Real-Variable Characterizations and Their Applications of Matrix-Weighted Triebel–Lizorkin Spaces

Qi Wang, Dachun Yang* and Yangyang Zhang

Abstract Let $\alpha \in \mathbb{R}$, $q \in (0, \infty]$, $p \in (0, \infty)$, and W be an $A_p(\mathbb{R}^n, \mathbb{C}^m)$-matrix weight. In this article, the authors characterize the matrix-weighted Triebel–Lizorkin space $\dot{F}^{\alpha,q}_p(W)$ via the Peetre maximal function, the Lusin area function, and the Littlewood–Paley g^*_λ-function. As applications, the authors establish the boundedness of Fourier multipliers on matrix-weighted Triebel–Lizorkin spaces under the generalized Hörmander condition. The main novelty of these results exists in that their proofs need to fully use both the doubling property of matrix weights and the reducing operator associated to matrix weights, which are essentially different from those proofs of the corresponding cases of classical Triebel–Lizorkin spaces that strongly depend on the Fefferman–Stein vector-valued maximal inequality on Lebesgue spaces.

1 Introduction

Lizorkin [21, 22] and Triebel [36] independently started to investigate Triebel–Lizorkin spaces $F^{\alpha,q}_p(\mathbb{R}^n)$ from 1970s. Furthermore, we mention the contributions [25, 26, 27] of Peetre who extended the range of the admissible parameters p and q to values less than one. We refer the reader to [37, 38, 39, 31, 32] for more studies of these function spaces and their history.

On the other hand, the real-variable theory of both function spaces and the boundedness of operators related to matrix weights on $\mathbb{R}^n$ has received increasing interest in recent years. In 1997, to solve some significative problems related to the multivariate random stationary process and the Toeplitz operators (see, for instance, [35]), Treil and Volberg [34] introduced the Muckenhoupt $A_2(\mathbb{R}^n, \mathbb{C}^m)$-matrix weights and generalized the Hunt–Muckenhoupt–Wheeden theorem to the vector-valued case, while Nazarov and Treil [23] introduced Muckenhoupt $A_p(\mathbb{R}^n, \mathbb{C}^m)$-matrix weights for any $p \in (1, \infty)$ (see also Definition 2.5 below for its definition), and obtained the boundedness of the Hilbert transform on the matrix-weighted Lebesgue space $L^p(W)$, which was proved again by Volberg [41] via a new approach involving the classical Littlewood–Paley theory. In 2016, Cruz-Uribe et al. [9] applied the theory of A_p matrix weights on $\mathbb{R}^n$ to study degenerate Sobolev spaces. See also, for instance, [5, 7, 8, 10] for more studies on matrix-weighted function spaces and their applications. Later, Frazier and Roudenko [15] introduced the matrix-weighted

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*Corresponding author, E-mail: dcyang@bnu.edu.cn/July 18, 2022/Final version.

homogeneous Triebel–Lizorkin space $\dot{F}_p^{\alpha,q}(W)$ via the discrete Littlewood–Paley g-function with $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, and $q \in (0,\infty]$ (see also Definition 2.18 below for its definition). For any given $p \in (1,\infty)$, Frazier and Roudenko [15] proved that $L^p(W) = \dot{F}_p^{0,2}(W)$ and, for any $k \in \mathbb{N}$, $F_p^{k,2}(W)$ coincides with the matrix-weighted Sobolev space $L_k^p(W)$. Frazier and Roudenko [15] also showed that a vector-valued function $\vec{f}$ belongs to $\dot{F}_p^{\alpha,q}(W)$ if and only if its φ−transform coefficients belong to the sequence space $\dot{f}_p^{\alpha,q}(W)$. As an application of the above results, Frazier and Roudenko [15] obtained the boundedness of Calderón–Zygmund operators on $\dot{F}_p^{\alpha,q}(W)$. However, *no other real-variable characterizations* of these Triebel–Lizorkin spaces are known so far. The main purpose of this article is try to fill this gap.

Let $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$, and W be an $A_p(\mathbb{R}^n, \mathbb{C}^m)$-matrix weight. In this article, we first consider other real-variable characterizations of $\dot{F}_p^{\alpha,q}(W)$, including its characterizations via the Peetre maximal function, the Lusin area function, and the Littlewood–Paley g_λ^*−function, respectively, in Theorems 3.1, 3.11, and 3.14 below. We should point out that the main strategy used in [40, 24] to establish these real-variable characterizations of classical Triebel–Lizorkin spaces is based on a technique of the application of the Fefferman–Stein vector-valued maximal inequality. However, since the matrix-weighted Fefferman–Stein vector-valued maximal inequality is still unknown so far, it follows that the approach used in [40, 24] is no longer feasible for matrix-weighted Triebel–Lizorkin spaces. To overcome these obstacles, we borrow some ideas from [15] and introduce both the Peetre maximal function and the Littlewood–Paley function quasi-norms in terms of reducing operators associated to W [see (3.8) and (3.35) below]. Then the problem can be reduced to study the equivalence between the quasi-norms of Triebel–Lizorkin spaces in terms of reducing operators of W in Definition 2.20 below and the corresponding Peetre maximal function or the corresponding Littlewood–Paley g_λ^*-function quasi-norm, respectively, in (3.8) and (3.35) below, which allows us to use the Fefferman–Stein vector-valued maximal inequality in $L^p(\mathbb{R}^n)$ to solve the problem. As an application of the Littlewood–Paley characterization of $\dot{F}_p^{\alpha,q}(W)$, we obtain, in Theorem 4.8 below, the boundedness of Fourier multipliers on $\dot{F}_p^{\alpha,q}(W)$ under the assumption of the Hörmander condition [see (4.1) below].

To be precise, the remainder of this article is organized as follows.

In Section 2, we first recall some concepts concerning the matrix weight W, the $A_p(\mathbb{R}^n, \mathbb{C}^m)$-matrix weight condition, and the reducing operator of W. Then we recall some known properties and also give some new properties of both $A_p(\mathbb{R}^n, \mathbb{C}^m)$-matrix weights and reducing operators of matrix weights, which play a key role in the proof of the whole article.

In Section 3, we establish some real-variable characterizations of $\dot{F}_p^{\alpha,q}(W)$. We first characterize the matrix-weighted Triebel–Lizorkin space $\dot{F}_p^{\alpha,q}(W)$ for any $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$, and $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$ in terms of the Peetre maximal function (see Theorem 3.1 below). By introducing the Peetre maximal function with the reducing operator relating to the matrix weight, we obtain both the Lusin area function and the Littlewood–Paley g_λ^*-function characterizations of matrix-weighted Triebel–Lizorkin spaces (see Theorems 3.11 and 3.14 below).

In Section 4, we prove the boundedness of Fourier multipliers on $\dot{F}_p^{\alpha,q}(W)$ (see Theorem 4.8 below) under the assumption of the Hörmander condition [see (4.1) below for its definition], which is an application of the Littlewood–Paley characterization of $\dot{F}_p^{\alpha,q}(W)$ with $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$, and W being an $A_p(\mathbb{R}^n, \mathbb{C}^m)$-matrix weight.

Finally, we make some conventions on notation. We use the symbol $f \lesssim g$ to denote that there exists a positive constant C such that $f \le Cg$. The symbol $f \sim g$ is used as an abbreviation of

$f \lesssim g \lesssim f$. If $f \le Cg$ and $g = h$ or $g \le h$, we then write $f \lesssim g \sim h$ or $f \lesssim g \lesssim h$, rather than $f \lesssim g = h$ or $f \lesssim g \le h$. Let $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$, and $\mathbb{Z}_+^n := (\mathbb{Z}_+)^n$. For any multi-index $\gamma := (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}_+^n$ and any $x := (x_1, \dots, x_n) \in \mathbb{R}^n$, let $|\gamma| := \gamma_1 + \cdots + \gamma_n$, $x^\gamma := x_1^{\gamma_1} \cdots x_n^{\gamma_n}$, and $\partial^\gamma := (\frac{\partial}{\partial x_1})^{\gamma_1} \cdots (\frac{\partial}{\partial x_n})^{\gamma_n}$. For any index $p \in [1, \infty]$, we use p' to denote its *conjugate index*, namely, $\frac{1}{p} + \frac{1}{p'} = 1$. In addition, for any measurable set $F \subset \mathbb{R}^n$, we denote by $\mathbf{1}_F$ its *characteristic function*. We use the notation $\langle f, g \rangle$ to denote a pairing which is linear in f and g; when this pairing is between a distribution f and a test function g, then $\langle f, g \rangle = f(g)$. We also use the notation $(\vec{x}, \vec{y})$ to denote the inner product of $\vec{x}, \vec{y} \in \mathbb{C}^m$. For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and any measurable set $E \subset \mathbb{R}^n$, let

$$\fint_E f(x)\, dx := \frac{1}{|E|} \int_E f(x)\, dx.$$

For any $s \in \mathbb{R}$, we use the symbol $\lfloor s \rfloor$ to denote the *largest integer not greater than s*. For any measurable function g and any $x \in \mathbb{R}^n$, let $\widetilde{g}(x) := g(-x)$. For any $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, let $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$ be the ball with center x and radius r. Furthermore, for any $a \in (0, \infty)$ and any ball $B := B(x_B, r_B)$ in $\mathbb{R}^n$ with $x_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$, let $aB := B(x_B, ar_B)$. We also use $\mathbf{0}$ to denote the origin of $\mathbb{R}^n$.

Let $\mathcal{S}(\mathbb{R}^n)$ be the *space of all Schwartz functions* on $\mathbb{R}^n$, equipped with the classical topology determined by a well-known countable family of norms, and $\mathcal{S}'(\mathbb{R}^n)$ its *topological dual space* [namely, the set of all continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$], equipped with the weak $*$-topology. Following Triebel, we let

$$\mathcal{S}_\infty(\mathbb{R}^n) := \left\{\varphi \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \varphi(x) x^\gamma\, dx = 0 \text{ for all multi-indices } \gamma \in \mathbb{Z}_+^n\right\}$$

and consider $\mathcal{S}_\infty(\mathbb{R}^n)$ as a subspace of $\mathcal{S}(\mathbb{R}^n)$, including its topology. Use $\mathcal{S}'_\infty(\mathbb{R}^n)$ to denote the *topological dual space* of $\mathcal{S}_\infty(\mathbb{R}^n)$, namely, the set of all continuous linear functionals on $\mathcal{S}_\infty(\mathbb{R}^n)$. We also equip $\mathcal{S}'_\infty(\mathbb{R}^n)$ with the weak $*$-topology. Let $\mathcal{P}(\mathbb{R}^n)$ be the *set of all polynomials* on $\mathbb{R}^n$. It is well known that $\mathcal{S}'_\infty(\mathbb{R}^n) = \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ as topological spaces; see, for instance, [18, 30].

2 Matrix-Weighted Triebel–Lizorkin Spaces $\dot{F}^{\alpha,q}_p(W)$

In this section, we present some basic definitions and results of matrix-weighted Triebel–Lizorkin spaces via two subsections. In Subsection 2.1, we recall the concepts of the matrix weight, the $A_p(\mathbb{R}^n, \mathbb{C}^m)$-matrix weight, and their properties. In Subsection 2.2, we present both the definition and also some basic properties of matrix-weighted Triebel–Lizorkin spaces.

2.1 $A_p(\mathbb{R}^n, \mathbb{C}^m)$-Matrix Weights

In this section, we recall the concepts of the matrix weight, the matrix $A_p(\mathbb{R}^n, \mathbb{C}^m)$ condition, and the reducing operator of W. Furthermore, we present their basic properties. We begin with recalling the concept of the matrix weight (see, for instance, [29, 34]). In what follows, for any $\vec{z} := (z_1, \dots, z_m)^T \in \mathbb{C}^m$, let $|\vec{z}| := (\sum_{j=1}^m |z_j|^2)^{1/2}$, where T denotes the *transpose* of the row vector.

Definition 2.1. Let $m \in \mathbb{N}$. An $m \times m$ complex-valued matrix A is said to be *nonnegative definite* if, for any $\vec{z} \in \mathbb{C}^m \setminus \{\mathbf{0}\}$, $(A\vec{z}, \vec{z}) \ge 0$. An $m \times m$ complex-valued matrix A is said to be *positive definite* if, for any $\vec{z} \in \mathbb{C}^m \setminus \{\mathbf{0}\}$, $(A\vec{z}, \vec{z}) > 0$. The set of all nonnegative definite $m \times m$ complex-valued matrices is denoted by $M_m(\mathbb{C})$. Furthermore, the *operator norm of a matrix* A is defined by setting

$$\|A\| := \sup_{\vec{z} \in \mathbb{C}^m \setminus \{\mathbf{0}\}} \frac{|A\vec{z}|}{|\vec{z}|}.$$

Definition 2.2. Let $m \in \mathbb{N}$ and $W : \mathbb{R}^n \to M_m(\mathbb{C})$ satisfy that every entry of W is a measurable function. The map W is called a *matrix weight* from $\mathbb{R}^n$ to $M_m(\mathbb{C})$ if $W(x)$ is invertible for almost every $x \in \mathbb{R}^n$.

The following definition is a part of [19, Definition 1.2].

Definition 2.3. Let $m \in \mathbb{N}$ and A be a positive definite $m \times m$ complex-valued matrix satisfying that there exists an invertible $m \times m$ complex-valued matrix P and a diagonal matrix $\operatorname{diag}(\lambda_1, \ldots, \lambda_m)$, with $\{\lambda_1, \ldots, \lambda_m\} \subset \mathbb{R}_+$, such that $A = P \operatorname{diag}(\lambda_1, \ldots, \lambda_m) P^{-1}$. Then, for any $\alpha \in \mathbb{R}$, let

$$A^\alpha := P \operatorname{diag}(\lambda_1^\alpha, \ldots, \lambda_m^\alpha) P^{-1}.$$

Remark 2.4. Let A be a positive definite $m \times m$ complex-valued matrix in Definition 2.3. By [20, Theorem 4.1.5], we find that the decomposition of A in Definition 2.3 exists. Furthermore, from [19, Problem 1.1] (see also [20, p. 407]), we deduce that, for any $\alpha \in (0, \infty)$, A^α in Definition 2.3 is independent of the choice of the invertible $m \times m$ complex-valued matrix P and the diagonal matrix $\operatorname{diag}(\lambda_1, \ldots, \lambda_m)$ with $\{\lambda_1, \ldots, \lambda_m\} \subset \mathbb{C}$.

In what follows, let $\mathcal{Q} := \{\text{all cubes } Q \subset \mathbb{R}^n\}$, here and thereafter, a *cube* means its edges parallel to the coordinate axis with a finite and positive edge length which is not necessary to be open. The following definition is just [28, Definition 3.2] and [14, p. 1226, (1.1)].

Definition 2.5. Let $m \in \mathbb{N}$ and $p \in (0, \infty)$. An $A_p(\mathbb{R}^n, \mathbb{C}^m)$*-matrix weight* W, denoted by $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$, is a matrix weight from $\mathbb{R}^n$ to $M_m(\mathbb{C})$ satisfying that, when $p \in (1, \infty)$,

$$\sup_{Q \in \mathcal{Q}} \frac{1}{|Q|} \int_Q \left[\frac{1}{|Q|} \int_Q \left\| W^{1/p}(x) W^{-1/p}(y) \right\|^{p'} dy \right]^{p/p'} dx < \infty,$$

where $\|\cdot\|$ denotes the operator norm of a matrix and, when $p \in (0, 1]$,

$$\sup_{Q \in \mathcal{Q}} \operatorname*{ess\,sup}_{y \in Q} \frac{1}{|Q|} \int_Q \left\| W^{1/p}(x) W^{-1/p}(y) \right\|^p dx < \infty.$$

Remark 2.6. When $p \in [1, \infty)$ and $m = 1$, the $A_p(\mathbb{R}^n, \mathbb{C}^m)$-matrix weight in Definition 2.5 coincides with the classical $A_p(\mathbb{R}^n)$-weight (see Definition 2.8 below for its definition).

The following result about the matrix weight is just [29, Corollary 3.3].

Lemma 2.7. *Let* W *be a matrix weight from* $\mathbb{R}^n$ *to* $M_m(\mathbb{C})$*,* $p \in (1, \infty)$*, and* $p' := p/(p-1)$*. Then the following statements are equivalent:*

(i) $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$*;*

(ii) $W^{-p'/p} \in A_{p'}(\mathbb{R}^n, \mathbb{C}^m)$.

Now, we recall the concept of the classical $A_p(\mathbb{R}^n)$-weight (see, for instance, [17]).

Definition 2.8. An $A_p(\mathbb{R}^n)$-*weight* ω, with $p \in [1,\infty)$, is a locally integrable and nonnegative function on $\mathbb{R}^n$ satisfying that, when $p \in (1,\infty)$,

$$\sup_{Q\in\mathcal{Q}}\left[\frac{1}{|Q|}\int_Q \omega(x)\,dx\right]\left[\frac{1}{|Q|}\int_Q \{\omega(x)\}^{\frac{1}{1-p}}\,dx\right]^{p-1} < \infty$$

and, when $p = 1$,

$$\sup_{Q\in\mathcal{Q}}\frac{1}{|Q|}\int_Q \omega(x)\,dx\left[\left\|\omega^{-1}\right\|_{L^\infty(Q)}\right] < \infty.$$

Define $A_\infty(\mathbb{R}^n) := \bigcup_{p\in[1,\infty)} A_p(\mathbb{R}^n)$.

From both [16, Corollary 2.2] and [14, Lemma 2.1], we deduce the following lemma; we omit the details here.

Lemma 2.9. *Let* $p \in (0,\infty)$, $W \in A_p(\mathbb{R}^n,\mathbb{C}^m)$, *and* $w_{\vec{y}}(x) := |W^{1/p}(x)\vec{y}|^p$ *for any* $x \in \mathbb{R}^n$ *and any given* $\vec{y} \in \mathbb{C}^m$. *Then, for any given* $\vec{y} \in \mathbb{C}^m \setminus \{\mathbf{0}\}$, $w_{\vec{y}} \in A_1(\mathbb{R}^n)$ *if* $p \in (0,1]$, *and* $w_{\vec{y}} \in A_p(\mathbb{R}^n)$ *if* $p \in (1,\infty)$.

