary 1.2. *Let* $(M, g, w\,\mathrm{dvol}_g)$ *be a smooth complete noncompact* n*-dimensional Riemannian manifold of smooth density* $w > 0$ *and asymptotically nonnegative Bakry-Émery Ricci curvature with respect to a base point* $o \in M$*. Then*

$$\int_{\partial\Omega} w \geq \Big((n+\alpha)\mathcal{V}_\alpha^{\frac{1}{n+\alpha}}\Big(\frac{1+b_0}{e^{2r_0 b_1 + b_0}}\Big)^{\frac{n+\alpha-1}{n+\alpha}} - 2(n+\alpha-1)b_1\big(\int_\Omega w\big)^{\frac{1}{n+\alpha}}\Big)\big(\int_\Omega w\big)^{\frac{n+\alpha-1}{n+\alpha}},$$

where $r_0 = \max\{d(o,x)|x \in \Omega\}$, $\mathcal{V}_\alpha$ *is the* α*-asymptotic volume ratio of* M *given by* (1.6)*,* b_0, b_1 *are defined in* (1.2) *and* (1.3)*.*

When $w = 1, b_0 = b_1 = \alpha = 0$, Corollary 1.2 was first given by V. Agostiniani, M. Fogagnolo, and L. Mazziari [1] in dimension 3 and obtained by S. Brendle [4] for any dimension. In recent years, the study of the isoperimetric problem in manifolds with density attracts much attention, see [15, 11, 5, 7]. For more about manifolds with density, we refer the reader to [12, 13] and references therein.

2 Preliminaries

In this section, we give a proof of the Bishop-Gromov volume comparison theorem for complete noncompact Riemannian manifold with density and asymptotically nonnegative Bakry-Émery Ricci curvature.

The following lemma is an almost verbatim combination of Lemma 2.1 and Corollary 2.2 in [14]. We should point out, however, that it deals with a different initial value case than [14].

Lemma 2.1. *Let* G *be a continuous nonnegative function on* $[0, +\infty)$ *and let* g, ψ *be solutions to the following problems*

$$\begin{cases} g' + g^2 \leq G, \quad t \in (0, +\infty), \\ g(t) = \dfrac{\beta}{t} + O(1), \ \textit{as } t \to 0^+, \end{cases} \qquad \begin{cases} \psi'' = G\psi, \quad t \in (0, +\infty), \\ \psi(0) = 0, \psi'(0) = 1, \end{cases} \tag{2.1}$$

where $0 < \beta \leq 1$*. Then we have*

$$g \leq \frac{\psi'}{\psi} \ \textit{on } (0, +\infty).$$

Proof. Observe that the initial conditions imply $\psi \geq 0$. Then the Fundamental Theorem of Calculus implies $\psi(t) \geq t$ and $\psi'(t) \geq 1$. Let $\phi(t) = t^{\beta} e^{\int_0^t (g - \frac{\beta}{\tau})\, d\tau} > 0$ for $t \in (0, +\infty)$. Similar to the proof of Lemma 2.1 and Corollary 2.2 in [14], it is easy to show that

$$\begin{aligned} & g = \frac{\phi'}{\phi}, \quad \phi'' \leq G\phi, \\ & \phi(t) = t^{\beta}(1 + O(1)), \text{ as } t \to 0^+, \\ & \lim_{t \to 0^+} \frac{\psi(t)}{t} = \frac{\psi'(0)}{1} = 1, \\ & \lim_{t \to 0^+} (\phi'\psi - \phi\psi') = \lim_{t \to 0^+} (g\phi\psi - \phi\psi') = 0. \end{aligned} \tag{2.2}$$

Using (2.1) and (2.2), we conclude that

$$(\phi'\psi - \psi'\phi)' = \phi''\psi - \phi\psi'' \leq G(t)\psi\phi - G(t)\psi\phi = 0$$

and

$$\phi'\psi - \psi'\phi \leq 0 \text{ on } (0, +\infty).$$

Thus,

$$g = \frac{\phi'}{\phi} \leq \frac{\psi'}{\psi} \text{ on } (0, +\infty).$$

□

The proof of the following theorem is a close adaption of Theorem A.1 in F. Johne [9].

