Parabolic Differential Equations with Bounded Delay

Marek Kryspin & Janusz Mierczyński Faculty of Pure and Applied Mathematics Wrocław University of Science and Technology Wybrzeże Wyspiańskiego 27 PL-50-370 Wrocław, Poland

Abstract

We show the continuous dependence of solutions of linear nonautonomous second order parabolic partial differential equations (PDEs) with bounded delay on coefficients and delay. The assumptions are very weak: only convergence in the weak-* topology of delay coefficients is required. The results are important in the applications of the theory of Lyapunov exponents to the investigation of PDEs with delay.

0 Introduction

The purpose of the present paper is to formulate and prove results on existence and continuous dependence on parameters of solutions of linear second order partial differential equations (PDEs) of parabolic type with bounded time delay. To be more specific, consider a rather simplified example, that is, an equation of the form

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + c_1(t,x)u(t-R(t),x), & t \in [0,T], \ x \in D\\ u(t,x) = 0 & t \in [0,T], \ x \in \partial D, \end{cases}$$
(0.1)

where $D \subset \mathbb{R}^N$ is a bounded domain with boundary ∂D , Δ is the Laplace operator in $x, T > 0, c_1: (0,T) \times D \to \mathbb{R}$ belongs to $L_{\infty}((0,T) \times D)$, and $R: [0,T] \to [0,1]$ is a function in $L_{\infty}((0,T))$.

The theory of Lyapunov exponents (or rather, more generally, the theory of skew-product dynamical systems) is a powerful tool in the applications of the theory of dynamical systems to the investigation of evolution equations (in a broad sense, containing but not excluded to, ordinary differential equations, parabolic partial differential equations, hyperbolic partial differential equations). That theory requires the (linear) equation to generate a skew-product dynamical system on some bundle whose base is the closure of the set of coefficients of the original equation. Let us consider two cases. We will remain in the simplified framework of (0.1).

• The nonautonomous case,

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + c_1(t,x)u(t-R(t),x), & t > 0, \ x \in D\\ u(t,x) = 0 & t > 0, \ x \in \partial D, \end{cases}$$
(0.2)

where c_1 is defined on $(-\infty, \infty) \times D$ and R is defined on $(-\infty, \infty)$. We take the closure, in an appropriate topology, of the set of all time-translates of c_1 (the so-called *hull*). The topology must be, on the one hand, coarse enough for the hull to be a compact (metrizable) space, and, on the other hand, fine enough for, first, the time translation operator on the hull to be continuous, and, second, the solution operator to depend continuously on parameters, that is, members of the hull. The paper [39] gives a survey of subsets of function spaces that can serve as hulls.

For the theory of linear skew-product (semi)flows on bundles whose fibers are Banach spaces and some of its applications, see, e.g., [40], [11], [12], [41], [10], [35], [36], [4] for a very incomplete list arranged in chronological order.

• The random case,

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + c_1(\theta_t \omega, x)u(t - R(\theta_t \omega), x), & t > 0, \ x \in D\\ u(t,x) = 0 & t > 0, \ x \in \partial D, \end{cases}$$
(0.3)

where c_1 is now defined on $\Omega \times D$ and R is defined on Ω , with $(\Omega, \mathfrak{F}, \mathbb{P})$ a probability space on which an ergodic measurable flow $\theta = (\theta_t)_{t \in \mathbb{R}}$ acts. Here the role of hull is played by Ω , and the measurability of the flow θ is one of the assumptions. In order to apply the theory of Lyapunov exponents in the measurable setting, as presented in, e.g., [30], [25], [26], [27], [9], one needs to show the measurable dependence of the solution operators on $\omega \in \Omega$

In the present paper we address the problem of continuous dependence on members of the hull. As the space of coefficients we take a closed and bounded subset of the Banach space of essentially bounded (Lebesgue-)measurable functions on $(0,T) \times D$, where T > 0, with the weak-* topology induced by the duality pairing between L_1 and L_{∞} . Regarding the zero order coefficients and delay terms, no additional assumption is made. In particular, the dependence on t can be quite weak.

Although there have been a lot of papers dealing with the issues of the existence of solutions of delay PDEs (many of them nonlinear, and admitting more general delay terms, employing various definitions of solutions, see, e.g., [42], [43], [44], [20], [21], [31], [32], [46], [23], [24], [6], [7], [8]), the only papers

we are aware of dealing explicitly with continuous dependence of solutions of delay PDEs on parameters are [37] and [38].

To give a flavor of our results, we formulate now some specializations of our main results to the case of (0.1). We assume 1 .

The first, a specialization of Theorem 3.1, establishes the existence and uniqueness of mild solutions.

Theorem. Let $c_1 \in L_{\infty}((0,T) \times D)$, $u_0 \in C([-1,0], L_p(D))$ and $R \in L_{\infty}((0,T))$ be such that $R(t) \in [0,1]$ for Lebesgue-a.e. $t \in (0,T)$. Then there exists a unique solution $u(\cdot; c_1, u_0, R) \in C([-1,T])$ of Eq. (0.1) with initial condition $u(t; c_1, u_0, R) = u_0$, $t \in [-1,0]$. The solution is understood in a suitable integral sense (a mild solution).

The second, a specialization of Theorem 5.1(ii), establishes the continuity, in a suitable sense, of a solution with respect to initial conditions and parameters.

Theorem. Assume that $(c_{1,m})_{m=1}^{\infty}$, $(u_{0,m})_{m=1}^{\infty}$ and $(R_m)_{m=1}^{\infty}$ are sequences satisfying the following:

- $c_{1,m}$ have their $L_{\infty}((0,T) \times D)$ -norms uniformly bounded, and converge in the weak-* topology to $c_1 \in L_{\infty}((0,T) \times D)$;
- $u_{0,m}$ converge in the norm topology of $C([-1,0], L_p(D)$ to $u_0;$
- R_m converge for Lebesgue-a.e. $t \in (0,T)$ to R.

Then

$$u(\cdot; c_{1,m}, u_{0,m}, R_m) \to u(\cdot; c_1, u_0, R)$$

in the $C([-1, 0], L_p(D))$ -norm.

The paper is organized as follows.

Section 1 presents the assumptions used throughout.

In Section 2 results concerning the existence and basic properties of (weak) solutions to linear parabolic PDEs without delay terms are gathered. They are for the most part taken from [35] and based on [13], though some of them (Proposition 2.18, for example), perhaps belonging to the folk lore, appear in print for the first time.

Section 3 is devoted to defining and proving the existence and uniqueness of (mild) solutions of PDEs with delay terms. Section 4 provides estimates of the solutions which are then used to prove the continuous dependence on initial conditions.

Section 5 can be considered the main part of the paper. Here the continuous dependence of solutions on coefficients and delay terms is proved under very weak assumptions: coefficients are required to converge in the weak-* topology only.

It should be mentioned that a similar approach has been successfully applied in the case of ordinary differential equations with delay in [36], [33], [34], see also [17, Chpt. 5], [18], [5].

0.1 General Notations

We write \mathbb{R}^+ for $[0, \infty)$, and \mathbb{Q} for the set of all rationals.

If $B \subset A$, we write $\mathbb{1}_B$ for the *indicator* of B: $\mathbb{1}_B(a) = 1$ if $a \in B$ and $\mathbb{1}_B(a) = 0$ if $a \in A \setminus B$.

For a metric space (Y, d), $\mathfrak{B}(Y)$ denotes the σ -algebra of all Borel subsets of Y.

For Banach spaces X_1, X_2 with norms $\|\cdot\|_{X_1}, \|\cdot\|_{X_2}$, we let $\mathcal{L}(X_1, X_2)$ stand for the Banach space of bounded linear mappings from X_1 into X_2 , endowed with the standard norm $\|\cdot\|_{X_1,X_2}$. Instead of $\mathcal{L}(X,X)$ we write $\mathcal{L}(X)$, and instead of $\|\cdot\|_{X,X}$ we write $\|\cdot\|_X$. $\mathcal{L}_s(X_1, X_2)$ denotes the space of bounded linear mappings from X_1 into X_2 equipped with the strong operator topology. Instead of $\mathcal{L}_s(X,X)$ we write $\mathcal{L}_s(X)$.

Throughout the paper, T>0 will be fixed. We set

$$\Delta := \{ (s,t) \in \mathbb{R}^2 : 0 \le s \le t \le T \}, \qquad \dot{\Delta} := \{ (s,t) \in \mathbb{R}^2 : 0 \le s < t \le T \}.$$

Throughout the paper, $D \subset \mathbb{R}^N$ stands for a bounded domain, with boundary ∂D .

By $\mathfrak{L}((0,T))$ we understand the σ -algebra of all Lebesgue-measurable subsets of (0,T). The notations $\mathfrak{L}(D)$ and $\mathfrak{L}((0,T) \times D)$ are defined in a similar way.

For u belonging to a Banach space of (equivalence classes of) functions defined on D we will denote by u[x] the value of u at $x \in D$.

 $L_p(D) = L_p(D, \mathbb{R})$ has the standard meaning, with the norm, for $1 \le p < \infty$, given by

$$||u||_{L_p(D)} := \left(\int_D |u[x]|^p \, \mathrm{d}x\right)^{\frac{1}{p}},$$

and for $p = \infty$ given by

$$\|u\|_{L_{\infty}(D)} := \operatorname{ess\,sup}_{D} u.$$

For $1 \leq p \leq \infty$ let p' stand for the Hölder conjugate of p. The duality pairing between $L_p(D)$ and $L_{p'}(D)$ is given, for 1 , or for <math>p = 1 and $p' = \infty$, by

$$\langle u, v \rangle_{L_p(D), L_{p'}(D)} = \int_D u[x] v[x] dx, \quad u \in L_p(D), \ v \in L_{p'}(D).$$

Let u be an equivalence class of functions defined for Lebesgue-a.e. $t \in (0, T)$ and taking values in $L_p(D)$, $1 \le p < \infty$ (in the sequel we will refer to such u simply as a function).

• u is said to be *measurable* if it is $(\mathfrak{L}((0,T)), \mathfrak{B}(L_p(D)))$ -measurable, meaning that the preimage under u of any open subset of $L_p(D)$ belongs to $\mathfrak{L}((0,T))$.

- u is strongly measurable (sometimes called Bochner measurable) if there exists a sequence $(u_m)_{m=1}^{\infty}$ of simple functions such that $\lim_{m\to\infty} ||u_m(t) u(t)||_{L_p(D)} = 0$ for Lebesgue-a.e. $t \in (0, T)$.
- *u* is weakly measurable if for any $v \in L_{p'}(D)$ the function

$$[t \mapsto \langle u(t), v \rangle_{L_p(D), L_{n'}(D)}]$$

is $(\mathfrak{L}((0,T)),\mathfrak{B}(\mathbb{R}))$ -measurable.

Theorem 0.1. For $u: (0,T) \to L_p(D)$ measurability, strong measurability and weak measurability are equivalent.

The equivalence of strong and weak measurability is a consequence of Pettis's Measurability Theorem (see, e.g., [16, Thm. 2.1.2]). For the fact that measurability implies strong measurability see, e.g., [45, Thm. 1], whereas the proof of the reverse implication is a simple exercise.

For our purposes we will use the following definitions (see, e.g. [2, Sect. X.4]). A measurable $u: (0,T) \to L_p(D)$ belongs to $L_r((0,T), L_p(D)), 1 \leq r < \infty$, if $||u(\cdot)||_{L_p(D)}$ belongs to $L_r((0,T))$, with

$$\|u\|_{L_r((0,T),L_p(D))} = \left(\int_0^T \|u(t)\|_{L_p(D)}^r \,\mathrm{d}t\right)^{1/r}.$$

Similarly, a measurable $u: (0,T) \to L_p(D)$ belongs to $L_{\infty}((0,T), L_p(D))$, if $||u(\cdot)||_{L_p(D)}$ belongs to $L_{\infty}((0,T))$, with

$$||u||_{L_{\infty}((0,T),L_{p}(D))} = \operatorname{ess\,sup}_{t \in (0,T)} ||u(t)||_{L_{p}(D)} .$$

The following result, a part of [19, Lemma III.11.16], will be used several times.

Lemma 0.1.

(a) If $u \in L_1((0,T), L_1(D))$ then the function

$$\left[(0,T) \times D \ni (t,x) \mapsto u(t)[x] \in \mathbb{R} \right]$$

belongs to $L_1((0,T) \times D, \mathbb{R})$.

(b) If w is $(\mathfrak{L}((0,T) \times D), \mathfrak{B}(\mathbb{R}))$ -measurable, and for Lebesgue-a.e. $t \in (0,T)$ the t-section $w(t, \cdot)$ belongs to $L_p(D)$, where $1 \le p < \infty$, then the function

$$\left[(0,T) \ni t \mapsto \left[D \ni x \mapsto w(t,x) \in \mathbb{R} \right] \right]$$

is $(\mathfrak{L}(0,T),\mathfrak{B}(L_p(D)))$ -measurable.

Remark 0.1. Regarding Lemma 0.1(b), we remark that in [19] the analog of w is assumed to be $(\mathfrak{L}((0,T)) \otimes \mathfrak{L}(D), \mathfrak{B}(\mathbb{R}))$ -measurable (no completion) rather than $(\mathfrak{L}((0,T) \times D), \mathfrak{B}(\mathbb{R}))$ -measurable. As an $(\mathfrak{L}((0,T) \times D), \mathfrak{B}(\mathbb{R}))$ -measurable function can be made into an $(\mathfrak{L}((0,T)) \otimes \mathfrak{L}(D), \mathfrak{B}(\mathbb{R}))$ -measurable function by changing its values on a set of (N+1)-dimensional Lebesgue measure zero (see, e.g., [22, Prop. 2.12]), our formulation follows.

1 Assumptions and Definitions

1.1 Main Equation

Consider a linear second order partial differential equation with bounded delay

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{N} a_{ij}(t, x) \frac{\partial u}{\partial x_j} + a_i(t, x) u \right) + \sum_{i=1}^{N} b_i(t, x) \frac{\partial u}{\partial x_i} + c_0(t, x) u$$
(ME)

$$+ c_1(t, x)u(t - R(t)); \quad 0 \le t \le T, \ x \in D.$$

The delay map $R \colon [0,T] \to \mathbb{R}$ is bounded from below by 0 and from above by 1, i.e.

$$0 \le R(t) \le 1, \qquad \forall t \in [0, T]$$

Sometimes the function $\xi \mapsto \xi - R(\xi)$ will be denoted by Φ . The function Φ will be called *relative time delay*. Further, $D \subset \mathbb{R}^N$ is a bounded domain with boundary ∂D . The equation (ME) will be complemented with boundary conditions

$$\mathcal{B}u = 0, \qquad 0 \le t \le T, \ x \in \partial D.$$
 (BC)

Later on, we will use the notation \mathcal{B}_a in other to exhibit dependence of the operator \mathcal{B} on a. The boundary conditions operator (BC) will be one of this form

$$\mathcal{B}u = \begin{cases} u & \text{(Dirichlet)} \\ \sum_{i=1}^{N} \left(\sum_{j=1}^{N} a_{ij}(t,x) \frac{\partial u(t)}{\partial x_j} + a_i(t,x)u \right) \nu_i & \text{(Neumann)} \\ \sum_{i=1}^{N} \left(\sum_{j=1}^{N} a_{ij}(t,x) \frac{\partial u(t)}{\partial x_j} + a_i(t,x)u \right) \nu_i + d_0(t,x)u & \text{(Robin).} \end{cases}$$

The vector $\boldsymbol{\nu} = (\nu_1, \dots, \nu_N)$ denotes the unit normal on the boundary ∂D pointing out of D, interpreted in a certain weak sense (in the regular sense if ∂D is sufficiently smooth [35]).