If $p \in (1,\infty)$, the following corollary is just [16, Corollary 2.3]. For the convenience of the reader, we present some details of its proof.

Corollary 2.10. *Let* $p \in (0,\infty)$ *and* $W \in A_p(\mathbb{R}^n,\mathbb{C}^m)$. *Then* $\|W^{1/p}\|^p \in A_1(\mathbb{R}^n)$ *if* $p \in (0,1]$, *and* $\|W^{1/p}\|^p \in A_p(\mathbb{R}^n)$ *if* $p \in (1,\infty)$.

Proof. By [29, Lemma 3.2], we conclude that, for any given $p \in (0,\infty)$ and for any $x \in \mathbb{R}^n$,

$$\left\|W^{1/p}(x)\right\|^p \sim \sum_{i=1}^m \left|W^{1/p}(x)\vec{e}_i\right|^p,$$

where $\{\vec{e}_1, \dots, \vec{e}_m\}$ is the standard unit basis of $\mathbb{C}^m$. Then, by Lemma 2.9, we conclude that, for any $i \in \{1, \dots, m\}$, $|W^{1/p}\vec{e}_i|^p$ is an $A_1(\mathbb{R}^n)$-weight if $p \in (0,1]$, and $|W^{1/p}\vec{e}_i|^p$ is an $A_p(\mathbb{R}^n)$-weight if $p \in (1,\infty)$, therefore, their finite sum is as well. This finishes the proof of Corollary 2.10. □

The following definition comes from [14, p. 1230].

Definition 2.11. Let $p \in (0,\infty)$. A non-zero matrix weight W is called *a doubling matrix weight of order* p if there exists a positive constant C such that, for any cube $Q \subset \mathbb{R}^n$ and any $\vec{z} \in \mathbb{C}^m$,

$$\int_{2Q}\left|W^{1/p}(x)\vec{z}\right|^p\,dx \le C\int_Q \left|W^{1/p}(x)\vec{z}\right|^p\,dx, \tag{2.1}$$

where $2Q$ denotes the cube concentric with Q and having twice the edge length of Q. Let

$$\beta := \min\left\{\beta \in (0,\infty): \quad (2.1) \text{ holds with } C = 2^\beta\right\}.$$

Then β is called the *doubling exponent of the doubling matrix weight* W *of order* p. For simplicity, such a β is also called the *doubling exponent of* W.

The following lemma can be found in [15, Lemma 2.2].

Lemma 2.12. *Let $p \in (0,\infty)$ and $W \in A_p(\mathbb{R}^n,\mathbb{C}^m)$. Then W is a doubling matrix weight of order p.*

Remark 2.13. It is easy to see that, if we replace any cube Q with any ball $B \subset \mathbb{R}^n$ in Definitions 2.5 and 2.11, then Lemmas 2.7 and 2.12 still hold true.

In what follows, for any $j \in \mathbb{Z}$ and $k := (k_1, \ldots, k_n) \in \mathbb{Z}^n$, let $Q_{jk} := \prod_{i=1}^n 2^{-j}[k_i, k_i + 1)$, $\mathcal{D} := \{Q_{jk} : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$, and

$$\mathcal{D}_j := \{Q_{jk} : k \in \mathbb{Z}^n\}. \tag{2.2}$$

Definition 2.14. Let $m \in \mathbb{N}$, $p \in (0,\infty)$, and W be a matrix weight from $\mathbb{R}^n$ to $M_m(\mathbb{C})$. A sequence $\{A_Q^{(W)}\}_{Q\in\mathcal{D}}$ of positive definite $m \times m$ matrices is called *a sequence of reducing operators of order p for W* if there exist positive constants C_1 and C_2 such that, for any $\vec{z} \in \mathbb{C}^m$ and $Q \in \mathcal{D}$,

$$C_1 \left|A_Q^{(W)}\vec{z}\right| \le \left[\frac{1}{|Q|}\int_Q \left|W^{1/p}(x)\vec{z}\right|^p dx\right]^{1/p} \le C_2 \left|A_Q^{(W)}\vec{z}\right|.$$

For simplicity, the sequence $\{A_Q^{(W)}\}_{Q\in\mathcal{D}}$ of reducing operators of order p for W is denoted by $\{A_Q\}_{Q\in\mathcal{D}}$.

Remark 2.15. Let $m \in \mathbb{N}$. From [16, Proposition 1.2] and [14, p. 1237], we deduce that, for any $p \in (0,\infty)$ and any matrix weight W from $\mathbb{R}^n$ to $M_m(\mathbb{C})$, a sequence of reducing operators of order p for W in Definition 2.14 exists.

The following lemmas are respectively a part of [15, Lemmas 3.2 and 3.3].

Lemma 2.16. *Let $p \in (1,\infty)$, $p' := p/(p-1)$, $W \in A_p(\mathbb{R}^n,\mathbb{C}^m)$, and $\{A_Q\}_{Q\in\mathcal{D}}$ be a sequence of reducing operators of order p for W. Then there exists a $\delta(W) \in (0,\infty)$ such that, for any $\eta \in (0, p' + \delta(W))$,*

$$\sup_{Q\in\mathcal{D}} \frac{1}{|Q|}\int_Q \left\|A_Q W^{-1/p}(x)\right\|^\eta dx < \infty.$$

Lemma 2.17. *Let $p \in (0,1]$, $W \in A_p(\mathbb{R}^n,\mathbb{C}^m)$, and $\{A_Q\}_{Q\in\mathcal{D}}$ be a sequence of reducing operators of order p for W. Then*

$$\sup_{Q\in\mathcal{D}} \operatorname*{ess\,sup}_{x\in Q} \left\|A_Q W^{-1/p}(x)\right\| < \infty.$$

2.2 Matrix-Weighted Triebel–Lizorkin Spaces

In this section, we begin with recalling the concepts of both matrix-weighted Triebel–Lizorkin spaces and sequence matrix-weighted Triebel–Lizorkin spaces. Then we prove the rationality of Definitions 2.18 and 2.20.

In what follows, for any $m \in \mathbb{N}$, let

$$[\mathcal{S}'_\infty(\mathbb{R}^n)]^m := \left\{\vec{f} := (f_1, \ldots, f_m)^T : \text{ for any } i \in \{1, \ldots, m\},\ f_i \in \mathcal{S}'_\infty(\mathbb{R}^n)\right\}.$$

For any $\vec{f} := (f_1, \ldots, f_m)^T \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$ and $\varphi \in \mathcal{S}_\infty(\mathbb{R}^n)$, let

$$\varphi * \vec{f} := (\varphi * f_1, \ldots, \varphi * f_m)^T$$

and $\varphi_j(\cdot) := 2^{jn}\varphi(2^j\cdot)$ with $j \in \mathbb{Z}$. For any $\phi \in \mathcal{S}(\mathbb{R}^n)$, $\widehat{\phi}$ denotes its *Fourier transform* which is defined by setting, for any $\xi \in \mathbb{R}^n$,

$$\widehat{\phi}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} \phi(x) e^{-ix\xi}\, dx.$$

For any $f \in \mathcal{S}'(\mathbb{R}^n)$, $\widehat{f}$ is defined by setting, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\langle \widehat{f}, \varphi\rangle := \langle f, \widehat{\varphi}\rangle$; also, for any $f \in \mathcal{S}(\mathbb{R}^n)$ [resp., $\mathcal{S}'(\mathbb{R}^n)$], $f^\vee$ denotes its *inverse Fourier transform*,

$$f^\vee(\cdot) := (2\pi)^{n/2} \int_{\mathbb{R}^n} \widehat{f}(x) e^{ix\cdot}\, dx$$

[resp., $\langle f^\vee, \varphi\rangle := \langle f, \varphi^\vee\rangle$ for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$]. For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\operatorname{supp}\widehat{\varphi} := \{x \in \mathbb{R}^n : \widehat{\varphi}(x) \neq 0\}$ and, for any $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$\operatorname{supp} f := \cap \left\{\text{closed set } K \subset \mathbb{R}^n : \langle f, \varphi\rangle = 0 \text{ if } \varphi \in \mathcal{S}(\mathbb{R}^n) \text{ and } \operatorname{supp}\varphi \subset \mathbb{R}^n \setminus K\right\},$$

which can be found in [17, Definition 2.3.16].

Definition 2.18. Let $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $p \in (0, \infty)$, $q \in (0, \infty]$, W be a matrix weight from $\mathbb{R}^n$ to $M_m(\mathbb{C})$, and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Furthermore, assume that

(T1) for any $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, there exists an $l \in \mathbb{Z}$ such that $\widehat{\varphi}(2^l x) \neq 0$,

(T2) $\overline{\operatorname{supp}\widehat{\varphi}} \subset \{x \in \mathbb{R}^n : |x| < \pi\}$ is bounded away from the origin.

Then the *matrix-weighted Triebel–Lizorkin space* $\dot{F}^{\alpha,q}_{p,\varphi}(W)$ is defined by setting

$$\dot{F}^{\alpha,q}_{p,\varphi}(W) := \left\{\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m : \ \left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_{p,\varphi}(W)} < \infty\right\},$$

where

$$\left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_{p,\varphi}(W)} := \left\|\left[\sum_{j\in\mathbb{Z}} \left|2^{j\alpha} W^{1/p} \left(\varphi_j * \vec{f}\right)\right|^q\right]^{1/q}\right\|_{L^p(\mathbb{R}^n)}$$

with suitable modification made when $q = \infty$.

Remark 2.19. (i) Observe that, if $\widehat{\varphi}(x) > 0$ for any $\{x \in \mathbb{R}^n : \epsilon \leq |x| \leq \pi - b\}$, where $b \in (0, \pi - 1]$ and $\epsilon \in (0, (\pi - b)/2)$, then φ automatically satisfies that, for any $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, there exists an $\ell \in \mathbb{Z}$ such that $\widehat{\varphi}(2^\ell x) \neq 0$ and hence, in this case, the assumption (T1) in Definition 2.18 is superfluous; see [42, Lemma 3.18 and Remark 3.19] for the details.

(ii) Let $\alpha \in \mathbb{R}$, $p \in (0, \infty)$, and $q \in (0, \infty]$ be the same as in Definition 2.18. If $m = 1$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies both

$$\operatorname{supp}\widehat{\varphi} \subset \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\} \tag{2.3}$$

and

$$\left|\widehat{\varphi}(\xi)\right| \geq c > 0 \tag{2.4}$$

when $3/5 \leq |\xi| \leq 5/3$ with c being a positive constant independent of ξ, then $\dot{F}^{\alpha,q}_{p,\varphi}(W)$ in Definition 2.18 is independent of the choice of φ and it coincides with the weighted Triebel–Lizorkin space $\dot{F}^{\alpha,q}_{p}(\omega)$. For more details on weighted Triebel–Lizorkin spaces, we refer the reader to [2, 4, 46].

(iii) Let $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, and $q \in (0,\infty]$ be the same as in Definition 2.18. If $m = 1$, $W = 1$, and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies both (2.3) and (2.4), then $\dot{F}^{\alpha,q}_{p,\varphi}(W)$ in Definition 2.18 is independent of the choice of φ (see, for instance, [13, Remark 2.6]) and it coincides with the Triebel–Lizorkin space $\dot{F}^{\alpha,q}_{p}$ in [13, p. 46].

Definition 2.20. Let $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$, $\{A_Q\}_{Q\in\mathcal{D}}$ be a sequence of $m \times m$ nonnegative definite matrices, and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy both (T1) and (T2) of Definition 2.18. The *$\{A_Q\}$-Triebel–Lizorkin space* $\dot{F}^{\alpha,q}_{p,\varphi}(\{A_Q\})$ is defined by setting

$$\dot{F}^{\alpha,q}_{p,\varphi}(\{A_Q\}) := \left\{\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m : \ \left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_{p,\varphi}(\{A_Q\})} < \infty\right\},$$

where

$$\left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_{p,\varphi}(\{A_Q\})} := \left\|\left[\sum_{j\in\mathbb{Z}}\sum_{Q\in\mathcal{D}_j}\left(2^{j\alpha}\left|A_Q\varphi_j * \vec{f}\right|\mathbf{1}_Q\right)^q\right]^{1/q}\right\|_{L^p(\mathbb{R}^n)}$$

with suitable modification made when $q = \infty$.

Remark 2.21. Let $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$, and $\{A_Q\}_{Q\in\mathcal{D}}$ be a sequence of nonnegative definite matrices in Definition 2.20. If $m = 1$, $A_Q = 1$ for any $Q \in \mathcal{D}$, and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies both (2.3) and (2.4), then $\dot{F}^{\alpha,q}_{p,\varphi}(\{A_Q\})$ in Definition 2.20 is independent of the choice of φ (see, for instance, [13, Remark 2.6]) and it coincides with the Triebel–Lizorkin space $\dot{F}^{\alpha,q}_{p}$ in [13, p. 46].

To prove the rationality of Definitions 2.18 and 2.20, we first recall some classical results (Lemmas 2.22, 2.23, and 2.24), which are just [42, Lemma 3.18], [44, Lemma 2.1] (see also [12, Lemma 2.1]), and [43, Lemma 2.1], respectively.

Lemma 2.22. *Let φ be a Schwartz function satisfying that, for any $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, there exists an $l \in \mathbb{Z}$ such that $\widehat{\varphi}(2^l x) \neq 0$. Then there exists a $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{\psi} \in C^\infty_c(\mathbb{R}^n)$ with its support away from origin, $\widehat{\varphi}\widehat{\psi} \geq 0$, and*

$$\sum_{j\in\mathbb{Z}} \widehat{\varphi}(2^{-j}x)\widehat{\psi}(2^{-j}x) = 1 \tag{2.5}$$

for any $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.

In what follows, for any $Q \in \mathcal{D}$, let $\ell(Q)$ denote its *edge length* and x_Q its *lower left corner*.

Lemma 2.23. *Let φ and ψ be the Schwartz functions such that $\overline{\operatorname{supp}\widehat{\varphi}}, \overline{\operatorname{supp}\widehat{\psi}} \subset \{x \in \mathbb{R}^n : |x| < \pi\}$ are bounded away from the origin and, furthermore, for any $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$,*

$$\sum_{j\in\mathbb{Z}} \widehat{\varphi}(2^{-j}x)\widehat{\psi}(2^{-j}x) = 1.$$

Then, for any $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$,

$$f = \sum_{j\in\mathbb{Z}} 2^{-jn} \sum_{k\in\mathbb{Z}^n} \varphi_j * f(2^{-j}k)\psi_j(\cdot - 2^{-j}k)$$

converges in $\mathcal{S}'_\infty(\mathbb{R}^n)$.

Lemma 2.24. *Let φ and ψ be the Schwartz functions satisfying* (2.5) *and that both* $\operatorname{supp}\widehat{\varphi}, \operatorname{supp}\widehat{\psi}$ *are compact and bounded away from the origin. Then, for any $f \in \mathcal{S}_\infty(\mathbb{R}^n)$,*

$$f = \sum_{j\in\mathbb{Z}} \varphi_j * \psi_j * f \tag{2.6}$$

holds true in $\mathcal{S}_\infty(\mathbb{R}^n)$. Moreover, for any $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$, (2.6) *also holds true in $\mathcal{S}'_\infty(\mathbb{R}^n)$.*

Lemma 2.25. *Let $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$, $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$, $\{A_Q^{(W)}\}_{Q\in\mathcal{D}}$ be a sequence of reducing operators of order p for W, and $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy both (T1) and (T2) of Definition 2.18. Then, for any $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$,*

$$\left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_{p,\varphi}(W)} \sim \left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_{p,\psi}(W)}$$

and

$$\left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_{p,\varphi}(\{A_Q\})} \sim \left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_{p,\psi}(\{A_Q\})},$$

where the positive equivalence constants are independent of $\vec{f}$.

Proof. Let φ and ψ be the same as in Definitions 2.18 and 2.20. By Lemma 2.22, we find that there exist $\gamma, \eta \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\widehat{\gamma}, \widehat{\eta} \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ with their supports away from the origin and (2.5) via φ and ψ replaced, respectively, by φ and γ or by ψ and η. Using this and Lemma 2.23, and repeating the proof of [15, Theorems 1.1 and 2.3], we then finish the proof of Lemma 2.25. □

For simplicity, the matrix-weighted Triebel–Lizorkin space is denoted by $\dot{F}^{\alpha,q}_p(W)$ and the $\{A_Q\}$-Triebel–Lizorkin space is denoted by $\dot{F}^{\alpha,q}_p(\{A_Q\})$. By a proof similar to that used in [15, Theorem 1.1], we conclude the following lemma; we omit the details here.

Lemma 2.26. *Let $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$, $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$, $\{A_Q\}_{Q\in\mathcal{D}}$ be a sequence of reducing operators of order p for W, and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy both (T1) and (T2) of Definition 2.18. Then, for any $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$,*

$$\|\vec{f}\|_{\dot{F}^{\alpha,q}_p(W)} \sim \|\vec{f}\|_{\dot{F}^{\alpha,q}_p(\{A_Q\})},$$

where the positive equivalence constants are independent of $\vec{f}$.

3 Real-Variable Characterizations of $\dot{F}^{\alpha,q}_p(W)$

In this section, we characterize the spaces $\dot{F}^{\alpha,q}_p(W)$ via the Peetre maximal function, the Lusin area function, and the Littlewood–Paley g^*_λ−function.