Theorem 2.2. *Let $(M, g, w\, \mathrm{dvol}_g)$ be a smooth complete noncompact n-dimensional Riemannian manifold of smooth density $w > 0$ and asymptotically nonnegative Bakery-Émery Ricci curvature with respect to a base point $o \in M$. Then the function*

$$r \mapsto \frac{\int_{B_r(o)} w}{(n + \alpha) \int_0^r h^{n+\alpha-1}(t)\, dt}$$

is nonincreasing.

Proof. Fix the base point $o \in M$ and $r > 0$, let $D_o = M \setminus \mathrm{cut(o)}$ be the domain of the normal geodesic coordinates centered at o. We define $B_r(o) = \{q \in M : d(o, q) < r\}$ and its boundary by $S_r(o) = \partial B_r(o)$. Denote the second fundamental form of the hypersurface $S_r(o) \cap D_o$ by B and the mean curvature of the geodesic sphere with an inward pointing normal vector by H.

Let $\gamma(t) := \exp_o(tv), t \in [0, r]$ be a normal geodesic such that $\gamma(t) \in S_t(o) \cap D_o$. We consider the variation set of hypersurfaces that have a constant distance from N. By (1.6) in [10], it is easy to know that

$$\frac{d}{dt} H = -|B|^2 - \mathrm{Ric}(\gamma', \gamma'), \quad t \in (0, r),$$

provided $\gamma(t) \in S_t(o) \cap D_o$. By the definition of Bakry-Émery Ricci curvature (1.1), we deduce that

$$
\begin{aligned}
&\frac{d}{dt}[H + \langle D\log w, \gamma'\rangle] \\
&= -\frac{1}{\alpha}\langle D\log w, \gamma'\rangle^2 - \mathrm{Ric}_\alpha^w(\gamma',\gamma') - |B|^2 \\
&\le -\frac{1}{n-1}H^2 - \frac{1}{\alpha}\langle D\log w, \gamma'\rangle^2 - \mathrm{Ric}_\alpha^w(\gamma',\gamma') \\
&= -\frac{1}{n+\alpha-1}(H + \langle D\log w, \gamma'\rangle)^2 - \frac{n-1}{\alpha(n+\alpha-1)}\Big(\frac{\alpha}{n-1}H - \langle D\log w, \gamma'\rangle\Big)^2 - \mathrm{Ric}_\alpha^w(\gamma',\gamma') \\
&\le -\frac{1}{n+\alpha-1}(H + \langle D\log w, \gamma'\rangle)^2 - \mathrm{Ric}_\alpha^w(\gamma',\gamma'). \qquad (2.3)
\end{aligned}
$$

Set $g_o = \frac{1}{n+\alpha-1}[H+\langle D\log w, \gamma'\rangle], t \in (0,r)$. Using $\lim\limits_{t\to 0^+} tH(t) = n-1$ and the smoothness of the density w, by (1.4) and (2.3), we can find

$$
\begin{cases}
g_o' + g_o^2 \le \lambda, \quad t \in (0,r), \\
g_o(t) = \dfrac{n-1}{(n+\alpha-1)t} + O(1), \text{ as } t \to 0^+.
\end{cases}
$$

Note that $0 < \frac{n-1}{n+\alpha-1} \le 1$, from (1.5) and Lemma 2.1, it follows

$$
g_o \le \frac{h'}{h},
$$

that is,

$$
H + \langle D(\log w), \gamma'\rangle \le (n+\alpha-1)\frac{h'}{h}.
$$

By the first variation formula for the manifold with density, we obtain

$$
\begin{aligned}
\frac{d}{dt}\Big(\int_{S_t(o)} w\Big) &= \frac{d}{dt}\Big(\int_{S_t(o)\cap D_o} w\Big) = \int_{S_t(o)\cap D_o} (H + \langle \nu, D\log w\rangle) w \\
&\le (n+\alpha-1)\frac{h'}{h}\int_{S_t(o)\cap D_o} w = (n+\alpha-1)\frac{h'}{h}\int_{S_t(o)} w,
\end{aligned}
$$

where ν is the unit outward vector field along $S_t(o) \cap D_o$. This implies that

$$
t \mapsto \frac{\int_{S_t(o)} w}{h^{n+\alpha-1}(t)}
$$

is a nonincreasing function. Following Lemma 2.2 in [17], we derive that

$$
\int_{B_r(o)} w = \int_0^r \frac{\int_{S_t(o)} w}{h^{n+\alpha-1}} h^{n+\alpha-1}\, dt \ge \frac{\int_{S_r(o)} w}{h^{n+\alpha-1}(r)} \int_0^r h^{n+\alpha-1}\, dt,
$$

