Regressive versions of Hindman's Theorem

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Abstract

When the Canonical Ramsey's Theorem by Erdős and Rado is applied to regressive functions, one obtains the Regressive Ramsey's Theorem by Kanamori and McAloon. Taylor proved a "canonical" version of Hindman's Theorem, analogous to the Canonical Ramsey's Theorem. We introduce the restriction of Taylor's Canonical Hindman's Theorem to a subclass of the regressive functions, the λ-regressive functions, relative to an adequate version of min-homogeneity and prove some results about the Reverse Mathematics of this Regressive Hindman's Theorem and of natural restrictions of it.

In particular we prove that the first non-trivial restriction of the principle is equivalent to Arithmetical Comprehension. We furthermore prove that the well-ordering-preservation principle for base-ω exponentiation is reducible to this same principle by a uniform computable reduction.

1 Introduction and motivation

Hindman's well-known Finite Sums Theorem [15] states that for any finite colouring of the natural numbers there exists an infinite subset of positive natural numbers such that all finite sums of distinct terms from that subset get the same colour.

The strength of Hindman's Theorem is a major open problem in Reverse Mathematics (see, e.g., [28]) since the seminal work of Blass, Hirst and Simpson [2]. They showed that Hindman's Theorem is provable in the system ACA_0^+ (axiomatized by closure under the ω-th Turing Jump) and in turn implies ACA_0 (axiomatized by closure under the Turing Jump) over the base system RCA_0. This leaves a huge gap between the upper and the lower bound (we refer to [31, 17] and to the recent [12] for Reverse Mathematics fundamentals).

Recently, substantial interest has been given to various restrictions of Hindman's Theorem (see [6] and [12] Section 9.9.3 for an overview and references). Dzhafarov, Jockusch, Solomon and Westrick [11] proved that the ACA_0 lower bound on Hindman's Theorem already applies to the restriction of the theorem to colourings in 3 colours and sums of at most 3 terms (denoted $\mathsf{HT}^{\leq 3}_3$) and that Hindman's Theorem restricted to colourings in 2 colours and sums of at most 2 terms (denoted $\mathsf{HT}^{\leq 2}_2$) is unprovable in RCA_0. The first author in joint work with Kołodziejczyk, Lepore and Zdanowski later showed that $\mathsf{HT}^{\leq 2}_4$ implies ACA_0 and that $\mathsf{HT}^{\leq 2}_2$ is unprovable in WKL_0 [5]. However, no upper bound other than the one for the full Hindman's Theorem is known for $\mathsf{HT}^{\leq 2}_k$, let alone $\mathsf{HT}^{\leq 3}_k$, for any $k > 1$. Indeed, it is an open question in Combinatorics whether Hindman's Theorem for sums of at most 2 terms is already equivalent to the full

Hindman's Theorem (see [16], Question 12). On the other hand some restrictions of Hindman's Theorem that are equivalent to ACA_0 have been isolated and called "weak yet strong" principles by the first author (see [4]). Theorem 3.3 in [5] shows that Hindman's Theorem restricted to colourings in 2 colours and sums of exactly 3 terms with an apartness condition on the solution set is a weak yet strong principle in this sense.

In this paper we isolate a new natural variant of Hindman's Theorem, called the *Regressive Hindman's Theorem*, modeled on Kanamori-McAloon's Regressive Ramsey's Theorem [25], and we investigate its strength in terms of provability over RCA_0 and in terms of computable reductions. In particular we prove that the weakest non-trivial restriction of the Regressive Hindman's Theorem is a weak yet strong principle in the sense of [4], being equivalent to ACA_0. We also show that the Range Existence Principle for injective functions is reducible to that same Regressive Hindman's Theorem by a uniform computable reduction (called Weihrauch reduction). Moreover, we show that the same is true of the Well-Ordering Preservation Principle for base-ω exponentiation. This principle states that, for any linear order $\mathcal{X}$, if $\mathcal{X}$ is well-ordered then $\omega^{\mathcal{X}}$ is well-ordered. It is known to be equivalent to ACA_0 (see [21]); well-ordering principles have received substantial attention in later years (see the recent survey by Michael Rathjen [30] for an overview and references). No direct connection to Hindman-type theorems has been drawn in previous works.

The theorems studied in this paper are Π^1_2-principles, i.e., principles that can be written in the following form:

$$\forall X (I(X) \rightarrow \exists Y S(X, Y))$$

where $I(X)$ and $S(X,Y)$ are arithmetical formulas and X and Y are set variables. For principles P of this form we call any X that satisfies I an *instance* of P and any Y that satisfies $S(X,Y)$ a *solution to* P *for* X. We will use the following notions of computable reducibility between two Π^1_2-principles P and Q, which have become of central interest in Computability Theory and Reverse Mathematics in recent years (see [12] for background and motivation).

1. Q is *strongly Weihrauch reducible* to P (denoted $\mathsf{Q} \leq_{\mathrm{sW}} \mathsf{P}$) if there exist Turing functionals Φ and Ψ such that for every instance X of Q we have that $\Phi(X)$ is an instance of P, and if $\hat{Y}$ is a solution to P for $\Phi(X)$ then $\Psi(\hat{Y})$ is a solution to Q for X.

2. Q is *Weihrauch reducible* to P (denoted $\mathsf{Q} \leq_{\mathrm{W}} \mathsf{P}$) if there exist Turing functionals Φ and Ψ such that for every instance X of Q we have that $\Phi(X)$ is an instance of P, and if $\hat{Y}$ is a solution to P for $\Phi(X)$ then $\Psi(X \oplus \hat{Y})$ is a solution to Q for X.

3. Q is *computably reducible* to P (denoted $\mathsf{Q} \leq_{\mathrm{c}} \mathsf{P}$) if every instance X of Q computes an instance $\hat{X}$ of P such that if $\hat{Y}$ is any solution to P for $\hat{X}$, then there is a solution Y to Q for X computable from $X \oplus \hat{Y}$.

The above reducibility notions are related by the following strict implications:

$$\leq_{\mathrm{sW}} \Longrightarrow \leq_{\mathrm{W}} \Longrightarrow \leq_{\mathrm{c}},$$

and make it possible to illuminate subtle differences in the intuitive idea of solving a problem Q algorithmically from a problem P. Note that $\mathsf{Q} \leq_{\mathrm{c}} \mathsf{P}$ implies that each ω-model of $\mathsf{RCA}_0 + \mathsf{P}$ is also a model of Q (the latter fact is usually denoted by $\mathsf{Q} \leq_{\omega} \mathsf{P}$). We refer the reader to [12] for examples witnessing how the three reducibility notions differ.

In the present paper we only establish positive reducibility results, indicating when implications of type $\mathsf{P} \rightarrow \mathsf{Q}$ over RCA_0 are witnessed by strongly Weihrauch, Weihrauch or computable reductions. A few non-reducibility results are obtained as simple corollaries of our reducibility results and non-reducibility results from the literature.

2 Canonical and Regressive Ramsey's Theorems

We review some definitions and known facts concerning Ramsey's Theorem and its canonical and regressive versions. We use $\mathbf{N}$ for the set of natural numbers and $\mathbf{N}^+$ for the set of positive integers. For $X \subseteq \mathbf{N}$ and $n \geq 1$ we denote by $[X]^n$ the set of subsets of X of cardinality n. For $k \in \mathbf{N}^+$ we identify k with $\{0, 1, \ldots, k-1\}$. Accordingly, for $S \subseteq \mathbf{N}$, $c : [S]^n \to k$ indicates a colouring of $[S]^n$ in k colours. Intervals are intervals in $\mathbf{N}$. We start by recalling the statement of the standard countable Ramsey's Theorem.

Definition 1 (Ramsey's Theorem)**.** *Let $n, k \in \mathbf{N}^+$. We denote by RT^n_k the following principle. For all $c : [\mathbf{N}]^n \to k$ there exists an infinite set $H \subseteq \mathbf{N}$ such that c is constant on $[H]^n$. The set H is called homogeneous or monochromatic for c. Also, we use RT^n to denote $(\forall k \geq 1)\, \mathsf{RT}^n_k$ and RT to denote $(\forall n \geq 1)\, \mathsf{RT}^n$.*

For $n \in \mathbf{N}^+$, $S \subseteq \{1, \ldots, n\}$, $I = \{i_1 < \cdots < i_n\} \subseteq \mathbf{N}$ and $J = \{j_1 < \cdots < j_n\} \subseteq \mathbf{N}$ we say that I and J *agree* on S if and only if for all $s \in S$, $i_s = j_s$. Note that if S is empty then all n-sized subsets of $\mathbf{N}$ agree on S.

The following generalization of Ramsey's Theorem to colourings in possibly infinitely many colours was established by Erdős and Rado [13].

Definition 2 (Erdős and Rado's Canonical Ramsey's Theorem)**.** *Let $n \in \mathbf{N}^+$. We denote by canRT^n the following principle. For all $c : [\mathbf{N}]^n \to \mathbf{N}$ there exists an infinite set $H \subseteq \mathbf{N}$ and a finite (possibily empty) set $S \subseteq \{1, \ldots, n\}$ such that for all $I, J \in [H]^n$ the equality $c(I) = c(J)$ holds if and only if I and J agree on S. The set H is called* canonical *for c. We use canRT to denote $(\forall n \geq 1)\mathsf{canRT}^n$.*

The Reverse Mathematics of canRT^n is studied in [27], where it is denoted by CAN^n.

As observed in [27] (Proposition 8.5), canRT^1 is equivalent to RT^1 over RCA_0.

Kanamori and McAloon [25] isolated a straightforward corollary of the Canonical Ramsey's Theorem inspired by Fodor's Lemma in Uncountable Combinatorics. To state the Kanamori-McAloon's principle we need the following definitions.

Definition 3 (Regressive function)**.** *Let $n \in \mathbf{N}^+$. A function $c : [\mathbf{N}]^n \to \mathbf{N}$ is called regressive if and only if, for all $I \in [\mathbf{N}]^n$, $c(I) < \min(I)$ if $\min(I) > 0$, else $c(I) = 0$.*

Definition 4 (Min-homogeneity)**.** *Let $n \in \mathbf{N}^+$, $c : [\mathbf{N}]^n \to \mathbf{N}$ and $H \subseteq \mathbf{N}$ an infinite set. The set H is min-homogeneous for c if and only if the following condition holds: for any $I, J \in [H]^n$, if $\min(I) = \min(J)$ then $c(I) = c(J)$.*

Definition 5 (Kanamori-McAloon's Regressive Ramsey's Theorem)**.** *Let $n \in \mathbf{N}^+$. We denote by regRT^n the following principle. For all regressive $c : [\mathbf{N}]^n \to \mathbf{N}$ there exists an infinite min-homogeneous set $H \subseteq \mathbf{N}$. We denote by regRT the principle $(\forall n \geq 1)\mathsf{regRT}^n$.*

The Reverse Mathematics of regRT^n is studied in [27], where it is denoted by REG^n. Note that regRT^1 is trivial. A finite first-order miniaturization of regRT was proved by Kanamori and McAloon [25] to be independent from Peano Arithmetic and is often considered as one of the most mathematically natural examples of statements independent from that system.

The following theorem summarizes the main known results about the Reverse Mathematics of the Canonical and Regressive versions of Ramsey's Theorem.

Theorem 1. *The following are equivalent over RCA_0.*

1. ACA_0.

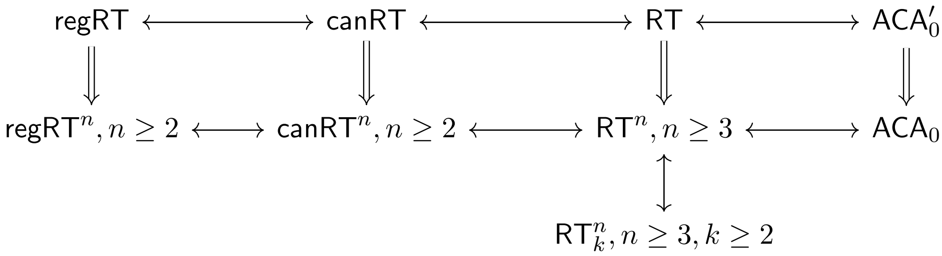

Figure 1: Implications over RCA_0. Double arrows indicate strict implications. The equivalences with ACA_0 are from Theorem 1. For the other implications we refer the reader to [27].

2. canRT^n, *for any fixed* $n \geq 2$.
3. regRT^n, *for any fixed* $n \geq 2$.
4. RT^n_k, *for any fixed* $n \geq 3$ *and* $k \geq 2$.