Now, with a modification of the classical Peetre-type maximal function in [26], we introduce the concept of the following *matrix-weighted Peetre-type maximal function*. Let $p \in (0,\infty)$, $m \in \mathbb{N}$, $W \in A_p(\mathbb{R}^n,\mathbb{C}^m)$, $\varphi \in \mathcal{S}_\infty(\mathbb{R}^n)$, and $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$. For any given $j \in \mathbb{Z}$ and $a \in (0,\infty)$, and for any $x \in \mathbb{R}^n$, let

$$(3.1)\qquad \left(\varphi^*_j \vec{f}\right)^{(W,p)}_a (x) := \sup_{y\in\mathbb{R}^n} \frac{|W^{1/p}(x)(\varphi_j * \vec{f})(y)|}{(1+2^j|x-y|)^a}.$$

Theorem 3.1. *Let $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$, $W \in A_p(\mathbb{R}^n,\mathbb{C}^m)$, and*

$$a \in (n/\min\{1,p,q\} + \beta/p, \infty),$$

where β is the doubling exponent of W. Assume that $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies both (T1) and (T2) of Definition 2.18. Then $\vec{f} \in \dot{F}^{\alpha,q}_p(W)$ if and only if $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$ and $\|\vec{f}\|^\star_{\dot{F}^{\alpha,q}_p(W)} < \infty$, where

$$(3.2)\qquad \left\|\vec{f}\right\|^\star_{\dot{F}^{\alpha,q}_p(W)} := \left\| \left\{ \sum_{j\in\mathbb{Z}} 2^{j\alpha q} \left[\left(\varphi^*_j \vec{f}\right)^{(W,p)}_a \right]^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)}$$

with usual modification made when $q = \infty$. Moreover, for any $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$,

$$(3.3)\qquad \left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(W)} \sim \left\|\vec{f}\right\|^\star_{\dot{F}^{\alpha,q}_p(W)},$$

where the positive equivalence constants are independent of $\vec{f}$.

Remark 3.2. (i) Theorem 3.1 when $m = 1$ is a part of [3, Theorem 3.1].

(ii) Theorem 3.1 when $m = 1$ and $W = 1$ is a part of [26, Theorem 3.1] which is the Peetre maximal function characterization of Triebel–Lizorkin spaces.

To show Theorem 3.1, we first recall the definitions of both strongly doubling and weakly doubling matrices, which can be found in [15, Definition 2.1].

Definition 3.3. Let $\{A_Q\}_{Q\in\mathcal{D}}$ be a sequence of positive definite matrices, $\beta \in (0,\infty)$, and $p \in (0,\infty)$. The sequence $\{A_Q\}_{Q\in\mathcal{D}}$ is said to be *strongly doubling of order* (β,p) if there exists a positive constant C such that, for any $Q, P \in \mathcal{D}$,

$$(3.4)\qquad \left\|A_Q A_P^{-1}\right\|^p \le C \max\left\{ \left[\frac{\ell(P)}{\ell(Q)}\right]^n, \left[\frac{\ell(Q)}{\ell(P)}\right]^{\beta-n} \right\} \left[1 + \frac{|x_Q - x_P|}{\max\{\ell(P),\ell(Q)\}}\right]^\beta.$$

The sequence $\{A_Q\}_{Q\in\mathcal{D}}$ is said to be *weakly doubling of order* $r \in (0,\infty)$ if there exists a positive constant C such that, for any $k, \ell \in \mathbb{Z}^n$ and $j \in \mathbb{Z}$,

$$(3.5)\qquad \left\|A_{Q_{jk}} A^{-1}_{Q_{j\ell}}\right\| \le C(1+|k-\ell|)^r,$$

where $Q_{jk} := \prod_{i=1}^n 2^{-j}[k_i, k_i+1)$ for any $j \in \mathbb{Z}$ and $k := (k_1, \dots, k_n) \in \mathbb{Z}^n$.

Remark 3.4. In Definition 3.3, a strongly doubling sequence of order (β, p) satisfying (3.4) is also weakly doubling of order $r := \beta/p$ satisfying (3.5) because, when $\ell(P) = \ell(Q)$, (3.5) coincides with (3.4).

The following lemma explains the connection between the doubling weight W and the doubling sequence $\{A_Q\}_{Q\in\mathcal{D}}$, which can be deduced from [15, Lemma 2.2], Lemma 2.12, and Remark 3.4; we omit the details.

Lemma 3.5. *Let $p \in (0,\infty)$, $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$, and $\{A_Q\}_{Q\in\mathcal{D}}$ be a sequence of reducing operators of order p for W. Then $\{A_Q\}_{Q\in\mathcal{D}}$ is weakly doubling of order $\frac{\beta}{p}$, where β is the doubling exponent of W.*

The following lemma is just [15, (2.8)].

Lemma 3.6. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (T2) of Definition 2.18. Suppose that $\{A_Q\}_{Q\in\mathcal{D}}$ is a weakly doubling sequence of order $r \in (0,\infty)$ of positive definite matrices. Then, for any given $A \in (0,1]$ and $R \in (0,\infty)$, there exists a positive constant C, depending on both A and R, such that, for any $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, and $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$,*

$$\begin{aligned}
&\sup_{x\in Q_{jk}} \left|A_{Q_{jk}}\left(\varphi_j * \vec{f}\right)(x)\right|^A \\
&\quad \le C \sum_{\ell\in\mathbb{Z}^n} (1+|k-\ell|)^{-A(R-r)}\, 2^{jn} \int_{Q_{j\ell}} \left|A_{Q_{j\ell}}\varphi_j * \vec{f}(s)\right|^A \, ds,
\end{aligned}$$

where $Q_{jk} := \prod_{i=1}^n 2^{-j}[k_i, k_i+1)$ for any $j \in \mathbb{Z}$ and $k := (k_1, \ldots, k_n) \in \mathbb{Z}^n$.

Recall that the *Hardy–Littlewood maximal operator* $\mathcal{M}$ is defined by setting, for any locally integrable function f and any $x \in \mathbb{R}^n$,

$$\mathcal{M}(f)(x) := \sup_{x\in B} \frac{1}{|B|} \int_B |f(y)|\, dy = \sup_{x\in B} ⨍_B |f(y)|\, dy, \tag{3.6}$$

where the supremum is taken over all the balls B of $\mathbb{R}^n$ containing x. Denote by the *symbol* $\mathscr{M}(\mathbb{R}^n)$ the set of all the complex-valued measurable functions on $\mathbb{R}^n$.

Lemma 3.7. *Let $\mathcal{M}$ be the maximal operator in* (3.6) *and $\eta > n$. Then there exists a positive constant C such that, for any $j \in \mathbb{Z}$ and $h \in \mathscr{M}(\mathbb{R}^n)$,*

$$\sum_{k\in\mathbb{Z}^n}\sum_{\ell\in\mathbb{Z}^n} (1+|k-\ell|)^{-\eta} 2^{jn} \int_{Q_{j\ell}} |h(s)|\, ds \mathbf{1}_{Q_{jk}} \le C\mathcal{M}(h).$$

Proof. Observe that, for any given $j \in \mathbb{Z}$ and any $x \in \mathbb{R}^n$, it is easy to see that there exists a unique $k_x \in \mathbb{Z}^n$ such that $x \in Q_{jk_x}$. Using this, we find that, for any $x \in \mathbb{R}^n$,

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^n}\sum_{\ell\in\mathbb{Z}^n} (1+|k-\ell|)^{-\eta} 2^{jn} \int_{Q_{j\ell}} |h(s)|\, ds \mathbf{1}_{Q_{jk}}(x) \\
&\quad = \sum_{\ell\in\mathbb{Z}^n} (1+|k_x-\ell|)^{-\eta} 2^{jn} \int_{Q_{j\ell}} |h(s)|\, ds
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\{\ell\in\mathbb{Z}^n:\,|\ell-k_x|\le 1\}} (1+|k_x-\ell|)^{-\eta} 2^{jn} \int_{Q_{j\ell}} |h(s)|\,ds + \sum_{m\in\mathbb{N}} \sum_{\{\ell\in\mathbb{Z}^n:\,2^{m-1}<|\ell-k_x|\le 2^m\}} \cdots \\
&\lesssim \sum_{\{\ell\in\mathbb{Z}^n:\,|\ell-k_x|\le 1\}} \int_{Q_{j\ell}} |h(s)|\,ds + \sum_{m\in\mathbb{N}} \sum_{\{\ell\in\mathbb{Z}^n:\,2^{m-1}<|\ell-k_x|\le 2^m\}} 2^{-m\eta} \int_{Q_{j\ell}} |h(s)|\,ds \\
&\sim \int_{\bigcup_{\{\ell\in\mathbb{Z}^n:\,|\ell-k_x|\le 1\}} Q_{j\ell}} |h(s)|\,ds + \sum_{m\in\mathbb{N}} 2^{-m\eta} \int_{\bigcup_{\{\ell\in\mathbb{Z}^n:\,2^{m-1}<|\ell-k_x|\le 2^m\}} Q_{j\ell}} |h(s)|\,ds \\
&\lesssim \sum_{m\in\mathbb{Z}_+} 2^{-m\eta} 2^{mn} \fint_{B_m} |h(s)|\,ds \lesssim \mathcal{M}(h)(x),
\end{aligned}
$$

where B_m for any $m \in \mathbb{Z}_+$ is the smallest ball containing both x and $\bigcup_{\{\ell\in\mathbb{Z}^n:\,|\ell-k_x|\le 2^m\}} Q_{j\ell}$. This finishes the proof of Lemma 3.7. □

Now, we recall the definition of the space $L^p(\ell^q)$, which can be found in [37, p. 14].

Definition 3.8. Let $p \in (0,\infty]$ and $q \in (0,\infty]$. Then the *space* $L^p(\ell^q)$ is defined by setting

$$
L^p(\ell^q) := \left\{ \{f_j\}_{j\in\mathbb{Z}} \subset \mathscr{M}(\mathbb{R}^n) : \ \left\| \{f_j\}_{j\in\mathbb{Z}} \right\|_{L^p(\ell^q)} < \infty \right\},
$$

where

$$
\left\| \{f_j\}_{j\in\mathbb{Z}} \right\|_{L^p(\ell^q)} := \left\{ \int_{\mathbb{R}^n} \left[\sum_{j\in\mathbb{Z}} \left| f_j(x) \right|^q \right]^{p/q} dx \right\}^{1/p}
$$

with suitable modifications made when $p = \infty$ or $q = \infty$.

The following lemma is just [15, Corollary 3.8].

Lemma 3.9. *Let $p \in (0,\infty)$, $q \in (0,\infty]$, $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$, and $\{A_Q\}_{Q\in\mathcal{D}}$ be a sequence of reducing operators of order p for W. For any $j \in \mathbb{Z}$, $x \in \mathbb{R}^n$, and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, let*

$$
\gamma_j(x) := \sum_{Q\in\mathcal{D}_j} \left\| W^{1/p}(x) A_Q^{-1} \right\| \mathbf{1}_Q(x)
$$

and

$$
E_j(f) := \sum_{Q\in\mathcal{D}_j} \left[\fint_Q f(y)\,dy \right] \mathbf{1}_Q.
$$

Then there exists a positive constant C such that, for any sequence $\{f_j\}_{j\in\mathbb{Z}}$ of measurable functions on $\mathbb{R}^n$,

$$
\left\| \left\{ \gamma_j E_j\left(f_j\right) \right\}_{j\in\mathbb{Z}} \right\|_{L^p(\ell^q)} \le C \left\| \left\{ E_j\left(f_j\right) \right\}_{j\in\mathbb{Z}} \right\|_{L^p(\ell^q)}.
$$

The following lemma is the famous Fefferman–Stein vector-valued maximal inequality, see [11, Theorem 1].

Lemma 3.10. *Let $p \in (1, \infty)$ and $q \in (1, \infty]$. Then there exists a positive constant C such that, for any sequence $\{f_j\}_{j\in\mathbb{Z}} \subset \mathscr{M}(\mathbb{R}^n)$,*

$$\left\| \left\{ \sum_{j\in\mathbb{Z}} \left[\mathcal{M}\left(f_j\right) \right]^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} \le C \left\| \left[\sum_{j\in\mathbb{Z}} |f_j|^q \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)},$$

where $\mathcal{M}$ is the same as in (3.6).

Proof of Theorem 3.1. Let all the symbols be the same as in the present theorem. Then, by the definition of $(\varphi_j^* \vec{f})_a^{(W,p)}$ in (3.1), we find that

$$W^{1/p}\left(\varphi_j * \vec{f}\right) \le \left(\varphi_j^* \vec{f}\right)_a^{(W,p)},$$

which implies that, for any $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$,

$$\left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(W)} \lesssim \left\|\vec{f}\right\|^{\star}_{\dot{F}^{\alpha,q}_p(W)}.$$

Thus, to show Theorem 3.1, it remains to prove that, for any $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$,

$$\left\|\vec{f}\right\|^{\star}_{\dot{F}^{\alpha,q}_p(W)} \lesssim \left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(W)}. \tag{3.7}$$

Let $\{A_Q\}_{Q\in\mathcal{D}}$ be a sequence of reducing operators of order p for W. For any $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$, let

$$\left\|\vec{f}\right\|^{\star}_{\dot{F}^{\alpha,q}_p(\{A_Q\})} := \left\| \left[\sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_j} 2^{j\alpha q} \sup_{y\in\mathbb{R}^n} \frac{|A_Q(\varphi_j * \vec{f})(y)|^q}{(1 + 2^j|\cdot - y|)^{aq}} \mathbf{1}_Q \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)}, \tag{3.8}$$

where, for any $j \in \mathbb{Z}$, $\mathcal{D}_j$ is the same as in (2.2). To prove (3.7), we first show that, for any $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$,

$$\left\|\vec{f}\right\|^{\star}_{\dot{F}^{\alpha,q}_p(\{A_Q\})} \lesssim \left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(\{A_Q\})}. \tag{3.9}$$

Indeed, by Lemma 3.5, a geometrical observation that $1 + 2^j|x - y| \sim 1 + |k - s|$ for any $x \in Q_{jk}$ and $y \in Q_{js}$, Lemma 3.6, and the fact that $1 + |k - \ell| \le (1 + |k - s|)(1 + |s - \ell|)$ for any $k, s, \ell \in \mathbb{Z}^n$, we obtain, for any given $A \in (0, 1]$ and for any $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$, and $x \in Q_{jk}$,

$$\begin{aligned} &\sup_{y\in\mathbb{R}^n} \frac{|A_{Q_{jk}}(\varphi_j * \vec{f})(y)|^A}{(1 + 2^j|x - y|)^{aA}} \\ &\quad = \sup_{s\in\mathbb{Z}^n} \sup_{y\in Q_{js}} \frac{|A_{Q_{jk}}(\varphi_j * \vec{f})(y)|^A}{(1 + 2^j|x - y|)^{aA}} \\ &\quad \le \sup_{s\in\mathbb{Z}^n} \sup_{y\in Q_{js}} \frac{\|A_{Q_{jk}}A^{-1}_{Q_{js}}\|^A |A_{Q_{js}}(\varphi_j * \vec{f})(y)|^A}{(1 + 2^j|x - y|)^{aA}} \end{aligned} \tag{3.10}$$

$$\begin{aligned}
&\lesssim \sup_{s\in\mathbb{Z}^n}\sup_{y\in Q_{js}} \frac{(1+|k-s|)^{Ar}|A_{Q_{js}}(\varphi_j*\vec{f})(y)|^A}{(1+2^j|x-y|)^{aA}}\\
&\sim \sup_{s\in\mathbb{Z}^n}(1+|k-s|)^{Ar}(1+|k-s|)^{-aA}\sup_{y\in Q_{js}}\left|A_{Q_{js}}(\varphi_j*\vec{f})(y)\right|^A\\
&\lesssim \sup_{s\in\mathbb{Z}^n}\sum_{\ell\in\mathbb{Z}^n}(1+|k-s|)^{-A(a-r)}(1+|\ell-s|)^{-A(R-r)}2^{jn}\int_{Q_{j\ell}}\left|A_{Q_{j\ell}}\left(\varphi_j*\vec{f}\right)(z)\right|^A\,dz\\
&\lesssim \sum_{\ell\in\mathbb{Z}^n}(1+|k-\ell|)^{-A(a-r)}2^{jn}\int_{Q_{j\ell}}\left|A_{Q_{j\ell}}\left(\varphi_j*\vec{f}\right)(z)\right|^A\,dz,
\end{aligned}$$

where $r:=\frac{\beta}{p}$ and, in the last step, we used the fact that $(1+|k-s|)(1+|\ell-s|)\geq(1+|k-\ell|)$ for any $k,\ell,s\in\mathbb{Z}^n$ and the fact that $a\in[r,\infty)$, and chose an $R\in[a,\infty)$. Let $A\in(0,1]$ satisfy $q/A>1$. Using this, (3.10), and the disjointness of the cubes Q_{jk} for any $k\in\mathbb{Z}^n$, we further find that

$$\begin{aligned}
&\sum_{Q\in\mathcal{D}_j}\left[2^{j\alpha}\sup_{y\in\mathbb{R}^n}\frac{|A_Q\left(\varphi_j*\vec{f}\right)(y)|}{(1+2^j|\cdot-y|)^a}\mathbf{1}_Q(\cdot)\right]^q\\
&\quad=\sum_{k\in\mathbb{Z}^n}2^{j\alpha q}\left[\sup_{y\in\mathbb{R}^n}\frac{|A_{Q_{jk}}\left(\varphi_j*\vec{f}\right)(y)|^A}{(1+2^j|\cdot-y|)^{aA}}\mathbf{1}_{Q_{jk}}(\cdot)\right]^{q/A}\\
&\quad\lesssim\left[\sum_{k\in\mathbb{Z}^n}\sum_{\ell\in\mathbb{Z}^n}(1+|k-\ell|)^{-A(a-r)}2^{jn}\int_{Q_{j\ell}}\left|2^{j\alpha}A_{Q_{j\ell}}\left(\varphi_j*\vec{f}\right)(z)\right|^A\,dz\mathbf{1}_{Q_{jk}}(\cdot)\right]^{q/A}.
\end{aligned}$$

