Generations of random hypergraphs and random simplicial complexes by the map algebra

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Abstract

We consider the random hypergraph on a finite vertex set by choosing each set of vertices as an hyperedge independently at random. We express the probability distributions of the (lower-)associated simplicial complex and the (lower-)associated independence hypergraph of the random hypergraph in terms of the probability distributions of certain random simplicial complex and certain random independence hypergraph of Erdös-Rényi type. We construct a graded structure of the map algebra explicitly and give algorithms to generate random hypergraphs and random simplicial complexes.

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1 Introduction

Let V be a finite vertex set. Let $\Delta[V]$ be the collection of all the nonempty subsets of V. Let $p : \Delta[V] \longrightarrow [0,1]$ be a function on $\Delta[V]$ with values in the unit interval. A hypergraph on V is a subset of $\Delta[V]$. A hypergraph is an (abstract) simplicial complex if any nonempty subset of any hyperedge is still a hyperedge in the hypergraph. For any simplicial complex $\mathcal{K}$ on V, an external face of $\mathcal{K}$ is a nonempty subset σ of V such that $\sigma \notin \mathcal{K}$ and $\tau \in \mathcal{K}$ for any proper subset τ of σ. Let $E(\mathcal{K})$ be the set of all the external faces of $\mathcal{K}$. We call a hypergraph an independence hypergraph if any superset, which is a subset of V, of any hyperedge is still a hyperedge in the hypergraph. The complement of a simplicial complex in $\Delta[V]$ is an independence hypergraph and vice versa. For any independence hypergraph $\mathcal{L}$ on V, a co-external face of $\mathcal{L}$ is a subset σ of V such that $\sigma \notin \mathcal{L}$ and $\tau \in \mathcal{L}$ for any proper superset τ of σ such that τ a subset of V. Let $\bar{E}(\mathcal{L})$ be the set of all the co-external faces of $\mathcal{L}$.

Let $\mathcal{H}$ be a hypergraph on V. The associated simplicial complex $\Delta\mathcal{H}$ of $\mathcal{H}$ is the smallest simplicial complex containing $\mathcal{H}$ (cf. [7, 9, 37]). The lower-associated simplicial complex $\delta\mathcal{H}$ of $\mathcal{H}$ is the largest simplicial complex contained in $\mathcal{H}$ (cf. [39]). We define the associated independence hypergraph $\bar{\Delta}\mathcal{H}$ of $\mathcal{H}$ as the smallest independence hypergraph containing $\mathcal{H}$ and define the lower-associated independence hypergraph $\bar{\delta}\mathcal{H}$ of $\mathcal{H}$ as the largest independence hypergraph contained in $\mathcal{H}$. Consider the random hypergraph whose probability is given by

$$\bar{\mathrm{P}}_p(\mathcal{H}) = \prod_{\sigma\in\mathcal{H}} p(\sigma) \prod_{\sigma\notin\mathcal{H}} \big(1-p(\sigma)\big),$$

where $\mathcal{H}$ is any hypergraph on V, the random simplicial complex whose probability is given by

$$\mathrm{P}_p(\mathcal{K}) = \prod_{\sigma\in\mathcal{K}} p(\sigma) \prod_{\sigma\in E(\mathcal{K})} \big(1-p(\sigma)\big),$$

where $\mathcal{K}$ is any simplicial complex on V, and the random independence hypergraph whose probability is given by

$$\mathrm{Q}_p(\mathcal{L}) = \prod_{\sigma\in\mathcal{L}} p(\sigma) \prod_{\sigma\in\bar{E}(\mathcal{L})} \big(1-p(\sigma)\big),$$

where $\mathcal{L}$ is any independence hypergraph on V. Note that $\bar{\mathrm{P}}_p$, P_p and Q_p are well-defined probability functions, i.e.

$$\sum_{\mathcal{H}} \bar{\mathrm{P}}_p(\mathcal{H}) = \sum_{\mathcal{K}} \mathrm{P}_p(\mathcal{K}) = \sum_{\mathcal{L}} \mathrm{Q}_p(\mathcal{L}) = 1.$$

The summations are over all the hypergraphs on V, all the simplicial complexes on V and all the independence hypergraphs on V respectively.

Theorem 1.1. *Let $\mathcal{H} \sim \bar{\mathrm{P}}_p$ be a randomly generated hypergraph on V. Then*

(1). the complement of $\mathcal{H}$ is a randomly generated hypergraph $\gamma\mathcal{H} \sim \bar{\mathrm{P}}_{1-p}$,

(2). the complement of the associated simplicial complex of $\mathcal{H}$ is a randomly generated independence hypergraph $\gamma\Delta\mathcal{H} \sim \mathrm{Q}_{1-p}$,

(3). the complement of the associated independence hypergraph of $\mathcal{H}$ is a randomly generated simplicial complex $\gamma\bar{\Delta}\mathcal{H} \sim \mathrm{P}_{1-p}$,

(4). the lower-associated simplicial complex of $\mathcal{H}$ is a randomly generated simplicial complex $\delta\mathcal{H} \sim \mathrm{P}_p$,

(5). the lower-associated independence hypergraph of $\mathcal{H}$ is a randomly generated independence hypergraph $\bar{\delta}\mathcal{H} \sim \mathrm{Q}_p$.

We will prove Theorem 1.1 in Section 3. In fact, Theorem 1.1 (1) follows from Lemma 3.2, Theorem 1.1 (2), (3) follow from Corollary 3.6 (1), (2) respectively, and Theorem 1.1 (4), (5) follow from Theorem 3.5 (3), (4) respectively. As by-products, we have the following two corollaries.

Corollary 1.2. *Let V' and V'' be two disjoint vertex sets. Let $p' : \Delta[V'] \longrightarrow [0,1]$ and let $p'' : \Delta[V''] \longrightarrow [0,1]$. Define $p' * p'' : \Delta[V' \sqcup V''] \longrightarrow [0,1]$ by letting $(p' * p'')(\sigma) = p'(\sigma\cap V')p''(\sigma\cap V'')$ for any $\sigma \in \Delta[V' \sqcup V'']$.*

(1). Let $\mathcal{H}' \sim \bar{\mathrm{P}}_{p'}$ and $\mathcal{H}'' \sim \bar{\mathrm{P}}_{p''}$ be randomly generated hypergraphs. Then their join is a randomly generated hypergraph $\mathcal{H}' * \mathcal{H}'' \sim \bar{\mathrm{P}}_{p'*p''}$;

(2). Let $\mathcal{K}' \sim \mathrm{P}_{p'}$ and $\mathcal{K}'' \sim \mathrm{P}_{p''}$ be randomly generated simplicial complexes. Then their join is a randomly generated simplicial complex $\mathcal{K}' * \mathcal{K}'' \sim \mathrm{P}_{p'*p''}$;

(3). Let $\mathcal{L}' \sim \mathrm{Q}_{p'}$ and $\mathcal{L}'' \sim \mathrm{Q}_{p''}$ be randomly generated independence hypergraphs. Then their join is a randomly generated independence hypergraph $\mathcal{L}' * \mathcal{L}'' \sim \mathrm{Q}_{p' * p''}$.

Corollary 1.2 (1) follows from Lemma 3.4 and Corollary 1.2 (2), (3) follow from Corollary 3.8 (1), (2) respectively.

Corollary 1.3. *Let $p', p'' : \Delta[V] \longrightarrow [0,1]$. Define $p' \cap p'', p' \cup p'' : \Delta[V] \longrightarrow [0,1]$ by letting $(p' \cap p'')(\sigma) = p'(\sigma)p''(\sigma)$ and $(p' \cup p'')(\sigma) = 1 - (1 - p'(\sigma))(1 - p''(\sigma))$ for any $\sigma \in \Delta[V]$.*

(1). Let $\mathcal{H}' \sim \bar{\mathrm{P}}_{p'}$ and let $\mathcal{H}'' \sim \bar{\mathrm{P}}_{p''}$. Then their intersection is a randomly generated hypergraph $\mathcal{H}' \cap \mathcal{H}'' \sim \bar{\mathrm{P}}_{p' \cap p''}$ and their union is a randomly generated hypergraph $\mathcal{H}' \cup \mathcal{H}'' \sim \bar{\mathrm{P}}_{p' \cup p''}$;

(2). Let $\mathcal{K}' \sim \mathrm{P}_{p'}$ and let $\mathcal{K}'' \sim \mathrm{P}_{p''}$. Then their intersection is a randomly generated simplicial complex $\mathcal{K}' \cap \mathcal{K}'' \sim \mathrm{P}_{p' \cap p''}$;

(3). Let $\mathcal{L}' \sim \mathrm{Q}_{p'}$ and let $\mathcal{L}'' \sim \mathrm{Q}_{p''}$. Then their intersection is a randomly generated independence hypergraph $\mathcal{L}' \cap \mathcal{L}'' \sim \mathrm{Q}_{p' \cap p''}$.

Corollary 1.3 (1) follows from [38, Lemma 4.5] (Lemma 3.3) and Corollary 1.3 (2), (3) follow from Corollary 3.7 (1), (2) respectively.

The generation of random hypergraphs as well as random simplicial complexes is useful in computer science (for example, [2]). The map algebra on the space of random sub-hypergraphs of a fixed simplicial complex was initially studied by C. Wu, J. Wu and the present author [38]. In Section 4 of this paper, we consider the map algebra on the space of random sub-hypergraphs of the complete complex $\Delta[V]$. By considering the compositions of Δ, $\bar{\Delta}$, δ, $\bar{\delta}$, γ, the intersections, the unions, the joins and the products, we construct a graded structure of the map algebra in (4.1). We consider the action of the map algebra on the random hypergraph $\mathcal{H} \sim \bar{\mathrm{P}}_p$, the random simplicial complex $\mathcal{K} \sim \mathrm{P}_p$ and the random independence hypergraph $\mathcal{L} \sim \mathrm{Q}_p$ in (4.2). Using the graded structure in (4.2), we give some algorithms generating random hypergraphs, random simplicial complexes and random independence hypergraphs. Each element in (4.2) will randomly generate a hypergraph and each element of certain particular forms will generate a random simplicial complex or a random independence hypergraph.