The initial condition is considered in the following way: for $u_0 \in C([-1, 0], L_p(D))$, where $1 \le p \le \infty$, find a solution of (ME)+(BC) satisfying

$$u(t) = u_0(t)$$
 for $t \in [-1, 0]$. (IC)

By (ME)+(BC) we understand equation (ME) equipped with boundary condition (BC). Later on, we will also use $(ME)_a + (BC)_a$ notation to indicate that parameters of (ME) + (BC) are fixed to be a.

Note that, without any additional assumptions on the delay map R, the initial data cannot be taken from $L_p(D) \oplus L_r((-1,0), L_p(D))$, as in [33] or [34].

The reason for this is that the delay map R can be constructed in such way that $t \mapsto t - R(t)$ would be a constant function. In such a situation the initial value problem (ME) + (BC) would be not meaningful. Under some additional assumptions the situation can change, for example a constant delay map allows us to introduce generalized initial data in $L_p(D) \oplus L_r((-1,0), L_p(D))$. However, in this paper we will not focus on that.

In order to clearly define the problem $(ME)_a + (BC)_a$, it is also necessary to set the delay map R. However, the assumptions on R will be given later. Moreover, we suppress the notation of R from $(ME)_a + (BC)_a$. We will present the solutions of $(ME)_a + (BC)_a$ in the form of $u(\cdot; a, u_0, R)$ and often suppress the notation of a, u_0 or R if it does not lead to confusion.

1.2 Main Assumptions

We introduce some assumptions on the domain $D \subset \mathbb{R}^N$ and the coefficients of the problem (ME)+(BC).

(DA1) (Boundary regularity) For Dirichlet boundary conditions, D is a bounded domain. For Neumann or Robin boundary conditions, D is a bounded domain with Lipschitz boundary.

In all expressions of the type "a.e." we consider 1-dimensional Lebesgue measure on (0,T), N-dimensional Lebesgue measure on D and (N-1)-dimensional Hausdorff measure on ∂D . The latter is, by (DA1), equal to surface measure on ∂D .

The notation $L_{\infty}(\partial D)$ [resp. $L_{\infty}((0,T) \times \partial D)$] corresponds to surface measure on ∂D [resp. to the product of 1-dimensional Lebesgue measure on (0,T) and surface measure on ∂D].

(DA2) (Boundedness) The functions

 $\begin{array}{l} \diamond \ a_{ij} \colon (0,T) \times D \to \mathbb{R} \ (i,j=1,\ldots,N), \\ \diamond \ a_i \colon (0,T) \times D \to \mathbb{R} \ (i=1,\ldots,N), \\ \diamond \ b_i \colon (0,T) \times D \to \mathbb{R} \ (i=1,\ldots,N), \\ \diamond \ c_0 \colon (0,T) \times D \to \mathbb{R} \ , \\ \diamond \ c_1 \colon (0,T) \times D \to \mathbb{R} \end{array}$

belong to $L_{\infty}((0,T) \times D)$. When the Robin boundary condition holds the function $d_0: (0,T) \times \partial D \to \mathbb{R}$ belongs to $L_{\infty}((0,T) \times \partial D)$.

It is worth noticing at this point that uniform $L_{\infty}(D)$ -boundedness of $a_{ij}(t, \cdot)$, $a_i(t, \cdot), b_i(t, \cdot), c_0(t, \cdot), c_1(t, \cdot)$ and uniform $L_{\infty}(\partial D)$ -boundedness of $d_0(t, \cdot)$ for a.e. $t \in (0, T)$ follow from the assumption (DA 2) and Fubini's theorem.

Definition 1.1 (Y coefficients space). Let Y be a subset of the Banach space $L_{\infty}((0,T) \times D, \mathbb{R}^{N^2+2N+2}) \oplus L_{\infty}((0,T) \times \partial D, \mathbb{R})$ satisfying the following assumptions

- (Y1) Y is norm-bounded and, moreover, it is closed (hence compact, via the Banach-Alaoglu theorem) in the weak-* topology,
- (Y2) the function $d_0 \ge 0$ if the Robin boundary condition holds. The function d_0 is interpreted as the zero function in the Dirichlet or Neumann cases.

Elements of Y will be denoted by

$$a := \left((a_{ij})_{i,j=1}^N, (a_i)_{i=1}^N, (b_i)_{i=1}^N, c_0, c_1, d_0 \right) \in Y$$

The weak-* topology of the space Y is understood in the standard sense, namely, as the weak-* topology induced via the isomorphism

$$L_{\infty}((0,T) \times D, \mathbb{R}^{N^{2}+2N+2}) \oplus L_{\infty}((0,T) \times \partial D, \mathbb{R})$$
$$\cong \left(L_{1}((0,T)) \times D, \mathbb{R}^{N^{2}+2N+2}\right) \oplus L_{1}((0,T) \times \partial D, \mathbb{R})\right)^{*}.$$

Definition 1.2 (Flattening Y to Y_0). The mapping defined on Y by

$$\left((a_{ij})_{i,j=1}^N, (a_i)_{i=1}^N, (b_i)_{i=1}^N, c_0, c_1, d_0\right) \widetilde{} := \left((a_{ij})_{i,j=1}^N, (a_i)_{i=1}^N, (b_i)_{i=1}^N, c_0, 0, d_0\right)$$

will be called the flattening of $a \in Y$.

The above mapping is obviously continuous. As a consequence, the image Y_0 of Y under that mapping shares properties analogous to (Y1) and (Y2).

(DA3) (Ellipticity) There exists a constant $\alpha_0 > 0$ such that for any $a_0 \in Y_0$ the inequality

$$\sum_{i,j=1}^{N} a_{ij}(t,x)\xi_i\xi_j \ge \alpha_0 \sum_{i=1}^{N} \xi_i^2,$$

holds for a.e. $(t, x) \in (0, T) \times D$ and all $\xi \in \mathbb{R}^N$, and the functions $a_{ij}(\cdot, \cdot)$ are symmetric in the indices, i.e. $a_{ij}(\cdot, \cdot) \equiv a_{ji}(\cdot, \cdot)$ for all i, j = 1, ..., N.

(DA4) (Sequential compactness of Y_0 with respect to convergence a.e.)

Any sequence $(a_{0,m})_{m=1}^{\infty}$ of elements of Y_0 , where

$$a_{0,m} := \left((a_{ij,m})_{i,j=1}^N, (a_{i,m})_{i=1}^N, (b_{i,m})_{i=1}^N, c_{0,m}, 0, d_{0,m} \right),\$$

convergent as $m \to \infty$ in the weak-* topology to $a_0 \in Y_0$ has the property that

- the sequence $((a_{ij,m})_{i,j=1}^N, (a_{i,m})_{i=1}^N, (b_{i,m})_{i=1}^N)$ converges to $((a_{ij})_{i,j=1}^N, (a_i)_{i=1}^N, (b_i)_{i=1}^N)$ pointwise a.e. on $(0,T) \times D$,
- the sequence $d_{0,m}$ converges to d_0 pointwise a.e. on $(0,T) \times \partial D$.

Occasionally we will use the following.

(DA5) Y_0 is a singleton.

For the purposes of studying continuous dependence on parameters and delay we introduce now the delay class and the relative delay class equipped with suitable topologies.

Definition 1.3. The delay class is defined as follows

$$\mathcal{R} := \{ R \in L_{\infty}((0,T)) : R(t) \in [0,1] \text{ for a.e. } t \in (0,T) \}.$$

The delay class is equipped with the weak-* topology.

At some moments we also use the following notation $\widetilde{\mathcal{R}} := \{[t \mapsto t - R(t)] : R \in \mathcal{R}\}$ especially in more abstract lemmas when general properties of the mapping $t \mapsto t - R(t)$ are important.

Remark 1.1. The delay class \mathcal{R} is a norm-bounded, convex and weak-* closed subset of $L_{\infty}((0,T))$, hence, by the Banach–Alaoglu theorem, it is compact in the weak-* topology.

The following assumption is a property of a subset $\mathcal{R}_0 \subset \mathcal{R}$.

(DA6) If $R \in \mathcal{R}_0$ is a weak-* limit of $(R_m)_{m=1}^{\infty} \subset \mathcal{R}_0$ then $(R_m)_{m=1}^{\infty}$ converge pointwise a.e. on (0,T) to R.

Remark 1.2. Note that the assumption (DA6) is naturally satisfied when \mathcal{R}_0 is compact in \mathcal{R} with respect to the norm topology. This fact follows from an observation that any Hausdorff topology weaker than the norm topology (such as the weak-* topology) is equal to the norm topology on a compact subset.

Remark 1.3. Note that the weak-* topology on Y is metrizable, see [35, (1.3.1)]. Similarly, the weak-* topology on \mathcal{R} is metrizable, see [14, Thm. 3.6.17 and Cor. 3.6.18].

2 Weak Solutions

In the present section we assume (DA1), (DA2) and that the flattening Y_0 of Y as in Definition 1.1 satisfies (DA3). Occasionally we will assume (DA4).

We start with a PDE parameterized by $a_0 \in Y_0$

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{N} a_{ij}(t, x) \frac{\partial u}{\partial x_j} + a_i(t, x) u \right)
+ \sum_{i=1}^{N} b_i(t, x) \frac{\partial u}{\partial x_i} + c_0(t, x) u; \quad 0 \le s \le t \le T, \ x \in D.$$
(ME)

Equation (ME) is complemented with boundary conditions

$$\mathcal{B}u = 0, \quad 0 \le s \le t \le T, \ x \in \partial D, \tag{BC}$$

where \mathcal{B} is either the Dirichlet or the Neumann or else the Robin boundary operator.

We are looking for solutions of the problem $(\widehat{\text{ME}})_{a_0} + (\widehat{\text{BC}})_{a_0}$ for initial condition $u_0 \in L_2(D)$. To define a solution we introduce the space H as follows. Let

$$V := \begin{cases} H_0^1(D) & \text{ for Dirichlet boundary condition} \\ H^1(D) & \text{ for Neumann or Robin boundary condition} \end{cases}$$

and

$$H = H(s,T;V) := \{ v \in L_2((s,T),V) : \dot{v} \in L_2((s,T),V^*) \}$$

equipped with the norm

$$\|v\|_{H} := \left(\int_{s}^{T} \|v(\zeta)\|_{V}^{2} \,\mathrm{d}\zeta + \int_{s}^{T} \|\dot{v}(\zeta)\|_{V^{*}}^{2} \,\mathrm{d}\zeta\right)^{\frac{1}{2}},$$

where $\dot{v} := dv/dt$ is the time derivative in the sense of distributions taking values in V^* (see [15, Chpt. XVIII] for definitions).

For $a_0 \in Y_0$ define a bilinear form

$$B_{a_0}[t; u, v] := \int_D \left(\sum_{i,j=1}^N a_{ij}(t, x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^N a_i(t, x) u \frac{\partial v}{\partial x_i} \right)$$
$$- \sum_{i=1}^N b_i(t, x) \frac{\partial u}{\partial x_i} v - c_0(t, x) u v dx$$

in the Dirichlet or Neumann case, and

$$B_{a_0}[t; u, v] := \int_D \left(\sum_{i,j=1}^N a_{ij}(t, x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^N a_i(t, x) u \frac{\partial v}{\partial x_i} \right)$$
$$- \sum_{i=1}^N b_i(t, x) \frac{\partial u}{\partial x_i} v - c_0(t, x) u v dx + \int_{\partial D} d_0(t, x) u v dH$$

in the Robin case, where H_{N-1} stands for the (N-1)-dimensional Hausdorff measure.

Definition 2.1 (Local Weak Solution). For $a_0 \in Y_0$, $0 \leq s \leq t \leq T$ and $u_0 \in L_2(D)$ a function $u \in L_2([s,t], V)$ such that $\dot{u} \in L_2([s,t], V^*)$ is a weak solution of $(\widehat{\text{ME}})_{a_0} + (\widehat{\text{BC}})_{a_0}$ on [s,t] with initial condition $u(s) = u_0$ if

$$-\int_{s}^{t} (u(\zeta), v)_{L_{2}(D)} \dot{\psi}(\zeta) \,\mathrm{d}\zeta + \int_{s}^{t} B_{a_{0}}[\zeta; u(\zeta), v] \psi(\zeta) \,\mathrm{d}\zeta = (u_{0}, v)_{L_{2}(D)} \,\psi(s)$$

for any $v \in V$ and any $\psi \in \mathcal{D}([s,t),\mathbb{R})$ where set $\mathcal{D}([s,t),\mathbb{R})$ is the space of all smooth real functions having compact support in [s,t) and $(\cdot,\cdot)_{L_2(D)}$ denotes the standard inner product in $L_2(D)$.

Definition 2.2 (Global Weak Solution). When t = T in definition 2.1 then a weak solution will be called a global weak solution.

Proposition 2.1 (Existence of global weak solution). For any initial condition $u_0 \in L_2(D)$ there exists a unique global weak solution of $(\widehat{\text{ME}})+(\widehat{\text{BC}})$.

Proof. See [13, Thm. 2.4] for a proof and [35, Prop. 2.1.5] for a unified theory of weak solutions. \Box

For $a_0 \in Y_0$ and $0 \leq s < T$ we write the unique global weak solution of $(\widehat{\text{ME}})_{a_0} + (\widehat{\text{BC}})_{a_0}$ with initial condition $u(s) = u_0$ as $U_{a_0}(t, s)u_0 := u(t)$. Below we present a couple of results from [35, Ch. 2].

below we present a couple of results from [55,

Proposition 2.2. The mappings

$$U_{a_0}(t,s)u_0 = u(t;a_0,u_0), \quad 0 \le s \le t \le T, \ a_0 \in Y_0, \ u_0 \in L_2(D)$$

have the following properties.

$$U_{a_0}(s,s) = \mathrm{Id}_{L_2(D)}, \quad a_0 \in Y_0, \ s \in [0,T], \tag{2.1}$$

$$U_{a_0}(t_2, t_1) \circ U_{a_0}(t_1, s) = U_{a_0}(t_2, s), \quad a_0 \in Y_0, \ 0 \le s \le t_1 \le t_2 \le T.$$
(2.2)

Proof. See [35, Props. 2.1.5 through 2.1.8].

Proposition 2.3.

(i) Let $1 \leq p < \infty$ and $0 \leq s < T$. For any $a_0 \in Y_0$ there exists $U_{a_0,p}(t) \in \mathcal{L}(L_p(D))$ such that

$$U_{a_0,p}(t,s)u_0 = U_{a_0}(t,s)u_0, \quad u_0 \in L_2(D) \cap L_p(D).$$

(ii) Let $1 and <math>a_0 \in Y_0$. Then the mapping

$$\left[[s,T] \ni t \mapsto U_{a_0,p}(t,s) \in \mathcal{L}_{s}(L_p(D)) \right]$$

is continuous.

Proof. See [13, Cor. 7.2] for part (i) and [13, Thm. 5.1] for part (ii). \Box

For p = 1 we have an analog of Proposition 2.3(ii).

Proposition 2.4. Let $1 \le p < \infty$, $0 \le s < T$ and $a_0 \in Y_0$. Then the mapping

$$|(s,T] \ni t \mapsto U_{a_0}(t,s) \in \mathcal{L}_{s}(L_p(D))|$$

 $is \ continuous.$

Proof. See [35, Prop. 2.2.6].

For $0 \leq s < T$ we write $U_{a_0,p}(s,s) = \operatorname{Id}_{L_p(D)}$ even if $p = 1, \infty$.

Proposition 2.5. For any $a_0 \in Y_0$, $0 \le s \le t_1 \le t_2 \le T$ and any $1 \le p \le \infty$

$$U_{a_0,p}(t_2,t_1) \circ U_{a_0,p}(t_1,s) = U_{a_0,p}(t_2,s)$$
(2.3)

Proof. See [35, Prop. 2.1.7] for the proof of p = 2 case. For $p \neq 2$ it suffices to use the fact that $U_{a_0}(t,s) \in \mathcal{L}(L_2(D))$ and the continuity of the mappings $[u \mapsto U_{a_0,p}(t_2,t_1) \circ U_{a_0,p}(t_1,s)u]$ and $[u \mapsto U_{a_0,p}(t_2,s)u]$, which is guaranteed by Proposition 2.3.