Proof. The equivalences concerning ACA_0 and Ramsey's Theorems are all due to Simpson (Theorem III.7.6 in [31]), based on the computability-theoretic analysis by Jockusch [24]. The fact that regRT^n implies ACA_0 is due to Hirst, see [20]. That ACA_0 implies canRT^n is due to Mileti, using a new proof of the Canonical Ramsey's Theorem [27]. The implications from canRT^n to RT^n and canRT^n to regRT^n are simple observations. □

Theorem 6.14 in Hirst's Ph.D. Thesis [20] gives an implication (and a strong Weihrauch reduction) from RT^{2n-1}_2 to regRT^n, for all $n \geq 2$.

There seems to be no direct and exponent-preserving proof of RT^n from regRT^n in the literature. A simple proof of this implication is in Proposition 7 below. As pointed out by one of the anonymous reviewers of the present paper, a simple forgetful function argument proves RT^n from regRT^{n+1}.

Also note that Ramsey's Theorem for pairs is strictly between RCA_0 and ACA_0 (see [17] for details). Moreover, the principles canRT, RT and regRT are all equivalent to ACA'_0, the system obtained by adding to RCA_0 the axiom $\forall n \forall X \exists Y (Y = (X)^{(n)})$ stating the closure of the set universe under the n-th Turing Jump for every n; see [27], Proposition 8.4. The main relations among Canonical, Regressive and standard Ramsey's Theorems with respect to implication over RCA_0 are visualized in Figure 1.

3 Canonical and Regressive Hindman's Theorems

We start by recalling Hindman's Finite Sums Theorem [15]. For a set $X \subseteq \mathbf{N}$ we denote by $\mathrm{FS}(X)$ the set of all finite non-empty sums of distinct elements of X.

Definition 6 (Hindman's Theorem)**.** *Let* $k \in \mathbf{N}^+$. *We denote by* HT_k *the following principle. For all* $c : \mathbf{N} \to k$ *there exists an infinite set* $H \subseteq \mathbf{N}$ *such that* c *is constant on* $\mathrm{FS}(H)$. *We denote by* HT *the principle* $(\forall k \geq 1)\, \mathsf{HT}_k$.

For technical convenience, Hindman's Theorem is usually stated with $\mathbf{N}^+$ instead of $\mathbf{N}$. Obviously we can always assume without loss of generality that H in the above definition is a subset of $\mathbf{N}^+$.

Taylor [32] proved the following "canonical" version of Hindman's Theorem, analogous to the Canonical Ramsey's Theorem by Erdős and Rado (Definition 2). We denote by $\mathrm{FIN}(\mathbf{N})$ the set of non-empty finite subsets of $\mathbf{N}$.

Definition 7 (Taylor's Canonical Hindman's Theorem)**.** *We denote by* canHT *the following principle. For all $c : \mathbf{N} \to \mathbf{N}$ there exists an infinite set $H = \{h_0 < h_1 < \cdots\} \subseteq \mathbf{N}$ such that one of the following holds:*

1. *For all $I, J \in \mathrm{FIN}(\mathbf{N})$, $c(\sum_{i\in I} h_i) = c(\sum_{j\in J} h_j)$.*
2. *For all $I, J \in \mathrm{FIN}(\mathbf{N})$, $c(\sum_{i\in I} h_i) = c(\sum_{j\in J} h_j)$ if and only if $I = J$.*
3. *For all $I, J \in \mathrm{FIN}(\mathbf{N})$, $c(\sum_{i\in I} h_i) = c(\sum_{j\in J} h_j)$ if and only if $\min(I) = \min(J)$.*
4. *For all $I, J \in \mathrm{FIN}(\mathbf{N})$, $c(\sum_{i\in I} h_i) = c(\sum_{j\in J} h_j)$ if and only if $\max(I) = \max(J)$.*
5. *For all $I, J \in \mathrm{FIN}(\mathbf{N})$, $c(\sum_{i\in I} h_i) = c(\sum_{j\in J} h_j)$ if and only if $\min(I) = \min(J)$ and $\max(I) = \max(J)$.*

The set H is called canonical *for c.*

None of the cases in Definition 7 can be omitted without falsifying Taylor's Theorem. For technical convenience, canHT is usually stated with $\mathbf{N}^+$ instead of $\mathbf{N}$. We can always assume without loss of generality that H in the above definition is a subset of $\mathbf{N}^+$.

We first observe how Taylor's Theorem implies the standard Hindman's Theorem just as the Canonical Ramsey's Theorem implies Ramsey's Theorem.

Proposition 1. canHT *implies* HT *over* RCA_0*. Moreover,* $\mathsf{canHT} \geq_{\mathrm{sW}} \mathsf{HT}$*.*

Proof. Let $c : \mathbf{N} \to k$ be a finite colouring of $\mathbf{N}$, with $k \in \mathbf{N}^+$. By canHT there exists an infinite set $H \subseteq \mathbf{N}^+$ such that one of the five canonical cases in Definition 7 occurs. It is easy to see that, since c is a colouring in k colours, only case (1) of Definition 7 can occur. Thus $\mathrm{FS}(H)$ is homogeneous for c. The argument obviously establishes a strong Weihrauch reduction. □

In the usual Finite Unions versions of Hindman's Theorem and of Taylor's Theorem the instance is a finite colouring of the finite subsets of $\mathbf{N}$ and the solution is an infinite sequence $(B_i)_{i\in\mathbf{N}}$ of finite subsets of $\mathbf{N}^+$ satisfying the so-called *block condition*: for all $i < j$, $\max(B_i) < \min(B_j)$; henceforth we will write $X < Y$ to indicate that this condition holds for the finite sets X and Y. When this condition is dropped, Hindman's Finite Unions Theorem becomes much weaker (in particular, provable in RCA_0) as shown by Hirst (see [6] for references). We introduce the corresponding property in the finite sums setting. This property is already implicit in Hindman's original proof [15] and was called *apartness* by the first author in [4]. Let $n \in \mathbf{N}^+$. If $n = 2^{t_1} + \cdots + 2^{t_p}$ with $0 \leq t_1 < \cdots < t_p$ let $\lambda(n) = t_1$ and $\mu(n) = t_p$ (the notation is from [2]). We set $\lambda(0) = \mu(0) = 0$.

Definition 8 (Apartness Condition)**.** *A set X satisfies the apartness condition if for all $x, x' \in X$ such that $x < x'$, we have $\mu(x) < \lambda(x')$. If X satisfies the apartness condition we say that X is apart.*

If P is a Hindman-type principle, we denote by P *with apartness* or $\mathsf{P}[\mathsf{ap}]$, the principle P with the apartness condition imposed on the solution set.

In Hindman's original proof the apartness condition is ensured by a simple counting argument (Lemma 2.2 in [14]) on any solution to the Finite Sums Theorem, i.e., an infinite $H \subseteq \mathbf{N}$ such that $\mathrm{FS}(H)$ is monochromatic (Lemma 2.3 in [14]). As noted in [2], the proof shows that a solution satisfying the apartness condition can be obtained computably in any such solution. In the Reverse Mathematics setting, one needs to be slightly more careful to establish that HT implies HT *with apartness* over RCA_0.

We first check that Lemma 2.2 in [14] holds in RCA_0.

Lemma 1. *The following is provable in RCA_0: For all ℓ, for all k, for all finite sets X, if X has cardinality 2^k and is such that $\lambda(x) = \ell$ for all $x \in X$, then there exists $Y \subseteq X$ such that $\lambda(\sum_{y \in Y} y) \geq \ell + k$.*

Proof. The Lemma is established by a straightforward induction on k. We give the details for completeness.

For the base case, let $k = 0$ and let $X = \{x\}$ be a finite set of cardinality 2^0 such that $\lambda(x) = \ell$. Obviously choosing $Y = X$ gives the desired solution.

For the inductive step, let $k \geq 0$ and let X be a set of cardinality 2^{k+1} such that for all $x \in X$ we have $\lambda(x) = \ell$. Let A and B be two disjoint subsets of X each of cardinality 2^k. By inductive hypothesis there exists $A' \subseteq A$ such that $\lambda(\sum_{a \in A'} a) \geq \ell + k$ and there exists $B' \subseteq B$ such that $\lambda(\sum_{b \in B'} b) \geq \ell + k$. We distinguish the following cases. If $\lambda(\sum_{a \in A'} a) = \ell + k$ and $\lambda(\sum_{b \in B'} b) = \ell + k$ then $\lambda(\sum_{c \in A' \cup B'} c) \geq \ell + k + 1$. If either $\lambda(\sum_{a \in A'} a) > \ell + k$ or $\lambda(\sum_{b \in B'} b) > \ell + k$ then we are done.

The argument can be carried out in RCA_0 since quantification over finite sets formally means quantification over their numerical codes and the set Y is a finite subset of the finite set X, so that the existential quantifier over Y is bounded. The induction predicate is then Π^0_1, and Π^0_1-induction holds in RCA_0. □

The following Lemma appears as Lemma 9.9.6 in Dzhafarov and Mummert [12]. As pointed out by one of the reviewers of the present paper, there is an error in the proof in [12] (where it is assumed that the element denoted by x_2 is in $\mathrm{FS}(I)$). We give an alternative argument, using Lemma 1.

Lemma 2. *The following is provable in $\mathsf{RCA}_0 + \mathsf{RT}^1$: For every $m \in \mathbf{N}$ and every infinite $I \subseteq \mathbf{N}$, there exists $x \in \mathrm{FS}(I)$ with $\lambda(x) \geq m$.*

Proof. Fix m and I and suppose that every $x \in \mathrm{FS}(I)$ satisfies $\lambda(x) < m$. In particular this implies that every $x \in I$ satisfies $\lambda(x) < m$, since $I \subseteq \mathrm{FS}(I)$. By RT^1 there exists an $\ell < m$ and an infinite set $J \subseteq I$ such that $\lambda(x) = \ell$ for all $x \in J$.

Since $\ell < m$ there exists k such that $\ell + k = m$. Pick a subset $X \subseteq J$ of cardinality 2^k. Then by Lemma 1 there exists a $Y \subseteq X$ such that $\lambda(\sum_{y \in Y} y) \geq \ell + k = m$. This contradicts the hypothesis that $\lambda(x) < m$ for all $x \in \mathrm{FS}(I)$.

□

As a corollary one obtains the following Proposition, which will be used to show that HT self-strengthens to $\mathsf{HT}[\mathsf{ap}]$ over RCA_0.

Proposition 2.

1. *The following is provable in $\mathsf{RCA}_0 + \mathsf{RT}^1$: For every infinite set $I \subseteq \mathbf{N}$, there is an infinite set J such that J is apart and $\mathrm{FS}(J) \subseteq \mathrm{FS}(I)$.*

2. *For all infinite set $I \subseteq$ of natural numbers there exists an infinite set J of natural numbers computable in I such that J is apart and $\mathrm{FS}(J) \subseteq \mathrm{FS}(I)$.*

Proof. Define a sequence of elements $x_0 < x_1 < \cdots$ in $\mathrm{FS}(I)$ recursively as follows. Let $x_0 = \min(I)$. Given x_i for some $i \in \mathbf{N}$, let x_{i+1} be the least element of $\mathrm{FS}(I \setminus [0, x_i])$ such that $\lambda(x_{i+1}) > \mu(x_i)$. The existence of x_{i+1} follows from Lemma 2. Let $J = \{x_i : i \in \mathbf{N}\}$. By construction J is apart and $\mathrm{FS}(J) \subseteq \mathrm{FS}(I)$. □

Proposition 2 is close in both statement and proof to Corollary 9.9.8 in [12] but ensures $\mathrm{FS}(J) \subseteq \mathrm{FS}(I)$ rather than $J \subseteq \mathrm{FS}(I)$ as in [12]. This stronger condition is indeed needed in the proof of the following corollary, which appears as Theorem 9.9.9 in [12]. The proof of the latter contains an error when it is claimed that $J \subseteq \mathrm{FS}(I)$ implies $\mathrm{FS}(J) \subseteq \mathrm{FS}(I)$.

Corollary 1. HT *implies* $\mathsf{HT}[\mathsf{ap}]$ *over* RCA_0. *Moreover* $\mathsf{HT} \geq_{\mathrm{sW}} \mathsf{HT}[\mathsf{ap}]$.

Proof. From Proposition 2 and the fact that HT trivially implies RT^1 over RCA_0. Let $c : \mathbf{N} \to k$. Let I be a solution to HT for c. By Proposition 2 there exists an infinite J such that $\mathrm{FS}(J) \subseteq \mathrm{FS}(I)$ and J is apart.

