WHY BOOTSTRAPPING FOR J-HOLOMORPHIC CURVES FAILS IN C^k

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Abstract. We present a simple example for the failure of the Calderón–Zygmund estimate for the $\overline{\partial}$-operator when the Sobolev (k, p)-norms are replaced by the C^k-norms. This example is discussed in the context of elliptic bootstrapping, Fredholm theory, and the regularity of J-holomorphic curves.

1. Introduction

In the theory of elliptic partial differential equations, with the Laplace equation as the prototype, it is well known (see [4, 6], for instance) that regularity results can be established for solutions lying in Sobolev or Hölder spaces, with the help of Calderón–Zygmund or Schauder estimates, respectively.

In symplectic topology, moduli spaces of J-holomorphic curves are described as solution sets of a nonlinear Cauchy–Riemann equation. Since the implicit function theorem fails in the Fréchet space of smooth maps, one needs to work with a Banach space of maps having lower regularity. A Calderón–Zygmund estimate for the $\overline{\partial}$-operator then is essential for two purposes:

(i) regularity results for J-holomorphic curves, in the sense that solutions of the nonlinear Cauchy–Riemann equation in the Sobolev space $W^{1,p}$ actually turn out to be of class C^∞;
(ii) existence of C^k-bounds for all k that guarantee compactness of the relevant moduli space of J-holomorphic curves.

The Calderón–Zygmund estimate allows one to bootstrap from $W^{k,p}$ to $W^{k+1,p}$; smoothness and C^k-bounds (for $p > 2$) then follow from the Sobolev embedding theorem and the corresponding Sobolev inequality.

Roughly speaking, the estimates say that a (weak) solution $u \in W^{k,p}$ of the inhomogeneous Laplace equation $\Delta u = f$ is two derivatives more regular than f. The literature abounds with examples that this statement fails in the smooth theory, that is, from Δu, understood in the distributional sense, of class C^k one cannot, in general, conclude that $u \in C^{k+2}$. This means that a C^k-analogue of these estimates cannot be formulated in a sensible way, because $\|u\|_{C^{k+2}}$ may not be defined.

However, we have not seen it emphasised that even when the correct order of differentiability is assumed *a priori*, the Calderón–Zygmund (or Schauder) estimates fail when the Sobolev (or Hölder) norms are replaced by C^k-norms. We allow that this may be fairly apparent to the more analytically inclined.

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In this expository note we adapt an example of Sikorav [9] (where the C^k-estimate cannot be formulated) to define an explicit family of solutions of the inhomogeneous Cauchy–Riemann equation, having the correct regularity (in the C^k-theory), but violating the Calderón–Zygmund estimate. We also place this in the context of the Fredholm property of the Cauchy–Riemann operator, which is essential for showing that the relevant moduli space of J-holomorphic curves is a smooth manifold.

As we shall explain, in the theory of J-holomorphic curves one often deals with these estimates in a setting where the maps are known to be smooth. As a consequence, the consideration of Sobolev norms on such maps, or the introduction of Sobolev spaces of J-holomorphic curves, may seem to lack motivation. Our example clarifies why one has to work with Sobolev completions.

2. The Calderón–Zygmund estimate

In this section we formulate the Calderón–Zygmund estimate for the inhomogeneous Cauchy–Riemann equation and briefly discuss its relevance for the bootstrapping of J-holomorphic curves.

Let $B_R \subset \mathbb{C}$ be the open disc of radius R centred at 0. We write $C^\infty_{\mathrm{c}}(B_R, \mathbb{C}^n)$ for the space of compactly supported smooth maps $B_R \to \mathbb{C}^n$, and $W^{k+1,p}_0(B_R, \mathbb{C}^n)$ for its closure in the Sobolev space of $k+1$ times weakly differentiable maps of finite Sobolev $(k+1,p)$-norm. The Cauchy–Riemann operator is

$$\overline{\partial} := \partial_{\overline{z}} := \frac{1}{2}\big(\partial_x + \mathrm{i}\partial_y\big).$$

Likewise, we are going to set $\partial := \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)$. For a proof of the following estimate see [5].