Proposition 2.6. For any $a_0 \in Y_0$ and any $0 \le s \le t_1 \le t_2 \le T$ the operator $U_{a_0}(t_2, t_1)$ has an a.e. nonnegative kernel.

Proof. See [3, Thm. 1.3] for the existence of a kernel, for nonnegativity see [13, Cor. 8.2]. \Box

Proposition 2.7.

- (i) For any $a_0 \in Y_0$, any $(s,t) \in \Delta$ and any $1 \le p \le q \le \infty$ there holds $U_{a_0}(t,0) \in \mathcal{L}(L_p(D), L_q(D)).$
- (ii) There are constants $M \geq 1$ and $\gamma \in \mathbb{R}$ such that

$$\|U_{a_0}(t,s)\|_{\mathcal{L}(L_p(D),L_q(D))} \le M(t-s)^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} e^{\gamma(t-s)}$$
(2.4)

for $1 \leq p \leq q \leq \infty$, $a_0 \in Y_0$ and $(s,t) \in \dot{\Delta}$.

Proof. See [13, Sect. 5 and Cor. 7.2].

In particular, setting p = q we have

$$||U_{a_0}(t,s)||_{\mathcal{L}(L_p(D))} \le M e^{\gamma(t-s)}.$$
 (2.5)

In the sequel we will frequently assume that $\gamma \ge 0$ in Proposition 2.7 and its derivates.

Proposition 2.8. Let $1 \le p \le \infty$ and $0 \le s < T$. Then for any $T_1 \in (s,T]$ there exists $\alpha \in (0,1)$ such that for any $a_0 \in Y_0$, any $u_0 \in L_p(D)$, and any compact subset $D_0 \subset D$ the function $[[T_1,T] \times D_0 \ni (t,x) \mapsto (U_{a_0}(t)u_0)[x]]$ belongs to $C^{\alpha/2,\alpha}([T_1,T] \times D_0)$. Moreover, for fixed T_1 , and D_0 , the $C^{\alpha/2,\alpha}([T_1,T] \times D_0)$ -norm of the above restriction is bounded above by a constant depending on $\|u_0\|_{L_p(D)}$ only.

Proof. It follows from Proposition 2.7 and from [29, Chpt. III, Thm. 10.1]. \Box

Proposition 2.9. For any $(s, T_1) \in \dot{\Delta}$, $1 \leq p < \infty$ and a bounded $E \subset L_p(D)$ the set

$$\{ \left[[T_1, T] \ni t \mapsto U_{a_0}(t, s) u_0 \in L_p(D) \right] : a_0 \in Y_0, u_0 \in E \}$$

is precompact in $C([T_1, T], L_p(D))$.

Proof. Fix $(s, T_1) \in \Delta$, $1 \leq p < \infty$ and a bounded $E \subset L_p(D)$. Let $(a_{0,m})_{m=1}^{\infty} \subset Y_0$ and $(u_{0,m})_{m=1}^{\infty} \subset E$. Put, for $m = 1, 2, \ldots$,

$$u_m(t) := U_{a_{0,m}}(t)u_{0,m}, \quad t \in [T_1, T].$$

It follows from Proposition 2.8 via the Ascoli–Arzelà theorem by diagonal process that, after possibly taking a subsequence, $(u_m)_{m=1}^{\infty}$ converges as $m \to \infty$ to some function \tilde{u} defined on $[T_1, T]$ and taking values in the set of continuous real functions on D in such a way that for any compact $D_0 \subset D$ the functions $[t \mapsto u_m(t)|_{D_0}]$ converge to $[t \mapsto \tilde{u}(t)|_{D_0}]$ in $C([T_1, T], C(D_0))$.

We claim that u_m converge to \tilde{u} in the $C([T_1, T], L_p(D))$ -norm. By Proposition 2.7, there is M > 0 such that $||u_m(t)||_{L_{\infty}(D)} \leq M$ and $||\tilde{u}(t)||_{L_{\infty}(D)} \leq M$ for all $m = 1, 2, \ldots$ and all $t \in [T_1, T]$. For $\epsilon > 0$ take a compact $D_0 \subset D$ such that $\lambda(D \setminus D_0) < (\epsilon/(4M))^p$, where λ denotes the N-dimensional Lebesgue measure. We have

$$\|(u_m(t) - \tilde{u}(t)) \mathbb{1}_{D \setminus D_0}\|_{L_p(D)} \le \frac{\epsilon}{2}$$

for all m = 1, 2, ... and all $t \in [T_1, T]$. Further, since $[t \mapsto u_m(t) \upharpoonright_{D_0}]$ converge to $[t \mapsto \tilde{u}(t) \upharpoonright_{D_0}]$ in the $C([T_1, T], C(D_0))$ -norm, there is m_0 such that

$$\|(u_m - \tilde{u}) \mathbb{1}_{D_0}\|_{C([T_1, T], L_p(D))} \le \frac{\epsilon}{2}$$

for all $m \geq m_0$ (here $\mathbb{1}_{D_0}$ stands for the function constantly equal to $\mathbb{1}_{D_0}$). Consequently,

$$\|u_m - \tilde{u}\|_{C([T_1,T],L_p(D))} \le \epsilon$$

for all $m \geq m_0$.

Corollary 2.1. Let $1 \le p < \infty$, $0 \le s < T$, $a_0 \in Y_0$. Then the mapping

$$[(s,T] \times L_p(D) \ni (t,u_0) \mapsto U_{a_0}(t,s)u_0 \in L_p(D)]$$

is continuous.

Proof. Let $(u_m)_{m=1}^{\infty}$ converge in $L_p(D)$ to u_0 and let $(t_m)_{m=1}^{\infty}$ converge to t > s. Take $\epsilon > 0$. It follows from Proposition 2.4 that there is m_1 such that $\|U_{a_0}(t_m, s)u_0 - U_{a_0}(t, s)u_0\|_{L_p(D)} < \epsilon/2$ for $m \ge m_1$, and it follows from Proposition 2.7(ii) that there is m_2 such that $\|U_{a_0}(t_m, s)u_m - U_{a_0}(t_m, s)u_0\|_{L_p(D)} < \epsilon/2$ for $m \ge m_2$. Consequently,

$$\begin{aligned} \|U_{a_0}(t_m,s)u_m - U_{a_0}(t,s)u_0\|_{L_p(D)} \\ &\leq \|U_{a_0}(t_m,s)u_m - U_{a_0}(t_m,s)u_0\|_{L_p(D)} \\ &+ \|U_{a_0}(t_m,s)u_0 - U_{a_0}(t,s)u_0\|_{L_p(D)} < \epsilon \end{aligned}$$
for $m \geq \max\{m_1, m_2\}.$

2.1 The Adjoint Operator

For a fixed $0 < s \leq T$ together with $(\overline{\text{ME}})_{a_0} + (\overline{\text{BC}})_{a_0}$ we consider the adjoint equations, that is the backward parabolic equations

$$-\frac{\partial u}{\partial t} = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{N} a_{ji}(t, x) \frac{\partial u}{\partial x_j} - b_i(t, x) u \right) - \sum_{i=1}^{N} a_i(t, x) \frac{\partial u}{\partial x_i} + c_0(t, x) u, \quad 0 \le t < s, \ x \in D,$$
(2.6)

complemented with the boundary conditions:

$$\mathcal{B}_{a_0}^* u = 0, \quad 0 \le t < s, \ x \in \partial D, \tag{2.7}$$

where $\mathcal{B}_{a_0}^* u = \mathcal{B}_{a_0^*} u$ with $a_0^* := ((a_{ji})_{i,j=1}^N, -(b_i)_{i=1}^N, -(a_i)_{i=1}^N, c_0, d_0)$ and $\mathcal{B}_{a_0^*}$ is as in $(\widehat{BC})_{a_0}$ with a_0 replaced by a_0^* .

Since all analogs of the assumptions (DA 1) and (DA 2) are satisfied for (2.6)+(2.7), we can define, for $u_0 \in L_2(D)$, a global (weak) solution of $(2.6)_{a_0^*}+(2.7)_{a_0^*}$, defined on [0, s], with the *final condition* $u(s) = u_0$. The following analog of Proposition 2.2 holds.

Proposition 2.10. For $a_0 \in Y_0$, $0 < s \le T$ and $u_0 \in L_2(D)$ there is precisely one global weak solution

$$\left[\left[0,s \right] \ni t \mapsto U_{a_0}^*(t,s) u_0 \in L_2(D) \right]$$

of $(2.6)_{a_0^*} + (2.7)_{a_0^*}$ satisfying the final condition $u^*(s; a_0, u_0) = u_0$. This mapping has the following properties

$$U_{a_0}^*(t,t) = \mathrm{Id}_{L_2(D)}, \quad a_0 \in Y_0, \ t \in [0,s],$$
(2.8)

$$U_{a_0}^*(t_1, t_2) \circ U_{a_0}^*(t_2, s) = U_{a_0}^*(t_1, s), \quad a_0 \in Y_0, \ 0 \le t_1 \le t_2 \le s.$$
(2.9)

From now on s and t will play a role as in the $(\widehat{ME})_{a_0} + (\widehat{BC})_{a_0}$. Below we formulate an analog of Proposition 2.3.

Proposition 2.11.

- (i) Let $1 \le p < \infty$ and $(s,t) \in \dot{\Delta}$. Then $U_{a_0}^*(s,t)$ extends to a linear operator in $\mathcal{L}(L_p(D))$.
- (ii) Let $1 , <math>0 < s \le T$ and $a_0 \in Y_0$. Then the mapping

$$\left| \left[0, s \right] \ni t \mapsto U_{a_0}^*(s, t) \in \mathcal{L}_{\mathbf{s}}(L_p(D)) \right|$$

is continuous.

The following analog of Proposition 2.7(i) holds.

Proposition 2.12. For any $a_0 \in Y_0$, any $0 \le t < s \le T$ and any $1 \le p \le q \le \infty$ there holds $U_{a_0}^*(s,t) \in \mathcal{L}_s(L_p(D), L_q(D))$.

Proposition 2.13. For $a_0 \in Y_0$ there holds

$$\langle U_{a_0}(t,s)u_0, v_0 \rangle_{L_2(D)} = \langle u_0, U_{a_0}^*(s,t)v_0 \rangle_{L_2(D)}$$

for any $0 \le s \le t \le T, \ u_0, v_0 \in L_2(D).$ (2.10)

Proposition 2.13 states that the linear operator $U_{a_0}^*(s,t) \in \mathcal{L}(L_2(D))$ is the dual (in the functional-analytic sense) of $U_{a_0}(t,s) \in \mathcal{L}(L_2(D))$. For a proof, see [35, Prop. 2.3.3].

Proposition 2.14. For $1 and <math>a_0 \in Y_0$ there holds

$$\langle U_{a_0}(t,s)u_0, v_0 \rangle_{L_p(D), L_{p'}(D)} = \langle u_0, U_{a_0}^*(s,t)v_0 \rangle_{L_p(D), L_{p'}(D)}$$
(2.11)

for any $(s,t) \in \dot{\Delta}$, $u_0 \in L_p(D)$ and $v_0 \in L_{p'}(D)$.

Proof. Fix $(s,t) \in \dot{\Delta}$, $u_0 \in L_p(D)$ and $v_0 \in L_{q'}(D)$. From Propositions 2.7(i) and 2.12 it follows that $U_{a_0}(\zeta, s)u_0, U^*_{a_0}(\zeta, t)v_0 \in L_2(D)$ for all $\zeta \in (s, t)$, consequently $\langle U_{a_0}(\zeta, s)u_0, U_{a_0}(\zeta, t)v_0 \rangle_{L_2(D)}$ is well defined for such ζ . An application of (2.2), Proposition 2.13 and (2.9) gives that for any $s < \zeta_1 \leq \zeta_2 < t$ there holds

$$\begin{aligned} \langle U_{a_0}(\zeta_2, s)u_0, U_{a_0}^*(\zeta_2, t)v_0 \rangle_{L_2(D)} \\ &= \langle U_{a_0}(\zeta_2, \zeta_1)U_{a_0}(\zeta_1, s)u_0, U_{a_0}^*(\zeta_2, t)v_0 \rangle_{L_2(D)} \\ &= \langle U_{a_0}(\zeta_1, s)u_0, U_{a_0}^*(\zeta_1, \zeta_2)U_{a_0}^*(\zeta_2, t)v_0 \rangle_{L_2(D)} \\ &= \langle U_{a_0}(\zeta_1, s)u_0, U_{a_0}^*(\zeta_1, s)u_0, U_{a_0}^*(\zeta_1, t)v_0 \rangle_{L_2(D)}. \end{aligned}$$

Therefore the assignment

$$(s,t) \ni \zeta \mapsto \langle U_{a_0}(\zeta,s)u_0, U_{a_0}^*(\zeta,t)v_0 \rangle_{L_2(D)} = \langle U_{a_0}(\zeta,s)u_0, U_{a_0}^*(\zeta,t)v_0 \rangle_{L_p(D),L_{p'}(D)} = \langle U_{a_0}(\zeta,s)u_0, U_{a_0}^*(\zeta,t)v_0 \rangle_{L_p(D),L_{p'}(D)}$$

is constant (denote its value by A). If we let $\zeta \nearrow t$, then $U_{a_0}(\zeta, s)u_0$ converges, by Proposition 2.4, in the $L_p(D)$ -norm to $U_{a_0}(t,s)u_0$ and $U_{a_0}^*(\zeta,t)v_0$ converges, by Proposition 2.11(ii), in the $L_{p'}(D)$ -norm to v_0 , consequently $\langle U_{a_0}(t,s)u_0, v_0\rangle_{L_p(D),L_{p'}(D)} = A$. If we let $\zeta \searrow s$, then $U_{a_0}(\zeta,s)u_0$ converges, by Proposition 2.3(ii), in the $L_p(D)$ -norm to u_0 and $U_{a_0}^*(\zeta,t)v_0$ converges, by Propositions 2.12 and 2.11(ii), in the $L_{p'}(D)$ -norm to $U_{a_0}^*(s,t)v_0$, consequently $\langle u_0, U_{a_0}^*(s,t)v_0 \rangle_{L_p(D),L_{p'}(D)} = A$. This concludes the proof. \Box

It follows from Proposition 2.14 that the linear operator $U_{a_0}^*(s,t) \in \mathcal{L}(L_{p'}(D))$ is the dual (in the functional-analytic sense) of $U_{a_0}(t,s) \in \mathcal{L}(L_p(D))$.

In the light of the above, the following counterpart to Proposition 2.7(ii) holds.

Proposition 2.15. There are constants $M \ge 1$ and $\gamma \in \mathbb{R}$, the same as in Proposition 2.7, such that

$$\|U_{a_0}^*(s,t)\|_{\mathcal{L}(L_p(D),L_q(D))} \le M(t-s)^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} e^{\gamma(t-s)}$$

for $1 \leq p \leq q \leq \infty$, $a_0 \in Y_0$ and $(s,t) \in \dot{\Delta}$.

2.2 Continuous Dependence of Weak Solutions

Lemma 2.1. Let $1 and <math>a_0 \in Y_0$. Then the mapping

$$\left[\Delta \ni (s,t) \mapsto U_{a_0}(t,s) \in \mathcal{L}_{s}(L_p(D)) \right]$$

 $is \ continuous.$

Proof. If $t_m \to t > s$ as $m \to \infty$, $U_{a_0}(t_m, s)u_0 \to U_{a_0}(t, s)u_0$ in $L_p(D)$, by Proposition 2.4.