1.1 Literature review on random simplicial complexes

Let V be a vertex set of cardinality N. Denote $\Delta[V]$ as Δ_N. Let $\mathrm{sk}_r(\Delta_N)$ be the r-skeleton of Δ_N, $0 \leq r \leq N-1$. Consider the space Ω_N^r consisting of all subcomplexes $\mathcal{K}$ of $\mathrm{sk}_r(\Delta_N)$. Consider the probability function $\mathrm{P}_{r,N,\mathbf{p}} : \Omega_N^r \longrightarrow \mathbb{R}$ given by

$$\mathrm{P}_{r,N,\mathbf{p}}(\mathcal{K}) = \prod_{\sigma \in \mathcal{K}} p(\sigma) \prod_{\sigma \in E(\mathcal{K})} \big(1 - p(\sigma)\big),$$

where $\mathbf{p} = (p_0, p_1, \ldots, p_r)$, $0 \leq p_0, p_1, \ldots, p_r \leq 1$ are constants, and $p(\sigma) = p_i$ for any i-simplex $\sigma \in \mathrm{sk}_r(\Delta_N)$, $0 \leq i \leq r$. This model $\mathrm{P}_{r,N,\mathbf{p}}$ was given by A. Costa and M. Farber in [15, 16, 19]. The connectivity, the fundamental group, the dimension, and the Betti number of the random simplicial complex with the probability $\mathrm{P}_{r,N,\mathbf{p}}$ have been studied by A. Costa and M. Farber in [15–19]. In particular, let $0 \leq p \leq 1$ be a constant. The followings are special cases of $\mathrm{P}_{r,N,\mathbf{p}}$:

(1). $\mathrm{P}_{1,N,\mathbf{p}}$ with $\mathbf{p} = (1, p)$ is the Erdös-Rényi model $G(N, p)$. P. Erdös and A. Rényi [20] and E.N. Gilbert [23] constructed the Erdös-Rényi model $G(N, p)$ by choosing each pair of vertices in V as an edge independently at random with probability p. Thresholds for the connectivity of $G(N, p)$ were proved by P. Erdös and A. Rényi in [21].

(2). $\mathrm{P}_{2,N,\mathbf{p}}$ with $\mathbf{p} = (1, 1, p)$ is the Linial-Meshulam model $Y_2(N, p)$. N. Linial and R. Meshulam [33] constructed a random 2-complex $Y_2(N, p)$ by taking the complete graph on V as the 1-skeleton and then choosing each 2-simplex independently at random with probability p. The fundamental group, the homology groups, and the asphericity as well as the hyperbolicity of $Y_2(N, p)$ were respectively studied in [5], [10, 11], and [12, 13].

(3). $\mathrm{P}_{d,N,\mathbf{p}}$ with $\mathbf{p} = (1, \ldots, 1, p)$ is the Meshulam-Wallach model $Y_d(N, p)$. R. Meshulam and N. Wallach [35] constructed a random d-complex $Y_d(N, p)$ by taking the $(d-1)$-skeleton of the complete complex on V and then choosing each d-simplex independently at random with probability p. The (co)homology groups, the phase transition of the homology groups, the eigenvalues of the Laplacian, the collapsibility and the topological minor were respectively studied in [4, 27, 31, 32], [34], [24], [3, 4] and [25].

(4). $\mathrm{P}_{N-1,N,\mathbf{p}}$ with $\mathbf{p} = (1, p, 1, \ldots, 1)$ is the random flag complex (random clique complex) $X(N, p)$ of $G(N, p)$. This random simplicial complex was studied by M. Kahle in [28, 30] and A. Costa, M. Farber and D. Horak in [14]. Sharp vanishing thresholds for the cohomology of $X(N, p)$ were proved by M. Kahle in [30].

(5). $\mathrm{P}_{2,N,\mathbf{p}}$ with $\mathbf{p} = (p_0, p_1, p_2)$ was considered by M. Farber and T. Nowik in [22]. The multi-parameter threshold for the property that every 2-dimensional simplicial complex admits a topological embedding into the $\mathrm{P}_{2,N,\mathbf{p}}$-generated random 2-complex asymptotically almost surely was established.

2 Simplicial models for hypergraphs

2.1 Hypergraphs and simplicial complexes

Let V be a finite set with a total order $\prec$. The elements of V are called *vertices*. Let 2^V be the power set of V. Let $\Delta[V] = 2^V \setminus \{\emptyset\}$. For any nonnegative integer n and any distinct vertices $v_0, v_1, \ldots, v_n \in V$ such that $v_0 \prec v_1 \prec \cdots \prec v_n$, we call the set $\sigma = \{v_0, v_1, \ldots, v_n\}$ an *n-hyperedge* on V. Let σ be an n-hyperedge on V. The *subset closure* $\Delta\sigma$ of σ is an n-dimensional simplicial complex whose set of simplices consists of all the non-empty subsets of σ. The *superset closure* $\bar{\Delta}\sigma$ of σ is $(\Delta[V] \setminus \Delta\sigma) \cup \sigma$, in other words, $\bar{\Delta}\sigma$ consists of all the supersets $\tau \supseteq \sigma$ such that $\tau \subseteq V$. Note that $\Delta\sigma$ does not depend on the choice of V while $\bar{\Delta}\sigma$ depends on the choice of V.

Definition 2.1. [6, 9, 37] A *hypergraph* $\mathcal{H}$ on V is a collection of hyperedges on V. In particular, (1). $\mathcal{H}$ is an *(abstract) simplicial complex* if $\tau \in \mathcal{H}$ for any $\sigma \in \mathcal{H}$ and any nonempty subset $\tau \subseteq \sigma$. A hyperedge of a simplicial complex is called a *simplex* (cf. [26, 36]);

(2). $\mathcal{H}$ is called an *independence hypergraph* if $\tau \in \mathcal{H}$ for any $\sigma \in \mathcal{H}$ and any superset $\tau \supseteq \sigma$ with $\tau \subseteq V$.

Let $\mathcal{H}$ be a hypergraph on V.

Definition 2.2. The *complement* of $\mathcal{H}$ is a hypergraph on V given by

$$\gamma\mathcal{H} = \{\sigma \in \Delta[V] \mid \sigma \notin \mathcal{H}\}.$$

By Definition 2.2, $\gamma\mathcal{H}$ is a simplicial complex (resp. an independence hypergraph) iff $\mathcal{H}$ is an independence hypergraph (resp. a simplicial complex). Note that $\gamma^2 = \mathrm{id}$.

Definition 2.3. The *associated simplicial complex* (cf. [7, 9, 37]) is

$$\Delta\mathcal{H} = \bigcup_{\sigma \in \mathcal{H}} \Delta\sigma,$$

which is the smallest simplicial complex containing $\mathcal{H}$. The *lower-associated simplicial complex* is (cf. [39])

$$\delta\mathcal{H} = \bigcup_{\Delta\sigma \subseteq \mathcal{H}} \{\sigma\},$$

which is the largest simplicial complex contained in $\mathcal{H}$. The *associated independence hypergraph* is

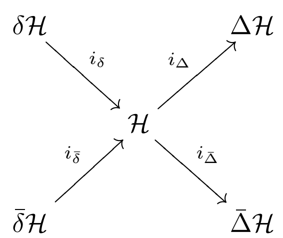
$$\bar{\Delta}\mathcal{H} = \bigcup_{\sigma \in \mathcal{H}} \bar{\Delta}\sigma,$$

which is the smallest independence hypergraph containing $\mathcal{H}$. The *lower-associated independence hypergraph* is

$$\bar{\delta}\mathcal{H} = \bigcup_{\bar{\Delta}\sigma \subseteq \mathcal{H}} \{\sigma\},$$

which is the largest independence hypergraph contained in $\mathcal{H}$.

By Definition 2.3, we have a diagram



such that each arrow is a canonical inclusion of hypergraphs.

Lemma 2.1. *$\bar{\Delta} = \gamma\delta\gamma$ and $\bar{\delta} = \gamma\Delta\gamma$.*

Proof. For any hypergraph $\mathcal{H}$ on V, we have

$$\begin{aligned}\gamma\bar{\Delta}\gamma\mathcal{H} &= \{\sigma\in\Delta[V]\mid \sigma\notin\bar{\Delta}\gamma\mathcal{H}\}\\ &= \{\sigma\in\Delta[V]\mid \sigma\not\supseteq\tau \text{ for any } \tau\in\gamma\mathcal{H}\}\\ &= \{\sigma\in\Delta[V]\mid \sigma\not\supseteq\tau \text{ for any } \tau\notin\mathcal{H},\ \text{where } \tau\in\Delta[V]\}\\ &= \{\sigma\in\Delta[V]\mid \tau\in\mathcal{H} \text{ for any } \tau\subseteq\sigma,\ \text{where } \tau\in\Delta[V]\}\\ &= \delta\mathcal{H}.\end{aligned}$$

Therefore, $\bar{\Delta}=\gamma\delta\gamma$. Moreover, For any hypergraph $\mathcal{H}$ on V, we have

$$\begin{aligned}\gamma\Delta\gamma\mathcal{H} &= \{\sigma\in\Delta[V]\mid \sigma\notin\Delta\gamma\mathcal{H}\}\\ &= \{\sigma\in\Delta[V]\mid \sigma\not\subseteq\tau \text{ for any } \tau\in\gamma\mathcal{H}\}\\ &= \{\sigma\in\Delta[V]\mid \sigma\not\subseteq\tau \text{ for any } \tau\notin\mathcal{H},\ \text{where } \tau\in\Delta[V]\}\\ &= \{\sigma\in\Delta[V]\mid \tau\in\mathcal{H} \text{ for any } \tau\supseteq\sigma,\ \text{where } \tau\in\Delta[V]\}\\ &= \bar{\delta}\mathcal{H}.\end{aligned}$$

Therefore, $\bar{\delta}=\gamma\Delta\gamma$. $\square$

By [38, Subsection 2.1] and Lemma 2.1, the following relations among γ, Δ, δ, $\bar{\Delta}$ and $\bar{\delta}$ hold: (1). $\gamma^2=\mathrm{id}$, (2). $\bar{\Delta}=\gamma\delta\gamma$ and $\bar{\delta}=\gamma\Delta\gamma$, (3). $\Delta\delta=\delta$, $\bar{\Delta}\bar{\delta}=\bar{\delta}$, $\delta\Delta=\Delta$ and $\bar{\delta}\bar{\Delta}=\bar{\Delta}$, (4). $\Delta^2=\Delta$, $\bar{\Delta}^2=\bar{\Delta}$, $\delta^2=\delta$ and $\bar{\delta}^2=\bar{\delta}$, (5). $(\delta\bar{\Delta})^2=(\delta\gamma\delta\gamma)^2=\delta\gamma\delta\gamma=\delta\bar{\Delta}$, $(\bar{\Delta}\delta)^2=(\gamma\delta\gamma\delta)^2=\gamma\delta\gamma\delta=\bar{\Delta}\delta$, $(\Delta\bar{\delta})^2=(\Delta\gamma\Delta\gamma)^2=\Delta\gamma\Delta\gamma=\Delta\bar{\delta}$ and $(\bar{\delta}\Delta)^2=(\gamma\Delta\gamma\Delta)^2=\gamma\Delta\gamma\Delta=\bar{\delta}\Delta$. Moreover, since the associated simplicial complex of any nonempty independence hypergraph is $\Delta[V]$, we have

$$\Delta\bar{\Delta}(\mathcal{H}) = \begin{cases}\Delta[V], & \mathcal{H}\neq\emptyset,\\ \emptyset, & \mathcal{H}=\emptyset,\end{cases}$$

$$\Delta\bar{\delta}(\mathcal{H}) = \begin{cases}\Delta[V], & \{V\}\in\mathcal{H},\\ \emptyset, & \{V\}\notin\mathcal{H}.\end{cases}$$

Thus the following relations hold as well: (6). $\Delta^2\bar{\Delta}=\bar{\Delta}\Delta\bar{\Delta}=\delta\Delta\bar{\Delta}=\bar{\delta}\Delta\bar{\Delta}=\Delta\bar{\Delta}$ and $\Delta^2\bar{\delta}=\bar{\Delta}\Delta\bar{\delta}=\delta\Delta\bar{\delta}=\bar{\delta}\Delta\bar{\delta}=\Delta\bar{\delta}$.