Proposition 1. *For any $k \in \mathbb{N}_0$ and real numbers $p > 1$ and $R > 0$, there is a positive constant $c = c(k,p,R)$ such that*

$$\|u\|_{k+1,p} \leq c\|\overline{\partial}u\|_{k,p} \quad \text{for all} \quad u \in W^{k+1,p}_0(B_R, \mathbb{C}^n).$$

Without the assumption on compact support, one needs to add the term $c\|u\|_{k,p}$ on the right-hand side, as can be seen by a partition of unity argument. This is the more common formulation of the Calderón–Zygmund estimate (and sometimes referred to as a semi-Fredholm estimate, cf. [5]).

One first has to prove the proposition for $u \in C^\infty_{\mathrm{c}}(B_R, \mathbb{C}^n)$, the stated version then follows by writing $u \in W^{k+1,p}_0(B_R, \mathbb{C}^n)$ as a limit of compactly supported smooth maps.

Remark. For $p = 2$, the proof of Proposition 1 simplifies considerably, see [1, Section 4.2] or [5, Section III.1.2], but for the subsequent application of the Sobolev embedding theorem one needs $p > 2$.

We are going to show by an example that there is no such uniform estimate $\|u\|_{C^{k+1}} \leq c\|\overline{\partial}u\|_{C^k}$.

Proposition 2. *For any $k \in \mathbb{N}_0$ there is a sequence (u_ν) in $C^{k+1}_{\mathrm{c}}(B_{1/2}, \mathbb{C})$ with $\|\overline{\partial}u_\nu\|_{C^k}$ bounded uniformly in ν, but $\|u_\nu\|_{C^{k+1}} \to \infty$ as $\nu \to \infty$.*

From Proposition 1, in [5] the regularity of J-holomorphic discs (with Lagrangian boundary condition) is established by a localisation argument and the difference quotient technique of Abbas and Hofer [1]. That difference quotient technique allows one to bootstrap from $W^{k,p}$ to $W^{k+1,p}$, but not from C^k to C^{k+1}, so a Calderón–Zygmund estimate in C^k would not be of help.

A different approach to the regularity of J-holomorphic curves can be found in [8, Section B.4]. Here the nonlinear Cauchy–Riemann equation $u_x + J(u)u_y = 0$ is reformulated as an inhomogeneous linear equation, and then one directly uses the regularity theory for the $\overline{\partial}$-operator. This approach would stumble at the first hurdle in the C^k-theory by Sikorav's example.

Much of the compactness theory of J-holomorphic curves as in [5] would go through in the smooth theory if one had a Calderón–Zygmund estimate (for smooth maps) in the C^k-norms. Our example shows why this hope is in vain.

3. The example

3.1. **Sikorav's example.** We begin with an example of a function $f\colon B_{1/2} \to \mathbb{C}$ that is not of class C^1, even though $\overline{\partial} f$ (in the distributional sense) is of class C^0. This is a slight modification (and correction) of an example presented by Sikorav [9], which is closely related to the standard example illustrating the corresponding phenomenon for the Laplace operator, see [4]. Set

$$f(z) = \begin{cases} z \log\log |z|^{-2} & \text{for } z \in B_{1/2} \setminus \{0\}, \\ 0 & \text{for } z = 0. \end{cases} \tag{1}$$

Then

$$\overline{\partial} f = -\frac{z}{\overline{z} \log |z|^{-2}} \quad \text{for } z \neq 0.$$

This extends continuously (with value 0) into $z = 0$, and this continuous extension is the distributional derivative $\overline{\partial} f$ on $B_{1/2}$ (see the discussion in Section 4).

On the other hand, we have

$$\partial f = \log\log |z|^{-2} - \frac{1}{\log |z|^{-2}} \quad \text{for } z \neq 0,$$

which does not extend continuously into $z = 0$.