Assume that $s_m \to s < t$ as $m \to \infty$. Fix $u_0 \in L_p(D)$ and $v_0 \in L_{p'}(D)$. We have, by Proposition 2.14 and the adjoint equation analog of Proposition 2.4,

$$\langle U_{a_0}(t, s_m) u_0, v_0 \rangle_{L_p(D), L_{p'}(D)} = \langle u_0, U^*_{a_0}(s_m, t) v_0 \rangle_{L_p(D), L_{p'}(D)} \rightarrow \langle u_0, U^*_{a_0}(s, t) v_0 \rangle_{L_p(D), L_{p'}(D)} = \langle U_{a_0}(t, s) u_0, v_0 \rangle_{L_p(D), L_{p'}(D)},$$

so $U_{a_0}(t, s_m)u_0 \rightharpoonup U_{a_0}(t, s)u_0$ in $L_p(D)$. As $\{U_{a_0}(t, s_m)u_0 : m \in \mathbb{N}\}$ is, by Proposition 2.9, precompact in $L_p(D)$, the convergence is in the norm.

Finally, assume that $s_m \to s$ and $t_m \to t$ with s < t, and fix $u_0 \in L_p(D)$. We can assume that $s_m < (s+t)/2 < t$ for all m. By the previous paragraph, $U_{a_0}((s+t)/2, s_m)u_0 \to U_{a_0}((s+t)/2, s)u_0$ in $L_p(D)$. Corollary 2.1 implies that

$$U_{a_0}(t_m, s_m)u_0 = U_{a_0}(t_m, \frac{1}{2}(s+t))(U_{a_0}(\frac{1}{2}(s+t), s_m)u_0$$

$$\to U_{a_0}(t, \frac{1}{2}(s+t))(U_{a_0}(\frac{1}{2}(s+t), s)u_0 = U_{a_0}(t, s)u_0,$$

where the convergence is in $L_p(D)$, too.

Proposition 2.16. Let $1 and <math>a_0 \in Y_0$. Then the mapping

$$\left[\dot{\Delta} \ni (s,t) \mapsto U_{a_0}(t,s) \in \mathcal{L}(L_p(D))\right]$$

is continuous.

Proof. Let $s_m \to s$ and $t_m \to t$ with s < t. Suppose to the contrary that there are $\epsilon > 0$ and $(u_m)_{m=1}^{\infty} \subset L_p(D)$, $||u_m||_{L_p(D)} = 1$, such that

$$||U_{a_0}(t_m, s_m)u_m - U(t, s)u_m||_{L_p(D)} \ge \epsilon, \quad m = 1, 2, 3, \dots$$

It follows from Proposition 2.9 that, after possibly taking a subsequence and relabelling, we can assume that $U_{a_0}(t_m, s_m)u_m$ converge to \tilde{u} and $U_{a_0}(t, s)u_m$ converge to \hat{u} , both in $L_p(D)$. For any $v_0 \in L_{p'}(D)$ we have, by Proposition 2.14,

$$\langle (U_{a_0}(t_m, s_m) - U_{a_0}(t, s)) u_m, v_0 \rangle_{L_p(D), L_{p'}(D)} = \langle u_m, (U_{a_0}^*(s_m, t_m) - U_{a_0}^*(s, t)) v_0 \rangle_{L_p(D), L_{p'}(D)}.$$

Since $||u_m||_{L_p(D)} = 1$, we conclude from the adjoint equation analog of Lemma 2.1 that the above expression converges to zero as $m \to \infty$. Consequently $\tilde{u} = \hat{u}$, a contradiction.

Proposition 2.17. Assume, in addition, (DA4). For 1 the mapping

$$\left[Y_0 \times \Delta \times L_p(D) \ni (a_0, s, t, u_0) \mapsto U_{a_0}(t, s)u_0 \in L_p(D)\right]$$

is continuous.

Proof. It follows from [35, Props. 2.2.12 and 2.2.13] that, for $2 \le p < \infty$, the mapping

$$\left[Y_0 \times \dot{\Delta} \times L_2(D) \ni (a_0, s, t, u_0) \mapsto U_{a_0}(t, s) u_0 \in L_p(D) \right]$$

is continuous, too.

To conclude the proof it suffices to show that for any 1 the mapping

$$\left[Y_0 \times \dot{\Delta} \times L_p(D) \ni (a_0, s, t, u_0) \mapsto U_{a_0}(t, s) u_0 \in L_2(D)\right]$$

is continuous. Observe that if we have $a_{0,m} \to a_0 \in Y_0$, $s_m \to s$, $t_m \to t$ with $s_m < t_n$ and s < t, and $u_{0,m} \to u_0 \in L_p(D)$, then from Proposition 2.9 it follows that, after possibly choosing a subsequence, there is $w \in L_2(D)$ such that $U_{a_{0,m}}(t_m, s_m)u_{0,m} \to w$ in $L_2(D)$. Consequently, $\langle U_{a_{0,m}}(t_m, s_m)u_{0,m}, v \rangle_{L_2(D)} \to \langle w, v \rangle_{L_2(D)}$ as $m \to \infty$, for any $v \in L_2(D)$. On the other hand, one has, by Proposition 2.14,

$$\langle U_{a_{0,m}}(t_m, s_m)u_{0,m}, v \rangle_{L_2(D)} = \langle u_{0,m}, U^*_{a_{0,m}}(s_m, t_m)v \rangle_{L_p(D), L_{p'}(D)}.$$

As $2 < p' < \infty$, an application of the result already obtained to the adjoint equation yields that $U^*_{a_{0,m}}(s_m, t_m)v$ converges, as $m \to \infty$, to $U^*_{a_0}(s, t)v$ in $L_{p'}(D)$. As $u_{0,m}$ converges to u_0 in $L_p(D)$, we have that $\langle u_{0,m}, U^*_{a_{0,m}}(s_m, t_m)v \rangle_{L_p(D), L_{p'}(D)}$ converges to $\langle u_0, U^*_{a_0}(s, t)v \rangle_{L_p(D), L_{p'}(D)}$, which is, by Proposition 2.14, equal to $\langle U_{a_0}(t, s)u_0, v \rangle_{L_2(D)}$. As $v \in L_2(D)$ is arbitrary, we have $w = U_{a_0}(t, s)u_0$.

Proposition 2.18. Assume, in addition, (DA4). For 1 the mapping

$$\left[Y_0 \times \dot{\Delta} \ni (a_0, s, t) \mapsto U_{a_0}(t, s) \in \mathcal{L}(L_p(D))\right]$$

is continuous.

Proof. In order not to overburden the notation we assume s = 0.

Let $(a_{0,m})_{m=1}^{\infty} \subset Y_0$ be a sequence converging to a_0 as $m \to \infty$, and let $(t_m)_{m=1}^{\infty} \subset (0,T]$ be a sequence converging to t > 0 as $m \to \infty$. Suppose to the contrary that $\|U_{a_{0,m}}(t_m,0) - U_{a_0}(t,0)\|_{\mathcal{L}(L_p(D))}$ does not converge to 0, that is, there exist $\epsilon > 0$ and a sequence $(u_m)_{m=1}^{\infty} \subset L_p(D)$, $\|u_m\|_{L_p(D)} = 1$ for all m, such that

$$||U_{a_{0,m}}(t_m, 0)u_m - U_{a_0}(t, 0)u_m||_{L_p(D)} \ge \epsilon$$

for all m.

It follows from Proposition 2.9 that, after possibly extracting a subsequence, we can assume that $U_{a_{0,m}}(t_m/2,0)u_m$ and $U_{a_0}(t/2,0)u_m$ converge, as $m \to \infty$, in the $L_p(D)$ -norm. We claim that both converge to the same \tilde{u} . Indeed, it suffices to check that the difference $(U_{a_{0,m}}(t_m/2,0) - U_{a_0}(t/2,0))u_m$ converges to zero in $L_p(D)$, which is, in light of the equalities

$$\begin{split} \langle (U_{a_{0,m}}(t_m/2,0) - U_{a_0}(t/2,0))u_m, v \rangle_{L_p(D), L_{p'}(D)} \\ &= \langle u_m, (U^*_{a_{0,m}}(0,t_m/2) - U^*_{a_0}(0,t/2))v \rangle_{L_p(D), L_{p'}(D)}, \quad v \in L_{p'}(D), \end{split}$$

a consequence of the analog for the adjoint equation of Proposition 2.17.

Proposition 2.17 implies that

$$\begin{aligned} \|U_{a_{0,m}}(t_m,0)u_m - U_{a_0}(t,t/2)\tilde{u}\|_{L_p(D)} \\ &= \|U_{a_{0,m}}(t_m,t_m/2)(U_{a_{0,m}}(t_m/2,0)u_m) - U_{a_0}(t,t/2)\tilde{u}\|_{L_p(D)} \to 0, \end{aligned}$$

and

$$\begin{split} \|U_{a_0}(t,0)u_m - U_{a_0}(t,t/2)\tilde{u}\|_{L_p(D)} \\ &= \|U_{a_0}(t,t/2)(U_{a_0}(t/2,0)u_m) - U_{a_0}(t,t/2)\tilde{u}\|_{L_p(D)} \\ &\leq \|U_{a_0}(t,t/2)\|_{\mathcal{L}(L_p(D))}\|U_{a_0}(t/2,0)u_m - \tilde{u}\|_{L_p(D)} \to 0 \end{split}$$

therefore $||U_{a_{0,m}}(t_m, 0)u_m - U_{a_0}(t, 0)u_m||_{L_p(D)}$, converges to zero, a contradiction.

3 Mild Solutions

In the present section we assume (DA1), (DA2) and that Y as in Definition 1.1 is such that its flattening Y_0 satisfies (DA3). Occasionally we will assume (DA5).

Definition 3.1 (Multiplication Operator). For $a \in Y$, $1 \le p \le \infty$ and $0 \le t \le T$ we define multiplication operator $C_a^1(t)$: $L_p(D) \to L_p(D)$ as follows

$$C_a^1(t)v = c_1(t,\cdot)v.$$

The $C_a^1(t)$ operator is well defined as long as assumption (DA 2) holds. To be more precise we use a corollary from assumption (DA 2) on t-sections of c_1 .

Lemma 3.1 (Boundedness of Multiplication Operator). The multiplication operator $C_a^1(t)$ is linear and bounded uniformly with respect to a.e. 0 < t < T and $a \in Y$.

It should be remarked that the exceptional sets can be different for different $a \in Y$.

Proof. Let K be the norm bound of Y (see assumption Y1). For any $v \in L_p(D)$ by the Hölder inequality we get

$$\begin{aligned} \|C_a^1(t)v\|_{L_p(D)} &= \|c_1(t,\cdot)v\|_{L_p(D)} \\ &\leq \|c_1(t,\cdot)\|_{L_\infty(D)} \|v\|_{L_p(D)} \\ &\leq K \|v\|_{L_p(D)} \end{aligned}$$

where above inequality holds for a.e. 0 < t < T, so the operator norm of $C_a^1(t)$ is bounded a.e. by $||c_1(t, \cdot)||_{L_{\infty}(D)}$ what can be bounded uniformly with respect to a.e. 0 < t < T by virtue of assumption (DA2). Since Y is bounded by K we also have uniform boundedness in $a \in Y$.

Below we present a series of lemmas to prove the measurability of individual parts of the mild solution. We will make frequent use of the Lemma 0.1 in this part of the work, in particular, Remark 0.1 on measurability will also be useful.

Lemma 3.2. For any $1 and any norm bounded set <math>E \subset C([-1,T], L_p(D))$ the set

$$\left\{ \left[(0,T) \ni t \mapsto (u \circ \Phi)(t) \in L_p(D) \right] : u \in E, \Phi \in \mathcal{R} \right\}$$
(3.1)

is bounded in $L_{\infty}((0,T), L_p(D))$.

Proof. Let $u \in E$ and $\Phi \in \widetilde{\mathcal{R}}$. The mapping $u \circ \Phi$ is $(\mathfrak{L}((0,T)), \mathfrak{B}(L_p(D)))$ -measurable, since for any fixed open set $V \subset L_p(D)$ the preimage $u^{-1}[V]$ is open, hence $(u \circ \Phi)^{-1}[V] \in \mathfrak{L}((0,T))$. Moreover, the $L_{\infty}((0,T), L_p(D))$ -norm of the map $u \circ \Phi$ is uniformly bounded with respect to u and Φ by the same constant as E due to the inequality

$$\|u \circ \Phi\|_{L_{\infty}((0,T),L_{p}(D))} = \underset{t \in (0,T)}{\operatorname{ess sup}} \|(u \circ \Phi)(t)\|_{L_{p}(D)}$$
$$\leq \underset{t \in [-1,T]}{\operatorname{sup}} \|u(t)\|_{L_{p}(D)} = \|u\|_{C([-1,T],L_{p}(D))}.$$

Lemma 3.3. For any $1 and any norm bounded set <math>\tilde{E} \subset L_{\infty}((0,T), L_p(D))$ the set

$$\left\{ \left[(0,T) \ni t \mapsto C_a^1(t)\tilde{u}(t) \in L_p(D) \right] : a \in Y, \tilde{u} \in \tilde{E} \right\}$$

is bounded in $L_{\infty}((0,T), L_p(D))$.

Proof. Let $\tilde{u} \in \tilde{E}$ and $a \in Y$. From Lemma 0.1(a) follows that the mapping

$$\left[(0,T) \times D \ni (t,x) \mapsto \tilde{u}(t)[x] \in \mathbb{R} \right]$$
(3.2)

is $(\mathfrak{L}((0,T)) \otimes \mathfrak{L}(D), \mathfrak{B}(\mathbb{R}))$ -measurable. Hence the function

$$\left[(0,T) \times D \ni (t,x) \mapsto \left(C_a(t)\tilde{u}(t) \right) [x] \in \mathbb{R} \right]$$
(3.3)

for any $a \in Y$ is $(\mathfrak{L}((0,T)) \otimes \mathfrak{L}(D), \mathfrak{B}(\mathbb{R}))$ -measurable, since it can be rewritten as the product of $(\mathfrak{L}((0,T)) \otimes \mathfrak{L}(D), \mathfrak{B}(\mathbb{R}))$ -measurable functions, namely

$$\left[(0,T) \times D \ni (t,x) \mapsto c_1(t,x) \,\tilde{u}(t)[x] \in \mathbb{R} \right]. \tag{3.4}$$

It suffices now to notice that for a.e. $t \in (0, T)$ the *t*-section of (3.3) belongs (by the definition of the multiplication operator) to $L_p(D)$. So from Lemma 0.1(b) it follows that the mapping

$$\left[(0,T) \ni t \mapsto C_a(t)\tilde{u}(t) \in L_p(D) \right]$$

is $(\mathfrak{L}((0,T)),\mathfrak{B}(L_p(D)))$ -measurable. By the norm estimate,

$$\begin{aligned} \|C_a^1(\cdot)\tilde{u}\|_{L_{\infty}((0,T),L_p(D))} &= \operatorname{ess\,sup}_{t\in(0,T)} \|c_1(t,\cdot)\tilde{u}(t)\|_{L_p(D)} \\ &\leq K \|\tilde{u}\|_{L_{\infty}((0,T),L_p(D))} \end{aligned}$$

we obtain the statement.

Lemma 3.4. Assume $1 , <math>a_0 \in Y_0$, and $u \in L_{\infty}((0,T), L_p(D))$. Then

(i) for any $0 < t \leq T$ the function

$$[\zeta \mapsto U_{a_0}(t,\zeta)u(\zeta), \text{ for a.e. } \zeta \in (0,t)]$$
(3.5)

belongs to $L_{\infty}((0,t), L_p(D))$ moreover, the linear operator assigning (3.5) to u belongs to $\mathcal{L}(L_{\infty}((0,1), L_p(D)), L_{\infty}((0,1), L_p(D)))$, with the norm bounded uniformly in $a_0 \in Y_0$,

(ii) the mapping

$$\left[[0,T] \ni t \mapsto \int_{0}^{t} U_{a_0}(t,\zeta)u(\zeta) \,\mathrm{d}\zeta \right]$$
(3.6)

belongs to $C([0,T], L_p(D))$.