which implies

$$\frac{d}{dr}\Big(\frac{\int_{B_r(o)} w}{\int_0^r h^{n+\alpha-1}\,dt}\Big) = \frac{h^{n+\alpha-1}(r)}{(\int_0^r h^{n+\alpha-1}\,dt)}\Big(\frac{\int_{S_r(o)} w}{h^{n+\alpha-1}(r)} - \frac{\int_{B_r(o)} w}{\int_0^r h^{n+\alpha-1}\,dt}\Big) \leq 0.$$

This proves the assertion. □

3 Proof of Theorem 1.1

Let $(M, g, w\,d\mathrm{vol}_g)$ be a complete noncompact n-dimensional Riemannian manifold with smooth density $w > 0$ and asymptotically nonnegative Bakry-Émery Ricci curvature with respect to a base point $o \in M$. Let Ω be a compact and connected domain in M with smooth boundary $\partial\Omega$, and f be a smooth positive function on Ω.

We only need to prove Theorem 1.1 in the case that Ω is connected. By scaling, we may assume that

$$\int_{\partial\Omega} wf + \int_\Omega w|Df| + 2(n+\alpha-1)b_1 \int_\Omega wf = (n+\alpha)\int_\Omega wf^{\frac{n+\alpha}{n+\alpha-1}}. \tag{3.1}$$

Due to (3.1) and the connectedness of Ω, we can find a solution to the following Neumann problem

$$\begin{cases} \mathrm{div}(wfDu) = (n+\alpha)wf^{\frac{n+\alpha}{n+\alpha-1}} - w|Df| - 2(n+\alpha-1)b_1 wf & \text{in } \Omega, \\ \\ \langle Du, \nu\rangle = 1, & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where ν is the outward unit normal vector field of $\partial\Omega$. By standard elliptic regularity theory (see Theorem 6.31 in [8]), we know that $u \in C^{2,\gamma}$ for each $0 < \gamma < 1$.

Following the notions in [4], we define

$$U := \{x \in \Omega \setminus \partial\Omega : |Du(x)| < 1\}.$$

For any $r > 0$, we denote A_r by

$$\{\bar{x} \in U : ru(x) + \frac{1}{2}d(x, \exp_{\bar{x}}(rDu(\bar{x})))^2 \geq ru(\bar{x}) + \frac{1}{2}r^2|Du(\bar{x})|^2, \forall x \in \Omega\}$$

and the transport map $\Phi_r : \Omega \to M$ by

$$\Phi_r(x) = \exp_x(rDu(x)), \quad \forall x \in \Omega.$$

Using the regularity of the solution u of the Neumann problem, we know that the transport map is of class $C^{1,\gamma}, 0 < \gamma < 1$.

We obtain the following lemma similar to Lemma 2.1 in [4].

Lemma 3.1. *Assume that $x \in U$. Then we have*

$$w\Delta u + \langle Dw, Du \rangle + 2(n+\alpha-1)b_1 w \leq (n+\alpha) w f^{\frac{1}{n+\alpha-1}}.$$

Proof. Using the Cauchy-Schwarz inequality and the property that $|Du| < 1$ for $x \in U$, we get

$$-\langle Df, Du \rangle \leq |Df|.$$

In terms of (3.2), we derive that

$$\begin{aligned} f(w\Delta u + \langle Dw, Du \rangle + 2(n+\alpha-1)b_1 w) &= (n+\alpha) w f^{\frac{n+\alpha}{n+\alpha-1}} - w(|Df| + \langle Df, Du \rangle) \\ &\leq (n+\alpha) w f^{\frac{n+\alpha}{n+\alpha-1}}. \end{aligned}$$

This proves the assertion. □

The proofs of the following three lemmas are identical to those of Lemmas 2.2-2.4 in [4] without any change for the case of asymptotically nonnegative Bakry-Émery Ricci curvature. So we omit them here.

Lemma 3.2 (cf. S. Brendle, Lemma 2.2 in [4])**.** *The set*

$$\{q \in M : d(x,q) < r, \forall x \in \Omega\}$$

is contained in $\Phi_r(A_r)$.