It is clear from the proof of Proposition 2 that there is a Turing functional that computes J from I uniformly. This is sufficient to establish the claimed strong Weihrauch reduction. □

It is natural to ask whether Taylor's Theorem satisfies a similar self-strengthening with respect to the apartness condition. A positive answer is expected by considering the finite unions version of the theorem. Yet to establish the result in RCA_0 the situation has to be analyzed more closely as we have done above for Hindman's Theorem. As observed by one of the reviewers of the present paper, the above argument does not immediately apply to the case of Taylor's Theorem. Indeed, what the min-term (or max-term) of a number is depends on whether that number is seen as a sum of elements of I or as a sum of elements of J, in the notation of Proposition 2 above. Nevertheless Taylor's Theorem *does* imply its own self-strenghtening with apartness, as we next prove.

Theorem 2. canHT *implies* $\mathsf{canHT}[\mathsf{ap}]$ *over* RCA_0. *Moreover,* $\mathsf{canHT} \geq_{\mathrm{sW}} \mathsf{canHT}[\mathsf{ap}]$.

Proof. Given $c : \mathbf{N} \to \mathbf{N}$, let $H = \{h_0 < h_1 < \cdots\}$ be a solution to canHT for c. Let $H' = \{h'_1 < h'_2 < \cdots\}$ be an infinite apart set such that $\mathrm{FS}(H') \subseteq \mathrm{FS}(H)$ (defined as the set J in the proof of Proposition 2.

For each $i \in \mathbf{N}$, let $A_i \in \mathrm{FIN}(\mathbf{N})$ be such that $\sum_{a \in A_i} h_a = h'_i$ and $h_{\min(A_i)} > h'_{i-1}$ if $i > 0$. A non-empty set with these properties exists by definition of H'. We fix a uniform computable method to select A_i if more than one choice exists (for instance, we take the set A that satisfies the conditions above and that minimizes $\sum_{a \in A} 2^a$). Then, we can state the following three Claims.

Claim 1. *For any set of indexes* $I = \{i_0 < i_1 < \cdots < i_m\} \in \mathrm{FIN}(\mathbf{N})$*, the following properties hold:*

(i) $A_{i_0} < A_{i_1} < \cdots < A_{i_m}$.

(ii) $\min(\bigcup_{i \in I} A_i) = \min(A_{i_0})$.

(iii) $\max(\bigcup_{i \in I} A_i) = \max(A_{i_m})$.

(iv) $\sum_{i \in I} h'_i = \sum_{s \in \bigcup_{i \in I} A_i} h_s$.

Proof. (i) derives from the fact that, for any $s \in (0, m]$, $h_{\min(A_{i_s})} > h'_{i_s - 1} \geq h'_{i_{s-1}} \geq h_{\max(A_{i_{s-1}})}$, which implies $\min(A_{i_s}) > \max(A_{i_{s-1}})$ because H is enumerated in increasing order.

(ii), (iii), and (iv) are trivial consequences of (i). □

Claim 2. *For any* $I = \{i_0 < i_1 < \cdots < i_m\} \in \mathrm{FIN}(\mathbf{N})$ *and* $J = \{j_0 < j_1 < \cdots < j_n\} \in \mathrm{FIN}(\mathbf{N})$, $\min(I) = \min(J)$ *if and only if* $\min(\bigcup_{i \in I} A_i) = \min(\bigcup_{j \in J} A_j)$.

Proof. ($\Longrightarrow$) By hypothesis, $i_0 = j_0$, hence $A_{i_0} = A_{j_0}$ and $\min(A_{i_0}) = \min(A_{j_0})$. Then, by Claim 1.(ii), $\min(\bigcup_{i \in I} A_i) = \min(\bigcup_{j \in J} A_j)$.

($\Longleftarrow$) By hypothesis, $\min(\bigcup_{i \in I} A_i) = \min(\bigcup_{j \in J} A_j)$ so, by Claim 1.(ii), $\min(A_{i_0}) = \min(A_{j_0})$ and then $h_{\min(A_{i_0})} = h_{\min(A_{j_0})}$. Thus, we can show that $i_0 = j_0$, i.e., $\min(I) = \min(J)$. Assume otherwise, and suppose $i_0 < j_0$ (the case $i_0 > j_0$ is analogous). By definition of A_{j_0}, we can derive $h_{\min(A_{j_0})} > h'_{j_0 - 1} \geq h'_{i_0} \geq h_{\min(A_{i_0})}$, hence contradicting $h_{\min(A_{i_0})} = h_{\min(A_{j_0})}$. □

Claim 3. *For any* $I = \{i_0 < i_1 < \cdots < i_m\} \in \mathrm{FIN}(\mathbf{N})$ *and* $J = \{j_0 < j_1 < \cdots < j_n\} \in \mathrm{FIN}(\mathbf{N})$, $\max(I) = \max(J)$ *if and only if* $\max(\bigcup_{i \in I} A_i) = \max(\bigcup_{j \in J} A_j)$.

Proof. ($\Longrightarrow$) By hypothesis, $i_m = j_n$, hence $A_{i_m} = A_{j_n}$ and $\max(A_{i_m}) = \max(A_{j_n})$. Then, by Claim 1.(iii), $\max(\bigcup_{i \in I} A_i) = \max(\bigcup_{j \in J} A_j)$.

($\Longleftarrow$) By hypothesis, $\max(\bigcup_{i \in I} A_i) = \max(\bigcup_{j \in J} A_j)$ so, by Claim 1.(iii), $\max(A_{i_m}) = \max(A_{j_n})$ and then $h_{\max(A_{i_m})} = h_{\max(A_{j_n})}$. Thus, we can show that $i_m = j_n$, i.e., $\max(I) = \max(J)$. Assume otherwise, and suppose $i_m < j_n$ (the case $i_m > j_n$ is analogous). By definition of A_{j_n}, we can derive $h_{\max(A_{j_n})} \geq h_{\min(A_{j_n})} > h'_{j_n - 1} \geq h'_{i_m} \geq h_{\max(A_{i_m})}$, hence contradicting $h_{\max(A_{i_m})} = h_{\max(A_{j_n})}$. □

Now we can show that H' is a solution to canHT for c by analyzing each case of Definition 7.

Case 1. For any $I, J \in \mathrm{FIN}(\mathbf{N})$, by homogeneity of H and by Claim 1.(iv), $c(\sum_{i \in I} h'_i) = c(\sum_{s \in \bigcup_{i \in I} A_i} h_s) = c(\sum_{t \in \bigcup_{j \in J} A_j} h_t) = c(\sum_{j \in J} h'_j)$.

Case 2. Let $I, J \in \mathrm{FIN}(\mathbf{N})$. If $I = J$, then $c(\sum_{i \in I} h'_i) = c(\sum_{j \in J} h'_j)$. Now assume $I \neq J$, as witnessed by $w \in I \setminus J$ (the case $w \in J \setminus I$ is analogous). By Claim 1.(i) applied to $J \cup \{w\}$, we have that $A_w \cap A_j = \emptyset$ for all $j \in J$, therefore $\bigcup_{i \in I} A_i \neq \bigcup_{j \in J} A_j$.

Then, $c(\sum_{i \in I} h'_i) = c(\sum_{s \in \bigcup_{i \in I} A_i} h_s) \neq c(\sum_{t \in \bigcup_{j \in J} A_j} h_t) = c(\sum_{j \in J} h'_j)$, where the two equalities hold by Claim 1.(iv), while the inequality holds by Case 2 of Definition 7, since c is applied to sums of different elements in H on the two sides of the equality, as we noted above.

Case 3. Let $I, J \in \mathrm{FIN}(\mathbf{N})$. If $\min(I) = \min(J)$, then $c(\sum_{i \in I} h'_i) = c(\sum_{s \in \bigcup_{i \in I} A_i} h_s) = c(\sum_{t \in \bigcup_{j \in J} A_j} h_t) = c(\sum_{j \in J} h'_j)$, where the first and the last equality hold by Claim 1.(iv), while the second equality holds by Case 3 of Definition 7, since in both sides of the equality, c is applied to sums of elements in H having the same minimum term by Claim 2. Similarly, if $\min(I') \neq \min(J')$, we have $c(\sum_{i \in I} h'_i) = c(\sum_{s \in \bigcup_{i \in I} A_i} h_s) \neq c(\sum_{t \in \bigcup_{j \in J} A_j} h_t) = c(\sum_{j \in J} h'_j)$.

Case 4. The proof is similar to the proof of Case 3, but using Claim 3 in place of Claim 2.

Case 5. The proof is analogous to the proof of Cases 3 and 4.

□

As observed in [25], when the Canonical Ramsey's Theorem is applied to regressive functions the Regressive Ramsey's Theorem is obtained. Similarly, a regressive version of Hindman's Theorem follows from Taylor's Theorem. We introduce the suitable versions of the notions of regressive function and min-homogeneous set.

Definition 9 (λ-regressive function)**.** *A function* $c : \mathbf{N} \to \mathbf{N}$ *is called* λ*-regressive if and only if, for all* $n \in \mathbf{N}$, $c(n) < \lambda(n)$ *if* $\lambda(n) > 0$ *and* $c(n) = 0$ *if* $\lambda(n) = 0$.

Obviously every λ-regressive function is regressive since $\lambda(n) < n$ for $n \in \mathbf{N}^+$.

Definition 10 (Min-term-homogeneity for FS)**.** *Let $c : \mathbf{N} \to \mathbf{N}$ and $H = \{h_0 < h_1 < \cdots\} \subseteq \mathbf{N}$. We call $\mathrm{FS}(H)$ min-term-homogeneous for c if and only if, for all $I, J \in \mathrm{FIN}(\mathbf{N})$, if $\min(I) = \min(J)$ then $c(\sum_{i\in I} h_i) = c(\sum_{j\in J} h_j)$.*

The following is an analogue of Kanamori-McAloon's Regressive Ramsey's Theorem in the spirit of Hindman's Theorem.

Definition 11 (Regressive Hindman's Theorem)**.** *We denote by $\lambda\mathsf{regHT}$ the following principle. For all λ-regressive $c : \mathbf{N} \to \mathbf{N}$ there exists an infinite $H \subseteq \mathbf{N}$ such that $\mathrm{FS}(H)$ is min-term-homogeneous.*

For technical convenience we will always assume that $H \subseteq \mathbf{N}^+$. In this paper we do not investigate optimal upper bounds on canHT and $\lambda\mathsf{regHT}$.

We start by observing how Taylor's Theorem implies the Regressive Hindman's Theorem just as the Canonical Ramsey's Theorem implies the Kanamori-McAloon Regressive Ramsey's Theorem.

Proposition 3. canHT *implies* $\lambda\mathsf{regHT}$ *over* RCA_0*. Moreover,* $\mathsf{canHT} \geq_{\mathrm{sW}} \lambda\mathsf{regHT}$*.*

Proof. Let $c : \mathbf{N} \to \mathbf{N}$ be a λ-regressive function. By canHT there exists an infinite set $H \subseteq \mathbf{N}^+$ such that one of the five canonical cases occurs for $\mathrm{FS}(H)$. It is easy to see that, since c is λ-regressive, only case (1) and case (3) of Definition 7 can occur. Thus $\mathrm{FS}(H)$ is min-term-homogeneous for c. □

Similarly to Hindman's Theorem and Taylor's Theorem, the Regressive Hindman's Theorem self-improves to its own version with apartness, as shown below. We first show that $\lambda\mathsf{regHT}$ implies the Infinite Pigeonhole Principle.

Lemma 3. $\lambda\mathsf{regHT}$ *implies* RT^1 *over* RCA_0*.*

Proof. Given $f : \mathbf{N} \to k$, with $k \geq 1$, let $g : \mathbf{N} \to \mathbf{N}$ be defined as follows:

$$g(n) = \begin{cases} \lambda'(n) & \text{if } \lambda'(n) < k, \\ f(n) & \text{otherwise,} \end{cases}$$

where $\lambda'(n) = \lambda(n) - 1$ if $\lambda(n) > 0$, otherwise $\lambda'(n) = 0$.

Clearly, g is f-computable and λ-regressive, so let $H = \{h_0 < h_1 < \cdots\}$ be a solution to $\lambda\mathsf{regHT}$ for g. First, we prove the following Claim.

Claim. *There exists an infinite $H' = \{h'_0 < h'_1 < \cdots\} \subseteq H$ such that $\lambda'(h'_{n_1}+h'_{n_2}+h'_{n_3}+h'_{n_4}) \geq k$ for all $n_1 < n_2 < n_3 < n_4$.*

Proof. Let us define $J = \{j \in H \mid \lambda'(j) < k\}$. If J contains finitely many elements, then $(H \setminus J)$ witnesses the existence of H'. Thus, let us assume $J = \{j_0 < j_1 < \cdots\}$ is infinite.