Proof. Fix $0 < t \leq T$. We show first that (3.5) defines a $(\mathfrak{L}((0,t)), \mathfrak{B}(L_p(D)))$ -measurable function. It is equivalent, by Theorem 0.1, to showing that for each $v \in L_{p'}(D)$ the function

$$[\zeta \mapsto \langle U_{a_0}(t,\zeta)u(\zeta),v\rangle_{L_p(D),L_{p'}(D)}]$$

is $(\mathfrak{L}((0,t)),\mathfrak{B}(\mathbb{R}))$ -measurable. By Proposition 2.14, for Lebesgue-a.e. $\zeta \in [0,t)$ there holds

$$\langle U_{a_0}(t,\zeta)u(\zeta),v\rangle_{L_p(D),L_{p'}(D)} = \langle u(\zeta),U_{a_0}^*(\zeta,t)v\rangle_{L_p(D),L_{p'}(D)}$$

It suffices now to notice that u is $(\mathfrak{L}((0,t)), \mathfrak{B}(L_p(D)))$ -measurable, by assumption, and that the function

$$\left[[0,t) \ni \zeta \mapsto U^*_{a_0}(\zeta,t)v \in L_{p'}(D) \right]$$

is continuous, by the adjoint equation analog of Lemma 2.1. It follows from Proposition 2.7(ii) that the function

$$\left[(0,t) \ni \zeta \mapsto \| U_{a_0}(t,\zeta) \|_{\mathcal{L}(L_p(D))} \right]$$

belongs to $L_{\infty}((0,t))$. Then the membership of (3.5) in $L_{\infty}((0,t), L_p(D))$ as well as the bound on its norm follow from the generalized Hölder inequality. The proof of part (i) is thus completed.

We proceed to the proof of part (ii). By part (i), the function (3.6) is well defined. Let $0 \le t_1 \le t_2 \le T$. We write

$$\int_{0}^{t_{2}} U_{a_{0}}(t_{2},\zeta)u(\zeta) \,\mathrm{d}\zeta - \int_{0}^{t_{1}} U_{a_{0}}(t_{1},\zeta)u(\zeta) \,\mathrm{d}\zeta$$
$$= \int_{0}^{t_{1}} \left(U_{a_{0}}(t_{2},\zeta) - Ua_{0}(t_{1},\zeta) \right)u(\zeta) \,\mathrm{d}\zeta + \int_{t_{1}}^{t_{2}} U_{a_{0}}(t_{2},\zeta)u(\zeta) \,\mathrm{d}\zeta.$$

Let $\epsilon > 0$. As (3.5) belongs to $L_{\infty}((0,t), L_p(D))$, it is a consequence of [16, Thm. II.2.4(i)] that the $L_p(D)$ -norm of the second term on the right-hand side can be made $\langle \epsilon/3 \rangle$ by taking t_1, t_2 sufficiently close to each other. Regarding the first term, we write

$$\int_{0}^{t_{1}} (U_{a_{0}}(t_{2},\zeta) - U_{a_{0}}(t_{1},\zeta)) u(\zeta) d\zeta$$

=
$$\int_{0}^{t_{1}-\eta} (U_{a_{0}}(t_{2},\zeta) - U_{a_{0}}(t_{1},\zeta)) u(\zeta) d\zeta + \int_{t_{1}-\eta}^{t_{1}} (U_{a_{0}}(t_{2},\zeta) - U_{a_{0}}(t_{1},\zeta)) u(\zeta) d\zeta.$$

Again by [16, Thm. II.2.4(i)], for $\eta > 0$ sufficiently small there holds

$$\left\|\int_{t_1-\eta}^{t_1} \left(U_{a_0}(t_2,\zeta) - U_{a_0}(t_1,\zeta)\right) u(\zeta) \,\mathrm{d}\zeta\right\|_{L_p(D)} < \frac{\epsilon}{3}.$$

It follows from Proposition 2.16 that the assignment

$$\left[\left\{\dot{\Delta}:\eta\leq\zeta+\eta\leq t\leq T\right\}\ni(\zeta,t)\mapsto U_{a_0}(t,\zeta)\in\mathcal{L}(L_p(D))\right]$$

is uniformly continuous, consequently there exists $\delta > 0$ such that if $\eta \leq \zeta + \eta \leq t_1 \leq t_2, t_2 - t_1 < \delta$, then

$$||U_{a_0}(t_2,\zeta) - U_{a_0}(t_1,\zeta)||_{\mathcal{L}(L_p(D))} < \frac{\epsilon}{3||u||_{L_1((0,T),L_p(D))}}.$$

Therefore

$$\left\| \int_{0}^{t_{1}-\eta} (U_{a_{0}}(t_{2},\zeta) - U_{a_{0}}(t_{1},\zeta)) u(\zeta) \,\mathrm{d}\zeta \right\|_{L_{p}(D)} < \frac{\epsilon}{3}.$$

This concludes the proof of part (ii).

Definition 3.2 (Mild Solution). For $1 \le p < \infty$, $a \in Y$, $0 \le s < T_0 \le T$ and $u_0 \in C([s-1,s], L_p(D))$ and $R \in \mathcal{R}$ the function $u \in C([s-1,T_0], L_p(D))$ such that

$$u(t) = u_0(t) \text{ for } t \in [s - 1, s],$$
 (3.7)

holds and the integral equation

$$u(t) = U_{\tilde{a}}(t,s)u_0(s) + \int_{s}^{t} U_{\tilde{a}}(t,\zeta)C_a^1(\zeta)u(\zeta - R(\zeta))\,\mathrm{d}\zeta$$
(3.8)

is satisfied in $L_p(D)$ on $[s, T_0]$ will be called a mild solution of $(ME)_a + (BC)_a$. For $T_0 = T$ we have a global mild solution.

At first note that the concept of mild solution, especially part (3.8), is well defined based on Lemma 3.3 and Lemma 3.4. At some moments we use the name "mild solution" to describe function $u \upharpoonright_{[0,T_0]}$ instead of $u \in C([-1,T_0], L_p(D))$ satisfying (3.7) and (3.8). This convention seems more natural especially in the context of continuous dependence on coefficients. A similar convention can be found in the literature [28].

3.1 Existence and Uniqueness of Global Mild Solutions

Proposition 3.1. There exists $\Theta_0 \in (0, T]$ such that for any $1 , <math>a \in Y$, $R \in \mathcal{R}$, $0 \le s \le T - \Theta_0$, $u_0 \in C([s - 1, s], L_p(D))$ and any $0 < \Theta \le \Theta_0$ there exist unique solution of $(ME)_a + (BC)_a$ on $[s - 1, s + \Theta]$ with initial condition u_0 .

Proof. The idea of the proof runs as follows. The solution is obtained as a fixed point of the contraction mapping \mathfrak{G} of $C([s, s+\Theta], L_p(D))$ into itself (see 2.3(ii), 3.4(ii) and 3.3) defined as

$$(\mathfrak{G}u)[t] := U_{\tilde{a}}(t,s)u_0(s) + \int_s^t U_{\tilde{a}}(t,\zeta)C_a^1(\zeta)u(\zeta - R(\zeta))\,\mathrm{d}\zeta, \qquad (3.9)$$

where $s \leq t \leq s + \Theta$ and $\Theta \in (0, T]$ is sufficiently small. Until revoking, u and v stand for generic functions in $C([s, s + \Theta], L_p(D))$. For such a u we interpret

 $u(\zeta - R(\zeta))$ (similarly v) as $u_0(\zeta - R(\zeta))$ when $\zeta - R(\zeta) \in (-1, 0)$.

$$\begin{split} \|(\mathfrak{G}u)[t] - (\mathfrak{G}v)[t]\|_{L_{p}(D)} &\leq \int_{s}^{t} \|U_{\bar{a}}(t,\zeta)C_{a}^{1}(\zeta)\left(u(\zeta - R(\zeta)) - v(\zeta - R(\zeta))\right)\|_{L_{p}(D)} \,\mathrm{d}\zeta \\ &\leq MKe^{\gamma t} \int_{s}^{t} \|u(\zeta - R(\zeta)) - v(\zeta - R(\zeta))\|_{L_{p}(D)} \,\mathrm{d}\zeta \\ &\leq MKe^{\gamma t} \int_{s}^{t} \sup_{s \leq \xi \leq t} \|u(\xi - R(\xi)) - v(\xi - R(\xi))\|_{L_{p}(D)} \,\mathrm{d}\zeta \\ &\leq MKe^{\gamma t} \int_{s}^{t} \sup_{s-1 \leq \xi \leq t} \|u(\xi) - v(\xi)\|_{L_{p}(D)} \,\mathrm{d}\zeta \\ &= MKe^{\gamma t} \int_{s}^{t} \sup_{s \leq \xi \leq t} \|u(\xi) - v(\xi)\|_{L_{p}(D)} \,\mathrm{d}\zeta \\ &\leq MKe^{\gamma t} \Theta \|u - v\|_{C([s,s+\Theta],L_{p}(D))}. \end{split}$$

By taking $0 < \Theta \leq \Theta_0 \coloneqq 1/(2MKe^{\gamma T})$ we obtain that the contraction coefficient is less than 1.

The Contraction Mapping Principle guarantees the existence and uniqueness of the fixed point u of \mathfrak{G} , which is then the unique mild L_p -solution of $(ME)_a + (BC)_a$ on $[s - 1, \Theta_0]$ satisfying the initial condition (IC).

Lemma 3.5. For any $1 , <math>0 < s_1 < s_2 \leq T$, $a \in Y$, $R \in \mathcal{R}$, $u_0 \in C([-1,0], L_p(D))$ and $v: [-1, s_2] \rightarrow L_p(D)$ the following statements are equivalent:

- (i) a function v is the mild solution of (ME)_a+(BC)_a on [-1, s₂] with initial condition u₀,
- (ii) a function v↾_[-1,s1] is the mild solution of (ME)_a+(BC)_a with initial condition u₀ and v↾_[s1-1,s2] is the mild solution of (ME)_a+(BC)_a with initial condition v↾_[s1-1,s1].

Proof. Let p, s_1, s_2, a, R, u_0 and v be as in the statement. To prove (i) \Rightarrow (ii) it suffices to see that for any $s_1 \leq t \leq s_2$, in view of (2.2) and [1, Lemma 11.45]

there holds

$$\begin{aligned} v(t) &= U_{\tilde{a}}(t,0)u_{0}(0) + \int_{0}^{t} U_{\tilde{a}}(t,\zeta)C_{a}^{1}(\zeta)v(\zeta - R(\zeta)) \,\mathrm{d}\zeta \\ &= U_{\tilde{a}}(t,s_{1}) \left(U_{\tilde{a}}(s_{1},0)u_{0}(0) + \int_{0}^{s_{1}} U_{\tilde{a}}(s_{1},\zeta)C_{a}^{1}(\zeta)v(\zeta - R(\zeta)) \,\mathrm{d}\zeta \right) \\ &+ \int_{s_{1}}^{t} U_{\tilde{a}}(t,\zeta)C_{a}^{1}(\zeta)v(\zeta - R(\zeta)) \,\mathrm{d}\zeta \\ &= U_{\tilde{a}}(t,s_{1})v_{\uparrow[s_{1}-1,s_{1}]}(s_{1}) + \int_{s_{1}}^{t} U_{\tilde{a}}(t,\zeta)C_{a}^{1}(\zeta)v_{\uparrow[s_{1}-1,s_{2}]}(\zeta - R(\zeta)) \,\mathrm{d}\zeta. \end{aligned}$$

$$(3.10)$$

In order to prove (ii) \Rightarrow (i) fix $t \in [-1, s_2]$ and consider the cases: $t \in [-1, 0]$, $t \in [0, s_1]$ or $t \in [s_1, s_2]$. Since the first two cases are straightforward and the third is a similar calculation to (3.10), the proof is finished.

Theorem 3.1. For any $1 , <math>a \in Y$, $u_0 \in C([-1, 0], L_p(D))$ and $R \in \mathcal{R}$ equation $(ME)_a + (BC)_a$ has a unique global mild solution on [0, T].

Proof. Fix a, u_0 and R as in the statement. Let

 $Q = \{q \in [0,T] : (ME)_a + (BC)_a \text{ has a unique mild solution on } [-1,q]\}.$

It suffices to prove that $T \in Q$. Suppose to the contrary that $T \notin Q$. Since $\Theta_0 \in Q$ (where Θ_0 stands for constant obtained in Proposition 3.1) and $Q \subset [0,T]$, $\sup Q < \infty$. It is straightforward that $0 \leq \sup Q - \Theta_0/2 < \sup Q \leq T$ hence there exists $s \in Q$ such that $s > \sup Q - \Theta_0/2$.

Let $v_1: [-1, s] \to L_p(D)$ be the unique mild solution with initial condition u_0 . From the definition of Θ it follows that there is a mild solution $v_2: [s - 1, \min\{s + \Theta_0, T\}] \to L_p(D)$ with initial condition $v_1 \upharpoonright_{[s-1,s]}$. Let

$$v(t) = \begin{cases} v_1(t) & \text{for } t \in [-1, s] \\ v_2(t) & \text{for } t \in [s - 1, \min\{s + \Theta_0, T\}]. \end{cases}$$

We claim that v is a unique mild solution of $(ME)_a + (BC)_a$ on $[-1, \min\{s + \Theta_0, T\}]$ with initial condition u_0 . From Lemma 3.5 it follows that v is in fact a mild solution. For uniqueness, assume $w \colon [-1, \min\{s + \Theta_0, T\}] \to L_p(D)$ is any mild solution. Then clearly $w \upharpoonright_{[-1,s]} = v_1$. Moreover, by Lemma 3.5, the function $w \upharpoonright_{[s-1,\min\{s+\Theta_0,T\}]}$ is a mild solution with initial condition $w \upharpoonright_{[s-1,s]} = v_1 \upharpoonright_{[s-1,s]}$, so by the uniqueness of v_2 we have that $w \upharpoonright_{[s-1,\min\{s+\Theta_0,T\}]} = v_2$. Hence v = w. The proof is completed by the following observation: if $\min\{s+\Theta_0,T\} = s+\Theta_0$ then $s + \Theta_0 \in Q$, so we get a contradiction with the fact that $s + \Theta_0 > \sup Q$: otherwise $T \in Q$, which contradicts the assumption.

The above result allows us to define a mild solution of $(ME)_a + (BC)_a$ on the whole of [-1, T] or [s - 1, T] if necessary.

For s = 0, to stress the dependence of the solution on a, u_0 , R we write $u(\cdot; a, u_0, R)$. For $t \in [-1, 0]$, $u(t; a, u_0, R)$ is interpreted as $u_0(t)$. Moreover, when it does not lead to confusion, we sometimes write $u(t; a, u_0, \Phi)$ instead of $u(t; a, u_0, R)$.

3.2 Compactness of Solution Operator

Lemma 3.6. Assume $1 and <math>0 < T_1 \leq T$. Then for any bounded $F \subset L_{\infty}((0,T), L_p(D))$ the set

$$\widehat{F} := \left\{ \int_{0}^{t} U_{a_0}(t,\zeta) u(\zeta) \, \mathrm{d}\zeta : a_0 \in Y_0, \ u \in F, \ t \in [T_1,T] \right\}$$

is precompact in $L_p(D)$.