3.2. **Proof of Proposition 2.** Here is the example for the failure of the Calderón–Zygmund estimate in the C^k-theory. Choose a smooth function $\psi\colon \mathbb{R}_0^+ \to [0,1]$ compactly supported in $[0, 1/4)$ and with $\psi \equiv 1$ on $[0, 1/16]$. For $\nu \in \mathbb{N}$ we define $f_\nu\colon B_{1/2} \to \mathbb{C}$ by

$$f_\nu(z) = \begin{cases} z|z|^{1/\nu} \log\log |z|^{-2} & \text{for } z \in B_{1/2} \setminus \{0\}, \\ 0 & \text{for } z = 0, \end{cases}$$

and we set $u_\nu(z) = \psi(|z|^2) \cdot f_\nu(z)$.

Since $|z|^{1/\nu} \log\log |z|^{-2} \to 0$ as $z \to 0$, the function f_ν is complex differentiable in $z = 0$ with $\partial f_\nu(0)$ equal to 0, and hence differentiable with $\overline{\partial} f_\nu(0)$ likewise equal to 0.

Writing $|z|^{1/\nu}$ as $(z\overline{z})^{1/2\nu}$ we see that

$$\overline{\partial} |z|^{1/\nu} = \frac{1}{2\nu} |z|^{\frac{1}{\nu} - 2} z \quad \text{for } z \neq 0.$$

We then compute

$$\overline{\partial} f_\nu(z) = \frac{1}{2\nu}|z|^{\frac{1}{\nu}-2}z^2 \log\log|z|^{-2} - \frac{z|z|^{1/\nu}}{\overline{z}\log|z|^{-2}} \quad \text{for } z \neq 0, \tag{2}$$

and

$$\partial f_\nu(z) = |z|^{1/\nu}\log\log|z|^{-2} + \frac{\overline{z}}{z}\,\overline{\partial} f_\nu(z) \quad \text{for } z \neq 0.$$

Since both $\overline{\partial} f_\nu(z)$ and $\partial f_\nu(z)$ go to 0 as $z \to 0$, we conclude that f_ν (and hence u_ν) is of class C^1.

The second summand in $\overline{\partial} f_\nu$ is bounded uniformly in ν on $B_{1/2} \setminus \{0\}$. Writing $|z| = r^\nu$ for $z \neq 0$ with $0 < r < 2^{-1/\nu} < 1$, we see that the first summand in $\overline{\partial} f_\nu$ is likewise bounded uniformly in ν, since

$$0 < \frac{1}{2\nu} r \log\log r^{-2\nu} < \frac{1}{2\nu} r \log r^{-2\nu} = -r\log r, \tag{3}$$

which extends continuously into $r = 0$. Clearly, these bounds also take care of the second summand in ∂f_ν.

On the other hand, the first summand of $\partial f_\nu(z)$ evaluated at $z = 2^{-\nu}$ yields $\frac{1}{2}\log\log 2^{2\nu}$, which goes to infinity as $\nu \to \infty$. It follows that $\|\overline{\partial} f_\nu\|_{C^0}$ is bounded uniformly in ν, whereas $\|f_\nu\|_{C^1}$ goes to infinity as $\nu \to \infty$.

The same is true for the compactly supported functions u_ν, since $\|f_\nu\|_{C^0}$ is bounded uniformly in ν, and

$$\overline{\partial} u_\nu(z) = \psi'(|z|^2) z f_\nu(z) + \psi(|z|^2)\overline{\partial} f_\nu(z),$$

with a similar expression for ∂u_ν, which means that the limiting behaviour of $\overline{\partial} u_\nu$ and ∂u_ν equals that of $\overline{\partial} f_\nu$ and ∂f_ν, respectively.

In order to get examples for the higher C^k-norms, simply start from the definition $f_\nu(z) = z^{k+1}|z|^{1/\nu}\log\log|z|^{-2}$ for $z \in B_{1/2} \setminus \{0\}$, for any $k \in \mathbb{N}_0$.

4. The Fredholm property of $\overline{\partial}$

The discussion so far shows that one cannot forgo Sobolev norms for bootstrapping arguments, but it still seems to leave room for the possibility to stay within the framework of C^k-maps. In the compactness theory of J-holomorphic curves one cannot simply work in a space of C^k-maps for some fixed k, since one typically relies on the Arzelà–Ascoli theorem and C^{k+1}-bounds to guarantee convergence in C^k. However, one might want to start with J-holomorphic curves of class C^1, interpret them as maps of class $W^{1,p}$, and then use elliptic bootstrapping (with respect to Sobolev norms) and Sobolev embedding to show that the curves are in fact smooth.