Notice that the sequence $\lambda'(j_0), \lambda'(j_1), \ldots$ never decreases: suppose otherwise by way of contradiction, and let $j, j' \in J$ be such that $j < j'$ and $\lambda'(j) > \lambda'(j')$. Then we have $g(j) = \lambda'(j) > \lambda'(j') = \lambda'(j + j') = g(j + j')$; this contradicts the min-term-homogeneity of $\mathrm{FS}(H)$. Hence λ' on J is a bounded non-decreasing function on an infinite set.

Then we have two cases. Either for any $j \in J$ there exists $j' > j$ in J such that $\lambda'(j') > \lambda'(j)$, or there exists $j \in J$ such that, for any $j' > j$ in J, $\lambda'(j) \geq \lambda'(j')$. The former case can not hold, since by definition of J, $\lambda'(j) < k$ for any $j \in J$.

In the latter case, instead, we have some $m \in J$ such that $\lambda'(m) \geq \lambda'(j)$ for any j in J. Since $\lambda'(j_0), \lambda'(j_1), \dots$ is non-decreasing, $\lambda'(j) = \lambda'(m)$ holds for each j in the infinite set $J' = J \setminus [0, m)$. Finally, we can show that J' witnesses the existence of H'. Assume otherwise by way of contradiction. Then, there exist $j, j', j'', j''' \in J'$ such that $j < j' < j'' < j'''$ and $\lambda'(j + j' + j'' + j''') < k$. Thus $g(j + j' + j'' + j''') = \lambda'(j + j' + j'' + j''')$ by definition of g. On the other hand, since $j \in J' \subseteq J$, $\lambda'(j) < k$ and therefore $g(j) = \lambda'(j)$ by definition of g. Moreover, $\lambda'(j) = \lambda'(j') = \lambda'(j'') = \lambda'(j''')$ since $j, j', j'', j''' \in J'$. Therefore we have the following inequality

$$g(j + j' + j'' + j''') = \lambda'(j + j' + j'' + j''') > \lambda'(j) = g(j),$$

contradicting the min-term-homogeneity of $\mathrm{FS}(H)$. This completes the proof of the Claim. Notice that, while $\lambda(x) = \lambda(y)$ implies $\lambda(x+y) > \lambda(x)$ for any $x, y \in \mathbf{N}^+$, the same implication does not hold when using λ': hence, sums of 4 elements are required in the argument above.

In order to prove the lemma, let $H' = \{h'_0 < h'_1 < \cdots\}$ be as in the previous Claim. Then, for any $n_0 < n_1 < n_2$ in $\mathbf{N}^+$, we have

$$\begin{aligned} f(h'_0 + h'_{n_0} + h'_{n_1} + h'_{n_2}) &= g(h'_0 + h'_{n_0} + h'_{n_1} + h'_{n_2}) \\ &= g(h'_0 + h'_1 + h'_2 + h'_3) \\ &= f(h'_0 + h'_1 + h'_2 + h'_3), \end{aligned}$$

where the first and the last equalities hold by the previous Claim and by definition of g, while the second equality holds by min-term-homogeneity of $\mathrm{FS}(H)$.

Hence $\{(h'_0 + h'_{n_0} + h'_{n_1} + h'_{n_2}) \mid 0 < n_1 < n_2 < n_3\}$ is an infinite homogeneous set for f. □

Proposition 4. $\lambda\mathsf{regHT}$ *implies* $\lambda\mathsf{regHT}[\mathsf{ap}]$ *over* RCA_0*. Moreover,* $\lambda\mathsf{regHT} \geq_{\mathrm{sW}} \lambda\mathsf{regHT}[\mathsf{ap}]$.

Proof. The proof of Theorem 2 adapts *verbatim* to the case of $\lambda\mathsf{regHT}$. Lemma 3 takes care of the use of RT^1 in that proof, which is only needed for the implication over RCA_0. □

It is easy to see that the proof of Lemma 3 uses only sums of at most 4 terms. However, this does not help in extending the previous Proposition to some restriction of $\lambda\mathsf{regHT}$ (see section 4), since the proof of Theorem 2 still requires sums of arbitrary length.

The following proposition shows that the Regressive Hindman's Theorem implies Hindman's Theorem.

Proposition 5. $\lambda\mathsf{regHT}$ *implies* HT *over* RCA_0.

Proof. Given $f : \mathbf{N} \to k$, with $k \geq 1$, and let $g : \mathbf{N} \to k$ be as follows:

$$g(n) = \begin{cases} f(n) & \text{if } f(n) < \lambda(n), \\ 0 & \text{otherwise.} \end{cases}$$

The function g is λ-regressive by construction and obviously f-computable. Let $H = \{h_0 < h_1 < \cdots\} \subseteq \mathbf{N}^+$ be an infinite set such that $FS(H)$ is min-term-homogeneous for g. By Proposition 4 we can assume that H is apart. Let i be the minimum such that $\lambda(h_i) > k$. Let $H^- = H \setminus \{h_0, \dots, h_i\}$. By choice of H^-, g behaves like f on $\mathrm{FS}(H^-)$. Let g^- be the k-colouring of numbers induced by g on H^-.

By RT^1_k (which we can assume by Lemma 3) let $H' = \{h'_0 < h'_1 < \cdots\}$ be an infinite subset of H^- homogeneous for g^-. Then, for $\{s_1, \ldots, s_m\}$ and $\{t_1, \ldots, t_n\}$ non-empty subsets of H', we have

$$\begin{aligned} f(s_1 + \cdots + s_m) &= g(s_1 + \cdots + s_m) \\ &= g(s_1) = g^-(s_1) \\ &= g^-(t_1) = g(t_1) \\ &= g(t_1 + \cdots + t_n) \\ &= f(t_1 + \cdots + t_n), \end{aligned}$$

since $\mathrm{FS}(H^-)$ is min-term-homogeneous for g and g coincides with f on $\mathrm{FS}(H^-)$. □

We do not know if the implication in Proposition 5 can be reversed. In the next section we will observe that RT^1_k can be Weihrauch-reduced to some restriction of $\lambda\mathsf{regHT}$ with apartness – hence, a fortiori, it can be Weihrauch-reduced to $\lambda\mathsf{regHT}$ (see Proposition 6 *infra*).

4 Restrictions of the Regressive Hindman's Theorem

Restrictions of Hindman's Theorem relaxing the monochromaticity requirement to particular families of finite sums received substantial attention in recent years (see [6] for an overview and bibliography). Two natural families of restrictions of Hindman's Theorem are obtained by restricting the number of terms in the monochromatic sums. We introduce the needed terminology. For $X \subseteq \mathbf{N}$ and $n \in \mathbf{N}^+$ we denote by $\mathrm{FS}^{\leq n}(X)$ the set of all non-empty sums of at most n distinct elements of X; we denote by $\mathrm{FS}^{=n}(X)$ the set of all sums of exactly n distinct elements of X.

Definition 12 (Bounded Hindman's Theorems)**.** *Let $n, k \in \mathbf{N}^+$. We denote by $\mathsf{HT}^{\leq n}_k$ (resp. $\mathsf{HT}^{=n}_k$) the following principle. For every $c : \mathbf{N} \to k$ there exists an infinite set $H \subseteq \mathbf{N}$ such that $\mathrm{FS}^{\leq n}(H)$ (resp. $\mathrm{FS}^{=n}(H)$) is monochromatic for c.*
We use $\mathsf{HT}^{\leq n}$ (resp. $\mathsf{HT}^{=n}$) to denote $(\forall k \geq 1)\,\mathsf{HT}^{\leq n}_k$ (resp. $(\forall k \geq 1)\,\mathsf{HT}^{=n}_k$).

Note that $\mathsf{HT}^{\leq 1}_k$, $\mathsf{HT}^{=1}_k$ and RT^1_k are all equivalent and strongly Weihrauch inter-reducible (by identity).

To formulate analogous restrictions of $\lambda\mathsf{regHT}$ we extend the definition of min-term-homogeneity in the natural way. For $n \geq 1$, we denote by $\mathrm{FIN}^{\leq n}(\mathbf{N})$ (resp. $\mathrm{FIN}^{=n}(\mathbf{N})$) the set of all non-empty subsets of $\mathbf{N}$ of cardinality at most n (resp. of cardinality n).

Definition 13 (Min-term-homogeneity for $\mathrm{FS}^{\leq n}, \mathrm{FS}^{=n}$)**.** *Let $n \in \mathbf{N}^+$. Let $c : \mathbf{N} \to \mathbf{N}$ be a colouring and $H = \{h_0 < h_1 < \cdots\}$ an infinite subset of $\mathbf{N}$. We call $\mathrm{FS}^{\leq n}(H)$ (resp. $\mathrm{FS}^{=n}(H)$) min-term-homogeneous for c if and only if, for all $I, J \in \mathrm{FIN}^{\leq n}(\mathbf{N})$ (resp. $I, J \in \mathrm{FIN}^{=n}(\mathbf{N})$), if $\min(I) = \min(J)$ then $c(\sum_{i \in I} h_i) = c(\sum_{j \in J} h_j)$.*

We can then formulate the natural restrictions of the Regressive Hindman's Theorem obtained by relaxing the min-term-homogeneity requirement from $\mathrm{FS}(H)$ to $\mathrm{FS}^{\leq n}(H)$ or $\mathrm{FS}^{=n}(H)$. For example, $\lambda\mathsf{regHT}^{\leq n}$ is defined as $\lambda\mathsf{regHT}$ with $\mathrm{FS}^{\leq n}(H)$ replacing $\mathrm{FS}(H)$.

Definition 14 (Bounded λ-Regressive Hindman's Theorems)**.** *Let $n \in \mathbf{N}^+$. We denote by $\lambda\mathsf{regHT}^{\leq n}$ (resp. $\lambda\mathsf{regHT}^{=n}$) the following principle. For all λ-regressive $c : \mathbf{N} \to \mathbf{N}$ there exists an infinite $H \subseteq \mathbf{N}$ such that $\mathrm{FS}^{\leq n}(H)$ (resp. $\mathrm{FS}^{=n}$) is min-term-homogeneous for c.*

Note that $\lambda\mathsf{regHT}^{\leq 1}$ and $\lambda\mathsf{regHT}^{=1}$ are trivial. We also point out the following obvious relations: $\lambda\mathsf{regHT}$ yields $\lambda\mathsf{regHT}^{\leq n}$ which yields $\lambda\mathsf{regHT}^{=n}$ for all n (both in RCA_0 and by strong Weihrauch reductions) and similarly for the versions with the apartness condition. Also, for $m > n$, $\lambda\mathsf{regHT}^{\leq m}$ obviously yields $\lambda\mathsf{regHT}^{\leq n}$, while $\lambda\mathsf{regHT}^{=m}$ yields $\lambda\mathsf{regHT}^{=n}$ if m is a multiple of n (see the analogous results for Hindman's Theorem for sums of exactly n terms in [5], Proposition 3.5).

4.1 Bounded regressive Hindman's Theorems and Ramsey-type principles

We compare the bounded versions of our regressive Hindman's Theorem with other prominent Ramsey-type and Hindman-type principles.

We start with the following simple Lemma showing that, for every $n \geq 2$, $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}]$ implies RT^1. Note that in Lemma 3 we established that $\lambda\mathsf{regHT}$ *without apartness* implies RT^1 and we later used this result to show that $\lambda\mathsf{regHT}$ implies $\lambda\mathsf{regHT}[\mathsf{ap}]$ (Proposition 4).

Lemma 4. *Let $n \geq 2$. Over RCA_0, $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}]$ implies RT^1. Moreover, for any $k \in \mathbf{N}^+$, we have $\mathsf{RT}^1_k \leq_{\mathrm{sW}} \lambda\mathsf{regHT}^{=n}[\mathsf{ap}]$.*

Proof. We give the proof for $n = 2$ for ease of readability. Let $f : \mathbf{N} \to k$ be given, with $k \geq 1$. Define $g : \mathbf{N} \to k$ as follows.

$$g(m) = \begin{cases} 0 & \text{if } \lambda(m) \leq k, \\ f(\mu(m)) & \text{otherwise.} \end{cases}$$

Clearly g is λ-regressive and f-computable in a uniform way. Let $H = \{h_0 < h_1 < \cdots\} \subseteq \mathbf{N}^+$ be an infinite apart set of positive integers such that $\mathrm{FS}^{=2}(H)$ is min-term-homogeneous for g.