Lemma 3.3 (cf. S. Brendle, Lemma 2.3 in [4])**.** *Assume that $\bar{x} \in A_r$, and let $\bar{\gamma}(t) := \exp_{\bar{x}}(tDu(\bar{x}))$ for all $t \in [0,r]$. If Z is a smooth vector field along $\bar{\gamma}$ satisfying $Z(r) = 0$, then*

$$(D^2 u)(Z(0), Z(0)) + \int_0^r \left(|D_t Z(t)|^2 - R(\bar{\gamma}'(t), Z(t), \bar{\gamma}'(t), Z(t)) \right) dt \geq 0,$$

where R is the Riemannian curvature tensor of M.

Lemma 3.4 (cf. S. Brendle, Lemma 2.2 in [4])**.** *Assume that $\bar{x} \in A_r$, and let $\bar{\gamma}(t) := \exp_{\bar{x}}(tDu(\bar{x}))$ for all $t \in [0,r]$. Moreover, let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $T_{\bar{x}}M$. Suppose that W is a Jacobi field along $\bar{\gamma}$ satisfying*

$$\langle D_t W(0), e_j \rangle = (D^2 u)(W(0), e_j), \quad 1 \leq j \leq n.$$

If $W(\tau) = 0$ for some $\tau \in (0,r)$, then W vanishes identically.

We now give the proof of Theorem 1.1. The strategy of the proof follows the work of S. Brendle [4] closely.

Proof of Theorem 1.1. For any $r > 0$ and $\bar{x} \in A_r$, let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space $T_{\bar{x}}M$. Choose the geodesic normal coordinates $(x^1, \ldots, x^n)$ around $\bar{x}$, such that $\frac{\partial}{\partial x^i} = e_i$ at $\bar{x}$. Let $\bar{\gamma}(t) := \exp_{\bar{x}}(tDu(\bar{x}))$ for all $t \in [0, r]$. For $1 \leq i \leq n$, let $E_i(t)$ be the parallel transport of e_i along $\bar{\gamma}$. For $1 \leq i \leq n$, let $X_i(t)$ be the Jacobi field along $\bar{\gamma}$ with the initial conditions of $X_i(0) = e_i$ and

$$\langle D_t X_i(0), e_j \rangle = (D^2 u)(e_i, e_j), \quad 1 \leq j \leq n.$$

Let $P(t) = (P_{ij}(t))$ be a matrix defined by

$$P_{ij}(t) = \langle X_i(t), E_j(t) \rangle, \quad 1 \leq i, j \leq n.$$

It follows from Lemma 3.4 that $X_1(t), \cdots, X_n(t)$ are linearly independent for each $t \in (0, r)$, which implies that the matrix $P(t)$ is invertible for each $t \in (0, r)$. It is obvious that $\det P(t) > 0$ if t is sufficiently small. Therefore, $|\det D\Phi_t(\bar{x})| = \det P(t) > 0$, for $t \in [0, r)$. Let $S(t) = (S_{ij}(t))$ be a matrix defined by

$$S_{ij}(t) = R(\bar{\gamma}'(t), E_i(t), \bar{\gamma}'(t), E_j(t)), \quad 1 \leq i, j \leq n,$$

where R denotes the Riemannian curvature tensor of M. By the Jacobi equation, we obtain

$$\begin{cases} P''(t) = -P(t)S(t), \quad t \in [0, r], \\ \\ P_{ij}(0) = \delta_{ij}, P'_{ij}(0) = (D^2 u)(e_i, e_j). \end{cases} \tag{3.3}$$

Let $Q(t) = P(t)^{-1}P'(t), t \in (0, r)$, which is symmetric showed by S. Brendle [4]. By (3.3), a simple computation yields

$$\frac{d}{dt}Q(t) = -S(t) - Q^2(t).$$

Recalling that

$$\mathrm{Ric}_w^\alpha := \mathrm{Ric} - D^2(\log w) - \frac{1}{\alpha} D\log w \otimes D\log w,$$