Proof. Compare [35, Thm. 6.1.3]. Fix p, T_1 and F as in the statement. Let $(t_m)_{m=1}^{\infty} \subset [T_1, T], (a_{0,m})_{m=1}^{\infty} \subset Y_0, (u_m)_{m=1}^{\infty} \subset F$. We claim that for any fixed $l \in \mathbb{N}$ the set

$$\widetilde{F}_{l} := \left\{ \int_{0}^{t_{m}-\frac{1}{t}} U_{a_{0,m}}(t_{m},\zeta) u_{m}(\zeta) \,\mathrm{d}\zeta : m \in \mathbb{N} \right\}.$$

is precompact in $L_p(D)$. Denote by $M_0 > 0$ the supremum of the $L_{\infty}((0,T), L_p(D))$ -norms of u_m , and put \check{F}_l to be the closure in $L_p(D)$ of the set

$$\left\{ U_{a_0}(s,0)\tilde{u}: a_0 \in Y_0, s \in \left[\frac{1}{l}, T\right], \|\tilde{u}\|_{L_p(D)} \le M_0 \right\}$$

The set \check{F}_l is balanced. We have $\check{F}_l \subset \check{F}_{l+1}$. By Proposition 2.9, \check{F}_l is compact. [16, Cor. II.2.8] implies that

$$\int_{0}^{t_m - \frac{1}{l}} U_{a_{0,m}}(t_m, \zeta) u_m(\zeta) \,\mathrm{d}\zeta \in T \cdot \overline{\mathrm{co}} \,\check{F}_l$$

where $\overline{\text{co}}$ denotes the closed convex hull in $L_p(D)$. As, by Mazur's theorem ([16, Thm. II.2.12]), $T \cdot \overline{\text{co}} \check{F}_l$ is compact for any $l \in \mathbb{N}$, this proves our claim that \widetilde{F}_l are precompact in $L_p(D)$. By a diagonal process we can assume without loss of generality that for each $l \in \mathbb{N}$ the integrals

$$\int_{0}^{t_m-\frac{1}{t}} U_{a_{0,m}}(t_m,\zeta) u_m(\zeta) \,\mathrm{d}\zeta$$

converge, as $m \to \infty$, in $L_p(D)$.

Lemma 3.4(i) guarantees that the functions $U_{a_{0,m}}(t_m, \cdot)u_m(\cdot)$ belong to $L_{\infty}((0, t_m), L_p(D))$, with their $L_{\infty}((0, t_m), L_p(D))$ -norms bounded uniformly in m. We estimate, via Hölder's inequality,

$$\left\| \int_{t_m - \frac{1}{t}}^{t_m} U_{a_{0,m}}(t_m, \zeta) u_m(\zeta) \, \mathrm{d}\zeta \right\|_{L_p(D)}$$

$$\leq \int_{t_m - \frac{1}{t}}^{t_m} \| U_{a_{0,m}}(t_m, \zeta) u_m(\zeta) \|_{L_p(D)} \, \mathrm{d}\zeta$$

$$\leq \| U_{a_{0,m}}(t_m, \cdot) u_m(\cdot) \|_{L_{\infty}((0,t_m), L_p(D))} \cdot (1/l).$$

It follows from Proposition 2.7(ii) that for any $\epsilon > 0$ there is $l_0 \in \mathbb{N}$ such that

$$\left\|\int_{t_m-\frac{1}{t_0}}^{t_m} U_{a_{0,m}}(t_m,\zeta)u_m(\zeta)\,\mathrm{d}\zeta\,\right\|_{L_p(D)}<\frac{\epsilon}{3}$$

uniformly in $m \in \mathbb{N}$. By the previous paragraph, there is m_0 such that if $m_1, m_2 \geq m_0$ then

$$\left\| \int_{0}^{t_{m_{1}}-\frac{1}{l_{0}}} U_{a_{0,m_{1}}}(t_{m_{1}},\zeta)u_{m_{1}}(\zeta) \,\mathrm{d}\zeta - \int_{0}^{t_{m_{2}}-\frac{1}{l_{0}}} U_{a_{0,m_{2}}}(t_{m_{2}},\zeta)u_{m_{2}}(\zeta) \,\mathrm{d}\zeta \,\right\|_{L_{p}(D)} < \frac{\epsilon}{3}.$$

Therefore

set

$$\left\| \int_{0}^{t_{m_{1}}} U_{a_{0,m_{1}}}(t_{m_{1}},\zeta) u_{m_{1}}(\zeta) \,\mathrm{d}\zeta - \int_{0}^{t_{m_{2}}} U_{a_{0,m_{2}}}(t_{m_{2}},\zeta) u_{m_{2}}(\zeta) \,\mathrm{d}\zeta \,\right\|_{L_{p}(D)} < \epsilon$$

for any $m_1, m_2 \ge m_0$.

From this it follows that

$$\left(\int_{0}^{t_m} \widetilde{U}_{a_{0,m}}(t_m,\zeta)u_m(\zeta)\,\mathrm{d}\zeta\right)_{m=1}^{\infty}$$

is a Cauchy sequence in $L_p(D)$. Therefore \widehat{F} is precompact in $L_p(D)$. \Box \Box Lemma 3.7. For any $1 and any bounded <math>F \subset L_{\infty}((0,T), L_p(D))$ the

$$\left\{ \left[[0,T] \ni t \mapsto \int_{0}^{t} U_{a_0}(t,\zeta) u(\zeta) \,\mathrm{d}\zeta \in L_p(D) \right] : a_0 \in Y_0, \ u \in F \right\}$$

is precompact in $C([0,T], L_p(D))$.

Proof. By the Ascoli–Arzelà theorem, it suffices, taking Lemma 3.6 into account, to show that for any $\epsilon > 0$ there is $\delta > 0$ such that, if $0 \le t_1 \le t_2 \le T$, $t_2 - t_1 < \delta$, then

$$\left\|\int_{0}^{t_{2}} U_{a_{0}}(t_{2},\zeta)u(\zeta)\,\mathrm{d}\zeta - \int_{0}^{t_{1}} U_{a_{0}}(t_{1},\zeta)u(\zeta)\,\mathrm{d}\zeta\right\|_{L_{p}(D)} < \epsilon$$

for all $a_0 \in Y_0$ and all $u \in F$. In order not to introduce too many constants we assume that F equals the unit ball in $L_{\infty}((0,T), L_p(D))$.

We write

$$\int_{0}^{t_{2}} U_{a_{0}}(t_{2},\zeta)u(\zeta) \,\mathrm{d}\zeta - \int_{0}^{t_{1}} U_{a_{0}}(t_{1},\zeta)u(\zeta) \,\mathrm{d}\zeta$$
$$= \int_{0}^{t_{1}} \left(U_{a_{0}}(t_{2},\zeta) - U_{a_{0}}(t_{1},\zeta) \right)u(\zeta) \,\mathrm{d}\zeta + \int_{t_{1}}^{t_{2}} U_{a_{0}}(t_{2},\zeta)u(\zeta) \,\mathrm{d}\zeta$$

By Proposition 2.7(ii),

$$\left\|\int_{t_1}^{t_2} U_{a_0}(t_2,\zeta) u(\zeta) \,\mathrm{d}\zeta\right\|_{L_p(D)} < \frac{\epsilon}{3},\tag{3.11}$$

provided $t_2 - t_1 < \epsilon/(3Me^{\gamma T})$.

Further, we write

$$\int_{0}^{t_{1}} \left(U_{a_{0}}(t_{2},\zeta) - U_{a_{0}}(t_{1},\zeta) \right) u(\zeta) \,\mathrm{d}\zeta$$

=
$$\int_{0}^{t_{1}-\eta} \left(U_{a_{0}}(t_{2},\zeta) - U_{a_{0}}(t_{1},\zeta) \right) u(\zeta) \,\mathrm{d}\zeta + \int_{t_{1}-\eta}^{t_{1}} \left(U_{a_{0}}(t_{2},\zeta) - U_{a_{0}}(t_{1},\zeta) \right) u(\zeta) \,\mathrm{d}\zeta.$$

By Proposition 2.7(ii), if $0 < \eta < \epsilon/(6Me^{\gamma T})$ then

$$\left\| \int_{t_1-\eta}^{t_1} \left(U_{a_0}(t_2,\zeta) - U_{a_0}(t_1,\zeta) \right) u(\zeta) \,\mathrm{d}\zeta \right\|_{L_q(D)} < \frac{\epsilon}{3}.$$
(3.12)

It follows from Proposition 2.18 that the assignment

$$\left[Y_0 \times [\eta, T] \ni (a_0, t) \mapsto U_{a_0}(t, 0) \in \mathcal{L}(L_p(D))\right]$$

is uniformly continuous, consequently there exists $\delta > 0$ such that if $\eta \leq s_1 < s_2$, $s_2 - s_1 < \delta$, then

$$||U_{a_0}(s_2,0) - U_{a_0}(s_1,0)||_{\mathcal{L}(L_p(D))} < \frac{\epsilon}{3T}.$$

Therefore

$$\left\|\int_{0}^{t_1-\eta} \left(U_{a_0}(t_2,\zeta)-U_{a_0}(t_1,\zeta)\right)u(\zeta)\,\mathrm{d}\zeta\right\|_{L_q(D)}<\frac{\epsilon}{3}.$$

The estimates (3.11), (3.12) and (3.2) do not depend on the choice of $a_0 \in Y_0$, so gathering them gives the required property.

Theorem 3.2. For any $0 < T_1 \leq T$, any $1 and any bounded <math>E \subset C([-1,0], L_p(D))$ the set

$$\left\{\left[\left[T_1,T\right]\ni t\mapsto u(t;a,u_0,R)\right]:a\in Y,u_0\in E,R\in\mathcal{R}\right\}$$

is precompact in $C([T_1, T], L_p(D))$.

Proof. We will use the notation $I_i(t; a, u_0, R)$, i = 0, 1 where

$$I_{0}(t) := U_{\tilde{a}}(t,0)u_{0}(0),$$

$$I_{1}(t) := \int_{0}^{t} U_{\tilde{a}}(t,\zeta)C_{a}^{1}(\zeta)u(\zeta - R(\zeta)) \,\mathrm{d}\zeta,$$
(3.13)

taking account of the parameter a and the initial value u_0 . The precompactness of the set

$$\left\{\left[\left[T_1,T\right]\ni t\mapsto I_0(t;a,u_0,R)\right]:a\in Y, u_0\in E, R\in\mathcal{R}\right\}$$

in $C([T_1, T], L_p(D))$ is a consequence of Proposition 2.9. In order to prove the precompactness in $C([T_1, T], L_p(D))$ of

$$\left\{ \left[\left[T_1, T \right] \ni t \mapsto I_1(t; a, u_0, R) \right] : a \in Y, u_0 \in E, R \in \mathcal{R} \right\},\$$

it suffices to use results from Lemma 3.3 and Lemma 3.7.

Theorem 3.2 leads to the following conclusion about precompactness of the solutions up to zero. Since under additional assumption (DA5) for a fixed $u_0 \in C([-1,0], L_p(D))$ the set

$$\left\{ \left[[0,T] \ni t \mapsto U_{\tilde{a}}(t,0)u_0(0) \right] : a \in Y \right\}$$

is simply a singleton, this observation combined with Lemma 3.7 leads to the following result.

Theorem 3.3. Assume additionally (DA5). For any $1 and any <math>u_0 \in C([-1,0], L_p(D))$ the set

$$\left\{\left[\left[0,T\right]\ni t\mapsto u(t;a,u_0,R)\right]:a\in Y,R\in\mathcal{R}\right\}$$

is precompact in $C([0,T], L_p(D))$.

4 Continuous Dependence on Initial Conditions

In the present section we assume (DA1), (DA2) and that Y as in Definition 1.1 is such that its flattening Y_0 satisfies (DA3). Further, 1 .

Definition 4.1. For $t \in [0,T]$, $a \in Y$, $u_0 \in C([-1,0], L_p(D))$ and $R \in \mathcal{R}$ we define

$$\begin{split} \delta(t; a, u_0, R) &= \sup_{\vartheta \in [-1, 0]} \| u(t + \vartheta; a, u_0, R) \|_{L_p(D)} \\ &= \| u(t + \cdot; a, u_0, R) \upharpoonright_{[-1, 0]} \|_{C([-1, 0], L_p(D))}. \end{split}$$

For notational simplicity we often write $u(t + \cdot)$ instead of $u(t + \cdot; a, u_0, R)$ and $\delta(t)$ instead of $\delta(t; a, u_0, R)$ when $a \in Y$ and $u_0 \in C([-1, 0], L_p(D))$ are fixed and this does not lead to confusion.

Lemma 4.1. For any $a \in Y$, $u_0 \in C([-1,0], L_p(D))$ and $R \in \mathcal{R}$ the function $\delta(\cdot; a, u_0, R) \colon [0,T] \to \mathbb{R}^+$ is continuous.

Proof. First note that the mapping

$$\left[[0,T] \times [-1,0] \ni (t,\vartheta) \mapsto \| u(t+\vartheta;a,u_0,R) \|_{L_n(D)} \in \mathbb{R}^+ \right]$$

is continuous as a composition of continuous mappings. Due to the compactness of [-1,0] the $\delta(\cdot, a, u_0, R)$ function is continuous when $a \in Y, R \in \mathcal{R}$ and $u_0 \in C([-1,0], L_p(D))$ are fixed.

Lemma 4.2. There are constants M_1, M_2 such that for any $\rho \in [0,T]$ the inequality

$$\|u(\rho; a, u_0, R)\|_{L_p(D)} \le M_1 \delta(0; a, u_0, R) + M_2 \int_0^\rho \delta(\zeta; a, u_0, R) \,\mathrm{d}\zeta$$

holds for all $a \in Y$, $u_0 \in C([-1,0], L_p(D))$ and $R \in \mathcal{R}$.

Proof. Fix $\rho \in [0, T]$ and note that

$$\begin{split} \|u(\rho)\|_{L_{p}(D)} &\leq \|U_{\bar{a}}(\rho)u_{0}(0)\|_{L_{p}(D)} + \int_{0}^{\rho} \|U_{\bar{a}}(\rho,\zeta)C_{a}^{1}(\zeta)(u\circ\Phi)(\zeta)\|_{L_{p}(D)} \,\mathrm{d}\zeta \\ &\leq Me^{\gamma\rho}\|u_{0}(0)\|_{L_{p}(D)} + Me^{\gamma\rho}K \int_{0}^{\rho} \|(u\circ\Phi)(\zeta)\|_{L_{p}(D)} \,\mathrm{d}\zeta \\ &\leq Me^{\gamma\rho}\|u_{0}\|_{C([-1,0],L_{p}(D))} + Me^{\gamma\rho}K \int_{0}^{\rho} \|u(\zeta+\cdot)\uparrow_{[-1,0]}\|_{C([-1,0],L_{p}(D))} \,\mathrm{d}\zeta \\ &= M_{1}\delta(0) + M_{2} \int_{0}^{\rho} \delta(\zeta) \,\mathrm{d}\zeta, \end{split}$$

where M is a uniform bound of the operator $U_{\bar{a}}(t)$ with respect to $a \in Y$ and $0 \leq t \leq T$ (see Proposition 2.7(ii)), the constant K is a uniform bound of the operator $C_a^1(\zeta)$ with respect to $a \in Y$ and $0 \leq t \leq T$ (see Lemma 3.1). Moreover, the bounds M and K are independent on initial condition u_0 . By setting $M_1 = Me^{\gamma T}$, $M_2 = MKe^{\gamma T}$ we end the proof.

From now on, throughout this section the constants M_1 and M_2 will be defined as in Lemma 4.2.

Proposition 4.1. For any sequence $(u_{0,m})_{m=1}^{\infty} \subset C([-1,0], L_p(D))$ convergent to zero, any $t \in [0,T]$, $R \in \mathcal{R}$ and $a \in Y$ the sequence $\delta_m(t) := \delta(t; a, u_{0,m}, R)$ converges to zero.

Proof. Fix $t \in [0, T]$ and $-1 \le \vartheta \le 0$ and let us consider two cases.

• If $0 \le t + \vartheta \le T$ then from Lemma 4.2 there holds

$$\|u(t+\vartheta)\|_{L_p(D)} \le M_1\delta(0) + M_2 \int_0^{t+\vartheta} \delta(\zeta) \,\mathrm{d}\zeta$$
$$\le M_1\delta(0) + M_2 \int_0^t \delta(\zeta) \,\mathrm{d}\zeta.$$

• If $-1 \le t + \vartheta \le 0$ then the inequality

$$\|u(t+\vartheta)\|_{L_p(D)} \le M_1\delta(0) + M_2 \int_0^t \delta(\zeta) \,\mathrm{d}\zeta$$

is straightforward, as even the stronger one $||u(t + \vartheta)||_{L_p(D)} \leq M_1 \delta(0)$ is true.