From $a\in(\frac{n}{\min\{1,p,q\}}+r,\infty)$, it follows that $\min\{1,p,q\}(a-r)>n$ and hence we can choose an $A\in(0,1]$ such that $A(a-r)>n$, $p/A>1$, and $q/A>1$. Thus, by Lemma 3.7 and the Fefferman–Stein vector-valued maximal inequality, we conclude that, for any $\vec{f}\in[\mathcal{S}'_\infty(\mathbb{R}^n)]^m$,

$$\begin{aligned}
&\left\|\left\{\sum_{j\in\mathbb{Z}}\sum_{Q\in\mathcal{D}_j}\left[2^{j\alpha}\sup_{y\in\mathbb{R}^n}\frac{|A_Q(\varphi_j*\vec{f})(y)|}{(1+2^j|\cdot-y|)^a}\mathbf{1}_Q\right]^q\right\}^{1/q}\right\|_{L^p(\mathbb{R}^n)}\\
&\quad\lesssim\left\|\left\{\sum_{j\in\mathbb{Z}}\left[\mathcal{M}\left(\sum_{Q\in\mathcal{D}_j}\left[2^{j\alpha}\left|A_Q\left(\varphi_j*\vec{f}\right)\right|\mathbf{1}_Q\right]^A\right)\right]^{q/A}\right\}^{A/q}\right\|_{L^{p/A}(\mathbb{R}^n)}^{1/A}\\
&\quad\lesssim\left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(\{A_Q\})},
\end{aligned}$$

which implies that (3.9) holds true for any $\vec{f}\in[\mathcal{S}'_\infty(\mathbb{R}^n)]^m$. Now, for any $\vec{f}\in[\mathcal{S}'_\infty(\mathbb{R}^n)]^m$, let

$$\left\|\vec{f}\right\|^{\star\star}_{\dot{F}^{\alpha,q}_p(\{A_Q\})}:=\left\|\left[\sum_{j\in\mathbb{Z}}\sum_{Q\in\mathcal{D}_j}2^{j\alpha q}\sup_{z\in Q}\sup_{y\in\mathbb{R}^n}\frac{|A_Q(\varphi_j*\vec{f})(y)|^q}{(1+2^j|z-y|)^{aq}}\mathbf{1}_Q\right]^{1/q}\right\|_{L^p(\mathbb{R}^n)}. \tag{3.11}$$

From (3.9) and Lemma 2.26, we infer that, for any $\vec{f}\in[\mathcal{S}'_\infty(\mathbb{R}^n)]^m$,

$$\left\|\vec{f}\right\|^{\star}_{\dot{F}^{\alpha,q}_p(\{A_Q\})}\lesssim\left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(\{A_Q\})}\sim\left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(W)}.$$

By this, to complete the proof of (3.7), we still need to prove that, for any $\vec{f}\in[\mathcal{S}'_\infty(\mathbb{R}^n)]^m$,

$$\left\|\vec{f}\right\|^{\star}_{\dot F^{\alpha,q}_p(W)}\lesssim\left\|\vec{f}\right\|^{\star\star}_{\dot F^{\alpha,q}_p(\{A_Q\})}\lesssim\left\|\vec{f}\right\|^{\star}_{\dot F^{\alpha,q}_p(\{A_Q\})}. \tag{3.12}$$

We first show the first inequality of (3.12). For any $j\in\mathbb{Z}$ and $x\in\mathbb{R}^n$, let

$$h_j(x):=2^{j\alpha}\sup_{y\in\mathbb{R}^n}\frac{|W^{1/p}(x)(\varphi_j*\vec{f})(y)|}{(1+2^j|x-y|)^a},$$

$$k_j(x):=\sum_{Q\in\mathcal{D}_j}|Q|^{-\alpha/n}\sup_{z\in Q}\sup_{y\in\mathbb{R}^n}\frac{|A_Q(\varphi_j*\vec{f})(y)|}{(1+2^j|z-y|)^a}\mathbf{1}_Q(x),$$

and

$$\gamma_j(x):=\sum_{Q\in\mathcal{D}_j}\left\|W^{1/p}(x)A_Q^{-1}\right\|\mathbf{1}_Q(x).$$

It is obvious that, for any $j\in\mathbb{Z}$ and $x\in\mathbb{R}^n$,

$$\begin{aligned}h_j(x)&=\sum_{Q\in\mathcal{D}_j}2^{j\alpha}\sup_{y\in\mathbb{R}^n}\frac{|W^{1/p}(x)A_Q^{-1}A_Q(\varphi_j*\vec{f})(y)|}{(1+2^j|x-y|)^a}\mathbf{1}_Q(x)\\&\le\sum_{Q\in\mathcal{D}_j}2^{j\alpha}\left\|W^{1/p}(x)A_Q^{-1}\right\|\sup_{y\in\mathbb{R}^n}\frac{|A_Q(\varphi_j*\vec{f})(y)|}{(1+2^j|x-y|)^a}\mathbf{1}_Q(x)\le\gamma_j(x)k_j(x).\end{aligned} \tag{3.13}$$

Notice that k_j is a constant on any given cube $Q\in\mathcal{D}_j$, which implies that

$$E_j(k_j)=k_j, \tag{3.14}$$

where E_j is the same as in Lemma 3.9. Then, by (3.2), (3.13), Lemma 3.9, (3.14), and (3.11), we have, for any $\vec{f}\in[\mathcal{S}'_\infty(\mathbb{R}^n)]^m$,

$$\begin{aligned}\left\|\vec{f}\right\|^{\star}_{\dot F^{\alpha,q}_p(W)}&=\left\|\left\{h_j\right\}_{j\in\mathbb{Z}}\right\|_{L^p(\ell^q)}\le\left\|\left\{\gamma_jE_j\left(k_j\right)\right\}_{j\in\mathbb{Z}}\right\|_{L^p(\ell^q)}\\&\lesssim\left\|\left\{E_j\left(k_j\right)\right\}_{j\in\mathbb{Z}}\right\|_{L^p(\ell^q)}\sim\left\|\left\{k_j\right\}_{j\in\mathbb{Z}}\right\|_{L^p(\ell^q)}\sim\left\|\vec{f}\right\|^{\star\star}_{\dot F^{\alpha,q}_p(\{A_Q\})},\end{aligned} \tag{3.15}$$

which is just the first inequality of (3.12). Next, we prove the second inequality of (3.12). Indeed, using a geometrical observation, we find that $1+2^j|x-y|\sim1+|s-k|\sim1+2^j|z-y|$ for any $x,z\in Q_{jk}$ and $y\in Q_{js}$. From this, we deduce that, for any $a\in(0,\infty)$, $j\in\mathbb{Z}$, $k\in\mathbb{Z}^n$, and $x\in Q_{jk}$,

$$\sup_{z\in Q_{jk}}\sup_{y\in\mathbb{R}^n}\frac{|A_{Q_{jk}}(\varphi_j*\vec{f})(y)|}{(1+2^j|z-y|)^a}\sim\sup_{y\in\mathbb{R}^n}\frac{|A_{Q_{jk}}(\varphi_j*\vec{f})(y)|}{(1+2^j|x-y|)^a},$$

which implies that, for any $\vec{f}\in[\mathcal{S}'_\infty(\mathbb{R}^n)]^m$,

$$\left\|\vec{f}\right\|^{\star\star}_{\dot F^{\alpha,q}_p(\{A_Q\})}\sim\left\|\vec{f}\right\|^{\star}_{\dot F^{\alpha,q}_p(\{A_Q\})}. \tag{3.16}$$

Thus, both (3.15) and (3.16) imply (3.12), and hence (3.3) holds true for any $\vec{f}\in[\mathcal{S}'_\infty(\mathbb{R}^n)]^m$, which completes the proof of Theorem 3.1. □

We now establish the Lusin-area function characterization of matrix-weighted Triebel–Lizorkin spaces.

Theorem 3.11. *Let $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$, and $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$. Assume that $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies both (T1) and (T2) of Definition 2.18. Then $\vec{f} \in \dot{F}^{\alpha,q}_p(W)$ if and only if $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$ with $\|\vec{f}\|^{\square}_{\dot{F}^{\alpha,q}_p(W)} < \infty$, where*

$$\left\|\vec{f}\right\|^{\square}_{\dot{F}^{\alpha,q}_p(W)} := \left\|\left[\sum_{j\in\mathbb{Z}} 2^{j\alpha q} \fint_{B(\cdot,2^{-j})} \left|W^{1/p}(\cdot)\left(\varphi_j * \vec{f}\right)(y)\right|^q \, dy\right]^{1/q}\right\|_{L^p(\mathbb{R}^n)}$$

with usual modification made when $q=\infty$. Moreover, for any $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$,

$$\left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(W)} \sim \left\|\vec{f}\right\|^{\square}_{\dot{F}^{\alpha,q}_p(W)}, \tag{3.17}$$

where the positive equivalence constants are independent of $\vec{f}$.

Proof. Let all the symbols be the same as in the present theorem. We now claim that, for any $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$,

$$\left\|\vec{f}\right\|^{\star}_{\dot{F}^{\alpha,q}_p(W)} \sim \left\|\vec{f}\right\|^{\square}_{\dot{F}^{\alpha,q}_p(W)},$$

if a is sufficiently large. Then, by Theorem 3.1, we conclude that the present theorem holds true.

First, we prove that, when $a \in (0,\infty)$, then, for any $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$,

$$\left\|\vec{f}\right\|^{\square}_{\dot{F}^{\alpha,q}_p(W)} \lesssim \left\|\vec{f}\right\|^{\star}_{\dot{F}^{\alpha,q}_p(W)}. \tag{3.18}$$

By the change of variables, the fact that $1+2^j|y| \sim 1$ for any $y \in B(\mathbf{0}, 2^{-j})$, and (3.1), we conclude that, for any given $q \in (0,\infty)$ and $a \in (0,\infty)$, and for any $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$,

$$\begin{aligned}
&\fint_{B(x,2^{-j})} \left|W^{1/p}(x)\left(\varphi_j * \vec{f}\right)(y)\right|^q \, dy \\
&\quad = \fint_{B(\mathbf{0},2^{-j})} \left|W^{1/p}(x)\left(\varphi_j * \vec{f}\right)(x+y)\right|^q \, dy \lesssim \sup_{y\in B(\mathbf{0},2^{-j})} \left|W^{1/p}(x)\left(\varphi_j * \vec{f}\right)(x+y)\right|^q \\
&\quad \sim \sup_{y\in B(\mathbf{0},2^{-j})} \frac{|W^{1/p}(x)(\varphi_j * \vec{f})(x+y)|^q}{(1+2^j|y|)^{aq}} \lesssim \left[\left(\varphi_j^* \vec{f}\right)_a^{(W,p)}(x)\right]^q,
\end{aligned}$$

which implies that (3.18) holds true.

Next, we show that, for any $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$,

$$\left\|\vec{f}\right\|^{\star}_{\dot{F}^{\alpha,q}_p(W)} \lesssim \left\|\vec{f}\right\|^{\square}_{\dot{F}^{\alpha,q}_p(W)}, \tag{3.19}$$

if a is sufficiently large. Using (3.12), to prove (3.19), we only need to show that, for any $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$,

$$\left\|\vec{f}\right\|^{\star}_{\dot{F}^{\alpha,q}_p(\{A_Q\})} \lesssim \left\|\vec{f}\right\|^{\square}_{\dot{F}^{\alpha,q}_p(W)}, \tag{3.20}$$

if a is sufficiently large. For any given $A \in (0,1]$ satisfying that $q/A > 1$ and $p/A > 1$, we choose an $a \in (0,\infty)$ sufficiently large such that $A(a-r) > n$, where $r := \frac{\beta}{p}$ and β is the doubling exponent of W. Then, by (3.10), the change of variables, the fact that, for any $z \in Q_{j\ell}$ and $s \in B(\mathbf{0}, 2^{-j})$, $z - s \in Q_{j(t+\ell)}$ for some $t := (t_1, \ldots, t_n) \in \mathbb{Z}^n$ satisfying $|t|_\infty := \max\{t_d : d \in \{1, \ldots, n\}\} \le 1$, and Lemma 3.5, we conclude that, for any $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, and $x \in Q_{jk}$,

$$\begin{aligned}
(3.21)\qquad & \sup_{y\in\mathbb{R}^n} \frac{|A_{Q_{jk}}(\varphi_j * \vec{f})(y)|^A}{(1+2^j|x-y|)^{aA}} \\
&\sim \sum_{\ell\in\mathbb{Z}^n} (1+|k-\ell|)^{-A(a-r)} 2^{jn} \fint_{B(\mathbf{0},2^{-j})} \int_{Q_{j\ell}} \left| A_{Q_{j\ell}} \left(\varphi_j * \vec{f}\right)(z)\right|^A dz\, ds \\
&\lesssim \sum_{\ell\in\mathbb{Z}^n} (1+|k-\ell|)^{-A(a-r)} 2^{jn} \\
&\quad\times \sum_{\{t\in\mathbb{Z}^n:\, |t|_\infty \le 1\}} \fint_{B(\mathbf{0},2^{-j})} \int_{Q_{j(\ell+t)}} \left| A_{Q_{j\ell}} \left(\varphi_j * \vec{f}\right)(s+z)\right|^A dz\, ds \\
&\lesssim \sum_{\ell\in\mathbb{Z}^n} (1+|k-\ell|)^{-A(a-r)} 2^{jn} \\
&\quad\times \sum_{\{t\in\mathbb{Z}^n:\, |t|_\infty \le 1\}} \int_{Q_{j(\ell+t)}} \fint_{B(\mathbf{0},2^{-j})} \left| A_{Q_{j\ell}} \left(\varphi_j * \vec{f}\right)(s+z)\right|^A ds\, dz \\
&\lesssim \sum_{\ell\in\mathbb{Z}^n} (1+|k-\ell|)^{-A(a-r)} 2^{jn} \\
&\quad\times \sum_{\{t\in\mathbb{Z}^n:\, |t|_\infty \le 1\}} \int_{Q_{j(\ell+t)}} \fint_{B(\mathbf{0},2^{-j})} \left| A_{Q_{j(\ell+t)}} \left(\varphi_j * \vec{f}\right)(s+z)\right|^A ds\, dz.
\end{aligned}$$

Now, we prove (3.20) by considering two cases on p.

Case 1) $p \in (0,1]$. In this case, noticing that $\mathbf{1}_{Q_{j(\ell+t)}} = \sum_{Q\in\mathcal{D}_j}(\mathbf{1}_Q \mathbf{1}_{Q_{j(\ell+t)}})$, we then have

$$\begin{aligned}
(3.22)\qquad & \int_{Q_{j(\ell+t)}} 2^{j\alpha} \fint_{B(\mathbf{0},2^{-j})} \left| A_{Q_{j(\ell+t)}} \left(\varphi_j * \vec{f}\right)(s+z)\right|^A ds\, dz \\
&= \int_{Q_{j(\ell+t)}} \sum_{Q\in\mathcal{D}_j} 2^{j\alpha} \fint_{B(\mathbf{0},2^{-j})} \left| A_Q \varphi_j * \vec{f}(s+z)\right|^A ds \mathbf{1}_Q(z)\, dz \\
&= \int_{Q_{j(\ell+t)}} g_j(z)\, dz,
\end{aligned}$$

where, for any $z \in \mathbb{R}^n$,

$$g_j(z) := \sum_{Q\in\mathcal{D}_j} 2^{j\alpha} \fint_{B(\mathbf{0},2^{-j})} \left| A_Q \varphi_j * \vec{f}(s+z)\right|^A ds \mathbf{1}_Q(z).$$

For any given $x \in Q_{jk}$, let $B_x := B(x_{k,\ell,t}, r_{k,\ell,t})$ be the smallest ball containing both x and the dyadic cube $Q_{j(\ell+t)}$. Then $r_{k,\ell,t} \sim 2^{-j}(1+|k-\ell-t|)$. Since $|t|_\infty \le 1$, it follows that

$$(3.23)\qquad r_{k,\ell,t} \sim 2^{-j}(1+|k-\ell|).$$

Using this and (3.22), we obtain, for any $x \in Q_{jk}$,

$$\begin{aligned}(3.24)\qquad &\int_{Q_{j(\ell+t)}} 2^{j\alpha} \fint_{B(\mathbf{0},2^{-j})} \left|A_{Q_{j(\ell+t)}}\left(\varphi_j * \vec{f}\right)(s+z)\right|^A \, ds\, dz \\ &\quad \le \int_{B_x} g_j(z)\, dz \lesssim 2^{-jn}(1+|k-\ell|)^n \mathcal{M}(g_j)(x).\end{aligned}$$

By both (3.21) and (3.24), we conclude that, for any $x \in \mathbb{R}^n$,

$$\begin{aligned}(3.25)\qquad &2^{j\alpha} \sum_{Q\in\mathcal{D}_j} \sup_{y\in\mathbb{R}^n} \frac{|A_Q(\varphi_j * \vec{f})(y)|^A}{(1+2^j|x-y|)^{aA}} \mathbf{1}_Q(x) \\ &\quad \lesssim \sum_{\ell\in\mathbb{Z}^n} (1+|k-\ell|)^{-A(a-r)+n} \mathcal{M}(g_j)(x) \lesssim \mathcal{M}(g_j)(x),\end{aligned}$$

where, in the last step, we used the assumption $A(a-r) > 2n$. From (3.25), we further deduce that, for any $x \in \mathbb{R}^n$,

$$(3.26)\qquad \sum_{Q\in\mathcal{D}_j} \left[2^{j\alpha} \sup_{y\in\mathbb{R}^n} \frac{|A_Q(\varphi_j * \vec{f})(y)|}{(1+2^j|x-y|)^{a}} \mathbf{1}_Q(x)\right]^q \lesssim \left[\mathcal{M}(g_j)(x)\right]^{q/A}.$$

By (3.26), the Fefferman–Stein vector-valued maximal inequality together with $p/A > 1$ and $q/A > 1$, the Hölder inequality, and Lemma 2.17, we find that, for any $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$,

$$\begin{aligned}\left\|\vec{f}\right\|^{\star}_{\dot{F}^{\alpha,q}_p(\{A_Q\})} &\lesssim \left\|\left\{\sum_{j\in\mathbb{Z}} \left[\mathcal{M}(g_j)\right]^{q/A}\right\}^{A/q}\right\|^{1/A}_{L^{p/A}(\mathbb{R}^n)} \\ &\lesssim \left\|\left\{\sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_j} 2^{j\alpha q} \left[\fint_{B(\mathbf{0},2^{-j})} \left|A_Q\varphi_j * \vec{f}(\cdot+z)\right|^A dz\right]^{q/A} \mathbf{1}_Q\right\}^{1/q}\right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \left\|\left[\sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_j} 2^{j\alpha q} \fint_{B(\mathbf{0},2^{-j})} \left|A_Q\varphi_j * \vec{f}(\cdot+z)\right|^q dz \mathbf{1}_Q\right]^{1/q}\right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \left\|\left[\sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_j} 2^{j\alpha q} \fint_{B(\mathbf{0},2^{-j})} \left\|A_Q W^{-1/p}(\cdot)\right\|^q \left|W^{1/p}(\cdot)\varphi_j * \vec{f}(\cdot+z)\right|^q dz \mathbf{1}_Q\right]^{1/q}\right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \left\|\vec{f}\right\|^{\square}_{\dot{F}^{\alpha,q}_p(W)}.\end{aligned}$$

Thus, (3.20) holds true when $p \in (0,1]$.