By the apartness condition, for all $h \in H \setminus \{h_0, h_1, \ldots, h_k\}$ we have $g(h) = f(\mu(h))$. Then it is easy to see that $M = \{\mu(h_{k+2}), \mu(h_{k+3}), \ldots\}$ is an infinite f-homogeneous set of colour $f(\mu(h_{k+2}))$ since, for any i, $f(\mu(h_{k+2+i})) = g(h_{k+1} + h_{k+2+i}) = g(h_{k+1} + h_{k+2}) = f(\mu(h_{k+2}))$. ☐

The following proposition relates the principles $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}]$ (respectively $\lambda\mathsf{regHT}^{\leq n}[\mathsf{ap}]$) with the principles $\mathsf{HT}^{=n}_k[\mathsf{ap}]$ (respectively $\mathsf{HT}^{\leq n}_k[\mathsf{ap}]$). The proof is essentially the same as the proof of Proposition 5.

Proposition 6. *Let $n \geq 2$.*

1. *$\lambda\mathsf{regHT}^{=n}[\mathsf{ap}]$ implies $\mathsf{HT}^{=n}[\mathsf{ap}]$ over RCA_0. Moreover, for any $k \in \mathbf{N}^+$, $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}] \geq_{\mathrm{c}} \mathsf{HT}^{=n}_k[\mathsf{ap}]$.*

2. *$\lambda\mathsf{regHT}^{\leq n}[\mathsf{ap}]$ implies $\mathsf{HT}^{\leq n}[\mathsf{ap}]$ over RCA_0. Moreover, for any $k \in \mathbf{N}^+$, $\lambda\mathsf{regHT}^{\leq n}[\mathsf{ap}] \geq_{\mathrm{c}} \mathsf{HT}^{\leq n}_k[\mathsf{ap}]$.*

Proof. We prove the second point, the proof of the first point being completely analogous. Given $f : \mathbf{N} \to k$, with $k \in \mathbf{N}^+$, let $g : \mathbf{N} \to k$ be as follows:

$$g(m) = \begin{cases} f(m) & \text{if } f(m) < \lambda(m), \\ 0 & \text{otherwise.} \end{cases}$$

The function g is λ-regressive and f-computable. By $\lambda\mathsf{regHT}^{\leq n}[\mathsf{ap}]$ let $H = \{h_0 < h_1 < \cdots\} \subseteq \mathbf{N}^+$ be an infinite apart set such that $\mathrm{FS}^{\leq n}(H)$ is min-term-homogeneous for g. Let $g' : H \setminus \{h_0, \ldots, h_{k-1}\} \to k$ be defined as $g'(h_i) = g(h_i + h_{i+1} + \cdots + h_{i+n-1})$.

By RT^1_k, let $H' \subseteq H$ be an infinite homogeneous set for g'. For the sake of establishing the implication over RCA_0, recall that RT^1 follows from $\lambda\mathsf{regHT}^{=2}[\mathsf{ap}]$ by Lemma 4 and therefore also from $\lambda\mathsf{regHT}^{\leq n}[\mathsf{ap}]$ for any $n \geq 2$. For the sake of the computable reduction result, just notice that for each fixed $k \in \mathbf{N}^+$, RT^1_k is computably true. Then, for $\{s_1, \dots, s_p\}$ and $\{t_1, \dots, t_q\}$ non-empty subsets of H', with $p, q \leq n$ and $s_1 < \dots < s_p$, $t_1 < \dots < t_q$, we have

$$\begin{aligned} f(s_1 + \dots + s_p) &= g(s_1 + \dots + s_p) \\ &\overset{(*)}{=} g(s_1) = g'(s_1) \\ &= g'(t_1) = g(t_1) \\ &\overset{(**)}{=} g(t_1 + \dots + t_q) \\ &= f(t_1 + \dots + t_q), \end{aligned}$$

where the equalities dubbed by $(*)$ and $(**)$ hold by the min-term-homogeneity of $\mathrm{FS}^{\leq n}(H)$ for g. This shows that H' is an apart solution to $\mathsf{HT}^{\leq n}_k$ for f. $\square$

Remark 1. *The previous proof gives us a hint as how to extend the reduction to* $\mathsf{HT}^{\leq n}[\mathsf{ap}]$*, i.e. to the universally-quantified principles* $(\forall k \geq 1)$ $\mathsf{HT}^{\leq n}_k[\mathsf{ap}]$*. In that case, the number of colours is not given as part of the instance, and it cannot be computably inferred from the instance* X *of the principle* $\mathsf{HT}^{\leq n}[\mathsf{ap}]$ *(see the discussion in [12] p. 54 for more details on this issue). Nevertheless, we can easily obtain a computable reduction by just observing that the proof of Proposition 6 provides us, for any* $k \geq 1$*, with both an* X*-computable procedure giving us an instance* $\widehat{X}$ *of* $\lambda\mathsf{regHT}^{\leq n}[\mathsf{ap}]$*, and an* $(X \oplus \widehat{Y})$*-computable procedure transforming a solution* $\widehat{Y}$ *for* $\widehat{X}$ *to a solution for* X*: so, even if we do not know the actual value of* k*, we know that the two procedures witnessing the computable reduction do exist. Thus, we can conclude that for any* $n \geq 2$*,* $\lambda\mathsf{regHT}^{\leq n}[\mathsf{ap}] \geq_{\mathrm{c}} \mathsf{HT}^{\leq n}[\mathsf{ap}]$*. It is not straightforward to improve this result to a Weihrauch reduction.*

The same argument also applies to the case of $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}]$*, so that we have that for any* $n \geq 2$*,* $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}] \geq_{\mathrm{c}} \mathsf{HT}^{=n}[\mathsf{ap}]$*.*

Also, we point out that a proof of $\lambda\mathsf{regHT}^{\leq 2}$ that does not also prove HT (or, more technically, a separation over RCA_0 of these two principles) would answer Question 12 from [16].

It is worth noticing that a further slight adaptation of the proof of Proposition 6 gives a direct proof of RT^n from regRT^n and also shows that $\mathsf{regRT}^n \geq_{\mathrm{c}} \mathsf{RT}^n_k$. The following definition can be used for computably reducing RT^n_k to regRT^n (for $n \geq 2$ and $k \in \mathbf{N}^+$). Given $c : [\mathbf{N}]^n \to k$, with $k \in \mathbf{N}^+$, let $c^+ : [\mathbf{N}]^n \to k$ be as follows:

$$c^+(x_1, \dots, x_n) = \begin{cases} 0 & \text{if } x_1 \leq k, \\ c(x_1, \dots, x_n) & \text{otherwise.} \end{cases}$$

We can thus state the following Proposition.

Proposition 7. *For any* $n \geq 2$ *and* $k \in \mathbf{N}^+$*,* $\mathsf{RT}^n_k \leq_{\mathrm{c}} \mathsf{regRT}^n$*.*

Note that by $\mathsf{HT}^{=n}_k[\mathsf{ap}] \leq_{\mathrm{sW}} \mathsf{RT}^n_k$ (see [5]), the above also implies $\mathsf{HT}^{=n}_k[\mathsf{ap}] \leq_{\mathrm{c}} \mathsf{regRT}^n$ for any $n \geq 2$ and $k \in \mathbf{N}^+$.

Equivalents of ACA_0. Proposition 6, coupled with the fact that $\mathsf{HT}^{=3}_2[\mathsf{ap}]$ implies ACA_0 (Theorem 3.3 in [5]), yields the following corollary.

Corollary 2. $\lambda\mathsf{regHT}^{=3}[\mathsf{ap}]$ *implies* ACA_0 *over* RCA_0*.*

Proof. From Theorem 3.3 in [5] and Proposition 6 above. □

We have the following reversal, showing that $\lambda\mathsf{regHT}^{=3}[\mathsf{ap}]$ is a "weak yet strong" restriction of Taylor's Theorem in the sense of [4]. The result is analogous to the implication from RT^n_k to $\mathsf{HT}^{=n}_k$ (see [5]).

Theorem 3. *Let $n \in \mathbf{N}^+$. ACA_0 proves $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}]$. Moreover, $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}] \leq_{\mathrm{sW}} \mathsf{regRT}^n$.*

Proof. We give the proof for $n = 2$ for ease of readability.

Let $f : \mathbf{N} \to \mathbf{N}$ be λ-regressive. Let $g : [\mathbf{N}]^2 \to \mathbf{N}$ be defined as follows: $g(x, y) = f(2^x + 2^y)$. The function g is regressive since f is λ-regressive. Recall that regRT^2 is provable in ACA_0. Let $H \subseteq \mathbf{N}^+$ be a min-homogeneous solution to regRT^2 for g. Let $\hat{H} = \{2^h : h \in H\}$. Obviously $\hat{H}$ is apart. It is easy to see that $\mathrm{FS}^{=2}(\hat{H})$ is min-term-homogeneous for f: let $2^h < 2^{h'} < 2^{h''}$ be elements of $\hat{H}$. Then

$$f(2^h + 2^{h'}) = g(h, h') = g(h, h'') = f(2^h + 2^{h''}).$$

□

We do not know if the reduction in Theorem 3 can be reversed.

We next show that $\lambda\mathsf{regHT}^{=2}[\mathsf{ap}]$ already implies Arithmetical Comprehension. The proof is reminiscent of the proof that $\mathsf{HT}^{\leq 2}_2[\mathsf{ap}]$ implies ACA_0 in [5], but the use of λ-regressive colourings allows us to avoid the parity argument used in that proof. As happens in the proofs of independence of combinatorial principles from Peano Arithmetic [25], in the present setting the use of regressive colourings simplifies the combinatorics.

Let RAN be the Π^1_2 principle stating that for every injective function $f : \mathbf{N} \to \mathbf{N}$ the range of f (denoted by $\rho(f)$) exists. It is well-known that RAN is equivalent to ACA_0 (see [31]).

Theorem 4. *Let $n \geq 2$. $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}]$ implies ACA_0 over RCA_0. Moreover,*

$$\lambda\mathsf{regHT}^{=n}[\mathsf{ap}] \geq_{\mathrm{W}} \mathsf{RAN}.$$

Proof. We give the proof for $n = 2$. The easy adaptation to larger values is left to the reader.

Let $f : \mathbf{N} \to \mathbf{N}$ be injective. For technical convenience and without loss of generality we assume that f never takes the value 0. We show, using $\lambda\mathsf{regHT}^{=2}[\mathsf{ap}]$, that $\rho(f)$ (the range of f) exists.

Define $c : \mathbf{N} \to \mathbf{N}$ as follows. If m is a power of 2 then $c(m) = 0$. Else $c(m) =$ the unique x such that $x < \lambda(m)$ and there exists $j \in [\lambda(m), \mu(m))$ such that $f(j) = x$ and for all $j < j' < \mu(m)$, $f(j') \geq \lambda(m)$. If no such x exists, we set $c(m) = 0$.

Intuitively c checks whether there are values below $\lambda(m)$ in the range of f restricted to $[\lambda(m), \mu(m))$. If any, it returns the latest one, i.e., the one obtained as image of the maximal $j \in [\lambda(m), \mu(m))$ that is mapped by f below $\lambda(m)$). In other words, x is the "last" element below $\lambda(m)$ in the range of f restricted to $[\lambda(m), \mu(m))$.

The function c is computable in f and λ-regressive.

Let $H = \{h_0 < h_1 < \cdots\} \subseteq \mathbf{N}^+$ be an apart solution to $\lambda\mathsf{regHT}^{=2}$ for c. Without loss of generality we can assume that $\lambda(h_0) > 1$, since H is apart. Let $h_i \in H$.

We claim that if $x < \lambda(h_i)$ and x is in the range of f then x is in the range of f restricted to $[0, \mu(h_{i+1}))$.

We prove the claim as follows. Suppose, by way of contradiction, that there exist $h_i \in H$ and $x < \lambda(h_i)$ such that $x \in \rho(f)$ but $x \notin f([0, \mu(h_{i+1}))$. Let b be the true bound for the elements in the range of f smaller than $\lambda(h_i)$, i.e., b is such that if $n < \lambda(h_i)$ and $n \in \rho(f)$, then $n < b$. The existence of b follows in RCA_0 from strong Σ^0_1-bounding (see [31], Exercise II.3.14):

$$\forall n \exists b \forall i < n(\exists j(f(j) = i) \to \exists j < b(f(j) = i)),$$

where we take $n = \lambda(h_i)$.

Let h_j in H be such that $h_j > h_{i+1}$ and $\mu(h_j) \geq b$. Such an h_j exists since H is infinite.