we follow the computation by F. Johne [9] to derive that

$$\begin{aligned}
&\frac{d}{dt}[\mathrm{tr}Q + \langle D\log w, \bar{\gamma}' \rangle] \\
&= -\frac{1}{\alpha}\langle D\log w, \bar{\gamma}' \rangle^2 - \mathrm{Ric}_\alpha^w(\bar{\gamma}', \bar{\gamma}') - \mathrm{tr}[Q^2] \\
&\leq -\frac{1}{n}[\mathrm{tr}Q]^2 - \frac{1}{\alpha}\langle D\log w, \bar{\gamma}' \rangle^2 - \mathrm{Ric}_\alpha^w(\bar{\gamma}', \bar{\gamma}') \\
&= -\frac{1}{n+\alpha}(\mathrm{tr}Q + \langle D\log w, \bar{\gamma}' \rangle)^2 - \frac{n}{\alpha(n+\alpha)}\left(\frac{\alpha}{n}\mathrm{tr}Q - \langle D\log w, \bar{\gamma}' \rangle\right)^2 - \mathrm{Ric}_\alpha^w(\bar{\gamma}', \bar{\gamma}') \\
&\leq -\frac{1}{n+\alpha}(\mathrm{tr}Q + \langle D\log w, \bar{\gamma}' \rangle)^2 - \mathrm{Ric}_\alpha^w(\bar{\gamma}', \bar{\gamma}').
\end{aligned}$$

Set $g = \frac{1}{n+\alpha}[\mathrm{tr}Q + \langle D(\log w), \bar{\gamma}'(t)\rangle]$. The assumption of asymptotic nonnegative Bakry-Émery Ricci curvature gives

$$g' + g^2 \leq \frac{n+\alpha-1}{n+\alpha}|Du(\bar{x})|^2\lambda(d(o, \bar{\gamma}(t))), \tag{3.4}$$

where o is the base point. By the triangle inequality, we get

$$d(o, \bar{\gamma}(t)) \geq |d(o, \bar{x}) - d(\bar{x}, \bar{\gamma}(t))| = |d(o, \bar{x}) - t|Du(\bar{x})||. \tag{3.5}$$

Set

$$\Lambda_{\bar{x}}(t) = \frac{n+\alpha-1}{n+\alpha}|Du(\bar{x})|^2\lambda(|d(o, \bar{x}) - t|Du(\bar{x})||).$$

Since λ is a nonincreasing function, it follows from (3.3), (3.4) and (3.5) that

$$\begin{cases} g'(t) + g(t)^2 \leq \Lambda_{\bar{x}}(t), & t \in (0, r), \\ g(0) = \dfrac{1}{n+\alpha}[\Delta u(\bar{x}) + \langle D(\log w)(\bar{x}), Du(\bar{x})\rangle]. \end{cases}$$

Let $\phi = e^{\int_0^t g(\tau)d\tau}$, then

$$\begin{cases} \phi'' \leq \Lambda_{\bar{x}}(t)\phi, & t \in (0, r), \\ \phi(0) = 1, \phi'(0) = g(0). \end{cases} \tag{3.6}$$

Set ψ_1, ψ_2 be solutions of the following problems

$$\begin{cases} \psi_1'' = \Lambda_{\bar{x}}(t)\psi_1, & t \in (0, r), \\ \psi_1(0) = 0, \psi_1'(0) = 1, \end{cases} \qquad \begin{cases} \psi_2'' = \Lambda_{\bar{x}}(t)\psi_2, & t \in (0, r), \\ \psi_2(0) = 1, \psi_2'(0) = 0. \end{cases} \tag{3.7}$$

By the assumption of (1.2), one knows that $\int_0^\infty \Lambda_{\bar{x}}(t)\ dt < \infty$. Similar to the proof of Lemma 2.6 in [6], we have

$$\frac{\psi_2}{\psi_1}(r) \leq \int_0^{+\infty} \Lambda_{\bar{x}}(t)\ dt + \frac{1}{r} \leq 2\frac{n+\alpha-1}{n+\alpha}b_1|Du(\bar{x})| + \frac{1}{r}. \tag{3.8}$$

Noting that $|Du(\bar{x})| < 1$, then

$$\frac{\psi_2}{\psi_1}(r) \leq 2\frac{n+\alpha-1}{n+\alpha}b_1 + \frac{1}{r}. \tag{3.9}$$

By Lemma 2.13 in [14] and (3.7), we derive that

$$\begin{aligned}\psi_1(t) &\le \int_0^t e^{\int_0^s \tau\Lambda_{\bar{x}}(\tau)\ d\tau}\ ds \le te^{\int_0^\infty \tau\Lambda_{\bar{x}}(\tau)\ d\tau} \\ &= te^{\frac{n+\alpha-1}{n+\alpha}\int_0^\infty v\lambda(|d(o,\bar{x})-v|)\ dv} \le te^{\frac{n+\alpha-1}{n+\alpha}(2r_0b_1+b_0)},\end{aligned} \tag{3.10}$$

where $r_0 = \max\{d(o,x)|x \in \Omega\}$.