Case 2) $p \in (1,\infty)$. In this case, from (3.21), the Hölder inequality, and Lemma 2.16, we infer that, for any $x \in Q_{jk}$,

$$(3.27)\qquad \sup_{y\in\mathbb{R}^n} \frac{|A_{Q_{jk}}(\varphi_j * \vec{f})(y)|^A}{(1+2^j|x-y|)^{aA}}$$

$$
\begin{aligned}
&\lesssim \sum_{\ell\in\mathbb{Z}^n}(1+|k-\ell|)^{-A(a-r)}\,2^{jn}\sum_{\{t\in\mathbb{Z}^n:\,|t|_\infty\le 1\}}\left[\int_{Q_{j(\ell+t)}}\left\|A_{Q_{j(\ell+t)}}W^{-1/p}(z)\right\|^A\right.\\
&\quad\left.\times\fint_{B(\mathbf{0},2^{-j})}\left|W^{1/p}(z)\left(\varphi_j*\vec{f}\right)(s+z)\right|^A\,ds\,dz\right]\\
&\lesssim \sum_{\ell\in\mathbb{Z}^n}(1+|k-\ell|)^{-A(a-r)}\,2^{jn}\sum_{\{t\in\mathbb{Z}^n:\,|t|_\infty\le 1\}}\left\{\left[\int_{Q_{j(\ell+t)}}\left\|A_{Q_{j(\ell+t)}}W^{-1/p}(z)\right\|^{p'}\,dz\right]^{A/p'}\right.\\
&\quad\left.\times\left[\int_{Q_{j(\ell+t)}}\left\{\fint_{B(\mathbf{0},2^{-j})}\left|W^{1/p}(z)\left(\varphi_j*\vec{f}\right)(s+z)\right|^A\,ds\right\}^{\frac{p'}{p'-A}}\,dz\right]^{\frac{p'-A}{p'}}\right\}\\
&\lesssim \sum_{\ell\in\mathbb{Z}^n}(1+|k-\ell|)^{-A(a-r)}\,2^{jn(1-\frac{A}{p'})}\\
&\quad\times\sum_{\{t\in\mathbb{Z}^n:\,|t|_\infty\le 1\}}\left\{\int_{Q_{j(\ell+t)}}\left[\fint_{B(\mathbf{0},2^{-j})}\left|W^{1/p}(z)\left(\varphi_j*\vec{f}\right)(s+z)\right|^A\,ds\right]^{\frac{p'}{p'-A}}\,dz\right\}^{\frac{p'-A}{p'}}.
\end{aligned}
$$

For any given $x\in\mathbb{R}^n$, let $B_x:=B(x_{k,\ell,t},r_{k,\ell,t})$ be the same as in Case 1). Notice that, for any $M>n$,

$$
\sup_{k\in\mathbb{Z}^n}\sum_{\ell\in\mathbb{Z}^n}(1+|k-\ell|)^{-M}=\sup_{k\in\mathbb{Z}^n}\sum_{k-\ell\in\mathbb{Z}^n}(1+|k-\ell|)^{-M}=\sum_{\ell\in\mathbb{Z}^n}(1+|\ell|)^{-M}\lesssim 1.
$$

By this, (3.27), (3.23), the Hölder inequality, and the disjointness of Q_{jk} for any $k\in\mathbb{Z}^n$, we conclude that, for any $x\in\mathbb{R}^n$,

$$
\begin{aligned}
(3.28)\qquad &\sum_{k\in\mathbb{Z}^n}\sup_{y\in\mathbb{R}^n}\frac{|A_{Q_{jk}}(\varphi_j*\vec{f})(y)|^q}{(1+2^j|x-y|)^{aq}}\mathbf{1}_{Q_{jk}}(x)\\
&\lesssim\left[\sum_{k\in\mathbb{Z}^n}\sum_{\ell\in\mathbb{Z}^n}(1+|k-\ell|)^{-A(a-r)}\,2^{jn(1-\frac{A}{p'})}\right.\\
&\quad\left.\times\left\{\int_{B_x}\left[\fint_{B(\mathbf{0},2^{-j})}\left|W^{1/p}(z)\left(\varphi_j*\vec{f}\right)(s+z)\right|^A\,ds\right]^{\frac{p'}{p'-A}}\,dz\right\}^{\frac{p'-A}{p'}}\mathbf{1}_{Q_{jk}}(x)\right]^{q/A}\\
&\lesssim\left\{\sum_{k\in\mathbb{Z}^n}\sum_{\ell\in\mathbb{Z}^n}(1+|k-\ell|)^{-A(a-r)+\frac{(p'-A)n}{p'}}\right.\\
&\quad\left.\times\left[\mathcal{M}\left(\left[\fint_{B(\mathbf{0},2^{-j})}\left|W^{1/p}(\cdot)\left(\varphi_j*\vec{f}\right)(\cdot+z)\right|^A\,dz\right]^{\frac{p'}{p'-A}}\right)(x)\right]^{\frac{p'-A}{p'}}\mathbf{1}_{Q_{jk}}(x)\right\}^{q/A}\\
&\lesssim\left\{\mathcal{M}\left(\left[\fint_{B(\mathbf{0},2^{-j})}\left|W^{1/p}(\cdot)\left(\varphi_j*\vec{f}\right)(\cdot+z)\right|^A\,dz\right]^{\frac{p'}{p'-A}}\right)(x)\right\}^{\frac{(p'-A)q}{Ap'}}
\end{aligned}
$$

$$
\begin{aligned}
&\quad \times\left[\sup_{k\in\mathbb{Z}^n}\sum_{\ell\in\mathbb{Z}^n}(1+|k-\ell|)^{-A(a-r)+\frac{(p'-A)n}{p'}}\right]^{q/A}\\
&\lesssim\left\{\mathcal{M}\left(\left[\fint_{B(\mathbf{0},2^{-j})}\left|W^{1/p}(\cdot)\left(\varphi_j*\vec{f}\right)(\cdot+z)\right|^A\,dz\right]^{\frac{p'}{p'-A}}\right)(x)\right\}^{\frac{(p'-A)q}{Ap'}},
\end{aligned}
$$

where, in the last step, we chose a sufficiently large $a\in(0,\infty)$ such that $A(a-r)-\frac{(p'-A)n}{p'}>n$. Noticing that $\frac{p(p'-A)}{Ap'}=\frac{A+(1-A)p}{A}>1$, choose $A\in(0,1)$ sufficiently small, and hence $\frac{(p'-A)q}{Ap'}>1$ and $q/A>1$. From this, (3.28), the Fefferman–Stein vector-valued maximal inequality, and the Hölder inequality, we deduce that, for any $\vec{f}\in[\mathcal{S}'_\infty(\mathbb{R}^n)]^m$,

$$
\begin{aligned}
&\left\|\vec{f}\right\|^{\star}_{\dot{F}^{\alpha,q}_p(\{A_Q\})}\\
&\quad\lesssim\left\|\left\{\sum_{j\in\mathbb{Z}}2^{j\alpha q}\left[\mathcal{M}\left(\left[\fint_{B(\mathbf{0},2^{-j})}\left|W^{1/p}(\cdot)\left(\varphi_j*\vec{f}\right)(\cdot+z)\right|^A\,dz\right]^{\frac{p'}{p'-A}}\right)\right]^{\frac{(p'-A)q}{Ap'}}\right\}^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^n)}\\
&\quad\sim\left\|\left\{\sum_{j\in\mathbb{Z}}\left[\mathcal{M}\left(\left[\fint_{B(\mathbf{0},2^{-j})}\left|2^{j\alpha}W^{1/p}(\cdot)\left(\varphi_j*\vec{f}\right)(\cdot+z)\right|^A\,dz\right]^{\frac{p'}{p'-A}}\right)\right]^{\frac{(p'-A)q}{Ap'}}\right\}^{\frac{Ap'}{(p'-A)q}}\right\|_{L^{\frac{p(p'-A)}{p'A}}(\mathbb{R}^n)}\\
&\quad\lesssim\left\|\left\{\sum_{j\in\mathbb{Z}}2^{j\alpha q}\left[\fint_{B(\mathbf{0},2^{-j})}\left|W^{1/p}(x)\left(\varphi_j*\vec{f}\right)(\cdot+z)\right|^A\,dz\right]^{q/A}\right\}^{1/q}\right\|_{L^p(\mathbb{R}^n)}\\
&\quad\lesssim\left\|\left\{\sum_{j\in\mathbb{Z}}2^{j\alpha q}\left[\fint_{B(\mathbf{0},2^{-j})}\left|W^{1/p}(x)\left(\varphi_j*\vec{f}\right)(\cdot+z)\right|^q\,dz\right]\right\}^{1/q}\right\|_{L^p(\mathbb{R}^n)}\\
&\quad\sim\left\|\vec{f}\right\|^{\square}_{\dot{F}^{\alpha,q}_p(W)}.
\end{aligned}
$$

Thus, (3.20) holds true when $p\in(0,\infty)$.

Combining both Cases 1) and 2), we conclude that (3.20) holds true. From (3.19), (3.12), and Theorem 3.1, we infer that (3.17) holds true for any $\vec{f}\in[\mathcal{S}'_\infty(\mathbb{R}^n)]^m$, which then completes the proof of Theorem 3.11. □

Remark 3.12. Theorem 3.11 when $m=1$ and $W=1$ is just [37, Theorem 2.12.1] which is the Lusin-area function characterization of Triebel–Lizorkin spaces.

In what follows, we establish the $g^*_\lambda-$ function characterization of $\dot{F}^{\alpha,q}_p(W)$. First, we give the following technical lemma.

Lemma 3.13. *Let $\varphi\in\mathcal{S}(\mathbb{R}^n)$ with $\operatorname{supp}\widehat{\varphi}$ being bounded and away from the origin. Then, for any $f\in\mathcal{S}'_\infty(\mathbb{R}^n)$, $\varphi*f\in C^\infty(\mathbb{R}^n)\cap\mathcal{S}'(\mathbb{R}^n)$ and $\operatorname{supp}(\varphi*f)^\wedge\subseteq\operatorname{supp}\widehat{\varphi}$.*

Proof. Since $\overline{\operatorname{supp}\widehat{\varphi}}$ is bounded and away from the origin, then we deduce that, for any $\psi \in \mathcal{S}(\mathbb{R}^n)$, $\operatorname{supp}(\widehat{\varphi}\widehat{\psi}) \subseteq \overline{\operatorname{supp}\widehat{\varphi}}$ is bounded and away from the origin, which implies that

$$\varphi * \psi \in \mathcal{S}_\infty(\mathbb{R}^n). \tag{3.29}$$

By an argument similar to that used in the proof of [17, Proposition 2.3.4(b)], we find that, if $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$, then there exists a positive constant C and $k, \ell \in \mathbb{Z}_+$ such that, for any $\phi \in \mathcal{S}_\infty(\mathbb{R}^n)$,

$$|\langle f, \phi\rangle| \le C \sum_{\{\mu,\,\nu\in\mathbb{Z}_+^n:\,|\mu|\le k,\,|\nu|\le\ell\}} \rho_{\mu,\nu}(\phi),$$

where

$$\rho_{\mu,\nu}(\phi) := \sup_{x\in\mathbb{R}^n} |x^\mu \partial^\nu \phi(x)|. \tag{3.30}$$

From this, $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$, and (3.29), we infer that there exist $k, \ell \in \mathbb{Z}_+$ such that, for any $\psi \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}
|\langle \varphi * f, \psi\rangle| &= |\langle f, \widetilde{\varphi} * \psi\rangle| \\
&\lesssim \sum_{\{\mu,\,\nu\in\mathbb{Z}_+^n:\,|\mu|\le k,|\nu|\le\ell\}} \rho_{\mu,\nu}(\widetilde{\varphi} * \psi) \\
&\lesssim \sum_{\{\mu,\,\nu\in\mathbb{Z}_+^n:\,|\mu|\le k,|\nu|\le\ell\}} \sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} \sum_{|\mu'|\le|\mu|} |x-y|^{|\mu'|}\,|y|^{|\mu|-|\mu'|} \left|\partial_x^\nu \psi(x-y)\right| |\widetilde{\varphi}(y)|\,dy \\
&\lesssim_\varphi \sum_{\{\mu,\,\nu\in\mathbb{Z}_+^n:\,|\mu|\le k,|\nu|\le\ell\}} \rho_{\mu,\nu}(\psi),
\end{aligned}$$

where, in the last inequality, the implicit positive constant depends on φ, which implies that $\varphi * f \in \mathcal{S}'(\mathbb{R}^n)$. By this, we conclude that, for any $\gamma \in \mathcal{S}(\mathbb{R}^n)$ with $\operatorname{supp}\gamma \subset (\mathbb{R}^n \setminus \operatorname{supp}\widehat{\varphi})$,

$$\left\langle (\varphi * f)^\wedge, \gamma\right\rangle = \langle \varphi * f, \widehat{\gamma}\rangle = \langle f, \widetilde{\varphi} * \widehat{\gamma}\rangle = \left\langle f, \widetilde{(\widehat{\varphi}\gamma)^\vee}\right\rangle = 0,$$

which implies that $\operatorname{supp}(\varphi * f)^\wedge \subseteq \operatorname{supp}\widehat{\varphi}$. Then, from [17, Theorem 2.3.21], we deduce that $\varphi * f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, which completes the proof of Lemma 3.13. □

Theorem 3.14. *Let $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$, $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$, and $\lambda \in (\frac{1}{\min\{1,p,q\}} + \frac{\beta}{np}, \infty)$, where β is the doubling exponent of W. Assume that $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies both (T1) and (T2) of Definition 2.18. Then $\vec{f} \in \dot{F}^{\alpha,q}_p(W)$ if and only if $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$ and $\|\vec{f}\|^{\clubsuit}_{\dot{F}^{\alpha,q}_p(W)} < \infty$, where*

$$\left\|\vec{f}\right\|^{\clubsuit}_{\dot{F}^{\alpha,q}_p(W)} := \left\| \left\{ \sum_{j\in\mathbb{Z}} 2^{j\alpha q} 2^{jn} \int_{\mathbb{R}^n} \frac{|W^{1/p}(\cdot)(\varphi_j * \vec{f})(y)|^q}{(1+2^j|\cdot - y|)^{\lambda n q}}\,dy \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} \tag{3.31}$$

with usual modification made when $q = \infty$. Moreover, for any $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$,

$$\left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(W)} \sim \left\|\vec{f}\right\|^{\clubsuit}_{\dot{F}^{\alpha,q}_p(W)}, \tag{3.32}$$

where the positive equivalence constants are independent of $\vec{f}$.

Proof. Let all the symbols be the same as in the present theorem. First, we prove that, for any $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$,

$$\left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(W)} \lesssim \left\|\vec{f}\right\|^{\clubsuit}_{\dot{F}^{\alpha,q}_p(W)}. \tag{3.33}$$

Indeed, by an observation that, for any $x \in \mathbb{R}^n$ and $y \in B(x, 2^{-j})$, $1 + 2^j|x - y| \sim 1$, we conclude that, for any $x \in \mathbb{R}^n$ and $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$,

$$\begin{aligned}
&\fint_{B(x,2^{-j})} \left|W^{1/p}(x)\left(\varphi_j * \vec{f}\right)(y)\right|^q dy \\
&\quad \sim 2^{jn} \int_{B(x,2^{-j})} \frac{|W^{1/p}(x)(\varphi_j * \vec{f})(y)|^q}{(1 + 2^j|x - y|)^{\lambda nq}}\, dy \lesssim 2^{jn} \int_{\mathbb{R}^n} \frac{|W^{1/p}(x)(\varphi_j * \vec{f})(y)|^q}{(1 + 2^j|x - y|)^{\lambda nq}}\, dy,
\end{aligned}$$

which implies that $\|\vec{f}\|^{\square}_{\dot{F}^{\alpha,q}_p(W)} \lesssim \|\vec{f}\|^{\clubsuit}_{\dot{F}^{\alpha,q}_p(W)}$. From this and Theorem 3.11, we infer that (3.33) holds true. Thus, to complete the proof of Theorem 3.14, it remains to show that, for any $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$,

$$\left\|\vec{f}\right\|^{\clubsuit}_{\dot{F}^{\alpha,q}_p(W)} \lesssim \left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(W)}. \tag{3.34}$$

Let $\{A_Q\}_{Q\in\mathcal{D}}$ be a sequence of reducing operators of order p for W. For any $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$, let

$$\left\|\vec{f}\right\|^{\clubsuit}_{\dot{F}^{\alpha,q}_p(\{A_Q\})} := \left\|\left\{\sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_j} 2^{j\alpha q} 2^{jn} \sup_{z\in Q} \int_{\mathbb{R}^n} \frac{|A_Q(\varphi_j * \vec{f})(y)|^q}{(1 + 2^j|z - y|)^{\lambda nq}}\, dy \mathbf{1}_Q\right\}^{1/q}\right\|_{L^p(\mathbb{R}^n)}. \tag{3.35}$$

To prove (3.34), we first show that, for any $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$,

$$\left\|\vec{f}\right\|^{\clubsuit}_{\dot{F}^{\alpha,q}_p(W)} \lesssim \left\|\vec{f}\right\|^{\clubsuit}_{\dot{F}^{\alpha,q}_p(\{A_Q\})}. \tag{3.36}$$