Then, by min-term-homogeneity of $\mathrm{FS}^{=2}(H)$, $c(h_i + h_{i+1}) = c(h_i + h_j)$. But by choice of h_i, x and h_j, and the definition of c, it must be the case that $c(h_i + h_{i+1}) \neq c(h_i + h_j)$. To see this, first note that, by apartness of H, the following equalities hold:

$$\lambda(h_i + h_{i+1}) = \lambda(h_i) = \lambda(h_i + h_j),\ \mu(h_i + h_{i+1}) = \mu(h_{i+1}),\ \mu(h_i + h_j) = \mu(h_j).$$

Then observe that $c(h_i+h_j) > 0$: by hypothesis $f^{-1}(x) \in [\mu(h_{i+1}), b)$ (recall that f is injective), therefore x is a value of f below $\lambda(h_i+h_j)$ whose pre-image under f is in $[\lambda(h_i+h_j), \mu(h_i+h_j))$, i.e. in $[\lambda(h_i), \mu(h_j))$. Suppose now that $c(h_i + h_{i+1}) = z > 0$. Then, by definition of c, it must be the case that $z < \lambda(h_i+h_{i+1})$, i.e., $z < \lambda(h_i)$, and $f^{-1}(z)$ is in $[\lambda(h_i+h_{i+1}), \mu(h_i+h_{i+1}))$, i.e. in $[\lambda(h_i), \mu(h_{i+1}))$. This z cannot be the value of $c(h_i + h_j)$, since by hypothesis and by choice of b, we have $x < \lambda(h_i)$ and $f^{-1}(x)$ is in $[\mu(h_{i+1}), b)$, hence in $[\lambda(h_i + h_j), \mu(h_i + h_j))$. Thus z cannot be the value of f below $\lambda(h_i)$ with maximal pre-image under f in $[\lambda(h_i+h_j), \mu(h_i+h_j))$ as the definition of $c(h_i+h_j)$ requires, since $f^{-1}(z) < \mu(h_{i+1}) \leq f^{-1}(x)$ and f is injective. This concludes our reasoning by way of contradiction and hence establishes the claim that values in the range of f below $\lambda(h_i)$ appear as values of f applied to arguments smaller than $\mu(h_{i+1})$.

In view of the just established claim it is easy to see that the range of f can be decided computably in H as follows. Given x, pick any $h_i \in H$ such that $x < \lambda(h_i)$ and check whether x appears in $f([0, \mu(h_{i+1}))$. ☐

Theorem 4 for the case of $n = 2$ should be contrasted with the fact that $\mathsf{HT}^{=2}_2[\mathsf{ap}]$ follows easily from RT^2_2 and is therefore strictly weaker than ACA_0, while $\mathsf{HT}^{=3}_2[\mathsf{ap}]$ implies ACA_0 as proved in [5]. The situation matches the one among regRT^2, RT^3_2 and RT^2_2 (see Theorem 1).

The proof of Theorem 4 can be recast in a straightforward way to show that there exists a computable λ-regressive colouring such that all apart solutions to $\lambda\mathsf{regHT}^{=2}$ for that colouring compute the first Turing Jump $\emptyset'$. Analogously, the reduction can be cast in terms of the Π^1_2-principle $\forall X \exists Y (Y = (X)')$ expressing closure under the Turing Jump, rather than in terms of RAN.

The next theorem summarizes the implications over RCA_0 for the Regressive Hindman's theorems for sums of exactly n elements, compared with other prominent Ramsey-theoretic principles (see Figure 2).

Theorem 5. *The following are equivalent over* RCA_0.

1. ACA_0.
2. regRT^n*, for any fixed* $n \geq 2$.
3. RT^n_k*, for any fixed* $n \geq 3$, $k \geq 2$.
4. $\mathsf{HT}^{=n}_k[\mathsf{ap}]$*, for any fixed* $n \geq 3$, $k \geq 2$.
5. $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}]$*, for any fixed* $n \geq 2$.

Proof. The equivalences between point (1), (2) and (3) are as in Theorem 1. The equivalence of (1) and (4) is from Proposition 3.4 in [5]. Then the equivalence of (5) with points from (1) to (4) follows from Theorem 3, Theorem 4 and Proposition 6. ☐

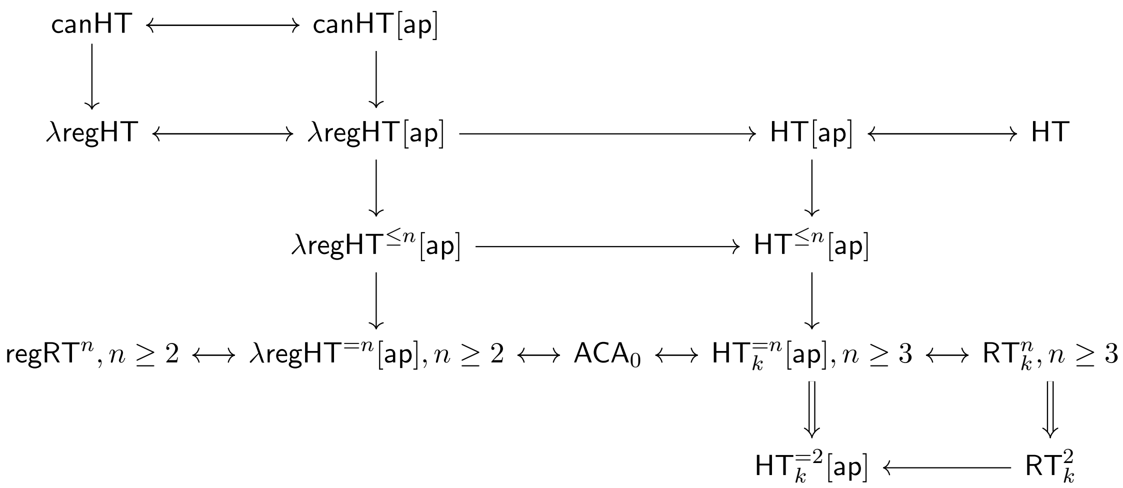

Figure 2: Implications over RCA_0. Double arrows indicate strict implications. The equivalence of $\mathsf{canHT}[\mathsf{ap}]$ and canHT is from Theorem 2. The implication from canHT to $\lambda\mathsf{regHT}$ is from Proposition 3 and similarly for the versions with apartness. The equivalence between $\lambda\mathsf{regHT}$ and $\lambda\mathsf{regHT}[\mathsf{ap}]$ is from Proposition 4. The implication from $\lambda\mathsf{regHT}$ to HT is from Proposition 5. The implication from $\lambda\mathsf{regHT}^{\leq n}[\mathsf{ap}]$ to $\mathsf{HT}^{\leq n}[\mathsf{ap}]$ is from Proposition 6. The equivalence of $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}]$ with ACA_0 (for $n \geq 2$) is from Theorem 3 and Theorem 4. The equivalence of $\mathsf{HT}_k^{=n}[\mathsf{ap}]$ with ACA_0 (for $n \geq 3, k \geq 2$) is from [5]. The equivalence of RT_k^n with ACA_0 (for $n \geq 3, k \geq 2$) is a classical result of Simpson, see Theorem III.7.6 in [31].

In terms of computable reductions we have the following, for $n \geq 2$ and $k \in \mathbf{N}^+$:

$$\mathsf{RT}_2^{2n-1} \geq_{\mathrm{sW}} \mathsf{regRT}^n \geq_{\mathrm{c}} \mathsf{RT}_k^n,$$

where the first inequality is due to Hirst [20] and the second inequality is from Proposition 7. Furthermore we have that

$$\mathsf{regRT}^n \geq_{\mathrm{W}} \lambda\mathsf{regHT}^{=n}[\mathsf{ap}] \geq_{\mathrm{c}} \mathsf{HT}_k^{=n}[\mathsf{ap}],$$

from Theorem 3 and Proposition 6.

Moreover, whereas $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}] \geq_{\mathrm{W}} \mathsf{RAN}$ for any $n \geq 2$ (Theorem 4), we have that $\mathsf{HT}_k^{=n}[\mathsf{ap}] \geq_{\mathrm{W}} \mathsf{RAN}$ only for $n \geq 3$ and $k \geq 2$ (by an easy adaptation of the proof of Theorem 3.3 in [5]). Also note that $\mathsf{RT}_k^n \geq_{\mathrm{sW}} \mathsf{HT}_k^{=n}[\mathsf{ap}]$ by a straightforward reduction (see [5]).

Some non-reducibility results can be gleaned from the above and known non-reducibility results from the literature. First, Dorais, Dzhafarov, Hirst, Mileti, and Shafer showed that $\mathsf{RT}_k^n \not\leq_{\mathrm{sW}} \mathsf{RT}_j^n$ (Theorem 3.1 of [11]). Then $\mathsf{RT}_k^n \not\leq_{\mathrm{W}} \mathsf{RT}_j^n$ was proved by Brattka and Rakotoniaina [3] and, independently, by Hirschfeldt and Jockusch [18]. Patey further improved this result by showing that the computable reduction does not hold either [29]; i.e., $\mathsf{RT}_k^n \not\leq_{\mathrm{c}} \mathsf{RT}_j^n$ for all $n \geq 2$, $k > j \geq 2$. We can derive, among others, the following corollaries.

Corollary 3. *For each $n, k \geq 2$, $\mathsf{regRT}^n \not\leq_{\mathrm{c}} \mathsf{RT}_k^n$.*

Proof. From Proposition 7 we know that $\mathsf{RT}_{k+1}^n \leq_{\mathrm{c}} \mathsf{regRT}^n$, so if we had $\mathsf{regRT}^n \leq_{\mathrm{c}} \mathsf{RT}_k^n$ we could transitively obtain $\mathsf{RT}_{k+1}^n \leq_{\mathrm{c}} \mathsf{RT}_k^n$, hence contradicting the fact that $\mathsf{RT}_{k+1}^n \not\leq_{\mathrm{c}} \mathsf{RT}_k^n$ proved by Patey [29]. □

Corollary 4. $\mathsf{RT}^3_3 \not\leq_{\mathrm{c}} \lambda\mathsf{regHT}^{=2}[\mathsf{ap}]$.

Proof. It is known from [29] that $\mathsf{RT}^3_3 \not\leq_{\mathrm{c}} \mathsf{RT}^3_2$. On the other hand $\lambda\mathsf{regHT}^{=2}[\mathsf{ap}] \leq_{\mathrm{W}} \mathsf{RT}^3_2$, since $\lambda\mathsf{regHT}^{=2}[\mathsf{ap}] \leq_{\mathrm{W}} \mathsf{regRT}^2$ (Theorem 3) and $\mathsf{regRT}^2 \leq_{\mathrm{sW}} \mathsf{RT}^3_2$ (from the proof of Theorem 6.14 in [20]) and since the involved reducibilities satisfy the following inclusions and are transitive: $\leq_{\mathrm{sW}} \subseteq \leq_{\mathrm{W}} \subseteq \leq_{\mathrm{c}}$. □

As proved in [5], restrictions of Hindman's Theorem have intriguing connections with the so-called Increasing Polarized Ramsey's Theorem for pairs IPT^2_2 of Dzhafarov and Hirst [10]. For example, $\mathsf{HT}^{=2}_2[\mathsf{ap}] \geq_{\mathrm{W}} \mathsf{IPT}^2_2$ (Theorem 4.2 in [5]). By this result and Proposition 6 we have the following corollary.

Corollary 5. $\mathsf{IPT}^2_2 \leq_{\mathrm{c}} \lambda\mathsf{regHT}^{=2}[\mathsf{ap}]$.

Note that IPT^2_2 is the strongest known lower bound for $\mathsf{HT}^{=2}_2[\mathsf{ap}]$ in terms of reductions. Some interesting lower bounds on $\mathsf{HT}^{=2}$ *without* apartness are in [8]. We haven't investigated $\lambda\mathsf{regHT}^{=n}$ *without* the apartness condition; we conjecture that the lower bounds on $\mathsf{HT}^{=2}$ (*without* apartness) from [8] can be adapted to $\lambda\mathsf{regHT}^{=2}$.