Let $\mathcal{H}$ and $\mathcal{H}'$ be two hypergraphs on V. We have the following observations:

(i). $\gamma(\mathcal{H}\cap\mathcal{H}')=\gamma\mathcal{H}\cup\gamma\mathcal{H}'$, $\gamma(\mathcal{H}\cup\mathcal{H}')=\gamma\mathcal{H}\cap\gamma\mathcal{H}'$,

(ii). $\Delta(\mathcal{H}\cap\mathcal{H}')\subseteq\Delta\mathcal{H}\cap\Delta\mathcal{H}'$, $\Delta(\mathcal{H}\cup\mathcal{H}')=\Delta\mathcal{H}\cup\Delta\mathcal{H}'$,

(iii). $\delta(\mathcal{H}\cap\mathcal{H}')=\delta\mathcal{H}\cap\delta\mathcal{H}'$, $\delta(\mathcal{H}\cup\mathcal{H}')\supseteq\delta\mathcal{H}\cup\delta\mathcal{H}'$,

(iv). $\bar{\Delta}(\mathcal{H}\cap\mathcal{H}')\subseteq\bar{\Delta}\mathcal{H}\cap\bar{\Delta}\mathcal{H}'$, $\bar{\Delta}(\mathcal{H}\cup\mathcal{H}')=\bar{\Delta}\mathcal{H}\cup\bar{\Delta}\mathcal{H}'$,

(v). $\bar{\delta}(\mathcal{H}\cap\mathcal{H}')=\bar{\delta}\mathcal{H}\cap\bar{\delta}\mathcal{H}'$, $\bar{\delta}(\mathcal{H}\cup\mathcal{H}')\supseteq\bar{\delta}\mathcal{H}\cup\bar{\delta}\mathcal{H}'$.

If both $\mathcal{H}$ and $\mathcal{H}'$ are simplicial complexes (resp. independence hypergraphs), then both $\mathcal{H}\cap\mathcal{H}'$ and $\mathcal{H}\cup\mathcal{H}'$ are simplicial complexes (resp. independence hypergraphs).

Let $\sigma=\{v_0,v_1,\ldots,v_n\}$ be an n-hyperedge on V and let $\tau=\{u_0,u_1,\ldots,u_m\}$ be an m-hyperedge on V'. The *box product* $\sigma\Box\tau$ is an $(mn+m+n+1)$-hyperedge

$$\sigma\Box\tau=\{(v,u)\mid v\in\sigma,u\in\tau\} \tag{2.1}$$

on the Cartesian product $V \times V'$. The right-hand side of (2.1) is up to a rearrangement of the vertices with respect to the lexicographic order. In addition, suppose $V \cap V' = \emptyset$. The *join* $\sigma * \tau$ is an $(n+m+1)$-hyperedge

$$\sigma * \tau = \{v_0, v_1, \ldots, v_n, u_0, u_1, \ldots, u_m\} \tag{2.2}$$

on the disjoint union $V \sqcup V'$, where the right-hand side of (2.2) is with respect to the total order on $V \sqcup V'$ given by $v \prec v'$ for any $v \in V$ and any $v' \in V'$. We observe

$$\begin{aligned} \Delta(\sigma * \tau) &= (\Delta\sigma) * (\Delta\tau), \\ \bar{\Delta}(\sigma * \tau) &= (\bar{\Delta}\sigma) * (\bar{\Delta}\tau), \end{aligned}$$

where the associated independence hypergraphs of $\sigma * \tau$, σ and τ are taken with respect to $V \sqcup V'$, V and V' respectively.

Definition 2.4. Let V and V' be two finite sets. Let $\mathcal{H}$ be a hypergraph on V and let $\mathcal{H}'$ be a hypergraph on V'.

(1). We define the *box product* of $\mathcal{H}$ and $\mathcal{H}'$ as

$$\mathcal{H}\Box\mathcal{H}' = \{\sigma\Box\sigma' \mid \sigma \in \mathcal{H}, \sigma' \in \mathcal{H}'\}.$$

Then $\mathcal{H}\Box\mathcal{H}'$ is a hypergraph on $V \times V'$;

(2). Suppose in addition $V \cap V' = \emptyset$. We define the *join* of $\mathcal{H}$ and $\mathcal{H}'$ as

$$\mathcal{H} * \mathcal{H}' = \{\sigma * \sigma' \mid \sigma \in \mathcal{H} \text{ and } \sigma' \in \mathcal{H}'\} \cup \mathcal{H} \cup \mathcal{H}'.$$

Then $\mathcal{H} * \mathcal{H}'$ is a hypergraph on $V \sqcup V'$.

By Definition 2.4, we observe the followings:

(i)'. $\Delta(\mathcal{H} * \mathcal{H}') = \Delta\mathcal{H} * \Delta\mathcal{H}'$,

(ii)'. $\delta(\mathcal{H} * \mathcal{H}') = \delta\mathcal{H} * \delta\mathcal{H}'$,

(iii)'. $\bar{\Delta}(\mathcal{H} * \mathcal{H}') = \bar{\Delta}\mathcal{H} * \bar{\Delta}\mathcal{H}'$,

(iv)'. $\bar{\delta}(\mathcal{H} * \mathcal{H}') = \bar{\delta}\mathcal{H} * \bar{\delta}\mathcal{H}'$,

(v)'. $\mathcal{H}_1 * (\mathcal{H}_2 \cup \mathcal{H}_3) = (\mathcal{H}_1 * \mathcal{H}_2) \cup (\mathcal{H}_1 * \mathcal{H}_3)$,

(vi)'. $\mathcal{H}_1 * (\mathcal{H}_2 \cap \mathcal{H}_3) = (\mathcal{H}_1 * \mathcal{H}_2) \cap (\mathcal{H}_1 * \mathcal{H}_3)$,

(vii)'. $\mathcal{H}_1\Box(\mathcal{H}_2 \cup \mathcal{H}_3) = (\mathcal{H}_1\Box\mathcal{H}_2) \cup (\mathcal{H}_1\Box\mathcal{H}_3)$,

(viii)'. $\mathcal{H}_1\Box(\mathcal{H}_2 \cap \mathcal{H}_3) = (\mathcal{H}_1\Box\mathcal{H}_2) \cap (\mathcal{H}_1\Box\mathcal{H}_3)$,

(ix)'. $\mathcal{H}_1\Box(\mathcal{H}_2 * \mathcal{H}_3) = (\mathcal{H}_1\Box\mathcal{H}_2) * (\mathcal{H}_1\Box\mathcal{H}_3)$,

where $\bar{\Delta}$ and $\bar{\delta}$ on the left-hand sides are with respect to $V \sqcup V'$ while $\bar{\Delta}$ and $\bar{\delta}$ on the right-hand sides are with respect to V and V'.

Let $\mathcal{K}$ be a simplicial complex on V and let $\mathcal{K}'$ be a simplicial complex on V'. Then $\mathcal{K} * \mathcal{K}'$ is a simplicial complex on $V \sqcup V'$ and both $\Delta(\mathcal{K}\Box\mathcal{K}')$ and $\delta(\mathcal{K}\Box\mathcal{K}')$ are simplicial complexes on $V \times V'$. Let $\mathcal{L}$ be an independence hypergraph on V and let $\mathcal{L}'$ be an independence hypergraph on V'. Then $\mathcal{L} * \mathcal{L}'$ is an independence hypergraph on $V \sqcup V'$ and both $\bar{\Delta}(\mathcal{L}\Box\mathcal{L}')$ and $\bar{\delta}(\mathcal{L}\Box\mathcal{L}')$ are independence hypergraphs on $V \times V'$.