Letting $\psi(t) = \psi_2(t) + g(0)\psi_1(t)$, using (3.6), (3.7) and Lemma 2.5 in [6], we obtain

$$\frac{1}{n+\alpha}[\mathrm{tr}Q + \langle D\log w, \bar{\gamma}'\rangle] = \frac{\phi'}{\phi} \le \frac{\psi'}{\psi}, \quad \forall t \in (0,r).$$

Consequently,

$$\frac{d}{dt}\log[w(\bar{\gamma}(t))\det P(t)] = \mathrm{tr}Q(t) + \langle D\log w(\bar{\gamma}(t)), \bar{\gamma}'(t)\rangle \le (n+\alpha)\frac{\psi'}{\psi}. \tag{3.11}$$

Through (3.11), we can get

$$\begin{aligned}&w(\Phi_t(\bar{x}))|\det D\Phi_t(\bar{x})| = w(\Phi_t(\bar{x}))\det P(t) \\ &\le w(\bar{x})\Big(\psi_2(t) + \frac{1}{n+\alpha}[\Delta u(\bar{x}) + \langle D\log w(\bar{x}), Du(\bar{x})\rangle]\psi_1(t)\Big)^{n+\alpha}\end{aligned}$$

for all $t \in [0,r]$. This implies

$$w(\Phi_r(\bar{x}))|\det D\Phi_r(\bar{x})| \le w(\bar{x})\Big(\frac{\psi_2(r)}{\psi_1(r)} + g(0)\Big)^{n+\alpha}\psi_1^{n+\alpha}(r)$$

for any $\bar{x} \in A_r$. Using (3.9), (3.10) and Lemma 3.1, it follows that

$$\begin{aligned}&w(\Phi_r(\bar{x}))|\det D\Phi_r(\bar{x})| \\ &\le w(\bar{x})\Big(\frac{n+\alpha-1}{n+\alpha}2b_1 + \frac{1}{r} + \frac{1}{n+\alpha}[\Delta u(\bar{x}) + \langle D(\log w)(\bar{x}), Du(\bar{x})\rangle]\Big)^{n+\alpha} \\ &\quad\cdot r^{n+\alpha}e^{(n+\alpha-1)(2r_0b_1+b_0)} \\ &\le w(\bar{x})\Big(\frac{1}{r} + f^{\frac{1}{n+\alpha-1}}(\bar{x})\Big)^{n+\alpha} r^{n+\alpha}e^{(n+\alpha-1)(2r_0b_1+b_0)}\end{aligned} \tag{3.12}$$

for any $\bar{x} \in A_r$. Moreover, by (1.5), we obtain $h(t) \ge t$ and

$$\lim_{t\to\infty} h'(t) = 1 + \int_0^\infty h(s)\lambda(s)\ ds \ge 1 + \int_0^\infty s\lambda(s)\ ds = 1 + b_0. \tag{3.13}$$

Combining Lemma 3.2, (3.12) with the formula for change of variables in multiple integrals, we conclude that

$$\begin{aligned}
&\int_{\{q\in M:d(x,q)<r \text{ for all } x\in\Omega\}} w \ d\mathrm{vol}_g(x)\\
&\leq \int_{A_r} |\det D\Phi_r(\bar{x})| w(\Phi_r(\bar{x})) \ d\mathrm{vol}_g(\bar{x}) \qquad (3.14)\\
&\leq \int_{A_r} \left(\frac{1}{r} + f^{\frac{1}{n+\alpha-1}}(\bar{x})\right)^{n+\alpha} r^{n+\alpha} e^{(n+\alpha-1)(2r_0b_1+b_0)} w(\bar{x}) \ d\mathrm{vol}_g(\bar{x}).
\end{aligned}$$