4.1. **Calderón–Zygmund estimate and Fredholm property.** As we want to explain now, it is not possible to avoid altogether the use of Sobolev spaces of maps. Typically, for a given geometric problem in symplectic topology, one describes a moduli space $\mathcal{M}$ of J-holomorphic curves in the form

$$\mathcal{M} = \{u \in \mathcal{B}\colon\ \overline{\partial}_J u = 0\},$$

where $\mathcal{B}$ is a Banach space of maps $u\colon\ \Sigma \to (M, J)$ from a compact Riemann surface Σ into an almost complex manifold (M, J) (with J tamed by some symplectic form ω), subject to (e.g. Lagrangian) boundary conditions when $\partial\Sigma \neq \emptyset$. The nonlinear Cauchy–Riemann equation $\overline{\partial}_J u := u_x + J(u)u_y = 0$ describes the J-holomorphicity of the map u.

In order to establish that $\mathcal{M}$ is a manifold of the expected dimension, one needs to verify that $\overline{\partial}_J$ is a Fredholm operator, so that one can apply the theorem of Sard–Smale. By a perturbation argument it may suffice to do this for the linear Cauchy–Riemann operator $\overline{\partial}$. This Fredholm property, as we shall see presently, holds for Sobolev spaces, but it is violated in the C^k-realm.

Consider a bounded linear operator $T\colon \mathbb{E} \to \mathbb{F}$ between Banach spaces with $\dim\ker T < \infty$. Let $\mathbb{E}_1$ be a closed complement of $\ker T$ in $\mathbb{E}$. It then follows from the open mapping theorem, applied to $T|_{\mathbb{E}_1}\colon \mathbb{E}_1 \to T(\mathbb{E})$, that T has a closed image if and only if we have an estimate $\|x\|_{\mathbb{E}} \le c\|Tx\|_{\mathbb{F}}$ for all $x \in \mathbb{E}_1$. The image $T(\mathbb{E})$ being closed is a necessary condition for $\operatorname{coker} T$ to be finite, i.e. the Fredholm property of T.

Thus, whether or not $\overline{\partial}$ has a closed image (when regarded as an operator between certain Banach spaces of functions) is equivalent to the existence or the failure of the Calderón–Zygmund estimate in the corresponding norms, provided $\ker\overline{\partial}$ is finite-dimensional in the given setting.

Here is an example how to use Proposition 2 to show the failure of the Fredholm property in the C^k-theory of J-holomorphic discs. Write $\mathbb{D} \subset \mathbb{C}$ for the closed unit disc, and $C^1_{\mathbb{R}}(\mathbb{D}, \mathbb{C})$ for the space of C^1-maps $\mathbb{D} \to \mathbb{C}$ with real boundary values. We may regard $C^1_{\mathrm{c}}(B_{1/2}, \mathbb{C})$ as a subspace of $C^1_{\mathbb{R}}(\mathbb{D}, \mathbb{C})$, and $C^0_{\mathrm{c}}(B_{1/2}, \mathbb{C})$ as a subspace of $C^0(\mathbb{D}, \mathbb{C})$. Notice that the ambient spaces are Banach spaces, but the subspaces are not closed.

Corollary 3. *The operator* $\overline{\partial}\colon C^1_{\mathbb{R}}(\mathbb{D}, \mathbb{C}) \to C^0(\mathbb{D}, \mathbb{C})$ *is not Fredholm.*

Proof. A function in $\ker\overline{\partial}$ can be extended by Schwarz reflection in the unit circle to a bounded holomorphic function on $\mathbb{C}$, which is constant by Liouville's theorem. This implies $\ker\overline{\partial} = \mathbb{R}$. Moreover, by Proposition 2, the operator $\overline{\partial}$ violates the Calderón–Zygmund estimate. Thus, $\operatorname{im}\overline{\partial}$ is not closed. □

4.2. **Failure of the Fredholm property in C^k.** We now want to use Sikorav's example to demonstrate by a specific example that the image of $\overline{\partial}\colon C^1_{\mathbb{R}}(\mathbb{D}, \mathbb{C}) \to C^0(\mathbb{D}, \mathbb{C})$ is not closed, and thus give a more concrete proof of Corollary 3.