Example 2.1. *Let* $V = \{v_0, v_1\}$ *and let* $V' = \{v'_0, v'_1, v'_2, v'_3\}$*. Let* $\mathcal{H} = \{\{v_0\}, \{v_0, v_1\}\}$ *be a hypergraph on* V *and let* $\mathcal{H}' = \{\{v'_0, v'_1\}, \{v'_0, v'_1, v'_2\}\}$ *be a hypergraph on* V'*. Let* $v_0 \prec v_1 \prec v'_0 \prec v'_1 \prec v'_2 \prec v'_3$*. Then*

$$\begin{aligned}\mathcal{H} * \mathcal{H}' \;=\; & \{\{v_0\}, \{v_0, v_1\}, \{v'_0, v'_1\}, \{v'_0, v'_1, v'_2\}, \{v_0, v'_0, v'_1\},\\ & \{v_0, v'_0, v'_1, v'_2\}, \{v_0, v_1, v'_0, v'_1\}, \{v_0, v_1, v'_0, v'_1, v'_2\}\}\end{aligned}$$

is a hypergraph on $V \sqcup V'$ *and*

$$\begin{aligned}\mathcal{H} \Box \mathcal{H}' \;=\; & \{\{(v_0, v'_0), (v_0, v'_1)\}, \{(v_0, v'_0), (v_0, v'_1), (v_0, v'_2)\},\\ & \{(v_0, v'_0), (v_0, v'_1), (v_1, v'_0), (v_1, v'_1)\},\\ & \{(v_0, v'_0), (v_0, v'_1), (v_0, v'_2), (v_1, v'_0), (v_1, v'_1), (v_1, v'_2)\}\}\end{aligned}$$

is a hypergraph on $V \times V'$*. Moreover,*

(1). $\Delta\mathcal{H} = \{\{v_0\}, \{v_1\}, \{v_0, v_1\}\}$, $\delta\mathcal{H} = \{\{v_0\}\}$, $\bar{\Delta}\mathcal{H} = \bar{\delta}\mathcal{H} = \mathcal{H}$,

(2). $\Delta\mathcal{H}' = \{\{v'_0\}, \{v'_1\}, \{v'_2\}, \{v'_0, v'_1\}, \{v'_0, v'_2\}, \{v'_1, v'_2\}, \{v'_0, v'_1, v'_2\}\}$, $\delta\mathcal{H}' = \emptyset$, $\bar{\Delta}\mathcal{H}' = \{\{v'_0, v'_1\}, \{v'_0, v'_1, v'_2\}, \{v'_0, v'_1, v'_2, v'_3\}\}$, $\bar{\delta}\mathcal{H}' = \emptyset$,

(3). $\Delta\mathcal{H} * \Delta\mathcal{H}' = \Delta\{v_0, v_1, v'_0, v'_1, v'_2\}$, $\delta\mathcal{H} * \delta\mathcal{H}' = \{\{v_0\}\}$, $\bar{\Delta}\mathcal{H} * \bar{\Delta}\mathcal{H}' = \{\{v_0, v'_0, v'_1\}, \{v_0, v'_0, v'_1, v'_2\}, \{v_0, v'_0, v'_1, v'_2, v'_3\}, \{v_0, v_1, v'_0, v'_1\}, \{v_0, v_1, v'_0, v'_1, v'_2\}, \{v_0, v_1, v'_0, v'_1, v'_2, v'_3\}\} \cup \mathcal{H} \cup \mathcal{H}'$, $\bar{\delta}\mathcal{H} * \bar{\delta}\mathcal{H}' = \mathcal{H}$.

2.2 Morphisms of hypergraphs and simplicial maps

Let V and V' be two finite sets. Let $\mathcal{H}$ be a hypergraph on V and let $\mathcal{H}'$ be a hypergraph on V'. A *morphism* $f : \mathcal{H} \longrightarrow \mathcal{H}'$ of hypergraphs is a map $f : V \longrightarrow V'$ such that for any hyperedge $\sigma = \{v_0, v_1, \ldots, v_n\}$ in $\mathcal{H}$, its image $f(\sigma)$ is a hyperedge in $\mathcal{H}'$ spanned by the (not necessarily distinct) vertices $f(v_0)$, $f(v_1)$, $\ldots$, $f(v_n)$. Let $\mathcal{K}$ be a simplicial complex on V and let $\mathcal{K}'$ be a simplicial complex on V'. We call a morphism $f : \mathcal{K} \longrightarrow \mathcal{K}'$ a *simplicial map.* Let $\mathcal{L}$ be an independence hypergraph on V and let $\mathcal{L}'$ be an independence hypergraph on V'. We call a morphism $f : \mathcal{L} \longrightarrow \mathcal{L}'$ a *morphism* of independence hypergraphs.

Let $f : \mathcal{H} \longrightarrow \mathcal{H}'$ be a morphism of hypergraphs. Then f canonically induces two simplicial maps

$$\begin{aligned}\Delta f : \quad & \Delta\mathcal{H} \longrightarrow \Delta\mathcal{H}',\\ \delta f : \quad & \delta\mathcal{H} \longrightarrow \delta\mathcal{H}'\end{aligned}$$

and two morphisms of independence hypergraphs

$$\begin{aligned}\bar{\Delta} f : \quad & \bar{\Delta}\mathcal{H} \longrightarrow \bar{\Delta}\mathcal{H}',\\ \bar{\delta} f : \quad & \bar{\delta}\mathcal{H} \longrightarrow \bar{\delta}\mathcal{H}'.\end{aligned}$$

We have the following naturalities:

(1). The canonical inclusions i_Δ and i_δ are natural. That is, for any morphism $f : \mathcal{H} \longrightarrow \mathcal{H}'$ we have the commutative diagrams

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\;f\;} & \mathcal{H}' \\ {\scriptstyle i_\Delta}\downarrow & & \downarrow{\scriptstyle i'_\Delta} \\ \Delta\mathcal{H} & \xrightarrow{\;\Delta f\;} & \Delta\mathcal{H}', \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{H} & \xrightarrow{\;f\;} & \mathcal{H}' \\ {\scriptstyle i_\delta}\downarrow & & \downarrow{\scriptstyle i'_\delta} \\ \delta\mathcal{H} & \xrightarrow{\;\delta f\;} & \delta\mathcal{H}'. \end{array}$$

(2). The join of hypergraphs is natural. That is, for any two morphisms $f : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ and $f' : \mathcal{H}'_1 \longrightarrow \mathcal{H}'_2$, we have a canonical induced morphism $f * f' : \mathcal{H}_1 * \mathcal{H}'_1 \longrightarrow \mathcal{H}_2 * \mathcal{H}'_2$ such that $(f * f') \mid_{\mathcal{H}_1} = f$, $(f * f') \mid_{\mathcal{H}'_1} = f'$ and for any $\sigma \in \mathcal{H}_1$ and any $\sigma' \in \mathcal{H}'_1$, the join $\sigma * \sigma'$ in $\mathcal{H}_1 * \mathcal{H}'_1$ is sent to the hyperedge $f(\sigma) * f'(\sigma')$ in $\mathcal{H}_2 * \mathcal{H}'_2$. The followings are direct

$$\Delta(f * f') = \Delta f * \Delta f', \qquad \delta(f * f') = \delta f * \delta f',$$
$$\bar{\Delta}(f * f') = \bar{\Delta} f * \bar{\Delta} f', \qquad \bar{\delta}(f * f') = \bar{\delta} f * \bar{\delta} f',$$

where $\bar{\Delta}$ and $\bar{\delta}$ on the left-hand sides are with respect to $V \sqcup V'$ while $\bar{\Delta}$ and $\bar{\delta}$ on the right-hand sides are with respect to V and V'.

(3). The box product of hypergraphs is natural. That is, for any two morphisms $f : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ and $f' : \mathcal{H}'_1 \longrightarrow \mathcal{H}'_2$, we have a canonical induced morphism $f \square f' : \mathcal{H}_1 \square \mathcal{H}'_1 \longrightarrow \mathcal{H}_2 \square \mathcal{H}'_2$ sending $\sigma \square \sigma'$ to $f(\sigma) \square f'(\sigma')$.

3 Random hypergraphs and random simplicial complexes

3.1 General random hypergraphs and random simplicial complexes

Let $\mathbf{H}(V)$ be the category whose objects are hypergraphs on V and whose morphisms are morphisms of hypergraphs. Let $\mathbf{K}(V)$ be the category whose objects are simplicial complexes on V and whose morphisms are simplicial maps. Let $\mathbf{L}(V)$ be the category whose objects are independence hypergraphs on V and whose morphisms are morphisms of independence hypergraphs. Both $\mathbf{K}(V)$ and $\mathbf{L}(V)$ are full subcategories of $\mathbf{H}(V)$.

A *random hypergraph* (resp. *random simplicial complex* and *random independence hypergraph*) on V is a probability function on $\mathbf{Obj}(\mathbf{H}(V))$ (resp. $\mathbf{Obj}(\mathbf{K}(V))$ and $\mathbf{Obj}(\mathbf{L}(V))$). Let $D(\mathbf{H}(V))$ (resp. $D(\mathbf{K}(V))$ and $D(\mathbf{L}(V))$) be the functional space of all the probability functions on $\mathbf{Obj}(\mathbf{H}(V))$ (resp. $\mathbf{Obj}(\mathbf{K}(V))$ and $\mathbf{Obj}(\mathbf{L}(V))$). For any map

$$f : \qquad \mathbf{H}(V) \longrightarrow \mathbf{H}(V)$$

there is an induced map

$$Df : \qquad D(\mathbf{H}(V)) \longrightarrow D(\mathbf{H}(V))$$

given by

$$(Df)(\varphi)(\mathcal{H}) = \sum_{f(\mathcal{H}')=\mathcal{H}} \varphi(\mathcal{H}'),$$

where $\varphi \in D(\mathbf{H}(V))$ and $\mathcal{H}, \mathcal{H}' \in \mathbf{Obj}(\mathbf{H}(V))$. Let f be $\Delta, \delta, \bar{\Delta}$ and $\bar{\delta}$ respectively. We have the induced maps

$$D\Delta, D\delta : \qquad D(\mathbf{H}(V)) \longrightarrow D(\mathbf{K}(V)),$$
$$D\bar{\Delta}, D\bar{\delta} : \qquad D(\mathbf{H}(V)) \longrightarrow D(\mathbf{L}(V)).$$

Moreover, for any map $\mu : \mathbf{Obj}(\mathbf{H}(V)) \times \mathbf{Obj}(\mathbf{H}(V)) \longrightarrow \mathbf{Obj}(\mathbf{H}(V))$ there is an induced map

$$D\mu : \qquad D(\mathbf{H}(V)) \times D(\mathbf{H}(V)) \longrightarrow D(\mathbf{H}(V))$$

given by

$$D\mu(\varphi', \varphi'')(\mathcal{H}) = \sum_{\mu(\mathcal{H}', \mathcal{H}'')=\mathcal{H}} \varphi'(\mathcal{H}')\varphi''(\mathcal{H}''),$$

where $\varphi', \varphi'' \in D(\mathbf{H}(V))$ and $\mathcal{H}, \mathcal{H}', \mathcal{H}'' \in \mathrm{Obj}(\mathbf{H}(V))$. Let $\mu(\mathcal{H}', \mathcal{H}'')$ be $\mathcal{H}' \cap \mathcal{H}''$ and $\mathcal{H}' \cup \mathcal{H}''$ respectively. We obtain the induced maps

$$D\cap, D\cup : \qquad D(\mathbf{H}(V)) \times D(\mathbf{H}(V)) \longrightarrow D(\mathbf{H}(V)).$$

The restrictions of $D\cap$ and $D\cup$ give the maps

$$D\cap, D\cup : \qquad D(\mathbf{K}(V)) \times D(\mathbf{K}(V)) \longrightarrow D(\mathbf{K}(V)),$$
$$D\cap, D\cup : \qquad D(\mathbf{L}(V)) \times D(\mathbf{L}(V)) \longrightarrow D(\mathbf{L}(V)).$$