4.2 Bounded regressive Hindman's Theorem and Well-ordering Principles

Let $(\mathcal{X}, <_{\mathcal{X}})$ be a linear ordering. We denote by $\omega^{\mathcal{X}}$ the collection of finite sequences of the form $(x_1, x_2, \ldots, x_s)$ such that, for all $i \in [1, s]$, $x_i \in \mathcal{X}$ and, for all $i, j \in [1, s]$ such that $i < j$, $x_i \geq_{\mathcal{X}} x_j$. We call the x_is the *components* of σ. We denote by $|\sigma|$ the *length of* σ, i.e. $|\sigma| = s$. We order $\omega^{\mathcal{X}}$ lexicographically. Then, if $\sigma, \tau \in \omega^{\mathcal{X}}$ and σ strictly extends τ, we have $\sigma > \tau$. If j is least such that $x_j = \sigma(j) \neq \tau(j) = x'_j$ and $x_j >_{\mathcal{X}} x'_j$ then $\sigma > \tau$. Otherwise $\tau \geq \sigma$.

If $(\mathcal{X}, <_{\mathcal{X}})$ is a well-ordering, then the just defined ordering on $\omega^{\mathcal{X}}$ is also a well-ordering (provably in sufficiently strong theories). In this case we can then identify an element $\sigma = (x_1, x_2, \ldots, x_s)$ of $\omega^{\mathcal{X}}$ with the ordinal $\omega^{x_1} + \omega^{x_2} + \cdots + \omega^{x_s}$. The lexicographic ordering of $\omega^{\mathcal{X}}$ coincides with the usual ordering of ordinals in Cantor Normal Form.

The well-ordering preservation principle (or well-ordering principle) for base-ω exponentiation is the following Π^1_2-principle:

$$\forall \mathcal{X}(\mathrm{WO}(\mathcal{X}) \to \mathrm{WO}(\omega^{\mathcal{X}})),$$

where $\mathrm{WO}(Y)$ is the standard Π^1_1-formula stating that Y is a well-ordering. We abbreviate the above well-ordering preservation principle by $\mathsf{WOP}(\mathcal{X} \mapsto \omega^{\mathcal{X}})$.

It is known that $\mathsf{WOP}(\mathcal{X} \mapsto \omega^{\mathcal{X}})$ is equivalent to ACA_0 by results of Girard and Hirst (see [21]). A direct combinatorial proof from RT^3_3 to $\mathsf{WOP}(\mathcal{X} \mapsto \omega^{\mathcal{X}})$ in RCA_0 was given by Carlucci and Zdanowski [7] (the proof yields a Weihrauch reduction as clear by inspection). On the other hand, we proved in Theorem 4 that, for any $n \geq 2$, $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}]$ implies ACA_0 over RCA_0. Therefore in RCA_0 we have that, for $n \geq 2$, $\lambda\mathsf{regHT}^{=n}$ with apartness implies $\mathsf{WOP}(\mathcal{X} \mapsto \omega^{\mathcal{X}})$. However, we can not use the same arguments to derive an analogous chain of reductions. In the next theorem we show that $\mathsf{WOP}(\mathcal{X} \mapsto \omega^{\mathcal{X}})$ is Weihrauch-reducible to $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}]$, while also giving a direct proof of the implication in RCA_0. This result relates for the first time, to the best of our knowledge, Hindman-type theorems and transfinite well-orderings.

To make the principle $\mathsf{WOP}(\mathcal{X} \mapsto \omega^{\mathcal{X}})$ amenable to questions of reducibility it is natural to consider its contrapositive form: an *instance* is an infinite descending sequence in $\omega^{\mathcal{X}}$ and a *solution* is an infinite descending sequence in $\mathcal{X}$ (in fact, one might require that the solution consists of terms already occurring as subterms of the elements of the instance sequence, as is the case in our argument below).

We briefly describe the idea in the proof of Theorem 6 below. Let $\mathcal{X}$ be a linear ordering. Let $\alpha = (\alpha_i)_{i\in\mathbf{N}}$ be an infinite decreasing sequence in $\omega^{\mathcal{X}}$. We show, using $\lambda\mathsf{regHT}^{=2}[\mathsf{ap}]$, that there exists an infinite decreasing sequence in $\mathcal{X}$. The proof uses ideas from our proof of the fact that $\lambda\mathsf{regHT}^{=2}$ with apartness implies ACA_0 (Theorem 4) adapted to the present context, based on the following analogy between deciding the first Turing jump $\emptyset'$ and computing an infinite descending sequence in $\mathcal{X}$. Given an enumeration of $\emptyset'$ and a number n, RCA_0 knows that there is a b such that all numbers in $\emptyset'$ below n appear within b steps of the enumeration, but is not able to compute this b. Similarly, given an ordinal α in an infinite decreasing sequence in $\omega^{\mathcal{X}}$, RCA_0 knows that there is a b such that if a term of α ever decreases, it will do so by the b-th term of the infinite descending sequence, but is unable to compute such a b. More precisely, while one can computably run through the given infinite descending sequence to find the first point at which an exponent of a component of α is decreased, we can not locate computably the leftmost such component. An appropriately designed colouring will ensure that the information about such a b can be read off from the elements of any apart solution to $\lambda\mathsf{regHT}^{=n}$.

We start with the following simple Lemma. For technical convenience in the rest of this section we index infinite sequences and sets starting from 1.

Lemma 5. *The following is provable in* RCA_0*: If* $\alpha = (\alpha_i)_{i\in\mathbf{N}^+}$ *is an infinite descending sequence in* $\omega^{\mathcal{X}}$*, then*

$$\forall n\ \exists n'\ \exists m \leq |\alpha_n|\ \big(n' > n\ \wedge m \leq |\alpha_{n'}|\ \wedge \alpha_{n,m} >_{\mathcal{X}} \alpha_{n',m}\big),$$

where $\alpha_{i,j}$ *denotes the* j*-th component of* α_i *for* $j \in [1, |\alpha_i|]$ *and is otherwise undefined.*

Proof. Assume by way of contradiction that the statement is false, as witnessed by n, and recall that for any distinct $\sigma, \tau \in \omega^{\mathcal{X}}$, we have $\sigma < \tau$ if and only if either (1.) σ is a proper initial segment of τ, or (2.) there exists m such that $\sigma(m) <_{\mathcal{X}} \tau(m)$ and $\sigma(m') = \tau(m')$ for each $m' < m$. Then we can show that:

$$\forall p\ (p \geq n \rightarrow (\alpha_{p+1} \text{ is a proper initial segment of both } \alpha_p \text{ and } \alpha_n))$$

by Δ^0_1-induction.

The case $p = n$ is trivial, since $\alpha_n >_{\mathcal{X}} \alpha_{n+1}$ and (2.) cannot hold by assumption.

For $p > n$, by induction hypothesis we know that α_p is a proper initial segment of α_n. Since $\alpha_{p+1} <_{\mathcal{X}} \alpha_p$, α_{p+1} must be a proper initial segment of α_p, otherwise the leftmost component differing between α_{p+1} and α_p – i.e. the component of α_{p+1} with index m witnessing (2.) – would contradict our assumption, for we would have $m \leq |\alpha_p|$ and $\alpha_{p+1,m} <_{\mathcal{X}} \alpha_{p,m} = \alpha_{n,m}$.

So α_{p+1} must be a proper initial segment of α_p and, by our assumption, it must be a proper initial segment of α_n as well.

The previous statement implies that:

$$\forall p\ (p \geq n \rightarrow |\alpha_p| > |\alpha_{p+1}|),$$

which contradicts $\mathrm{WO}(\omega)$. This concludes the proof. □

Theorem 6. *Let* $n \geq 2$. $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}]$ *implies* $\mathsf{WOP}(\mathcal{X} \mapsto \omega^{\mathcal{X}})$ *over* RCA_0*. Moreover,* $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}] \geq_{\mathrm{W}} \mathsf{WOP}(\mathcal{X} \mapsto \omega^{\mathcal{X}})$*.*

Proof. Let $\alpha = (\alpha_n)_{n\in\mathbf{N}^+}$ be an infinite descending sequence in $\omega^{\mathcal{X}}$. We say that $\alpha_{n,m}$ is *decreasible* if there exists a $n' > n$ such that $\alpha_{n',m} <_{\mathcal{X}} \alpha_{n,m}$. In this case we say that $\alpha_{n',m}$ *decreases* $\alpha_{n,m}$. With this terminology Lemma 5 says that RCA_0 knows that for all $i \geq 1$ there exists $j \in [1, |\alpha_i|]$ such that $\alpha_{i,j}$ is decreasible. If $\alpha_{n',m}$ decreases $\alpha_{n,m}$ and no $\alpha_{k,m}$ with $k < n'$ decreases $\alpha_{n,m}$ we call $\alpha_{n',m}$ the *least decreaser* of $\alpha_{n,m}$.

Now suppose that $f : \mathbf{N} \to \mathbf{N}$ is a function with the following property:

Property P: For all $i \in \mathbf{N}^+$ for all $j \in [1, |\alpha_i|]$ if $\alpha_{i,j}$ is decreasible then $\alpha_{i,j}$ is decreased by $\alpha_{k,j}$ for some $k \leq f(i)$.

We first show that given such an f we can compute (in f and α) an infinite descending sequence $(\sigma_i)_{i\in\mathbf{N}^+}$ in $\mathcal{X}$ as follows.

Step 1. Pick the leftmost decreasible component of α_1 (which exists by Lemma 5). This can be done by inspecting all components in α up through $\alpha_{f(1)}$, since f has Property P.

Let p_1 be the position of the leftmost decreasible component of α_1. Pick the smallest $d_1 \leq f(1)$ such that α_{d_1,p_1} decreases α_{1,p_1}. We set $\sigma_1 = \alpha_{d_1,p_1}$ and observe that all decreasible components in α_{d_1} occur at positions $\geq p_1$. Suppose otherwise and let $1 \leq p^* < p_1$ be such that α_{d_1,p^*} is decreasible. Let $d^* > d_1$ such that α_{d^*,p^*} decreases α_{d_1,p^*}. Then $\alpha_{d^*,p^*} <_{\mathcal{X}} \alpha_{d_1,p^*}$ by definition of decreasible. On the other hand, by choice of d_1 and p_1, and since $p^* < p_1$, it must be the case that $\alpha_{d_1,p^*} = \alpha_{1,p^*}$. Hence α_{1,p^*} is a decreasible component in α_1 on the left of position p_1, which contradicts the choice of p_1.

Step $i+1$ **(**$i > 0$**)**. Suppose d_i, p_i, σ_i are defined so that $\sigma_i = \alpha_{d_i,p_i}$, $(\sigma_j)_{1\leq j\leq i}$ is decreasing in $\mathcal{X}$ and all decreasible components in α_{d_i} occur at positions $\geq p_i$.

Pick the leftmost decreasible component in α_{d_i} (which exists by Lemma 5). This can be done by inspecting all components in α up to $\alpha_{f(d_i)}$, since f has Property P. Let $\alpha_{d_i,\ell}$ be the chosen component. Set $p_{i+1} = \ell$ and note that necessarily $p_{i+1} \geq p_i$.

Pick $d \leq f(d_i)$ minimal such that $\alpha_{d,p_{i+1}}$ decreases $\alpha_{d_i,p_{i+1}}$. Set $d_{i+1} = d$. Let $\sigma_{i+1} = \alpha_{d_{i+1},p_{i+1}}$. Obviously $\sigma_i >_{\mathcal{X}} \sigma_{i+1}$, since $\sigma_i = \alpha_{d_i,p_i} \geq \alpha_{d_i,p_{i+1}} >_{\mathcal{X}} \alpha_{d_{i+1},p_{i+1}} = \sigma_{i+1}$ (note that $p_i \leq p_{i+1}$).

We observe that also the last part of the inductive invariant is guaranteed, since no decreasible component in $\alpha_{d_{i+1}}$ occurs on the left of p_{i+1}. Suppose otherwise as witnessed by $1 \leq p^* < p_{i+1}$. Let $d^* > d_{i+1}$ such that α_{d^*,p^*} decreases α_{d_{i+1},p^*}. Then α_{d^*,p^*} also decreases α_{d_i,p^*} since $\alpha_{d_{i+1},p^*} = \alpha_{d_i,p^*}$, where the latter is due to the fact that α is decreasing and p^* is less than p_{i+1}, which is the position of the leftmost decreasible component in α_{d_i}. This contradicts the choice of p_{i+1}.

We next show how to obtain a function satisfying Property P from a solution of $\lambda\mathsf{regHT}^{=n}[\mathsf{ap}]$ for a suitable colouring. The argument is similar to the proof of Theorem 4.