Let $r > r_0$, the triangle inequality implies that

$$B_{r-r_0}(o) \subset \{q \in M : d(x,q) < r \text{ for all } x \in \Omega\} \subset B_{r+r_0}(o). \qquad (3.15)$$

Using $\lim\limits_{r\to\infty} \frac{\int_0^{r-r_0} h(t)^{n+\alpha-1} \ dt}{\int_0^r h(t)^{n+\alpha-1} \ dt} = \lim\limits_{r\to\infty} \frac{h(r-r_0)^{n+\alpha-1}}{h(r)^{n+\alpha-1}}$ by the L'Hospital's rule, and combining (1.6), (3.15) with Lemma 2.7 in [6], we have

$$\begin{aligned}
\mathcal{V}_\alpha &= \mathcal{V}_\alpha \lim_{r\to\infty} \frac{h(r-r_0)^{n+\alpha-1}}{h(r)^{n+\alpha-1}}\\
&= \lim_{r\to\infty} \frac{\int_{B_{r-r_0}(o)} w \ d\mathrm{vol}_g}{(n+\alpha)\int_0^{r-r_0} h(t)^{n+\alpha-1} \ dt} \frac{\int_0^{r-r_0} h(t)^{n+\alpha-1} \ dt}{\int_0^r h(t)^{n+\alpha-1} \ dt}\\
&\leq \lim_{r\to\infty} \frac{\int_{\{q\in M:d(x,q)<r \text{ for all } x\in\Omega\}} w \ d\mathrm{vol}_g}{(n+\alpha)\int_0^r h(t)^{n+\alpha-1} \ dt}\\
&\leq \lim_{r\to\infty} \frac{\int_{B_{r+r_0}(o)} w \ d\mathrm{vol}_g}{(n+\alpha)\int_0^{r+r_0} h(t)^{n-1} \ dt} \frac{\int_0^{r+r_0} h(t)^{n+\alpha-1} \ dt}{\int_0^r h(t)^{n+\alpha-1} \ dt}\\
&= \mathcal{V}_\alpha \lim_{r\to\infty} \frac{h(r+r_0)^{n+\alpha-1}}{h(r)^{n+\alpha-1}}\\
&= \mathcal{V}_\alpha,
\end{aligned}$$

which implies that

$$\mathcal{V}_\alpha = \lim_{r\to\infty} \frac{\int_{\{q\in M:d(x,q)<r \text{ for all } x\in\Omega\}} w \ d\mathrm{vol}_g}{(n+\alpha)\int_0^r h(t)^{n+\alpha-1} \ dt}. \qquad (3.16)$$

Dividing both side of (3.14) by $(n+\alpha)\int_0^r h(t)^{n+\alpha-1}dt$ and letting $r \to \infty$, using (3.13) and (3.16), one can find that

$$\begin{aligned}\mathcal{V}_\alpha &\leq e^{(n+\alpha-1)(2r_0b_1+b_0)}\int_\Omega wf^{\frac{n+\alpha}{n+\alpha-1}}\lim_{r\to\infty}\frac{r^{n+\alpha}}{(n+\alpha)\int_0^r h(t)^{n+\alpha-1}\,dt}\\ &= e^{(n+\alpha-1)(2r_0b_1+b_0)}\int_\Omega wf^{\frac{n+\alpha}{n+\alpha-1}}\lim_{r\to\infty}\frac{1}{h'(t)^{n+\alpha-1}}\\ &\leq \left(\frac{e^{2r_0b_1+b_0}}{1+b_0}\right)^{n+\alpha-1}\int_\Omega wf^{\frac{n+\alpha}{n+\alpha-1}}.\end{aligned}$$

Under our scaling assumption (3.1), we obtain

$$\begin{aligned}&\int_{\partial\Omega} wf + \int_\Omega w|Df| + (n+\alpha-1)2b_1\int_\Omega wf = (n+\alpha)\int_\Omega wf^{\frac{n+\alpha}{n+\alpha-1}}\\ &\geq (n+\alpha)\mathcal{V}_\alpha^{\frac{1}{n+\alpha}}\left(\frac{1+b_0}{e^{2r_0b_1+b_0}}\right)^{\frac{n+\alpha-1}{n+\alpha}}\left(\int_\Omega wf^{\frac{n+\alpha}{n+\alpha-1}}\right)^{\frac{n+\alpha-1}{n+\alpha}}.\end{aligned}$$

□

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