Furthermore, let V' and V'' be two finite sets. Then the box product

$$\Box : \qquad \mathbf{Obj}(\mathbf{H}(V')) \times \mathbf{Obj}(\mathbf{H}(V'')) \longrightarrow \mathbf{Obj}(\mathbf{H}(V' \times V''))$$

sending $(\mathcal{H}', \mathcal{H}'')$ to $\mathcal{H}' \Box \mathcal{H}''$ induces a map

$$D\Box : \qquad D(\mathbf{H}(V')) \times D(\mathbf{H}(V'')) \longrightarrow D(\mathbf{H}(V' \times V''))$$

given by

$$(D\Box)(\varphi', \varphi'')(\mathcal{H}) = \sum_{\mathcal{H}' \Box \mathcal{H}''=\mathcal{H}} \varphi'(\mathcal{H}')\varphi''(\mathcal{H}''). \tag{3.1}$$

Here in (3.1), we take $\varphi' \in D(\mathbf{H}(V'))$, $\varphi'' \in D(\mathbf{H}(V''))$, $\mathcal{H}' \in \mathbf{Obj}(\mathbf{H}(V'))$, $\mathcal{H}'' \in \mathbf{Obj}(\mathbf{H}(V''))$ and $\mathcal{H} \in \mathbf{Obj}(\mathbf{H}(V' \times V''))$. Suppose in addition $V' \cap V'' = \emptyset$. Then the join

$$* : \qquad \mathbf{Obj}(\mathbf{H}(V')) \times \mathbf{Obj}(\mathbf{H}(V'')) \longrightarrow \mathbf{Obj}(\mathbf{H}(V' \sqcup V''))$$

sending $(\mathcal{H}', \mathcal{H}'')$ to $\mathcal{H}' * \mathcal{H}''$ induces a map

$$D* : \qquad D(\mathbf{H}(V')) \times D(\mathbf{H}(V'')) \longrightarrow D(\mathbf{H}(V' \sqcup V''))$$

given by

$$(D*)(\varphi', \varphi'')(\mathcal{H}) = \sum_{\mathcal{H}' * \mathcal{H}''=\mathcal{H}} \varphi'(\mathcal{H}')\varphi''(\mathcal{H}''). \tag{3.2}$$

Here in (3.2), we take $\varphi' \in D(\mathbf{H}(V'))$, $\varphi'' \in D(\mathbf{H}(V''))$, $\mathcal{H}' \in \mathbf{Obj}(\mathbf{H}(V'))$, $\mathcal{H}'' \in \mathbf{Obj}(\mathbf{H}(V''))$ and $\mathcal{H} \in \mathbf{Obj}(\mathbf{H}(V' \sqcup V''))$.

Lemma 3.1. *(1).* $(D\cup)(D\Delta, D\Delta) = (D\Delta)(D\cup)$ *and* $(D\cup)(D\bar{\Delta}, D\bar{\Delta}) = (D\bar{\Delta})(D\cup)$,

(2). $(D\cap)(D\delta, D\delta) = (D\delta)(D\cap)$ *and* $(D\cap)(D\bar{\delta}, D\bar{\delta}) = (D\bar{\delta})(D\cap)$*;*

(3). $(D*)(D\Delta, D\Delta) = (D\Delta)(D*)$, $(D*)(D\delta, D\delta) = (D\delta)(D*)$, $(D*)(D\bar{\Delta}, D\bar{\Delta}) = (D\bar{\Delta})(D*)$ *and* $(D*)(D\bar{\delta}, D\bar{\delta}) = (D\bar{\delta})(D*)$.

Proof. For any $\varphi', \varphi'' \in D(\mathbf{H}(V))$ and any $\mathcal{K} \in \mathbf{K}(V)$, with the help of Subsection 2.1 (ii),

$$\begin{aligned}
(D\cup)(D\Delta, D\Delta)(\varphi', \varphi'')(\mathcal{K}) &= \sum_{\mathcal{K}'\cup\mathcal{K}''=\mathcal{K}} \Big((D\Delta)(\varphi')(\mathcal{K}')\Big)\Big((D\Delta)(\varphi'')(\mathcal{K}'')\Big) \\
&= \sum_{\mathcal{K}'\cup\mathcal{K}''=\mathcal{K}} \Big(\sum_{\Delta\mathcal{H}'=\mathcal{K}'} \varphi'(\mathcal{H}')\Big)\Big(\sum_{\Delta\mathcal{H}''=\mathcal{K}''} \varphi''(\mathcal{H}'')\Big) \\
&= \sum_{\Delta\mathcal{H}'\cup\Delta\mathcal{H}''=\mathcal{K}} \varphi'(\mathcal{H}')\varphi''(\mathcal{H}'') \\
&= \sum_{\Delta(\mathcal{H}'\cup\mathcal{H}'')=\mathcal{K}} \varphi'(\mathcal{H}')\varphi''(\mathcal{H}'') \\
&= \sum_{\Delta\mathcal{H}=\mathcal{K}} \sum_{\mathcal{H}'\cup\mathcal{H}''=\mathcal{H}} \varphi'(\mathcal{H}')\varphi''(\mathcal{H}'') \\
&= (D\Delta)(D\cup)(\varphi', \varphi'')(\mathcal{K}).
\end{aligned}$$

We obtain $(D\cup)(D\Delta, D\Delta) = (D\Delta)(D\cup)$. Other identities can be proved analogously. The second identity in (1) is proved with the help of Subsection 2.1 (iv). The two identities in (2) are respectively proved with the help of Subsection 2.1 (iii) and (v). The identities in (3) are proved with the help of Subsection 2.1 (i)' - (iv)'. □

3.2 Random hypergraphs and random simplicial complexes of Erdös-Rényi type

Let $p : \Delta[V] \longrightarrow [0, 1]$. Let $\mathcal{K}$ be a simplicial complex on V. An *external face* of $\mathcal{K}$ is a hyperedge $\sigma \in \Delta[V]$ such that $\sigma \notin \mathcal{K}$ and $\tau \in \mathcal{K}$ for any nonempty proper subset $\tau \subsetneq \sigma$. Let $E(\mathcal{K})$ be the collection of all the external faces of $\mathcal{K}$. Let $\mathcal{L}$ be an independence hypergraph on V. A *co-external face* of $\mathcal{L}$ is a hyperedge $\sigma \in \Delta[V]$ such that $\sigma \notin \mathcal{L}$ and $\tau \in \mathcal{L}$ for any proper superset $\tau \supsetneq \sigma$, where $\tau \in \Delta[V]$. Let $\bar{E}(\mathcal{L})$ be the collection of all the co-external faces of $\mathcal{L}$.

(1). Consider the Erdös-Rényi-type model $\bar{\mathrm{P}}_p$ of random hypergraphs given by

$$\bar{\mathrm{P}}_p(\mathcal{H}) = \prod_{\sigma\in\mathcal{H}} p(\sigma) \prod_{\sigma\notin\mathcal{H}} \big(1 - p(\sigma)\big)$$

for any $\mathcal{H} \in \mathbf{Obj}(\mathbf{H}(V))$. We choose each element $\sigma \in \Delta[V]$ to be a hyperedge of $\mathcal{H}$ independently at random with probability $p(\sigma)$. This randomly generated hypergraph $\mathcal{H}$ satisfies the probability distribution $\bar{\mathrm{P}}_p$, written $\mathcal{H} \sim \bar{\mathrm{P}}_p$.

(2). Consider the Erdös-Rényi-type model P_p of random simplicial complexes given by

$$\mathrm{P}_p(\mathcal{K}) = \prod_{\sigma\in\mathcal{K}} p(\sigma) \prod_{\sigma\in E(\mathcal{K})} \big(1 - p(\sigma)\big),$$

where $\mathcal{K} \in \mathrm{Obj}(\mathbf{K}(V))$. We generate the 0-skeleton of $\mathcal{K}$ by choosing each 0-hyperedge $\{v\} \in \Delta[V]$ independently at random with probability $p(\{v\})$. For any nonnegative integer k, once the k-skeleton of $\mathcal{K}$ is randomly generated, we generate the $(k+1)$-skeleton of $\mathcal{K}$ by choosing each external $(k+1)$-face σ of the k-skeleton of $\mathcal{K}$ independently at random with probability $p(\sigma)$. By an induction on k, the final randomly generated simplicial complex $\mathcal{K}$ satisfies the probability distribution P_p, written $\mathcal{K} \sim \mathrm{P}_p$.

(3). Consider the Erdös-Rényi-type model Q_p of random independence hypergraphs given by

$$\mathrm{Q}_p(\mathcal{L}) = \prod_{\sigma\in\mathcal{L}} p(\sigma) \prod_{\sigma\in\bar{E}(\mathcal{L})} \big(1-p(\sigma)\big),$$

where $\mathcal{L} \in \mathrm{Obj}(\mathbf{L}(V))$. We choose the $(|V|-1)$-hyperedge $V \in \Delta[V]$ at random with probability $p(V)$ to be a hyperedge of $\mathcal{L}$. Once all the i-hyperedges, $i = k+1, k+2, \dots, |V|-1$, of $\mathcal{L}$ are randomly generated, we generate the k-hyperedges of $\mathcal{L}$ by choosing each co-external face σ of the collection of all the $(k+1)$-hyperedges of $\mathcal{L}$ independently at random with probability $p(\sigma)$. By an induction on k in reverse, The final randomly generated independence hypergraph $\mathcal{L}$ satisfies the probability distribution Q_p, written $\mathcal{L} \sim \mathrm{Q}_p$.