Applying sup with respect to ϑ on both sides give us that

$$\sup_{\theta \in [-1,0]} \|u(t+\vartheta, a, u_0, R)\|_{L_p(D)} \le M_1 \delta(0) + M_2 \int_0^t \delta(\zeta) \,\mathrm{d}\zeta,$$

what can be rewritten in terms of the δ function as

$$\delta(t) \le M_1 \delta(0) + M_2 \int_0^t \delta(\zeta) \,\mathrm{d}\zeta.$$

The function δ is nonnegative and continuous on the compact domain, hence it is integrable. Using the Grönwall lemma we get

$$\delta(t) \le M_1 \delta(0) \exp\left(M_2 \int_0^t d\zeta\right). \tag{4.1}$$

The above Lemma 4.2 and Proposition 4.1 lead to global L_p -norm estimation of the mild solution of $(ME)_a + (BC)_a$ in terms of initial conditions.

Proposition 4.2. There is constant $\overline{M} > 0$ such that inequality

$$||u(t; a, u_0, R)||_{L_p(D)} \le \overline{M} ||u_0||_{C([-1,0], L_p(D))}$$

holds for any $1 , <math>t \in [0,T]$, $a \in Y$, $R \in \mathcal{R}$ and $u_0 \in C([-1,0], L_p(D))$.

Proof. Let $\overline{M} = M_1 \exp(M_2 T)$, where M_1, M_2 are constants as in Lemma 4.2. Then by Proposition 4.1 we can write

$$\|u(t; a, u_0, R)\|_{L_p(D)} \le \delta(t; a, u_0, R)$$

$$\le \overline{M} \|u_0\|_{C([-1,0], L_p(D))}.$$

Proposition 4.3. For any $a \in Y$ and $R \in \mathcal{R}$ the mapping

$$\left[C([-1,0], L_p(D)) \ni u_0 \mapsto u(\cdot; a, u_0, R) \in C([-1,T], L_p(D)) \right]$$

is continuous.

Proof. Let $a \in Y$, $R \in \mathcal{R}$ be fixed. Then in the spirit of Cauchy's definition we can find that

$$\begin{aligned} \|u(\cdot; a, u_{0,1}, R) - u(\cdot; a, u_{0,2}, R)\|_{C([-1,T], L_p(D))} \\ &\leq \sup_{t \in [0,T]} \delta(t; a, u_{0,1} - u_{0,2}, R) \\ &\leq M_1 \exp(M_2 T) \|u_{0,1} - u_{0,2}\|_{C([-1,0], L_p(D))} \end{aligned}$$

for any initial conditions $u_{0,1}, u_{0,2} \in C([-1,0], L_p(D))$. The first inequality results from the linearity of the problem $(ME)_a + (BC)_a$ and the second inequality follows from (4.1).

5 Continuous Dependence on Coefficients and Delay

In the present section we assume (DA1), (DA2) and that Y as in Definition 1.1 is such that its flattening Y_0 satisfies (DA3) and (DA4). As in Section 4, 1 .

Proposition 5.1. For any $0 < T_1 \leq T$, $R \in \mathcal{R}$ and $u_0 \in C([-1,0], L_p(D))$ the mapping

$$|Y \ni a \mapsto u(\cdot; a, u_0, R)|_{[T_1, T]} \in C([T_1, T], L_p(D))|$$

is continuous.

Proof. Fix p, T_1, R and u_0 as in the Proposition. Let $(a_m)_{m=1}^{\infty} \subset Y$ converge to a. Put $u_m(\cdot)$ for $u(\cdot; a_m, u_0, R)$ and $u(\cdot)$ for $u(\cdot; a, u_0, R)$. It suffices to prove that there is a subsequence $(a_{m_k})_{k=1}^{\infty} \subset Y$ such that $u_{m_k}(\cdot)$ converges to $u(\cdot)$ on $[T_1, T]$ uniformly. By Theorem 3.2 via diagonal process, we can find a subsequence u_{m_k} such that $u_{m_k} \upharpoonright_{(0,T]}$ converge to some continuous $\hat{u}: (0,T] \to L_p(D)$ and the convergence is uniform on compact subsets of (0,T]. The function \hat{u} is clearly bounded by Proposition 4.2. Moreover, we extend the map \hat{u} to the whole [-1,T] by u_0 on [-1,0], i.e., now

$$\tilde{u}(t) := \begin{cases} u_0(t) & \text{for } t \in [-1,0] \\ \lim_{k \to \infty} u_{m_k}(t) & \text{for } t \in (0,T]. \end{cases}$$

It remains to prove that $\tilde{u} = u$. In order not to overburden the notation we write u_m instead of u_{m_k} .

Our first step is to show that, for each $t \in (0, T]$,

$$U_{\tilde{a}_m}(t,0)u_0(0) \to U_{\tilde{a}}(t,0)u_0(0)$$
(5.1)

$$\int_{0}^{t} U_{\tilde{a}_{m}}(t,\zeta) C_{a_{m}}^{1}(\zeta) u_{m}(\zeta - R(\zeta)) \,\mathrm{d}\zeta \to \int_{0}^{t} U_{\tilde{a}}(t,\zeta) C_{a}^{1}(\zeta) \tilde{u}(\zeta - R(\zeta)) \,\mathrm{d}\zeta.$$
(5.2)

in the $L_p(D)$ -norm as $m \to \infty$. The convergence in (5.1) is a consequence of Proposition 2.17. The convergence in (5.2) can be shown by showing the convergence of the difference

$$\int_{0}^{t} U_{\tilde{a}_{m}}(t,\zeta) C_{a_{m}}^{1}(\zeta) u_{m}(\zeta - R(\zeta)) \, \mathrm{d}\zeta - \int_{0}^{t} U_{\tilde{a}}(t,\zeta) C_{a}^{1}(\zeta) \tilde{u}(\zeta - R(\zeta)) \, \mathrm{d}\zeta$$

$$= \int_{0}^{t} (U_{\tilde{a}_{m}}(t,\zeta) - U_{\tilde{a}}(t,\zeta)) C_{a_{m}}^{1}(\zeta) u_{m}(\zeta - R(\zeta)) \, \mathrm{d}\zeta$$

$$+ \int_{0}^{t} U_{\tilde{a}}(t,\zeta) C_{a_{m}}^{1}(\zeta) (u_{m}(\zeta - R(\zeta)) - \tilde{u}(\zeta - R(\zeta))) \, \mathrm{d}\zeta$$

$$+ \int_{0}^{t} U_{\tilde{a}}(t,\zeta) (C_{a_{m}}^{1}(\zeta) - C_{a}^{1}(\zeta)) \tilde{u}(\zeta - R(\zeta)) \, \mathrm{d}\zeta$$
(5.3)

to zero. Write $J_m^{(i)}(t)$, i = 1, 2, 3, for the *i*-th term on the right-hand side of (5.3). The convergence of $J_m^{(1)}(t)$ follows from the Lebesgue dominated convergence theorem for Bochner integral: since the integrand

$$(0,t) \ni \zeta \mapsto (U_{\tilde{a}_m}(t,\zeta) - U_{\tilde{a}}(t,\zeta))C^1_{a_m}(\zeta)u_m(\zeta - R(\zeta)) \in L_p(D)$$

is $(\mathfrak{L}((0,t)), \mathfrak{B}(L_p(D)))$ -measurable for all $m \in \mathbb{N}$ (see Lemmas 3.4(i) and 3.3) and bounded uniformly (see Proposition 4.2) in $m \in \mathbb{N}$:

$$\begin{aligned} \| (U_{\tilde{a}_m}(t,\zeta) - U_{\tilde{a}}(t,\zeta)) C_{a_m}^1(\zeta) u_m(\zeta - R(\zeta)) \|_{L_p(D)} \\ &\leq 2M e^{\gamma T} K \overline{M} \| u_0 \|_{C([-1,0],L_p(D))} \end{aligned}$$

it suffices to check that for a.e. $\zeta \in (0, t)$ the integrand converges to zero, which follows from the estimate

$$\begin{aligned} \| (U_{\tilde{a}_m}(t,\zeta) - U_{\tilde{a}}(t,\zeta)) C_{a_m}^1(\zeta) u_m(\zeta - R(\zeta)) \|_{L_p(D)} \\ & \leq \| (U_{\tilde{a}_m}(t,\zeta) - U_{\tilde{a}}(t,\zeta)) \|_{\mathcal{L}(L_p(D))} K \overline{M} \| u_0 \|_{C([-1,0],L_p(D))} \end{aligned}$$

and Proposition 2.18.

In order to prove $J_m^{(2)}(t) \to 0$ as $m \to \infty$ we proceed similarly. We see that the mapping

$$\left[(0,t) \ni \zeta \mapsto U_{\tilde{a}}(t,\zeta) C^{1}_{a_{m}}(\zeta) (u_{m}(\zeta - R(\zeta)) - \tilde{u}(\zeta - R(\zeta))) \in L_{p}(D) \right]$$

is $(\mathfrak{L}((0,t)), \mathfrak{B}(L_p(D)))$ -measurable for all $m \in \mathbb{N}$, as a consequence of Lemmas 3.4(i) and 3.3, and bounded uniformly in $m \in \mathbb{N}$, since, by Proposition 2.7(ii), Lemma 3.1 and Proposition 4.2,

$$\begin{aligned} \|U_{\tilde{a}}(t,\zeta)C_{a_{m}}^{1}(\zeta)(u_{m}(\zeta-R(\zeta))-\tilde{u}(\zeta-R(\zeta)))\|_{L_{p}(D)} \\ &\leq 2MKe^{\gamma T}\overline{M}\|u_{0}\|_{C([-1,0],L_{p}(D))}. \end{aligned}$$

Further, the convergence, for a.e. $\zeta \in (0, t)$,

$$U_{\tilde{a}}(t,\zeta)C^{1}_{a_{m}}(\zeta)(u_{m}(\zeta-R(\zeta))-\tilde{u}(\zeta-R(\zeta)))\to 0$$

in $L_p(D)$ is due to the pointwise convergence of u_m to \tilde{u} on [-1,T] and the estimate (by Proposition 2.7(ii) and Lemma 3.1)

$$\begin{aligned} \|U_{\tilde{a}}(t,\zeta)C^{1}_{a_{m}}(\zeta)(u_{m}(\zeta-R(\zeta))-\tilde{u}(\zeta-R(\zeta)))\|_{L_{p}(D)} \\ &\leq MKe^{\gamma T}\|u_{m}(\zeta-R(\zeta))-\tilde{u}(\zeta-R(\zeta))\|_{L_{p}(D)}. \end{aligned}$$

The convergence of $J_m^{(3)}(t)$ follows from the facts that the set

$$\{J_m^{(3)}(t):m\in\mathbb{N}\}\$$

is precompact in $L_p(D)$ (see Lemma 3.6, Lemma 3.3, Proposition 4.2) and that $J_m^{(3)}(t)$ converge weakly to zero, i.e.,

$$\langle J_m^{(3)}(t), v \rangle \to 0 \text{ as } m \to \infty,$$
 (5.4)

for any $v \in L_{p'}(D)$, where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $L_p(D)$ and $L_{p'}(D)$. By the Hille theorem ([16, Thm. II.2.6]) and Proposition 2.14,

$$\langle J_m^{(3)}(t), v \rangle = \int_0^t \langle U_{\tilde{a}}(t,\zeta) (C_{a_m}^1(\zeta) - C_a^1(\zeta)) \tilde{u}(\zeta - R(\zeta)), v \rangle \,\mathrm{d}\zeta$$

=
$$\int_0^t \langle (C_{a_m}^1(\zeta) - C_a^1(\zeta)) \tilde{u}(\zeta - R(\zeta)), U_{\tilde{a}}^*(t,\zeta) v \rangle \,\mathrm{d}\zeta.$$

Now we need to use a subtler approach based on the convergence $c_{1,m}$ to c_1 in the weak-* topology of $L_{\infty}((0,t) \times D)$. First note that the mappings

$$\left[(0,t) \ni \zeta \mapsto \tilde{u}(\zeta - R(\zeta)) \in L_p(D) \right] \quad \& \quad \left[(0,t) \ni \zeta \mapsto U_{\tilde{a}}^*(t,\zeta)v \in L_{p'}(D) \right]$$

belong to $L_{\infty}((0,t), L_p(D))$ and $L_{\infty}((0,t), L_{p'}(D))$ respectively. Therefore the mapping (the product of the above maps)

$$\left[(0,t) \times D \ni (\zeta, x) \mapsto \tilde{u}(\zeta - R(\zeta))[x](U_{\tilde{a}}^*(t,\zeta)v)[x] \in \mathbb{R} \right]$$

belong to $L_1((0,t) \times D)$ see Lemma 0.1(a). It suffices now to note that from Fubini's theorem we have

$$\int_{0}^{t} \langle (C_{a_{m}}^{1}(\zeta) - C_{a}^{1}(\zeta))\tilde{u}(\zeta - R(\zeta)), U_{\tilde{a}}^{*}(t,\zeta)v\rangle \,\mathrm{d}\zeta$$
$$= \int_{0}^{t} \int_{D} (c_{1,m}(\zeta, x) - c_{1}(\zeta, x)) \,\tilde{u}(\zeta - R(\zeta))[x] \, (U_{\tilde{a}}^{*}(\zeta, t)v)[x] \,\mathrm{d}x \,\mathrm{d}\zeta,$$

so the integral tends to zero as $m \to \infty$.

We have thus proved that

$$\tilde{u}(t) = U_{\tilde{a}}(t,0)u_0(0) + \int_0^t U_{\tilde{a}}(t,\zeta)C_a^1(\zeta)\tilde{u}(\zeta - R(\zeta))\,\mathrm{d}\zeta, \quad t \in [0,T].$$

Now we prove the continuity of the extension \tilde{u} . Note that the only point where it can fail is t = 0. However, this is not the case since the mappings

$$\left[[0,T] \ni t \mapsto U_{\tilde{a}}(t,0)u_0(0) \in L_p(D) \right]$$
$$\left[[0,T] \ni t \mapsto \int_0^t U_{\tilde{a}}(t,\zeta)C_a^1(\zeta)\tilde{u}(\zeta - R(\zeta)) \,\mathrm{d}\zeta \in L_p(D) \right]$$

are continuous (see Lemmas 2.3(ii) and 3.4(ii)), so the mapping \tilde{u} is continuous on the whole [-1, T]. Also, $\tilde{u} = u_0$ on [-1, 0], hence \tilde{u} is in fact the mild solution of $(ME)_{\tilde{a}} + (BC)_{\tilde{a}}$, therefore, by uniqueness, $\tilde{u}(t) = u(t; \tilde{a}, u_0, R)$ for any $t \in [-1, T]$. **Proposition 5.2.** Assume additionally (DA5). For any $u_0 \in C([-1, 0], L_p(D))$ and $\mathcal{R}_0 \subset \mathcal{R}$ satisfying the assumption (DA6) the mapping

$$\left[Y \times \mathcal{R}_0 \ni (a, R) \mapsto u(\cdot; a, u_0, R) \upharpoonright_{[0,T]} \in C([0,T], L_p(D))\right]$$

is continuous.

Sketch of proof. Fix p, u_0 and \mathcal{R}_0 . Let $(a_m)_{m=1}^{\infty} \subset Y$ converge to a and $(R_m)_{m=1}^{\infty} \subset \mathcal{R}_0$ converge to R, and put $u_m(\cdot)$ for $u(\cdot; a_m, u_0, R_m)$ and $u(\cdot)$ for $u(\cdot; a, u_0, R)$. We will proceed as in Proposition 5.1. In particular, \tilde{u} has the same meaning. However, we have in fact more: as we assume (DA5), we can apply Theorem 3.3 to show that u_m converge to \hat{u} uniformly on [0, T], from which it follows in particular that \tilde{u} is continuous on the whole of [-1, T].