For this purpose it is convenient to use a sequence β of all the components of the terms α_n in α, enumerated in order of appearance: more precisely, $(\beta_h)_{h\in\mathbf{N}^+}$ is the ordered sequence $\alpha_{1,1}, \alpha_{1,2}, \dots, \alpha_{1,|\alpha_1|}, \alpha_{2,1}, \alpha_{2,2}, \dots, \alpha_{2,|\alpha_2|}, \dots$. This sequence is obviously easily computable from α. Formally we construct such a sequence by first defining a function $\iota : \mathbf{N}^+ \times \mathbf{N}^+ \to \mathbf{N}^+$ as follows: $\iota(n,m) = \sum_{1\leq k<n} |\alpha_k| + m$, for all $n \in \mathbf{N}^+$ and all $m \in [1, |\alpha_n|]$, while $\iota(n,m) = 0$ in all other cases. We correspondigly fix functions $t : \mathbf{N}^+ \to \mathbf{N}^+$ and $p : \mathbf{N}^+ \to \mathbf{N}^+$ such that for each $n \in \mathbf{N}^+$ we have $\iota(t(n), p(n)) = n$. The sequence $(\beta_h)_{h\in\mathbf{N}^+}$ of all components appearing in α is then defined by setting $\beta_h = \alpha_{t(h),p(h)}$.

Define $c : \mathbf{N} \to \mathbf{N}$ as follows: $c(x) =$ the unique $i < \lambda(x)$ satisfying the following conditions:

1. There exists j such that $\lambda(x) \leq j < \mu(x)$ and β_j is the least decreaser of β_i, and
2. For all j' such that $j < j' < \mu(x)$, if $\beta_{j'}$ is the least decreaser of $\beta_{i'}$ then $i' \geq \lambda(x)$.

If no such i exists, we set $c(x) = 0$.

The function c is computable in α and λ-regressive. Let $H = \{h_1 < h_2 < h_3 < \dots\}$ be an apart solution to $\lambda\mathsf{regHT}^{=n}$ for c. The following Claim ensures the existence of an $(\alpha \oplus H)$-computable function with Property P.

Claim 4. *For each $h_k \in H$ and each $\alpha_{\ell,m}$ such that $\iota(\ell,m) < \lambda(h_k)$, if there exists $\alpha_{\ell',m}$ such that $\alpha_{\ell',m}$ decreases $\alpha_{\ell,m}$ then there exists such an $\alpha_{\ell',m}$ with $\iota(\ell',m) < \mu(h_{k+n-1})$.*

Proof of Claim 4. Assume by way of contradiction that there is some $h_k \in H$ and some $\alpha_{\ell,m}$ with $\iota(\ell,m) = i < \lambda(h_k)$ such that $\alpha_{\ell,m}$ is decreasible but not by any $\alpha_{\ell',m}$ with $\iota(\ell',m) < \mu(h_{k+n-1})$.

Let b be such that if $\alpha_{\ell'',m}$ is decreasible and $\iota(\ell'',m) < \lambda(h_k)$, then there exists ℓ', m such that $\iota(\ell',m) < b$ and $\alpha_{\ell',m}$ decreases $\alpha_{\ell'',m}$. The existence of b can be proved in RCA_0 using the following instance of strong Σ^0_1-bounding (similarly as in the proof of Theorem 4):

$$\forall n \exists b \forall i < n(\exists j(\alpha_{t(j),p(j)} \text{ decreases } \alpha_{t(i),p(i)}) \to \exists j < b(\alpha_{t(j),p(j)} \text{ decreases } \alpha_{t(i),p(i)}).$$

Since H is infinite, there is an $h_{k'} \in H$ such that $h_{k'} > h_{k+n-1}$ and $\mu(h_{k'}) \geq b$. Then, by min-term-homogeneity, $c(h_k + \cdots + h_{k+n-1}) = c(h_k + \cdots + h_{k+n-2} + h_{k'})$. But by choice of h_k, $h_{k'}$ and the definition of c, we can show that $c(h_k + \cdots + h_{k+n-1}) \neq c(h_k + \cdots + h_{k+n-2} + h_{k'})$, yielding a contradiction.

To see this we reason as follows. First observe that, by apartness of H, the following identities hold:

$$\lambda(h_k + \cdots + h_{k+n-1}) = \lambda(h_k + \cdots + h_{k+n-2} + h_{k'}) = \lambda(h_k),$$

and

$$\mu(h_k + \cdots + h_{k+n-2} + h_{k'}) = \mu(h_{k'}).$$

Let $j \in [\mu(h_{k+n-1}), \mu(h_{k'}))$ be such that $\alpha_{t(j),p(j)}$ is the least decreaser of $\alpha_{\ell,m}$. Such a j exists by choice of $\alpha_{\ell,m}$, h_k and $h_{k'}$. In fact, by hypothesis, $\alpha_{\ell,m}$ is decreasible but not by any component with ι-index below $\mu(h_{k+n-1})$. By choice of h'_k the least decreaser of $\alpha_{\ell,m}$ must have ι-index smaller than $\mu(h_{k'})$, since $\iota(\ell,m) < \lambda(h_k)$.

First note that $c(h_k + \cdots + h_{k+n-2} + h_{k'})$ cannot be 0, since this occurs if and only if there is no $i^* < \lambda(h_k)$ such that for some $j^* \in [\lambda(h_k), \mu(h_{k'}))$, $\alpha_{t(j),p(j)}$ decreases $\alpha_{t(i^*),p(i^*)}$; but the latter is false by choice of h_k and $h_{k'}$.

If $c(h_k + \cdots + h_{k+n-1})$ takes some non-zero value $i^* < \lambda(h_k)$, then this same value cannot be taken by $c(h_k + \cdots + h_{k+n-2} + h_{k'})$ under our assumptions. If it were, it would mean that $\alpha_{t(i^*),p(i^*)}$ is decreased for the first time by some $\alpha_{t(j^*),p(j^*)}$ with $j^* < \mu(h_{k'})$ such that j^* is also maximal below $\mu(h_{k'})$ such that $\alpha_{t(j^*),p(j^*)}$ is the least decreaser of some $\alpha_{t(q),p(q)}$ with $q < \lambda(h_k)$. This is impossible since the least decreaser of $\alpha_{t(i^*),p(i^*)}$, by the hypothesis that $c(h_k + \cdots + h_{k+n-1}) = i^*$, occurs earlier in the sequence of the β_h's than the least decreaser of $\alpha_{t(i),p(i)}$ since, by the definition of c, it must be that $j^* < \mu(h_k + \cdots + h_{k+n-1})$ and the latter value, by apartness, equals $\mu(h_{k+n-1})$, as noted above. On the other hand, j is in $[\mu(h_{k+n-1}), \mu(h_{k'}))$, so that $j^* < j$. Thus j^* cannot be maximal below $\mu(h_{k'})$ such that $\alpha_{t(j^*),p(j^*)}$ is the least decreaser of some $\alpha_{\ell'',m}$ with $\iota(\ell'',m)$ below $\lambda(h_k)$, as required by the definition of c, since $\alpha_{t(j),p(j)}$ is such a least decreaser of $\alpha_{t(i),p(i)}$, and $i < \lambda(h_k)$.

This proves the Claim. □

Now it is sufficient to observe that the $(\alpha \oplus H)$-computable function f defined as follows has the Property P: on input n, pick the least k such that $\sum_{1 \leq n' \leq n} |\alpha_{n'}| < \lambda(h_k)$ and let $f(n)$ be the α-index of the $\mu(h_{k+n-1})$-th element in the sequence β of all components appearing in α, i.e., $f(n) = t(\mu(h_{k+n-1}))$. That this choice of f satisfies Property P is implied by Claim 4 above. This concludes the proof of the theorem. □

The proof of Proposition 1 in [7] shows that $\mathsf{WOP}(\mathcal{X} \to \omega^{\mathcal{X}}) \leq_{\mathrm{W}} \mathsf{RT}^3_3$. The proof of Theorem 6 can be adapted to show that $\mathsf{WOP}(\mathcal{X} \to \omega^{\mathcal{X}}) \leq_{\mathrm{W}} \mathsf{HT}^{=3}_2[\mathsf{ap}]$. Details will be reported elsewhere.

The main reductions between restrictions of HT, restrictions of $\lambda\mathsf{regHT}$ and other principles of interest are visualized in Figure 3.

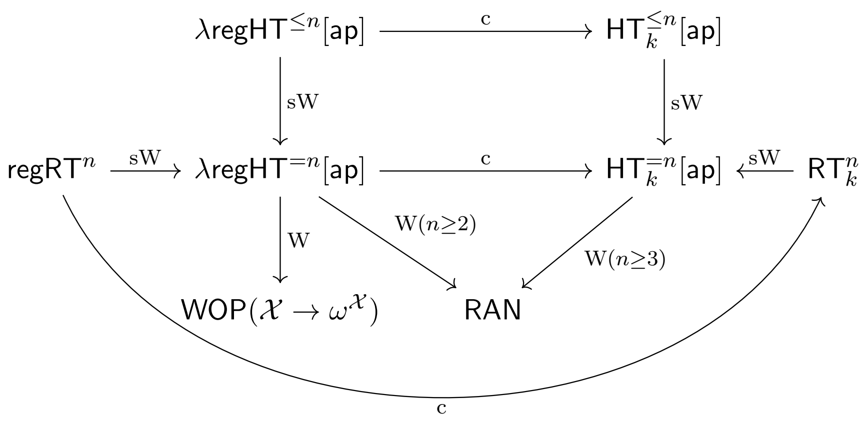

Figure 3: Diagram of reductions. $\mathsf{HT}^{\leq n}[\mathsf{ap}] \leq_{\mathrm{c}} \lambda\mathsf{regHT}^{\leq n}[\mathsf{ap}]$ is from Proposition 6. That the versions with sums of exactly n terms reduce to the corresponding versions for sums of $\leq n$ terms is a trivial observation. The reduction $\mathsf{WOP}(\mathcal{X} \to \omega^{\mathcal{X}}) \leq_{\mathrm{W}} \lambda\mathsf{regHT}^{=n}$ for $n \geq 2$ is Theorem 6. The reduction $\mathsf{RAN} \leq_{\mathrm{W}} \lambda\mathsf{regHT}^{=n}$ for $n \geq 2$ is Theorem 4. The reduction $\mathsf{RAN} \leq_{\mathrm{W}} \mathsf{HT}^{=n}_{k}[\mathsf{ap}]$ for $n \geq 3, k \geq 2$ is from [5]. The reduction $\mathsf{RT}^n_k \leq_{\mathrm{c}} \mathsf{regRT}^n$ is from Proposition 7. The reduction $\mathsf{HT}^{=n}_{h} \leq_{\mathrm{sW}} \mathsf{RT}^n_k$ is folklore.

5 Conclusion and open questions

In analogy with Kanamori-McAloon's Regressive Ramsey's Theorem [25] we obtained a Regressive Hindman's Theorem as a straightforward corollary of Taylor's Canonical Hindman's Theorem [32] restricted to a suitable class of regressive functions and relative to an appropriate variant of min-homogeneity. We studied the strength of this principle and of its restrictions in terms of provability over RCA_0 and computable reductions.

In particular we showed that the seemingly weakest (non-trivial) restriction of our Regressive Hindman's Theorem ($\lambda\mathsf{regHT}^{=2}$), with a natural apartness condition on the solution set, is equivalent to ACA_0. This restriction ensures that sums of two numbers from the solution set get the same colour if they have the same minimum term. For the restrictions of the standard Hindman's Theorem to sums of exactly n elements, the level of ACA_0 is reached only when we consider sums of exactly 3 elements. This situation is analogous to that of regRT^2 when compared to RT^3_2. Furthermore, we proved that the well-ordering preservation principle that characterizes ACA_0 ($\mathsf{WOP}(\mathcal{X} \to \omega^{\mathcal{X}})$) is Weihrauch-reducible to $\lambda\mathsf{regHT}^{=2}$ with apartness.

Many open questions remain concerning the strength of the Regressive Hindman's Theorem, of its restrictions, and of related principles. Here are some natural ones.

Question 1. *What are the optimal upper bounds for* canHT*, for* $\lambda\mathsf{regHT}$ *and for* $\lambda\mathsf{regHT}^{\leq n}$*?*

Question 2. *Is* $\lambda\mathsf{regHT}$ *implied by/reducible to* HT *(and similarly for bounded versions)?*

Question 3. *What is the strength of* $\lambda\mathsf{regHT}^{=2}$ *without apartness? More generally, how do the bounded Regressive Hindman's Theorems behave with respect to apartness?*

Question 4. *Can the reductions in Proposition 6 and Theorem 6 be improved to stronger reductions?*

Very recently, Hirschfeldt and Reitzes [19] investigated Hindman-type variants of the Thin Set Theorem which, as is the case for our Regressive Hindman's Theorem, deals with colourings with unboundedly many colours. It would be interesting to investigate possible relations between the two families.

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