Lemma 3.2. (cf. [38, Lemma 4.4]). *$(D\gamma)(\bar{\mathrm{P}}_p) = \bar{\mathrm{P}}_{1-p}$.*

Lemma 3.3. (cf. [38, Lemma 4.5]). *Let $p', p'' : \Delta[V] \longrightarrow [0,1]$. Write $(p' \cap p'')(\sigma) = p'(\sigma)p''(\sigma)$ and $(p' \cup p'')(\sigma) = 1 - (1-p'(\sigma))(1-p''(\sigma))$ for any $\sigma \in \Delta[V]$. Then $(D\cap)(\bar{\mathrm{P}}_{p'}, \bar{\mathrm{P}}_{p''}) = \bar{\mathrm{P}}_{p'\cap p''}$ and $(D\cup)(\bar{\mathrm{P}}_{p'}, \bar{\mathrm{P}}_{p''}) = \bar{\mathrm{P}}_{p'\cup p''}$. In other words, if $\mathcal{H}' \sim \bar{\mathrm{P}}_{p'}$ and $\mathcal{H}'' \sim \bar{\mathrm{P}}_{p''}$, then $\mathcal{H}' \cap \mathcal{H}'' \sim \bar{\mathrm{P}}_{p'\cap p''}$ and $\mathcal{H}' \cup \mathcal{H}'' \sim \bar{\mathrm{P}}_{p'\cup p''}$.*

Lemma 3.4. *Let V' and V'' be two disjoint vertex sets. Let $p' : \Delta[V'] \longrightarrow [0,1]$ and $p'' : \Delta[V''] \longrightarrow [0,1]$. Write $(p' * p'')(\sigma) = p'(\sigma \cap V')p''(\sigma \cap V'')$ for any $\sigma \in \Delta[V' \sqcup V'']$. Then $(D*)(\bar{\mathrm{P}}_{p'}, \bar{\mathrm{P}}_{p''}) = \bar{\mathrm{P}}_{p'*p''}$. In other words, if $\mathcal{H}' \sim \bar{\mathrm{P}}_{p'}$ and $\mathcal{H}'' \sim \bar{\mathrm{P}}_{p''}$, then $\mathcal{H}' * \mathcal{H}'' \sim \bar{\mathrm{P}}_{p'*p''}$.*

Proof. Consider the following two independent trials: (1). generate $\mathcal{H}'$, (2). generate $\mathcal{H}''$. Then for any $\sigma \in \Delta[V' \sqcup V'']$, both of the followings hold:

(1). $\sigma \in \mathcal{H}' * \mathcal{H}''$ iff both $(\sigma \cap V') \in \mathcal{H}'$ in trial (1) and $(\sigma \cap V'') \in \mathcal{H}''$ in trial (2). Thus the event $\sigma \in \mathcal{H}' * \mathcal{H}''$ is the product event of the two independent events $(\sigma \cap V') \in \mathcal{H}'$ and $(\sigma \cap V'') \in \mathcal{H}''$. This event has the probability $p'(\sigma \cap V')p''(\sigma \cap V'')$;

(2). $\sigma \notin \mathcal{H}' * \mathcal{H}''$ iff either $(\sigma \cap V') \notin \mathcal{H}'$ in trial (1) or $(\sigma \cap V'') \notin \mathcal{H}''$ in trial (2). Thus the event $\sigma \notin \mathcal{H}' * \mathcal{H}''$ has the probability $1 - p'(\sigma \cap V')p''(\sigma \cap V'')$.

Therefore, $\mathcal{H}' * \mathcal{H}''$ is the randomly generated hypergraph on $V' \sqcup V''$ satisfying the probability distribution $\bar{\mathrm{P}}_{p'*p''}$. ☐

Theorem 3.5. *For any $\mathcal{H} \in \mathrm{Obj}(\mathbf{H}(V))$, any $\mathcal{K} \in \mathrm{Obj}(\mathbf{K}(V))$ and any $\mathcal{L} \in \mathrm{Obj}(\mathbf{L}(V))$,*

(1). $\big((D\Delta)(\bar{\mathrm{P}}_p)\big)(\mathcal{K}) = \mathrm{Q}_{1-p}(\gamma\mathcal{K})$,

(2). $\big((D\bar{\Delta})(\bar{\mathrm{P}}_p)\big)(\mathcal{L}) = \mathrm{P}_{1-p}(\gamma\mathcal{L})$,

(3). $\big((D\delta)(\bar{\mathrm{P}}_p)\big)(\mathcal{K}) = \mathrm{P}_p(\mathcal{K})$,

(4). $\big((D\bar\delta)(\bar{\mathrm{P}}_p)\big)(\mathcal{L}) = \mathrm{Q}_p(\mathcal{L})$.

We will prove Theorem 3.5 in the next subsection. The following corollary is a restatement of Theorem 3.5 (1) and (2).

Corollary 3.6. *For any* $\mathcal{H} \in \mathrm{Obj}(\mathbf{H}(V))$*, any* $\mathcal{K} \in \mathrm{Obj}(\mathbf{K}(V))$ *and any* $\mathcal{L} \in \mathrm{Obj}(\mathbf{L}(V))$*,*

(1). $\big((D\gamma)(D\Delta)(\bar{\mathrm{P}}_p)\big)(\mathcal{L}) = \mathrm{Q}_{1-p}(\mathcal{L})$*,*

(2). $\big((D\gamma)(D\bar\Delta)(\bar{\mathrm{P}}_p)\big)(\mathcal{K}) = \mathrm{P}_{1-p}(\mathcal{K})$*.*

The following two corollaries follow from Theorem 3.5.

Corollary 3.7. *Let* $p', p'' : \Delta[V] \longrightarrow [0,1]$*. Then*

(1). $(D\cap)(\mathrm{P}_{p'}, \mathrm{P}_{p''}) = \mathrm{P}_{p'\cap p''}$*,*

(2). $(D\cap)(\mathrm{Q}_{p'}, \mathrm{Q}_{p''}) = \mathrm{Q}_{p'\cap p''}$*.*

Proof. By Lemma 3.1, Lemma 3.3 and Theorem 3.5,

$$\begin{aligned} (D\cap)(\mathrm{P}_{p'}, \mathrm{P}_{p''}) &= (D\cap)(D\delta, D\delta)(\bar{\mathrm{P}}_{p'}, \bar{\mathrm{P}}_{p''}) \\ &= (D\delta)(D\cap)(\bar{\mathrm{P}}_{p'}, \bar{\mathrm{P}}_{p''}) \\ &= (D\delta)(\bar{\mathrm{P}}_{p'\cap p''}) \\ &= \mathrm{P}_{p'\cap p''} \end{aligned}$$

and

$$\begin{aligned} (D\cap)(\mathrm{Q}_{p'}, \mathrm{Q}_{p''}) &= (D\cap)(D\bar\delta, D\bar\delta)(\bar{\mathrm{P}}_{p'}, \bar{\mathrm{P}}_{p''}) \\ &= (D\bar\delta)(D\cap)(\bar{\mathrm{P}}_{p'}, \bar{\mathrm{P}}_{p''}) \\ &= (D\bar\delta)(\bar{\mathrm{P}}_{p'\cap p''}) \\ &= \mathrm{Q}_{p'\cap p''}. \end{aligned}$$

□

Corollary 3.8. *Let* V' *and* V'' *be two disjoint vertex sets. Let* $p' : \Delta[V'] \longrightarrow [0,1]$ *and* $p'' : \Delta[V''] \longrightarrow [0,1]$*. Then*

(1). $(D*)(\mathrm{P}_{p'}, \mathrm{P}_{p''}) = \mathrm{P}_{p'*p''}$*,*

(2). $(D*)(\mathrm{Q}_{p'}, \mathrm{Q}_{p''}) = \mathrm{Q}_{p'*p''}$*.*

Proof. By Lemma 3.1, Lemma 3.4 and Theorem 3.5,

$$\begin{aligned} (D*)(\mathrm{P}_{p'}, \mathrm{P}_{p''}) &= (D*)(D\delta, D\delta)(\bar{\mathrm{P}}_{p'}, \bar{\mathrm{P}}_{p''}) \\ &= (D\delta)(D*)(\bar{\mathrm{P}}_{p'}, \bar{\mathrm{P}}_{p''}) \\ &= (D\delta)(\bar{\mathrm{P}}_{p'*p''}) \\ &= \mathrm{P}_{p'*p''} \end{aligned}$$

and

$$\begin{aligned} (D*)(\mathrm{Q}_{p'}, \mathrm{Q}_{p''}) &= (D*)(D\bar\delta, D\bar\delta)(\bar{\mathrm{P}}_{p'}, \bar{\mathrm{P}}_{p''}) \\ &= (D\bar\delta)(D*)(\bar{\mathrm{P}}_{p'}, \bar{\mathrm{P}}_{p''}) \\ &= (D\bar\delta)(\bar{\mathrm{P}}_{p'*p''}) \\ &= \mathrm{Q}_{p'*p''}. \end{aligned}$$

□

3.3 Proof of Theorem 3.5

Consider a map $p : \Delta[V] \longrightarrow [0,1]$. For any hypergraph $\mathcal{H}$ on V and any hyperedge $\sigma \in \mathcal{H}$, we call σ a *maximal hyperedge* of $\mathcal{H}$ if there does not exist any $\tau \in \mathcal{H}$ such that $\sigma \subsetneq \tau$ and call σ a *minimal hyperedge* of $\mathcal{H}$ if there does not exist any $\tau \in \mathcal{H}$ such that $\sigma \supsetneq \tau$. Let $\max(\mathcal{H})$ be the collection of all the maximal hyperedges of $\mathcal{H}$ and let $\min(\mathcal{H})$ be the collection of all the minimal hyperedges of $\mathcal{H}$.

Lemma 3.9. (cf. [38, Theorem 1.5 (2)]). *For any simplicial complex $\mathcal{K}$ on V,*

$$\big((D\Delta)(\bar{\mathrm{P}}_p)\big)(\mathcal{K}) = \prod_{\tau\in\max(\mathcal{K})} p(\tau) \prod_{\substack{\tau\in\Delta[V]\\ \tau\notin\mathcal{K}}} \big(1-p(\tau)\big). \tag{3.3}$$

The proof of Lemma 3.9 is in [38, Lemma 4.2].