It follows again from (DA5) that $\tilde{a}_m = \tilde{a}$ for all $m \in \mathbb{N}$, so

$$U_{\tilde{a}_m}(t,0)u_0(0) \to U_{\tilde{a}}(t,0)u_0(0)$$

holds trivially. We start by showing that, for each $t \in [0, T]$,

$$\int_{0}^{t} U_{\tilde{a}}(t,\zeta) C_{a_{m}}^{1}(\zeta) u_{m}(\zeta - R_{m}(\zeta)) \,\mathrm{d}\zeta \to \int_{0}^{t} U_{\tilde{a}}(t,\zeta) C_{a}^{1}(\zeta) \tilde{u}(\zeta - R(\zeta)) \,\mathrm{d}\zeta.$$
(5.5)

in the $L_p(D)$ -norm as $m \to \infty$. The above convergence can be proved by showing convergence of the terms

$$\int_{0}^{t} U_{\tilde{a}}(t,\zeta) C_{a_{m}}^{1}(\zeta) u_{m}(\zeta - R_{m}(\zeta)) \, \mathrm{d}\zeta - \int_{0}^{t} U_{\tilde{a}}(t,\zeta) C_{a}^{1}(\zeta) \tilde{u}(\zeta - R(\zeta)) \, \mathrm{d}\zeta$$

$$= \int_{0}^{t} U_{\tilde{a}}(t,\zeta) C_{a_{m}}^{1}(\zeta) (u_{m}(\zeta - R_{m}(\zeta)) - \tilde{u}(\zeta - R(\zeta))) \, \mathrm{d}\zeta \qquad (5.6)$$

$$+ \int_{0}^{t} U_{\tilde{a}}(t,\zeta) (C_{a_{m}}^{1}(\zeta) - C_{a}^{1}(\zeta)) \tilde{u}(\zeta - R(\zeta)) \, \mathrm{d}\zeta.$$

to zero. Write $K_m^{(i)}(t)$, i = 1, 2, for the *i*-th term on the right-hand side of (5.6).

Regarding the convergence of $K_m^{(1)}(t)$ to zero, we show the $(\mathfrak{L}((0,t)), \mathfrak{B}(L_p(D)))$ -measurability, for all $m \in \mathbb{N}$, of the integrand

$$\left[(0,t) \ni \zeta \mapsto U_{\tilde{a}}(t,\zeta) C^{1}_{a_{m}}(\zeta) (u_{m}(\zeta - R_{m}(\zeta)) - \tilde{u}(\zeta - R(\zeta))) \in L_{p}(D) \right]$$

in the same way as in the proof of the convergence of $J_m^{(2)}(t)$ in Proposition 5.1. The fact that for a.e. $\zeta \in (0,T)$ we have that

$$U_{\tilde{a}}(t,\zeta)C^{1}_{a_{m}}(\zeta)(u_{m}(\zeta-R_{m}(\zeta))-\tilde{u}(\zeta-R(\zeta)))\to 0$$

in $L_p(D)$ is due, in view of (DA6), to the uniform convergence of u_m to \tilde{u} on [-1,T] together with the estimate

$$\begin{aligned} \|U_{\tilde{a}}(t,\zeta)C^{1}_{a_{m}}(\zeta)(u_{m}(\zeta-R_{m}(\zeta))-\tilde{u}(\zeta-R(\zeta)))\|_{L_{p}(D)} \\ &\leq MKe^{\gamma T}\|u_{m}(\zeta-R_{m}(\zeta))-\tilde{u}(\zeta-R(\zeta))\|_{L_{p}(D)}. \end{aligned}$$

The proof of the convergence of $K_m^{(2)}(t)$ to zero is just a copy, word for word, of the proof of the convergence of $J_m^{(3)}(t)$ in Proposition 5.1.

Theorem 5.1.

(i) For any $0 < T_1 \leq T$ and $R \in \mathcal{R}$ the mapping

$$\left[Y \times C([-1,0], L_p(D)) \ni (a, u_0) \\ \mapsto u(\cdot; a, u_0, R) \upharpoonright_{[T_1,T]} \in C([T_1,T], L_p(D))\right]$$

 $is \ continuous.$

(ii) Under (DA5), if $\mathcal{R}_0 \subset \mathcal{R}$ is a subset such that the assumption (DA6) holds then the mapping

$$\begin{bmatrix} Y \times C([-1,0], L_p(D)) \times \mathcal{R}_0 \ni (a, u_0, R) \\ \mapsto u(\cdot; a, u_0, R) \upharpoonright_{[0,T]} \in C([0,T], L_p(D)) \end{bmatrix}$$

is continuous.

Proof. Fix $1 . We start by proving (i), so fix also <math>T_1$, R as in the statement. Let a sequence $(a_m)_{m=1}^{\infty} \subset Y$ converge to $a \in Y$ and $(u_{0,m})_{m=1}^{\infty} \subset C([-1,0], L_p(D))$ converge to $u_0 \in C([-1,0], L_p(D))$. The main idea of the proof is based on the estimation

$$\begin{aligned} &|u(\cdot; a_m, u_{0,m}, R)|_{[T_1,T]} - u(\cdot; a, u_0, R)|_{[T_1,T]} \|_{C([T_1,T], L_p(D))} \\ &\leq \|u(\cdot; a_m, u_{0,m}, R)|_{[T_1,T]} - u(\cdot; a_m, u_0, R)|_{[T_1,T]} \|_{C([T_1,T], L_p(D))} \\ &\quad + \|u(\cdot; a_m, u_0, R)|_{[T_1,T]} - u(\cdot; a, u_0, R)|_{[T_1,T]} \|_{C([T_1,T], L_p(D))}. \end{aligned}$$
(5.7)

Proposition 4.2 implies

$$\begin{aligned} \|u(\cdot;a_m,u_{0,m},R)|_{[T_1,T]} - u(\cdot;a_m,u_0,R)|_{[T_1,T]}\|_{C([T_1,T],L_p(D))} \\ &\leq \overline{M}\|u_{0,m} - u_0\|_{C([-1,0],L_p(D))}. \end{aligned}$$

Therefore the first part of the right-hand side of (5.7) converges to zero as $m \to \infty$. The second part of (5.7) converges to zero by Proposition 5.1. Item (ii) can be proved similarly. So, assume additionally (DA5) and, instead of fixing

the delay $R \in \mathcal{R}$ take a sequence $(R_m)_{m=1}^{\infty} \subset \mathcal{R}_0$ convergent to some $R \in \mathcal{R}_0$. By similar estimation,

$$\begin{aligned} \|u(\cdot; a_m, u_{0,m}, R_m)|_{[0,T]} &- u(\cdot; a, u_0, R)|_{[0,T]} \|_{C([0,T], L_p(D))} \\ &\leq \|u(\cdot; a_m, u_{0,m}, R_m)|_{[0,T]} - u(\cdot; a_m, u_0, R_m)|_{[0,T]} \|_{C([0,T], L_p(D))} \\ &+ \|u(\cdot; a_m, u_0, R_m)|_{[0,T]} - u(\cdot; a, u_0, R)|_{[0,T]} \|_{C([0,T], L_p(D))} \end{aligned}$$
(5.8)

together with Propositions 4.2 and 5.2 concludes the proof. \Box

References

- [1] C. D. Aliprantis and K. C. Border, "Infinite Dimensional Analysis. A Hitchhiker's Guide," third edition, Springer, Berlin, 2006. MR 2008m:46001
- [2] H. Amann and J. Escher, "Analysis. III," translated from the 2001 German original by S. Levy and M. Cargo, Birkhäuser, Basel, 2009. MR 2010d:00001
- [3] W. Arendt and A. V. Bukhvalov, Integral representations of resolvents and semigroups, Forum Math. 6 (1994), no. 1, 111–135.
- [4] L. Barreira, D. Dragičević and C. Valls, Nonuniform spectrum on Banach spaces, Adv. Math. 321 (2017), 547–591.
- [5] L. Barreira and C. Valls, "Hyperbolicity in Delay Equations," Ser. Appl. Comput. Math., 4, World Scientific, Hackensack, NJ, 2021.
- [6] A. Bátkai and S. Piazzera, Semigroups and linear partial differential equations with delay, J. Math. Anal. Appl. 264 (2001), no. 1, 1–20.
- [7] A. Bátkai and S. Piazzera, A semigroup method for delay equations with relatively bounded operators in the delay term, Semigroup Forum 64 (2002), no. 1, 71–89.
- [8] A. Bátkai and S. Piazzera, "Semigroups for Delay Equations," Res. Notes Math., 10, A K Peters, Wellesley, MA, 2005.
- [9] A. Blumenthal, A volume-based approach to the multiplicative ergodic theorem on Banach spaces, Discrete Contin. Dyn. Syst. 36 (2016), no. 5, 2377– 2403.
- [10] C. Chicone and Y. Latushkin, "Evolution Semigroups in Dynamical Systems and Differential Equations," Math. Surveys Monogr., 70, American Mathematical Society, Providence, RI, 1999.
- [11] S.-N. Chow and H. Leiva, Existence and roughness of the exponential dichotomy for skew-product semiflow in Banach spaces, J. Differential Equations 120 (1995), no. 2, 429–477.

- [12] S.-N. Chow and H. Leiva, Two definitions of exponential dichotomy for skew-product semiflow in Banach spaces, Proc. Amer. Math. Soc. 124 (1996), no. 4, 1071–1081.
- [13] D. Daners, Heat kernel estimates for operators with boundary conditions, Math. Nachr. 217 (2000), 13–41.
- [14] Z. Denkowski, S. Migórski and N. S. Papageorgiou, "An Introduction to Nonlinear Analysis: Theory," Kluwer, Boston, MA, 2003.
- [15] R. Dautray and J.-L. Lions, "Mathematical Analysis and Numerical Methods for Science and Technology. Vol. 5: Evolution Problems. I," with the collaboration of M. Artola, M. Cessenat and H. Lanchon, translated from the French by A. Craig, Springer, Berlin, 1992.
- [16] J. Diestel and J. J. Uhl, Jr., "Vector Measures," with a foreword by B. J. Pettis, Mathematical Surveys, No. 15, American Mathematical Society, Providence, R.I., 1977.
- [17] T. S. Doan, Lyapunov exponents for random dynamical systems, Ph. D. dissertation, Technische Universität Dresden, 2009.
- [18] T. S. Doan and S. Siegmund, Differential equations with random delay, in: Infinite Dimensional Dynamical Systems, Fields Inst. Commun., 64, Springer, New York, 2013, 279–303.
- [19] N. Dunford and J. T. Schwartz, "Linear Operators. I. General Theory," with the assistance of W. G. Bade and R. G. Bartle, Pure and Applied Mathematics, Vol. 7, Interscience, New York and London, 1958.
- [20] W. E. Fitzgibbon, Stability for abstract nonlinear Volterra equations involving finite delay, J. Math. Anal. Appl. 60 (1977), no. 2, 429–434.
- [21] W. E. Fitzgibbon, Semilinear functional differential equations in Banach space, J. Differential Equations 29 (1978), no. 1, 1–14.
- [22] G. B. Folland, "Real Analysis. Modern Techniques and Their Applications," second edition, Pure and Applied Mathematics, Wiley, 1984.
- [23] G. Fragnelli, A spectral mapping theorem for semigroups solving PDEs with nonautonomous past, Abstr. Appl. Anal. 2003, no. 16, 933–951.
- [24] G. Fragnelli and G. Nickel, Partial functional differential equations with nonautonomous past in L^p-phase spaces, Differential Integral Equations 16 (2003), no. 3, 327–348.
- [25] G. Froyland, S. Lloyd, and A. Quas, A semi-invertible Oseledets theorem with applications to transfer operator cocycles, Discrete Contin. Dyn. Syst. 33 (2013), no. 9, 3835–3860.

- [26] C. González-Tokman and A. Quas, A semi-invertible operator Oseledets theorem, Ergodic Theory Dynam. Systems 34 (2014), no. 4, 1230–1272.
- [27] C. González-Tokman and A. Quas, A concise proof of the multiplicative ergodic theorem on Banach spaces, J. Mod. Dyn. 9 (2015), 237–255.
- [28] J. K. Hale and S. M. Verduyn Lunel, "Introduction to Functional Differential Equations," Appl. Math. Sci., 99, Springer, New York, 1993.
- [29] O. A. Ladyzhenskaya [O. A. Ladyženskaja], V. A. Solonnikov and N. N. Ural'tseva [N. N. Ural'ceva], "Linear and Quasilinear Equations of Parabolic Type," translated from the Russian by S. Smith, Transl. Math. Monogr., Vol. 23, American Mathematical Society, Providence, RI, 1967.
- [30] Z. Lian and K. Lu, "Lyapunov Exponents and Invariant Manifolds for Random Dynamical Systems in a Banach Space," Mem. Amer. Math. Soc. 206 (2010), no. 967.
- [31] R. H. Martin, Jr., and H. L. Smith, Abstract functional-differential equations and reaction-diffusion systems, Trans. Amer. Math. Soc. 321 (1990), no. 1, 1–44.
- [32] R. H. Martin, Jr., and H. L. Smith, Reaction-diffusion systems with time delays: monotonicity, invariance, comparison and convergence, J. Reine Angew. Math. 413 (1991), 1–35.
- [33] J. Mierczyński, S. Novo and R. Obaya, Principal Floquet subspaces and exponential separations of type II with applications to random delay differential equations., Discrete Contin. Dyn. Syst. 38 (2018), no. 12, 6163–6193.
- [34] J. Mierczyński, S. Novo and R. Obaya, Lyapunov exponents and Oseledets decomposition in random dynamical systems generated by systems of delay differential equations, Commun. Pure Appl. Anal. 19 (2020), no. 4, 2235– 2255
- [35] J. Mierczyński and W. Shen, "Spectral Theory for Random and Nonautonomous Parabolic Equations and Applications," Chapman & Hall/CRC Monogr. Surv. Pure Appl. Math., Chapman & Hall/CRC, Boca Raton, FL, 2008.
- [36] J. Mierczyński and W. Shen, Principal Lyapunov exponents and principal Floquet spaces of positive random dynamical systems. III. Parabolic equations and delay systems, J. Dynam. Differential Equations 28 (2016), no. 3–4, 1039– 1079.
- [37] S. Novo, C. Núñez, R. Obaya and A. M. Sanz, Skew-product semiflows for non-autonomous partial functional differential equations with delay, Discrete Contin. Dyn. Syst. 34 (2014), no. 10, 4291–4321.

- [38] R. Obaya and A. M. Sanz, Persistence in non-autonomous quasimonotone parabolic partial functional differential equations with delay, Discrete Contin. Dyn. Syst. Ser. B 24 (2019), no. 8, 3947–3970.
- [39] R. J. Sacker, A new metric yielding a richer class of unbounded functions having compact hulls in the shift flow, J. Dynam. Differential Equations 33 (2021), no. 2, 833–848.
- [40] R. J. Sacker and G. R. Sell, Dichotomies for linear evolutionary equations in Banach spaces, J. Differential Equations 113 (1994), no. 1, 17–67.
- [41] W. Shen and Y. Yi, "Almost Automorphic and Almost Periodic Dynamics in Skew-product Semiflows," Mem. Amer. Math. Soc. 136 (1998), no. 647.
- [42] C. C. Travis and G. F. Webb, Existence and stability for partial functional differential equations, Trans. Amer. Math. Soc. 200 (1974), 395–418.
- [43] C. C. Travis and G. F. Webb, Partial differential equations with deviating arguments in the time variable, J. Math. Anal. Appl. 56 (1976), no. 2, 397– 409.
- [44] C. C. Travis and G. F. Webb, Existence, stability, and compactness in the α -norm for partial functional differential equations, Trans. Amer. Math. Soc. **240** (1978), 129–143.
- [45] V. S. Varadarajan, On the convergence of sample probability distributions, Sankhyā 19 (1958), no. 1–2, 23–26. MR 20 #1348
- [46] J. Wu, "Theory and Applications of Partial Functional-Differential Equations," Appl. Math. Sci., 119, Springer, New York, 1996.