Proposition 4. *The weak derivative* $\overline{\partial}(\psi f) \in C^0(\mathbb{D}, \mathbb{C})$ *— with* f *as in* (1)*, and* ψ *the cut-off function from Section 3.2 — is in the closure of* $\operatorname{im}\overline{\partial}$*, but not itself an element of that image.*

We first present a 'classical' argument using mollification, and then an alternative approach using the sequence (f_ν) introduced in Section 3.2.

4.3. **Proof by mollification.** We begin by analysing Sikorav's example a little more carefully.

Lemma 5. *The function* f *defined in* (1) *is an element of* $W^{1,p}(B_{1/2}, \mathbb{C})$ *for any* $p \in [1, \infty)$*, and the weak derivatives* $\overline{\partial} f$ *and* ∂f *may be assumed to coincide with the actual derivatives on* $B_{1/2} \setminus \{0\}$*.*

Proof. (i) First we are going to show that the weak ∂- and $\overline{\partial}$-derivatives of f are as claimed. We consider ∂f; for $\overline{\partial} f$ the argument is completely analogous. For $\varepsilon > 0$, let $\chi_\varepsilon \in C^\infty(\mathbb{C})$ be a cut-off function with $\chi_\varepsilon \equiv 0$ on B_ε, and $\chi_\varepsilon \equiv 1$ outside $B_{2\varepsilon}$.

Set $f_\varepsilon = \chi_\varepsilon f \in C^\infty(B_{1/2})$. For any test function $\varphi \in C^\infty_{\mathrm{c}}(B_{1/2})$, integration by parts gives

$$\int_{B_{1/2}} (\partial f_\varepsilon)\varphi = -\int_{B_{1/2}} f_\varepsilon(\partial\varphi). \tag{4}$$

We may assume that

$$|\partial\chi_\varepsilon| \leq \frac{c}{2\varepsilon}$$

for some constant c. For $|z|\leq\varepsilon$ and $|z|\geq 2\varepsilon$, the derivative $\partial\chi_\varepsilon$ vanishes identically. It follows that

$$|\partial\chi_\varepsilon(z)| \leq \frac{c}{|z|} \quad \text{for } z\in B_{1/2}\setminus\{0\}.$$

From

$$\partial f_\varepsilon = \chi_\varepsilon(\partial f) + (\partial\chi_\varepsilon)f$$

we then conclude that

$$|\partial f_\varepsilon(z)| \leq |\partial f(z)| + c\left|\frac{f(z)}{z}\right| \quad \text{for } z\in B_{1/2}\setminus\{0\}.$$

Now, ∂f_ε converges pointwise on $B_{1/2}\setminus\{0\}$ to ∂f. Hence, provided the functions f/z and ∂f are integrable, we can take the limit $\varepsilon\searrow 0$ in (4) and conclude with the Lebesgue dominated convergence theorem that

$$\int_{B_{1/2}} (\partial f)\varphi = -\int_{B_{1/2}} f(\partial\varphi),$$

so ∂f constitutes the weak ∂-derivative of f.

(ii) It remains to show that the functions f/z, ∂f and $\overline{\partial} f$ are in $L^p(B_{1/2},\mathbb{C})$ for any $p\in[1,\infty)$. Both $\overline{\partial}f$ and the function $z\mapsto 1/\log|z|^{-2}$ extend continuously to $\overline{B}_{1/2}$, so we need only show that

$$z\longmapsto \log\log|z|^{-2}, \quad z\neq 0,$$

is in $L^p(B_{1/2})$.