Lemma 3.10. *For any independence hypergraph $\mathcal{L}$ on V,*

$$\big((D\bar{\Delta})(\bar{\mathrm{P}}_p)\big)(\mathcal{L}) = \prod_{\tau\in\min(\mathcal{L})} p(\tau) \prod_{\substack{\tau\in\Delta[V]\\ \tau\notin\mathcal{L}}} \big(1-p(\tau)\big). \tag{3.4}$$

Proof. Let $\mathcal{L}$ be an independence hypergraph on V. Let $S = \{\sigma_1, \sigma_2, \ldots, \sigma_s\}$ be any set (allowed to be the emptyset) of distinct hyperedges in $\mathcal{L}$ such that for each σ_i, $i = 1, 2, \ldots, s$, there exists $\tau \in \min(\mathcal{L})$ such that $\sigma_i \supsetneq \tau$. Suppose S runs over all such sets of hyperedges in $\mathcal{L}$. Then $\mathcal{H} = \min(\mathcal{L}) \sqcup S$ runs over all the hypergraphs on V such that $\bar{\Delta}\mathcal{H} = \mathcal{L}$. Consequently,

$$\begin{aligned}
\big((D\bar{\Delta})(\bar{\mathrm{P}}_p)\big)(\mathcal{L}) &= \sum_{\bar{\Delta}\mathcal{H}=\mathcal{L}} \bar{\mathrm{P}}_p(\mathcal{H}) \\
&= \sum_{\bar{\Delta}\mathcal{H}=\mathcal{L}} \prod_{\sigma\in\mathcal{H}} p(\sigma) \prod_{\sigma\notin\mathcal{H}} \big(1-p(\sigma)\big) \\
&= \sum_{\mathcal{H}=\min(\mathcal{L})\sqcup S} \prod_{\sigma\in\mathcal{H}} p(\sigma) \prod_{\substack{\sigma\in\Delta[V]\\ \sigma\notin\mathcal{H}}} \big(1-p(\sigma)\big) \\
&= \prod_{\sigma\in\min(\mathcal{L})} p(\sigma) \prod_{\substack{\sigma\in\Delta[V]\\ \sigma\notin\mathcal{L}}} \big(1-p(\sigma)\big) \\
&\quad \Big(\sum_{S\subseteq\mathcal{L}\setminus\min(\mathcal{L})} \prod_{\sigma\in S} p(\sigma) \prod_{\substack{\sigma\in\mathcal{L}\setminus\min(\mathcal{L})\\ \sigma\notin S}} \big(1-p(\sigma)\big)\Big) \\
&= \prod_{\sigma\in\min(\mathcal{L})} p(\sigma) \prod_{\substack{\sigma\in\Delta[V]\\ \sigma\notin\mathcal{L}}} \big(1-p(\sigma)\big) \\
&\quad \prod_{\sigma\in\mathcal{L}\setminus\min(\mathcal{L})} \Big(p(\sigma) + \big(1-p(\sigma)\big)\Big) \\
&= \prod_{\tau\in\min(\mathcal{L})} p(\tau) \prod_{\substack{\tau\in\Delta[V]\\ \tau\notin\mathcal{L}}} \big(1-p(\tau)\big).
\end{aligned} \tag{3.5}$$

We obtain (3.4). ∎

Lemma 3.11. *For any simplicial complex $\mathcal{K}$ on V,*

$$\big((D\delta)(\bar{\mathrm{P}}_p)\big)(\mathcal{K}) = \prod_{\tau\in\min(\gamma\mathcal{K})} \big(1-p(\tau)\big) \prod_{\tau\in\mathcal{K}} p(\tau).$$

Proof. By a direct calculation,

$$\begin{aligned}
\big((D\delta)(\bar{\mathrm{P}}_p)\big)(\mathcal{K}) &= \big((D\gamma)\circ(D\bar{\Delta})\circ(D\gamma)(\bar{\mathrm{P}}_p)\big)(\mathcal{K})\\
&= \sum_{\gamma\mathcal{L}=\mathcal{K}} (D\bar{\Delta}\circ D\gamma)(\bar{\mathrm{P}}_p)(\mathcal{L})\\
&= \sum_{\gamma\mathcal{L}=\mathcal{K}}\ \sum_{\bar{\Delta}\mathcal{H}=\mathcal{L}} (D\gamma)(\bar{\mathrm{P}}_p)(\mathcal{H})\\
&= \sum_{\gamma\mathcal{L}=\mathcal{K}}\ \sum_{\bar{\Delta}\mathcal{H}=\mathcal{L}}\ \sum_{\gamma\mathcal{H}'=\mathcal{H}} \bar{\mathrm{P}}_p(\mathcal{H}')\\
&= \sum_{\bar{\Delta}\mathcal{H}=\gamma\mathcal{K}} \bar{\mathrm{P}}_p(\gamma\mathcal{H})\\
&= \prod_{\tau\in\min(\gamma\mathcal{K})} \big(1-p(\tau)\big) \prod_{\tau\in\mathcal{K}} p(\tau).
\end{aligned}$$

The last equality follows by a analogous calculation of (3.5). ☐

Lemma 3.12. *For any independence hypergraph $\mathcal{L}$ on V,*

$$\big((D\bar{\delta})(\bar{\mathrm{P}}_p)\big)(\mathcal{L}) = \prod_{\tau\in\max(\gamma\mathcal{L})} \big(1-p(\tau)\big) \prod_{\tau\in\mathcal{L}} p(\tau).$$

Proof. By a direct calculation,

$$\begin{aligned}
\big((D\bar{\delta})(\bar{\mathrm{P}}_p)\big)(\mathcal{L}) &= \big((D\gamma)\circ(D\Delta)\circ(D\gamma)(\bar{\mathrm{P}}_p)\big)(\mathcal{L})\\
&= \sum_{\gamma\mathcal{K}=\mathcal{L}} (D\Delta\circ D\gamma)(\bar{\mathrm{P}}_p)(\mathcal{K})\\
&= \sum_{\gamma\mathcal{K}=\mathcal{L}}\ \sum_{\Delta\mathcal{H}=\mathcal{K}} (D\gamma)(\bar{\mathrm{P}}_p)(\mathcal{H})\\
&= \sum_{\gamma\mathcal{K}=\mathcal{L}}\ \sum_{\Delta\mathcal{H}=\mathcal{K}}\ \sum_{\gamma\mathcal{H}'=\mathcal{H}} \bar{\mathrm{P}}_p(\mathcal{H}')\\
&= \sum_{\Delta\mathcal{H}=\gamma\mathcal{L}} \bar{\mathrm{P}}_p(\gamma\mathcal{H})\\
&= \prod_{\tau\in\max(\gamma\mathcal{L})} \big(1-p(\tau)\big) \prod_{\tau\in\mathcal{L}} p(\tau).
\end{aligned}$$

The last equality follows by a analogous calculation of [38, Lemma 4.2]. ☐

Proof of Theorem 3.5. For any simplicial complex $\mathcal{K}$ on V,

$$E(\mathcal{K}) = \min(\gamma\mathcal{K}).$$

For any independence hypergraph $\mathcal{L}$ on V,

$$\bar{E}(\mathcal{L}) = \max(\gamma\mathcal{L}).$$

Therefore, Theorem 3.5 follows from Lemma 3.9 - Lemma 3.12. ☐

4 Generations of random hypergraphs and random simplicial complexes by the map algebra

4.1 A graded construction of the map algebra

Let $\mathbf{H}$ be the category such that each object is a pair $(V, \mathcal{H})$, where V is a finite set and $\mathcal{H}$ is a hypergraph on V, and each morphism from $(V, \mathcal{H})$ to $(V', \mathcal{H}')$ is a morphism $f : V \longrightarrow V'$ of hypergraphs from $\mathcal{H}$ to $\mathcal{H}'$. Let G^1 be the semi-group generated by γ, δ and Δ, where the unit element is the identity map and the multiplication is the composition of maps. Each element of G^1 is a map from $\mathbf{Obj}(\mathbf{H})$ to itself.

Proposition 4.1. *Each of the triples* $\{\gamma, \delta, \Delta\}$, $\{\gamma, \bar{\delta}, \bar{\Delta}\}$, $\{\gamma, \Delta, \bar{\Delta}\}$ *and* $\{\gamma, \delta, \bar{\delta}\}$ *is a set of multiplicative generators of* G^1.

Proof. By Lemma 2.1, each of the triples $\{\gamma, \Delta, \delta\}$, $\{\gamma, \bar{\Delta}, \bar{\delta}\}$, $\{\gamma, \Delta, \bar{\Delta}\}$ and $\{\gamma, \delta, \bar{\delta}\}$ could multiplicatively generate γ, Δ, δ, $\bar{\Delta}$ and $\bar{\delta}$. Therefore, each of the triple could multiplicatively generate G^1. □

For any positive integer k, Let G^k be the collection of all the words

$$(\cdots(w_1 \bullet w_2) \bullet \cdots \bullet w_k),$$

where $w_1, w_2, \ldots, w_k \in G$ and $\bullet = \cap, \cup, *$ or $\square$, with the binary operation $\bullet$ for $k-1$ times and $k-2$ brackets giving the order of operations. For any $w \in G^k$, we call an element $(\cdots, (V_1, \mathcal{H}_1), (V_2, \mathcal{H}_2), \cdots)$ in $\mathbf{Obj}(\mathbf{H})^{\times k}$ *w-admissible* if it is sent to an element in $\mathbf{Obj}(\mathbf{H})$ by w, i.e. all the binary operations $w_1(\mathcal{H}_1) \bullet w_2(\mathcal{H}_2)$ in w are well-defined. For example, any element in $\mathbf{Obj}(\mathbf{H})^{\times 2}$ is $\square$-admissible, a $*$-admissible element in $\mathbf{Obj}(\mathbf{H})^{\times 2}$ is of the form $((V_1, \mathcal{H}_1), (V_2, \mathcal{H}_2))$ where $V_1 \cap V_2 = \emptyset$, and a $\cap$-admissible element as well as a $\cup$-admissible element in $\mathbf{Obj}(\mathbf{H})^{\times 2}$ is of the form $((V, \mathcal{H}_1), (V, \mathcal{H}_2))$. Note that w is a map from the set of all w-admissible elements, which is a subset of $\mathbf{Obj}(\mathbf{H})^{\times k}$, to $\mathbf{Obj}(\mathbf{H})$. We define the *map algebra* to be the union

$$G = \bigcup_{k \geq 1} G^k. \tag{4.1}$$

Let T be a binary tree. Let x_0 be the root of T. We label x_0 with $\cap$, $\cup$, $*$ or $\square$ if $\deg x_0 = 2$ and label x_0 with Δ, δ or γ if $\deg x_0 = 0, 1$. For any vertex x of T with $x \neq x_0$, we label x with $\cap$, $\cup$, $*$ or $\square$ if $\deg x = 3$ and label x with Δ, δ or γ if $\deg x = 1, 2$. Let V_T be the vertex set of T. Consider a map

$$\alpha : \qquad V_T \longrightarrow \{\cap, \cup, *, \square, \Delta, \delta, \gamma\}$$

sending each vertex of T to its label satisfying the above labeling rule. Let $k(T) = 1$ if T is the empty binary tree or T has a single vertex x_0 and let $k(T)$ be the number of vertices $x \neq x_0$ such that $\deg x = 1$ if T has at least two vertices. Then each pair (T, α) represents an element in $G^{k(T)}$. For each nonnegative integer k, each element in G^k has such a representation (T, α), while two pairs (T, α) and (T', α') may represent one element in G^k. The pair $(\emptyset, \alpha)$, where $\emptyset$ is the empty binary tree, represents $\mathrm{id} \in G^1$. A pair (T, α), where T is a binary tree such that $\deg x_0 = 1$ and $\deg x = 1, 2$ for any $x \neq x_0$, represents an element in G^1.