Indeed, for any given $p \in (0, \infty)$ and $q \in (0, \infty]$, and for any $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$, let

$$\gamma_j(x) := \sum_{Q\in\mathcal{D}_j} \left\|W^{1/p}(x) A_Q^{-1}\right\| \mathbf{1}_Q(x),$$

$$h_j(x) := 2^{j\alpha} 2^{jn/q} \left[\int_{\mathbb{R}^n} \frac{|W^{1/p}(x)(\varphi_j * \vec{f})(y)|^q}{(1 + 2^j|x - y|)^{\lambda nq}}\, dy\right]^{1/q},$$

and

$$f_j(x) := \sum_{Q\in\mathcal{D}_j} |Q|^{-\alpha/n}\, 2^{jn/q} \left[\sup_{z\in Q} \int_{\mathbb{R}^n} \frac{|A_Q(\varphi_j * \vec{f})(y)|^q}{(1 + 2^j|z - y|)^{\lambda nq}}\, dy\right]^{1/q} \mathbf{1}_Q(x).$$

It is obvious that, for any $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$,

$$h_j(x) = \sum_{Q\in\mathcal{D}_j} 2^{j\alpha} 2^{jn/q} \left[\int_{\mathbb{R}^n} \frac{|W^{1/p}(x) A_Q^{-1} A_Q(\varphi_j * \vec{f})(y)|^q}{(1 + 2^j|x - y|)^{\lambda nq}}\, dy\right]^{1/q} \mathbf{1}_Q(x) \tag{3.37}$$

$$
\begin{aligned}
&\leq \sum_{Q\in\mathcal{D}_j} |Q|^{-\alpha/n}\, 2^{jn/q} \left\|W^{1/p}(x)A_Q^{-1}\right\| \left[\int_{\mathbb{R}^n} \frac{|A_Q(\varphi_j * \vec{f})(y)|^q}{(1+2^j|x-y|)^{\lambda nq}}\,dy\right]^{1/q} \mathbf{1}_Q(x)\\
&\leq \sum_{Q\in\mathcal{D}_j} |Q|^{-\alpha/n}\, 2^{jn/q} \left\|W^{1/p}(x)A_Q^{-1}\right\| \left[\sup_{z\in Q}\int_{\mathbb{R}^n} \frac{|A_Q(\varphi_j * \vec{f})(y)|^q}{(1+2^j|z-y|)^{\lambda nq}}\,dy\right]^{1/q} \mathbf{1}_Q(x)\\
&\leq \gamma_j(x) f_j(x).
\end{aligned}
$$

Notice that f_j is a constant on any given $Q \in \mathcal{D}_j$, which implies that $E_j(f_j) = f_j$, where E_j is the same as in Lemma 3.9. By this, (3.31), (3.37), Lemma 3.9, and (3.35), we conclude that, for any $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$,

$$
\begin{aligned}
\left\|\vec{f}\right\|^{\clubsuit}_{\dot{F}^{\alpha,q}_p(W)} &= \left\|\left\{h_j\right\}_{j\in\mathbb{Z}}\right\|_{L^p(\ell^q)} \leq \left\|\left\{\gamma_j E_j(f_j)\right\}_{j\in\mathbb{Z}}\right\|_{L^p(\ell^q)}\\
&\lesssim \left\|\left\{E_j(f_j)\right\}_{j\in\mathbb{Z}}\right\|_{L^p(\ell^q)} \sim \left\|\left\{f_j\right\}_{j\in\mathbb{Z}}\right\|_{L^p(\ell^q)} \sim \left\|\vec{f}\right\|^{\clubsuit}_{\dot{F}^{\alpha,q}_p(\{A_Q\})}.
\end{aligned}
$$

Thus, (3.36) holds true for any $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$.

Next, we prove that, for any $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$,

$$
\left\|\vec{f}\right\|^{\clubsuit}_{\dot{F}^{\alpha,q}_p(\{A_Q\})} \lesssim \left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(W)}. \tag{3.38}
$$

Let $\lambda \in (0,\infty)$ and $j \in \mathbb{Z}$. Then we claim that, for any $z \in \mathbb{R}^n$ and $A \in (0,q]$,

$$
\sup_{v\in\mathbb{R}^n} \frac{|A_Q(\varphi_j * \vec{f})(v)|}{(1+2^j|z-v|)^{\lambda n}} \lesssim \left[2^{jn}\int_{\mathbb{R}^n} \frac{|A_Q(\varphi_j * \vec{f})(y)|^A}{(1+2^j|z-y|)^{\lambda nA}}\,dy\right]^{1/A}. \tag{3.39}
$$

Then we prove (3.39) by considering the following two cases on A.

Case 1) $A \in (0,1]$. In this case, using the assumption that φ satisfies (T2) of Definition 2.18, we can then easily prove that there exists a $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\operatorname{supp}\widehat{\psi}$ is bounded away from the origin and $\widehat{\psi} = 1$ on $\operatorname{supp}\widehat{\varphi}$. By this and Lemma 3.13, we find that

$$
\varphi_j * \vec{f} = \psi_j * \varphi_j * \vec{f} \tag{3.40}
$$

on $\mathbb{R}^n$. From this and the estimate that

$$
(1+2^j|z-v|)^{-1} \leq (1+2^j|z-y|)^{-1}(1+2^j|v-y|) \tag{3.41}
$$

for any $j \in \mathbb{Z}$ and $z, v, y \in \mathbb{R}^n$, we deduce that, for any given $A \in (0,\min\{1,q\}]$ and for any $z \in \mathbb{R}^n$,

$$
\begin{aligned}
&\sup_{v\in\mathbb{R}^n} \frac{|A_Q(\varphi_j * \vec{f})(v)|}{(1+2^j|z-v|)^{\lambda n}} \tag{3.42}\\
&\quad\leq \sup_{v\in\mathbb{R}^n} \frac{\int_{\mathbb{R}^n} |A_Q(\varphi_j * \vec{f})(y)2^{jn}\psi(2^j(v-y))|\,dy}{(1+2^j|z-v|)^{\lambda n}}\\
&\quad\leq \sup_{v\in\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|A_Q(\varphi_j * \vec{f})(y)2^{jn}\psi(2^j(v-y))|(1+2^j|v-y|)^{\lambda n}}{(1+2^j|z-y|)^{\lambda n}}\,dy
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim 2^{jn}\int_{\mathbb{R}^n}\frac{|A_Q(\varphi_j*\vec{f})(y)|}{(1+2^j|z-y|)^{\lambda n}}\,dy\\
&\lesssim \sup_{v\in\mathbb{R}^n}\left[\frac{|A_Q(\varphi_j*\vec{f})(v)|}{(1+2^j|z-v|)^{\lambda n}}\right]^{1-A}\int_{\mathbb{R}^n}2^{jn}\frac{|A_Q(\varphi_j*\vec{f})(y)|^A}{(1+2^j|z-y|)^{\lambda nA}}\,dy.
\end{aligned}
$$

When $a\in(n/\min\{1,p,q\}+\beta/p,\infty)$, by $\vec{f}\in\dot{F}^{\alpha,q}_p(W)$, (3.9), and Lemma 2.26, we find that

$$
\left\{\sum_{Q\in\mathcal{D}_j}2^{j\alpha}\sup_{v\in\mathbb{R}^n}\frac{|A_Q(\varphi_j*\vec{f})(v)|}{(1+2^j|\cdot-v|)^a}\mathbf{1}_Q\right\}_{j\in\mathbb{Z}}\in L^p(\ell^q),
$$

which implies that $\sup_{v\in\mathbb{R}^n}\frac{|A_Q(\varphi_j*\vec{f})(v)|}{(1+2^j|\cdot-v|)^a}<\infty$ almost everywhere on $\mathbb{R}^n$. Using this, we find that there exists a measurable set $F\subset\mathbb{R}^n$ satisfying that $|F|=0$ and, for any $x\in\mathbb{R}^n\setminus F$,

$$
\sup_{v\in\mathbb{R}^n}\frac{|A_Q(\varphi_j*\vec{f})(v)|}{(1+2^j|x-v|)^a}<\infty. \tag{3.43}
$$

Then, for any $e\in F$, there exists an $x_e\in\mathbb{R}^n\setminus F$ such that $2^j|x_e-e|<1/2$. From this, we deduce that

$$
\begin{aligned}
\sup_{v\in\mathbb{R}^n}\frac{|A_Q(\varphi_j*\vec{f})(v)|}{(1+2^j|e-v|)^a}&\le\sup_{v\in\mathbb{R}^n}\frac{|A_Q(\varphi_j*\vec{f})(v)|}{(1+2^j|x_e-v|-2^j|x_e-e|)^a}\\
&\le\sup_{v\in\mathbb{R}^n}\frac{|A_Q(\varphi_j*\vec{f})(v)|}{(1/2+2^j|x_e-v|)^a}<\infty,
\end{aligned}
$$

which, combined with (3.43), implies that

$$
\sup_{v\in\mathbb{R}^n}\frac{|A_Q(\varphi_j*\vec{f})(v)|}{(1+2^j|\cdot-v|)^{\lambda n}}<\infty
$$

on $\mathbb{R}^n$ if $\lambda\in(\frac{1}{\min\{1,p,q\}}+\frac{\beta}{np},\infty)$. By this and (3.42), we conclude that (3.39) holds true for $A\in(0,1]$.

Case 2) $A\in(1,\infty)$. In this case, from (3.40), (3.41), the Hölder inequality, and the change of variables, we infer that, for any given $A\in(1,q]$ and for any $z\in\mathbb{R}^n$,

$$
\begin{aligned}
&\sup_{v\in\mathbb{R}^n}\frac{|A_Q(\varphi_j*\vec{f})(v)}{(1+2^j|z-v|)^{\lambda n}}\\
&\quad\le\sup_{v\in\mathbb{R}^n}\frac{\int_{\mathbb{R}^n}|A_Q(\varphi_j*\vec{f})(y)2^{jn}\psi(2^j(v-y))|\,dy}{(1+2^j|z-v|)^{\lambda n}}\\
&\quad\le\sup_{v\in\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|A_Q(\varphi_j*\vec{f})(y)2^{jn}\psi(2^j(v-y))|(1+2^j|v-y|)^{\lambda n}}{(1+2^j|z-y|)^{\lambda n}}\,dy\\
&\quad\le\left[2^{jn}\int_{\mathbb{R}^n}\frac{|A_Q(\varphi_j*\vec{f})(y)|)^A}{(1+2^j|z-y|)^{\lambda nA}}\,dy\right]^{1/A}\\
&\qquad\times\sup_{v\in\mathbb{R}^n}\left\{\left[2^{jn}\int_{\mathbb{R}^n}|\psi(2^j(v-y))|^{A'}(1+2^j|v-y|)^{\lambda nA'}\,dy\right]^{1/A'}\right\}
\end{aligned}
$$

$$\sim \left[2^{jn}\int_{\mathbb{R}^n}\frac{|A_Q(\varphi_j*\vec{f}(y))|^A}{(1+2^j|z-y|)^{\lambda nA}}\,dy\right]^{1/A},$$

which implies that (3.39) holds true for $q\in(1,\infty)$.

Thus, (3.39) holds true for any $A\in(0,q]$. Using (3.39), we obtain, for any given $A\in(0,q]$ and any $z\in\mathbb{R}^n$,

$$\begin{aligned}
&2^{jn}\int_{\mathbb{R}^n}\frac{|A_Q(\varphi_j*\vec{f})(y)|^q}{(1+2^j|z-y|)^{\lambda nq}}\,dy\\
&\quad\le \sup_{v\in\mathbb{R}^n}\left[\frac{|A_Q(\varphi_j*\vec{f})(v)|}{(1+2^j|z-v|)^{\lambda n}}\right]^{q-A}2^{jn}\int_{\mathbb{R}^n}\frac{|A_Q(\varphi_j*\vec{f})(y)|^A}{(1+2^j|z-y|)^{\lambda nA}}\,dy\\
&\quad\lesssim \left[2^{jn}\int_{\mathbb{R}^n}\frac{|A_Q(\varphi_j*\vec{f})(y)|^A}{(1+2^j|z-y|)^{\lambda nA}}\,dy\right]^{q/A}.
\end{aligned}\tag{3.44}$$

Notice that, for any $k,u\in\mathbb{Z}^n$, $j\in\mathbb{Z}$, $z\in Q_{ju}$, and $y\in Q_{jk}$,

$$(1+2^j|z-y|)\sim 1+|k-u|.\tag{3.45}$$

Since $\lambda\in(1/\min\{1,p,q\}+\beta/(np),\infty)$, then it follows that there exists an $A\in(0,\min\{1,p,q\})$ such that $A[\lambda-\frac{\beta}{np}]>1$. By this, (3.45), (3.44), Lemma 3.5, the disjointness of Q_{ju} for any $u\in\mathbb{Z}^n$, and Lemma 3.7, we conclude that, for any $x\in\mathbb{R}^n$,

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\sum_{Q\in\mathcal{D}_j}2^{j\alpha q}2^{jn}\sup_{z\in Q}\int_{\mathbb{R}^n}\frac{|A_Q(\varphi_j*\vec{f})(y)|^q}{(1+2^j|z-y|)^{\lambda nq}}\,dy\mathbf{1}_Q(x)\\
&\quad\lesssim\sum_{j\in\mathbb{Z}}\sum_{u\in\mathbb{Z}^n}\sup_{z\in Q_{ju}}\left[2^{jn}\sum_{k\in\mathbb{Z}^n}\int_{Q_{jk}}\frac{\|A_{Q_{ju}}A_{Q_{jk}}^{-1}\|^A|2^{j\alpha}A_{Q_{jk}}(\varphi_j*\vec{f})(y)|^A}{(1+2^j|z-y|)^{\lambda nA}}\,dy\right]^{q/A}\mathbf{1}_{Q_{ju}}(x)\\
&\quad\lesssim\sum_{j\in\mathbb{Z}}\left[\sum_{u\in\mathbb{Z}^n}\sum_{k\in\mathbb{Z}^n}(1+|k-u|)^{-\lambda nA+\frac{\beta A}{p}}2^{jn}\int_{Q_{jk}}\left|2^{j\alpha}A_{Q_{jk}}\left(\varphi_j*\vec{f}\right)(y)\right|^A dy\mathbf{1}_{Q_{ju}}(x)\right]^{q/A}\\
&\quad\lesssim\sum_{j\in\mathbb{Z}}\left[\mathcal{M}\left(\sum_{Q\in\mathcal{D}_j}\left(\left|2^{j\alpha}A_Q\left(\varphi_j*\vec{f}\right)\right|\mathbf{1}_Q\right)^A\right)(x)\right]^{q/A}.
\end{aligned}$$

From this, $A\in(0,\min\{p,q\})$, the Fefferman–Stein vector-valued maximal inequality, and Lemma 2.26, we deduce that, for any $\vec{f}\in[\mathcal{S}_\infty'(\mathbb{R}^n)]^m$,

$$\begin{aligned}
\left\|\vec{f}\right\|_{\dot{F}_p^{\alpha,q}(\{A_Q\})}^{\clubsuit}&\lesssim\left\|\left\{\sum_{j\in\mathbb{Z}}\left[\mathcal{M}\left(\sum_{Q\in\mathcal{D}_j}\left[\left|2^{j\alpha}A_Q\varphi_j*\vec{f}\right|\mathbf{1}_Q\right]^A\right)\right]^{q/A}\right\}^{1/q}\right\|_{L^p(\mathbb{R}^n)}\\
&\sim\left\|\left[\mathcal{M}\left(\sum_{Q\in\mathcal{D}_j}\left[\left|2^{j\alpha}A_Q\varphi_j*\vec{f}\right|\mathbf{1}_Q\right]^A\right)\right]_{j\in\mathbb{Z}}\right\|_{L^{p/A}(\ell^{q/A})}^{1/A}
\end{aligned}$$

$$\lesssim \left\|\vec{f}\right\|_{\dot{F}_p^{\alpha,q}(\{A_Q\})} \sim \left\|\vec{f}\right\|_{\dot{F}_p^{\alpha,q}(W)},$$

which implies that (3.38) holds true. By both (3.36) and (3.38), we obtain (3.34). Using (3.34) and (3.33), we find that (3.32) holds true for any $\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$, which completes the proof of Theorem 3.14. □

Remark 3.15. (i) Theorem 3.14 when $m = 1$, $W = 1$ and $q \in (0, \infty)$ is a part of [6, Theorem 3.2] which is the g_λ^*-function characterization of Triebel–Lizorkin spaces $\dot{F}_p^{\alpha,q}(\mathbb{R}^n)$.

(ii) Let $p \in (0, \infty)$, $q \in (0, \infty)$, $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$, and β be the doubling exponent of W. We should point out that the range of λ in Theorem 3.14 does not coincide with the one in [6, Theorem 3.2], namely, $(\frac{1}{\min\{p,q\}}, \infty)$. It is still unclear whether or not Theorem 3.14 still holds true when $\lambda \in (\frac{1}{\min\{p,q\}}, \frac{1}{\min\{1,p,q\}} + \frac{\beta}{np}]$.

4 Fourier Multiplier

In this section, we study the mapping property on $\dot{F}_p^{\alpha,q}(W)$ for a class of Fourier multipliers, which was originally introduced by Cho [6].

First, we denote by the *symbol* $C(\mathbb{R}^n \setminus \{\mathbf{0}\})$ the space of all continuous functions on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ and recall the definition of the space $C^\ell(\mathbb{R}^n \setminus \{\mathbf{0}\})$. For any $\ell \in \mathbb{N}$, let

$$C^\ell(\mathbb{R}^n \setminus \{\mathbf{0}\}) := \{f \in C(\mathbb{R}^n \setminus \{\mathbf{0}\}) : \ \partial^\sigma f \in C(\mathbb{R}^n \setminus \{\mathbf{0}\}),\ \forall\, \sigma \in \mathbb{Z}^n \text{ and } |\sigma| \leq \ell\}.$$

For a given $\ell \in \mathbb{N}$ and a given $s \in \mathbb{R}$, assume that $m \in C^\ell(\mathbb{R}^n \setminus \{\mathbf{0}\})$ satisfies that, for any $\sigma \in \mathbb{Z}_+^n$ and $|\sigma| \leq \ell$,

$$\sup_{R\in(0,\infty)} \left[R^{-n+2s+2|\sigma|} \int_{R\leq|\xi|<2R} \left|\partial^\sigma m(\xi)\right|^2 \, d\xi\right] \leq A_\sigma < \infty. \tag{4.1}$$

Remark 4.1. When $s = 0$ and $\ell \in \mathbb{N}$, (4.1) is known as the Hörmander condition (see, for instance, [33, p. 263]). Typical examples are given by the kernels of the Riesz transforms $R^{(d)}$, where

$$(\widehat{R^{(d)} f})(\xi) := -i(\xi_d/|\xi|)\widehat{f}(\xi)$$

for any $\xi := (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $f \in \mathcal{S}(\mathbb{R}^n)$, and $d \in \{1, \ldots, n\}$. When $s \neq 0$ and $\ell \in \mathbb{N}$, a typical example satisfying (4.1) is given by $m(\xi) := |\xi|^{-s}$ for any $\xi \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.