For $r\in(0,1/2)$ we have

$$0<\log\log r^{-2} < \log r^{-2} = 2\log r^{-1},$$

so it suffices to show that $r\mapsto\log r^{-1}$ is an L^p-function on the interval $(0,1/2)$. In fact, this function is even L^p-integrable on $(0,1)$, as can be seen by the substitution $t=\log r^{-1}$, $t\in(0,\infty)$, which yields a transformation to the Γ-function. For with $r=\mathrm{e}^{-t}$ and $\mathrm{d}r = -\mathrm{e}^{-t}\,\mathrm{d}t$ we have

$$\int_0^1 \left(\log r^{-1}\right)^p \mathrm{d}r = \int_0^\infty t^p\mathrm{e}^{-t}\,\mathrm{d}t = \Gamma(p+1).$$

We conclude that the function $z\mapsto\log\log|z|^{-2}$ is in $L^p(B_{1/2},\mathbb{C})$. □

For the basic theory of mollifiers we use presently in the first proof of Proposition 4, see [3, Sections C.5 and 5.3].

First proof of Proposition 4. Let $\rho\in C^\infty(\mathbb{C})$ be the standard mollifier,

$$\rho(z) = \begin{cases} C\cdot\exp\left(\frac{1}{|z|^2-1}\right) & \text{for } |z|<1,\\ 0 & \text{for } |z|\geq 1,\end{cases}$$

with $C\in\mathbb{R}^+$ chosen such that $\int_{\mathbb{R}^2}\rho = 1$. Set $\rho_\varepsilon(z) = \frac{1}{\varepsilon^2}\rho\left(\frac{z}{\varepsilon}\right)$.

The function f can be defined on $B_{0.6}$, so for $\varepsilon < 0.1$ we can define the mollification $f^\varepsilon = \rho_\varepsilon * f \in C^\infty(\overline{B}_{1/2})$ of f, that is,

$$f^\varepsilon(z) := \int_{B_{0.6}} \rho_\varepsilon(z-w) f(w)\, \mathrm{d}\lambda_w^2 = \int_{B_\varepsilon} \rho_\varepsilon(w) f(z-w)\, \mathrm{d}\lambda_w^2 \quad \text{for } z \in \overline{B}_{1/2},$$

where $\mathrm{d}\lambda_w^2$ denotes the 2-dimensional Lebesgue measure with respect to the variable w. As $\varepsilon \searrow 0$, the function f^ε converges to f in $W^{1,p}(B_{1/2}, \mathbb{C})$, and for $p > 2$ this convergence is uniform in $C^0(\overline{B}_{1/2}, \mathbb{C})$ by the Sobolev embedding theorem.

Since $\overline{\partial} f$ is continuous on $\overline{B}_{1/2}$, we have

$$\overline{\partial} f^\varepsilon = \overline{\partial}(\rho_\varepsilon * f) = \rho_\varepsilon * \overline{\partial} f \longrightarrow \overline{\partial} f$$

uniformly on $\overline{B}_{1/2}$ as $\varepsilon \searrow 0$.

Now let ψ be the cut-off function defined in Section 3.2, and set

$$u^\varepsilon(z) = \psi(|z|^2) f^\varepsilon(z).$$

This function is compactly supported in $B_{1/2}$, and we may regard it as an element of $C^1_{\mathbb{R}}(\mathbb{D}, \mathbb{C})$. Then

$$\overline{\partial} u^\varepsilon(z) = \psi'(|z|^2) z f^\varepsilon(z) + \psi(|z|^2) \overline{\partial} f^\varepsilon(z) \longrightarrow \psi'(|z|^2) z f(z) + \psi(|z|^2) \overline{\partial} f(z)$$

uniformly on $\mathbb{D}$, and this limit equals the weak $\overline{\partial}$-derivative of ψf.