Let $\mathbf{DH}$ be the category such that each object is a triple $(V, \mathcal{H}, \varphi)$, where V is a finite set, $\mathcal{H}$ is a hypergraph on V and $\varphi \in D(\mathbf{H}(V))$ is a probability distribution on $\mathbf{Obj}(\mathbf{H}(V))$, and each morphism from $(V, \mathcal{H}, \varphi)$ to $(V', \mathcal{H}', \varphi')$ is a morphism $f : V \longrightarrow V'$ of hypergraphs from $\mathcal{H}$ to $\mathcal{H}'$ satisfying $(Df)(\varphi) = \varphi'$. Let S be the subset of $\mathbf{Obj}(\mathbf{DH})$ consisting of all the elements $(V, \mathcal{H}, \bar{\mathrm{P}}_p) \in \mathbf{Obj}(\mathbf{DH})$, where $p : \Delta[V] \longrightarrow [0, 1]$. For any $w_1, w_2 \in G^1$, define $D(w_1 w_2) = (Dw_1)(Dw_2)$. For each $w \in G^1$, define

$$w(V, \mathcal{H}, \varphi) = (w(V, \mathcal{H}), (Dw)(\varphi)).$$

Then w is a map from $\mathbf{Obj}(\mathbf{DH})$ to itself. Let $G^1(S)$ be the union of $w(S)$ for all $w \in G_1$, where $w(S)$ is the image of S under w.

Proposition 4.2. *For any $p : \Delta[V] \longrightarrow [0, 1]$, P_p and and Q_p are in $G^1(S)$.*

Proof. By Theorem 1.1 (4) or Theorem 3.5 (3), $\mathrm{P}_p \in G^1(S)$. By Theorem 1.1 (5) or Theorem 3.5 (4), $\mathrm{Q}_p \in G^1(S)$. $\square$

Define $D\square$ to be the map from $\mathbf{Obj}(\mathbf{DH}) \times \mathbf{Obj}(\mathbf{DH})$ to $\mathbf{Obj}(\mathbf{DH})$ by

$$(D\square)((V, \mathcal{H}, \varphi), (V', \mathcal{H}', \varphi')) = (V \times V', \mathcal{H} \square \mathcal{H}', (D\square)(\varphi, \varphi')).$$

Define $D*$ to be the map from the collection of $*$-admissible elements in $\mathbf{Obj}(\mathbf{DH}) \times \mathbf{Obj}(\mathbf{DH})$ to $\mathbf{Obj}(\mathbf{DH})$ by

$$(D*)((V, \mathcal{H}, \varphi), (V', \mathcal{H}', \varphi')) = (V \sqcup V', \mathcal{H} * \mathcal{H}', (D*)(\varphi, \varphi')),$$

where $V \cap V' = \emptyset$. Define $D\cap$ and $D\cup$ to be maps from the collection of $\cap$-admissible (equivalently, $\cup$-admissible) elements in $\mathbf{Obj}(\mathbf{DH}) \times \mathbf{Obj}(\mathbf{DH})$ to $\mathbf{Obj}(\mathbf{DH})$ by

$$\begin{aligned}(D\cap)((V, \mathcal{H}, \varphi), (V, \mathcal{H}', \varphi')) &= (V, \mathcal{H} \cap \mathcal{H}', (D\cap)(\varphi, \varphi')),\\ (D\cup)((V, \mathcal{H}, \varphi), (V, \mathcal{H}', \varphi')) &= (V, \mathcal{H} \cup \mathcal{H}', (D\cup)(\varphi, \varphi')).\end{aligned}$$

Define $D(w_1 \bullet w_2) = (D\bullet)(Dw_1, Dw_2)$ for $\bullet = \cap, \cup, *$ or $\square$. Then each element in G^k is a map from an admissible subset of $\mathbf{Obj}(\mathbf{DH})^{\times k}$ to $\mathbf{Obj}(\mathbf{DH})$. Let

$$\begin{aligned}G^k(S^{\times k}) = \{&w((V_1, \mathcal{H}_1, \bar{\mathrm{P}}_*), \cdots, (V_k, \mathcal{H}_k, \bar{\mathrm{P}}_*)) \mid w \in G^k \text{ and}\\ &((V_1, \mathcal{H}_1, \bar{\mathrm{P}}_*), \cdots, (V_k, \mathcal{H}_k, \bar{\mathrm{P}}_*)) \in S^{\times k} \text{ is } w\text{-admissible}\}.\end{aligned}$$

Take the union

$$G(S) = \bigcup_{k \geq 1} G^k(S^{\times k}). \tag{4.2}$$

Each element in $G(S)$ is of the form $(V, \mathcal{H}, \varphi)$, where V is a finite vertex set, $\mathcal{H}$ is a hypergraph on V and φ is a probability function on $\mathbf{Obj}(\mathbf{H}(V))$. The probability function φ is given by the action of certain Dw, where $w \in G$, on multiple probability functions of the form $\bar{\mathrm{P}}_*$, P_* and Q_*.

4.2 Algorithms for the generations of random hypergraphs and random simplicial complexes

Algorithm generating random hypergraphs. **Step 1**. Choose a positive integer k. Choose k finite sets V_1, V_2, …, V_k as the vertex sets without mutual intersections.

Step 2. For each $i = 1, 2, \ldots, k$, choose a positive integer n_i. Use the Erdös-Rényi-type model $\bar{\mathrm{P}}_p$ to give n_i randomly generated hypergraphs $\mathcal{H}_{1,i}$, $\mathcal{H}_{2,i}$, …, $\mathcal{H}_{n_i,i}$ on V_i.

Step 3. For each $i = 1, 2, \ldots, k$ and each $j = 1, 2, \ldots, n_i$, choose an element $w_{j,i} \in G^1$. Apply $w_{j,i}$ to $\mathcal{H}_{j,i}$ and give a randomly generated hypergraph $w_{j,i}(\mathcal{H}_{j,i})$ on V_i.

Step 4. For each $i = 1, 2, \ldots, k$, **(4.a)**. Choose two randomly generated hypergraphs from $w_{j,i}(\mathcal{H}_{j,i})$, $j = 1, 2, \ldots, n_i$, and apply $\cap$ or $\cup$ to these two randomly generated hypergraphs. **(4.b)**. Choose a randomly generated hypergraph from the remaining $n_i - 2$ randomly generated hypergraphs $w_{j,i}(\mathcal{H}_{j,i})$, $j = 1, 2, \ldots, n_i$, and apply $\cap$ or $\cup$ to this chosen randomly generated hypergraph and the randomly generated hypergraph given in (4.a). **(4.c)**. Repeat (4.b) for $n_i - 2$ times. Denote this final randomly generated hypergraph on V_i as $\mathcal{H}_i$.

Step 5. **(5.a)**. Choose two randomly generated hypergraphs from $\mathcal{H}_i$, $i = 1, 2, \ldots, k$, and apply $\square$ or $*$ to these two randomly generated hypergraphs. **(5.b)**. Choose a randomly generated hypergraph from the remaining $k - 2$ randomly generated hypergraphs $\mathcal{H}_i$, $i = 1, 2, \ldots, k$, and apply $\square$ or $*$ to this chosen randomly generated hypergraph and the randomly generated hypergraph given in (5.a). **(5.c)**. Repeat (5.b) for $k - 2$ times. This gives a randomly generated hypergraph $\mathcal{H}$.

Step 6. Choose $w \in G^1$. Apply w to $\mathcal{H}$. The final randomly generated hypergraph is $w(\mathcal{H})$. $\square$

Some algorithms that generate random simplicial complexes and random independence hypergraphs follow immediately.

Algorithm generating random simplicial complexes. **Step 1**. Same as Step 1 in the algorithm generating random hypergraphs.

Step 2. For each $i = 1, 2, \ldots, k$, choose a positive integer n_i. Use the algorithm generating random hypergraphs to give n_i randomly generated hypergraphs $\mathcal{H}_{1,i}$, $\mathcal{H}_{2,i}$, …, $\mathcal{H}_{n_i,i}$ on V_i.

Step 3. For each $i = 1, 2, \ldots, k$ and each $j = 1, 2, \ldots, n_i$, apply Δ or δ to $\mathcal{H}_{j,i}$. This gives a randomly generated simplicial complex $\mathcal{K}_{j,i}$ on V_i.

Step 4. For each $i = 1, 2, \ldots, k$, similar to Step 4 of the previous algorithm, apply the binary operations $\cap$ and $\cup$ to $\mathcal{K}_{1,i}$, $\mathcal{K}_{2,i}$, …, $\mathcal{K}_{n_i,i}$. After $n_i - 1$ times of the binary operations, we obtain a randomly generated simplicial complex $\mathcal{K}_i$ on V_i.

Step 5. Similar to Step 5 of the previous algorithm, apply the binary operations $*$, $\Delta\square$ and $\delta\square$ to $\mathcal{K}_1$, $\mathcal{K}_2$, …, $\mathcal{K}_k$. After $k - 1$ times of the binary operations, we obtain a randomly generated simplicial complex $\mathcal{K}$. $\square$

Algorithm generating random independence hypergraphs. **Steps 1-2**. Same as Steps 1-2 in the algorithm generating random simplicial complexes.

Step 3. For each $i = 1, 2, \ldots, k$ and each $j = 1, 2, \ldots, n_i$, apply $\bar{\Delta}$ or $\bar{\delta}$ to $\mathcal{H}_{j,i}$ with respect to V_i. This gives a randomly generated independence hypergraph $\mathcal{L}_{j,i}$ on V_i.

Step 4. For each $i = 1, 2, \ldots, k$, similar to Step 4 of the previous algorithms, apply the binary operations $\cap$ and $\cup$ to $\mathcal{L}_{1,i}$, $\mathcal{L}_{2,i}$, $\ldots$, $\mathcal{L}_{n_i,i}$. After $n_i - 1$ times of the binary operations, we obtain a randomly generated independence hypergraph $\mathcal{L}_i$ on V_i.

Step 5. Similar to Step 5 of the previous algorithms, apply the binary operations $*$, $\bar{\Delta}\Box$ and $\bar{\delta}\Box$ to $\mathcal{L}_1$, $\mathcal{L}_2$, $\ldots$, $\mathcal{L}_k$. After $k - 1$ times of the binary operations, we obtain a randomly generated independence hypergraph $\mathcal{L}$. $\Box$

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