Let K be a compact set of $\mathbb{R}^n$. Then $\vec{f} := (f_1, \ldots, f_m)^T \in [\mathcal{S}(\mathbb{R}^n)]^m$ or $\vec{f} \in [\mathcal{S}'(\mathbb{R}^n)]^m$ is said to have compact support set K, denoted by $\operatorname{supp} \vec{f} \subset K$, if, for any $d \in \{1, \ldots, m\}$, $\operatorname{supp} f_d \subset K$. From [17, Theorem 2.3.21], it follows that, for any $\vec{f} \in [\mathcal{S}'(\mathbb{R}^n)]^m$ with $\operatorname{supp} \widehat{\vec{f}} \subset K$, $\vec{f} \in [L^1_{\mathrm{loc}}(\mathbb{R}^n)]^m$. The following lemma is a part of [1, Corollary 6.13].

Lemma 4.2. *Let K be a compact subset of $\mathbb{R}^n$ and $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$. If $p \in (0, 1)$ and $N \in (\beta/p + n, \infty) \cap \mathbb{Z}_+$, or if $p \in (1, \infty)$ and $N \in (\beta/p, \infty) \cap \mathbb{Z}_+$, where β is the doubling exponent of W, then there exists a positive constant C such that, for any $\vec{f} \in [\mathcal{S}'(\mathbb{R}^n)]^m$ with* $\operatorname{supp} \widehat{\vec{f}} \subset K$,

$$\sup_{x\in\mathbb{R}^n} \frac{|\vec{f}(x)|}{(1 + |x|)^N} \leq C \left[\int_{\mathbb{R}^n} \left|\left(W^{1/p} \vec{f}\right)(x)\right|^p \, dx\right]^{1/p}. \tag{4.2}$$

Moreover, if $\operatorname{supp}\widehat{\vec{f}} \subset \{x \in \mathbb{R}^n : |x| \le 2^j\}$*, where* $j \in \mathbb{Z}_+$*, then, for any* N *as above, there exists a positive constant* C *such that*

$$\sup_{x\in\mathbb{R}^n} \frac{|\vec{f}(x)|}{(1+b|x|)^N} \le C4^{jn/p}\left[\int_{\mathbb{R}^n}\left|\left(W^{1/p}\vec{f}\right)(x)\right|^p\,dx\right]^{1/p}. \tag{4.3}$$

Lemma 4.3. *Let* $\ell \in \mathbb{N}$ *and* $m \in C^\ell(\mathbb{R}^n \setminus \{\mathbf{0}\})$ *be the same as in* (4.1) *with* $s \in \mathbb{R}$*. Then* $m \in \mathcal{S}'(\mathbb{R}^n)$*.*

Proof. By $m \in C^\ell(\mathbb{R}^n \setminus \{\mathbf{0}\})$, we conclude that, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}
\mathrm{I} :&= \int_{\mathbb{R}^n} |m(x)\varphi(x)|\,dx \\
&= \sum_{j\in\mathbb{Z}_+}\int_{2^j<|x|\le 2^{j+1}} |m(x)\varphi(x)|\,dx + \int_{|x|\le 1}\cdots \\
&\lesssim \rho_{\mu,0}(\varphi)\left[\sum_{j\in\mathbb{Z}_+}\int_{2^j<|x|\le 2^{j+1}} \frac{|m(x)|}{(1+|x|)^N}\,dx + \max\{|m(x)| :\ 0<|x|\le 1\}\right] \\
&\lesssim \rho_{\mu,0}(\varphi)\left[\sum_{j\in\mathbb{Z}_+}\int_{2^j<|x|\le 2^{j+1}} \frac{|m(x)|}{2^{jN}}\,dx + \max\{|m(x)| :\ 0<|x|\le 1\}\right],
\end{aligned} \tag{4.4}$$

where $\mu \in \mathbb{Z}_+^n$ with $|\mu| \le N$, N can be chosen as any positive integer, and $\rho_{\mu,0}(\varphi)$ is the same as in (3.30). Notice that, if $-n+2s \in [0,\infty)$, then $2^{j(-N+1)} \le 2^{j(-n+2s)}$ for any $j, N \in \mathbb{Z}_+$; if $-n+2s \in (-\infty,0)$, by choosing an $N \in (n-2s+1,\infty)$, we then have $2^{j(-N+1)} \le 2^{j(-n+2s)}$ for any $j, N \in \mathbb{Z}_+$. From this, (4.4), and (4.1), we deduce that, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}
\mathrm{I} &\lesssim \rho_{N,0}(\varphi)\left[\sum_{j\in\mathbb{Z}_+} 2^{-j}\int_{2^j<|x|\le 2^{j+1}} |m(x)|2^{j(-N+1)}\,dx + \max\{|m(x)| :\ 0<|x|\le 1\}\right] \\
&\lesssim \rho_{N,0}(\varphi)\left[\sum_{j\in\mathbb{Z}_+} 2^{-j}2^{j(-n+2s)}\int_{2^j<|x|\le 2^{j+1}} |m(x)|\,dx + \max\{|m(x)| :\ 0<|x|\le 1\}\right] \\
&\lesssim \rho_{N,0}(\varphi)\left[\sum_{j\in\mathbb{Z}_+} 2^{-j}A_0 + \max\{|m(x)| :\ 0<|x|\le 1\}\right] \sim \rho_{N,0}(\varphi),
\end{aligned}$$

which implies that $m \in \mathcal{S}'(\mathbb{R}^n)$. This finishes the proof of Lemma 4.3. □

Let $\ell \in \mathbb{N}$ and $m \in C^\ell(\mathbb{R}^n \setminus \{\mathbf{0}\})$ be the same as in (4.1) with $s \in \mathbb{R}$. By Lemma 4.3, we can define the *Fourier multiplier* T_m by setting, for any $\vec{f} \in [\mathcal{S}_\infty(\mathbb{R}^n)]^m$,

$$\left(T_m\vec{f}\right)^\wedge := m\widehat{\vec{f}}. \tag{4.5}$$

Furthermore, let K be the distribution whose Fourier transform is m.

Then we show that, via a suitable way, T_m can be defined on the space $\dot{F}^{\alpha,q}_p(W)$. To this end, let $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy both (T2) of Definition 2.18 and (2.5). For any $\vec{f} \in \dot{F}^{\alpha,q}_p(W)$ and $\phi \in \mathcal{S}_\infty(\mathbb{R}^n)$, let

$$\left\langle T_m\vec{f}, \phi\right\rangle := \sum_{j\in\mathbb{Z}} \vec{f} * \varphi_j * \psi_j * \widetilde{\phi} * K(\mathbf{0}), \tag{4.6}$$

where $\widetilde{\phi} = \phi(-\cdot)$. It is obvious that, when $\vec{f} \in [\mathcal{S}_\infty(\mathbb{R}^n)]^m$, then both $T_m\vec{f}$ in (4.5) and $T_m\vec{f}$ in (4.6) coincide in $[\mathcal{S}'_\infty(\mathbb{R}^n)]^m$. The following result shows that the right-hand side of (4.6) converges and $T_m\vec{f}$ in (4.6) is well defined for any $\vec{f} \in \dot{F}^{\alpha,q}_p(W)$.

Lemma 4.4. *Let $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$, $W \in A_p(\mathbb{R}^n,\mathbb{C}^m)$, and $\vec{f} \in \dot{F}^{\alpha,q}_p(W)$. Let $\ell \in (\beta/p + n/\min\{1,p\} + \frac{n}{2}, \infty)$, where β is the doubling exponent of W, and let $m \in C^\ell(\mathbb{R}^n \setminus \{\mathbf{0}\})$ be the same as in* (4.1) *with $s \in \mathbb{R}$. Then T_m in* (4.6) *is independent of the choice of the pair (φ,ψ) of Schwartz functions satisfying both (T2) of Definition 2.18 and* (2.5)*. Moreover, $T_m\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$ and $T_m\vec{f}$ in* (4.6) *is well defined.*

To show Lemma 4.4, we need the following lemmas, which are just [6, Lemma 4.1(i)] and [43, Lemma 2.2], respectively.

Lemma 4.5. *Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy that $\widehat{\psi}$ has compact support away from the origin. Let $\lambda \in (0,\infty)$, $\ell \in (\lambda + n/2, \infty)$, and m be the same as in* (4.1) *with $s \in \mathbb{R}$. Let K be the distribution whose Fourier transform is m. Then there exists a positive constant C such that, for any $j \in \mathbb{Z}$,*

$$\int_{\mathbb{R}^n} \left(1 + 2^j|z|\right)^\lambda \left|\left(K * \psi_j\right)(z)\right| \, dz \le C 2^{-js}.$$

Lemma 4.6. *For any given $M \in \mathbb{N}$, there exists a positive constant C such that, for any $\varphi, \psi \in \mathcal{S}_\infty(\mathbb{R}^n)$, $j, \ell \in \mathbb{Z}$, and $x \in \mathbb{R}^n$,*

$$\left|\varphi_j * \psi_\ell(x)\right| \le C \|\varphi\|_{S_{M+1}} \|\psi\|_{S_{M+1}} 2^{-|\ell-j|M} \frac{2^{-\min\{j,\ell\}M}}{(2^{-\min\{j,\ell\}} + |x|)^{n+M}},$$

where, for any $\varphi \in \mathcal{S}_\infty(\mathbb{R}^n)$,

$$\|\varphi\|_{S_M} := \sup_{\{\gamma \in \mathbb{Z}^n_+ : |\gamma| \le M\}} \sup_{x \in \mathbb{R}^n} \left|\partial^\gamma \varphi(x)\right| (1 + |x|)^{n+M+|\gamma|}.$$

Proof of Lemma 4.4. Let φ and ψ be a pair of Schwartz functions satisfying both (T2) of Definition 2.18 and (2.5). Let $\varphi^\star$ and $\psi^\star$ be another pair of Schwartz functions satisfying both (T2) of Definition 2.18 and (2.5). By this, Lemma 2.24, and $\phi \in \mathcal{S}_\infty(\mathbb{R}^n)$, we find that

$$\widetilde{\phi} = \sum_{t \in \mathbb{Z}} \varphi^\star_t * \psi^\star_t * \widetilde{\phi} \quad \text{in} \quad \mathcal{S}_\infty(\mathbb{R}^n). \tag{4.7}$$

Since φ and $\varphi^\star$ satisfy (T2) of Definition 2.18, it follows that there exists an $L \in \mathbb{N}$ such that, for any $|j - t| > L$,

$$\varphi_j * \varphi^\star_t = 0. \tag{4.8}$$

Let α, p, q, W, and m be the same as in this lemma. Let $\vec{f} \in \dot{F}^{\alpha,q}_p(W)$. To prove

$$\sum_{j \in \mathbb{Z}} \vec{f} * \varphi_j * \psi_j * \widetilde{\phi} * K(\mathbf{0})$$

converges, where K is the distribution whose Fourier transform is m, we first show that

$$\sum_{j \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \vec{f} * \varphi_j * \psi_j * \varphi^\star_t * \psi^\star_t * \widetilde{\phi} * K(\mathbf{0})$$

converges. To this end, by both (4.3) of Lemma 4.2 and Definition 2.18, we find that, for any $N\in\mathbb{Z}_+$ satisfying Lemma 4.2 and for any $\vec{f}\in\dot{F}^{\alpha,q}_p(W)$ and $j\in\mathbb{Z}_+$,

$$\sup_{x\in\mathbb{R}^n}\frac{|\varphi_j*\vec{f}(x)|}{(1+2^j|x|)^N}\lesssim 2^{2jn/p}\left\{\int_{\mathbb{R}^n}\left|W^{1/p}(x)\left(\varphi_j*\vec{f}\right)(x)\right|^p\,dx\right\}^{1/p}\lesssim 2^{j(n/p-\alpha)}\left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(W)}. \tag{4.9}$$

From this, $\varphi_j^\star*\psi_j^\star=(\varphi^\star*\psi^\star)_j\in\mathcal{S}_\infty(\mathbb{R}^n)$, Lemma 4.6, and the estimate that

$$1+2^j|z|\le\left(1+2^j|y|\right)\left(1+2^j|z-y|\right)$$

for any $j\in\mathbb{Z}$ and $y,z\in\mathbb{R}^n$, and Lemma 4.5 with $\ell>N+n/2$, we infer that

$$\begin{aligned}
&\sum_{j=0}^{\infty}\left|\vec{f}*\varphi_j*\psi_j*\varphi_j^\star*\psi_j^\star*\widetilde{\phi}*K(\mathbf{0})\right|\\
&\quad\le\sum_{j=0}^{\infty}\int_{\mathbb{R}^n}\left|\vec{f}*\varphi_j(-z)\right|\left|\left(\varphi^\star*\psi^\star\right)_j*\widetilde{\phi}*\psi_j*K(z)\right|\,dz\\
&\quad\lesssim\sum_{j=0}^{\infty}2^{j(2n/p-\alpha)}\left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(W)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(1+2^j|z|\right)^N\\
&\qquad\times\left|\left(\varphi^\star*\psi^\star\right)_j*\widetilde{\phi}(z-y)\right|\left|\psi_j*K(z)\right|\,dy\,dz\\
&\quad\lesssim\sum_{j=0}^{\infty}2^{j(2n/p-\alpha-M)}\left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(W)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(1+2^j|z|)^N}{(1+|z-y|)^{n+M}}\left|\psi_j*K(y)\right|\,dy\,dz\\
&\quad\lesssim\sum_{j=0}^{\infty}2^{j(N+2n/p-\alpha-M)}\left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(W)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(1+2^j|y|)^N}{(1+|z-y|)^{n+M-N}}\left|\psi_j*K(y)\right|\,dy\,dz\\
&\quad\sim\sum_{j=0}^{\infty}2^{j(N+2n/p-\alpha-M)}\left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(W)}\int_{\mathbb{R}^n}\left(1+2^j|y|\right)^N\left|\psi_j*K(y)\right|\,dy\\
&\quad\lesssim\sum_{j=0}^{\infty}2^{j(N+2n/p-\alpha-M-s)}\left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(W)}\sim\left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(W)},
\end{aligned}$$

where the implicit positive constants depend on $\psi^\star,\varphi^\star$, and ϕ, and where $M\in\mathbb{N}$ is chosen to be sufficiently large such that $M>\max\{N,N+2n/p-s-\alpha\}$. On the other hand, by (4.9), Lemma 4.6, and the estimate that $1+2^j|z|\le(1+2^j|y|)(1+2^j|z-y|)$ for any $j\in\mathbb{Z}$ and $y,z\in\mathbb{R}^n$, and Lemma 4.5, we find that

$$\begin{aligned}
&\sum_{j=-\infty}^{-1}\left|\vec{f}*\varphi_j*\psi_j*\varphi_j^\star*\psi_j^\star*\widetilde{\phi}*K(\mathbf{0})\right|\\
&\quad\le\sum_{j=-\infty}^{-1}\int_{\mathbb{R}^n}\left|\vec{f}*\varphi_j(-z)\right|\left|\left(\varphi^\star*\psi^\star\right)_j*\widetilde{\phi}*\psi_j*K(z)\right|\,dz
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{j=-\infty}^{-1} 2^{j(-N-\alpha)} \left\|\vec{f}\right\|_{\dot{F}_p^{\alpha,q}(W)} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(1+2^j|z|\right)^N \\
&\quad\times \left|\left(\varphi^\star * \psi^\star\right)_j * \widetilde{\phi}(z-y)\right| \left|\psi_j * K(z)\right| \,dy\,dz \\
&\lesssim \sum_{j=-\infty}^{-1} 2^{j(-N-\alpha+n+M)} \left\|\vec{f}\right\|_{\dot{F}_p^{\alpha,q}(W)} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(1+2^j|z|)^N}{(1+2^j|z-y|)^{n+M}} \\
&\quad\times \left|\psi_j * K(y)\right| \,dy\,dz \\
&\lesssim \sum_{j=-\infty}^{-1} 2^{j(-N-\alpha+n+M)} \left\|\vec{f}\right\|_{\dot{F}_p^{\alpha,q}(W)} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(1+2^j|y|)^N}{(1+2^j|z-y|)^{n+M-N}} \\
&\quad\times \left|\psi_j * K(y)\right| \,dy\,dz \\
&\sim \sum_{j=-\infty}^{-1} 2^{j(-N-\alpha+M)} \left\|\vec{f}\right\|_{\dot{F}_p^{\alpha,q}(W)} \int_{\mathbb{R}^n} \left(1+2^j|y|\right)^N \left|\psi_j * K(y)\right| \,dy \\
&\lesssim \sum_{j=-\infty}^{-1} 2^{j(-N-\alpha+M-s)} \left\|\vec{f}\right\|_{\dot{F}_p^{\alpha,q}(W)} \sim \left\|\vec{f}\right\|_{\dot{F}_p^{\alpha,q}(W)},
\end{aligned}
$$

where the implicit positive constants depend on $\psi^\star, \varphi^\star$, and ϕ, and where $M \in \mathbb{N}$ is chosen to be sufficiently large such that $M > \max\{N, s+\alpha+N\}$. From an argument similar to that used in the above two estimations, we deduce that

$$
\sum_{j\in\mathbb{Z}} \sum_{t=j-L}^{j+L} \left|\vec{f} * \varphi_j * \psi_j * \varphi_t^\star * \psi_t^\star * \widetilde{\phi} * K(\mathbf{0})\right| < \infty. \tag{4.10}
$$