But $\overline{\partial}(\psi f)$ — regarded as an element of $C^0(\mathbb{D}, \mathbb{C})$ — does not equal $\overline{\partial} h$ for any $h \in C^1_{\mathbb{R}}(\mathbb{D}, \mathbb{C})$, for otherwise we would have $\overline{\partial}(\psi f - h) = 0$ (in the distributional sense), which by the regularity of the $\overline{\partial}$-operator would entail that $\psi f - h$ is of class C^∞, contradicting the fact that ψf is not of class C^1. Indeed, any weak solution $u \in L^1_{\mathrm{loc}}$ of the equation $\overline{\partial} u = 0$ is also a weak solution of the Laplace equation $\Delta u = 0$, and hence harmonic (and, in particular, smooth) by Weyl's lemma. □

4.4. **Proof using the sequence (f_ν).** We now give a more explicit construction of a sequence in $\operatorname{im} \overline{\partial} \subset C^0(\mathbb{D}, \mathbb{C})$ with limit not contained in that image, based directly on the sequence (f_ν) presented in Section 3.2.

The idea is to interpolate between f_ν (or rather $4f_\nu$) near $z = 0$ and f outside a neighbourhood of $z = 0$ that is shrinking as $\nu \to \infty$. To do so, consider the continuous function

$$t \longmapsto \begin{cases} 4t^{\frac{1}{2\nu}} & \text{for } 0 \le t \le 16^{-\nu}, \\ 1 & \text{for } 16^{-\nu} \le t < 1/2, \end{cases} \tag{5}$$

which is smooth away from $t = 0$ and $t = 16^{-\nu}$. Let $\phi_\nu \colon [0, 1/2) \to [0, 1]$ be a smoothening of this function at $t = 16^{-\nu}$. Specifically, ϕ_ν is a function that coincides with (5) outside the interval $\left[\frac{1}{16^\nu + 1}, \frac{1}{16^\nu}\right]$, and such that ϕ'_ν is pointwise at most double the slope of (5), that is,

$$0 \le \phi'_\nu < \frac{4}{\nu} t^{\frac{1}{2\nu} - 1} \quad \text{for } t \in (0, 16^{-\nu}), \tag{6}$$

and $\phi'_\nu(t) = 0$ for $t \ge 16^{-\nu}$.

Now set $g_\nu(z) = \phi_\nu(|z|^2) f(z)$. This function coincides with f for $|z| \ge 4^{-\nu}$, and with $4f_\nu$ near $z = 0$. In particular, it is of class C^1. We have

$$\overline{\partial} g_\nu(z) = \phi'_\nu(|z|^2) z f(z) + \phi_\nu(|z|^2) \overline{\partial} f(z) \quad \text{for } z \in B_{1/2} \setminus \{0\},$$

and $\overline{\partial} g_\nu(0) = 4 \overline{\partial} f_\nu(0) = 0 = \overline{\partial} f(0)$.

We claim that $\overline{\partial} g_\nu \to \overline{\partial} f$ uniformly on $B_{1/2}$. To this end we estimate

$$|\overline{\partial} g_\nu(z) - \overline{\partial} f(z)| \leq |\phi'_\nu(|z|^2) z f(z)| + |(\phi_\nu(|z|^2) - 1)\overline{\partial} f(z)|. \tag{7}$$

Thanks to (6), the first summand is at most equal to 8 times the norm of the first summand in (2). With (3) and the observation that the function $r \mapsto -r \log r$ is monotone increasing on the interval $(0, \mathrm{e}^{-1})$, we see that the first summand on the right-hand side of (7) is bounded uniformly by $-8 \cdot 16^{-\nu} \log 16^{-\nu}$, which goes to 0 as $\nu \to \infty$.

Regarding the second summand in (7), notice that $|\phi_\nu(|z|^2) - 1|$ is bounded by 1, and it is identically equal to 0 for $z \geq 16^{-\nu}$. It follows that

$$|(\phi_\nu(|z|^2) - 1)\overline{\partial} f(z)| \leq \max_{|\zeta| \leq 16^{-\nu}} |\overline{\partial} f(\zeta)| \quad \text{for } z \in B_{1/2}.$$

Since $\overline{\partial} f$ is continuous with $\overline{\partial} f(0) = 0$, this bound on the right-hand side goes to 0 as $\nu \to \infty$. This proves the claim.

Similarly, $\overline{\partial}(\psi g_\nu) \to \overline{\partial}(\psi f)$ uniformly in $C^0(\mathbb{D}, \mathbb{C})$. This second proof of Proposition 4 now concludes just like the first one.

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