By (4.10) and (4.8) with $|j-t| > L$, we conclude that

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}} \sum_{t\in\mathbb{Z}} \left|\vec{f} * \varphi_j * \psi_j * \varphi_t^\star * \psi_t^\star * \widetilde{\phi} * K(\mathbf{0})\right| \\
&\quad = \sum_{j\in\mathbb{Z}} \sum_{t=j-L}^{j+L} \left|\vec{f} * \varphi_j * \psi_j * \varphi_t^\star * \psi_t^\star * \widetilde{\phi} * K(\mathbf{0})\right| < \infty.
\end{aligned}
$$

Using (4.2) of Lemma 4.2, Definition 2.18, Lemma 4.6, the estimate that

$$
1 + 2^t|x| \le \left(1+2^t|y|\right)\left(1+2^t|x-y|\right)
$$

for any $x, y \in \mathbb{R}^n$ and $t \in \mathbb{Z}$, and Lemma 4.5, we find that, for any fixed $j \in \mathbb{Z}$,

$$
\begin{aligned}
&\sum_{t\in\mathbb{Z}} \int_{\mathbb{R}^n} \left|\vec{f} * \varphi_j * \psi_j(-x)\left(\varphi_t^\star * \psi_t^\star * \widetilde{\phi} * K\right)(x)\right| \,dx \\
&\quad\lesssim \left\|\vec{f}\right\|_{\dot{F}_p^{\alpha,q}(W)} \sum_{t\in\mathbb{Z}} \int_{\mathbb{R}^n} (1+|x|)^N \int_{\mathbb{R}^n} \left|\psi_t^\star * \widetilde{\phi}(x-y)\right| \left|\varphi_t^\star * K(y)\right| \,dy\,dx \\
&\quad\lesssim \left\|\vec{f}\right\|_{\dot{F}_p^{\alpha,q}(W)} \left[\sum_{t=0}^{\infty} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{2^{-tR}(1+|x|)^N}{(1+|x-y|)^{n+R}} \left|\varphi_t^\star * K(y)\right| \,dy\,dx\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{t=-\infty}^{-1}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{2^{t(n+R)}(1+|x|)^N}{(1+2^t|x-y|)^{n+R}}\left|\varphi_t^\star * K(y)\right|\,dy\,dx\Bigg]\\
&\lesssim \left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(W)}\Bigg[\sum_{t=0}^{\infty}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{2^{-tR}(1+|x|)^N}{(1+|x-y|)^{n+R}}\left|\varphi_t^\star * K(y)\right|\,dy\,dx\\
&+\sum_{t=-\infty}^{-1}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{2^{t(n+R-N)}(1+2^t|x|)^N}{(1+2^t|x-y|)^{n+R}}\left|\varphi_t^\star * K(y)\right|\,dy\,dx\Bigg]\\
&\lesssim \left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(W)}\Bigg[\sum_{t=0}^{\infty}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{2^{-tR}(1+|y|)^N}{(1+|x-y|)^{n+R-N}}\left|\varphi_t^\star * K(y)\right|\,dy\,dx\\
&+\sum_{t=-\infty}^{-1}2^{t(n+R-N)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(1+2^t|y|)^N}{(1+2^t|x-y|)^{n+R-N}}\left|\varphi_t^\star * K(y)\right|\,dy\,dx\Bigg]\\
&\lesssim \left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(W)}\Bigg[\sum_{t=0}^{\infty}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{2^{-tR}(1+2^t|y|)^N}{(1+|x-y|)^{n+R-N}}\left|\varphi_t^\star * K(y)\right|\,dy\,dx\\
&+\sum_{t=-\infty}^{-1}2^{t(n+R-N)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(1+2^\ell|y|)^N}{(1+2^t|x-y|)^{n+R-N}}\left|\varphi_t^\star * K(y)\right|\,dy\,dx\Bigg]\\
&\sim \left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(W)}\Bigg[\sum_{t=0}^{\infty}2^{-tR}\int_{\mathbb{R}^n}\left(1+2^t|y|\right)^N\left|\varphi_t^\star * K(y)\right|\,dy\\
&+\sum_{t=-\infty}^{-1}2^{t(R-N)}\int_{\mathbb{R}^n}\left(1+2^t|y|\right)^N\left|\varphi_t^\star * K(y)\right|\,dy\Bigg]\\
&\sim \left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(W)}\Bigg[\sum_{t=0}^{\infty}2^{-tR}\int_{\mathbb{R}^n}\left(1+2^t|y|\right)^N\left|\varphi_t^\star * K(y)\right|\,dy\\
&+\sum_{t=-\infty}^{-1}2^{t(R-N)}\int_{\mathbb{R}^n}\left(1+2^t|y|\right)^N\left|\varphi_t^\star * K(y)\right|\,dy\Bigg]\\
&\lesssim \left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(W)}\left[\sum_{t=0}^{\infty}2^{-t(R+s)}+\sum_{t=-\infty}^{-1}2^{t(R-N-s)}\right]\sim\left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(W)},
\end{aligned}
$$

where the implicit positive constants depend on j,α,p, and n, and where $R\in\mathbb{N}$ is chosen to be sufficiently large such that $R>\max\{N,-s,N+s\}$ with N the same as in (4.9). From this and (4.7), we deduce that

$$
\begin{aligned}
&\vec{f}*\varphi_j*\psi_j*\widetilde{\phi}*K(\mathbf{0})\\
&\quad=\int_{\mathbb{R}^n}\vec{f}*\varphi_j*\psi_j(-x)\widetilde{\phi}*K(x)\,dx\\
&\quad=\int_{\mathbb{R}^n}\vec{f}*\varphi_j*\psi_j(-x)\left\langle K,\widetilde{\phi}(x-\cdot)\right\rangle\,dx\\
&\quad=\int_{\mathbb{R}^n}\vec{f}*\varphi_j*\psi_j(-x)\sum_{t\in\mathbb{Z}}\left\langle K,\varphi_t^\star*\psi_t^\star*\widetilde{\phi}(x-\cdot)\right\rangle\,dx
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{t\in\mathbb{Z}} \int_{\mathbb{R}^n} \vec{f} * \varphi_j * \psi_j(-x) \left\langle K, \varphi_t^\star * \psi_t^\star * \widetilde{\phi}(x-\cdot) \right\rangle dx \\
&= \sum_{t\in\mathbb{Z}} \vec{f} * \varphi_j * \psi_j * \varphi_t^\star * \psi_t^\star * \widetilde{\phi} * K(\mathbf{0}).
\end{aligned}
$$

By this and (4.8), we find that

$$
\sum_{j\in\mathbb{Z}} \vec{f} * \varphi_j * \psi_j * \widetilde{\phi} * K(\mathbf{0}) = \sum_{j\in\mathbb{Z}} \sum_{t\in\mathbb{Z}} \vec{f} * \varphi_j * \psi_j * \varphi_t^\star * \psi_t^\star * \widetilde{\phi} * K(\mathbf{0}),
$$

which, together with (4.7), $K \in \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{S}'_\infty(\mathbb{R}^n)$, and Lemma 2.24, further implies that

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}} \vec{f} * \varphi_j * \psi_j * \widetilde{\phi} * K(\mathbf{0}) \\
&\quad= \sum_{t\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \vec{f} * \varphi_t^\star * \psi_t^\star * \varphi_j * \psi_j * \widetilde{\phi} * K(\mathbf{0}) \\
&\quad= \sum_{t\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} \vec{f} * \varphi_t^\star * \psi_t^\star(-x) \left\langle K, \varphi_j * \psi_j * \widetilde{\phi}(\cdot - x) \right\rangle dx \\
&\quad= \sum_{t\in\mathbb{Z}} \int_{\mathbb{R}^n} \vec{f} * \varphi_t^\star * \psi_t^\star(-x) \sum_{j\in\mathbb{Z}} \left\langle K, \varphi_j * \psi_j * \widetilde{\phi}(\cdot - x) \right\rangle dx \\
&\quad= \sum_{t\in\mathbb{Z}} \int_{\mathbb{R}^n} \vec{f} * \varphi_t^\star * \psi_t^\star(-x) \left\langle K, \widetilde{\phi}(\cdot - x) \right\rangle dx \\
&\quad= \sum_{t\in\mathbb{Z}} \vec{f} * \varphi_t^\star * \psi_t^\star * \widetilde{\phi} * K(\mathbf{0}).
\end{aligned}
$$

Thus, $T_m\vec{f}$ in (4.6) is independent of the choice of the pair (φ, ψ) satisfying both (T2) of Definition 2.18 and (2.5). Moreover, the above argument also implies that $T_m\vec{f} \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m$ is well defined, which completes the proof of Lemma 4.4. □

Lemma 4.7. *Let $p \in (0,\infty)$ and $W \in A_p(\mathbb{R}^n, \mathbb{C}^m)$. Assume that $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy both (T1) and (T2) of Definition 2.18 and* (2.5)*. Let $\lambda \in (\beta/p + n/\min\{1, p\}, \infty)$, where β is the doubling exponent of W, and let $\ell \in (\lambda + n/2, \infty)$ and $m \in C^\ell(\mathbb{R}^n \setminus \{\mathbf{0}\})$ be the same as in* (4.1) *with $s \in \mathbb{R}$. Let $\phi := \varphi * \psi$ and T_m be the same as in* (4.6)*. Then there exists a positive constant C such that, for any $\vec{f} \in \dot{F}^{\alpha,q}_p(W)$ and $x \in \mathbb{R}^n$,*

$$
\left| W^{1/p}(x) \left(T_m\vec{f} * \phi_j \right)(x) \right| \le C 2^{-js} \sup_{y\in\mathbb{R}^n} \frac{|W^{1/p}(x)(\varphi_j * \vec{f})(y)|}{(1 + 2^j|x-y|)^\lambda}.
$$

Proof. Let all the symbols be the same as in the present lemma. Let K be the distribution whose Fourier transform is m, $\phi = \varphi * \psi$, and $\vec{f} \in \dot{F}^{\alpha,q}_p(W)$. We first show that, for any $j \in \mathbb{Z}$,

$$
\vec{f} * \varphi_j * \psi_j * K \in [\mathcal{S}'_\infty(\mathbb{R}^n)]^m.
$$

Indeed, by (4.9), the estimate that $1 + 2^j|x-y| \le (1 + 2^j|x|)(1 + 2^j|y|)$ for any $x, y \in \mathbb{R}^n$, and Lemma 4.5, we conclude that, for any $\gamma \in \mathcal{S}_\infty(\mathbb{R}^n)$,

$$
\left| \left\langle \vec{f} * \varphi_j * \psi_j * K, \gamma \right\rangle \right|
$$

$$
\begin{aligned}
&\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|\vec{f}*\varphi_j(x-y)\right|\left|\psi_j*K(y)\right||\gamma(x)|\,dy\,dx\\
&\lesssim \left\|\vec{f}\right\|_{\dot{F}_p^{\alpha,q}(W)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(1+2^j|x-y|\right)^N\left|\psi_j*K(y)\right||\gamma(x)|\,dy\,dx\\
&\lesssim \left\|\vec{f}\right\|_{\dot{F}_p^{\alpha,q}(W)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(1+2^j|x|\right)^N\left(1+2^j|y|\right)^N\left|\psi_j*K(y)\right||\gamma(x)|\,dy\,dx\\
&\lesssim \left\|\vec{f}\right\|_{\dot{F}_p^{\alpha,q}(W)}\int_{\mathbb{R}^n}\left(1+2^j|x|\right)^N|\gamma(x)|\,dx\\
&\lesssim \left\|\vec{f}\right\|_{\dot{F}_p^{\alpha,q}(W)}\rho_{N+2n,0}(\gamma),
\end{aligned}
$$

where the implicit positive constants depend on $\alpha\in\mathbb{R}$, $j\in\mathbb{Z}$, and $s\in\mathbb{R}$, and where N is the same as in (4.9) and $\rho_{N+2n,0}$ is the same as in (3.30), which implies that the above claim holds true. Using this claim and Lemma 2.24, we find that

$$
\sum_{\ell\in\mathbb{Z}}\vec{f}*\varphi_j*\psi_j*K*\varphi_\ell*\psi_\ell
$$

converges in $[\mathcal{S}'_\infty(\mathbb{R}^n)]^m$. From (4.6) and Lemmas 4.4 and 2.24, we deduce that, for any $\vec{f}\in\dot{F}_p^{\alpha,q}(W)$, $T_m\vec{f}\in[\mathcal{S}'_\infty(\mathbb{R}^n)]^m$ and, for any $\eta\in\mathcal{S}_\infty(\mathbb{R}^n)$,

$$
\begin{aligned}
\langle T_m\vec{f}*\phi_j,\eta\rangle &= \langle T_m\vec{f},\eta*\widetilde{\phi_j}\rangle\\
&=\sum_{\ell\in\mathbb{Z}}\vec{f}*\varphi_\ell*\psi_\ell*\widetilde{\eta}*\varphi_j*\psi_j*K(\mathbf{0})\\
&=\sum_{\ell\in\mathbb{Z}}\left\langle\vec{f}*\varphi_\ell*\psi_\ell*\varphi_j*\psi_j*K,\eta\right\rangle\\
&=\left\langle\sum_{\ell\in\mathbb{Z}}\vec{f}*\varphi_\ell*\psi_\ell*\varphi_j*\psi_j*K,\eta\right\rangle\\
&=\left\langle\vec{f}*\varphi_j*\psi_j*K,\eta\right\rangle.
\end{aligned}
\tag{4.11}
$$

Let $\gamma\in\mathcal{S}(\mathbb{R}^n)$ satisfy both $\widehat{\gamma}=1$ on $\operatorname{supp}\widehat{\phi}$ and $\operatorname{supp}\gamma\subseteq\{x\in\mathbb{R}^n:0<|x|<\pi\}$. For any $j\in\mathbb{Z}$ and $x\in\mathbb{R}^n$, let $\gamma_j(x):=2^{jn}\gamma(2^jx)$. Then, by (4.11) with $\gamma\in\mathcal{S}_\infty(\mathbb{R}^n)$, [17, Theorem 2.3.21], and $\phi=\varphi*\psi$ with $\operatorname{supp}\widehat{\phi}\subset\{x\in\mathbb{R}^n:|x|<\pi\}$, we find that

$$
\begin{aligned}
T_m\vec{f}*\phi_j(x) &= T_m\vec{f}*\phi_j*\gamma_j(x)=\vec{f}*\varphi_j*\psi_j*K*\gamma_j(x)\\
&=\vec{f}*\varphi_j*\psi_j*K(x)
\end{aligned}
$$

for any $x\in\mathbb{R}^n$. From this and Lemma 4.5, we infer that, for any $x\in\mathbb{R}^n$,

$$
\begin{aligned}
&\left|W^{1/p}(x)\left(T_m\vec{f}*\phi_j\right)(x)\right|\\
&\quad\le\int_{\mathbb{R}^n}\frac{|W^{1/p}(x)(\varphi_j*\vec{f})(x-y)|}{(1+2^j|y|)^\lambda}\left(1+2^j|y|\right)^\lambda\left|\left(K*\psi_j\right)(y)\right|\,dy\\
&\quad\lesssim 2^{-js}\sup_{y\in\mathbb{R}^n}\frac{|W^{1/p}(x)(\varphi_j*\vec{f})(y)|}{(1+2^j|x-y|)^\lambda}.
\end{aligned}
$$

This finishes the proof of Lemma 4.7. ◻

Theorem 4.8. *Let* $\alpha \in \mathbb{R}$, $p \in (0,\infty)$, $q \in (0,\infty]$, *and* $W \in A_p(\mathbb{R}^n,\mathbb{C}^m)$. *Let* $\ell \in (\frac{n}{\min\{1,p,q\}}+\frac{\beta}{p}+\frac{n}{2},\infty)$ *and* $m \in C^\ell(\mathbb{R}^n\setminus\{\mathbf{0}\})$ *be the same as in* (4.1) *with* $s \in \mathbb{R}$, *where* β *is the doubling exponent of* W. *Let* T_m *be the same as in* (4.6). *Then there exists a positive constant* C *such that, for any* $\vec{f} \in \dot{F}^{\alpha,q}_p(W)$,

$$\left\|T_m\vec{f}\right\|_{\dot{F}^{\alpha+s,q}_p(W)} \le C\left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(W)}.$$

Proof. Let ϕ, φ, and ψ be the same as in the present lemma. Since $\phi = \varphi * \psi$, it follows that ϕ satisfies both (T1) and (T2) of Definition 2.18. Using this, Definition 2.18, Lemmas 2.26 and 4.7, and Theorem 3.1, we find that

$$\begin{aligned}\left\|T_m\vec{f}\right\|_{\dot{F}^{\alpha+s,q}_p(W)} &= \left\|\left[\sum_{j\in\mathbb{Z}}\left|2^{j(\alpha+s)}W^{1/p}\left(T_m\vec{f}*\phi_j\right)\right|^q\right]^{1/q}\right\|_{L^p(\mathbb{R}^n)}\\ &\lesssim \left\|\left[\sum_{j\in\mathbb{Z}}\left|2^{j\alpha}\sup_{y\in\mathbb{R}^n}\frac{|W^{1/p}(\cdot)(\varphi_j*\vec{f})(y)|}{(1+2^j|\cdot-y|)^\lambda}\right|^q\right]^{1/q}\right\|_{L^p(\mathbb{R}^n)}\\ &\sim \left\|\vec{f}\right\|_{\dot{F}^{\alpha,q}_p(W)},\end{aligned}$$

which completes the proof of Theorem 4.8. □

Remark 4.9. Theorem 4.8 when $m = 1$ and $W = 1$ is a part of [45, Theorem 1.5(i)].

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Qi Wang, Dachun Yang (Corresponding author), and Yangyang Zhang

Laboratory of Mathematics and Complex Systems (Ministry of Education of China), School of Mathematical Sciences, Beijing Normal University, Beijing 100875, The People's Republic of China

E-mails: wqmath@mail.bnu.edu.cn (Q. Wang)
dcyang@bnu.edu.cn (D. Yang)
yangyzhang@mail.bnu.edu.cn (Y. Zhang)